Rainbow Vacua of Colored Higher Spin (A)dS$_3$ Gravity

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Abstract: We study the color-decoration of higher spin (anti)-de Sitter gravity in three dimensions. We show that the rainbow vacua, which we found recently for the colored gravity theory, also exist in the colored higher spin theory. The color singlet spin-two plays the role of metric. The difference is that when spontaneous breaking of color symmetry takes place, the Goldstone modes of massless spin-two combine with all other spins and become the maximal depth partially massless fields of the highest spin in the theory.

Contents

Whenever a theory appears to you as the only possible one,
take this as a sign that you have neither understood the theory
nor the problem which the theory was intended to solve.
Carl Popper, 'Objective Knowledge: Evolutionary Approach' (1972)

1 Introduction

In recent years, higher spin gravity had much progress related to the AdS/CFT duality [1, 2]. In particular, the breakthroughs in three dimensional theories took place in several places consecutively: first, the asymptotic symmetry of higher spin gravity has been identified as nonlinear W-algebras [3, 4], then it led to the conjecture of the W_N minimal models as dual CFTs [5]. Blackhole-like exact solutions were constructed [6]. Most of these results are about the $hs(\lambda) \oplus hs(\lambda)$ Chern-Simons HS gravity or the Prokushkin-Vasiliev (PV) theory [7] which contains the former as the gauge sector. Many variant models of three-dimensional higher spin gravity have been considered later on and many other interesting features were discovered (see [8–13] for a non-complete list of references).

In the companion paper [14], we proposed an extension of the three-dimensional gravity to multi-graviton system, which we called the color-decorated three-dimensional gravity. The purpose of this paper is to extend this analysis to higher-spins. More precisely, we consider the Chern-Simons formulation of higher-spin (A)dS gravity, leaving aside the matter coupling issue of the PV theory. In fact, the possibility of color decoration appears as a rather natural generalization in the Vasiliev's approach to higher-spin gravity, where the higher spin algebra plays the key role for the consistency of the theory and the color decoration is one of the simplest extension of the algebra. This was first pointed out in [15, 16, 18]. Actually, all the Vasiliev's nonlinear higher spin equations can be consistently color decorated with the same mechanism. Judging from this and also from the dual CFT considerations, we anticipate the issue of color decoration might be more consequential in the context of higher-spin gravity.

We analyze the colored higher spin gravity in three dimensions and show, in particular, that the salient aspect of the colored gravity persists in the higher spin extension: when the higher spin gravity is color decorated, there appears a staircase potential with a number of extrema. Each of these extrema provides an (A)dS background solution with a different cosmological constant. For $SU(N)$ color symmetry, there exist $[\frac{N+1}{2}]$ different vacua — henceforth, referred to as *rainbow vacua* — which spontaneously break the color symmetry down to $SU(N-k) \times SU(k)$ $(k = 0, 1, \ldots, [\frac{N-1}{2}])$. When this symmetry breaking happens, the goldstone modes — the spin two fields corresponding to the broken part of the symmetries — combine with all other spins and become a long spectrum. For the models of higher spin gravity involving massless spins up to M, that is the $\mathfrak{gl}_M \oplus \mathfrak{gl}_M$ Chern-Simons model, the symmetry broken part of the spectrum forms maximal depth partially massless spin M. The degrees of freedom (DoF) of this field are as many as those of self-dual massive spin M up to a scalar one. Therefore, for a generic model of colored higher spin gravity — involving arbitrary higher spins — we end up with a rather exotic spectrum: maximal depth partially massless fields of infinite spin. In three dimensions, the (partially) massless fields have only boundary DoF, but in the limit of infinite spin, the dimension of phase space becomes as large as that of a bulk propagation. The physical meaning of this intriguing result is not yet clear to the authors, and we shall revisit this issue in the forthcoming paper. We would like to emphasize that the color-decorated higher spin gravity, or any variant/generalization of the latter, is a plausible bulk theory when considering a matrix-type free CFT and looking for its bulk dual in the context of AdS/CFT. In this set-up, the rainbow vacua and the spontaneous symmetry breaking are generic and unneglectable phenomena.

The organization of the paper is as follows. In Section 2, we review how the consistent color decoration works in the models of higher spin gravity. In Section 3, the rainbow vacua with different cosmological constants are identified. In Section 4, we expand the theory around one of the rainbow vacua by solving the torsionless condition. In Section 5, we analyze the spectrum resulting from the symmetry breaking and show it corresponds to the maximal depth partially massless field. In Section 6, we provide an account for the partially massless representations in three dimensions. Finally, Section 7 contains further discussion.

2 Color Decoration of Higher-Spin (A)dS$_3$ Gravity

2.1 Color Decoration

We first recapitulate how one can consistently color-decorate a given higher-spin theory, extending our previous work [14].

Suppose we are given an uncolored (higher-spin) gravity theory, defined either by an action or by a set of field equations. Assume that elementary fields take values in an (higher-spin) isometry Lie algebra $\mathfrak{g}_i$. The idea is that we color-decorate this theory can by attaching Chan-Paton factors to these fields, which amounts to requiring the fields to take values in the tensor product algebra $\mathfrak{g}_i \otimes \mathfrak{g}_c$. The $\mathfrak{g}_c$ is the color symmetry algebra. However, though we may start with Lie algebras $\mathfrak{g}_i$ and $\mathfrak{g}_c$, the tensor product algebra does not automatically provide a Lie algebra $\mathfrak{g}_i \otimes \mathfrak{g}_c$ because the anticommutators are not defined. This point should be clear from

$$[\, M_X \otimes \boldsymbol{T}_I, M_Y \otimes \boldsymbol{T}_J \,] = \frac{1}{2}\,[\, M_X, M_Y \,] \otimes \{\, \boldsymbol{T}_I, \boldsymbol{T}_J \,\} + \frac{1}{2}\,\{\, M_X, M_Y \,\} \otimes [\, \boldsymbol{T}_I, \boldsymbol{T}_J \,]\,. \tag{2.1}$$

We conclude that, if an associative product can be defined in $\mathfrak{g}_i$ and $\mathfrak{g}_c$, the color-decoration through the Chan-Paton factor can be achieved.

One can always take $\mathfrak{g}_c = \mathfrak{u}(N)$ as color symmetry, and hence the associativity of $\mathfrak{g}_c$ is satisfied. However, the relevant isometry algebras $\mathfrak{so}(d,2)$ or $\mathfrak{so}(d+1,1)$ for (A)dS$_d$ space do not have an associative structure for general d. The way out of this problem is to consider a larger algebra $\mathfrak{g}_i$ which contains the isometry algebra. In this way, the $\mathfrak{g}_i$ would contain more generators, and so the corresponding uncolored theory would involve more fields than the pure (A)dS$_d$ Einstein gravity. For instance, in the previous work [14], we considered the three-dimensional (anti)-de Sitter gravity, where the original isometry algebra $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ was first extended to $\mathfrak{g}_i = \mathfrak{gl}_2 \oplus \mathfrak{gl}_2$. The other generators than those of the (A)dS$_3$ isometries corresponded to the spin-one field whose dynamics is described by the Chern-Simons action.

The higher-spin theories are particularly suited for the color-decoration. The higher-spin algebra in which the higher-spin fields take values is typically an associative algebra unless one deliberately truncates the theory to the so-called minimal spectrum, containing only spins of even integers. In fact, the color-decoration necessarily requires fields of odd integer spins in the spectrum (at least spin-one for the pure Chern-Simons (A)dS$_3$ gravity as studied in the previous work [14]). As such, it is not possible to truncate the spectrum of the colored higher-spin theory to even spins only.

It was also noticed in [16] that including fermion generators necessarily requires non-trivial color algebra, therefore realistic models of higher-spin theory in four dimensions should be given by color-decorated theories, possibly with additional color symmetry breaking pattern that leaves only one massless graviton in the spectrum. This is not, however, our concern here.

