SCUOLA NORMALE SUPERIORE

ENS | PSL RESEARCH UNIVERSITY PARIS

Some functional inequalities and spectral properties of metric measure spaces with curvature bounded below

Tesi di Perfezionamento in Matematica
in cotutela SNS - PSL (ENS)

Candidato

David Tewodrose

Relatore

Prof. **Luigi Ambrosio**
Prof. **Thierry Coulhon**

Anno accademico 2017/2018

Contents

Chapter 1

Introduction

The aim of this thesis is to study metric measure spaces with a synthetic notion of Ricci curvature bounded below. We study them from the point of view of Sobolev/Nash type functional inequalities in the non-compact case, and from the point of view of spectral analysis in the compact case. The heat kernel links the two cases: in the first one, the goal is to get new estimates on the heat kernel of some associated weighted structure; in the second one, the heat kernel is the basic tool to establish our results.

The topic of synthetic Ricci curvature bounds has known a constant development over the past few years. In this introduction, we shall give some historical account on this theory, before explaining in few words the content of this work. The letter K will refer to an arbitrary real number and N will refer to any finite number greater or equal than 1.

Ricci curvature

Ricci curvature is one fundamental way to express how a smooth Riemannian manifold differs from being flat. Finding its roots in the tensorial calculus - "calcolo assoluto" - developed at the turn of the twentieth century by G. Ricci-Curbastro and T. Levi-Civita [RL01, Ric02], it came especially into light in 1915, when A. Einstein used it to formalize in a concise way its celebrated equations modelling how a spacetime is curved by the presence of local energy and momentum encoded in the so-called stress-energy tensor [E15] (see also [To54] for historical details).

Throughout the second half of the XXth century, many mathematicians have studied the implications of a lower bound on the Ricci curvature. Indeed, it was soon realized that such a bound grants powerful comparison theorems giving a control of several analytic quantities, like the Hessian and the Laplacian of distance functions $\mathrm{d}(x,\cdot)$, or the volume of geodesic balls and spheres, in terms of the corresponding ones in the model spaces with constant curvature (see e.g. [GHL04]).

Following this path, J. Cheeger and D. Gromoll provided in 1971 their well-known splitting theorem [CG71], extending V. Topogonov's previous result [To64] established under the weaker assumption of non-negative sectional curvature: if a complete non-negatively Ricci curved Riemannian manifold contains a line, then it splits into the Riemannian product of $\mathbb{R}$ and a submanifold of codimension 1.

On a more analytical side, P. Li and S.-T. Yau proved in 1986 a striking global Harnack inequality for positive solutions of the equation $(\Delta - q(x) - \frac{\partial}{\partial t})u(x,t) = 0$ on complete Riemannian manifolds with Ricci curvature bounded below, where q is a C^2 potential with controlled gradient and bounded Laplacian [LY86]. Their results apply especially to the case of the heat equation $(\Delta - \frac{\partial}{\partial t})u(x,t) = 0$, i.e. when $q \equiv 0$. In the Euclidean space, the Harnack inequality for parabolic differential equations was estabilished in 1964 by J. Moser

[Mo64]. It implies Hölder regularity of positive weak solutions of the equation and sharp upper and lower Gaussian bounds for the associated Green's kernel. Therefore, Li-Yau's Harnack inequality extended these two results to the setting of Riemannian manifolds with Ricci curvature bounded below.

Moreover, the assumption $\mathrm{Ric} \geq K$ implies two important results, namely the Bishop-Gromov theorem ([Bis63], see also [GHL04]) and the local L^2 Poincaré inequality ([Bus82], see also [CC96, Th. 2.11 and Rk. 2.82]). The former implies the local doubling condition (2.1.8) which is a useful tool to extend classical analytic results to the setting of metric measure spaces. Let us mention that L. Saloff-Coste proved that on Riemannian manifolds, the local doubling condition and the local L^2 Poincaré inequality are equivalent to the parabolic Harnack inequality [Sa92]. K.-T. Sturm extended this result to the context of Dirichlet spaces [St96].

Further consequences of Ricci curvature bounded below are eigenvalue estimates, isoperimetric inequalities, Sobolev inequalities, etc., for which we refer e.g. to [L12].

Looking for a synthetic notion

All these results focused the attention on the set of Riemannian manifolds with Ricci curvature bounded below by $K \in \mathbb{R}$, seen as particular subclass of the general collection of metric spaces. In this regard, M. Gromov provided in 1981 an important result, known as the precompactness theorem, which states that for any given $n \in \mathbb{N}$ and $D > 0$, the set $\mathcal{M}(n, K, D)$ of all n-dimensional compact Riemannian manifolds with Ricci curvature bounded below by K and diameter bounded above by D, is precompact for the Gromov-Hausdorff topology (Theorem 2.4.2). Let us recall that the Gromov-Hausdorff distance between compact metric spaces (X, d_X) and (Y, d_Y) is $\mathrm{d}_{GH}((X, \mathrm{d}_X), (Y, \mathrm{d}_Y)) := \inf \mathrm{d}_{H,Z}(i(X), j(Y))$, where the infimum is taken over the triples (Z, i, j) such that Z is a metric space, $i : X \hookrightarrow Z$, $j : Y \hookrightarrow Z$ are isometric embeddings, and $\mathrm{d}_{H,Z}$ stands for the Hausdorff distance in Z.

In other words, Gromov's precompactness theorem states that from any sequence of elements in $\mathcal{M}(n, K, D)$, one can extract a subsequence converging, in terms of the Gromov-Hausdorff distance, to a metric space. This result extends in an appropriate way to the set $\mathcal{M}(n, K)$ of all complete n-dimensional Riemannian manifolds with Ricci curvature bounded below by K, see [Gro07, Th. 5.3].

Metric spaces arising as such limits of manifolds, nowadays called Ricci limits, are in general not Riemannian manifolds. Nevertheless, they possess some structural properties which were investigated at the end of the nineties by J. Cheeger and T. Colding [CC97, CC00a, CC00b]. These two authors proved that several results from the smooth world of Ricci curvature bounded below hold also on Ricci limits. Indeed, as a suitable consequence of Ascoli-Arzelà theorem, it is always possible to construct a limit measure ν_∞ on any such space $X = \lim_i M_i$ so that the Gromov-Hausdorff convergence $M_i \to X$ upgrades to the measured Gromov-Hausdorff convergence, due to K. Fukaya [F87], for which we have $\mathrm{vol}_i(B_i) \to \nu_\infty(B_\infty)$ whenever $B_i \to B_\infty$ in the Gromov-Hausdorff sense for any ball B_∞ with ν_∞-negligible boundary. In particular, the Bishop-Gromov theorem passes automatically to the limit, and J. Cheeger and T. Colding also proved that the local L^2 Poincaré inequality is still true on Ricci limits.

Driven by these observations, they asked the following question [CC97, Appendix 2] which was also stated in [Gro91, p. 84]: calling *synthetic* a set of conditions defining a class of metric spaces without refering to any notion of smoothness, can one provide a synthetic notion of having Ricci curvature bounded below?

Note that in the case in which Ricci curvature is replaced by sectional curvature, the theory of Alexandrov spaces [Ale51, Ale57, BGP92] provides an interesting answer to this question: indeed, the class $\mathcal{A}(K,N)$ of Alexandrov spaces with dimension bounded above by N and curvature bounded below by K is defined in a synthetic way, and it contains the Gromov-Hausdorff closure $\overline{\mathcal{S}}(K,N)$ of the collection of Riemannian manifolds with dimension lower than N and sectional curvature bounded below by K. Whether the inclusion $\overline{\mathcal{S}}(K,N) \subset \mathcal{A}(K,N)$ is strict or not is still an open question, the conjecture being that it is not [Kap05].

A first natural direction towards an answer to Cheeger-Colding's question could have been provided by the class of PI doubling spaces (Definition 2.2.17), namely those metric measure spaces $(X,\mathrm{d},\mathfrak{m})$ satisfying the local doubling condition and a local Poincaré inequality. In his seminal paper [Ch99], J. Cheeger constructed a first-order weak differential structure on such spaces based on the following functional:

$$\mathrm{Ch}(f) = \inf_{f_n \to f} \left\{ \liminf_{n\to+\infty} \int_X |\nabla f_n|^2 \,\mathrm{d}\mathfrak{m} \right\} \in [0,+\infty]$$

defined for any $f \in L^2(X,\mathfrak{m})$, where the infimum is taken over all the sequences $(f_n)_n \subset L^2(X,\mathfrak{m}) \cap \mathrm{Lip}(X,\mathrm{d})$ such that $\|f_n - f\|_{L^2(X,\mathfrak{m})} \to 0$ and where $|\nabla f_n|$ is the slope of f_n. Setting $H^{1,2}(X,\mathrm{d},\mathfrak{m}) := \{\mathrm{Ch} < +\infty\}$ as an extension of the classical Sobolev space $H^{1,2}$ to this setting, J. Cheeger showed that for functions $f \in H^{1,2}(X,\mathrm{d},\mathfrak{m})$ it is possible to define a suitable notion of norm of the gradient, called minimal relaxed slope, and denoted by $|\nabla f|_*$. Out of this, he built a vector bundle $T\tilde{X} \to \tilde{X}$ over a set of full measure $\tilde{X} \subset X$ with possibly varying dimension of the fibers. Local trivializations of this bundle are given by uples $(U, f_1, \ldots, f_k)$ where $U \subset \tilde{X}$ is Borel and $f_1, \ldots, f_k : U \to \mathbb{R}$ are Lipschitz maps, and any Lipschitz function $f : X \to \mathbb{R}$ admits on U a differential representation $\mathrm{d}f = \sum_{i=1}^k (\alpha_i, f_i)$, the Borel functions α_i being understood as the local coordinates of $\mathrm{d}f$ (Theorem 2.2.18), and the pairs (α_i, f_i) being usually denoted in the more intuitive manner $\alpha_i \,\mathrm{d}f_i$.

However, the class of PI doubling spaces is too large to be regarded as a synthetic definition of Ricci curvature bounded below: for instance, it was proved by N. Juillet [Ju09] that for any $n \in \mathbb{N}\backslash\{0\}$, the n-dimensional Heisenberg group does not belong to the Gromov-Hausdorff closure of $\mathcal{M}(n,K)$ even though they do belong to the class of PI doubling spaces.

Nevertheless, what follows from this discussion is that any tentative synthetic definition of Ricci curvature bounded below should single out a subclass of the collection of PI doubling spaces.

$\mathrm{RCD}^*(K,N)$ spaces

R standing for "Riemannian", *C* for "curvature" and *D* for "dimension", the $\mathrm{RCD}^*(K,N)$ condition for complete, separable, geodesic metric measure spaces $(X,\mathrm{d},\mathfrak{m})$ amounts to the seminal works of K.-T. Sturm [St06a] and J. Lott and C. Villani [LV09] in which were introduced similar but slightly different conditions giving a meaning to Ricci curvature bounded below and dimension bounded above on such spaces and denoted by "$\mathrm{CD}(K,N)$". Note that in the second cited article only the cases $\mathrm{CD}(K,\infty)$ and $\mathrm{CD}(0,N)$ with $N < +\infty$ were considered. Afterwards, two main requirements were added to the theory. The first one is the CD^* condition, due to K. Bacher and K.-T. Sturm [BS10], which ensures better tensorization and globalization properties of the identified spaces. The second one is the infinitesimally Hilbertian condition added to the $\mathrm{CD}(K,\infty)$ condition by L. Ambrosio, N. Gigli and G. Savaré in [AGS14b], providing the class of $\mathrm{RCD}(K,\infty)$ spaces which rules out non-Riemannian Finsler structures. The addition of the infinitesimally Hilbertian condition to the $\mathrm{CD}(K,N)$ condition with $N<+\infty$ was suggested by N. Gigli in [G15, p. 75], and provides especially a splitting theorem similar to Cheeger-Gromoll's original result [G13].

The original formulation of the CD and CD^* conditions involves optimal transportation, and gradient flow theory for the infinitesimally Hilbertian condition, but we can now adopt the equivalent characterization provided a posteriori by M. Erbar, K. Kuwada and K.-T. Sturm [EKS15] based on Γ-calculus and built upon the study of the infinite dimensional case $\mathrm{RCD}^*(K,\infty)$ carried out by L. Ambrosio, N. Gigli and G. Savaré [AGS15], saying that a space $(X,\mathrm{d},\mathfrak{m})$ is $\mathrm{RCD}^*(K,N)$ if:

(i) balls grow at most exponentially i.e. $\mathfrak{m}(B_r(\bar{x}))\le c_1\exp(c_2r^2)$ for some (and thus any) $\bar{x}\in X$;

(ii) Cheeger's energy is quadratic (and this it provides a strongly regular Dirichlet form with a Γ operator);

(iii) the Sobolev-to-Lipschitz property holds, namely any $f\in H^{1,2}(X,\mathrm{d},\mathfrak{m})$ with $\Gamma(f)\le 1$; $\mathfrak{m}$-a.e. admits a 1-Lipschitz representative;

(iv) Bochner's inequality

$$\frac{1}{2}\Delta\Gamma(f)-\Gamma(f,\Delta f)\ge\frac{(\Delta f)^2}{N}+K\Gamma(f)$$

holds in the class of functions $f\in\mathrm{Lip}_b(X,\mathrm{d})\cap H^{1,2}(X,\mathrm{d},\mathfrak{m})$ such that $\Delta f\in H^{1,2}(X,\mathrm{d},\mathfrak{m})$ which, a posteriori, turns out to form an algebra [S14].

Note that quadraticity of Ch allows us to adopt the standard definition of Laplacian, namely

$$\mathcal{D}(\Delta):=\{f\in H^{1,2}(X,\mathrm{d},\mathfrak{m})\ :\text{there exists } h\in L^2(X,\mathfrak{m})\text{ such that}$$
$$\int_X\Gamma(f,g)\,\mathrm{d}\mathfrak{m}=-\int_X hg\,\mathrm{d}\mathfrak{m}\ \text{ for all } g\in H^{1,2}(X,\mathrm{d},\mathfrak{m})\,\}$$

and $\Delta f:=h$ for any $f\in\mathcal{D}(\Delta)$.

The $\mathrm{RCD}^*(K,N)$ condition holds on n-dimensional Riemannian manifolds (M,g) with $\mathrm{Ric}_g\ge Kg$ and $n\le N$ and is stable with respect to measured Gromov-Hausdorff convergence. Moreover, the collection of $\mathrm{RCD}^*(K,N)$ spaces is a subclass of the set of PI doubling spaces, and it has been shown to contain the Gromov-Hausdorff closure of $\mathcal{M}(n,K)$ for any $n\le N$. For these reasons, over the past few years the collection of $\mathrm{RCD}^*(K,N)$ spaces has appeared as an interesting class of possibly non-smooth spaces on which one could study Riemannian type properties, strictly contained in the class of $\mathrm{CD}^*(K,N)$ spaces. Chapter 2 is devoted to a detailed review on $\mathrm{RCD}^*(K,N)$ spaces and on their properties, starting from the optimal transportation context from which it originates.

Weighted Sobolev inequalities via patching

To test the validity of the $\mathrm{RCD}^*(K,N)$ condition as a good synthetic notion of Ricci curvature bounded below and dimension bounded above, many works in the recent years have aimed at proving classical results from Riemannian geometry on $\mathrm{RCD}^*(K,N)$ spaces. N. Gigli's splitting theorem [G13] and the Li-Yau Harnack inequality established in this context by R. Jiang, L. Huaiqian and H. Zhang [JLZ16] are particularly relevant examples. But, some results hold true in the broader context of $\mathrm{CD}(K,N)$ spaces, for which the Ch energy might not be quadratic and the synthetic notion of Ricci curvature bounded below by K and dimension bounded above by N is expressed in terms of K-convexity of the Rényi entropy along Wasserstein geodesics, see Section 2.1 for details. For instance, the Bishop-Gromov inequality and the local L^2 Poincaré inequality hold true on $\mathrm{CD}(K,N)$ spaces, see Theorem 2.1.14 and Theorem 2.1.16 respectively. Therefore, any result from Riemannian geometry whose proof requires only these two ingredients can be performed on $\mathrm{CD}(K,N)$ spaces, provided the smooth structure of the spaces can be forgotten in the proof.

We follow this path in Chapter 3 to establish weighted Sobolev inequalities on $\mathrm{CD}(0,N)$ spaces satisfying a suitable growth condition at infinity. The content of this chapter is taken from the submitted note [T17a] and the work in progress [T17b]. Our approach is based on an abstract patching procedure due to A. Grigor'yan and L. Saloff-Coste, which permits to glue local Sobolev inequalities into a global one via a discrete Poincaré inequality formulated on a suitable discretization of the space [GS05]. The validity of the local Sobolev inequalities follows from the doubling and Poincaré properties, and the discrete Poincaré inequality requires the additional volume growth condition.

Such a procedure was already performed in 2009 by V. Minerbe on Riemannian manifolds with Ricci curvature bounded below [Mi09]. Let us spend a few words on V. Minerbe's motivation in this context. It is well-known that any n-dimensional non-negatively Ricci curved Riemannian manifold (M,g) with volume measure vol having maximal volume growth, i.e. such that $\mathrm{vol}(B_r(x))r^{-n}$ tends as $r \to +\infty$ to some positive number $\Theta > 0$ for some (and then any) $x \in X$, satisfies the classical global Sobolev inequality:

$$\sup\left\{ \left(\int_M |f|^{2n/(n-2)} \,\mathrm{dvol}\right)^{1-2/n} \left(\int_M |\nabla f|^2 \,\mathrm{dvol}\right)^{-1} \,:\, f \in C^\infty(M)\backslash\{0\}\right\} < +\infty.$$

However, this inequality is not satisfied if the manifold has non-maximal volume growth. V. Minerbe's idea consisted in putting a weight on the volume measure vol to absorb the lack of maximal volume growth, providing an adapted weighted Sobolev inequality from which he deduced several rigidity results. H.-J. Hein subsequently extended Minerbe's result to smooth Riemannian manifolds with an appropriate polynomial growth condition and quadratically decaying lower bound on the Ricci curvature, deducing existence results and decay estimates for bounded solutions of the Poisson equation [He11].

We show that this procedure also applies on $\mathrm{CD}(0,N)$ spaces, and deduce from our weighted Sobolev inequality a weighted Nash inequality and a uniform control on the heat kernel of the associated weighted structure.

Weyl's law on $\mathrm{RCD}^*(K,N)$ spaces

Coming back to $\mathrm{RCD}^*(K,N)$ spaces, until recently the best structural result in this context was the so-called Mondino-Naber decomposition stating that any $\mathrm{RCD}^*(K,N)$ space $(X,\mathrm{d},\mathfrak{m})$ could be written, up to a $\mathfrak{m}$-negligible set, as a countable partition of bi-Lipschitz charts (U_i,φ_i) where $\varphi_i(U_i)$ is a Borel set of $\mathbb{R}^{k_i}$ and the uniformly bounded dimensions $k_i \le N$ might be varying [MN14]. Further independent works subsequently proved the absolute continuity of $\mathfrak{m}\llcorner U_i$ with respect to the corresponding Hausdorff measure $\mathscr{H}^{k_i}$ [GP16, DePhMR16, KM17]. For Ricci limits, it was known from the work of T. Colding and A. Naber [CN12] that the dimensions k_i are all the same. This result was conjectured to hold true also on general $\mathrm{RCD}^*(K,N)$ spaces, and a recent work of E. Brué and D. Semola solved positively this conjecture [BS18]. To be precise, it is by now known that for some $n =: \dim_{\mathrm{d},\mathfrak{m}}(X)$, we have $\mathfrak{m}(X\backslash\mathcal{R}_n) = 0$ where the so-called set of n-regular points $\mathcal{R}_n$ is defined to be the set of points $x \in X$ for which the limit of the sequence of rescalings $\{(X, r^{-1}\mathrm{d}, \mathfrak{m}(B_r(x))^{-1}\mathfrak{m}, x)\}_{r>0}$ is the Euclidean space $(\mathbb{R}^n, \mathrm{d}_{eucl}, \hat{\mathscr{H}}^n, 0_n)$ where $\hat{\mathscr{H}}^n = \mathscr{H}^n/\omega_n$ where ω_n is the volume of the n-dimensional Euclidean unit ball.

Motivated by this conjecture, we studied compact $\mathrm{RCD}^*(K,N)$ spaces $(X,\mathrm{d},\mathfrak{m})$ from the point of view of spectral theory. As for closed Riemannian manifolds, the doubling and Poincaré properties ensure the existence of a discrete spectrum for the Laplace operator Δ of $(X,\mathrm{d},\mathfrak{m})$ which can be represented by a non-decreasing sequence $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \ldots$ such that $\lambda_i \to +\infty$ when $i \to +\infty$. A classical result of geometric analysis is Weyl's asymptotic formula, which states that for any closed n-dimensional Riemannian manifold (M, g) one has

$$\frac{N(\lambda)}{\lambda^{n/2}} \sim \frac{\omega_n}{(2\pi)^n}\mathcal{L}^n(\Omega) \qquad \lambda \to +\infty$$

where $N(\lambda) = \sharp\{i \in \mathbb{N} : \ \lambda_i \le \lambda\}$ and $\mathcal{L}^n(\Omega)$ is the n-dimensional Lebesgue measure of Ω.

We proved in the article [AHT18] that this result holds true also on $(X,\mathrm{d},\mathfrak{m})$, but with the Lebesgue measure appearing in the above right-hand side replaced by the top-dimensional Hausdorff measure appearing in Mondino-Naber's decompotion. To this purpose, we establish a pointwise convergence result for the heat kernels of a measured Gromov-Hausdorff converging sequence of $\mathrm{RCD}^*(K,N)$ spaces. We present these results in Chapter 4: our presentation slightly differs from the published paper, since we can take into account the simplifications stemming from Brué-Semola's theorem.

Our proof is based on Karamata's theorem, which rephrases Weyl's law into a short-time asymptotic formula for the trace of the heat kernel of $(X,\mathrm{d},\mathfrak{m})$, namely

$$\int_X p(x,x,t)\mathrm{d}\mathfrak{m}(x) \sim (4\pi t)^{-n/2}\mathcal{H}^n(\mathcal{R}_n) \qquad t \to 0$$

where $n = \dim_{\mathrm{d},\mathfrak{m}}(X)$ is the dimension of $(X,\mathrm{d},\mathfrak{m})$. For $\mathfrak{m}$-a.e. point $x \in X$, the rescaled spaces $(X, r^{-1}\mathrm{d}, \mathfrak{m}(B_r(x))^{-1}\mathfrak{m}, x)$ converge to $(\mathbb{R}^n, \mathrm{d}_{eucl}, \omega_n^{-1}\mathscr{L}^n, 0)$ when $r \to 0$, and the heat kernels p^r of these rescaled spaces satisfy the scaling formula $p^{\sqrt{t}}(x,x,1) = \mathfrak{m}(B_{\sqrt{t}}(x))p(x,x,t)$ for any $t > 0$. Let us provide an informal computation in order to make clear the idea of our proof, with p^e denoting the heat kernel on $\mathbb{R}^n$:

$$\begin{aligned}
\lim_{t\to 0} t^{n/2}\int_X p(x,x,t)\,\mathrm{d}\mathfrak{m}(x) &= \lim_{t\to 0}\int_X \mathfrak{m}(B_{\sqrt{t}}(x))p(x,x,t)\frac{t^{n/2}}{\mathfrak{m}(B_{\sqrt{t}}(x))}\,\mathrm{d}\mathfrak{m}(x)\\
&= \lim_{t\to 0}\int_X p^{\sqrt{t}}(x,x,1)\frac{t^{n/2}}{\mathfrak{m}(B_{\sqrt{t}}(x))}\,\mathrm{d}\mathfrak{m}(x)\\
&= \int_X \left(\lim_{t\to 0} p^{\sqrt{t}}(x,x,1)\right)\omega_n^{-1}\left(\lim_{t\to 0}\frac{\omega_n t^{n/2}}{\mathfrak{m}(B_{\sqrt{t}}(x))}\right)\mathrm{d}\mathfrak{m}(x)\\
&= \int_X p^e(0,0,1)\frac{\mathrm{d}\mathscr{H}^n}{\mathrm{d}\mathfrak{m}}(x)\,\mathrm{d}\mathfrak{m}(x)\\
&= (4\pi)^{-n/2}\mathscr{H}^n(X).
\end{aligned}$$

Our proof consists in turning this informal computation in rigorous terms. Note that in order to justify the third equality we assume a criterion which turns out to be satisfied on all known examples of $\mathrm{RCD}^*(K,N)$ spaces (and, in particular, in all doubling spaces). The fourth equality requires a careful study of the "reverse" absolute continuity property $\mathscr{H}^n \ll \mathfrak{m}$ which is achieved using a reduced n-dimensional regular set.

Embedding $\mathrm{RCD}^*(K,N)$ spaces into a Hilbert space

In the last chapter of this thesis, we present the results of the paper [AHPT17] which shall be finalized this summer (after adding a few more extensions, but the paper is already essentially complete).

In 1994, P. Bérard, G. Besson and S. Gallot studied the asymptotic properties of a family $(\Psi_t)_{t>0}$ of embeddings of a closed n-dimensional Riemannian manifold (M,g) into the space of square-integrable real-valued sequences [BBG94]. Constructed with the eigenvalues and eigenfunctions of the Laplace-Beltrami operator, these embeddings tend to be isometric when $t \downarrow 0$, in the sense that they provide a family of pull-back metrics $(g_t)_{t>0}$ such that

$$g_t = g + A(g)t + O(t^2) \qquad t \downarrow 0 \tag{1.0.1}$$

where the smooth function $A(g)$ involves the Ricci and scalar curvatures of (M,g).

In [AHPT17], we start the study of an extension of this result, replacing (M,g) with a generic compact $\mathrm{RCD}^*(K,N)$ space $(X,\mathrm{d},\mathfrak{m})$. For convenience, we work with the family of embeddings $\Phi_t : x \mapsto p(x,\cdot,t)$, $t>0$, which take values in the space $L^2(X,\mathfrak{m})$. Thanks to the heat kernel expansion (4.0.21), this approach is equivalent to Bérard-Besson-Gallot's one and it allows us to refine the blow-up techniques which were already used in [AHT18]. Note that we provide in Proposition 5.2.1 a first-order differentiation formula for the functions Φ_t which does not appear in [AHPT17].

To provide a meaningful version of (1.0.1) on $(X,\mathrm{d},\mathfrak{m})$, we use N. Gigli's formalism [G18], and in particular the Hilbert module $L^2T(X,\mathrm{d},\mathfrak{m})$ which plays for $(X,\mathrm{d},\mathfrak{m})$ the role of an abstract space of L^2-vector fields, to provide a genuine notion of RCD metrics on $\mathrm{RCD}^*(K,N)$ spaces. Shortly said, RCD metrics are functions $\bar{g} : L^2T(X,\mathrm{d},\mathfrak{m}) \times L^2T(X,\mathrm{d},\mathfrak{m}) \to L^1(X,\mathfrak{m})$ retaining the main algebraic features of Riemannian metrics seen as functions $C^\infty(TM) \times C^\infty(TM) \to C^\infty(M)$. Among these objects, we single out a canonical element g which is characterized by the property

$$\int_X g(\nabla f_1, \nabla f_2)\,\mathrm{d}\mathfrak{m} = \int_X \Gamma(f_1,f_2)\,\mathrm{d}\mathfrak{m} \qquad \forall f_1, f_2 \in H^{1,2}(X,\mathrm{d},\mathfrak{m}),$$

where the objects $\nabla f_1, \nabla f_2$ are the analogues of L^2 gradient vector fields in Gigli's formalism - note that one can equivalently understand these objects as L^2-derivations, in which case we have $g(V,V) = |V|^2$ for any $V \in L^2T(X,\mathrm{d},\mathfrak{m})$ where $|V|$ is the local norm of the derivation V, see Remark 5.2.11.

Afterwards we show that for any $t>0$, an integrated version of the pointwise expression of the Riemannian pull-back metric g_t written in the appropriate language on $(X,\mathrm{d},\mathfrak{m})$, namely

$$\int_X \Phi_t^* g_{L^2}(V_1,V_2)(x)\,\mathrm{d}\mathfrak{m}(x) = \int_X \Big(\int_X \langle \nabla_x p(x,y,t), V_1(x)\rangle \langle \nabla_x p(x,y,t), V_2(x)\rangle\,\mathrm{d}\mathfrak{m}(y)\Big)\,\mathrm{d}\mathfrak{m}(x), \quad \forall V_1,\, V_2 \in L^2T(X,\mathrm{d},\mathfrak{m}),$$

defines a RCD metric g_t on $(X,\mathrm{d},\mathfrak{m})$.

A natural partial order $\le$ holds on the set of RCD metrics of $(X,\mathrm{d},\mathfrak{m})$ allowing to define on the space of metrics $\bar{h}$ such that $\bar{h} \le Cg$ for some $C>0$ a notion of L^2-weak convergence $\bar{g}_i \to \bar{g}$ by requiring that $\bar{g}_i(V,V) \to \bar{g}(V,V)$ holds in the weak topology of $L^1(X,\mathfrak{m})$ for any $V \in L^2T(X,\mathrm{d},\mathfrak{m})$. To define L^2-strong convergence, we rely again on Gigli's formalism, this time using the tensor products

$$L^2T(X,\mathrm{d},\mathfrak{m}) \otimes L^2T(X,\mathrm{d},\mathfrak{m}) \quad \text{and} \quad L^2T^*(X,\mathrm{d},\mathfrak{m}) \otimes L^2T^*(X,\mathrm{d},\mathfrak{m})$$

which are easily shown to be dual one to another. Any RCD metric $\bar{g}$ is then associated to a $(0,2)$ tensor $\bar{\mathbf{g}}$, and we can define the local Hilbert-Schmidt norm $|\cdot|_{HS}$ of any (difference of) tensors by duality with the Hilbert-Schmidt norm considered in [G18]. Then L^2-strong convergence $\bar{g}_i \to \bar{g}$ is defined as L^2-weak convergence plus convergence of the norms $\||\bar{\mathbf{g}}_{\mathbf{i}} - \bar{\mathbf{g}}|_{HS}\|_{HS} \to 0$.