2.2 Color-Decorated (A)dS$_3$ Higher-Spin Theory

In this work, we shall consider the simplest class of higher-spin (A)dS$_3$ theory and study their color-decoration. The theory we shall study is the colored version of the Chern-Simons formulation of the higher-spin (A)dS$_3$ theory whose gauge algebra is given by the infinite-dimensional algebra labelled by a continuous parameter λ:

$$\mathfrak{g}_i = hs(\lambda) \oplus hs(\lambda)\,. \tag{2.2}$$

To render the conceptual problem simpler, we shall often restrict ourselves to the truncated algebras,

$$\mathfrak{g}_i = \mathfrak{gl}_M \oplus \mathfrak{gl}_M \qquad (M = 2, 4, \cdots) \tag{2.3}$$

or even to the simplest higher-spin algebra, the $M = 3$ case. The gauge algebra of spin-two, leading to $(A)dS_3$ Einstein gravity, corresponds to $M = 2$.

Let us further discuss aspects of the colored higher-spin (A)dS$_3$ gravity in the Chern-Simons formulation. The theory is based on the gauge field taking value in $\mathfrak{g}_i \otimes \mathfrak{g}_c$:

$$\mathcal{A} = A^{X,I}\, M_X \otimes \boldsymbol{T}_I\,, \tag{2.4}$$

where M_X are the generators of higher-spin algebra $\mathfrak{g}_i$ and the index X is the shorthand notation for the set of indices $A_1B_1,\ldots,A_rB_r$ of higher-spin generators. The color algebra $\mathfrak{g}_c$ is spanned by the generators $\boldsymbol{T}_I$. The gauge field strength is given by

$$\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = F^{X,I}\, M_X \otimes \boldsymbol{T}_I\,. \tag{2.5}$$

Up to this point, it is clear that all elements of the theory can be straightforwardly color-decorated by adjoining the Chan-Paton indices. Hence, if a theory can be defined solely in terms of $\mathcal{A}$ and $\mathcal{F}$ as in the Chern-Simons formulation — or together with more elements which can be equally well color-decorated — then the theory can be consistently generalized to a color-decorated version.

The action of the (A)dS$_3$ higher-spin theory is given in the Chern-Simons formulation by

$$S = \frac{\kappa}{4\pi} \int \mathrm{Tr}\Big(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3}\,\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}\Big)\,, \tag{2.6}$$

where the constant κ is the Chern-Simons level. Concretely, we take the full associative algebra on which the theory will be based on as

$$\mathfrak{g} = (hs(\lambda) \oplus hs(\lambda)) \otimes \mathfrak{u}(N) \ \ominus\ \mathrm{id} \otimes \boldsymbol{I}\,. \tag{2.7}$$

Note that we subtracted the $\mathrm{id} \otimes \boldsymbol{I}$ — where id and $\boldsymbol{I}$ are the centers of $hs(\lambda) \oplus hs(\lambda)$ and $\mathfrak{u}(N)$, respectively. This generator corresponds to an Abelian Chern-Simons field which does not interact with other fields in the theory. Since gauge field $\mathcal{A}$ takes value in the subspace of the tensor product space, the trace Tr of (2.6) should be defined in the tensor product space and it is given by the product of two traces as $\mathrm{Tr}(\mathfrak{g}_i \otimes \mathfrak{g}_c) = \mathrm{Tr}(\mathfrak{g}_i)\,\mathrm{Tr}(\mathfrak{g}_c)$.

3 Color Symmetry Breaking and Rainbow Vacua

In the previous work [14], we showed that the dynamics of color-decorated (A)dS$_3$ gravity rich enough to trigger a spontaneous color symmetry breaking as the colored spin-two fields take expectation values proportional to the color-singlet metric $g_{\mu\nu}$. By analyzing the scalar potential of these colored spin-two fields, we identified multiple local vacua – named as *rainbow vacua* — having different values of effective cosmological constants.

The existence of panoramic rainbow vacua is not a feature unique to the three-dimensional gravity theory. The feature actually holds true for color-decorated (higher-spin) (A)dS$_3$ theory for general D-dimensional spacetime. In the following, we shall make this point clear, while keeping full generality of our discussion.

We look for solutions to the equations of motion of the color-decorated (A)dS$_3$ higher-spin theory. We will uncover the rainbow vacua using the Chern-Simons formulation unique to (A)dS$_3$ space but we can extend the analysis to the Vasiliev formulation, which we relegate in the forthcoming work [17].

Among possible solutions, we focus on a vacuum configurtion for which the only non-trivial component is the spin two component of the one-form gauge field $\mathcal{A}$. All other components — the one-forms of the other spins and also the zero forms of the Vasiliev's equation— are set to zero. In this situation, the ansatz of the one-form gauge field takes the form:

$$\mathcal{A} = (\omega^{ab}\,\boldsymbol{I} + \boldsymbol{\Omega}^{ab})\,M_{ab} + \frac{1}{\ell}\,(e^a\,\boldsymbol{I} + \boldsymbol{E}^a)\,P_a\,, \tag{3.1}$$

where $\boldsymbol{\Omega}^{ab}$ and $\boldsymbol{E}^a$ take values in the Chan-Paton algebra, $\mathfrak{su}(N)$.

The idea for finding classical solutions is to require that the configuration does not lead to back-reaction onto the other components of the one-form field, viz. either spin-one or higher-spin and the zero-form field. It is straightforward to see that these requirements are met if we impose the conditions

$$\boldsymbol{\Omega}^{ab} \wedge \boldsymbol{\Omega}^{cd}\,\big\{M_{ab}, M_{cd}\big\} = 0\,, \quad \big\{\,\boldsymbol{\Omega}^{ab} \overset{\wedge}{,}\, \boldsymbol{E}^c\,\big\}\big\{M_{ab}, P_c\big\} = 0\,, \quad \boldsymbol{E}^a \wedge \boldsymbol{E}^b\,\big\{P_a, P_b\big\} = 0\,. \tag{3.2}$$

Were if these conditions not met, anti-commutators $\{M_{AB}, M_{CD}\}$ would contribute[1] and give rise to the generators of fields with other values of spin than two.

We take the ansatz, corresponding to the tensor product structure, for (3.2) as

$$\boldsymbol{\Omega}^{ab} = \Omega^{ab}\, \boldsymbol{X} \qquad \text{and} \qquad \boldsymbol{E}^a = E^a\, \boldsymbol{X}\,, \tag{3.3}$$

Here, $\boldsymbol{X}$ is a particular element of the $\mathfrak{su}(N)$ to be determined. We further consider a stronger ansatz

$$\Omega^{ab} = 0 \qquad \text{and} \qquad E^a = e^a\,. \tag{3.4}$$

Here, note that we could add a factor in the latter equation but it can be simply absorbed into the matrix $\boldsymbol{X}$. So, our final ansatz for the gauge field takes the form

$$\mathcal{A} = \omega^{ab}\, M_{ab}\, \boldsymbol{I} + \frac{1}{\ell}\, e^a\, P_a\, (\boldsymbol{I} + \boldsymbol{X})\,. \tag{3.5}$$

Now, the field equation of the Chern-Simons action is the zero curvature equation, $\mathcal{F} = 0$, where the associative product of $\mathfrak{g}_i \otimes \mathfrak{g}_c$ is assumed between $\mathcal{A}$'s. Other equations are trivially solved with the assumption that all other components than the spin-two vanish. With (3.5), the equation reads

$$\left(d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}\right) \boldsymbol{I} + \frac{\sigma}{\ell^2}\, e^a \wedge e^b\, (\boldsymbol{I} + \boldsymbol{X})^2 = 0\,, \qquad (de^a + \omega^a{}_c \wedge e^c)\, (\boldsymbol{I} + \boldsymbol{X}) = 0\,, \tag{3.6}$$

where σ is a $\pm$ sign, positive for AdS_3 and negative for dS_3.

The first equation in (3.6) clearly shows that the Chan-Paton gauge symmetry acts as a new source to the spacetime curvature. To see this more explicitly, let us decompose the above $N \times N$ matrix equations into the singlet $\mathfrak{u}(1)$ part and the $\mathfrak{su}(N)$ part, corresponding to the *closed string* and *open string* parts. The *closed string* part of the equation of motion reads

$$\mathrm{d}\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} - \frac{2}{(D-1)(D-2)}\, \Lambda\, e^a \wedge e^b = 0\,, \qquad \mathrm{d}e^a + \omega^a{}_c \wedge e^c = 0\,, \tag{3.7}$$

where the cosmological constant Λ (measured in unit of D-dimensional Newton's constant) is given by

$$\Lambda = -\frac{(D-1)(D-2)\,\sigma}{2\,N\,\ell^2}\, \mathrm{Tr}(\boldsymbol{I} + \boldsymbol{X})^2\,, \tag{3.8}$$

On the other hand, the *open string* part of the equation of motion is given by

$$e^a \wedge e^b \left(2\,\boldsymbol{X} + \boldsymbol{X}^2 - \frac{\mathrm{Tr}(2\,\boldsymbol{X} + \boldsymbol{X}^2)}{N}\, \boldsymbol{I}\right) = 0\,. \tag{3.9}$$

We note that, for nondegenerate e^a's, (3.9) is identical to the condition for a critical point of the scalar potential,

$$V(\boldsymbol{X}) = -\frac{(D-1)(D-2)\,\sigma}{N\,\ell^2}\, \mathrm{Tr}\left(\boldsymbol{I} + 3\,\boldsymbol{X}^2 + \boldsymbol{X}^3\right), \tag{3.10}$$

[1] We denote by M_{AB} generators of (A)dS algebra while M_{ab} and P_a are Lorenz and translation generators respectively.

and that the cosmological constant in (3.8) is given by $\Lambda = V(\boldsymbol{X})/2$ evaluated at extremum points. Putting $D = 3$, these are precisely what we fnd in the analysis of the color-decorated (A)dS$_3$ gravity in [14]. We conclude that the stairwell potential of the color-decorated (A)dS$_3$ gravity persists to exist in the generic higher-spin (A)dS$_D$ theory.

The complete set of solutions to (3.9) can be found by precisely the same way as in the color-decorated (A)dS$_3$ gravity analyzed in [14]. We simply state the result:

$$\boldsymbol{X}_k = \frac{N}{\mathrm{Tr}(\boldsymbol{Z}_k)}\,\boldsymbol{Z}_k - \boldsymbol{I}\,, \tag{3.11}$$

with

$$\boldsymbol{Z}_k = \begin{bmatrix} \boldsymbol{I}_{(N-k)\times(N-k)} & 0 \\ 0 & -\boldsymbol{I}_{k\times k} \end{bmatrix} \tag{3.12}$$

modulo a $SU(N)$ rotation. Extrema of the potential are labelled by $k = 0, 1, \ldots, [\frac{N-1}{2}]$. Moreover, $\boldsymbol{X}_k$ at the k-th vacuum, which shifts the gauge field $\mathcal{A}$ from the (A)dS$_3$ vacuum by (3.5), spontaneously breaks the $\mathfrak{sy}(N)$ color symmetry down to $\mathfrak{su}(N-k) \oplus \mathfrak{su}(k)$. We thus conclude by (3.5) that the colored spin-two fields act as the order parameter of the color symmetry breaking.

4 Metric-like Formulation

The appearance of the non-trivial potential with multiple extrema and the field contents at such vacua are better treatable in the metric form. The exact expression of the staircase potential can be computed along the same lines as in [14], so we shall not aim to repeat the derivation. Rather, we shall focus on the identification of perturbative spectrum around each extremum.

4.1 Decomposition of Associative Algebra

Once the frame-like formulation of higher spin gravity is given, rewriting it in the metric form is in principle possible. However, it is technically cumbersome, if not impossible, to get exact expressions in the metric-like variables. Here, we shall reformulate the Chern-Simons action to metric form but the reformulation will concern only the genuine gravity leaving aside all other field contents. For this task, it is convenient to decompose the algebra $\mathfrak{g}$ (2.7) into two pieces $\mathfrak{b}$ and $\mathfrak{c}$:

$$\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{c}\,, \qquad \mathrm{Tr}(\mathfrak{b}\,\mathfrak{c}) = 0\,, \tag{4.1}$$

in a proper way. The rule of the decomposition is that $\mathfrak{b}$ forms a subalgebra under which $\mathfrak{c}$ carries an adjoint representation, that is,

$$[\mathfrak{b}, \mathfrak{b}] \subset \mathfrak{b}\,, \qquad [\mathfrak{b}, \mathfrak{c}] \subset \mathfrak{c}\,. \tag{4.2}$$

Correspondingly to this decomposition of the algebra, we also split the one-form gauge field into two parts,

$$\mathcal{A} = \mathcal{B} + \mathcal{C}\,, \tag{4.3}$$

where $\mathcal{B}$ and $\mathcal{C}$ take values in $\mathfrak{b}$ and $\mathfrak{c}$ respectively. In terms of $\mathcal{B}$ and $\mathcal{C}$, the Chern-Simons action (2.6) reduces to

$$S = \frac{\kappa}{4\pi} \int \mathrm{Tr}\left(\mathcal{B} \wedge d\mathcal{B} + \frac{2}{3}\mathcal{B}\wedge\mathcal{B}\wedge\mathcal{B} + \mathcal{C}\wedge D_{\mathcal{B}}\mathcal{C} + \frac{2}{3}\mathcal{C}\wedge\mathcal{C}\wedge\mathcal{C}\right), \tag{4.4}$$

with $D_{\mathcal{B}}\mathcal{C} = d\mathcal{C} + \mathcal{B}\wedge\mathcal{C} + \mathcal{C}\wedge\mathcal{B}$. Properly selecting $\mathfrak{b}$ and $\mathfrak{c}$ from the full algebra $\mathfrak{g}$ (2.7), we can conveniently handle the manifest covariance with respect to diffeomorphism and non-Abelian gauge transformation.

4.2 Higher-Spin Algebra

In the uncolored case, the Chern-Simons theory with the algebra $hs(\lambda) \oplus hs(\lambda)$ can be interpreted as a theory of massless fields with spins $s = (1), 2, 3, 4, \ldots$, where spin 1 may or may not be there depending on whether $hs(\lambda)$ includes the identity or not. When the parameter λ takes an integer value, say M, then the $hs(M)$ develops an ideal. The quotient of $hs(M)$ by the ideal is the finite-dimensional algebra $\mathfrak{gl}_M$, whose generators can be organized as

$$\mathfrak{gl}_M = \mathrm{Span}\{\, J\,,\ J_a\,,\ J_{a_1a_2}\,,\ \ldots\,,\ J_{a_1\cdots a_{M-1}}\,\}\,, \tag{4.5}$$

whereas the other $\mathfrak{gl}_M$ is spanned by $\tilde{J}_{a_1\cdots a_n}$ $(n = 0, \ldots, M-1)$. As in $M = 2$ case, J and $\tilde{J}$ are the identities of two copies of $\mathfrak{gl}_M$. The choice of the basis is to make manifest that the Chern-Simons system with $\mathfrak{gl}_M \oplus \mathfrak{gl}_M$ describe a system of massless spins $2, 3, \ldots, M$.

In order to simplify the multi indices, let us employ the following notation,

$$\begin{aligned} A_{a(n)} \equiv A_{a\cdots a} \quad &\leftrightarrow \quad A_{a_1\cdots a_n}\,, \\ A_{a(n)}\, B_{a(m)} \quad &\leftrightarrow \quad A_{\{a_1\cdots a_n}\, B_{a_{n+1}\cdots a_{n+m}\}}\,, \end{aligned} \tag{4.6}$$

where the index operation $\{-\}$ means the *traceless* symmetrization:

$$\begin{aligned} &A_{\{a_1\cdots a_n}\, B_{a_{n+1}\cdots a_{n+m}\}} = A_{(a_1\cdots a_n}\, B_{a_{n+1}\cdots a_{n+m})} - (\mathrm{trace})\,, \\ &\eta^{a_1a_2}\, A_{\{a_1\cdots a_n}\, B_{a_{n+1}\cdots a_{n+m}\}} = 0\,. \end{aligned} \tag{4.7}$$