With these basic definitions in hand, we prove L^2-strong convergence results for suitable rescalings $\mathrm{sc}_t g_t$ of g_t. Two natural scalings can be chosen. The first one is $\mathrm{sc}_t \equiv t^{(n+2)/2}$ with $n = \dim_{\mathrm{d},\mathfrak{m}}(X)$, in direct analogy with the Riemannian context, but the most natural one in the $\mathrm{RCD}^*(K,N)$ context is $\mathrm{sc}_t = t\mathfrak{m}(B_{\sqrt{t}}(\cdot))$, which takes into account the fact that $\mathrm{RCD}^*(K,N)$ spaces are indeed closer to weighted Riemannian manifolds. Thus we prove the L^2 strong convergence $t\mathfrak{m}(B_{\sqrt{t}}(\cdot)) \to c_n g$ when $t \downarrow 0$, where c_n is a positive dimensional constant. We also prove that $t^{(n+2)/2} g \to F_n g$ L^2-strongly when $t \downarrow 0$ where F_n is a $\mathfrak{m}$-measurable function which involves notably the inverse of the density of $\mathfrak{m}$ with respect to $\mathscr{H}^n$; as shown in [AHT18], this inverse is well-defined on a suitable reduced regular set $\mathcal{R}_n^*$ whose complement is $\mathfrak{m}$-negligible in X.

Let us explain in few words the strategy of the proofs. First of all, one can show that the L^2-weak convergence $\hat{g}_t := t\mathfrak{m}(B_{\sqrt{t}}(\cdot))g_t \to c_n g$ follows from the property $\int_A \hat{g}_t(V,V)\,\mathrm{d}\mathfrak{m} \to c_n \int_A g(V,V)^2\,\mathrm{d}\mathfrak{m}$ for any Borel set $A \subset X$ and any given $V \in L^2T(X,\mathrm{d},\mathfrak{m})$. By Fubini's theorem,

$$\int_A \hat{g}_t(V,V)\,\mathrm{d}\mathfrak{m} = \int_X \int_A t\mathfrak{m}(B_{\sqrt{t}}(x))\langle \nabla_x p(x,y,t), V(x)\rangle^2\,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}\mathfrak{m}(y),$$

and therefore we are left with understanding the behavior of

$$\int_A t\mathfrak{m}(B_{\sqrt{t}}(x))\langle \nabla_x p(x,y,t), V(x)\rangle^2\,\mathrm{d}\mathfrak{m}(x) \tag{1.0.2}$$

when $t \downarrow 0$ for $\mathfrak{m}$-a.e. $y \in X$, a careful application of the dominated convergence theorem leading eventually to the result. To proceed, we introduce a notion of harmonic points z of L^2-vector fields which allows us to replace V in (1.0.2) by ∇f for some $f \in H^{1,2}(X,\mathrm{d},\mathfrak{m})$ such that for any tangent space $(Y,\mathrm{d}_Y,\mathfrak{m}_Y,y) = \lim_{r_i\to 0}(X, r_i^{-1}\mathrm{d}, \mathfrak{m}(B_{r_i}(z))^{-1}\mathfrak{m}, z)$, the rescaled functions $f_{r_i,z} \in H^{1,2}(X, r_i^{-1}\mathrm{d}, \mathfrak{m}(B_{r_i}(z))^{-1}\mathfrak{m})$ converge in some suitable sense ($H^{1,2}_{loc}$-strongly) to a Lipschitz and harmonic function $\hat{f} : Y \to \mathbb{R}$. We show that the set $H(V)$ of such points has full measure in X. Assuming without any loss of generality that harmonic points are also Lebesgue points of $|\nabla f|^2$ (it will be part of the definition), we can restrict the attention to points $z \in H(V) \cap \mathcal{R}_n$ for which the following heuristic computation can be made rigorous (where $\mathrm{d}_t := \sqrt{t}^{-1}\mathrm{d}$, $\mathfrak{m}(B_{\sqrt{t}}(z))^{-1}\mathfrak{m}$, and $\hat{p}^e$ is the heat kernel of $(\mathbb{R}^n, \mathrm{d}_{eucl}, \hat{\mathscr{H}}^n)$): for any $L > 0$,

$$\begin{aligned}
&\int_{B_{L\sqrt{t}}(z)} t\mathfrak{m}(B_{\sqrt{t}}(x))\langle \nabla_x p(x,z,t), \nabla f(x)\rangle^2\,\mathrm{d}\mathfrak{m}(x)\\
=\ &\int_{B_L^{\mathrm{d}_t}(z)} \mathfrak{m}_t(B_1^{\mathrm{d}_t}(x))\langle \nabla_x p^{\sqrt{t}}(x,z,1), \nabla f_{\sqrt{t},z}(x)\rangle^2\,\mathrm{d}\mathfrak{m}_t(x)\\
\xrightarrow{t\downarrow 0}\ &\int_{B_L(0_n)} \hat{\mathscr{H}}^n(B_1(x))\langle \nabla_x \hat{p}^e(x,0_n,1), \nabla \hat{f}(x)\rangle^2\,\mathrm{d}\mathscr{H}^n(x)\\
=\ &c_n(L)\sum_{j=1}^n \left|\frac{\partial \hat{f}}{\partial x_j}\right|^2 = c_n(L)(|\nabla f|^2)^*(z)
\end{aligned}$$

for some constant $c_n(L) > 0$ which is such that $c_n(L) \to c_n$ when L $\to +\infty$ and where $(|\nabla f|^2)^*(z) = \lim_{r\to 0} \fint_{B_r(z)} |\nabla f|^2\,\mathrm{d}\mathfrak{m}$ is well-defined as z is a Lebesgue point of $|\nabla f|^2$.

This computation is at the core of Proposition 5.3.9 which contains most of the technical ingredients leading to the convergence

$$\int_X \int_A t\mathfrak{m}(B_{\sqrt{t}}(x))\langle \nabla_x p(x,y,t), V(x)\rangle^2 \, d\mathfrak{m}(x) \, d\mathfrak{m}(y) \to c_n \int_A |V|^2 \, d\mathfrak{m} \qquad t \to 0.$$

In order to improve the convergence $\hat{g}_t \to c_n g$ from L^2-weak to L^2-strong, we need to prove convergence for the Hilbert-Schmidt local norm which in our case can be translated into the following estimate:

$$\limsup_{t\downarrow 0} \int_X \left(t\mathfrak{m}(B_{\sqrt{t}}(x))\right)^2 \left| \int_X \nabla_x p(x,y,t) \otimes \nabla_x p(x,y,t) \, d\mathfrak{m}(y) \right|^2_{HS} d\mathfrak{m}(x) \le n c_n^2 \mathfrak{m}(X).$$

The proof of this estimate requires a more delicate blow-up procedure. We refer to Section 5.4 for the details. We prove the L^2-weak/strong convergence $\tilde{g}_t := t^{(n+2)/2} g_t \to F_n g$ in a similar way.

Finally, building on stability results of [AH17a] and extending classical estimates on eigenvalues and eigenfunctions of the Riemannian Laplace-Beltrami operator to the $\mathrm{RCD}^*(K,N)$ setting, we show that for any measured Gromov-Hausdorff convergent sequence of compact $\mathrm{RCD}^*(K,N)$ spaces $(X_j, \mathrm{d}_j, \mathfrak{m}_j) \to (X, \mathrm{d}, \mathfrak{m})$ and any $t_j \to t > 0$, we have Gromov-Hausdorff convergence $\Phi_{t_j}(X_j) \to \Phi_t(X)$, where the distances are induced by the corresponding L^2 scalar products.

Further directions and open questions

The work described in this introduction leads to several interesting questions which could be future research projects.

A first issue concerns applications of the weighted Sobolev inequalities we have established on $\mathrm{CD}(0,N)$ spaces. For instance, is there a way to exploit it in order to get existence, boundedness and decay estimates of the solutions of Poisson's equation, as done by H.-J. Hein in [He11] on Riemannian manifolds?

A second issue concerns Weyl's law. In 1980, V. Ivrii proved [Ivr80] that for any compact Riemannian manifolds (M,g) with non-empty boundary, under a rather mild assumption,

$$N(\lambda) = \frac{\omega_n}{(2\pi)^n} \mathscr{H}^n(M) \lambda^{n/2} \pm \frac{\pi}{2} \frac{\omega_{n-1}}{(2\pi)^{n-1}} \mathscr{H}^{n-1}(\partial M) \lambda^{(n-1)/2} + o(\lambda^{(n-1)/2}) \qquad \lambda \to +\infty. \tag{1.0.3}$$

An appropriate notion of boundary in $\mathrm{RCD}^*(K,N)$ spaces is still a topic of research. However, the subclass of $\mathrm{RCD}^*(K,N)$ spaces made of stratified spaces [BKMR18] which are, loosely speaking, manifolds with conical singularities of codimension at least two, might suit well a study of further terms in the expansion (1.0.3).

Finally, the work [AHPT17] opens several questions. The first one is: how to turn the RCD metrics g_t into distances d_t converging to d when $t \downarrow 0$? A possible way would be to study the functionals $\mathrm{Ch}_t(f) := \int_X \int_X |\langle \nabla_x p(x,y,t), \nabla f(x)\rangle|^2 \, d\mathfrak{m}(y) \, d\mathfrak{m}(x)$ from which one can define the intrinsic pseudo-distances: for any $y, z \in X$,

$$\mathrm{d}_t(y,z) := \sup\{|f(y) - f(z)| \;:\; f \in \mathrm{Lip}(X) \text{ s.t. } \int_X |\langle \nabla_x p(x,\cdot,t), \nabla f(x)\rangle|^2 \, d\mathfrak{m} \le 1 \text{ for } \mathfrak{m}\text{-a.e. } x \in X\}.$$

It seems doable to show that for some constant C depending only on K and N, we have $\mathrm{d}_t \le C\mathrm{d}$ for any $t > 0$ sufficiently small. However, in our attempts to prove a reverse estimate, we need a bound from below for the norm of the gradient of the heat kernel. It does not seem that such a bound has been studied yet, not even on Riemannian manifolds. Another approach involving a family of Wasserstein distances $W_{2,t}$ suitable to define the distances d_t has been proposed by L. Ambrosio.

A second question is: we know that Φ_t provides an homeomorphism from X to $L^2(X,\mathfrak{m})$, but to what extent the family $(g_t)_t$ provides a regularization of the space $(X,\mathrm{d},\mathfrak{m})$? Indeed, as the heat kernel has nice regularizing properties on functions (L^1 to C^∞ on Riemannian manifolds, L^1 to Lipschitz on $\mathrm{RCD}^*(K,N)$ spaces), one motivation for the extension of Bérard-Besson-Gallot's theorem to the setting of $\mathrm{RCD}^*(K,N)$ spaces was to produce an approximation scheme of any compact $\mathrm{RCD}^*(K,N)$ space $(X,\mathrm{d},\mathfrak{m})$ with more regular spaces $(X,\mathrm{d}_t,\mathfrak{m})$, say spaces which can be embedded by bi-Lipschitz maps into the Euclidean space. Note that another approximation has been proposed by N. Gigli and C. Mantegazza in [GM14], however M. Erbar and N. Juillet proved on cones that it has not the desired regularizing properties [EKS16]. In any case, it may be interesting to compare the two approaches: indeed, on Riemannian manifolds, Bérard-Beeson-Gallot's family of metrics is tangent to the gradient flow of the Hilbert-Einstein functional, whereas Gigli-Mantegazza's family is, in a weak sense, tangent to the Ricci flow.

Finally, a last research direction could be to start from the Riemannian expansion (1.0.1) (or (5.1.6)) together with the notion of measure-valued Ricci tensor $\mathrm{Ric}_{\mathrm{d},\mathfrak{m}}$ proposed by B.-X. Han [Ha17] to provide a notion of scalar curvature bounded below/above on compact $\mathrm{RCD}^*(K,N)$ spaces $(X,\mathrm{d},\mathfrak{m})$:

$$\liminf_{t\downarrow 0} 2\left(3\frac{\hat{g}_t - g}{t}\mathfrak{m} + \mathrm{Ric}_{\mathrm{d},\mathfrak{m}}\right) \geq K\mathfrak{m},$$

$$\limsup_{t\downarrow 0} 2\left(3\frac{\hat{g}_t - g}{t}\mathfrak{m} + \mathrm{Ric}_{\mathrm{d},\mathfrak{m}}\right) \leq K\mathfrak{m}.$$

Chapter 2

Preliminaries

This first chapter is dedicated to the background knowledge on the RCD theory.

2.1 Curvature-dimension conditions via optimal transport

In this section, we present Sturm's and Lott-Villani's optimal transport curvature-dimension conditions, nowadays known as $\mathrm{CD}(K,N)$ conditions where C stands for curvature, D for dimension, $K \in \mathbb{R}$ for a lower bound on the curvature and $N \in [1,+\infty]$ for an upper bound on the dimension. According to M. Ledoux, the first occurence of the notation "$\mathrm{CD}(K,N)$" goes back to [Ba91] in which D. Bakry denoted a curvature-dimension condition previously introduced by D. Bakry himself and M. Émery [BE85] in the setting of Markov diffusion operators. We will return on Bakry-Émery's notion in Section 2.3.

Preliminaries in optimal transport theory

Let us start with recalling some notions from optimal transport theory. We refer to [Vi03, Ch. 7] or [Vi09, Ch. 6] for a more detailed treatement and proofs of the statements.

Let (X,d) be a Polish (meaning complete and separable) metric space. We denote by $\mathcal{P}(X)$ the set of probability measures on (X,d), i.e. positive Borel measures μ such that $\mu(X)=1$.

If (X,d) is compact, we equip $\mathcal{P}(X)$ with the Wasserstein distance W_2 defined by:

$$\mathrm{W}_2(\mu_0,\mu_1) := \inf_{\pi\in\mathrm{TP}(\mu_0,\mu_1)} \left(\int_X \mathrm{d}^2(x_0,x_1)\,\mathrm{d}\pi(x_0,x_1)\right)^{1/2} \qquad \forall \mu_0,\mu_1 \in \mathcal{P}(X),$$

where $\mathrm{TP}(\mu_0,\mu_1)$ is the set of transport plans between μ_0 and μ_1, namely probability measures $\pi \in \mathcal{P}(X\times X)$ with first marginal μ_0 and second marginal μ_1. The above infimum is always achieved, and any minimizer is called optimal transport plan between μ_0 and μ_1. The distance W_2 metrizes the weak topology. Moreover, the space $(\mathcal{P}(X),W_2)$ is compact, with diameter equals to the diameter of X, as one can easily see from the embedding $X \ni x \mapsto \delta_x \in \mathcal{P}_2(X)$.

If (X,d) is noncompact, without any further assumptions on μ_0 and μ_1, the quantity $W_2(\mu_0,\mu_1)$ might be infinite, as one can check by applying Kantorovitch duality formula ([Vi03, Th. 1.3], [Vi09, Th. 5.10]) to $(X,\mathrm{d}) = (\mathbb{R},\mathrm{d}_{eucl})$ with $\mu_0 = |\cdot|^{-2}1_{(-\infty,-1]}$ and $\mu_1 = |\cdot|^{-2}1_{[1,+\infty)}$. Therefore, we restrict W_2 to the set $\mathcal{P}_2(X)$ of probability measures μ with finite second moment, meaning that $\int_X \mathrm{d}(x,x_0)^2\,\mathrm{d}\mu(x) < +\infty$ for some $x_0 \in X$. It can be easily checked that W_2 takes only finite values on $\mathcal{P}_2(X)\times\mathcal{P}_2(X)$. Note that having finite second moment does not depend on the base point x_0, as one can immediately verified from the triangle inequality.

When (X, d) is compact, $\mathcal{P}_2(X)$ and $\mathcal{P}(X)$ coincide. Therefore, regardless of compactness properties of (X, d), we will always consider W_2 defined on $\mathcal{P}_2(X)$.

Let us finally point out that the Polish structure of (X, d) transfers to the metric space $(\mathcal{P}_2(X), W_2)$, and that in the non-compact case, the W_2-convergence $\mu_n \to \mu$ is equivalent to weak convergence of μ_n to μ together with convergence of the second moments $\int_X \mathrm{d}(x, x_o)^2 \, \mathrm{d}\mu_n(x) \to \int_X \mathrm{d}(x, x_o)^2 \, \mathrm{d}\mu(x)$ taken with respect to any fixed base point $x_o \in X$.

Geodesics

By definition, a geodesic on (X, d) is a continuous curve $\gamma : [0, 1] \to X$ such that $\mathrm{d}(\gamma_s, \gamma_t) = |s - t| \mathrm{d}(\gamma_0, \gamma_1)$ for any $s, t \in [0, 1]$. The set of all geodesics on (X, d) is denoted by $\mathrm{Geo}(X, \mathrm{d})$, or more simply $\mathrm{Geo}(X)$ whenever the distance d is clear from the context. Any generic geodesic on (X, d) is often called a d-geodesic. Note that if (X, d) is a Riemannian manifold equipped with its canonical Riemannian distance, such geodesics γ coincide with smooth constant speed curves which locally minimize the energy (or equivalently, the length) functional, see [GHL04, Section 2.C.3].

The metric space (X, d) is called geodesic whenever for any $x, y \in X$ there exists a geodesic γ such that $\gamma_0 = x$ and $\gamma_1 = y$. If (X, d) is geodesic, then $(\mathcal{P}_2(X), W_2)$ is geodesic too. In particular, if (M, d) is a Riemannian manifold equipped with its canonical Riemannian distance, then $(\mathcal{P}_2(M), W_2)$ is geodesic. Any W_2-geodesic is sometimes also called Wasserstein geodesic. Finally, let us recall the following important proposition (which is a consequence of a more general characterization of Wasserstein geodesics, see [AG13, Th. 2.10] for instance).

Proposition 2.1.1. *Let (X, d) be a Polish geodesic space, $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ and $(\mu_t)_{t \in [0,1]}$ be the Wasserstein geodesic between μ_0, μ_1. Then*

$$\mathrm{supp}(\mu_t) \subset \{\gamma(t) \, : \, \gamma \in \mathrm{Geo}(X) \textit{ s.t. } \gamma_0 \in \mathrm{supp}(\mu_0) \textit{ and } \gamma_1 \in \mathrm{supp}(\mu_1)\}.$$

Reference measure

Throughout the whole thesis, we will deal with metric measure spaces, namely triples $(X, \mathrm{d}, \mathfrak{m})$ where (X, d) is a metric space, and $\mathfrak{m}$ is a non-negative Borel measure on (X, d) which will always be assumed finite and non-zero on balls with finite and non-zero radius.

We denote by $\mathcal{P}_2^a(X, \mathfrak{m})$ the set of probability measures μ on (X, d) which are absolutely continuous with respect to $\mathfrak{m}$, i.e. such that $\mu(A) = 0$ whenever $\mathfrak{m}(A) = 0$ for any Borel set $A \subset X$. Recall that for any $\mu \in \mathcal{P}_2^a(X, \mathfrak{m})$, the Radon-Nikodym theorem ensures the existence of a $\mathfrak{m}$-measurable function $\rho : X \to [0, +\infty)$ called density of μ with respect to $\mathfrak{m}$ such that $\mu(A) = \int_A \rho \, \mathrm{d}\mathfrak{m}$ for any Borel set $A \subset X$.

Displacement convexity

In [Mc97], R. McCann studied the existence of unique minimizers for energy functionals modelling a gas in $\mathbb{R}^n$ interacting only with itself. Such functionals, defined on $\mathcal{P}(\mathbb{R}^n)$, required an appropriate notion of convexity in order to be treated by classical means of convex analysis. Indeed, for the simple example

$$F(\mu) = \iint_{\mathbb{R} \times \mathbb{R}} |x - y|^2 \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(y),$$

whose convex interaction density $A(x) = |x|^2$ suggests a convex behavior of the functional, one can check that $F((1-t)\delta_0 + t\delta_1) = 2t(1-t)$ for any $t \in [0, 1]$, and the map $t \mapsto 2t(1-t)$ is concave. In other words, F is not convex for the traditional linear structure of $\mathcal{P}(\mathbb{R}^n)$.

To deal with this difficulty, R. McCann proposed to interpolate measures in $\mathcal{P}(\mathbb{R}^n)$ - to be fair, in $\mathcal{P}_2^a(\mathbb{R}^n)$, the reference measure being tacitly the n-dimensional Lebesgue measure - using Wassertein geodesics, and to study convexity of functionals F along these geodesics. This led him to introduce the crucial notion of displacement convexity, that we phrase here in the context of a general metric measure space $(X, \mathrm{d}, \mathfrak{m})$.

Definition 2.1.2 (displacement convexity)**.** We say that a functional $F : \mathcal{P}_2^a(X, \mathfrak{m}) \to \mathbb{R} \cup \{+\infty\}$ is displacement convex if for any W_2-geodesic $(\mu_t)_{t\in[0,1]}$,

$$F(\mu_t) \leq (1-t)F(\mu_0) + tF(\mu_1) \qquad \forall t \in [0,1].$$

As a matter of fact, R. McCann showed that any internal-energy functional:

$$F(\mu) = \int_{\mathbb{R}^n} A\left(\frac{\mathrm{d}\mu}{\mathrm{d}\mathscr{L}^n}\right) \mathrm{d}\mathscr{L}^n \qquad \forall \mu \in \mathcal{P}_2^a(\mathbb{R}^n),$$

is displacement convex as soon as its density $A : [0, +\infty) \to \mathbb{R} \cup \{+\infty\}$ satisfies

$$\lambda \mapsto \lambda^n A(\lambda^{-n}) \text{ be convex non-increasing on } (0, +\infty) \text{ and } A(0) = 0.$$

The set of such functions A is called the n-dimensional displacement convex class, and is usually denoted by $\mathcal{DC}_n$.

Distorted displacement convexity

In order to extend the Euclidean Borell-Brascamp-Lieb inequality to complete connected Riemannian manifolds, D. Cordero-Erausquin, R. McCann and M. Schmuckenschläger introduced in [CMS01] the following so-called distortion coefficients.

Definition 2.1.3. Let (M, g) be a smooth Riemannian manifold with canonical volume measure vol. Then the distortion coefficients of (M, g) are the non-negative functions $\{\beta_t : M \times M \to \mathbb{R}\}_{t\in[0,1]}$ defined as follows:

- set β_1 constantly equal to 1;
- for $t \in (0,1)$, for any $x, y \in M$,
  - (i) if x and y are joined by a unique geodesic, set
  $$\beta_t(x,y) := \lim_{r\to 0} \frac{\mathrm{vol}(Z_t(x, B_r(y)))}{\mathrm{vol}(B_{tr}(y))}$$
  where $Z_t(x, B_r(y))$ is the set of t-barycenters between $\{x\}$ and $B_r(y)$, namely
  $$\bigcup_{\tilde{y}\in B_r(y)} \{z \in M \,:\, \mathrm{d}(x,z) = t\mathrm{d}(x,\tilde{y}) \text{ and } \mathrm{d}(z,\tilde{y}) = (1-t)\mathrm{d}(x,\tilde{y})\},$$
  - (ii) if x and y are joined by several geodesics, set
  $$\beta_t(x,y) := \inf_{\gamma} \limsup_{s\to 1^-} \beta_t(x, \gamma_s)$$
  where the infimum is taken over all geodesics γ such that $\gamma_0 = x$ and $\gamma_1 = y$;
- for any $x, y \in M$, set $\beta_0(x,y) := \lim_{t\to 0} \beta_t(x,y)$.

Note that $\beta_t(x,y) = +\infty$ if and only if x and y are conjugate points.

The physical meaning of this distortion coefficients is explained at length in [Vi09, p. 394-395]. Armed with it, one can deal with energy functionals defined over any non-flat Riemannian manifold, by upgrading displacement convexity into a notion taking into account the distorted geometry of the manifold. This is the content of the next definition, which we write directly in the context of a general metric measure space $(X, \mathrm{d}, \mathfrak{m})$, and where β stands for a family of non-negative functions $\{\beta_t : X \times X \to \mathbb{R}\}_{t\in[0,1]}$, coinciding with the above coefficients when X is a Riemannian manifold.

Definition 2.1.4 (distorted displacement convexity)**.** Let $F : \mathcal{P}_2^a(X, \mathfrak{m}) \to \mathbb{R} \cup \{+\infty\}$ be an internal-energy functional with continuous and convex density A satisfying $A(0) = 0$. We say that F is displacement convex with distortion β if for any W_2-geodesic $(\mu_t)_{t\in[0,1]}$, there exists an optimal transport plan π between μ_0 and μ_1, such that

$$F(\mu_t) \le (1-t)\int_{X\times X} A\left(\frac{\rho_0(x_0)}{\beta_{1-t}(x_0,x_1)}\right)\beta_{1-t}(x_0,x_1)\,\mathrm{d}\pi_{\mu_0}(x_1)\,\mathrm{d}\mathfrak{m}(x_0) \tag{2.1.1}$$

$$+\, t\int_{X\times X} A\left(\frac{\rho_1(x_1)}{\beta_t(x_0,x_1)}\right)\beta_t(x_0,x_1)\,\mathrm{d}\pi_{\mu_1}(x_0)\,\mathrm{d}\mathfrak{m}(x_1) \tag{2.1.2}$$

for all $t \in [0, 1]$, where ρ_0 (resp. ρ_1) is the density of μ_0 (resp. μ_1) with respect to $\mathfrak{m}$, and π_{μ_0} (resp. π_{μ_1}) denotes the disintegration w.r.t. μ_0 (resp. μ_1) of the optimal transport plan π between μ_0 and μ_1.

Remark 2.1.5. Introducing for all $t \in [0, 1]$ the distorted functionals

$$\tilde{F}_\pi^{\beta_t}(\mu) := \int_{X\times X} A\left(\frac{1}{\beta_t(x_0,x_1)}\frac{\mathrm{d}\mu}{\mathrm{d}\mathfrak{m}}\right)\beta_t(x_0,x_1)\,\mathrm{d}\pi_{\mu_0}(x_1)\,\mathrm{d}\mathfrak{m}(x_0) \quad \forall \mu \in \mathcal{P}_2^a(X,\mathfrak{m})$$

and

$$\tilde{F}_{\breve{\pi}}^{\beta_t}(\mu) := \int_{X\times X} A\left(\frac{1}{\beta_t(x_0,x_1)}\frac{\mathrm{d}\mu}{\mathrm{d}\mathfrak{m}}\right)\beta_t(x_0,x_1)\,\mathrm{d}\pi_{\mu_1}(x_0)\,\mathrm{d}\mathfrak{m}(x_1) \quad \forall \mu \in \mathcal{P}_2^a(X,\mathfrak{m}),$$

then (2.1.1) writes in the more concise way

$$F(\mu_t) \le (1-t)\tilde{F}_\pi^{\beta_{1-t}}(\mu_0) + t\tilde{F}_{\breve{\pi}}^{\beta_t}(\mu_1).$$

Reference distortion coefficients

For the three reference spaces $\mathbb{R}^n$, $\mathbb{S}^n$ and $\mathbb{H}^n$, which have constant Ricci curvature equal to 0, $n-1$ and $-(n-1)$ respectively, the distortion coefficients are explicitly computable. Indeed, for $\mathbb{R}^n$, it is immediatly checked that $\beta_t \equiv 1$ for any $t \in [0, 1]$. Let us explain in few words how to do the computation on $\mathbb{S}^n$, refering to [GHL04] for the basic notions of Riemannian geometry involved. We only treat the simple case $t \in (0, 1]$, and assume that x and y are not conjugate, the other cases not being much more difficult to handle, but more lengthy. As x and y are not conjugate, there exists a unique geodesic γ such that $\gamma(0) = x$ and $\gamma(1) = y$. Let $(e_1, \dots, e_n)$ be an orthonormal basis of $T_y\mathbb{S}^n$ such that $e_n = \gamma'(1)$, and for any $t \in [0, 1]$ and $1 \le i \le n-1$, let $U_i(t) \in T_{\gamma(t)}\mathbb{S}^n$ be the parallel transport of e_i along γ. Then for any $1 \le i \le n-1$, the unique Jacobi field J_i along γ such that $J_i(0) = 0$ and $J_i(1) = e_i$ is given by

$$J_i(t) = \frac{\sin(t\mathrm{d}(x,y))}{\sin(\mathrm{d}(x,y))}U_i(t) \qquad \forall t \in [0,1].$$

Then it follows immediately from [Vi09, Prop. 14.18] that

$$\beta_t(x,y) = \left(\frac{\sin(t\mathrm{d}(x,y))}{t\sin(\mathrm{d}(x,y))}\right)^{n-1}$$

as $\mathbf{J}^{0,1}(t)$ there is precisely the matrix formed by $J_1(t), \ldots, J_{n-1}(t), t\gamma'(t)$.

A similar computation can be performed for $\mathbb{H}^n$, as well as for the scaled sphere with Ricci curvature constantly equal to $(n-1)K$ for some $1 \neq K > 0$ and for the scaled hyperbolic space with Ricci curvature constantly equal to $-(n-1)K$. This motivates the introduction of the reference coefficients $\{\beta_t^{(K,N)} : [0,+\infty) \to [0,+\infty]\}_{t\in[0,1]}$, defined for any $K \in \mathbb{R}$ and $N \in [1,+\infty]$ as follows:

- $\beta_0^{(K,N)}$ constantly equal to 1;
- if $0 < t \leq 1$ and $1 < N < +\infty$,

$$\beta_t^{(K,N)}(\alpha) = \begin{cases} +\infty & \text{if } K > 0 \text{ and } \alpha > \pi, \\ \left(\frac{\sin(t\sqrt{K/(N-1)}\alpha)}{t\sin(\sqrt{K/(N-1)})} \right)^{N-1} & \text{if } K > 0 \text{ and } 0 \leq \alpha \leq \pi, \\ 1 & \text{if } K = 0, \\ \left(\frac{\sinh(t\sqrt{|K|/(N-1)}\alpha)}{t\sinh(\sqrt{|K|(N-1)})} \right)^{N-1} & \text{if } K < 0. \end{cases}$$

- if $0 < t \leq 1$ and $N = 1$, modify the above expressions as follows:

$$\beta_t^{(K,1)}(\alpha) = \begin{cases} +\infty & \text{if } K > 0, \\ 1 & \text{if } K \leq 0. \end{cases}$$

- if $0 < t \leq 1$ and $N = +\infty$, modify only

$$\beta_t^{(K,\infty)}(\alpha) = e^{\frac{K}{6}(1-t^2)\alpha^2}.$$

We will eventually use $\alpha = \mathrm{d}(x,y)$.

Sturm and Lott-Villani conditions : infinite dimensional case

Now that the appropriate language is set up, we are in a position to express how to read curvature using optimal transportation.

Although similar in spirit, Sturm's and Lott-Villani's approaches are slightly different; nevertheless, as we shall explain later, they coincide on a large class of spaces. Let us start with an informal explanation, inspired by [Vi03, p. 445], to motivate Sturm's definition. Let us consider the sphere $S^n \subset \mathbb{R}^{n+1}$, whose curvature is known to be constant and equal to $n-1$, and rescale it to work with the sphere with constant curvature equal to 1. Imagine that a gas made of non-interacting particles is supported in a region U_0 of the sphere, close to the equator but lying completely inside the north hemisphere, and that we want to let it evolve into, say, the symmetric U_1 of this region with respect to the equator. To do so in the most efficient way (efficiency being measured here by the Wasserstein distance), particles must follow geodesics from U_0 to U_1. Starting from U_0, such geodesics first move away one to another, before drawing near back when approching U_1. Consequently, during the transportation of the gas, its denstiy ρ lowers constantly until an intermediate time before constantly re-increasing. This implies a concave behavior for the entropy $S(\rho) = -\int \rho \log \rho$, which measures the spreading of the gas, or a convex behavior of the Boltzmann entropy

$$\mathrm{Ent}(\rho) := \int \rho \log \rho,$$

which in turn measures the concentration of the gas.

Remark 2.1.6. Let us propose a simple example to illustrate how Ent measures the concentration of a density ρ. Set $L^1(\mathbb{R},\mathscr{L}^1) \ni \rho_n : x \mapsto (\int_{\mathbb{R}} e^{-n|x|}\,\mathrm{d}x)^{-1}e^{-n|x|}$ and notice that ρ_n concentrates around the origin for n high (more rigorously, $\rho_n \to \delta_0$ in $\mathcal{D}'(\mathbb{R})$). A direct computation shows that $\mathrm{Ent}(\rho_n) = 2\log(n/2) - 2$, then $\mathrm{Ent}(\rho_n) \to +\infty$ when $n \to +\infty$.

According to this observation, the following conjecture, due to F. Otto and C. Villani [OV00], sounds natural: non-negativity of the Ricci curvature of a manifold implies displacement convexity of the functional $\mathrm{Ent}_{\mathrm{vol}}$. This conjecture was proved true by D. Cordero-Erausquin, R. McCann and M. Schmuckenschläger [CMS01, Th. 6.2]. Afterwards, K.-T. Sturm and M.-K. Von Renesse dramatically improved this result: calling K-displacement convex any functional $F : \mathcal{P}_2^a(X,\mathfrak{m}) \to \mathbb{R}\cup\{+\infty\}$ such that for any W_2-geodesic $(\mu_t)_{t\in[0,1]}$,

$$F(\mu_t) \le (1-t)F(\mu_0) + tF(\mu_1) - K\frac{t(1-t)}{2}W_2(\mu_0,\mu_1) \qquad \forall t\in[0,1],$$

they showed the following characterization of Ricci curvature bounded below [RS05, Th. 1].

Theorem 2.1.7. *Let (M,g) be a smooth connected Riemannian manifold with canonical Riemannian volume measure denoted by* vol. *Then the following two properties are equivalent:*

(i) $\mathrm{Ric}_g \ge Kg$*;*

(ii) the functional $\mathrm{Ent}_{\mathrm{vol}} : \mathcal{P}_2^a(M,\mathrm{vol}) \to \mathbb{R}$ *defined by:*

$$\mathrm{Ent}_{\mathrm{vol}}(\mu) := \int_M \frac{\mathrm{d}\mu}{\mathrm{dvol}}\log\left(\frac{\mathrm{d}\mu}{\mathrm{dvol}}\right)\mathrm{dvol} \qquad \forall \mu\in\mathcal{P}_2^a(M,\mathrm{vol}),$$

is K-displacement convex.

Note that condition (ii) can be formulated on any Polish geodesic metric measure space $(X,\mathrm{d},\mathfrak{m})$ satisfying the following exponential estimate on the volume growth of balls: for some $x\in X$, there exist $c_0, c_1 > 0$ such that:

$$\mathfrak{m}(B_r(x)) \le c_0 e^{c_1 r} \qquad \forall r>0. \tag{2.1.3}$$

Indeed, such a condition implies that for any $\mu\in\mathcal{P}_2^a(X,\mathfrak{m})$, the negative part of $A(\mu) := \frac{\mathrm{d}\mu}{\mathrm{d}\mathfrak{m}}\log\left(\frac{\mathrm{d}\mu}{\mathrm{d}\mathfrak{m}}\right)$ is integrable, and then the functional $\mathrm{Ent}_{\mathfrak{m}} : \mathcal{P}_2^a(X,\mathfrak{m}) \ni \mu \mapsto \int_X A(\mu)\,\mathrm{d}\mathfrak{m}$ cannot take the value $-\infty$, implying meaningfulness of the K-displacement convexity assumption. Sturm's $\mathrm{CD}(K,\infty)$ condition is built on this observation. However, K-displacement convexity is a too strong requirement for non-smooth spaces, especially because of convergence issues that we shall describe in Section 2.4. One needs therefore to introduce the notion of weak K-displacement convexity.

Definition 2.1.8. We say that a functional $F : \mathcal{P}_2^a(X,\mathfrak{m}) \to \mathbb{R}\cup\{+\infty\}$ is weakly K-displacement convex if between any $\mu_0,\mu_1\in\mathcal{P}_2^a(X,\mathfrak{m})$, there exists at least one W_2-geodesic $(\mu_t)_{t\in[0,1]}$ such that $F(\mu_t) \le (1-t)F(\mu_0) + tF(\mu_1) - Kt(1-t)W_2(\mu_0,\mu_1)/2$ for all $t\in[0,1]$.

In order to emphasize the difference with the weak notion, K-displacement convexity is often refers as "strong K-displacement convexity" in the literature. We are now in a position to state Sturm's $\mathrm{CD}(K,\infty)$ condition.

Definition 2.1.9. (Sturm's $\mathrm{CD}(K,\infty)$ condition) A Polish geodesic metric measure space $(X,\mathrm{d},\mathfrak{m})$ is called Sturm $\mathrm{CD}(K,\infty)$ if $\mathrm{Ent}_{\mathfrak{m}}$ is weakly K-displacement convex.

In [LV09], J. Lott and C. Villani followed a slightly different path, in continuity of McCann's works, by considering the functionals

$$F(\mu)=\int_X A\left(\frac{\mathrm{d}\mu}{\mathrm{d}\mathfrak{m}}\right)\mathrm{d}\mathfrak{m} \tag{2.1.4}$$

with density $A:[0,+\infty)\to\mathbb{R}\cup\{+\infty\}$ satisfying

$$\lambda\mapsto e^{\lambda}A(e^{-\lambda}) \text{ be convex non-increasing on } (0,+\infty) \text{ and } A(0)=0.$$

The set of such densities A is called the ∞-dimensional displacement convexity class, and denoted by $\mathcal{DC}_\infty$. Note that for every $A\in\mathcal{DC}_\infty$, the limit

$$A'_+(r)=\lim_{s\to r^+}\frac{A(s)-A(r)}{s-r}$$

makes sense for any $r>0$. Here is Lott-Villani's condition.

Definition 2.1.10. (Lott-Villani's $\mathrm{CD}(K,\infty)$ condition) A Polish geodesic metric measure space $(X,\mathrm{d},\mathfrak{m})$ is called Lott-Villani $\mathrm{CD}(K,\infty)$ if for any $\mu_0,\mu_1\in\mathcal{P}_2^a(X,\mathfrak{m})$, there exists a W_2-geodesic $(\mu_t)_{t\in[0,1]}$ such that for all $A\in\mathcal{DC}_\infty$, denoting by $F:\mathcal{P}_2^a(X,\mathfrak{m})\to\mathbb{R}$ the corresponding functional (2.1.4), one has:

$$F(\mu_t)\le(1-t)F(\mu_0)+tF(\mu_1)-\frac{1}{2}\lambda_K(A)t(1-t)W_2(\mu_0,\mu_1)^2 \qquad \forall t\in[0,1],$$

where $\lambda_K(A):=\inf_{r>0}K(A'_+(r)-A(r)/r)$.

Note that Lott-Villani's condition is often refered as weak because of the requirement of convexity along at least only one W_2-geodesic.

The bridge between Sturm's and Lott-Villani's conditions is provided by the following observation. For any continuous and convex function $A:(0,+\infty)\to\mathbb{R}$, let us define the associated pressure p^A and iterated pressure p_2^A by $p^A(r):=rA'_+(r)-A(r)$ and $p_2^A(r)=rp'(r)-p(r)$ for any $r>0$. Then $A\in\mathcal{DC}_\infty$ if and only if $p_2^A\ge0$, and $p_2^A\equiv0$ if and only if $A(r)=r\log r$. In this regard, the entropy $\mathrm{Ent}_{\mathfrak{m}}$ can be seen as a borderline case in the class of functionals (2.1.4) with density A belonging to $\mathcal{DC}_\infty$.

Sturm's and Lott-Villani's conditions: finite dimensional case

Let us present now the $\mathrm{CD}(K,N)$ conditions for $N<+\infty$. Recall that $K\in\mathbb{R}$ is kept fixed. Here again, Sturm's and Lott-Villani's approaches differ a little bit. Let us present first Lott-Villani's condition, formulated with the displacement convex class $\mathcal{DC}_N$. Here if $F:\mathcal{P}_2^a(X,\mathfrak{m})\to\mathbb{R}$ is an internal-energy functional with density A, we stress the dependance in A of F by means of the notation F_A.

Definition 2.1.11. A Polish geodesic metric measure space $(X,\mathrm{d},\mathfrak{m})$ is called Lott-Villani $\mathrm{CD}(K,N)$ if the family of functionals $\{F_A:\mathcal{P}_2^a(X,\mathfrak{m})\to\mathbb{R}\cup\{+\infty\}\}_{A\in\mathcal{DC}_N}$ is jointly K-displacement convex with distortion β, meaning that for any $\mu_0,\mu_1\in\mathcal{P}(X)$ with $\mathrm{supp}(\mu_0),\mathrm{supp}(\mu_1)\subset\mathrm{supp}(\mathfrak{m})$ compact, there exists a W_2-geodesic $(\mu_t)_{t\in[0,1]}$ and an optimal transport plan π between μ_0 and μ_1 such that for any $A\in\mathcal{DC}_N$ and any $t\in[0,1]$,

$$F_A(\mu_t)\le(1-t)\int_{X\times X}A\left(\frac{\rho_0(x_0)}{\beta_{1-t}(x_0,x_1)}\right)\beta_{1-t}(x_0,x_1)\,\mathrm{d}\pi_{\mu_0}(x_1)\,\mathrm{d}\mathfrak{m}(x_0) \tag{2.1.5}$$

$$+\,t\int_{X\times X}A\left(\frac{\rho_1(x_1)}{\beta_t(x_0,x_1)}\right)\beta_t(x_0,x_1)\,\mathrm{d}\pi_{\mu_1}(x_0)\,\mathrm{d}\mathfrak{m}(x_1), \tag{2.1.6}$$

or using the notation introduced in Remark 2.1.5,

$$F_A(\mu_t) \le (1-t)[\tilde{F_A}]_\pi^{\beta_{1-t}}(\mu_0) + t[\tilde{F_A}]_{\check{\pi}}^{\beta_t}(\mu_1).$$

Let us present now Sturm's $\mathrm{CD}(K,N)$ condition. To this purpose, we introduce the N-dimensional Rényi entropy $S_{\mathfrak{m}}^N$, which is defined as

$$S_{\mathfrak{m}}^N(\mu) := -\int_X \rho^{1-1/N}\,\mathrm{d}\mathfrak{m}$$

for any measure $\mu \in \mathcal{P}_2^a(X,\mathfrak{m})$ with density ρ. We also need the modified distortion coefficients $\tau^{(K,N)} := \{\tau_t^{(K,N)} : \mathbb{R}_+ \to [0,+\infty]\}_{t\in[0,1]}$ given as follows: for any $\theta \ge 0$, if $N > 1$,

$$\tau_t(\theta) := \begin{cases} t^{\frac{1}{N}}\left(\frac{\sinh(t\theta\sqrt{-K/(N-1)})}{\sinh(\theta\sqrt{-K/(N-1)})}\right)^{1-\frac{1}{N}} & \text{if } K<0 \\ t & \text{if } K=0 \\ t^{\frac{1}{N}}\left(\frac{\sin(t\theta\sqrt{K/(N-1)})}{\sin(\theta\sqrt{K/(N-1)})}\right)^{1-\frac{1}{N}} & \text{if } K>0 \text{ and } 0<\theta<\sqrt{K/(N-1)} \\ \infty & \text{if } K>0 \text{ and } \theta \ge \sqrt{K/(N-1)} \end{cases}$$

and if $N=1$, $\tau_t(\theta) = t$.

Definition 2.1.12. A Polish geodesic metric measure space $(X,\mathrm{d},\mathfrak{m})$ is called Sturm $\mathrm{CD}(K,N)$ if for all $N' \ge N$, $S_{\mathfrak{m}}^{N'}$ is weakly displacement convex with distortion $\tau^{(K,N)}$, meaning that for any $\mu_0,\mu_1 \in \mathcal{P}_2^a(X,\mathfrak{m})$ with respective densities ρ_0,ρ_1, there exists at least one W_2-geodesic $(\mu_t)_{t\in[0,1]}$ and an optimal transport plan π between μ_0 and μ_1 such that:

$$S_{\mathfrak{m}}^{N'}(\mu_t) \le (1-t)[\tilde{S}_{\mathfrak{m}}^{N'}]_\pi^{\beta_{1-t}}(\mu_0) + t[\tilde{S}_{\mathfrak{m}}^{N'}]_{\check{\pi}}^{\beta_t}(\mu_1) \qquad \forall t \in [0,1].$$

Note that in particular, Sturm $\mathrm{CD}(0,N)$ spaces are those metric measure spaces for which all Rényi entropies $S_{\mathfrak{m}}^{N'}$, with $N' \ge N$, are weakly displacement convex.

As for the infinite dimensional case, one can introduce the pressure and iterated pressure of any density $A \in \mathcal{DC}_N$. Then $A \in \mathcal{DC}_N \iff p_2^A + \frac{(p^A)^2}{N} \ge 0$, and the borderline case $p_2^A + \frac{(p^A)^2}{N} = 0$ is provided by the density $A(r) = -N(r^{1-1/N} - r)$ which gives rise to the functional $\tilde{S}_{\mathfrak{m}}^N = N + NS_{\mathfrak{m}}^N$. For convenience, Sturm preferred to work with $S_{\mathfrak{m}}^N$ instead of $\tilde{S}_{\mathfrak{m}}^N$; he accordingly needed the modified distortion $\tau^{(K,N)}$ in place of $\beta^{(K,N)}$.

Non-branching and essentially non-branching spaces

It turns out that Sturm's and Lott-Villani's $\mathrm{CD}(K,N)$ conditions (including the case $N=+\infty$) coincide on non-branching metric measure spaces, as proved, for instance, in [Vi03, Th. 30.32]. Let us recall that a metric space (X,d) is called non-branching if two geodesics γ_1 and γ_2 which coincide on $[0,t_0]$ for some $0<t_0<1$ coincide actually on $[0,1]$. Riemannian manifolds and Alexandrov spaces are non-branching, but whether general Ricci limit spaces are always non-branching is still an open question. It is then natural to consider the following weaker notion.

Definition 2.1.13. A Polish geodesic metric measure space $(X,\mathrm{d},\mathfrak{m})$ is called essentially non-branching if any optimal transport plan $\pi \in \mathcal{P}(\mathrm{Geo}(X))$ between two generic $\mu_0,\mu_1 \in \mathcal{P}_2^a(X,\mathfrak{m})$ is concentrated on a set of non-branching geodesics.

In other words, a space is essentially non-branching if any optimal transport on the space is carried out over non-branching geodesics. Essentially non-branching spaces were introduced in [RS14] by T. Rajala and K.-T. Sturm who proved the equivalence between Sturm's and Lott-Villani's $\mathrm{CD}(K,\infty)$ conditions on such spaces, as well as the inclusion within this framework of the class of $\mathrm{RCD}(K,N)$ spaces, on which we shall focus later.

Doubling property of finitely dimensional $\mathrm{CD}(K,N)$ spaces

The classical Bishop-Gromov inequality (see e.g. [Gro07, Lem. 5.3bis] for a proof) asserts that if (M,g) is a complete n-dimensional Riemannian manifold with $\mathrm{Ric} \geq (n-1)Kg$ for some $K \in \mathbb{R}$, then for any $p \in M$,

$$\frac{\mathrm{vol}(B_R(p))}{\mathrm{vol}_K(B_R(p))} \leq \frac{\mathrm{vol}(B_r(p))}{\mathrm{vol}_K(B_r(p))} \qquad \forall\, 0 < r \leq R,$$

where p_K is any point of the unique complete simply connected n-dimensional Riemanian manifold (M^n_K, g^K) with Ricci curvature constantly equal to $(n-1)Kg^K$, and vol_K is the corresponding volume measure of (M^n_K, g^K). Let us point out that a direct computation provides

$$\mathrm{vol}_K(B_r(p)) = \begin{cases} \int_0^r \sin(t\sqrt{K/(n-1)})^{n-1}\,\mathrm{d}t & \text{if } K>0,\\ \omega_n r^n & \text{if } K=0,\\ \int_0^r \sinh(t\sqrt{|K|/(n-1)})^{n-1}\,\mathrm{d}t & \text{if } K<0. \end{cases}$$

for any $r > 0$, where ω_n denotes the n-dimensional Lebesgue measure of the unit Euclidean ball. Note that the above right-hand sides are independant of $p_K \in M^n_K$, and they still make sense if one replaces $n \in \mathbb{N}$ by any real number $N \geq 1$. Therefore, we can set

$$\mathrm{vol}_{K,N}(r) := \begin{cases} \int_0^r \sin(t\sqrt{K/(N-1)})^{N-1}\,\mathrm{d}t & \text{if } K>0,\\ \omega_n r^n & \text{if } K=0,\\ \int_0^r \sinh(t\sqrt{|K|/(N-1)})^{N-1}\,\mathrm{d}t & \text{if } K<0, \end{cases}$$

for any $r > 0$ and $N \geq 1$.

Lott and Villani extended Bishop-Gromov inequality to the case of $\mathrm{CD}(0,N)$ spaces in [LV09, Prop. 5.27], while Sturm proved it directly for Sturm $\mathrm{CD}(K,N)$ spaces [St06a, Th. 2.3], whatever be $K \in \mathbb{R}$. In [Vi03, Th. 30.11], Villani provided Bishop-Gromov inequality for any Lott-Villani $\mathrm{CD}(K,N)$ spaces.

Theorem 2.1.14 (Bishop-Gromov inequality)**.** *Let $(X,\mathrm{d},\mathfrak{m})$ be a $\mathrm{CD}(K,N)$ space. Then for any $x \in \mathrm{supp}(\mathfrak{m})$,*

$$\frac{\mathfrak{m}(B_R(x))}{\mathfrak{m}(B_r(x))} \leq \frac{\mathrm{vol}_{K,N}(R)}{\mathrm{vol}_{K,N}(r)} \qquad \forall\, 0 < r \leq R.$$

An immediate corollary of Bishop-Gromov inequality is the (local) doubling condition, which is crucial to apply several analytic means.

Corollary 2.1.15 (Doubling condition)**.** *Let $(X,\mathrm{d},\mathfrak{m})$ be a $\mathrm{CD}(K,N)$ space. If $K \geq 0$, then the measure $\mathfrak{m}$ is doubling with constant $\leq 2^N$, meaning that*

$$\mathfrak{m}(B_{2r}(x)) \leq 2^N \mathfrak{m}(B_r(x)) \qquad \forall x \in \mathrm{supp}(\mathfrak{m}),\ \forall r > 0. \tag{2.1.7}$$

If $K < 0$, then $\mathfrak{m}$ is locally doubling, meaning that for any $x \in \mathrm{supp}(\mathfrak{m})$, there exists $C_D = C_D(K,N) > 0$ and $r_o = r_o(x) > 0$ such that

$$\mathfrak{m}(B_{2r}(x)) \leq C_D \mathfrak{m}(B_r(x)) \qquad \forall\, 0 < r < r_0. \tag{2.1.8}$$

Local Poincaré inequality

The local Poincaré inequality is a common assumption in the study of metric measure spaces. Coupled with the doubling condition, it gives access to a large spectrum of analytic tools, as revealed by the works of J. Heinonen and P. Koskela [HeK98], J. Cheeger [Ch99] and P. Hajlasz and P. Koskela [HK00], see Definition 2.2.17 and Theorem 2.2.18 in the next section. Therefore, the following result, due to T. Rajala, is of the outmost importance for the CD theory.

Theorem 2.1.16 (Rajala's Poincaré type inequalities [Raj12]). *1. Any Lott-Villani* $\mathrm{CD}(K,\infty)$ *space with* $K \leq 0$ *supports the weak local* $(1,1)$*-Poincaré type inequality*

$$\fint_B |u - u_B| \,\mathrm{d}\mathfrak{m} \leq 4re^{|K|r^2} \fint_{2B} g \,\mathrm{d}\mathfrak{m}$$

holding for any ball $B \subset X$ *with radius* $r > 0$*, any locally integrable function* u *defined on* B*, and any integrable upper gradient* g *of* u*.*

2. Any Lott-Villani $\mathrm{CD}(K,N)$ *space with* $K \leq 0$ *and* $N < +\infty$ *supports the weak local* $(1,1)$*-Poincaré type inequality*

$$\fint_B |u - u_B| \,\mathrm{d}\mathfrak{m} \leq 2^{N+2} re^{\sqrt{(N-1)|K|}2r} \fint_{2B} g \,\mathrm{d}\mathfrak{m}$$

holding for any ball $B \subset X$ *with radius* $r > 0$*, any locally integrable function* u *defined on* B*, and any integrable upper gradient* g *of* u*.*

Note that we prefer to call these inequalities "Poincaré type" inequalities because of the exponential term which does not appear in the definition of Poincaré inequality we give in Definition 2.2.17.

Remark 2.1.17. In particular, any Lott-Villani $\mathrm{CD}(0,N)$ space supports the weak local $(1,1)$-Poincaré inequality:

$$\fint_B |u - u_B| \,\mathrm{d}\mathfrak{m} \leq 2^{N+2} r \fint_{2B} g \,\mathrm{d}\mathfrak{m}$$

for any ball $B \subset X$ with radius $r > 0$, any locally integrable function u defined on B, and any integrable upper gradient g of u.

Local-to-global property

The local-to-global issue for $\mathrm{CD}(K,N)$ spaces asked whether a space $(X,\mathrm{d},\mathfrak{m})$ is $\mathrm{CD}(K,N)$ whenever it is $\mathrm{CD}_{loc}(K,N)$, meaning that there exists a countable partition $X = \amalg_i X_i$ such that each $(X_i, \mathrm{d}_{|X_i \times X_i}, \mathfrak{m} \llcorner X_i)$ is $\mathrm{CD}(K',N')$ for some $K' \geq K$ and $N' \geq N$. The answer is obviously yes for Riemannian manifolds, but in full generality, the problem is much more involved. K.-T. Sturm [St06a, Th. 4.17] and C. Villani [Vi09, Th. 30.37] proved respectively that under the non-branching assumption, compact $\mathrm{CD}(K,\infty)$ and $\mathrm{CD}(0,N)$ satisfy the local-to-global property. Nonetheless, T. Rajala provided a final negative answer to the general question by constructing a (highly branching) $\mathrm{CD}_{loc}(0,4)$ space which is not $\mathrm{CD}(K,N)$, whatever $K \in \mathbb{R}$ and $N \in [1,+\infty)$ be [Raj16].

In the meantime, K. Bacher and K.-T. Sturm had introduced the so-called reduced curvature dimension condition $\mathrm{CD}^*(K,N)$, for $N < +\infty$, by considering the reduced distortion $\sigma^{(K,N)} = \{\sigma_t^{(K,N)} : \mathbb{R}_+ \to [0,+\infty]\}_{t\in[0,1]}$ defined by:

$$\sigma_t^{(K,N)}(\theta) := t^{-1/N}[\tau_t^{(K,N+1)}(\theta)]^{1+1/N} \qquad \forall t \in [0,1].$$

Definition 2.1.18. A Polish geodesic metric measure space $(X, \mathrm{d}, \mathfrak{m})$ is called $\mathrm{CD}^*(K, N)$ if for any $\mu_0, \mu_1 \in \mathcal{P}_2^a(X, \mathfrak{m})$ with respective densities ρ_0, ρ_1, there exists a W_2-geodesic $(\mu_t)_{t\in[0,1]}$ and an optimal transport plan π between μ_0 and μ_1 such that for all $N' \geq N$, $S_{\mathfrak{m}}^{N'}$ is displacement convex with distortion $\sigma^{(K,N)}$.

Note that the coefficients $\sigma_t^{(K,N)}$ are slightly smaller than the coefficients $\tau_t^{(K,N)}$, implying that the $\mathrm{CD}^*(K, N)$ condition is slightly weaker than the $\mathrm{CD}(K, N)$. However, $\mathrm{CD}^*(K, N)$ spaces advantageously satisfy the local-to-global property, and the condition $\mathrm{CD}^*_{loc}(K, N)$ is equivalent to $\mathrm{CD}_{loc}(K, N)$.

Finally, let us point out that for essentially non-branching spaces with finite mass, the conditions $\mathrm{CD}(K, N)$ and $\mathrm{CD}^*(K, N)$ are equivalent, as shown by F. Cavalletti and E. Milman [CMi16].

Examples

Let us conclude this section with some examples of $\mathrm{CD}(K, N)$ spaces.

1. As proved by K.-T. Sturm and M.-T. von Renesse [RS05], any complete connected n-dimensional Riemannian manifold (M, g) with $\mathrm{Ric}_g \geq (n-1)Kg$ equipped with its canonical Riemannian distance d_g and volume measure vol_g is a $\mathrm{CD}(K, \infty)$ space.

2. For any given $N > n$ and $V \in C^2(M)$, the weighted Riemannian manifold $(M^n, \mathrm{d}_g, e^{-V}\mathfrak{m})$ satisfies the $\mathrm{CD}(K, N)$ condition if and only if

$$\mathrm{Ric} + \mathrm{Hess}_f - \frac{1}{N-n}\nabla f \otimes \nabla f \geq Kg,$$

see [Vi09, Th. 29.9] for a proof. In particular for $V \equiv 0$, we get that $(M, \mathrm{d}_g, \mathrm{vol}_g)$ is $\mathrm{CD}(K, n)$ if and only if $\mathrm{Ric}_g \geq (n-1)Kg$.

3. When $(M^n, \mathrm{d}_g, \mathrm{vol}_g) = (\mathbb{R}^n, \mathrm{d}_{eucl}, \mathscr{L}^n)$ and V is only C^0 on $\mathbb{R}^n$, it can be shown that the space $(\mathbb{R}^n, \mathrm{d}_{eucl}, e^{-V}\mathscr{L}^n)$ is $\mathrm{CD}(0, \infty)$ if and only if V is convex. For any $K \in \mathbb{R}$, a similar statement holds if one replaces $\mathrm{CD}(0, \infty)$ with $\mathrm{CD}(K, \infty)$ and convexity of V with K-convexity, meaning that

$$V((1-t)x + ty) \leq (1-t)V(x) + tV(y) - K\frac{(1-t)t}{2}|x-y| \qquad (2.1.9)$$

holds for any $x, y \in \mathbb{R}^n$ and $0 \leq t \leq 1$, cf. [Vi09, Ex. 29.13].

4. Let (M^n, g) be a smooth compact Riemannian manifold with non-negative Ricci curvature, and G a compact group acting on M by isometries, meaning that there exists a map $\Psi : G \to \mathrm{Isom}(G)$ such that $\Psi(e_G) = \mathrm{Id}$ and $\Psi(g \cdot g') = \Psi(g) \circ \Psi(g')$ for all $g, g' \in G$. Then the quotient M/G, which might not be a smooth manifold (it could typically have singularities at fixed points of the action), is $\mathrm{CD}(0, n)$.