The algebraic structure of $hs(\lambda)$ is given by the product,

$$J_{a(m)}\, J_{b(n)} = \eta_{a(m),b(n)}\, J + \frac{1}{l!}\, c_{a(m),b(n),c(l)}\, J^{c(l)}\,, \tag{4.8}$$

where $c_{a(m),b(n),c(l)}$ are the structure constants and $\eta_{a(m),b(n)}$ is defined by

$$\eta_{a(m)}{}^{,b(n)} = \delta_{mn}\, \left(\delta_a^b\right)^n. \tag{4.9}$$

For our analysis, it is not necessary to identify all explicit forms of $c_{a(m),b(n),c(l)}$. It is sufficient to know the following product

$$J_{a(n)}\, J_b = c_n\, \eta_{ba}\, J_{a(n-1)} + \epsilon_{ab}{}^{c}\, J_{ca(n-1)} + c_{n+1}\, J_{ba(n)}\,, \tag{4.10}$$

where c_n is

$$c_n = \sqrt{\frac{n(\lambda^2 - n^2)}{2n+1}}\,. \tag{4.11}$$

Under the Hermitian conjugation, we get

$$(J_{a(n)},\tilde{J}_{a(n)})^\dagger=(-1)^n\begin{cases}(J_{a(n)},\tilde{J}_{a(n)}) & [\sigma=+1]\\ (\tilde{J}_{a(n)},J_{a(n)}) & [\sigma=-1]\end{cases}. \tag{4.12}$$

We also define the trace of the identity element as

$$\mathrm{Tr}(J)=2\sqrt{\sigma}=-\mathrm{Tr}(\tilde{J})\,. \tag{4.13}$$

Here, we have deliberately chosen the definition of trace different from the value of matrix trace. This convention is related to the quantization properties of the Chern-Simons level κ. In the uncolored Chern-Simons (higher-spin) gravity, it is unclear whether the level κ in (2.6) needs to be quantized because the the gauge group is non-compact. In the situation the higher-spin algebra is color-decorated, the level κ ought to take a discrete value for the consistency of the spin-one part under large $SU(N)$ gauge transformations. It turns out the definition (4.13) is consistent with this requirement when the level κ is integer-valued.

4.3 Color Decoration

In order to color-decorate the theory, we need to take a proper decomposition (4.1) and the choice of the decomposition reflects the symmetry of the background around which we are expanding the theory. Instead of analysing the spectrum separately for the singlet vacuum and the colored vacua, we directly consider the latter case since it also covers the former as a special case. In order to begin with the proper decomposition (4.1), we first split the isometry algebra into the Lorentz part $\mathcal{M}$ and the translation part $\mathcal{P}$ as

$$\mathfrak{iso}=\mathcal{M}\oplus\mathcal{P}\,. \tag{4.14}$$

For the color algebra, we take the k-th extremum which breaks the $\mathfrak{su}(N)$ symmetry down to $\mathfrak{su}(k)\oplus\mathfrak{su}(N-k)\oplus\mathfrak{u}(1)$. In accordance with this symmetry breaking, we decompose the space of the $\mathfrak{su}(N)$ as

$$\mathfrak{su}(N)\simeq\mathfrak{su}(N-k)\oplus\mathfrak{su}(k)\oplus\mathfrak{u}(1)\oplus\mathfrak{bs}\,, \tag{4.15}$$

where $\mathfrak{bs}$ is the $2k(N-k)$ dimensional vector space corresponding to the *broken gauge symmetry* generators. Then, the background matrix $\boldsymbol{Z}_k$ (3.12) enjoys either commutation or anti-commutation properties with each of these generators:

$$\big[\,\boldsymbol{Z}_k\,,\,\mathfrak{su}(N-k)\oplus\mathfrak{su}(k)\oplus\mathfrak{u}(1)\,\big]=0\,,\qquad\big\{\,\boldsymbol{Z}_k\,,\,\mathfrak{bs}\,\big\}=0\,. \tag{4.16}$$

Taking advantage of these two decompositions (4.14) and (4.15) of $\mathfrak{g}_i$ and $\mathfrak{g}_c$, we now decompose the full algebra $\mathfrak{g}$ (2.7) according to (4.1): the gravity plus gauge sector $\mathfrak{b}$ and the matter sector $\mathfrak{c}$.

The gravity plus gauge sector has two parts $\mathfrak{b}=\mathfrak{b}_{\rm GR}\oplus\mathfrak{b}_{\rm Gauge}$:

$$\mathfrak{b}_{\rm GR}=\big(\mathcal{M}\otimes\boldsymbol{I}\big)\oplus\big(\mathcal{P}\otimes\boldsymbol{Z}_k\big)\,,\qquad\mathfrak{b}_{\rm Gauge}=\mathrm{id}\otimes\Big(\mathfrak{su}(N-k)\oplus\mathfrak{su}(k)\oplus\mathfrak{u}(1)\Big)\,. \tag{4.17}$$

In the gravity sector, the isometry algebra is deformed by $\boldsymbol{Z}_k$ as

$$\boldsymbol{M}_{ab} = M_{ab}\,\boldsymbol{I}\,, \qquad \boldsymbol{P}_a = P_a\,\boldsymbol{Z}_k\,, \tag{4.18}$$

but still satisfies the same commutation relations as the undeformed one. We now specify the fields corresponding to the $\mathfrak{b}$ sector as

$$\mathcal{B}_{\text{GR}} = \frac{1}{2}\left(\omega^{ab} + \Omega^{ab}\right)\boldsymbol{M}_{ab} + \frac{1}{\ell_k}\,e^a\,\boldsymbol{P}_a\,, \qquad \mathcal{B}_{\text{Gauge}} = \boldsymbol{A}_+ + \boldsymbol{A}_- + \tilde{\boldsymbol{A}}_+ + \tilde{\boldsymbol{A}}_- + (A+\tilde{A})\,\boldsymbol{Y}_k\,, \tag{4.19}$$

where the traceless matrix

$$\boldsymbol{Y}_k = \frac{k\,\boldsymbol{I}_+ - (N-k)\,\boldsymbol{I}_-}{N}\,, \tag{4.20}$$

corresponds to $\mathfrak{u}(1)$ symmetry. The tensor Ω^{ab} has been introduced so that the spin connection ω^{ab} is determined only by e^a when solving the torsionless condition. Consequently, the Ω^{ab} contains the contributions from other matter and higher-spin fields. The gauge fields $\boldsymbol{A}_\pm$ and $\tilde{\boldsymbol{A}}_\pm$ take value in $\mathfrak{su}(N-k)$ for the subscript $+$ and $\mathfrak{su}(k)$ for the subscript $-$. Finally, the deformed radius ℓ_k is related to the undeformed one as

$$\ell_k = \frac{N-2k}{N}\,\ell\,. \tag{4.21}$$

The matter sector $\mathfrak{c}$ has four parts:

$$\mathfrak{c} = \mathfrak{c}_{\text{CM}} \oplus \mathfrak{c}_{\text{NM}} \oplus \mathfrak{c}_{\text{BS}} \oplus \mathfrak{c}_{\text{HS}}\,. \tag{4.22}$$

For the introduction of each element, we need to define first the deformed generators analogously to the spin-two sector (4.18) as

$$\boldsymbol{J}_{a(n)} = J_{a(n)}\,\boldsymbol{I}_+ + \tilde{J}_{a(n)}\,\boldsymbol{I}_-\,, \qquad \tilde{\boldsymbol{J}}_{a(n)} = J_{a(n)}\,\boldsymbol{I}_- + \tilde{J}_{a(n)}\,\boldsymbol{I}_+\,, \tag{4.23}$$

where $\boldsymbol{I}_\pm$ are the identities associated with $\mathfrak{u}(N-k)$ and $\mathfrak{u}(k)$, respectively:

$$\boldsymbol{I}_\pm = \frac{1}{2}\left(\boldsymbol{I} \pm \boldsymbol{Z}_k\right). \tag{4.24}$$

Analogous to the deformation of the spin-two part (4.18), the deformed higher-spin generators (4.23) still form $hs(\lambda) \oplus hs(\lambda)$ algebra. In terms of these generators, we define the one form fields corresponding to the colored and the color-neutral matter $\mathcal{C}_{\text{CM}}$ and $\mathcal{C}_{\text{NM}}$ as

$$\mathcal{C}_{\text{CM}} = \frac{1}{\ell_k}\sum_{n\geq 1}\frac{1}{n!}\left[\left(\boldsymbol{\varphi}_+^{a(n)} + \boldsymbol{\varphi}_-^{a(n)}\right)\boldsymbol{J}_{a(n)} + \left(\tilde{\boldsymbol{\varphi}}_+^{a(n)} + \tilde{\boldsymbol{\varphi}}_-^{a(n)}\right)\tilde{\boldsymbol{J}}_{a(n)}\right], \tag{4.25}$$

$$\mathcal{C}_{\text{NM}} = \frac{1}{\ell_k}\sum_{n\geq 1}\frac{1}{n!}\left(\psi^{a(n)}\,\boldsymbol{J}_{a(n)} + \tilde{\psi}^{a(n)}\,\tilde{\boldsymbol{J}}_{a(n)}\right)\boldsymbol{Y}_k\,\boldsymbol{Z}_k\,, \tag{4.26}$$

where the fields $\boldsymbol{\varphi}_{+/-}^{a(n)}$ and $\tilde{\boldsymbol{\varphi}}_{+/-}^{a(n)}$ take value in $\mathfrak{su}(N-k)/\mathfrak{su}(k)$. The last matrix factor $\boldsymbol{Y}_k$ has been introduced so that $\text{Tr}(\mathfrak{b}_{\text{GR}}\,\mathfrak{c}_{\text{NM}}) = 0$, equivalently,

$$\text{Tr}\big(\boldsymbol{J}\,\boldsymbol{Y}_k\boldsymbol{Z}_k\big) = 0 = \text{Tr}\big(\tilde{\boldsymbol{J}}\,\boldsymbol{Y}_k\boldsymbol{Z}_k\big)\,. \tag{4.27}$$

The one-form corresponding to the $\mathfrak{c}_{\rm BS}$ sector is given by

$$\mathcal{C}_{\rm BS} = \frac{1}{\ell_k}\sum_{n\geq 0}\frac{1}{n!}\left(\boldsymbol{\phi}^{a(n)}\,\boldsymbol{J}_{a(n)} + \tilde{\boldsymbol{\phi}}^{a(n)}\,\tilde{\boldsymbol{J}}_{a(n)}\right), \tag{4.28}$$

where $\boldsymbol{\phi}^{a(n)}$ and $\tilde{\boldsymbol{\phi}}^{a(n)}$ take values in $\mathfrak{bs}$ (4.15). Notice that the summation starts from $n=0$ so it involves not only higher spin generators but also the identity piece corresponding to spin one. Lastly, we have the singlet higher spin sector:

$$\mathcal{C}_{\rm HS} = \frac{1}{\ell_k}\sum_{n\geq 2}\frac{1}{n!}\left(\varphi^{a(n)}\,\boldsymbol{J}_{a(n)} + \tilde{\varphi}^{a(n)}\,\tilde{\boldsymbol{J}}_{a(n)}\right), \tag{4.29}$$

which, in principle, could be treated together with the gravity plus gauge sector. But, since we do not know any natural form of higher spin covariant interactions in metric-like form, they are treated here as other matter fields.

4.4 Action in Metriclike Formulation

Putting all the above results into the Chern-Simons action, we get

$$S = S_{\rm CS} + S_{\rm HSG} + S_{\rm Matter}\,, \tag{4.30}$$

where the first term $S_{\rm CS}$ is given by two copies of the Chern-Simons gauge theory whose gauge algebra is given by $\mathfrak{su}(N-k)\oplus\mathfrak{su}(k)$.

The second term $S_{\rm HSG}$ is the metric-like — only for the gravity part — action for higher spin gravity given by

$$S_{\rm HSG} = \frac{1}{16\pi\,G}\int\left[d^3x\sqrt{|g|}\left(R+\frac{2\,\sigma}{\ell_k^2}\right)+L(\varphi,\tilde{\varphi},\ell_k)\right] \tag{4.31}$$

where the three form L is given by

$$\begin{aligned}
&L[\varphi,\tilde{\varphi},\ell] = L_+[\varphi,\ell]-L_-[\tilde{\varphi},\ell]\,,\\
&L_\pm[\varphi,\ell] = \frac{2\sqrt{\sigma}}{N-2k}\sum_n\frac{1}{n!}\,{\rm Tr}\left[\frac{1}{\ell}\,\varphi^{a(n)}\wedge\left(D\,\varphi_{a(n)}\pm\frac{1}{\sqrt{\sigma}\,\ell}\,c_{a(n)bc(n)}\,e^b\wedge\varphi^{c(n)}\right)+\right.\\
&\qquad\qquad\left.+\frac{2}{3\,\ell^2}\,c_{a(m)b(n)c(l)}\,\varphi^{a(m)}\wedge\varphi^{b(n)}\wedge\varphi^{c(l)}\right],
\end{aligned} \tag{4.32}$$

where the covariant derivative D is both with respect to Lorentz transformations and $\mathfrak{su}(N)$ gauge transformations. At quadratic order, components with different n are independent and describe massless spin $n+1$. The gravitational constant G is fixed in terms of the Chern-Simons level by

$$\kappa = \frac{\ell}{4\,N\,G}\,. \tag{4.33}$$

Finally, the matter action takes the form,

$$\begin{aligned}
S_{\rm Matter} = \frac{1}{16\,\pi\,G}\int\frac{1}{N-2k}\,&{\rm Tr}\Big(L[\boldsymbol{\varphi}_+,\tilde{\boldsymbol{\varphi}}_+,\ell_k]-L[\boldsymbol{\varphi}_-,\tilde{\boldsymbol{\varphi}}_-,\ell_k]\Big)\\
&-\frac{k(N-k)}{N^2}\,L[\psi,\tilde{\psi},\ell_k]+\frac{1}{N-2k}\,L_{\rm BS}[\boldsymbol{\phi},\tilde{\boldsymbol{\phi}},\ell_k]+L_{\rm int}\,,
\end{aligned} \tag{4.34}$$

where the new three form $L_{\rm BS}$ has fully correlated components as opposed to L (4.32):

$$L_{\rm BS}[\boldsymbol{\phi},\tilde{\boldsymbol{\phi}},\ell]=\frac{4\sqrt{\sigma}}{\ell}\sum_n\frac{1}{n!}\,{\rm Tr}\left[\tilde{\boldsymbol{\phi}}_{a(n)}\wedge\left(D\,\boldsymbol{\phi}^{a(n)}+\frac{c_n}{\sqrt{\sigma}\,\ell}\,e^a\wedge\boldsymbol{\phi}^{a(n-1)}+\frac{c_{n+1}}{\sqrt{\sigma}\,\ell}\,e_a\wedge\boldsymbol{\phi}^{a(n+1)}\right)\boldsymbol{Z}_k\right]. \tag{4.35}$$

The term $L_{\rm int}$ in the second line of (4.35) concerns exclusively the interaction terms and it contains the cross couplings from the Chern-Simons cubic interaction and the quadratic terms in Ω^{ab} (which itself is quadratic in fields, hence these terms represent quartic couplings).

In describing the action of colored higher spin gravity, we have not provided the explicit expression for the structure constants $c_{a(m),b(n),c(l)}$, $L_{\rm cross}$ and Ω^{ab} . Identifying their form is straightforward in principle but not necessary for our purpose: we are interested in the nature of spectra around rainbow vacua and the qualitative structure of interactions. In the following, we elaborate more on these aspects.

We have the spin-one Chern-Simons gauge fields, spin-two gravity, and higher-spin fields, whose dynamics are governed by $S_{\rm CS}$ and $S_{\rm HSG}$. Especially, the latter $S_{\rm HSG}$ coincides with the action of the uncolored Chern-Simons higher-spin theory in three-dimensions. The colored higher-spin fields $\boldsymbol{\varphi}_{\pm}^{a(n)}$ and the color-neutral $\psi^{a(n)}$ (and their tilde counter parts) share the same structure of the quadratic Lagrangian L (4.32). As such, both describe massless spin-$(n+1)$ fields, which are not propagating in three dimensions. The rest are the fields $\boldsymbol{\phi}^{a(n)}$ and $\tilde{\boldsymbol{\phi}}^{a(n)}$ corresponding to the broken part of the color symmetries.