5. Non-negatively curved Alexandrov spaces of dimension $n \leq N$ are $\mathrm{CD}(0, N)$, as proved by A. Petrunin [Pet11].

6. K. Bacher and K.-T. Sturm proved in [BS14] that Euclidean and spherical cones over smooth complete non-negatively curved Riemannian manifolds, which might not be neither smooth Riemannian manifolds nor Alexandrov spaces, satisfy a $\mathrm{CD}(K, N)$ condition for some right $K \in \mathbb{N}$ and $N \in \mathbb{N}$. C. Ketterer extended these results to warped products over complete Finsler manifolds [K13a].

7. Anticipating on Section 2.4, any Ricci limit space is $\mathrm{CD}(K,N)$, whenever the approximating sequence has dimension constantly equal to $n \le N$ and Ricci curvature uniformly bounded below by $(n-1)K$. This follows from the stability of Lott-Villani's $\mathrm{CD}(K,N)$ condition with respect to measured Gromov-Hausdorff convergence, see Th. 2.4.9.

8. The space $(\mathbb{R}^n, \|\cdot\|_\infty, \mathscr{L}^n)$, where $\|x\|_\infty = \sup_i |x_i|$ for any $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, is Lott-Villani $\mathrm{CD}(0,n)$. This result is due to D. Cordero-Erausquin, K.T.-Sturm and C. Villani, but we point out that it doesn't appear in any publication. Nevertheless, the reader can find a detailed sketch of proof in [Vi09, p. 912].

   This example has been a cornerstone in the theory of synthetic Ricci curvature bounds, because it is not a Ricci limit space. Indeed, if it were so, Cheeger-Colding's almost splitting theorem [CC96, Th. 6.64] would imply the isometric splitting of $(\mathbb{R}^n, \|\cdot\|_\infty)$ into $\mathbb{R} \times X$ for some $(n-1)$-dimensional manifold X. This is impossible, as one can easily check from the case in which the splitting line coincides with $\{x_2 = \cdots = x_n = 0\}$: take $x = (x_1, x'), y = (y_1, y') \in \mathbb{R} \times X$ such that $|x_1 - y_1| = 1$, $x' \neq y'$ and $|x_i - y_i| < 1$ for any $2 \le i \le n$. Then $|x_1 - y_1| + \mathrm{d}_X(x', y') = \|x - y\|_\infty = 1$, implying $\mathrm{d}_X(x', y') = 0$, what is impossible.

9. More generally, it follows from S.-T. Ohta's work [Oh09, Th. 2] that any compact smooth Finsler manifold is $\mathrm{CD}(K,N)$ for some appropriate $K \in \mathbb{R}$ and $1 \le N < +\infty$. Let us recall that a smooth Finsler manifold is a smooth manifold M equipped with a positive definite, homogeneous and subadditive function $F : TM \to [0, +\infty]$ smooth on the complement of the zero section in TM. The space $(\mathbb{R}^n, \|\cdot\|_\infty, \mathscr{L}^n)$, with F constantly equal to $\|\cdot\|_\infty$, is one simple example.

2.2 Calculus tools on metric measure spaces and Riemannian curvature-dimension conditions

The aim of this section is to present the Riemannian curvature-dimension conditions $\mathrm{RCD}(K,\infty)$ and its refinements, and to give the abstract calculus tools that we shall frequently use throughout the thesis. From now on, unless explicitely mentioned, by Polish metric measure space we mean a triple $(X,\mathrm{d},\mathfrak{m})$ where (X,d) is a Polish (i.e. complete and separable) metric space and $\mathfrak{m}$ is a Borel regular measure on (X,d) finite and non-zero on balls with finite and non-zero radius.

Cheeger energy and the Sobolev space $H^{1,2}(X,\mathrm{d},\mathfrak{m})$

Although previous works had already dealt with the extension of Euclidean analytic tools to the setting of general metric measure spaces (e.g. [CW77], [Ha95], [HK95], [KM96], [Se96],[HeK98], [St98]), J. Cheeger's celebrated article [Ch99] is nowadays refered as the starting point of the modern theory of analysis on metric measure spaces spaces $(X,\mathrm{d},\mathfrak{m})$. Aiming at a generalized Rademacher's theorem on such spaces, J. Cheeger developed a robust first-order differential structure embodied by the space $H^{1,2}(X,\mathrm{d},\mathfrak{m})$ of weakly differentiable functions defined by means of approximation. Recall that when Ω is an open set of $\mathbb{R}^n$, the space $H^{1,2}(\Omega)$ is defined as the closure of $C^\infty(\Omega)$ with respect to the $\|\cdot\|_{1,2}$ norm. Replacing Ω with $(X,\mathrm{d},\mathfrak{m})$, smooth functions are, of course, not available. The starting point of J. Cheeger's analysis consisted in replacing those missing smooth functions with Lipschitz functions. Recall that by definition, the slope (also called sometimes local Lipschitz constant) of a Lipschitz function $f:X\to\mathbb{R}$ is

$$|\nabla f|(x) := \begin{cases} \limsup\limits_{y\to x} \frac{|f(x)-f(y)|}{\mathrm{d}(x,y)} & \text{if } x\in X \text{ is not isolated,}\\ 0 & \text{otherwise.}\end{cases}$$

Definition 2.2.1. (Cheeger's $H^{1,2}$ space) A function $f\in L^2(X,\mathfrak{m})$ is called Sobolev if there exists a sequence of Lipschitz functions $(f_n)_n$ such that $\|f_n-f\|_{L^2(X,\mathfrak{m})}\to 0$ and $\sup_n \||\nabla f_n|\|_{L^2(X,\mathfrak{m})}<+\infty$. The real vector space of such Sobolev functions is denoted by $H^{1,2}(X,\mathrm{d},\mathfrak{m})$.

It is worth point out that in this thesis, we only work with the Sobolev space $H^{1,2}$ with exponent 2, even if Cheeger's original definition, as well as several other notions discussed in this section, was given for any exponent $p\in(1,+\infty)$.

Related to this construction is the crucial Cheeger energy which is defined for any $f\in L^2(X,\mathfrak{m})$ by

$$\mathrm{Ch}(f)=\inf_{f_n\to f}\left\{\liminf_{n\to+\infty}\int_X |\nabla f_n|^2\,\mathrm{d}\mathfrak{m}\right\}\in[0,+\infty], \tag{2.2.1}$$

where the infimum is taken over all the sequences $(f_n)_n\subset L^2(X,\mathfrak{m})\cap\mathrm{Lip}(X,\mathrm{d})$ such that $\|f_n-f\|_{L^2(X,\mathfrak{m})}\to 0$.

Although Ch can reasonably be understood as an extension of the classical Dirichlet energy, there is no guarantee that it defines a Dirichlet form (Definition 2.3.1) on $(X,\mathfrak{m})$: this is one of the argument put forward by N. Gigli [G15] in favour of the name "Cheeger energy" instead of "Dirichlet energy", the second argument being that an equivalent definition of Ch where slopes of Lipschitz functions are replaced with upper gradients (Definition 2.2.7 below) of L^2 functions goes back to [Ch99], see (2.2.6) below.

Main properties of Ch are gathered in the next proposition.

Proposition 2.2.2. *The function* $\mathrm{Ch} : L^2(X, \mathfrak{m}) \to [0, +\infty]$ *is convex and lower semicontinuous. Morover, its finitness domain coincides with* $H^{1,2}(X, \mathrm{d}, \mathfrak{m})$ *which is a Banach space dense in* $L^2(X, \mathfrak{m})$*.*

Proof. Convexity can be directly shown using the following simple algebraic property of the slope, holding for any $f, g \in \mathrm{Lip}(X)$ and $a, b > 0$:

$$|\nabla(af + bg)| \le a|\nabla f| + b|\nabla g| \qquad \mathfrak{m}\text{-a.e. on } X. \tag{2.2.2}$$

By definition, $H^{1,2}(X, \mathrm{d}, \mathfrak{m})$ coincides with the finiteness domain of Ch. Lower semicontinuity of Ch is a straigthforward consequence of the definition, and implies that $H^{1,2}(X, \mathrm{d}, \mathfrak{m})$ endowed with the norm $\| \cdot \|_{H^{1,2}} = \sqrt{\| \cdot \|_{L^2} + \mathrm{Ch}}$ is a Banach space, following the lines of [Ch99, Th. 2.7]. Finally, density of $H^{1,2}(X, \mathrm{d}, \mathfrak{m})$ in $L^2(X, \mathfrak{m})$ follows from the density of Lipschitz functions $f \in L^2(X, \mathfrak{m})$ with $|\nabla f| \in L^2(X, \mathfrak{m})$ (see [?, Prop. 4.1]). □

As classical Sobolev spaces $H^{1,2}(\Omega)$ defined over open sets $\Omega \in \mathbb{R}^n$ are Hilbert, the next definition sounds rather natural.

Definition 2.2.3. (Infinitesimally Hilbertianity) A Polish metric measure space $(X, \mathrm{d}, \mathfrak{m})$ is called infinitesimally Hilbertian if $H^{1,2}(X, \mathrm{d}, \mathfrak{m})$ endowed with the norm $\| \cdot \|_{H^{1,2}}$ is a Hilbert space, or equivalently, if Ch is a quadratic form.

Recall that a quadratic form over a real vector space E is a map $q : E \to \mathbb{R}$ such that $q(\lambda f) = \lambda^2 q(f)$ for any $\lambda \in \mathbb{R}$ and $f \in E$ (we say that q is 2-homogeneous), and for which the map $\tilde{q} : (f, g) \mapsto \frac{1}{4}(q(f + g) - q(f - g))$[1] defines a real-valued bilinear symmetric map on $E \times E$. The application $q \mapsto \tilde{q}$ defines an isomorphism of real vector spaces between the space of quadratic forms over E and the space of bilinear symmetric maps $E \times E \to \mathbb{R}$, and the inverse of such an application associates $p : f \mapsto \tilde{p}(f, f)$ to any given bilinear symmetric map $\tilde{p} : E \times E \to \mathbb{R}$.

Note that Ch is obviously 2-homogeneous, and that as such Ch is quadratic if and only if it satisfies the parallelogram rule:

$$\mathrm{Ch}(f + g) + \mathrm{Ch}(f - g) = 2\mathrm{Ch}(f) + 2\mathrm{Ch}(g) \qquad \forall f, g \in H^{1,2}(X, \mathrm{d}, \mathfrak{m}).$$

It turns out that infinitesimal Hilbertianity is a powerful requirement which can be reformulated in various ways and provides a simple manner to state the Riemannian curvature-dimension conditions.

Definition 2.2.4. ($\mathrm{RCD}(K, N)$ and $\mathrm{RCD}^*(K, N)$ conditions) Let $K \in \mathbb{R}$ and $1 \le N \le +\infty$ be fixed. A Polish metric measure space is called $\mathrm{RCD}(K, N)$ (resp. $\mathrm{RCD}^*(K, N)$) if it is both $\mathrm{CD}(K, N)$ (resp. $\mathrm{CD}^*(K, N)$) and infinitesimally Hilbertian.

To fully appreciate the structural power of such conditions, we need to introduce further first-order calculus tools.

Absolutely continuous curves

We start with the basic notion of absolutely continuous curve on a metric space (X, d). Classically, a function $f : [a, b] \to \mathbb{R}$ is called absolutely continuous if there exists $g \in L^1(a, b)$ such that

$$f(x) - f(y) = \int_y^x g(t)\, \mathrm{d}t \quad \forall x, y \in (a, b),$$

[1] Or equivalently $\tilde{q} : (f, g) \mapsto \frac{1}{2}(q(f + g) - q(f) - q(g)$ or $\tilde{q} : (f, g) \mapsto \frac{1}{2}(q(f) + q(g) - q(f - g)$

in which case such a g is easily shown to be unique (as equivalent class of L^1 functions), and the derivative $f'(x) = \lim_{y\to x}(x-y)^{-1}(f(x)-f(y))$ exists for $\mathscr{L}^1$-a.e. $x \in (a,b)$, with equality $f'(x) = g(x)$. Absolutely continuous curves are an extension of this notion for functions taking values in a metric space (X, d).

Definition 2.2.5 (Absolutely continuous curves)**.** We say that $\gamma : [a,b] \to X$ is an absolutely continuous curve if there exists $g \in L^1(a,b)$ non-negative such that

$$\mathrm{d}(\gamma(x),\gamma(y)) \leq \int_x^y g(t)\,\mathrm{d}t \qquad \forall\, a \leq x \leq y \leq b. \tag{2.2.3}$$

The set of such curves is denoted by $AC([a,b];X)$. The set of absolutely continuous curves γ such that there exists a function $g \in L^p(a,b)$ satisfying (2.2.3) is denoted by $AC^p([a,b];X)$.

Notice that in case $X = \mathbb{R}^n$, Definition 2.2.5 is equivalent to the classical notion of absolute continuity which is also equivalent to Vitali's formulation: $f : [a,b] \to \mathbb{R}$ is absolutely continuous if and only if for any $\epsilon > 0$, there exists $\delta > 0$ such that for any collection of disjoint intervals $\{(a_i,b_i)\}_i$ in $[a,b]$ with $\sum_i(b_i - a_i) < \delta$, we have $\sum_i |f(b_i) - f(a_i)| < \epsilon$. In the context of general metric spaces, Definition 2.2.5 and Vitali's formulation still make sense and are equivalent.

Absolutely continuous curves are often used to construct a first-order differential structure on metric spaces because they possess a weak notion of velocity, called metric derivative.

Theorem 2.2.6 (Metric derivative)**.** *For any $\gamma \in AC([a,b];X)$ the limit*

$$\lim_{h\to 0} \frac{d(\gamma(t),\gamma(t+h))}{|h|} =: |\gamma'|(t)$$

exists for $\mathscr{L}^1$-a.e. $t \in (a,b)$. In addition, up to $\mathscr{L}^1$-negligible sets, $|\gamma'|$ coincide with the $\mathscr{L}^1$-a.e. minimal function g that we can choose in Definition 2.2.5. It is called the metric derivative of γ.

Upper gradients

The fundamental theorem of calculus implies that whenever $f : \mathbb{R}^n \to \mathbb{R}$ is a C^1 function and $\gamma : [0,1] \to \mathbb{R}^n$ is a C^1 curve, $f(\gamma(1)) - f(\gamma(0)) = \int_0^1 \langle \nabla f(\gamma(t)), \gamma'(t)\rangle\,\mathrm{d}t$. Taking absolute values and applying Cauchy-Schwarz inequality, one gets

$$|f(\gamma(1)) - f(\gamma(0))| \leq \int_0^1 |\nabla f|(\gamma(t))|\gamma'(t)|\,\mathrm{d}t.$$

In case f is defined over a metric space (X, d) and $\gamma \in AC([0,1],X)$, the term $|\gamma'(t)|$ in the above inequality still makes sense, at least for $\mathscr{L}^1$-a.e. $t \in [0,1]$, as metric derivative of γ. This motivates the introduction of upper gradients, which should be seen as extensions of the norm of the gradient for general functions $f : X \to \mathbb{R}$.

Definition 2.2.7 (Upper gradient)**.** For any $f : X \to \mathbb{R}$, we say that a function $g : X \to [0,+\infty]$ is an upper gradient of f, and we write $g \in \mathrm{UG}(f)$, if for any $\gamma \in AC([0,1];X)$,

$$|f(\gamma(1)) - f(\gamma(0))| \leq \int_0^1 g(\gamma(s))|\gamma'|(s)\,\mathrm{d}s. \tag{2.2.4}$$

If moreover, $g \in L^p(X,\mathfrak{m})$, we write $g \in UG^p(f)$.

Remark 2.2.8. We usually write $\int_{\partial\gamma} f$ for $f(\gamma(1)) - f(\gamma(0))$ and $\int_\gamma g$ for $\int_0^1 g(\gamma(s))|\gamma'|(s)\,\mathrm{d}s$, so that (2.2.4) becomes

$$\left|\int_{\partial\gamma} f\right| \leq \int_\gamma g. \tag{2.2.5}$$

The notion of upper gradient goes back to Heinonen-Koskela's work [HeK98] in which it was called "very weak gradient" and formulated with rectifiable curves instead of absolute continuous curves. But any rectifiable curve can be reparametrized into an absolutely continuous one with constant metric derivative, see e.g. [AT03, Th. 4.2.1], so that the definition we gave is equivalent to Heinonen-Koskela's one.

Note that a simple example of upper gradient is provided by the slope of a Lipschitz function. In this regard, let us point out that Cheeger's original approach, defining Ch as

$$\mathrm{Ch}(f) := \inf_{f_n \to f,\, g_n} \left\{ \liminf_{n\to+\infty} \int_X g_n^2 \,\mathrm{d}\mathfrak{m} \right\} \qquad \forall f \in L^2(X,\mathfrak{m}), \tag{2.2.6}$$

where the infimum is taken over all the sequences $(f_n)_n, (g_n)_n \subset L^2(X,\mathfrak{m})$ such that $\|f_n - f\|_{L^2(X,\mathfrak{m})} \to 0$ and g_n is an upper gradient of f_n for any n, provides an a priori smaller functional compared to (2.2.1), as we are taking the infimum over a bigger set. However, L. Ambrosio, N. Gigli and G. Savaré proved in [?, Th. 6.2, Th. 6.3] that the two functionals coincide.

Minimal relaxed slope

A suitable diagonal argument applied to optimal approximating sequences in (2.2.1) or (2.2.6) (see for instance [Ch99, Th. 2.10]) provides for any $f \in H^{1,2}(X,\mathrm{d},\mathfrak{m})$ the existence of a L^2-function $|\nabla f|_*$, called *minimal relaxed slope* or *minimal generalized upper gradient* of f which gives integral representation of Ch:

$$\mathrm{Ch}(f) = \int_X |\nabla f|_*^2 \,\mathrm{d}\mathfrak{m} \qquad \forall f \in H^{1,2}(X,\mathrm{d},\mathfrak{m}).$$

The minimal relaxed slope is a local object, meaning that

$$|\nabla f|_* = |\nabla g|_* \qquad \mathfrak{m}\text{-a.e. on } \{f = g\}$$

for any $f, g \in H^{1,2}(X,\mathrm{d},\mathfrak{m})$. This, combined with the integral representation property, ensures that $|\nabla f|_*$ is unique as class of L^2-equivalent functions.

Locality here is closely related to the theory of Dirichlet forms whose connection with our discussion follows from the forthcoming Proposition 2.2.10. Let us state a preliminary lemma with a sketch of proof, refering to [?] for the details.

Lemma 2.2.9. *For any $f, g \in H^{1,2}(X,\mathrm{d},\mathfrak{m})$, the limit*

$$\lim_{\varepsilon\downarrow 0} \frac{|\nabla(f+\varepsilon g)|_*^2 - |\nabla f|_*^2}{2\varepsilon}$$

exists in $L^1(X,\mathfrak{m})$.

Sketch of proof. Take $f, g \in H^{1,2}(X,\mathrm{d},\mathfrak{m})$, and define $F : [0,+\infty) \to L^1(X,\mathfrak{m})$ by $F(\varepsilon) = |\nabla(f+\varepsilon g)|_*^2$ for any $\varepsilon > 0$. Convexity of the slope of Lipschitz functions follows from (2.2.2) and implies by approximation the convexity of $|\nabla \cdot|_*$, which in turns gives convexity of F. Consequently, the growth rate $G_x : (0,+\infty) \ni \varepsilon \mapsto \varepsilon^{-1}(F(\varepsilon) - F(0))(x)$ is nondecreasing for $\mathfrak{m}$-a.e. $x \in X$, whence the existence of the limit $\lim_n G_x(\varepsilon_n)$ for any such x and any infinitesimal decreasing sequence $(\varepsilon_n) \subset (0,+\infty)$. Of course F can be defined also for $\varepsilon' < 0$, and the L^1 function $\varepsilon'^{-1}(F(\varepsilon') - F(0))$ provides a bound by below for the sequence of L^1 functions $G(\varepsilon_n)$ - such functions are integrable because $|\nabla(f+\varepsilon g)|_*, |\nabla f|_* \in L^2(X,\mathfrak{m})$ for all ε. Then monotone convergence theorem for Lebesgue integrals provides the result. □

As stated in the next proposition, one of the main properties of infinitesimally Hilbertian spaces is the existence of a unique strongly local Dirichlet form $\mathcal{E}$ such that $\mathcal{E}(f,f) = \mathrm{Ch}(f)$ for any $f \in H^{1,2}(X,\mathrm{d},\mathfrak{m})$ (see [AGS14b, Sect. 4.3]). We will spend further words on Dirichlet forms in the next section. Let us also point out that strong locality of $\mathcal{E}$ stems from locality of the minimal relaxed slope.

Proposition 2.2.10. *Assume that* $(X,\mathrm{d},\mathfrak{m})$ *is infinitesimally Hilbertian. Then the function*

$$C(f,g) := \lim_{\varepsilon\downarrow 0} \frac{|\nabla(f+\varepsilon g)|_*^2 - |\nabla f|_*^2}{2\varepsilon} \qquad \forall f,g \in H^{1,2}(X,\mathrm{d},\mathfrak{m})$$

provides a symmetric bilinear form on $H^{1,2}(X,\mathrm{d},\mathfrak{m}) \times H^{1,2}(X,\mathrm{d},\mathfrak{m})$ *with values in* $L^1(X,\mathfrak{m})$*, and*

$$\mathcal{E}(f,g) := \int_X C(f,g)\,\mathrm{d}\mathfrak{m}, \qquad \forall f,g \in H^{1,2}(X,\mathrm{d},\mathfrak{m})$$

defines a strongly local Dirichlet form.

In the sequel, we will often use the notation $\langle \nabla f, \nabla g\rangle$ instead of $C(f,g)$, because of the obvious case of the Euclidean space. For consistency with the theory of Dirichlet forms, one sometimes writes $\Gamma(f,g)$ instead of $C(f,g)$, in which case the operator Γ is called *carré du champ*.

Heat flow

A further important consequence of infinitesimal Hilbertiannity is linearity of the heat flow, which we are going to define in a moment.

First of all, recall that if $(H,\|\cdot\|)$ is a Hilbert space, then any absolutely continuous curve $(x(t))_{t>0}$ in H is differentiable $\mathscr{L}^1$-a.e. with norm of the derivative equal to the metric derivative. In the classical theory of gradient flows, the Komura-Brezis theorem ensures that, given a K-convex (2.1.9) and lower semicontinuous function $F : H \to [-\infty,+\infty]$, for any "starting point" $\bar{x} \in \overline{\{f < +\infty\}}$, there exists a unique absolutely continuous curve $(x(t))_{t>0} \subset H$ such that

$$\begin{cases} x'(t) \in -\partial_K F(x(t)) & \text{for } \mathscr{L}^1\text{-a.e. } t > 0, \\ \lim\limits_{t\to 0} \|\bar{x} - x(t)\| = 0, \end{cases}$$

where $\partial_K F(x)$ denotes the K-subdifferential of F at $x \in H$, defined as

$$\partial_K F(x) := \left\{ p \in H : \forall y \in H,\ F(y) \geq F(x) + \langle p, y-x\rangle + \frac{K}{2}\|y-x\|^2 \right\}.$$

Such a curve $(x(t))_{t>0}$ is called gradient flow of F starting from $\bar{x}$. It satisfies the remarkable contraction property $\|x(t) - y(t)\| \leq e^{-2Kt}\|\bar{x} - \bar{y}\|$ for any $t > 0$. Moreover, for $\mathscr{L}^1$-a.e. t>0, the derivative $x'(t)$ equals the element in $\partial_K F(x(t))$ with minimal norm.

As Ch is convex and lower semicontinuous, the Komura-Brezis theorem provides a family of maps $\{P_t : D \subset L^2(X,\mathfrak{m}) \to L^2(X,\mathfrak{m})\}_{t>0}$ defined by $P_t(f) = f_t$ for any $f \in L^2(X,\mathfrak{m})$, with $(f_t)_{t>0}$ being the gradient flow of Ch starting from f. Here D is the set of functions $f \in L^2(X,\mathfrak{m})$ such that $\partial_0 \mathrm{Ch}(f) \neq 0$. This family is called heat flow of $(X,\mathrm{d},\mathfrak{m})$, because if one writes $\frac{\mathrm{d}}{\mathrm{d}t}P_t f$ for the derivative of the absolutely continuous curve $(P_t f)_{t>0}$ and $-\Delta P_t f$ for the element of minimal norm in $\partial_0 \mathrm{Ch}(P_t f)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}P_t f = \Delta P_t f \quad \text{for } \mathscr{L}^1\text{-a.e. } t > 0.$$

In full generality, the maps P_t might fail to be linear, as it is the case for instance on smooth Finsler manifolds [OS09]: in this context, the heat flow is linear if and only if the Finsler structure is actually Riemannian. Historically, this observation was at the core of the definition of the $\mathrm{RCD}(K,\infty)$ condition, as one can easily understand from the next proposition.

Proposition 2.2.11. *The Polish metric measure space $(X, \mathrm{d}, \mathfrak{m})$ is infinitesimally Hilbertian if and only if the heat flow is linear, meaning that all maps P_t, $t > 0$, are linear.*

Moreover, linearity of the heat flow implies linearity of the operator Δ, which coincides with the infinitesimal generator of the semi-group $(P_t)_{t>0}$, granting several results from spectral theory or from the theory of diffusion processes, like the celebrated Bakry-Émery estimate on which we will spend more time in Section 2.3.

Gradient flow of the relative entropy

As we shall explain in this paragraph, it turns out that in a quite general context, the heat flow produces the same evolution as the gradient flow of the relative entropy $\mathrm{Ent}_{\mathfrak{m}}$. Obviously, as $(\mathcal{P}_2(X), W_2)$ is not necessarily an Hilbert space, we cannot apply the Komura-Brezis theory to define the gradient flow of $\mathrm{Ent}_{\mathfrak{m}}$; we need a fully metric characterization. The next definition, due to E. De Giorgi, applies in wide generality and fit our context well.

Definition 2.2.12. Let (E, d_E) be a metric space, $f : E \to \mathbb{R} \cup \{+\infty\}$, and $\bar{x} \in \{f < +\infty\}$. An EDI (Energy Dissipation Inequality) gradient flow of f starting from $\bar{x}$ is a locally absolutely continuous curve $(x(t))_{t\geq 0} \subset E$ such that $x(0) = \bar{x}$ and

$$f(\bar{x}) \geq f(x(t)) + \frac{1}{2}\int_0^t |x'|^2(s)\,\mathrm{d}s + \frac{1}{2}\int_0^t |\nabla^- f|^2(x(s))\,\mathrm{d}s \tag{2.2.7}$$

for any $t > 0$, where $|\nabla^- f|$ is the descending slope of f, defined as

$$|\nabla^- f|(x) := \begin{cases} \limsup_{y\to x} \frac{\max(f(y)-f(x),0)}{\mathrm{d}_E(x,y)} & \text{if } x \in X \text{ is not isolated,} \\ 0 & \text{otherwise.} \end{cases}$$

When equality holds for all $t > 0$ in (2.2.7), the curve $(x(t))_{t>0}$ is called EDE (Energy Dissipation Equality) gradient flow.

The identification between the heat flow and the EDE gradient flow of the relative entropy was originally observed at a formal level by R. Jordan, D. Kinderlehrer and F. Otto in [JK098] and stated rigorously for the first time in the Euclidean space in [AGS08]. Afterwards N. Gigli, K. Kuwada and S. Ohta extended it to the purely metric setting of compact Alexandrov spaces [GKO13], replacing the Dirichlet energy with the Cheeger energy. The next statement is taken from [**?**, Th. 8.5], and as a byproduct, it justifies the existence of the EDE gradient flow of the relative entropy on $\mathrm{CD}(K,\infty)$ spaces.

Theorem 2.2.13. *Let $(X, \mathrm{d}, \mathfrak{m})$ be a $\mathrm{CD}(K,\infty)$ space. Then for all $f_0 \in L^2(X, \mathfrak{m})$ such that $\mu_0 = f_0\mathfrak{m} \in \mathcal{P}_2(X)$,* *(i) if* $(f_t)_{t>0}$ *is the heat flow of* $(X, \mathrm{d}, \mathfrak{m})$ *starting from* f_0*, then* $(\mu_t := f_t\mathfrak{m})_{t>0}$ *is the* EDE *gradient flow of* $\mathrm{Ent}_\mathfrak{m}$ *starting from* μ_0*,* $t \mapsto \mathrm{Ent}_\mathfrak{m}(\mu_t)$ *is locally absolutely continuous in* $(0, +\infty)$ *and*

$$-\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Ent}_\mathfrak{m}(\mu_t) = |\dot{\mu}_t|^2 \qquad \textit{for a.e. } t \in (0, +\infty);$$

(ii) if $(\mu_t)_{t>0}$ *is the* EDE *gradient flow of* $\mathrm{Ent}_\mathfrak{m}$ *starting from* μ_0*, then for any* $t > 0$ *we have* $\mu_t \ll \mathfrak{m}$*, and the family of densities* $(f_t := \frac{\mathrm{d}\mu_t}{\mathrm{d}\mathfrak{m}})_{t>0}$ *is the heat flow of* $(X, \mathrm{d}, \mathfrak{m})$ *starting from* f_0*.*

Remark 2.2.14. The heat flow can also be identified with the so-called EVI_K-gradient flow of $\mathrm{Ent}_\mathfrak{m}$. This property is particularly useful in the study of convergent sequences of $\mathrm{RCD}(K, \infty)$ spaces, see [AGS14b].

The Newtonian space $N^{1,2}(X, \mathrm{d}, \mathfrak{m})$

Another way to characterize a Sobolev function f on a given open set Ω in $\mathbb{R}^n$ is to require that for almost any line γ in Ω, the function $f \circ \gamma$ is absolutely continuous. Such an approach goes back to 1901 and the pioneering work of B. Levi [Le01]. It was later on pushed forward by B. Fuglede [Fu57] who formulated the quantification over almost all lines via an outer measure called 2-Modulus. This observation inspired N. Shanmugalingam who proposed in [Sh00] another possible definition of Sobolev space over $(X, \mathrm{d}, \mathfrak{m})$, replacing lines in Levi-Fuglede approch with absolutely continuous curves: this gives rise to the so-called Newtonian space $N^{1,2}(X, \mathrm{d}, \mathfrak{m})$, which is by definition the space of functions $\overline{f} : X \to [-\infty, +\infty]$ with $\int_X \overline{f}^2 \,\mathrm{d}\mathfrak{m} < +\infty$ for which there exist a function $\tilde{f} : X \to \mathbb{R}$ such that $\tilde{f} = \overline{f}$ $\mathfrak{m}$-a.e. on X, and a function $g \in L^2(X, \mathfrak{m})$ called 2-weak upper gradient, such that

$$\left| \int_{\partial\gamma} \tilde{f} \right| \leq \int_\gamma g$$

holds for Mod_2-a.e. curve γ. Here Mod_2 is the outer measure defined on the set of paths, i.e. absolutely continuous curves taking values in X, by

$$\mathrm{Mod}_2(\Gamma) := \inf \left\{ \|g\|^2_{L^2(X,\mathfrak{m})} \; : \; g : X \to [0, +\infty] \;\; \text{Borel such that } \int_\gamma g \geq 1 \text{ for all } \gamma \in \Gamma \right\}$$

for any family of paths Γ. The set of L^2-weak upper gradients of f being convex and closed, whenever it is non-empty there exists an element with minimal $L^2(X, \mathfrak{m})$ norm, denoted by $|\nabla \overline{f}|_S$ and called minimal 2-weak upper gradient of $\overline{f}$.

Using Cheeger's formalism, Shanmugalingam proved in [Sh00, Th. 4.1] that $H^{1,2}(X, \mathrm{d}, \mathfrak{m})$ coincides with $N^{1,2}(X, \mathrm{d}, \mathfrak{m})$ in the following sense: an equivalent class $f \in L^2(X, \mathfrak{m})$ belongs to $H^{1,2}(X, \mathrm{d}, \mathfrak{m})$ if and only if there exists a representative $\overline{f} : X \to [-\infty, +\infty]$ of f belonging to $N^{1,2}(X, \mathrm{d}, \mathfrak{m})$, in which case the minimal 2-weak upper gradient of $\overline{f}$ is denoted by $|\nabla f|_S$. Furthermore, $|\nabla f|_* = |\nabla f|_S$ $\mathfrak{m}$-a.e. on X for any $f \in H^{1,2}(X, \mathrm{d}, \mathfrak{m})$, see [AGS13, Th. 7] (see also [?] in the context of extended metric measure spaces).

The Sobolev class $S^2(X, \mathrm{d}, \mathfrak{m})$

A different way to quantify over almost every curves was proposed by L. Ambrosio, N. Gigli and G. Savaré in [?, Sect. 5] using the notion of test plan. For any $s \in [0, 1]$, let us denote by $e_s : C([0, 1], X) \to \mathbb{R}$ the evaluation map defined by $e_s(\gamma) = \gamma(s)$ for any $\gamma \in C([0, 1], X)$. We call test plan on $(X, \mathrm{d}, \mathfrak{m})$ any outer measure π on $C([0, 1], X)$

such that $\pi(C([0,1],X)) = 1$, concentrated in $AC^2([0,1],X)$, with bounded compression (meaning that there exists $C > 0$ such that $(e_s)_\# \pi \le C\mathfrak{m}$ for all $s \in [0,1]$) and with finite energy, i.e. $\int \int_0^1 |\gamma'|(s)^2 \, \mathrm{d}s \, \mathrm{d}\pi(\gamma) < +\infty$. Then a set of curves $\Gamma \subset C([0,1],X)$ is called 2-negligible whenever $\pi(\Gamma) = 0$ for any test plan π. A property is said to hold for 2-a.e. curve if the set of curves on which it doesn't hold is 2-negligible.

The Sobolev class $S^2(X, \mathrm{d}, \mathfrak{m})$ is by definition the set of Borel functions $f : X \to \mathbb{R}$ admitting a L^2 weak upper gradient, namely a non-negative function $G \in L^2(X, \mathfrak{m})$ such that

$$\int_{C([0,1],X)} |f(\gamma(1)) - f(\gamma(0))| \, \mathrm{d}\pi(\gamma) \le \int_{C([0,1],X)} \int_0^1 G(\gamma(t)) |\dot\gamma|(t) \, \mathrm{d}t \, \mathrm{d}\pi(\gamma) \tag{2.2.8}$$

holds for any test plan π. It can be shown (see [?, Def. 5.1] or [AGS13, Sec. 4.3]) that for every $f \in S^2(X, \mathrm{d}, \mathfrak{m})$ there exists a unique (up to $\mathfrak{m}$-negligible sets) mimimal (in the $\mathfrak{m}$-a.e. sense) L^2 weak upper gradient, denoted by $|Df|$. The intersection $S^2(X, \mathrm{d}, \mathfrak{m}) \cap L^2(X, \mathfrak{m})$ coincides with $H^{1,2}(X, \mathrm{d}, \mathfrak{m})$, and the minimal weak upper gradient $|Df|$ coincides with the minimal relaxed slope $|\nabla f|$, up to $\mathfrak{m}$-negligible sets, see [ACDM15] for a nice presentation of this fact.

The link between functions in the Sobolev class and their regularity along curves lies in the following statement, see [G18, Th. 6.4] for a proof.

Proposition 2.2.15. *Let $f : X \to \mathbb{R}$ and $G : X \to [0, +\infty]$ be Borel functions with $G \in L^2(X, \mathfrak{m})$. Then the following are equivalent.*

1. *$f \in S^2(X, \mathrm{d}, \mathfrak{m})$ and G is a 2-weak upper gradient of f;*

2. *for 2-a.e. curve γ, the function $f \circ \gamma$ coincides $\mathscr{L}^1$-a.e. in $[0,1]$ and in $\{0,1\}$ with an absolutely continuous map $f_\gamma : [0,1] \to \mathbb{R}$, and $\left| \int_{\partial\gamma} f \right| \le \int_\gamma G$.*

Remark 2.2.16. Several other definitions of Sobolev spaces on metric measure spaces exist in the literature. For instance, N.J. Korevaar and R.M. Schoen considered in [KS93] the space of functions $u : X \to \overline{\mathbb{R}}$ such that

$$\sup \left\{ \limsup_{\varepsilon \to 0} \int_X f(x) \left[\fint_{B_\varepsilon(x)} \varepsilon^{-p} |u(x) - u(y)|^p \, \mathrm{d}\mathfrak{m}(y) \right] \mathrm{d}\mathfrak{m}(x) \ : \ f \in C_c(X, [0,1]) \right\} < +\infty.$$

In [Ha95], P. Hajlasz set as Sobolev function any $f \in L^p(X, \mathfrak{m})$ for which there exists a function $g \in L^p(X, \mathfrak{m})$ and a $\mathfrak{m}$-negligible set $N \subset X$ such that $|f(x) - f(y)| \le \mathrm{d}(x,y)(g(x) + g(y))$ holds for any $x, y \in X \backslash N$. Both spaces are equal when $(X, \mathrm{d}, \mathfrak{m})$ satisfies a local $(1,q)$-Poincaré inequality with $1 \le q < p$, see [KM98, Th. 4.5], and of course they coincide with the usual Sobolev space $H^{1,p}(\Omega)$ when Ω is an open subset of $\mathbb{R}^n$.

Cheeger's first-order differential structure

Recall that Rademacher's theorem states that any Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable $\mathscr{L}^n$-a.e. on $\mathbb{R}^n$. In [Ch99], J. Cheeger extended this result to PI doubling spaces which are defined as follows.

Definition 2.2.17. (PI doubling spaces) Let (X, d) be a metric space and $\mathfrak{m}$ a Borel regular measure on (X, d) which is finite and non-zero on balls with finite and non-zero radius. Then the metric measure space $(X, \mathrm{d}, \mathfrak{m})$ is called PI doubling if the two following conditions hold:

(i) (local doubling condition) for all $R > 0$ there exists $C_D = C_D(R) > 0$ such that for all $x \in X$ and $0 < r < R$,

$$\mathfrak{m}(B_{2r}(x)) \le C_D \mathfrak{m}(B_r(x));$$

(ii) (local weak $(1,p)$-Poincaré inequality) for some $1 \le p < +\infty$, there exists $R > 0$, $1 < \lambda < +\infty$ and $C_P = C_P(p,R) > 0$ such that for all $f \in L^0(X,\mathfrak{m})$ and $g \in \mathrm{UG}^p(f)$,

$$\fint_{B_r} |f - f_{B_r}| \, \mathrm{d}\mathfrak{m} \le C_P r \left(\fint_{B_r} g^p \, \mathrm{d}\mathfrak{m} \right)^{1/p}$$

holds for all ball B_r with radius $0 < r < R$.

In addition to the generalized Rademacher's theorem, which provides a first-order differential structure on PI doubling spaces, J. Cheeger also proved that Sobolev functions asymptotically satisfy the Dirichlet principle, i.e. they tend to minimize the local Cheeger energy on every ball with radius converging to 0.

Theorem 2.2.18. *Let $(X, \mathrm{d}, \mathfrak{m})$ be a PI doubling space. Then:*

(1) (asymptotic minimizers of the Cheeger energy) for all $f \in H^{1,2}(X,\mathrm{d},\mathfrak{m})$, for $\mathfrak{m}$-a.e. $x \in X$ one has $\mathrm{Dev}(f, B_r(x)) = o(\mathfrak{m}(B_r(x)))$ as $r \downarrow 0$, where

$$\mathrm{Dev}(f, B_r(x)) = \int_{B_r(x)} |\nabla f|^2 \, \mathrm{d}\mathfrak{m} - \inf \left\{ \int_{B_r(x)} |\nabla h|^2 \, \mathrm{d}\mathfrak{m} : \ f - h \in \mathrm{Lip}_c(B_r(x)) \right\}; \tag{2.2.9}$$

(2) (generalized Rademacher's theorem) there exist two constants $0 < M < +\infty$ and $N \in \mathbb{N}$ depending only on C_D and C_P such that $\mathfrak{m}$-almost all of X can be covered by a sequence of Borel sets C with the following property: there exist a family of $k = k(C)$ Lipschitz functions $F_1, \ldots, F_k \in H^{1,2}(X,\mathrm{d},\mathfrak{m})$, with $k \le N$, such that for all $f \in \mathrm{Lip}(X,\mathrm{d}) \cap H^{1,2}(X,\mathrm{d},\mathfrak{m})$, one has

$$\mathrm{lip}(f(\cdot) - \sum_{i=1}^{k} \chi_i(x_0) F_i(\cdot))(x_0) = 0 \qquad \text{for } \mathfrak{m}\text{-a.e. } x_0 \in C \tag{2.2.10}$$

for suitable $\chi_i \in L^2(C,\mathfrak{m})$ with $\sum_i \chi_i^2 \le M |\nabla f|^2$ $\mathfrak{m}$-a.e. on C.

Sobolev spaces and Laplacians on open sets

Following a standard approach, let us localize some of the concepts previously introduced in this chapter. These notions will be used only in Chapter 5. First of all, let us introduce the Sobolev space $H^{1,2}(B_R(x), \mathrm{d}, \mathfrak{m})$ on a $\mathrm{RCD}^*(K,N)$-space $(X,\mathrm{d},\mathfrak{m})$. See also [Ch99, Sh00] for the definition of Sobolev space $H^{1,p}(U,\mathrm{d},\mathfrak{m})$ for any $p \in [1,\infty)$ and any open subset U of X. Our working definition is the following.

Definition 2.2.19. Let $U \subset X$ be open.

1. ($H_0^{1,2}$-Sobolev space) We denote by $H_0^{1,2}(U,\mathrm{d},\mathfrak{m})$ the $H^{1,2}$-closure of $\mathrm{Lip}_c(U,\mathrm{d})$, the subspace of $\mathrm{Lip}(U,\mathrm{d})$ of compactly supported functions.

2. (Sobolev space on an open set U) We say that $f \in L^2_{loc}(U,\mathfrak{m})$ belongs to $H^{1,2}_{\mathrm{loc}}(U,\mathrm{d},\mathfrak{m})$ if $\varphi f \in H^{1,2}(X,\mathrm{d},\mathfrak{m})$ for any $\varphi \in \mathrm{Lip}_c(U,\mathrm{d})$. In case $X \backslash U \neq \emptyset$, if, in addition, $|\nabla f| \in L^2(U,\mathfrak{m})$, we say that $f \in H^{1,2}(U,\mathrm{d},\mathfrak{m})$.

Notice that $f \in H^{1,2}_{\rm loc}(U, \mathrm{d}, \mathfrak{m})$ if and only if for any $V \Subset U$ there exists $\tilde{f} \in H^{1,2}(X, \mathrm{d}, \mathfrak{m})$ with $\tilde{f} \equiv f$ on V. The global condition $|\nabla f| \in L^2(U, \mathfrak{m})$ in the definition of $H^{1,2}(U, \mathrm{d}, \mathfrak{m})$ is meaningful, since the locality properties of the minimal relaxed slope ensure that $|\nabla f|$ makes sense $\mathfrak{m}$-a.e. in X for all functions f such that $\varphi f \in H^{1,2}(X, \mathrm{d}, \mathfrak{m})$ for any $\varphi \in \mathrm{Lip}_c(U, \mathrm{d})$. Indeed, choosing $\varphi_n \in \mathrm{Lip}_c(U, \mathrm{d})$ with $\{\varphi_n = 1\} \uparrow U$ and defining

$$|\nabla f| := |\nabla (f\varphi_n)| \qquad \mathfrak{m}\text{-a.e. on } \{\varphi_n = 1\} \setminus \{\varphi_{n-1} = 1\}$$

we obtain an extension of the minimal relaxed gradient to $H^{1,2}(U, \mathrm{d}, \mathfrak{m})$ (for which we keep the same notation, being also independent of the choice of φ_n) which retains all bilinearity and locality properties.

Accordingly, for $U \subset X$ open we can define the Cheeger energy $\mathrm{Ch}_U : L^2(U, \mathfrak{m}) \to [0, \infty]$ on U by

$$\mathrm{Ch}_U(f) := \begin{cases} \mathrm{Ch}(f) & \text{if } f \in H^{1,2}_0(U, \mathrm{d}, \mathfrak{m}); \\ +\infty & \text{otherwise} \end{cases} \tag{2.2.11}$$

and put $\mathrm{Ch}_{(x,R)} := \mathrm{Ch}_{B_R(x)}$.

We introduce the Dirichlet Laplacian acting only on $H^{1,2}_0$-functions as follows:

Definition 2.2.20 (Dirichlet Laplacian on an open set U)**.** Let $D_0(\Delta, U)$ denote the set of all $f \in H^{1,2}_0(U, \mathrm{d}, \mathfrak{m})$ such that there exists $h := \Delta_U f \in L^2(U, \mathfrak{m})$ satisfying

$$\int_U hg \,\mathrm{d}\mathfrak{m} = -\int_U \langle \nabla f, \nabla g\rangle \,\mathrm{d}\mathfrak{m} \qquad \forall g \in H^{1,2}_0(U, \mathrm{d}, \mathfrak{m}).$$

We also set $\Delta_{x,R} := \Delta_{B_R(x)}$ when $U = B_R(x)$ for some $x \in X$ and $R \in (0, \infty)$.

Strictly speaking, the Dirichlet Laplacian Δ_U should not be confused with the operator Δ, even if the two operators agree on functions compactly supported on U; for this reason we adopted a distinguished symbol. Notice that $\lambda_1^D(B_R(x)) > 0$ whenever if $\mathfrak{m}(X \setminus B_r(x)) > 0$, as a direct consequence of the local Poincaré inequality.

Definition 2.2.21 (Laplacian on an open set U)**.** For $f \in H^{1,2}(U, \mathrm{d}, \mathfrak{m})$, we write $f \in D(\Delta, U)$ if there exists $h := \Delta_U f \in L^2(U, \mathfrak{m})$ satisfying

$$\int_U hg \,\mathrm{d}\mathfrak{m} = -\int_U \langle \nabla f, \nabla g\rangle \,\mathrm{d}\mathfrak{m} \qquad \forall g \in H^{1,2}_0(U, \mathrm{d}, \mathfrak{m}).$$

Since for $f \in H^{1,2}_0(U, \mathrm{d}, \mathfrak{m})$ one has $f \in D(\Delta, U)$ iff $f \in D_0(\Delta, U)$ and the Laplacians are the same, we retain the same notation Δ_U of Definition 2.2.20. It is easy to check that for any $f \in D(\Delta, U)$ and any $\varphi \in D(\Delta) \cap \mathrm{Lip}_c(U, \mathrm{d})$ with $\Delta\varphi \in L^\infty(X, \mathfrak{m})$ one has (understanding $\varphi \Delta_U f$ to be null out of U) $\varphi f \in D(\Delta)$ with

$$\Delta(\varphi f) = f\Delta\varphi + 2\langle \nabla\varphi, \nabla f\rangle + \varphi\Delta_U f \qquad \mathfrak{m}\text{-a.e. in } X. \tag{2.2.12}$$

Such notions allow to define harmonic functions on an open set U as follows.

Definition 2.2.22. Let $U \subset X$ be open. We say that $f \in H^{1,2}_{\rm loc}(U, \mathrm{d}, \mathfrak{m})$ is harmonic in U if $f \in \mathcal{D}(\Delta, V)$ with $\Delta f = 0$ for any open set $V \Subset U$, namely

$$\int_U \langle \nabla f, \nabla g\rangle \,\mathrm{d}\mathfrak{m} = 0 \qquad \forall g \in \mathrm{Lip}_c(U, \mathrm{d}).$$

Let us denote by $\mathrm{Harm}(U, \mathrm{d}, \mathfrak{m})$ the set of harmonic functions on U.

In Chapter 5, we will consider mainly globally defined harmonic functions. It is worth pointing out that, in general, these functions do not belong to $H^{1,2}(X, \mathrm{d}, \mathfrak{m})$ but, by definition, they belong to $H^{1,2}_{\mathrm{loc}}(X, \mathrm{d}, \mathfrak{m})$.

Harmonic replacement

Let us conclude this section by introducing the notion of harmonic replacement which will play a key role in Chapter 5. As we already remarked, the assumption that the first Dirichlet eigenvalue $\lambda_1^D(B_R(x))$ for the ball $B_R(x)$ is strictly positive is valid for sufficently small balls, indeed it holds as soon as $\mathfrak{m}(X \setminus B_R(x)) > 0$.

Proposition 2.2.23. *Assume that* $\mathfrak{m}(B_R(x)) > 0$ *and* $\lambda_1^D(B_R(x)) > 0$*. Then for any* $f \in H^{1,2}(B_R(x), \mathrm{d}, \mathfrak{m})$*, there exists a unique* $\hat{f} \in D(\Delta, B_R(x))$*, called harmonic replacement of* f*, such that*

$$\begin{cases} \Delta_{x,R}\hat{f} = 0 \\ \\ f - \hat{f} \in H^{1,2}_0(B_R(x), \mathrm{d}, \mathfrak{m}). \end{cases} \tag{2.2.13}$$

Moreover,

$$\||\nabla \hat{f}|\|_{L^2(B_R(x))} \leq 2\||\nabla f|\|_{L^2(B_R(x))}, \tag{2.2.14}$$

$$\|\hat{f}\|_{L^2(B_R(x))} \leq \|f\|_{L^2(B_R(x))} + \frac{1}{\lambda_1^D(B_R(x))}\||\nabla f|\|_{L^2(B_R(x))}. \tag{2.2.15}$$

Finally, $\hat{f} - f$ *is the unique minimizer of the functional*

$$\psi \in H^{1,2}_0(B_R(x), \mathrm{d}, \mathfrak{m}) \mapsto \int_X |f + \psi|^2 \, \mathrm{d}\mathfrak{m}.$$

2.3 Main properties of spaces with Riemannian curvature-dimension conditions

In this section, we explain how $\mathrm{RCD}^*(K,N)$ spaces, with $N<+\infty$, admit a unique heat kernel satisfying sharp Gaussian bounds. Afterwards we present the relationship between the $\mathrm{RCD}^*(K,N)$ conditions (with N possibly infinite) and Bakry-Émery's conditions stated in the context of diffusion processes. We conclude with deep results concerning the structure of finite-dimensional $\mathrm{RCD}^*(K,N)$ spaces.

Heat kernel in the setting of diffusion processes

In the context of diffusion processes, K.-T. Sturm established in [St94, St95, St96] several results concerning existence, uniqueness and regularity properties of the heat kernel associated with a Dirichlet form, under the assumption that this Dirichlet form defines a metric on the state space with nice properties. Let us give the precise setting in which these results hold.

Let (X,τ) be a locally compact, separable, Hausdorff topological space and $\mathfrak{m}$ a positive Radon measure on the Borel σ-algebra of (X,τ) such that $\operatorname{supp}\mathfrak{m}=X$. We shall use the following notation: $C(X)$ is the set of continuous real-valued functions on X, $C_c(X)$ (resp. $C_b(X)$) is the subset of $C(X)$ made of compactly supported (resp. bounded) functions, Rad stands for the set of signed Radon measures defined on the Borel σ-algebra of (X,τ).

We start with the definition of Dirichlet form, following Sturm's approach.

Definition 2.3.1. (Dirichlet form) A Dirichlet form $\mathcal{E}$ on $L^2(X,\mathfrak{m})$ with domain $\mathcal{D}(\mathcal{E})$ is a positive definite bilinear map $\mathcal{E}:\mathcal{D}(\mathcal{E})\times\mathcal{D}(\mathcal{E})\to\mathbb{R}$, with $\mathcal{D}(\mathcal{E})$ being a dense subset of $L^2(X,\mathfrak{m})$, satisfying closedness, i.e. the set $\{(f,g,\mathcal{E}(f,g)) : f,g\in\mathcal{D}(\mathcal{E})\}$ (the graph of $\mathcal{E}$) is closed in $\mathcal{D}(\mathcal{E})\times\mathcal{D}(\mathcal{E})\times\mathbb{R}$, and the Markov property: $\mathcal{E}(f_0^1,f_0^1)\le\mathcal{E}(f,f)$ for any $f\in\mathcal{D}(\mathcal{E})$, where $f_0^1=\min(\max(f,0),1)$.

We refer to [BH91, FOT10] for equivalent formulations of the Markov property, and more generally for the basics on Dirichlet forms.

Note that instead of requiring closedness, we can equivalently assume $L^2(X,\mathfrak{m})$-lower semicontinuity of $\mathcal{E}$. This fact is important as it enables to apply the general theory of gradient flows to any Dirichlet form $\mathcal{E}$, producing a semi-group of operators $(P_t)_{t>0}$: $L^2(X,\mathfrak{m})\to L^2(X,\mathfrak{m})$.

We will only consider symmetric Dirichlet forms, i.e. such that $\mathcal{E}(f,g)=\mathcal{E}(g,f)$ for any $f,g\in\mathcal{D}(\mathcal{E})$, and always assume that the space $\mathcal{D}(\mathcal{E})$ is a Hilbert space once equipped with

$$\langle f,g\rangle := \int_X fg\,\mathrm{d}\mathfrak{m}+\mathcal{E}(f,g)\qquad \forall f,g\in\mathcal{D}(\mathcal{E}).$$

Usually Dirichlet forms are studied with further assumptions.

Definition 2.3.2. Let $\mathcal{E}$ be a Dirichlet form on $L^2(X,\mathfrak{m})$ with domain $\mathcal{D}(\mathcal{E})$.

1. (locality) We say that $\mathcal{E}$ is local if $\mathcal{E}(f,g)=0$ for any $f,g\in\mathcal{D}(\mathcal{E})$ with disjoint supports;

2. (regularity) we say that $\mathcal{E}$ is regular if $C_c(X)\cap\mathcal{D}(\mathcal{E})$ contains a subset which is both dense in $C_c(X)$ for $\|\cdot\|_\infty$ and dense in $\mathcal{D}(\mathcal{E})$ for $\|\cdot\|_{\mathcal{E}}$;

3. (strong locality) we say that $\mathcal{E}$ is strongly local if $\mathcal{E}(f,g)=0$ for any $f,g\in\mathcal{D}(\mathcal{E})$ such that f is constant on a neighborhood of $\operatorname{supp} g$;

4. (irreducibility) denoting by $\mathcal{D}_{loc}(\mathcal{E})$ the set of $\mathfrak{m}$-measurable functions f on X such that for any compact set K there exists $g \in \mathcal{D}(\mathcal{E})$ such that $f = g$ $\mathfrak{m}$-a.e. on K, we say that $\mathcal{E}$ is irreductible if any $f \in \mathcal{D}_{loc}(\mathcal{E})$ such that $\mathcal{E}(f,f) = 0$ is constant on X.

Under suitable assumptions, any Dirichlet form admits an important representation formula established by A. Beurling and J. Deny [BD59].

Proposition 2.3.3. *(Representation of $\mathcal{E}$ with Radon measures) Assume that $\mathcal{E}$ is a strongly local regular symmetric Dirichlet form on $L^2(X,\mathfrak{m})$. Then there exists a nonnegative definite and symmetric bilinear map $\Gamma : \mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E}) \to \mathrm{Rad}$ such that*

$$\mathcal{E}(f,g) = \int_X \mathrm{d}\Gamma(f,g) \qquad \forall f,g \in \mathcal{D}(\mathcal{E})$$

where $\int_X \mathrm{d}\Gamma(f,g)$ denotes the total mass of the measure $\Gamma(f,g)$.

It is worth point out that the above map Γ is concretely given as follows: for any $f \in \mathcal{D}(\mathcal{E}) \cap L^\infty(X,\mathfrak{m})$, the measure $\Gamma(f) = \Gamma(f,f)$ is defined through its action on test functions:

$$\int_X \varphi \,\mathrm{d}\Gamma(f) := \mathcal{E}(f, f\varphi) - \frac{1}{2}\mathcal{E}(f^2,\varphi) \qquad \forall \varphi \in \mathcal{D}(\mathcal{E}) \cap C_c(X). \tag{2.3.1}$$

Regularity of $\mathcal{E}$ allows to extend (2.3.1) to the set of functions $C_c(X)$, providing a well-posed definition of $\Gamma(f)$ by duality between $C_c(X)$ and Rad. The general expression of $\Gamma(f,g)$ for any $f,g \in \mathcal{D}(\mathcal{E})$ is then obtained by polarization:

$$\Gamma(f,g) := \frac{1}{4}(\Gamma(f+g,f+g) - \Gamma(f-g,f-g)).$$

Locality of Γ allows to extend $\mathcal{E}$ to the set $\mathcal{D}_{loc}(\mathcal{E})$ consisting of all $\mathfrak{m}$-measurable functions f such that for any compact set $K \in X$ there exists a function $g \in \mathcal{D}(\mathcal{E})$ such that $f = g$ $\mathfrak{m}$-a.e. on K.

Thanks to Γ, we can associate with $\mathcal{E}$ an (extended) pseudo-distance which shall be of outmost importance in the sequel.

Definition 2.3.4. (Intrinsic extended pseudo-distance) The intrinsic extended pseudo-distance ρ associated to $\mathcal{E}$ is defined by

$$\rho(x,y) := \sup\{|f(x) - f(y)| \,:\, f \in \mathcal{D}_{loc}(\mathcal{E}) \cap L^\infty(X,\mathfrak{m}) \text{ s. t. } [f] \le \mathfrak{m}\} \qquad \forall x,y \in X.$$

Here $[f] \le \mathfrak{m}$ means that $[f]$ is absolutely continuous with respect to $\mathfrak{m}$ with $\left|\frac{\mathrm{d}[f]}{\mathrm{d}\mathfrak{m}}\right| \le 1$ $\mathfrak{m}$-a.e. on X, and "extended" refers to the fact that $\rho(x,y)$ may be infinite.

Recall finally that any Dirichlet form is associated with a non-positive definite self-adjoint operator L with dense domain $\mathcal{D}(L) \subset L^2(X,\mathfrak{m})$ characterized by the following property:

$$\begin{cases} \mathcal{D}(L) \subset \mathcal{D}(\mathcal{E}), \\ \mathcal{E}(f,g) = -\int_X (Lf)g \,\mathrm{d}\mathfrak{m} \quad \forall f \in \mathcal{D}(L), g \in \mathcal{D}(E). \end{cases}$$

We are now in a position to present Sturm's results.