Postponing the analysis of this spectrum to the next section, we conclude this section with the discussion on the qualitative nature of the interaction. The structure of interaction in the color non-singlet vacua is analogous to the colored (A)dS$_3$ gravity we studied in the previous work [14]. All the fields are coupled to gravity in the diffeomorphism invariant manner. All the colored fields — adjoints $\boldsymbol{\varphi}_{\pm}$ and bi-fundamentals $\boldsymbol{\phi}$ — have covariant gauge couplings to the Chern-Simons gauge fields. There are self-couplings among the matter fields with coupling constants controlled by N and k (4.34). These interactions become strong for small k (small symmetry breaking) and as weak as gravity for large k (large symmetry breaking).

5 Mass Spectrum of the Broken Color Symmetries

We already noted that the color breaking triggers the mass generation as well. Here, we analyze spectrum of these massive components.

5.1 General Structure

We now analyze the action for the field $\boldsymbol{\phi}^{a(n)}$ and $\tilde{\boldsymbol{\phi}}^{a(n)}$ corresponding the the broken part of the color symmetries. For definiteness, we concentrate on the AdS space. To get the result of dS space, we simply relate the AdS radius to $\sqrt{i}$ times the dS radius.

It turns out all these fields with different n are entangled and even the left and right movers have cross-couplings in the quadratic action. However, we can always diagonalize the action and reduce it to a collection of $S_{\rm BS}$ given by

$$S_{\rm BS}[\phi,\phi^a,\dots,\phi^{a(M-1)}] = \\ = \sum_{n=0}^{M-1}\int \phi_{a(n)}\wedge\left[D\,\phi^{a(n)}+\frac{1}{\ell}\left(c_n\,e^a\wedge\phi^{a(n-1)}+c_{n+1}\,e_a\wedge\phi^{a(n+1)}\right)\right]. \quad (5.1)$$

Here, we have assumed the $\mathfrak{gl}_M$ case for clarity of the analysis. Notice that the one-form fields contributing to the action are truncated to the first M fields.

The above action also admits gauge symmetries with parameters $(\varepsilon,\varepsilon^a,\dots,\varepsilon^{a(M-1)})$ as

$$\delta\,\phi^{a(n)} = D\,\varepsilon^{a(n)}+\frac{1}{\ell}\left(c_n\,e^a\,\varepsilon^{a(n-1)}+c_{n+1}\,e_a\,\varepsilon^{a(n+1)}\right). \quad (5.2)$$

For the analysis of equation of motion, we consider the decomposition of $\phi^{a(n)}$ into

$$\begin{aligned} \bar h_{\mu(n+1)} &= (e_\mu{}^a)^n\,\phi_{\mu\,a(n)}-(trace)\,,\\ h'_{\mu(n-1)} &= (e_\mu{}^a)^{n-1}\,e^{\mu b}\,\phi_{\mu\,a(n-1)b}\,,\\ f_{\mu(n),\nu} &= (e_\mu{}^a)^n\,\phi_{\nu\,a(n)}+e_\nu{}^a\,(e_\mu{}^a)^{n-1}\,\phi_{\mu\,a(n)}\,, \end{aligned} \quad (5.3)$$

where $\bar h_{\mu(n+1)}$ and $h'_{\mu(n-1)}$ are totally symmetric traceless fields and $f_{\mu(n),\nu}$'s are the traceless fields of the Young-symmetry type $\{n,1\}$. The procedure of the analysis can be summarized as the following steps:

- We first gauge fix $\bar h_{\mu(n)}$ from $n=1$ to $M-1$ by making use of the gauge transformations (5.2) with the parameters $\varepsilon_{\mu(n)}$ from $n=1$ to $M-1$.
- Upon using the equations of motions, all the hook fields $f_{\mu(n),\nu}$ can be algebraically determined in terms of the rest. At this stage, the residual field content is

  $$\bar h_{\mu(M)}\,,\qquad h'_{\mu(n)}\quad [n=0,\dots,M-2]\,, \quad (5.4)$$

  and these fields combine to form two traceful fields of spin M and $M-3$, respectively. This is the field content of massive higher-spin fields along the lines taken by Singh-Hagen [19], but in this case we also have a gauge symmetry with the scalar parameter ε. This already suggests that the spectrum described by this system corresponds to the maximal depth partially-massless spin M.
- Other equations can be used to algebraically determine $h'_{\mu(n)}$ from $n=1$ to $M-2$. Hence, after this step, we end up only with $\bar h_{\mu(M)}$ and h', modulo the gauge equivalence given by the scalar parameter ε. In the $M=2$ case, $\bar h_{\mu\nu}$ and h' can combine to a single traceful field $h_{\mu\nu}$.

The final equation is of first-order type and involves the fields $\bar h_{\mu(M)}$ and h'. These fields have gauge symmetries involving M derivatives for $\bar h_{\mu(M)}$ and of second-order for h'. Instead of proceeding with the generic value of M, we shall consider the $M=3$ example in detail. The analysis for generic values of M is a straightforward generalization and they will be presented in a forthcoming paper [17] along with the analysis of the colored Vasiliev's equation.

5.1.1 Example: $\mathfrak{gl}_3 \oplus \mathfrak{gl}_3$

For more concrete understanding, let us explicitly analyze the $M = 3$ case. From (5.3), we get seven fields

$$\bar{h}_{\mu\nu\rho}\,, \quad \bar{h}_{\mu\nu}\,, \quad \bar{h}_{\mu}\,, \quad f_{\mu\nu,\rho}\,, \quad f_{\mu\nu}\,, \quad h'_{\mu}\,, \quad h'\,. \tag{5.5}$$

They admit the equations of motions,

$$\nabla_{[\mu}\,\bar{h}_{\nu]}{}^{\rho\sigma} + \frac{4}{3}\,\nabla_{[\mu}\,f^{\rho\sigma}{}_{,\nu]} - \frac{3}{5}\,\delta^{\{\rho}_{[\mu}\,\nabla_{\nu]}\,h'^{\sigma\}} + \frac{2\sqrt{2}}{\ell}\,\delta^{\{\rho}_{[\mu}\,\bar{h}_{\nu]}{}^{\sigma\}} = 0\,, \tag{5.6}$$

$$\nabla_{[\mu}\bar{h}_{\nu]\rho} + \nabla_{[\mu}f_{\nu]\rho} + g_{\rho[\mu}\left(\frac{1}{\ell}\,\bar{h}_{\nu]} - \frac{1}{3}\,\partial_{\nu]}h'\right) = \frac{2\sqrt{2}}{3\,\ell}\left(f_{\rho[\nu,\mu]} - \frac{3}{4}\,g_{\rho[\nu}\,h'_{\mu]}\right), \tag{5.7}$$

$$\partial_{[\mu}\bar{h}_{\nu]} = \frac{8}{3\,\ell}\,f_{\mu\nu}\,, \tag{5.8}$$

where (5.7) and (5.8) simply imply that $f_{\mu\nu,r}$, $f_{\mu\nu}$ and h'_{μ} are not independent. The rest of the fields have the gauge symmetries,

$$\begin{aligned} \delta\bar{h}_{\mu\nu\rho} &= \nabla_{\{\mu}\,\varepsilon_{\nu\rho\}}\,, \qquad & \delta\bar{h}_{\mu\nu} &= \nabla_{\{\mu}\,\varepsilon_{\nu\}} + \frac{1}{\sqrt{2}\,\ell}\,\varepsilon_{\mu\nu}\,, \\ \delta\bar{h}_{\mu} &= \partial_{\mu}\,\varepsilon + \frac{8}{3\,\ell}\,\varepsilon_{\mu}\,, \qquad & \delta h' &= \nabla^{\rho}\varepsilon_{\rho} + \frac{3}{\ell}\,\varepsilon\,. \end{aligned} \tag{5.9}$$

One can first gauge fix $\bar{h}_{\mu\nu}$ and $\bar{h}_{\mu}$ using the gauge transformations with the parameters $\varepsilon_{\mu\nu}$ and ε_{ν}. This gauge fixing will relate the latter gauge parameters to the scalar one ε as

$$\varepsilon_{\mu\nu} = \frac{3\,\ell^2}{4\sqrt{2}}\,\nabla_{\{\mu}\partial_{\nu\}}\,\varepsilon\,, \qquad \varepsilon_{\mu} = -\frac{3\,\ell}{8}\,\partial_{\mu}\,\varepsilon\,. \tag{5.10}$$