Theorem 2.3.5. *Let $(P_t)_{t>0}$ be the semi-group of operators associated to $\mathcal{E}$. Assume that:*

(A) the intrinsic pseudo-distance ρ associated to $\mathcal{E}$ is actually a distance which induces the topology τ;

(B) *the local doubling property* (2.1.8) *holds for* $(X, \rho, \mathfrak{m})$*;*

(C) *a local weak* L^2*-Poincaré inequality holds: there exists a constant* $C_P > 0$ *such that for all* $f \in H^{1,2}(X, \mathrm{d}, \mathfrak{m})$,

$$\int_{B_r} |f - f_{B_r}|^2 \, \mathrm{d}\mathfrak{m} \leq C_P r^2 \int_{B_{2r}} \mathrm{d}\Gamma(f, f)$$

for all relatively compact balls $B_{2r} \subset X$*, with* f_{B_r} *denoting the mean-value of* f *over the ball* B_r*.*

Then there exists a measurable function $p : X \times X \times (0, +\infty) \to (0, +\infty]$ *called heat kernel of* $(X, \mathfrak{m}, \mathcal{E})$ *such that:*

(1) p *is symmetric with respect to the first two variables;*

(2) *([St95, Prop. 2.3]) for any* $t > 0$ *and* $f \in \mathcal{D}(L)$,

$$P_t f(x) = \int_X p(x, y, t) f(y) \, \mathrm{d}\mathfrak{m}(y) \qquad \text{for } \mathfrak{m}\text{-a.e. } x \in X;$$

(3) *([St96, Th. 3.5 and Cor. 3.3]) there exists a locally Hölder continous representative of* p*, which from now on we will always use to state formulae: for any relatively compact set* $Y \subset X$*, there exists constants* $\alpha = \alpha(Y) \in (0, 1)$ *and* $C = C(Y) > 0$ *such that for all balls* $B_{2r} \subset Y$*, all* $T > 0$*, and any* $x \in X$*,*

$$|p(x, y_1, t_1) - p(x, y_2, t_2)| \leq C (\sup_{Q_2} u) \left(\frac{|t_1 - t_2|^{1/2} + |y_1 - y_2|}{r} \right)^{\alpha} \tag{2.3.2}$$

for any $0 < r < \sqrt{T}/2$*, where* $Q_2 = (T - 4r^2, T) \times B_{2r}$ *and* $(y_1, t_1), (y_2, t_2)$ *are in* $Q_1 := (T - r^2, T) \times B_r$*;*

(4) *([St95, Prop. 2.3]) the Chapman-Kolmogorov formula*

$$p(x, y, t_1 + t_2) = \int_X p(x, z, t_1) p(z, y, t_2) \, \mathrm{d}\mathfrak{m} \tag{2.3.3}$$

holds for any $x, y \in X$ *and any* $t_1, t_2 > 0$*;*

(5) *([St95, Eq. (2.27)]) for any* $x, y \in X$*, the function* $t \mapsto p(x, y, t)$ *is* C^∞ *on* $(0, +\infty)$*, and for every* $\varepsilon > 0$ *and* $j \in \mathbb{N}$*, there exists a constant* $C_1 = C_1(\varepsilon, j, C_D, C_P) > 0$ *such that*

$$\left[\frac{\mathrm{d}}{\mathrm{d}t} \right]^j p(x, y, t) \leq \frac{C_1}{\mathfrak{m}(B_{\sqrt{t}}(x))} e^{-\frac{\rho^2(x,y)}{(4+\varepsilon)t}} \qquad \forall t > 0; \tag{2.3.4}$$

(6) *([St96, Cor. 4.10]) there exists a constant* $C_2 = C_2(C_D, C_P) > 0$ *such that*

$$p(x, y, t) \geq \frac{C_2^{-1}}{\mathfrak{m}(B_{\sqrt{t}}(x))} e^{-C_2 \frac{\rho^2(x,y)}{t}} \tag{2.3.5}$$

for all $t > 0$ *and* $x, y \in X$*.*

Several remarks are in order.

Remark 2.3.6. 1. Let us point out that Sturm's results hold in a more general context in which $\mathcal{E}$ is replaced by a family of Dirichlet forms $(\mathcal{E}_s)_{s\in\mathbb{R}}$ with common domain $\mathcal{F}$, the estimate (2.3.2), (2.3.4) and (2.3.5) being then suitably modified. In this case, further requirements are needed, namely uniform parabolicity and local strong uniform parabolicity with respect to a reference strongly local regular Dirichlet form $\mathcal{E}$, but they are automatically satisfied when $\mathcal{E}_s \equiv \mathcal{E}$.

2. The proof of the existence of p presented in [St95, Prop. 2.3] relies on precise L^p estimates for sub- and supersolutions of the equation $(L+\alpha)u = \partial_t u$ with $\alpha \in \mathbb{R}$, obtained by Moser iteration technique which in turn requires a local Sobolev inequality. On smooth Riemannian manifolds, such an inequality is implied by doubling condition and local Poincaré inequality, as proved indepedently by A. Grigor'yan [Gr92] and L. Saloff-Coste [Sa92]. K.-T. Sturm pointed out that Saloff-Coste's proof is valid also in the present setting [St96, Th. 2.6].

3. The previously mentioned local Sobolev inequality used by Sturm involves a dimension N which might depend on the relatively compact open set on which it holds.

4. The local Hölder continuity of a representative of the heat kernel is a consequence of the parabolic Harnack inequality [St96, Prop. (II)] which was proved equivalent in the smooth setting to the local doubling and Poincaré properties by L. Saloff-Coste [Sa92]. Here again, K.-T. Sturm noticed that one side of the equivalence in Saloff-Coste's proof could be carried over line by line in his context, the other side following from the previously mentioned L^p estimates.

5. The constants in the Gaussian estimates (2.3.4) and (2.3.5) are not sharp.

Heat kernel on $\mathrm{RCD}^*(K,N)$ spaces

Theorem 2.3.5 applies to any given $\mathrm{RCD}^*(K,N)$ space $(X,\mathrm{d},\mathfrak{m})$ - note that here for convenience we assume $\mathrm{supp}(\mathfrak{m}) = X$. Indeed, the intrinsic distance associated to the strongly local and regular Dirichlet form identified in Proposition 2.2.10, that we shall also denote by Ch in the sequel, coincides with the original distance d on $X \times X$, see [AGS14b, Th. 6.10], and recall that by Corollary 2.1.15 and Theorem 2.1.16, the local doubling condition and a local L^1-Poincaré inequality hold true even in the more general context of $\mathrm{CD}(K,N)$ spaces. Whence the existence of the heat kernel p on $(X,\mathrm{d},\mathfrak{m})$.

Note that by properties of the heat flow, for any $y \in X$, $t > 0$ we have $p(\cdot,y,t) \in H^{1,2}(X,\mathrm{d},\mathfrak{m})$, and in the sequel we adopt the notation $|\nabla_x p(x,y,t)| := |\nabla p(\cdot,y,t)|(x)$ and $\langle \nabla_x p(x,y,t), g\rangle = \langle \nabla p(\cdot,y,t), \nabla g\rangle(x)$ for any $g \in H^{1,2}(X,\mathrm{d},\mathfrak{m})$.

The Gaussian bounds (2.3.4) and (2.3.5) can be sharpened in the $\mathrm{RCD}^*(K,N)$ setting, thanks to the Laplacian comparison theorem due to N. Gigli [G15, Th. 5.14] and the parabolic Harnack inequality for the heat flow established by N. Garofalo and A. Mondino [GM14, Th. 1.4] for the case $\mu(X) < +\infty$ and R. Jiang [J16, Th. 1.3] for the case $\mu(X) = +\infty$. Combining these sharp estimates with the Li-Yau gradient estimate proved by the same authors ([GM14, Th. 1.1], [J16, Th. 1.1]), one can derive a sharp bound for the gradient of the heat kernel. These results are due to R. Jiang, H. Li and H. Zhang [JLZ16, Th. 1.2 and Cor. 1.2].

Theorem 2.3.7. *Let $(X,\mathrm{d},\mathfrak{m})$ be a $\mathrm{RCD}^*(K,N)$ space with $K < 0$ and $N \in [1,+\infty)$. Then for any $\varepsilon > 0$, there exist positive constants $C_1, C_2, C_3, C_4 > 0$ depending only on K, N and ε such that for any $t > 0$ and $x,y \in X$,*

$$\frac{C_1^{-1}}{\mathfrak{m}(B_{\sqrt{t}}(x))} \exp\left(-\frac{\mathrm{d}^2(x,y)}{(4-\varepsilon)t} - C_2 t\right) \le p(x,y,t) \le \frac{C_1}{\mathfrak{m}(B_{\sqrt{t}}(x))} \exp\left(-\frac{\mathrm{d}^2(x,y)}{(4+\varepsilon)t} + C_2 t\right), \tag{2.3.6}$$

and for any $y \in X$ *and* $t > 0$,

$$|\nabla_x p(x, y, t)| \leq \frac{C_3}{\sqrt{t}\mathfrak{m}(B_{\sqrt{t}}(x))} \exp\left(-\frac{\mathrm{d}^2(x, y)}{(4+\varepsilon)t} + C_4 t\right) \tag{2.3.7}$$

for $\mathfrak{m}$*-a.e.* $x \in X$.

Moreover by [D97, Thm. 4] with (2.3.6) the inequality

$$\left|\frac{\mathrm{d}}{\mathrm{d}t} p(x, y, t)\right| = |\Delta_x p(x, y, t)| \leq \frac{C_5}{t\mathfrak{m}(B_{t^{1/2}}(x))} \exp\left(-\frac{\mathrm{d}(x, y)^2}{(4+\epsilon)t} + C_6 t\right) \tag{2.3.8}$$

holds on all $t > 0$ and $\mathfrak{m} \times \mathfrak{m}$-a.e. $(x, y) \in X \times X$, where $C_5, C_6 > 1$ depend only on ϵ, K, N (see also [JLZ16, (3.11)]).

We will mainly apply these estimates in the case $\varepsilon = 1$. Note that (2.3.7) implies a quantitative local Lipschitz bound on p, i.e., for any $z \in X$, any $R > 0$ and any $0 < t_0 \leq t_1 < \infty$ there exists $C := C(K, N, R, t_0, t_1) > 0$ such that

$$|p(x, y, t) - p(\hat{x}, \hat{y}, t)| \leq \frac{C}{\mathfrak{m}(B_{t_0^{1/2}}(z))} \mathrm{d}((x, y), (\hat{x}, \hat{y})) \tag{2.3.9}$$

for all x, y, $\hat{x}$, $\hat{y} \in B_R(z)$ and any $t \in [t_0, t_1]$.

Relation between $\mathrm{RCD}^*(K, N)$ **spaces and Bakry-Émery's curvature-dimension condition** $\mathrm{BE}(K, N)$

Recall that a Markov semi-group on a σ-finite measure space $(X, \mathfrak{m})$ is a family of operators $(P_t)_{t>0}$ acting on $L^2(X, \mathfrak{m})$ such that $P_{t+s} = P_t \circ P_s$ for any $t, s > 0$ and $P_t f \geq 0, \|P_t f\|_{L^1(X,\mathfrak{m})} = 1$ whenever $f \geq 0$ and $\|f\|_{L^1(X,\mathfrak{m})} = 1$ respectively. As one can easily check, a simple example of Markov semi-group is the one associated to a Dirichlet form. Any Markov semi-group $(P_t)_{t>0}$ is called hypercontractive whenever there exists $\lambda > 0$ such that $\|P_t f\|_{L^q(X,\mathfrak{m})} \leq \|f\|_{L^p(X,\mathfrak{m})}$ for any $p, q \geq 1$ and $t > 0$ such that $q - 1 \leq (p-1)e^{\lambda t}$. Hypercontractivity plays a central role in the theory of diffusion processes because it provides crucial tools to prove many estimates and functional inequalities like logarithmic Sobolev, Poincaré, or Talagrand ones, see e.g. [Ba94] for an overview on this topic.

At the end of the eighties, D. Bakry and M. Émery introduced a sufficient condition for a Markov semi-group to be hypercontractive [BE85] which was later on extensively studied as a curvature-dimension condition for measure spaces equipped with a suitable Dirichlet form, see e.g. [CS86, Ba91, Le00, BGL14].

In full generality, the objects under consideration in Bakry-Émery's condition are a set X, an algebra $\mathcal{A}$ of functions $f : X \to \mathbb{R}$, and a linear map $L : \mathcal{A} \to \mathcal{A}$. Associated to this linear map are the symmetric bilinear operators $\Gamma, \Gamma_2 : \mathcal{A} \times \mathcal{A} \to \mathbb{R}$, respectively called *carré du champ* and *iterated carré du champ*, which are defined as follows:

$$\Gamma(f, g) := \frac{1}{2}(L(fg) - fLg - gLf),$$
$$\Gamma_2(f, g) := \frac{1}{2}(L(\Gamma(f, g)) - \Gamma(Lf, g) - \Gamma(f, Lg)),$$

for all $f, g \in \mathcal{A}$.

Definition 2.3.8. For $K \in \mathbb{R}$ and $1 \leq N < +\infty$, we say that the triple $(X, \mathcal{A}, L)$ satisfies Bakry-Émery's curvature-dimension condition $\mathrm{BE}(K, N)$ if

$$\Gamma_2(f) \geq K\Gamma(f) + \frac{(Lf)^2}{N} \qquad \forall f \in \mathcal{A}.$$

For $N = \infty$, the requirement is $\Gamma_2(f) \geq K\Gamma(f)$ for all $f \in \mathcal{A}$.

Let us give a simple class of $\mathrm{BE}(K,N)$ spaces. Recall Bochner's formula

$$\frac{1}{2}\Delta|\nabla f|^2 = \mathrm{Ric}(\nabla f,\nabla f) + |\mathrm{Hess} f|_{HS}^2 + \langle \nabla f, \nabla \Delta f\rangle, \tag{2.3.10}$$

holding for any smooth and compactly supported function f defined over a smooth complete Riemannian manifold (M^n,g). Assume $\mathrm{Ric}_g \geq (n-1)Kg$, and note that $|\mathrm{Hess} f|_{HS}^2 \geq (\Delta f)^2/n$ as a consequence of the inequality of Cauchy-Schwartz inequality. Considering the Laplace-Beltrami operator $L = \Delta$, one can easily check that $\Gamma_2(f) = \frac{1}{2}\Delta|\nabla f|^2 - \langle \nabla f, \nabla \Delta f\rangle$ for any $f \in C^\infty(M)$, from what follows thanks to (2.3.10) that $(M, C^\infty(M), \Delta)$ satisfies the $\mathrm{BE}((n-1)K,n)$ condition.

Moreover, it was immediately realized [AGS14b, Th. 6.2] that infinitesimal Hilbertianity provides the following gradient estimate for the heat flow of any $\mathrm{RCD}^*(K,\infty)$ space $(X,\mathrm{d},\mathfrak{m})$: for any $f \in H^{1,2}(X,\mathrm{d},\mathfrak{m})$,

$$|\nabla(P_t f)|_*^2 \leq e^{-2Kt}P_t(|\nabla f|_*^2) \quad \mathfrak{m}\text{-a.e. on } X, \quad \forall t>0. \tag{2.3.11}$$

It can be shown that this estimate implies the weak Bochner inequality

$$\frac{1}{2}\int_X (\Delta\varphi)|\nabla f|_*^2 \,\mathrm{d}\mathfrak{m} - \int_X \varphi\langle \nabla\Delta f, \nabla f\rangle \,\mathrm{d}\mathfrak{m} \geq K\int_X \varphi|\nabla f|_*^2\,\mathrm{d}\mathfrak{m} \tag{2.3.12}$$
$$\forall\,\varphi \in \mathcal{D}(\Delta)\cap L^\infty(X,\mathfrak{m}) \text{ nonnegative with } \Delta\varphi \in L^\infty(X,\mathfrak{m}),$$

holding for any $f \in \mathcal{D}(\Delta)$ with $\Delta f \in H^{1,2}(X,\mathrm{d},\mathfrak{m})$, see [AGS14b, Rk. 6.3]. Building on this, G. Savaré proved that the class of so-called test functions $\mathrm{TestF}(X,\mathrm{d},\mathfrak{m}) := \{f \in \mathrm{Lip}_b(X,\mathrm{d}) \cap H^{1,2}(X,\mathrm{d},\mathfrak{m}) : \Delta f \in H^{1,2}(X,\mathrm{d},\mathfrak{m})\}$ is an algebra [S14]. Therefore, the inequality (2.3.12) can be understood as a weak $\mathrm{BE}(K,\infty)$ condition, which can be written in the more enlightning way

$$\Gamma_2(f;\cdot) \geq K\Gamma(f;\cdot) \qquad \forall f \in \mathrm{TestF}(X,\mathrm{d},\mathfrak{m}), \tag{2.3.13}$$

by setting $\Gamma(f,\varphi) = \int_X \varphi|\nabla f|_*^2 \,\mathrm{d}\mathfrak{m}$ and $\Gamma_2(f,\varphi) = \frac{1}{2}\int_X(\Delta\varphi)|\nabla f|_*^2\,\mathrm{d}\mathfrak{m} - \int_X \varphi\langle\nabla\Delta f,\nabla f\rangle\,\mathrm{d}\mathfrak{m}$ for any $\varphi \in \mathcal{D}(\Delta)\cap L^\infty(X,\mathfrak{m})$ nonnegative with $\Delta\varphi \in L^\infty(X,\mathfrak{m})$ and $\Gamma_2(f,\cdot) \geq K\Gamma(f,\cdot)$ if and only if $\Gamma_2(f,\varphi) \geq K\Gamma(f,\varphi)$ for any φ as above. In other words, $\mathrm{RCD}^*(K,\infty) \Rightarrow$ weak $\mathrm{BE}(K,\infty)$.

Note that on Riemannian manifolds, (2.3.13) and therefore (2.3.11) implies the bound $\mathrm{Ric} \geq K$, so we immediately get equivalence between $\mathrm{RCD}^*(K,\infty)$ and $\mathrm{BE}(K,\infty)$ in this context.

In general, the converse property weak $\mathrm{BE}(K,\infty) \Rightarrow \mathrm{RCD}^*(K,\infty)$ was established in an appropriate way by L. Ambrosio, N. Gigli and G. Savaré [AGS15, Th. 4.17].

The finite dimensional picture was studied by M. Erbar, K. Kuwada and K.-T. Sturm in [EKS15]. Armed with a dimensional modification of $\mathrm{Ent}_\mathfrak{m}$, namely $U_N := \exp(-N^{-1}\mathrm{Ent}_\mathfrak{m})$, thanks to which they reformulated the $\mathrm{CD}(K,N)$ condition into the suitable so-called entropic curvature-dimension condition $\mathrm{CD}^e(K,N)$, and introducing a new notion of EVI gradient flow, which they called $\mathrm{EVI}_{K,N}$, taking into account both the curvature and the dimension, they proved the following equivalence result.

Theorem 2.3.9. *A geodesic Polish metric measure space* $(X,\mathrm{d},\mathfrak{m})$ *is* $\mathrm{RCD}^*(K,N)$ *if and only if it is infinitesimally Hilbertian with no more than exponential volume growth (meaning that* (2.1.3) *holds), satisfies the weak* $\mathrm{BE}(K,N)$ *condition:*

$$\frac{1}{2}\int \Gamma(f)\Delta\varphi \mathrm{d}\mathfrak{m} \geq \int \varphi(\Gamma(f,\Delta f) + \frac{1}{N}(\Delta f)^2 + K\Gamma(f))\mathrm{d}\mathfrak{m} \tag{2.3.14}$$
$$\forall\,\varphi \in \mathcal{D}(\Delta)\cap L^\infty(X,\mathfrak{m}) \textit{ nonnegative with } \Delta\varphi \in L^\infty(X,\mathfrak{m})$$

and the so-called Sobolev-to-Lipschitz property, stating that any $f \in H^{1,2}(X, \mathrm{d}, \mathfrak{m})$ *with* $|\nabla f|_* \leq 1$ $\mathfrak{m}$*-a.e. on* X *admits a Lipschitz representative* $\tilde{f}$ *with Lipschitz constant smaller or equal than* 1*.*

Therefore, several consequences of the Bakry-Émery condition weak $\mathrm{BE}(K, N)$ are available on $\mathrm{RCD}^*(K, N)$ spaces.

Note that for the particular case of weighted Riemannian manifolds, the last statement takes a simpler form.

Theorem 2.3.10. *Let* $(M, \mathrm{d}, e^{-f}\mathcal{H}^n)$ *be a smooth weighted Riemannian manifold. Then the Bakry-Emery condition* $\mathrm{BE}(K, N)$ *(or equivalently the weak* $\mathrm{BE}(K, N)$ *condition) reads as the following Bochner's tensorial inequality*

$$\mathrm{Ric} + \mathrm{Hess}_f - \frac{\mathrm{d}f \otimes \mathrm{d}f}{N - n} \geq Kg \tag{2.3.15}$$

and is equivalent to the $\mathrm{RCD}^*(K, N)$ *condition.*

Structure of $\mathrm{RCD}^*(K, N)$ spaces

Spaces with Riemannian curvature-dimension bounds enjoy strong structural properties, strenghtening the relevance of the RCD conditions as synthetic notion of Ricci curvature bounded below. Let us first recall that T. Colding and A. Naber proved in [CN12] that Ricci limit spaces have constant dimension, up to a negligible set. Their technique was carried out on $\mathrm{RCD}^*(K, N)$ spaces by A. Mondino and A. Naber who established that any $\mathrm{RCD}^*(K, N)$ space could be partitioned in Borel sets, each bi-Lipschitz equivalent to a Borel subset of an Euclidean space, with possibly varying dimension [MN14, Th. 1.1]. To state their result with better accuracy, let us introduce some preliminary notions.

Definition 2.3.11 (Rectifiable sets)**.** Let (E, d) be a metric space and $k \geq 1$ be an integer.

(1) We say that $S \subset E$ is countably k-rectifiable if there exist at most countably many bounded sets $B_i \subset \mathbb{R}^k$ and Lipschitz maps $f_i : B_i \to E$ such that $S \subset \cup_i f_i(B_i)$.

(2) For a nonnegative Borel measure μ in E (not necessarily σ-finite), we say that S is (μ, k)-rectifiable if there exists a countably k-rectifiable set $S' \subset S$ such that $\mu^*(S \setminus S') = 0$, i.e. $S \setminus S'$ is contained in a μ-negligible Borel set.

Definition 2.3.12 (Tangent metric measure spaces)**.** Let $(X, \mathrm{d}, \mathfrak{m})$ be a Polish metric measure space. For any $x \in X$, we denote by $\mathrm{Tan}(X, \mathrm{d}, \mathfrak{m}, x)$ the set of tangents to $(X, \mathrm{d}, \mathfrak{m})$ at x, that is to say the collection of all pointed metric measure spaces $(Y, \mathrm{d}_Y, \mathfrak{m}_Y, y)$ such that

$$\left(X, \frac{1}{r_i}\mathrm{d}, \frac{\mathfrak{m}}{\mathfrak{m}(B_{r_i}(x))}, x\right) \overset{mGH}{\to} (Y, \mathrm{d}_Y, \mathfrak{m}_Y, y)$$

for some infinitesimal sequence $(r_i) \subset (0, \infty)$.

Definition 2.3.13. For any $k \geq 1$, the k-dimensional regular set $\mathcal{R}_k$ of a $\mathrm{RCD}^*(K, N)$ space $(X, \mathrm{d}, \mathfrak{m})$ is by definition the set of points $x \in X$ such that

$$\mathrm{Tan}(X, \mathrm{d}, \mathfrak{m}, x) = \left\{ (\mathbb{R}^k, \mathrm{d}_{\mathbb{R}^k}, \frac{\mathcal{L}^k}{\omega_k}, 0) \right\},$$

where ω_k is the k-dimensional volume of the unit ball in $\mathbb{R}^k$.

We are now in a position to state Mondino-Naber's result.

Proposition 2.3.14. *Let* $(X, \mathrm{d}, \mathfrak{m})$ *be a* $\mathrm{RCD}^*(K, N)$ *space. Then*

$$\mathfrak{m}(X \setminus \bigcup_{k=1}^{[N]} \mathcal{R}_k) = 0,$$

where $[N]$ *is the integer part of* N*.*

Having in mind the case of Ricci limit spaces, it was conjectured that there is a unique k between 1 and $[N]$ such that $\mathfrak{m}(\mathcal{R}_k) > 0$. This conjecture was proved true in the recent work of E. Brué and D. Semola [BS18], building upon the careful analysis made by several independent groups of researchers [DePhMR16, GP16, KM17] on the relationship between the reference measure $\mathfrak{m}$ restricted to each $\mathcal{R}_k$ and the corresponding Hausdorff measure $\mathscr{H}^k$.

Theorem 2.3.15 (Constant dimension of $\mathrm{RCD}^*(K, N)$ spaces)**.** *For any* $\mathrm{RCD}^*(K, N)$ *space* $(X, \mathrm{d}, \mathfrak{m})$*, there exists a unique integer* n*, also denoted by* $\dim_{\mathrm{d},\mathfrak{m}}(X)$*, between* 1 *and* $[N]$ *such that* $\mathfrak{m}(\mathcal{R}_n) > 0$*; in particular* $\mathfrak{m}(X \setminus \mathcal{R}_n) = 0$*. Moreover,* $\mathfrak{m} \llcorner \mathcal{R}_n \ll \mathscr{H}^n$*.*

The converse absolute continuity has been studied in [AHT18] in which the next theorem was proved. To some extent, it is a generalization of [CC00b, Theorem 4.6] to the RCD setting. Note that we slightly rewrite the original statement of [AHT18] to take Brué-Semola's theorem into account.

Theorem 2.3.16 (Weak Ahlfors regularity)**.** *Let* $(X, \mathrm{d}, \mathfrak{m})$ *be a* $\mathrm{RCD}^*(K, N)$ *space with* $K \in \mathbb{R}$*,* $N \in (1, +\infty)$ *and set* $n = \dim_{\mathrm{d},\mathfrak{m}}(X)$*. Define*

$$\mathcal{R}_n^* := \left\{ x \in \mathcal{R}_n : \ \exists \lim_{r \to 0^+} \frac{\mathfrak{m}(B_r(x))}{\omega_n r^n} \in (0, +\infty) \right\}. \tag{2.3.16}$$

Then $\mathfrak{m}(\mathcal{R}_n \setminus \mathcal{R}_n^*) = 0$*,* $\mathfrak{m} \llcorner \mathcal{R}_n^*$ *and* $\mathcal{H}^n \llcorner \mathcal{R}_n^*$ *are mutually absolutely continuous and*

$$\lim_{r \to 0^+} \frac{\mathfrak{m}(B_r(x))}{\omega_n r^n} = \frac{\mathrm{d}\mathfrak{m} \llcorner \mathcal{R}_n^*}{\mathrm{d}\mathcal{H}^n \llcorner \mathcal{R}_n^*}(x) \qquad \text{for } \mathfrak{m}\text{-a.e. } x \in \mathcal{R}_n^*. \tag{2.3.17}$$

Moreover, one has

$$\lim_{r \to 0^+} \frac{\omega_n r^n}{\mathfrak{m}(B_r(x))} = \chi_{\mathcal{R}_n^*}(x) \frac{\mathrm{d}\mathcal{H}^n \llcorner \mathcal{R}_n^*}{\mathrm{d}\mathfrak{m} \llcorner \mathcal{R}_n^*}(x) \qquad \text{for } \mathfrak{m}\text{-a.e. } x \in X. \tag{2.3.18}$$

Proof. Let S_n be a countably n-rectifiable subset of $\mathcal{R}_n$ with $\mathfrak{m}(\mathcal{R}_n \setminus S_n) = 0$. From (4.0.4) we obtain that the set $\mathcal{R}_n^* \setminus S_n$ is $\mathcal{H}^n$-negligible, hence $\mathcal{R}_n^*$ is $(\mathcal{H}^n, n)$-rectifiable. We denote $\mathfrak{m}_n = \mathfrak{m} \llcorner \mathcal{R}_n$ and recall that $\mathfrak{m}_n \ll \mathcal{H}^n$ thanks to Proposition 2.3.15. We denote by $f : X \to [0, +\infty)$ a Borel function such that $\mathfrak{m}_n = f \mathcal{H}^n \llcorner \mathcal{R}_n^*$ (whose existence is ensured by the Radon-Nikodym theorem, being $\mathcal{R}_n^*$ σ-finite w.r.t. $\mathcal{H}^n$) and recall that (4.0.5) gives

$$\exists \lim_{r \to 0} \frac{\mathfrak{m}_n(B_r(x))}{\omega_k r^n} = f(x) \qquad \text{for } \mathcal{H}^n\text{-a.e. } x \in \mathcal{R}_n^*. \tag{2.3.19}$$

Now, in (2.3.19) we can replace $\mathfrak{m}_n$ by $\mathfrak{m}$ for $\mathcal{H}^n$-a.e. $x \in \mathcal{R}_n^*$; this is a direct consequence of (4.0.3) with $\mu = \mathfrak{m} - \mathfrak{m}_n$ and $S = \mathcal{R}_n^*$.

Calling then N_n the $\mathcal{H}^n$-negligible (and then $\mathfrak{m}_n$-negligible) subset of $\mathcal{R}_n^*$ where the equality

$$\lim_{r \to 0} \frac{\mathfrak{m}(B_r(x))}{\omega_n r^n} = f(x)$$

fails, we obtain existence and finiteness of the limit on $\mathcal{R}_n^* \setminus N_n$; since f is a density, it is also obvious that the limit is positive $\mathfrak{m}_n$-a.e., and that $\mathcal{H}^n \llcorner \mathcal{R}_n^* \cap \{f > 0\}$ is absolutely continuous w.r.t. $\mathfrak{m}_n$.

This proves that $\mathfrak{m}(\mathcal{R}_n \setminus \mathcal{R}_n^*) = 0$ and that $\mathfrak{m} \llcorner \mathcal{R}_n^*$ and $\mathcal{H}^n \llcorner \mathcal{R}_n^*$ are mutually absolutely continuous. The last statement (2.3.18) is straigthforward. □

2.4 Convergence of metric measure spaces and stability results

In this section, we provide the stability results related to the curvature-dimension conditions $\mathrm{CD}\,/\,\mathrm{RCD}(K,N)$ for $K\in\mathbb{R}$ and $N\in[1,+\infty]$.

Measured Gromov-Hausdorff convergence

There are several ways to express convergence of metric measure spaces, the most common one being probably the measured Gromov-Hausdorff convergence. Let us first recall the definition of Gromov-Hausdorff distance introduced by M. Gromov in [Gro81].

Definition 2.4.1. The Gromov-Hausdorff distance between two compact metric spaces (X,d_X) and (Y,d_Y) is

$$\mathrm{d}_{GH}((X,\mathrm{d}_X),(Y,\mathrm{d}_Y)) := \inf\{\mathrm{d}_{H,Z}(i(X),j(Y)) \,:\, ((Z,\mathrm{d}_Z),i,j)\}$$

where the infimum is taken over all the triples $((Z,\mathrm{d}_Z),i,j)$ where (Z,d_Z) is a complete metric space and $i:X\to Z$, $j:Y\to Z$ are isometric embeddings, and $\mathrm{d}_{H,Z}$ stands for the Hausdorff distance in Z.

The Gromov-Hausdorff distance is invariant by replacing the metric spaces under consideration by isometric copies; in particular, we can replace all the spaces by their completion. Therefore, without loss of generality, we will always work with complete spaces.

Gromov-Hausdorff convergence is convergence with respect to d_{GH}, usually denoted with $\xrightarrow{GH}$. It can be reformulated in terms of functions called ε-isometries: $(X_n,\mathrm{d}_n)\xrightarrow{GH}(X,\mathrm{d})$ if and only if there exists a sequence $\varepsilon_n\downarrow 0$ and functions $\varphi_n:X_n\to X$ such that the ε_n-neighborhood of $\varphi_n(X_n)$ coincides with X, and $|\mathrm{d}_X(\varphi_n(x),\varphi_n(x'))-\mathrm{d}_{X_n}(x,x')|\le\varepsilon_n$ for all $x,x'\in X_n$. Recall that for any $\varepsilon>0$, the ε-neighborhood of a subset $Y\subset X$ is by definition $\bigcup_{x\in Y}B_\varepsilon(x)$.

In the context of non-compact metric spaces, this notion adapts by distinguishing reference points on the spaces and requiring Gromov-Hausdorff convergence to hold on balls centered at these reference points: $(X_n,\mathrm{d}_n,x_n)\xrightarrow{GH}(X,\mathrm{d},x)$ whenever $B_r(x_n)\subset X_n$ Gromov-Hausdorff converges to $B_r(x)\subset X$ for any $r>0$. We usually talk about pointed Gromov-Hausdorff convergence, and a triple (X,d,x) is called a pointed metric space. Here again ε-isometries provides a user-friendly characterization: $(X_n,\mathrm{d}_n,x_n)\xrightarrow{GH}(X,\mathrm{d},x)$ if and only if there exists two sequences $\varepsilon_n\downarrow 0$ and $r_n\uparrow+\infty$ and functions $\varphi_n:B_{r_n}(x_n)\to X$ such that for any n, $\varphi_n(x_n)=\varphi(x)$, the ε_n-neighborhood of $\varphi_n(B_{r_n}(x_n))$ contains $B_{r_n-\varepsilon_n}(x_n)$ and $|\mathrm{d}_X(\varphi_n(x),\varphi_n(x'))-\mathrm{d}_{X_n}(x,x')|\le\varepsilon_n$ for all $x,x'\in X_n$.

To deal with compact metric *measure* spaces, K. Fukaya introduced in [F87] the measured Gromov-Hausdorff convergence $(X_n,\mathrm{d}_n,\mathfrak{m}_n)\xrightarrow{mGH}(X,\mathrm{d},\mathfrak{m})$, which is by definition $(X_n,\mathrm{d}_n,x_n)\xrightarrow{GH}(X,\mathrm{d},x)$ with the further condition $(\varphi_n)_\#\mathfrak{m}_n\overset{C_{bs}(X)}{\rightharpoonup}\mathfrak{m}$, where $C_{bs}(X)$ denotes the set of continuous functions $f:X\to\mathbb{R}$ with bounded support. Such a notion extends in a natural way to pointed metric measure spaces: $(X_n,\mathrm{d}_n,\mathfrak{m}_n,x_n)\xrightarrow{mGH}(X,\mathrm{d},\mathfrak{m},x)$ whenever $(B_r(x_n),(\mathrm{d}_n)_{|B_r(x_n)\times B_r(x_n)},\mathfrak{m}_n\llcorner B_r(x_n))$ converges in the measure Gromov-Hausdorff sense to $(B_r(x),\mathrm{d}_{|B_r(x)\times B_r(x)},\mathfrak{m}\llcorner B_r(x))$ for any $r>0$.

Here is Gromov's well-known precompactness theorem [Gro07, Th. 5.3] and two refinements.

Theorem 2.4.2 (Gromov's precompactness theorem). *1. Let* CMS *be the set of all compact metric spaces. Then for any $n \in \mathbb{N}^*$, $K \in \mathbb{R}$ and $0 < D < +\infty$, the set $\mathcal{M}(n, K, D)$ of compact n-dimensional Riemannian manifolds (M, d_g) with Ricci curvature bounded below by K and diameter bounded above by D is a precompact subset of $(\mathrm{CMS}, \mathrm{d}_{GH})$.*

2. *For any $n \in \mathbb{N}^*$ and $K \in \mathbb{R}$, any sequence of pointed complete n-dimensional Riemannian manifolds (M, d_g, x_n) with Ricci curvature uniformly bounded below by K admits a subsequence GH converging to some pointed metric space (X, d, x).*

3. *For any $n \in \mathbb{N}^*$ and $K \in \mathbb{R}$, any sequence of pointed complete n-dimensional Riemannian manifolds $(M, \mathrm{d}_g, \mathrm{vol}_g, x_n)$ with Ricci curvature uniformly bounded below by K admits a subsequence mGH converging to some pointed metric measure space $(X, \mathrm{d}, \mathfrak{m}, x)$.*

Stability of Lott-Villani's $\mathrm{CD}(K, \infty)$ condition under mGH convergence of compact spaces

The next result justifies that compact Ricci limit spaces are Lott-Villani's $\mathrm{CD}(K, \infty)$ spaces for an appropriate $K \in \mathbb{R}$.

Theorem 2.4.3 (Stability of Lott-Villani $\mathrm{CD}(K, \infty)$ condition for compact spaces). *Let $K \in \mathbb{R}$ and $\{(X_i, \mathrm{d}_i, \mathfrak{m}_i)\}_i$ be a sequence of compact Lott-Villani $\mathrm{CD}(K, \infty)$ spaces such that $(X_i, \mathrm{d}_i, \mathfrak{m}_i) \xrightarrow{mGH} (X, \mathrm{d}, \mathfrak{m})$ for some compact metric measure space $(X, \mathrm{d}, \mathfrak{m})$. Then $(X, \mathrm{d}, \mathfrak{m})$ is a Lott-Villani's $\mathrm{CD}(K, \infty)$ space.*

Theorem 2.4.3 is a direct corollary of the following proposition. Recall that a metric space (X, d) is called a length space if for any $x, y \in X$,

$$\mathrm{d}(x, y) = \inf \left\{ \int_0^1 |\gamma'|(t) \, \mathrm{d}t \ : \ \gamma \in AC([0, 1], X) \text{ s.t. } \gamma(0) = x \text{ and } \gamma(1) = y \right\},$$

and that geodesic metric spaces are particular examples of length spaces.

Proposition 2.4.4. *Let $\{(X_i, \mathrm{d}_i, \mathfrak{m}_i)\}_i$ be a sequence of compact metric measure length spaces. Assume that $(X_i, \mathrm{d}_i, \mathfrak{m}_i) \xrightarrow{mGH} (X, \mathrm{d}, \mathfrak{m})$ for some compact measured length space $(X, \mathrm{d}, \mathfrak{m})$. Let us assume that for some convex continuous function $A : [0, +\infty) \to [0, +\infty)$ with $A(0) = 0$, for any $i \in \mathbb{N}$ the associated functional $F_i : \mathcal{P}(X_i) \ni \mu \mapsto \int_{X_i} A(\mu) \, \mathrm{d}\mathfrak{m}_i$ is weakly K-displacement convex on $(X_i, \mathrm{d}_i, \mathfrak{m}_i)$. Then $F : \mathcal{P}(X) \ni \mu \mapsto \int_X A(\mu) \, \mathrm{d}\mathfrak{m}$ is weakly K-displacement convex on $(X, \mathrm{d}, \mathfrak{m})$.*

We will provide a proof of Proposition 2.4.4. To this purpose, we need three lemmas and a theorem. We shall give a proof only for the theorem and provide suitable references for the lemmas. The first lemma states that it is enough to consider probability measures with continuous densities to check the validity of weak K-displacement convexity of an internal-energy functional. See [OV00, Lem. 3.24].

Lemma 2.4.5. *Let $(X, \mathrm{d}, \mathfrak{m})$ be a compact length space. Let $A : [0, +\infty) \to \mathbb{R}$ be a convex continuous function with $A(0) = 0$ and $F : \mathcal{P}^a(X, \mathfrak{m}) \to \mathbb{R} \cup \{+\infty\}$ be the associated internal energy functional. Assume that for all $\mu_0, \mu_1 \in \mathcal{P}^a(X, \mathfrak{m})$ with continuous densities, there exists at least one geodesic $(\mu_t)_{t \in [0,1]}$ joining μ_0 to μ_1 such that*

$$F(\mu_t) \leq (1 - t) F(\mu_0) + t F(\mu_1) - K \frac{t(1 - t)}{2} W_2^2(\mu_0, \mu_1) \qquad \forall t \in [0, 1].$$

Then F is weakly K-displacement convex.

The second lemma is a kind of Ascoli-Arzelà theorem for functions defined over a convergent sequence of compact metric spaces. See [Gro81] for details. Note that any ε-approximation φ between two compact metric spaces (X, d_X) and (Y, d_Y) has a so-called approximate inverse $\varphi' : Y \to X$ defined as follows: for any $y \in Y$, choose $x \in X$ so that $\mathrm{d}_Y(\varphi(x), y) \leq \varepsilon$ and put $\varphi'(x) = y$. It is easily seen that φ' is then a 3ε-approximation between Y and X.

Lemma 2.4.6. *Let* $(X_i, \mathrm{d}_{X_i}) \xrightarrow{GH} (X, \mathrm{d}_X)$ *and* $(Y_i, \mathrm{d}_{Y_i}) \xrightarrow{GH} (Y, \mathrm{d}_Y)$ *be two convergent sequences of compact metric spaces. Let* $\{\alpha_i : X_i \to Y_i\}_i$ *be an asymptotically equicontinuous family of maps, meaning that for any* $\varepsilon > 0$, *there exists* $\delta_\varepsilon > 0$ *and* $i_\varepsilon \in \mathbb{N}$ *such that for all* $i \geq i_\varepsilon$,

$$\mathrm{d}_{X_i}(x, x') \leq \delta_\varepsilon \quad \Rightarrow \mathrm{d}_{Y_i}(\alpha_i(x), \alpha_i(x')) \leq \varepsilon$$

for all $x, x \in X_i$. *Let* $\varphi_i : X_i \to X$ *and* $\psi_i : Y_i \to Y$ *be* ε_i*-approximations for some infinitesimal sequence* $(\varepsilon_i)_i \subset (0, +\infty)$, *and* $\varphi_i' : X \to X_i$ *be an approximate inverse of* φ_i *for any* i.

$$\begin{array}{ccc} X_i & \xrightarrow{\varphi_i} & X \\ {\scriptstyle \alpha_i}\downarrow & & \downarrow{\scriptstyle \psi_i \circ \alpha_i \circ \varphi_i'} \\ Y_i & \xrightarrow[\psi_i]{} & Y \end{array}$$

Then after passing to a subsequence, the maps $\psi_i \circ \alpha_i \circ \varphi_i'$ *converge uniformly to some continuous map* $\alpha : X \to Y$.

Note that the above maps $\psi_i \circ \alpha_i \circ \varphi_i'$ may not be continuous.

The third lemma states that Wasserstein spaces are stable under measured Gromov-Hausdorff convergence. We refer to [OV00, Cor. 4.3] for a proof.

Lemma 2.4.7. *Let* $(X_i, \mathrm{d}_i) \xrightarrow{GH} (X, \mathrm{d})$ *be a convergent sequence of compact metric spaces. Then* $(\mathcal{P}_2(X_i), W_2) \xrightarrow{GH} (\mathcal{P}_2(X), W_2)$. *More specifically, if* $\varphi_i : X_i \to X$ *are* ε_i*-isometries for some infinitesimal sequence* $(\varepsilon_i)_i \subset (0, +\infty)$, *then* $(\varphi_i)_\# : \mathcal{P}_2(X_i) \to \mathcal{P}_2(X)$ *are* ε_i*-isometries.*

We will finally need a theorem whose proof is given for completeness.

Theorem 2.4.8. *Let* (X, τ) *be a compact Hausdorff topological space. Let* $A : [0, +\infty) \to \mathbb{R}$ *be a continuous and convex density with* $A(0) = 0$ *and* $A(r)/r \to +\infty$ *when* $r \to +\infty$, *and for any* $\nu \in \mathcal{P}(X)$, *let*

$$F_\nu(\mu) = \begin{cases} \int_X A\left(\frac{\mathrm{d}\mu}{\mathrm{d}\nu}\right) \mathrm{d}\nu & \text{if } \mu \ll \nu, \\ +\infty & \text{otherwise,} \end{cases}$$

be the associated internal energy functional. Then giving $\mathcal{P}(X)$ *the weak* topology, the function*

$$\begin{array}{ccc} \mathcal{P}(X) \times \mathcal{P}(X) & \to & (-\infty, +\infty] \\ (\mu, \nu) & \mapsto & F_\nu(\mu) \end{array}$$

is lower semicontinuous. Moreover, if (Y, d_Y) *is a compact Hausdorff metric space, if* $f : X \to Y$ *is a Borel map, then for any* $\mu, \nu \in \mathcal{P}(X)$, *we have* $F_{f_\#\nu}(f_\#\mu) \leq F_\nu(\mu)$.

Proof. Let us recall that the topological dual $C(X)^*$ of $C(X)$ equipped with the weak* topology is the space of linear and continuous functionals defined on $C(X)$, and that any finite Borel measure μ defines an element $\varphi_\mu \in C(X)^*$ by

$$\varphi_\mu(f) = \int_X f \, \mathrm{d}\mu \qquad \forall f \in C(X).$$

For any $L \in C(X)$, we denote by $L^{**} \in C(X)^{**}$ its bidual element. The proof of the Theorem is based on the representation formula [LV09, Th. B.2] which can be written

$$F_\nu(\mu) = \sup_{(L_1,L_2)\in\mathcal{L}} \{L_1^{**}(\varphi_\mu) + L_2^{**}(\varphi_\nu)\} \qquad \forall(\mu,\nu) \in \mathcal{P}(X) \times \mathcal{P}(X),$$

where $\mathcal{L}$ is a subset of $C(X) \times C(X)$. For any $(L_1, L_2) \in \mathcal{L}$, the function $(\mu,\nu) \mapsto L_1^{**}(\varphi_\mu) + L_2^{**}(\varphi_\nu)$ is continuous, then a fortiori lower semicontinuous. As the supremum of a family of lower semicontinuous functions is lower semicontinuous, we get the first statement of the theorem. For the second statement, we refer to [AGS08, Lem. 9.4.5]. □

We are now in a position to prove Proposition 2.4.4. Recall that we are given a convergent sequence of compact metric measure length spaces $(X_i, \mathrm{d}_i, \mathfrak{m}_i) \xrightarrow{mGH} (X, \mathrm{d}, \mathfrak{m})$, and that for some convex continuous function $A : [0, +\infty) \to [0, +\infty)$ with $A(0) = 0$, for any $i \in \mathbb{N}$ the associated functional $F_i : \mathcal{P}(X_i) \ni \mu \mapsto \int_{X_i} A(\mu) \, \mathrm{d}\mathfrak{m}_i$ is weakly K-displacement convex on $(X_i, \mathrm{d}_i, \mathfrak{m}_i)$. We want to prove that $F : \mathcal{P}(X) \ni \mu \mapsto \int_X A(\mu) \, \mathrm{d}\mathfrak{m}$ is weakly K-displacement convex on $(X, \mathrm{d}, \mathfrak{m})$.

Proof. Let $\mu_0, \mu_1 \in \mathcal{P}(X)$. Assume that $\mu_0 \ll \mathfrak{m}$ and $\mu_1 \ll \mathfrak{m}$, otherwise there is nothing to prove, and write $\mu_0 = \rho_0\mathfrak{m}$ and $\mu_1 = \rho_1\mathfrak{m}$. Thanks to Lemma 2.4.5, we can assume $\rho_0, \rho_1 \in C(X)$.

Step 1: Construction of good geodesics on $(\mathcal{P}(X_n), W_2)$.

As $(X_i, \mathrm{d}_i, \mathfrak{m}_i) \xrightarrow{GH} (X, \mathrm{d}, \mathfrak{m})$, there exists an infinitesimal sequence $(\varepsilon_i)_i \subset (0, +\infty)$ and ε_i-approximations $\varphi_i : X_i \to X$ such that $(\varphi_i)_\#\mathfrak{m}_i \overset{C(X)}{\rightharpoonup} \mathfrak{m}$, from which we get that for i large enough, $\int_X \rho_0 \, \mathrm{d}[(\varphi_i)_\#\mu_0] > 0$ and $\int_X \rho_1 \, \mathrm{d}[(\varphi_i)_\#\mu_1] > 0$. For such i, define $\mu_{i,0}, \mu_{i,1} \in \mathcal{P}(X)$ by

$$\mu_{i,0} := \frac{\rho_0 \circ \varphi_i}{\int_X \rho_0 \, \mathrm{d}[(\varphi_i)_\#\mu_0]}\mathfrak{m}_i \quad \text{and} \quad \mu_{i,1} := \frac{\rho_1 \circ \varphi_i}{\int_X \rho_1 \, \mathrm{d}[(\varphi_i)_\#\mu_1]}\mathfrak{m}_i.$$

As F_i is displacement convex, there exists a W_2-geodesic $(\mu_{i,t})_{t\in[0,1]} \subset \mathcal{P}(X)$ joining $\mu_{i,0}$ and $\mu_{i,1}$ such that

$$F_i(\mu_{i,t}) \leq (1-t)F_i(\mu_{i,0}) + tF_i(\mu_{i,1}) - K\frac{t(1-t)}{2}W_2^2(\mu_{i,0}, \mu_{i,1}) \tag{2.4.1}$$

for all $t \in [0, 1]$. We claim that

$$W_2(\mu_{i,0}, \mu_{i,1}) \to W_2(\mu_0, \mu_1). \tag{2.4.2}$$

To justify this, let us first prove that $(\varphi_i)_\#\mu_{i,0}$ weakly converges to μ_0. Take $h \in C(X)$. As $\rho_0 \in C(X)$, $\int_X \rho_0 \, \mathrm{d}[(\varphi_i)_\#\mathfrak{m}_i] \to \int_X \rho_0 \, \mathrm{d}\mathfrak{m} = 1$, so

$$\int_X h d[(\varphi_i)_\#\mu_{i,0}] = \int_X h \frac{\rho_0}{\int_X \rho_0 \, \mathrm{d}[(\varphi_i)_\#\mathfrak{m}_i]} \mathrm{d}[(\varphi_i)_\#\mathfrak{m}_i] \to \int_X h\rho_0 \, \mathrm{d}\mathfrak{m} = \int_X h \, \mathrm{d}\mu_0.$$

One can similarly show that $(\varphi_i)_\# \mu_{i,1} \rightharpoonup \mu_1$. Moreover, it follows from Lemma 2.4.7 that for any i,

$$|W_2(\mu_{i,0}, \mu_{i,1}) - W_2((\varphi_i)_\#\mu_{i,0}, (\varphi_i)_\#\mu_{i,1})| \leq \varepsilon_i.$$

The convergence (2.4.2) follows from the triangle inequality:

$$\begin{aligned}|W_2(\mu_{i,0}, \mu_{i,1}) - W_2(\mu_0, \mu_1)| \leq &|W_2(\mu_{i,0}, \mu_{i,1}) - W_2((\varphi_i)_\#\mu_{i,0}, (\varphi_i)_\#\mu_{i,1})| \\ &+ |W_2((\varphi_i)_\#\mu_{i,0}, (\varphi_i)_\#\mu_{i,1}) - W_2(\mu_0, \mu_1)|.\end{aligned}$$

Step 2: Construction of a good geodesic between μ_0 and μ_1 by limiting argument.

Let us apply Lemma 2.4.6 to the case $(X_i, \mathrm{d}_{X_i}) \equiv ([0,1], \mathrm{d}_{eucl})$, $(Y_i, \mathrm{d}_{Y_i}) = (\mathcal{P}(X_i), W_2)$ and $\alpha_i : t \mapsto \mu_{i,t}$, in which case asymptotical equicontinuity follows directly from the equilipschitz property

$$W_2(\alpha_i(t), \alpha_i(t')) = |t - t'| W_2(\mu_{i,0}, \mu_{i,1}) \leq C|t - t'| W_2(\mu_0, \mu_1) \qquad \forall t, t' \in [0,1] \tag{2.4.3}$$

for i large enough and some $C \geq 1$, which is an automatic consequence of the fact that $(\mu_{i,t})_t$ are geodesics and (2.4.2). We get a continuous map $\alpha : [0,1] \mapsto \mathcal{P}(X)$ with uniform convergence $\alpha_i \to \alpha$. Let us write $\mu_t := \alpha(t)$ for any $t \in [0,1]$.

Step 3: Passing to the limit in (2.4.1).

Let us show that $F_i(\mu_{i,0}) \to F(\mu_0)$ (and similarly, $F_i(\mu_{i,1}) \to F(\mu_1)$). As

$$F(\mu_{i,0}) = \int_X A\left(\frac{\rho_0 \circ \varphi_i}{\int_X \rho_0 \,\mathrm{d}[(\varphi_i)_\#\mathfrak{m}_i]}\right) \mathrm{d}\mathfrak{m}_i = \int_X A\left(\frac{\rho_0}{\int_X \rho_0 \,\mathrm{d}[(\varphi_i)_\#\mathfrak{m}_i]}\right) \mathrm{d}[(\varphi_i)_\#\mathfrak{m}_i],$$

we can compute

$$\begin{aligned}|F(\mu_{i,0}) - F(\mu_0)| \leq &\left|\int_X A\left(\frac{\rho_0}{\int_X \rho_0 \,\mathrm{d}[(\varphi_i)_\#\mathfrak{m}_i]}\right) \mathrm{d}[(\varphi_i)_\#\mathfrak{m}_i] - \int_X A(\rho_0)\,\mathrm{d}[(\varphi_i)_\#\mathfrak{m}_i]\right| \\ &+ |\int_X A(\rho_0)\,\mathrm{d}[(\varphi_i)_\#\mathfrak{m}_i] - \int_X A(\rho_0)\,\mathrm{d}\mathfrak{m}| \\ \leq &\underbrace{\|A\left(\left(\frac{\rho_0}{\int_X \rho_0 \,\mathrm{d}[(\varphi_i)_\#\mathfrak{m}_i]}\right)\right) - A(\rho_0)\|_\infty}_{\to 0} \underbrace{\int_X \mathrm{d}[(\varphi_i)_\#\mathfrak{m}_i]}_{=1} \\ &+ \underbrace{|\int_X A(\rho_0)\,\mathrm{d}[(\varphi_i)_\#\mathfrak{m}_i] - \int_X A(\rho_0)\,\mathrm{d}\mathfrak{m}|}_{\to 0 \text{ since } (\varphi_i)_\#\mathfrak{m}_i \rightharpoonup \mathfrak{m}}.\end{aligned}$$

This, combined with (2.4.2), implies that the right-hand side in (2.4.1) converges to $(1-t)F(\mu_0) + tF(\mu_1) - K\frac{t(1-t)}{2}W_2^2(\mu_0, \mu_1)$ when $i \to +\infty$. To conclude the proof, apply Theorem 2.4.8 to get

$$F(\mu_t) \leq \liminf_{i \to +\infty} F_i((\varphi_i)_\#\mu_{i,t}) \leq \liminf_{i \to +\infty} F_i(\mu_{i,t})$$

for all $t \in [0,1]$.

□

Stability of Lott-Villani's CD(K,N), $N < +\infty$, condition under pointed mGH convergence

Lott-Villani's finite dimensional conditions CD(K,N), $N < +\infty$, are also stable under measured Gromov Hausdorff convergence. This statement holds for locally compact complete (possibly non-compact) Polish metric measure spaces. As the proof is long and involved, we omit it, and refer to [Vi09, Th. 29.25].

Theorem 2.4.9 (Stability of Lott-Villani's $\mathrm{CD}(K,N)$ conditions)**.** *Let $\{(X_i,\mathrm{d}_i,\mathfrak{m}_i,x_i)\}_i$ and $(X,\mathrm{d},\mathfrak{m},x)$ be locally compact complete separable metric measure spaces with σ-finite reference measure. Assume that the spaces $(X_i,\mathrm{d}_i,\mathfrak{m}_i,x_i)$ are all Lott-Villani's $\mathrm{CD}(K,N)$ for some $K\in\mathbb{R}$ and $N\in[1,+\infty)$, and that $(X_i,\mathrm{d}_i,\mathfrak{m}_i,x_i)\xrightarrow{mGH}(X,\mathrm{d},\mathfrak{m},x)$. Then $(X,\mathrm{d},\mathfrak{m},x)$ is Lott-Villani's $\mathrm{CD}(K,N)$.*

Sturm's $\mathbb{D}$-convergence

For completeness, we present now Sturm's $\mathbb{D}$-convergence. Let us recall that $(X,\mathrm{d}_X,\mathfrak{m}_X)$ and $(Y,\mathrm{d}_Y,\mathfrak{m}_Y)$ are isomorph if there exists an isometry $f:\mathrm{supp}(\mathfrak{m}_X)\to\mathrm{supp}(\mathfrak{m}_Y)$ such that $f_\#\mathfrak{m}_X=\mathfrak{m}_Y$; notably, $(X,\mathrm{d}_X,\mathfrak{m}_X)$ and $(\mathrm{supp}(\mathfrak{m}_X),\mathrm{d}_X,\mathfrak{m}_X)$ are isomorph. When considering pointed metric measure spaces $(X,\mathrm{d}_X,\mathfrak{m}_X,x)$ and $(Y,\mathrm{d}_Y,\mathfrak{m}_Y,y)$, we require in addition $f(x)=y$.

K.-T. Sturm proposed in [St06a] an alternative notion of convergence of metric measure spaces, introducing the distance $\mathbb{D}$ which is a kind of extension of the Wasserstein distance to the set $\mathfrak{X}$ of all isomorphism classes of normalized Polish metric measure spaces $(X,\mathrm{d},\mathfrak{m})$ with finite second moment. Here by normalized we mean $\mathfrak{m}(X)=1$. Two normalized Polish metric measure spaces $(X,\mathrm{d}_X,\mathfrak{m}_X)$ and $(Y,\mathrm{d}_Y,\mathfrak{m}_Y)$ being given, a coupling between d_X and d_Y is by definition a pseudo-distance $\tilde{\mathrm{d}}$ on $X\sqcup Y$ such that $\tilde{\mathrm{d}}_{|X}=\mathrm{d}_X$ and $\tilde{\mathrm{d}}_{|Y}=\mathrm{d}_Y$, and a coupling between $\mathfrak{m}_X$ and $\mathfrak{m}_Y$ is a probability measure $\gamma\in\mathcal{P}(X\sqcup Y)$ such that $\gamma(A\sqcup Y)=\mathfrak{m}_X(A)$ and $\gamma(X\sqcup A')=\mathfrak{m}_Y(A')$ for any measurable sets $A\subset X$, $A'\subset Y$. Sturm's distance between two isomorphism classes of metric measure spaces $[X,\mathrm{d}_X,\mathfrak{m}_X],[Y,\mathrm{d}_Y,\mathfrak{m}_Y]\in\mathfrak{X}$ is then defined as

$$\mathbb{D}([X,\mathrm{d}_X,\mathfrak{m}_X],[Y,\mathrm{d}_Y,\mathfrak{m}_Y]):=\inf_{(\tilde{\mathrm{d}},\gamma)}\left\{\int_{X\times Y}\tilde{\mathrm{d}}^2(x,y)\,\mathrm{d}\gamma(x,y)\right\},$$

the infimum being taken over all couplings $\tilde{\mathrm{d}}$ between d_X and d_Y and γ between $\mathfrak{m}_X$ and $\mathfrak{m}_Y$. Such an infimum is always achieved [St06a, Lem. 3.3]. The space $(\mathfrak{X},\mathbb{D})$ is a Polish length space [St06a, Th. 3.6]. Moreover, when restricted to the class $\mathfrak{X}(D,C)$ of doubling normalized Polish metric measure spaces with full support and diameter and doubling constant bounded above by D and C respectively, the distance $\mathbb{D}$ metrizes the measured Gromov-Hausdorff convergence [St06a, Lem. 3.18].

In this context, the following stability property holds. The infinite dimensional case is taken from [St06a, Th. 4.20] and the finite dimensional case from [St06b, Th. 3.1].