Finally, the remaining equations of motions and gauge transformations are given by

$$\nabla_{[\mu}\,\bar{h}_{\nu]\rho\sigma} + \frac{\sqrt{2}\,\ell}{5}\,\nabla_{[\mu}\,g_{\nu]\{\rho}\,\nabla_{\sigma\}}\,h' = 0\,, \tag{5.11}$$

and

$$\delta\bar{h}_{\mu\nu\rho} = \ell^2\;\nabla_{\{\mu}\nabla_{\nu}\nabla_{\rho\}}\,\varepsilon\,, \qquad \delta h' = -\frac{\ell}{\sqrt{2}}\left(\nabla^2 - \frac{8}{\ell^2}\right)\varepsilon\,. \tag{5.12}$$

These gauge transformations precisely coincide with those of the maximal depth partially-massless fields, which has been studied e.g. in [20] and [21]. Hence, this $M = 3$ example demonstrates that the spectrum of colored higher spin gravity corresponding to its broken part of color symmetry is indeed the maximal depth partially massless fields of the highest spin in the theory.

We conclude this section with the comments on our symmetry breaking mechanism compared to the standard Higgs one. The role of the Higgs field with the Mexican hat potential is played by the colored spin two fields. When their backgrounds take non-trivial values, the fields corresponding to the broken part of the symmetries — which are analogous to the Goldstone bosons — combine with all the other spins to form partially-massless fields.

6 Partially-Massless Fields in Three Dimensions

Partially-massless (PM) fields carry irreducible representations of the isometry algebra of non-vanishing constant curvature background. They are unitary only in dS whereas in AdS they involve negative norm states even though their energy is bounded from below. For a given spin s, there are s different PM fields labelled by depth $t = 0, 1, \ldots, s-1$ where $t = 0$ case corresponds to the massless field. In the flat limit, depth t PM field reduces to a set of massless fields with helicities $s, s-1, \ldots, s-t$. This pattern manifests the number of degrees of freedom (DoF) they have: the number interpolates between those of massless and massive fields. In the case of maximal depth with $t = s-1$, the PM field contains one less DoF — corresponding to a scalar mode — than a massive spin-s field.

In three dimensions, the decomposition of the isometry algebra — $\mathfrak{so}(2,2) \simeq \mathfrak{so}(2,1) \oplus \mathfrak{so}(2,1)$ and $\mathfrak{so}(1,3) \simeq \mathfrak{so}(3) \oplus \overline{\mathfrak{so}(3)}$ — simplifies the analysis of PM representations. In the following, we provide a brief account for partially-massless (PM) representations in three dimensions.

Let us first consider AdS case $\mathfrak{so}(2,d)$ and its lowest weight representation (LWR) $\mathcal{V}_{\mathfrak{so}(2,d)}(\Delta, s)$ labeled by the lowest energy Δ and spin s. The unitarity bound $\Delta = s+d-2$ corresponds to the massless field (for $s \geq 1$) and the depth t PM fields corresponds to $\Delta = s+d-2-t$. In these cases, we have to factor out invariant subspaces corresponding to gauge modes in order to describe irreducible representations.

In three dimensions, the LWR $\mathcal{V}_{\mathfrak{so}(2,2)}(\Delta, s)$ is decomposed into the LWRs of $\mathfrak{so}(2,1) \oplus \mathfrak{so}(2,1)$ as

$$\mathcal{V}_{\mathfrak{so}(2,2)}(h_1+h_2, h_1-h_2) = \left[\,\mathcal{V}_{\mathfrak{so}(2,1)}(h_1) \otimes \mathcal{V}_{\mathfrak{so}(2,1)}(h_2)\,\right] \oplus \left[\,\mathcal{V}_{\mathfrak{so}(2,1)}(h_2) \otimes \mathcal{V}_{\mathfrak{so}(2,1)}(h_1)\,\right], \quad (6.1)$$

where we identify the LW and spin of $\mathfrak{so}(2,2)$ with the LWs of $\mathfrak{so}(2,1) \oplus \mathfrak{so}(2,1)$ as $\Delta = h_1+h_2$ and $s = |h_1-h_2|$. Here, we focus on the parity-invariant representations so included both of the $\pm s$ helicities assuming $s \neq 0$. If $s = 0$, then we simply get $\mathcal{V}_{\mathfrak{so}(2,2)}(2h, 0) = \mathcal{V}_{\mathfrak{so}(2,1)}(h) \otimes \mathcal{V}_{\mathfrak{so}(2,1)}(h)$. In terms of $V_h := \mathcal{V}_{\mathfrak{so}(2,1)}(h)$, it is simpler to understand the appearance of invariant subspace: when h takes a non-positive half-integer value, the representation splits as

$$V_{-h} = R_h \oplus V_{h+1} \qquad [\,2h \in \boldsymbol{N}\,], \quad (6.2)$$

where V_{h+1} is the infinite-dimensional invariant subspace and R_h is the $(2h+1)$-dimensional representation. Now considering the LW of PM fields, $\Delta = s-t$, the LWR $\mathcal{V}_{\mathfrak{so}(2,2)}(\Delta, s)$ reduces to

$$\begin{aligned}\mathcal{V}_{\mathfrak{so}(2,2)}(s-t, s) &= \left(V_{s-\frac{t}{2}} \otimes V_{-\frac{t}{2}}\right) \oplus \left(V_{-\frac{t}{2}} \otimes V_{s-\frac{t}{2}}\right) \\ &= \left[V_{s-\frac{t}{2}} \otimes \left(R_{\frac{t}{2}} \oplus V_{\frac{t}{2}+1}\right)\right] \oplus \left[\left(R_{\frac{t}{2}} \oplus V_{\frac{t}{2}+1}\right) \otimes V_{s-\frac{t}{2}}\right], \end{aligned} \quad (6.3)$$

which involve an invariant subspace,

$$\mathcal{V}_{\mathfrak{so}(2,2)}(s+1, s-t-1) = \left(V_{s-\frac{t}{2}} \otimes V_{\frac{t}{2}+1}\right) \oplus \left(V_{\frac{t}{2}+1} \otimes V_{s-\frac{t}{2}}\right). \quad (6.4)$$

corresponding to the gauge modes. After factoring out this, the remaining representation corresponds to the PM ones:

$$\mathcal{D}_{\mathfrak{so}(2,2)}(s-t,s) = \Big(V_{s-\frac{t}{2}} \otimes R_{\frac{t}{2}}\Big) \oplus \Big(R_{\frac{t}{2}} \otimes V_{s-\frac{t}{2}}\Big)\,. \tag{6.5}$$

Differently from the massless case where R_0 is the trivial representation, the PM field cannot be decomposed neatly into left and right moving (or holomorphic and anti-holomorphic) parts due to $R_{t/2}$. Moreover, the finite-dimensional representation $R_{t/2}$ is not unitary apart from the trivial one corresponding to $t=0$. Hence, all PM fields with $t\neq 0$ are non-unitary around AdS background.

Note that the PM field $\mathcal{D}_{\mathfrak{so}(2,2)}(s-t,s)$ does not have any bulk DoF (as one of two $\mathfrak{so}(2,1)$'s has a finite-dimensional representation $R_{t/2}$) but it has $2(t+1)$ boundary DoFs. On the other hand, the gauge modes $\mathcal{V}_{\mathfrak{so}(2,2)}(s+1,s-t-1)$ has two bulk DoFs as it has infinite-dimensional representation for both of $\mathfrak{so}(2,1)$. The maximal depth case with $t=s-1$ is special here. Even though the PM field $\mathcal{D}_{\mathfrak{so}(2,2)}(1,s)$ follows the same pattern as generic t , its gauge mode $(V_{(s+1)/2}\otimes V_{(s+1)/2})^{\otimes 2}$ is given by *two copies* of a parity-invariant scalar mode, $[\mathcal{V}_{\mathfrak{so}(2,2)}(s+1,0)]^{\otimes 2}$. Hence, compared to other depths, the maximal depth field has doubled gauge modes in a sense. This particularity of the maximal depth can be understood as well in the field-theoretical viewpoint. For more clear understanding, let us consider the simplest example of spin one. The maximal depth partially massless coincides with the massless case for vector field. The massive spin one is usually described by Proca action,

$$S_{\rm Proca}[A] = \int d^3x\sqrt{g}\left(\frac{1}{4}\,F_{\mu\nu}\,F^{\mu\nu} + \frac{m^2}{2}\,A_\mu\,A^\mu\right), \tag{6.6}$$

but in three dimensions it can also be described as a *topologically massive* (TM) action,

$$S_{\rm TM}[A^\pm] = \int d^3x\left(\frac{1}{2}\,\epsilon^{\mu\nu\rho}\,A^\pm_\mu\,\partial_\nu A^\pm_\rho \pm \sqrt{g}\,\frac{m}{2}\,A^\pm_\mu\,A^{\pm\mu}\right), \tag{6.7}$$

where $A^\pm$ separately describe the $\pm$ helicity modes. In the massless limit, both actions acquire gauge symmetries and loose DoFs.

- The TM action (6.7) acquires two gauge symmetries — one for A^+ and the other for A^- — and each removes $V_{\frac{s+1}{2}}\otimes V_{\frac{s+1}{2}}$ ending up with two copies of Abelian Chern-Simons, describing $\mathcal{D}_{\mathfrak{so}(2,2)}(1,1)$ with only boundary degrees of freedom.

- The Proca action (6.6) acquires one gauge symmetry removing only one mode $\mathcal{V}_{\mathfrak{so}(2,2)}(2,0)$ from $\mathcal{V}_{\mathfrak{so}(2,2)}(1,1)$ (6.3) leaving $\mathcal{D}_{\mathfrak{so}(2,2)}(1,1)\oplus\mathcal{V}_{\mathfrak{so}(2,2)}(2,0)$. Hence, together with two boundary modes, it also describes a bulk scalar mode [**?**].

In the higher-spin cases, one can still construct a one-derivative TM action for a massive field that is parity odd. For $t\neq s-1$ PM field there is also an equivalent two-derivative parity preserving description. For $t=s-1$ the situation is similar to Maxwell field — the one-derivative description is not equivalent to the two-derivative one. In the PM limit (including $t=s-1$) of TM action, each gauge symmetry eliminates either $V_{s-\frac{t}{2}}\otimes V_{\frac{t}{2}+1}$ or $V_{\frac{t}{2}+1}\otimes V_{s-\frac{t}{2}}$, so completely removes the parity-invariant gauge mode $\mathcal{V}_{\mathfrak{so}(2,2)}(s+1,s-t-1)$

(6.4) leaving only the boundary DoF $\mathcal{D}_{\mathfrak{so}(2,2)}(s+1,s-t-1)$ (6.5). While beginning with a two-derivative massive action (see e.g. [21]), the PM limit attains one *parity-invariant* gauge symmetry. When the depth is not maximal, $t \neq s-1$, it again removes the parity-invariant combination of gauge modes $\mathcal{V}_{\mathfrak{so}(2,2)}(s+1,s-t-1)$. On the contrary, in the maximal depth case with $t=s-1$, the gauge symmetry removes only one mode among two $\mathcal{V}_{\mathfrak{so}(2,2)}(s+1,0)$'s. Hence, the left-over DoFs $\mathcal{D}_{\mathfrak{so}(2,2)}(1,s)\oplus\mathcal{V}_{\mathfrak{so}(2,2)}(s+1,0)$ contain bulk scalar.

In dS case, the isometry group is given by $\mathfrak{so}(1,3)\simeq\mathfrak{so}(3)\oplus\overline{\mathfrak{so}(3)}$, and we begin with the representations of $\mathfrak{so}(3)$ and $\overline{\mathfrak{so}(3)}$. Differently from AdS case, we do not assume that these representations are of LW type because in dS we do not have an invariant notion of energy to which we can impose a bound condition. Still, the representations can be labelled by $\mathbb{C}$ numbers h and h^* (for the moment, h^* is different from the complex conjugate $\bar{h}$) which parameterize the Casimir operators $\mathfrak{so}(3)$ and $\overline{\mathfrak{so}(3)}$ as

$$C=h(h+1)\,,\qquad C^*=h^*(h^*+1)\,. \tag{6.8}$$

For the compactness of $\widetilde{SO(3)}\simeq SU(2)$ in $\widetilde{SO(1,3)}\simeq SL(2,\mathbb{C})$, we get the quantization condition:

$$h-h^*=s\in\tfrac{1}{2}\,\mathbb{Z}\,, \tag{6.9}$$

which is related to the spin of a particle in dS. For convenience, let us define the other combination of h and h^* as

$$h+h^*+1=\mu\,, \tag{6.10}$$

and μ is an arbitrary complex number for the moment.

Since $\overline{\mathfrak{so}(3)}$ is the complex conjugate of $\mathfrak{so}(3)$, for unitarity, their representations should also be related by complex conjugate:

$$\bar{C}=C^*\quad\Leftrightarrow\quad(\bar{h}-h^*)(\bar{h}+h^*+1)=0\,, \tag{6.11}$$

and there are two options:

$$\bar{h}+h^*+1=0\quad\Rightarrow\quad\text{any } s\,,\quad \mu\in i\,\mathbb{R}\,, \tag{6.12}$$
$$\bar{h}-h^*=0\quad\Rightarrow\quad s=0\,,\quad \mu\in\mathbb{R}\,. \tag{6.13}$$

The first case (6.12) corresponds to the usual massive spin s representation with mass-squared given by μ^2, whereas the second case (6.13) corresponds to the special mass region only allowed for the scalar. According to the scalar representation analysis in general dimensions, the unitary value for μ in the second case should be further restricted to some discrete values. Since the representation space does not develop any invariant subspace in both of the cases (6.12, 6.13), we do not find any unitary short representation in dS background. In a sense, this is consistent with the fact that dS space does not have any Lorentzian boundary where the short representation can live.

7 Discussions

In this paper, we have analyzed the theory of colored higher spin gravity in three dimensions. We showed that the theory can be viewed as a theory of higher spin gravity and Chern-Simons gauge fields coupled to matter fields consisting of again massless higher spins. The matter fields introduce more saddle point vacua with different cosmological constants to the theory, exactly like in the case of 3d multigravity [14]. On each of these vacua, the symmetry breaking takes place giving rise to a new spectrum.

The mechanism of symmetry breaking and the resulting spectrum are interesting. First, the goldstone modes, which are spin-two fields corresponding to the broken part of the color symmetry, are not simply eaten by one of the other fields but by all other fields. In a sense, it is more correct to describe this as if goldstone mode devours all other spectrum or more neutrally they combine altogether to become a single irreducible spectrum — the maximal depth partially massless field.

The nature of partially massless field is also intriguing. For the algebra $\mathfrak{gl}_M \oplus \mathfrak{gl}_M$, it is the spin M maximal depth partially massless field, which contains the mode of massless spin from 1 to M . In other words, it behaves almost like a massive spin M field but lacks only one DoF, the scalar mode. However, in three dimensions, all (partially) massless fields with spin greater or equal to one (considering Chern-Simons as spin one) do not have propagating DoF but only boundary ones. The scalar mode is special as it is the only propagating DoF in the three dimensional bulk. Interestingly, when considering a generic $hs(\lambda)$ rather than $\mathfrak{gl}_M$ we do not have any bound on the highest spin, meaning that the maximal depth partially massless fields, appearing in the symmetry-broken phase of colored higher spin gravity, have an infinite spin. Clearly this spectrum is exotic one, but it is interesting to remark that the infinite spin limit may somehow compensate with the absence of the bulk DoF so that the field could eventually describe a curious bulk DoF. It is not yet clear to the authors what the implication of this exotic spectrum is, but hopefully it will become more clear when revising it in the context of the PV theory (they correspond to the so-called twisted sector). In the forthcoming papers, we shall study the color decoration of Vasiliev equations in various dimensions as well as several generalizations of the color-decoration mechanism.

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