Theorem 2.4.10 (Stability of Sturm's $\mathrm{CD}(K,N)$ conditions)**.** *Let $K\in\mathbb{R}$ and $N\in[1,+\infty]$. Let $\{(X_i,\mathrm{d}_i,\mathfrak{m}_i)\}_i$ be a sequence of Sturm's $\mathrm{CD}(K,N)$ normalized metric measure spaces with uniformly bounded diameter. If $[X_i,\mathrm{d}_i,\mathfrak{m}_i]\xrightarrow{\mathbb{D}}[X,\mathrm{d},\mathfrak{m}]$ for some normalized metric measure space $(X,\mathrm{d},\mathfrak{m})$, then $(X,\mathrm{d},\mathfrak{m})$ is Sturm's $\mathrm{CD}(K,N)$.*

Pointed Gromov convergence and extrinsic approach

Lott-Villani's stability results need the spaces to be proper, meaning that any closed ball must be compact. This assumption is automatically satisfied when the spaces are $\mathrm{CD}(K,N)$ with $N<+\infty$, but it might fail to be true on non-compact $\mathrm{CD}(K,\infty)$ spaces. On the other hand, Sturm's approach restricts his stability results to the class of normalized spaces with finite variance. In [GMS15], N. Gigli, A. Mondino and G. Savaré introduced a notion of convergence of pointed metric measure spaces which does not require any compactness assumption on the spaces nor particular restriction on the reference measures to imply stability of Ricci curvature bounds. They call their notion *pointed measured Gromov convergence*, "pmG" for short. It is based on the next theorem, proved by M. Gromov [Gro07] in the case of spaces with finite mass and extended in [GMS15] to spaces with infinite mass, which provides a characterization of the equivalent classes of isomorphic metric measure spaces in terms of suitable test functions. Let us introduce the set TestG made of functions $\varphi : \mathbb{R}^{N^2} \to \mathbb{R}$ for some $N \geq 2$ which are continuous and have bounded support. For any $\varphi \in \mathrm{TestG}$ and any pointed metric measure space $\mathbb{X} = (X, \mathrm{d}, \mathfrak{m}, x)$, set

$$\varphi^*(\mathbb{X}) := \int_{X^N} \varphi(D(x_1, \ldots, x_n))\, \mathrm{d}\delta_x(x_1)\, \mathrm{d}\mathfrak{m}^{\otimes^{N-1}}(x_2, \ldots, x_n)$$

where $D(x_1, \ldots, x_n)$ is the N^2-uple formed by the elements $\{\mathrm{d}(x_i, x_j)\}_{1 \leq i,j \leq N}$.

Theorem 2.4.11 (Gromov's reconstruction theorem)**.** *Two pointed metric measure spaces $\mathbb{X}_1 = (X_1, \mathrm{d}_1, \mathfrak{m}_1, x_1)$ and $\mathbb{X}_2 = (X_2, \mathrm{d}_2, \mathfrak{m}_2, x_2)$ are isomorph if and only if:*

$$\varphi^*(\mathbb{X}_1) = \varphi^*(\mathbb{X}_2) \qquad \forall \varphi \in \mathrm{TestG}.$$

Therefore, the next definition provides a notion of convergence of equivalent classes of pointed metric measure spaces [GMS15, Def. 3.8].

Definition 2.4.12 (pmG convergence)**.** Let $\{\mathbb{X}_i = (X_i, \mathrm{d}_i, \mathfrak{m}_i, x_i)\}_{i \in \mathbb{N}}, \mathbb{X} = (X, \mathrm{d}, \mathfrak{m}, x)$ be pointed metric measure spaces. We say that $\mathbb{X}_i$ converge in the pointed measured Gromov sense to $\mathbb{X}$, and we write $(X_i, \mathrm{d}_i, \mathfrak{m}_i, x_i) \xrightarrow{pmG} (X, \mathrm{d}, \mathfrak{m}, x)$, if for any $\varphi \in \mathrm{TestG}$,

$$\lim_{i \to +\infty} \varphi^*(\mathbb{X}_i) = \varphi^*(\mathbb{X}).$$

N. Gigli, A. Mondino and G. Savaré proved that this notion of convergence is equivalent to the classical so-called extrinsic approach of convergence.

Definition 2.4.13 (Extrinsic approach)**.** Let $\{(X_i, \mathrm{d}_i, \mathfrak{m}_i, x_i)\}_{i \in \mathbb{N}}, (X, \mathrm{d}, \mathfrak{m}, x)$ be pointed metric measure spaces. We say that $(X_i, \mathrm{d}_i, \mathfrak{m}_i, x_i)$ converge to $(X, \mathrm{d}, \mathfrak{m}, x)$ in the extrinsic sense if there exists a complete and separable metric space (Y, d_Y) and isometric embeddings $\varphi_i : X_i \to Y$ and $\varphi : X \to Y$ such that $\mathrm{d}_Y(\varphi_i(x_i), \varphi(x)) \to 0$ and $(\varphi_i)_\# \mathfrak{m}_i \overset{C_{\mathrm{bs}}(Y)}{\rightharpoonup} \varphi_\# \mathfrak{m}$ when $i \to \infty$.

Proposition 2.4.14 (Equivalence pmG/extrinsic approach)**.** *Let $\{(X_i, \mathrm{d}_i, \mathfrak{m}_i, x_i)\}_{i \in \mathbb{N}}$ and $(X, \mathrm{d}, \mathfrak{m}, x)$ be pointed metric measure spaces. Then $(X_i, \mathrm{d}_i, \mathfrak{m}_i, x_i) \xrightarrow{pmG} (X, \mathrm{d}, \mathfrak{m}, x)$ if and only if $(X_i, \mathrm{d}_i, \mathfrak{m}_i, x_i) \to (X, \mathrm{d}, \mathfrak{m}, x)$ in the extrinsic sense.*

Note that [GMS15, Th. 3.15] states also the equivalence of pointed measured Gromov convergence with two other notions of convergence, one being a variant of Sturm's $\mathbb{D}$ convergence.

With this notion in hand, N. Gigli, A. Mondino and G. Savaré proved stability of Sturm's $\mathrm{CD}(K, \infty)$ condition [GMS15, Th. 4.9].

Theorem 2.4.15. *Let $K \in \mathbb{R}$. Let $\{(X_i, \mathrm{d}_i, \mathfrak{m}_i, x_i)\}_{i \in \mathbb{N}}$ be a sequence of Sturm's $\mathrm{CD}(K, \infty)$ spaces converging in the pointed measured Gromov sense to a space $(X, \mathrm{d}, \mathfrak{m}, x)$. Then the space $(X, \mathrm{d}, \mathfrak{m}, x)$ is Sturm's $\mathrm{CD}(K, \infty)$.*

Using the extrinsic approach, N. Gigli, A. Mondino and G. Savaré also proved stability of the (possibly non linear) heat flows [GMS15, Th. 5.7] and stability of the Cheeger's energies [GMS15, Th. 6.8] (in the sense of Mosco convergence [Mo69]). The stability of heat flows will be a key result for us in Chapter 4.

It is easily seen that pointed measured Gromov-Hausdorff convergence implies pmG-convergence [GMS15, Prop. 3.30], but the converse implication might fail, see for instance [GMS15, Ex. 3.31]. Nevertheless, the next proposition ensures that the two notions coincide for a large class of spaces, namely those satisfying a uniform doubling condition.

Proposition 2.4.16. *[GMS15, Prop. 3.33] Let $\{(X_i, \mathrm{d}_i, \mathfrak{m}_i, x_i)\}_{i\in\mathbb{N}}$ be a sequence of uniformly doubling pointed metric measure spaces. Then $(X_i, \mathrm{d}_i, \mathfrak{m}_i, x_i)$ pmG-converge to some pointed metric measure space $(X, \mathrm{d}, \mathfrak{m}, x)$ if and only if it converges in the pointed measured Gromov-Hausdorff sense. Moreover, using the extrinsic approach of convergence, we can assume the common metric space (Y, d) in which all the spaces $(X_i, \mathrm{d}_i), (X, \mathrm{d})$ are embedded to be doubling.*

Note that complete and doubling spaces are proper (i.e. bounded closed sets are compact), hence separable.

Thanks to Bishop-Gromov's theorem (Theorem 2.1.14), Proposition 2.4.16 applies especially in case $(X_i, \mathrm{d}_i, \mathfrak{m}_i)$ are all Lott-Villani's $\mathrm{CD}(K, N)$ spaces, or all $\mathrm{RCD}(K, N)$ spaces. Therefore, Theorem 2.4.9 can be reformulated assuming pmG convergence of the spaces instead of pointed mGH convergence, and we can equivalently use the extrinsic formulation of convergence.

Convergence of functions defined on varying spaces and related stability results

The extrinsic approach is convenient to formulate various notions of convergence of functions and to avoid the use of ϵ-isometries, see Remark 2.4.19 below. However, it should be handled with care: for instance, if $f \in \mathrm{Lip}_b(Y, \mathrm{d})$ is viewed as a sequence of bounded Lipschitz functions in the spaces $(X_i, \mathrm{d}_i, \mathfrak{m}_i)$, then the sequence need not be strongly convergent in $H^{1,2}$ in the sense of Definition 2.4.21 below (see [AST17, Ex. 6.3] for a simple example).

Let us give now the definition of L^2-weak/strong convergence of functions defined on pmG-convergent sequences of spaces, following the formulation in [GMS15] and [AST17]. Other good formulations of L^2-convergence, in connection with mGH-convergence, can be found in [H15, KS11]. Note that in the setting of $\mathrm{RCD}^*(K, N)$ spaces these formulations are equivalent by the volume doubling condition (see e.g. [H16, Proposition 3.3]).

Definition 2.4.17 (L^2-weak convergence of functions with respect to variable measures)**.**

1. (L^2-weak convergence) We say that $f_i \in L^2(X_i, \mathfrak{m}_i)$ *L^2-weakly converge to* $f \in L^2(X, \mathfrak{m})$ if $\sup_i \|f_i\|_{L^2} < \infty$ and $f_i\mathfrak{m}_i \overset{C_{\mathrm{bs}}(X)}{\rightharpoonup} f\mathfrak{m}$.
2. (L^2_{loc}-weak convergence) We say that $f_i \in L^2_{\mathrm{loc}}(X_i, \mathfrak{m}_i)$ *L^2_{loc}-weakly converge to* $f \in L^2_{\mathrm{loc}}(X, \mathfrak{m})$ if ζf_i L^2_{loc}-weakly converge to ζf for any $\zeta \in C_{\mathrm{bs}}(X)$.

Note that it was proven in [GMS15] (see also [AST17], [AH17a]) that any L^2-bounded sequence has an L^2-weak convergent subsequence in the above sense.

The analogy with the usual weak convergence in Hilbert spaces is immediate by writing $f_i\mathfrak{m}_i \overset{C_{\mathrm{bs}}(X)}{\rightharpoonup} f\mathfrak{m}$ as

$$\lim_{i\to 0}\langle f_i, \varphi\rangle_{L^2(X,\mathfrak{m})} = \langle f, \varphi\rangle_{L^2(X,\mathfrak{m})} \qquad \forall \varphi \in C_{\mathrm{bs}}(X).$$

Moreover, for any L^2-weak convergent sequence $L^2(X_i, \mathfrak{m}_i) \ni f_i \to f \in L^2(X, \mathfrak{m})$, it can be shown that $\liminf_{i\to\infty} \|f_i\|_{L^2(X_i,\mathfrak{m}_i)} \geq \|f\|_{L^2(X,\mathfrak{m})}$. Following the classical property of weak convergence in Hilbert spaces stating that strong convergence follows from weak convergence and convergence of norms, we can define L^2-strong convergence of functions defined on varying spaces in the following natural way.

Definition 2.4.18 (L^2-strong convergence of functions with respect to variable measures)**.**

1. We say that $f_i \in L^2(X_i, \mathfrak{m}_i)$ *L^2-strongly converge to* $f \in L^2(X, \mathfrak{m})$ if f_i L^2-weakly converge to f with $\limsup_{i\to\infty} \|f_i\|_{L^2} \le \|f\|_{L^2}$.
2. We say that $f_i \in L^2_{\text{loc}}(X_i, \mathfrak{m}_i)$ *L^2_{loc}-strongly converge to* $f \in L^2_{\text{loc}}(X, \mathfrak{m})$ if ζf_i L^2_{loc}-strongly converge to ζf for any $\zeta \in C_{\text{bs}}(X)$.

Remark 2.4.19. Note that a naive way to define L^2 convergence of functions $L^2(X_i, \mathfrak{m}) \ni f_i \to f \in L^2(X, \mathfrak{m})$ for a pointed measured Gromov-Hausdorff convergent sequence $(X_i, \mathrm{d}_i, \mathfrak{m}_i, x_i) \xrightarrow{mGH} (X, \mathrm{d}, \mathfrak{m}, x)$ would be the following: if $\varphi_i : X_i \to X$ are ε_i-isometries for some infinitesimal sequence $(\varepsilon_i)_i \subset (0, +\infty)$, we could require $\|f_i - f \circ \varphi_i\|_{L^2(X_i, \mathfrak{m}_i)}$ to tend to zero when $i \to \infty$. But in case $X_i \equiv X = [0, 1]$, $f_i \equiv 1$ and $f = 1_{\mathbb{Q}}$, taking ε_i-isometries φ_i with values in $\mathbb{Q}$ we get $\|f_i - f \circ \varphi_i\|_{L^2(0,1)} \equiv 0$, but $\|f_i\|_{L^2(0,1)} \equiv 1$ whereas $\|f\|_{L^2(0,1)} = 0$. So we cannot formulate a relevant notion of L^2-convergence in this way.

We are now in a position to state an important stability result concerning the heat flow for a pmG-convergent sequence of $\mathrm{CD}(K, \infty)$ spaces [GMS15, Th. 6.11].

Theorem 2.4.20 (Stability of the heat flow for $\mathrm{CD}(K, \infty)$ spaces)**.** *Let* $(X_i, \mathrm{d}_i, \mathfrak{m}_i, x_i) \xrightarrow{pmG} (X, \mathrm{d}, \mathfrak{m}, x)$ *be a converging sequence of* $\mathrm{CD}(K, \infty)$ *spaces. For any* i*, let* $(P^i_t)_{t>0}$ *be the heat flow of* $(X_i, \mathrm{d}_i, \mathfrak{m}_i)$ *and* $(P_t)_{t>0}$ *the one of* $(X, \mathrm{d}, \mathfrak{m})$*. Then for any L^2-strongly convergent sequence* $L^2(X_i, \mathfrak{m}_i) \ni f_i \to f \in L^2(X, \mathfrak{m})$*, we have L^2-strong convergence of the functions* $P^i_t f_i$ *to* $P_t f$ *for any* $t > 0$*.*

Let us conclude by mentioning that N. Gigli, A. Mondino and G. Savaré also established in [GMS15] stability of the $\mathrm{RCD}(K, \infty)$ condition with respect to pmG-convergence and convergence of the eigenvalues of the Laplacian (defined by Courant's min-max procedure, see (4.0.2)) for pmG-convergent sequences of metric measure spaces satisfying a uniform weak logarithmic Sobolev-Talagrand inequality which holds easily for sequences of $\mathrm{RCD}^*(K, N)$ spaces.

Still following [GMS15], let us now define weak and strong convergence of Sobolev functions defined on varying metric measure spaces. To that purpose, let us fix a pmG-convergent sequence of $\mathrm{CD}(K, N)$ spaces $(X_i, \mathrm{d}_i, \mathfrak{m}_i, x) \xrightarrow{pmG} (X, \mathrm{d}, \mathfrak{m}, x)$. For convenience, we shall denote by $\mathrm{Ch}^i = \mathrm{Ch}_{\mathfrak{m}_i}$, $\langle \cdot, \cdot \rangle_i$, Δ_i, etc. the various objects associated to the i-th metric measure structure.

Definition 2.4.21 ($H^{1,2}$-convergence of functions defined on varying spaces)**.** We say that $f_i \in H^{1,2}(X_i, \mathrm{d}_i, \mathfrak{m}_i)$ are weakly convergent in $H^{1,2}$ to $f \in H^{1,2}(X_i, \mathrm{d}_i, \mathfrak{m}_i)$ if f_i are L^2-weakly convergent to f and $\sup_i \mathrm{Ch}^i(f_i)$ is finite. Strong convergence in $H^{1,2}$ is defined by requiring L^2-strong convergence of the functions, and $\mathrm{Ch}(f) = \lim_i \mathrm{Ch}^i(f_i)$.

We can also introduce the local counterpart of these concepts.

Definition 2.4.22 (Local $H^{1,2}$-convergence on varying spaces)**.** We say that the functions $f_i \in H^{1,2}(B_R(x_i), \mathrm{d}_i, \mathfrak{m}_i)$ are weakly convergent in $H^{1,2}$ to $f \in H^{1,2}(B_R(x), \mathrm{d}, \mathfrak{m})$ on $B_R(x)$ if f_i are L^2-weakly convergent to f on $B_R(x)$ with $\sup_i \|f_i\|_{H^{1,2}} < \infty$. Strong convergence in $H^{1,2}$ on $B_R(x)$ is defined by requiring strong L^2 convergence and $\lim_i \||\nabla f_i|\|_{L^2(B_R(x_i))} = \||\nabla f|\|_{L^2(B_R(x))}$.

We say that $g_i \in H^{1,2}_{\rm loc}(X_i, \mathrm{d}_i, \mathfrak{m}_i)$ $H^{1,2}_{\rm loc}$-weakly (or strongly, resp.) convergent to $g \in H^{1,2}_{\rm loc}(X, \mathrm{d}, \mathfrak{m})$ if $g_i|_{B_R(x_i)}$ $H^{1,2}$-weakly (or strongly, resp.) convergent to $g|_{B_R(x)}$ for all $R > 0$.

The following fundamental properties of local convergence of functions have been established in [AH17b]. They imply, among other things, that in the definition of local $H^{1,2}$-weak convergence one may equivalently require L^2-weak or L^2-strong convergence of the functions.

Theorem 2.4.23 (Compactness of local Sobolev functions)**.** *Let $R > 0$ and let $f_i \in H^{1,2}(B_R(x_i), \mathrm{d}_i, \mathfrak{m}_i)$ with $\sup_i \|f_i\|_{H^{1,2}} < \infty$. Then there exist $f \in H^{1,2}(B_R(x), \mathrm{d}, \mathfrak{m})$ and a subsequence $f_{i(j)}$ such that $f_{i(j)}$ L^2-strongly converge to f on $B_R(x)$ and*

$$\liminf_{j\to\infty} \int_{B_R(x_{i(j)})} |\nabla f_{i(j)}|^2_{i(j)} \, \mathrm{d}\mathfrak{m}_{i(j)} \geq \int_{B_R(x)} |\nabla f|^2 \, \mathrm{d}\mathfrak{m}.$$

Theorem 2.4.24 (Stability of Laplacian on balls)**.** *Let $f_i \in D(\Delta, B_R(x_i))$ with*

$$\sup_i (\|f_i\|_{H^{1,2}(B_R(x_i))} + \|\Delta_{x_i,R} f_i\|_{L^2(B_R(x_i))}) < \infty,$$

and with f_i L^2-strongly convergent to f on $B_R(x)$ (so that, by Theorem 2.4.23, $f \in H^{1,2}(B_R(x), \mathrm{d}, \mathfrak{m})$). Then:

(1) $f \in D(\Delta, B_R(x))$;

(2) $\Delta_{x_i,R} f_i$ L^2-weakly converge to $\Delta_{x,R} f$ on $B_R(x)$;

(3) $|\nabla f_i|_i$ L^2-strongly converge to $|\nabla f|$ on $B_r(x)$ for any $r < R$.

We shall also use in Chapter 5 the following local compactness theorem under BV bounds for sequences of Sobolev functions. Note that for any $p \in (1, +\infty)$, one can define L^p-weak/strong convergence out of the definition of L^2-weak/strong convergence by replacing 2 by p everywhere.

Theorem 2.4.25. *Assume that $f_i \in H^{1,2}(B_2(x_i), \mathrm{d}_i, \mathfrak{m}_i)$ satisfy*

$$\sup_i \left(\|f_i\|_{L^\infty(B_2(x_i))} + \int_{B_2(x_i)} |\nabla f_i| \, \mathrm{d}\mathfrak{m}_i \right) < \infty.$$

Then (f_i) has a subsequence L^p-strong convergent on $B_1(x)$ for all $p \in [1, \infty)$.

Proof. The proof of the compactness w.r.t. L^1-strong convergence can be obtained arguing as in [AH17a, Prop. 7.5] (where the result is stated in global form, for normalized metric measure spaces, even in the BV setting), using good cut-off functions, see also [H15, Prop. 3.39] where a uniform L^p bound on gradients, for some $p > 1$ is assumed. Then, because of the uniform L^∞ bound, the convergence is L^p-strong for any $p \in [1, \infty)$, see [AH17a, Prop. 1.3.3(e)]. □

Let us conclude with sufficient conditions under which harmonic replacements (recall Proposition 2.2.23) are continuous with respect to measured Gromov-Hausdorff convergence. This last result is a consequence of [AH17b, Thm. 3.4].

Proposition 2.4.26 (Continuity of harmonic replacements)**.** *Assume that* $\mathfrak{m}(B_R(x)) > 0$, $\lambda_1(B_R(x)) > 0$ *and that*

$$H_0^{1,2}(B_R(x), \mathrm{d}, \mathfrak{m}) = \bigcap_{\epsilon>0} H_0^{1,2}(B_{R+\epsilon}(x), \mathrm{d}, \mathfrak{m}). \tag{2.4.4}$$

Let $f_i \in H^{1,2}(B_R(x_i), \mathrm{d}_i, \mathfrak{m}_i)$ *be a weak* $H^{1,2}$*-convergent sequence to* $f \in H^{1,2}(B_R(x), \mathrm{d}, \mathfrak{m})$ *on* $B_R(x)$*. Then the harmonic replacements* $\hat{f}_i$ *of* f_i *on* $B_R(x_i)$ *exist for* i *large enough and* L^2*-strongly converge to the harmonic replacement* $\hat{f}$ *of* f *on* $B_R(x)$*.*

Notice that a simple separability argument shows that, given $x \in X$, the condition (2.4.4) is satisfied for all $R > 0$ with $\mathfrak{m}(B_R(x)) > 0$, with at most countably many exceptions (see [AH17b, Lem. 2.12]).

Chapter 3

Weighted Sobolev inequalities via patching

In this chapter, we present the results of the notes [T17a] and [T17b], namely weighted Sobolev inequalities on non-compact metric measure spaces satisfying a growth assumption on the volume of large balls, following an approach due to A. Grigor'yan and L. Saloff-Coste [GS05] and applied successfully by V. Minerbe [Mi09] to get a weighted L^2-Sobolev inequality on smooth Riemannian manifolds with non-negative Ricci curvature satisfying a suitable reverse doubling condition.

Unless explicitely mentioned, in the whole chapter the triple $(X, \mathrm{d}, \mathfrak{m})$ stands for a complete and separable metric space (X, d) equipped with a reference measure $\mathfrak{m}$ defined on the Borel σ-algebra of (X, d).

Several constants appear in this section. For better readability, if a constant C depends only on parameters $a_1, a_2, \ldots$ we will always write $C = C(a_1, a_2, \ldots)$ for its first occurrence, and then write more simply C if there is no ambiguity.

Weighted Sobolev inequalities in CD(0,N) spaces and consequences

Let us present immediately the main results of [T17a], postponing the proofs later in this section.

Theorem 3.0.1 (Weighted Sobolev inequalities)**.** *Let $(X, \mathrm{d}, \mathfrak{m})$ be a* $\mathrm{CD}(0, N)$ *space with $N > 2$. Assume that there exists $1 < \eta < N$ such that*

$$0 < \Theta_{inf} := \liminf_{r \to +\infty} \frac{V(o,r)}{r^\eta} \leq \Theta_{sup} := \limsup_{r \to +\infty} \frac{V(o,r)}{r^\eta} < +\infty \tag{3.0.1}$$

for some $o \in X$. Then for any $1 \leq p < \eta$, there exists a constant $C = C(N, \eta, \Theta_{inf}, \Theta_{sup}, p) > 0$, depending only on N, η, Θ_{inf}, Θ_{sup} and p, such that for any Borel function $u : X \to \mathbb{R}$ admitting an upper gradient $g \in L^p(X, \mathfrak{m})$,

$$\left(\int_X |u|^{p^*} d\mu \right)^{\frac{1}{p^*}} \leq C \left(\int_X g^p d\mathfrak{m} \right)^{\frac{1}{p}} \tag{3.0.2}$$

where $p^* = Np/(N-p)$ and μ is the measure absolutely continuous with respect to $\mathfrak{m}$ with density $w_o = V(o, \mathrm{d}(o, \cdot))^{p/(N-p)} \mathrm{d}(o, \cdot)^{-Np/(N-p)}$.

Note that the growth condition (3.0.1) imposes a restriction on the dimension at infinity of the space.

We shall deduce from Theorem 3.0.1 the following weighted Nash inequality holding in the context of $\mathrm{RCD}(0, N)$ spaces. Let us point out that some weighted Nash inequalities were also considered in [BBGL12], but they seem unrelated to ours.

Theorem 3.0.2 (Weighted Nash inequality)**.** *Assume that* $(X, \mathrm{d}, \mathfrak{m})$ *is a* $\mathrm{RCD}(0, N)$ *space, with* $N > 2$*, satisfying* (3.0.1) *with* $\eta > 2$*. Then there exists a constant* $C = C(N, \eta, \Theta_{inf}, \Theta_{sup}) > 0$ *such that for any* $u \in L^1(X, \mu) \cap H^{1,2}(X, \mathrm{d}, \mathfrak{m})$*,*

$$\|u\|_{L^2(X,\mu)}^{2+\frac{4}{N}} \le C \|u\|_{L^1(X,\mu)}^{\frac{4}{N}} \mathrm{Ch}(u).$$

Finally, using Theorem 3.0.2, we can deduce a uniform bound on the corresponding weighted heat kernel of $\mathrm{RCD}(0, N)$ spaces provided a N-Ahlfors regularity property holds for balls with small radii. Recall that for any integer k, a space $(X, \mathrm{d}, \mathfrak{m})$ is called k-Ahlfors regular if there exists a constant $C > 1$ such that

$$C^{-1} \le \frac{V(x,r)}{r^k} \le C, \qquad \forall r > 0 \tag{3.0.3}$$

holds for all $x \in X$. Note that if $1 \le p, q \le +\infty$ and L is a bounded operator from $L^p(X, \mu)$ to $L^q(X, \mu)$, we denote by $\|L\|_{L^p(X,\mu) \to L^q(X,\mu)}$ its norm.

Theorem 3.0.3 (Bound of the weighted heat kernel)**.** *Assume that* $(X, \mathrm{d}, \mathfrak{m})$ *is a* $\mathrm{RCD}(0, N)$ *space with* $\dim_{\mathrm{d},\mathfrak{m}}(X) = N \ge 3$ *satisfying the growth condition* (3.0.1) *for some* $\eta > 2$ *and such that*

$$C_o^{-1} \le \frac{\mathfrak{m}(B_r(x))}{r^N} \le C_o \qquad \forall x \in X, \ \forall\, 0 < r < r_o \tag{3.0.4}$$

for some $C_o > 1$ *and* $r_o > 0$*. Let* $(h_t^\mu)_{t>0}$ *be the semi-group generated by the Dirichlet form* Q *defined on* $L^2(X, \mu)$ *by*

$$Q(f) = \begin{cases} \int_X |\nabla f|_*^2 \,\mathrm{d}\mathfrak{m} & \textit{if } f \in H^{1,2}_{loc}(X, \mathrm{d}, \mathfrak{m}) \textit{ with } |\nabla f|_* \in L^2(X, \mathfrak{m}) \\ +\infty & \textit{otherwise.} \end{cases}$$

Then there exists $C = C(N, \eta, \Theta_{inf}, \Theta_{sup}) > 0$ *such that*

$$\|h_t^\mu\|_{L^1(X,\mu) \to L^\infty(X,\mu)} \le \frac{C}{t^{N/2}}, \qquad \forall t > 0,$$

or equivalently, for any $t > 0$*,* h_t^μ *admits a kernel* p_t^μ *with respect to* μ *such that for every* $x, y \in X$*,*

$$p_t^\mu(x, y) \le \frac{C}{t^{N/2}}.$$

Preliminary notions

Let us start with recalling two technical notions taken from [HK00]. First, it is well-known that the classical gradient of a Lipschitz function on, say, a smooth manifold vanishes on the open sets on which the function is constant. The following truncation property is an extension of this fact to the context of metric measure spaces.

Definition 3.0.4. (Truncation property) Let $u : X \to [-\infty, +\infty]$ and $g : X \to [0, +\infty]$ be two measurable functions. For any $0 < t_1 < t_2$ and any function $v : X \to \mathbb{R}$, we denote by $v_{t_1}^{t_2}$ the truncated function $\min(\max(0, u - t_1), t_2 - t_1)$. We say that (u, g) satisfies the truncation property if for any $0 < t_1 < t_2$, any $b \in \mathbb{R}$ and any $\varepsilon \in \{-1, 1\}$, $g\chi_{t_1 < u < t_2}$ is an upper gradient of $(\varepsilon(u - b))_{t_1}^{t_2}$.

It can be easily checked that the couple (u, g) made of a function u and any of its upper gradients g satisfies the truncation property.

Next notion will be useful to turn weak inequalities into strong inequalities.

Definition 3.0.5 (John domains)**.** Let Ω be a bounded open set of X. Ω is called a John domain if there exists $x_0 \in \Omega$ and $C > 0$ such that for every $x \in \Omega$, there exists a Lipschitz curve $\gamma : [0, L] \to \Omega$ parametrized by arc-length such that $\gamma(0) = x$, $\gamma(L) = x_0$ and for any $t \in [0, L]$,

$$C \leq \frac{d(\gamma(t), X\backslash\Omega)}{t}. \tag{3.0.5}$$

Let us point out that condition (3.0.5) prevents John domains to have cusps on their boundary, as one can easily understand from a simple example. Take $\Omega = \{(x, y) \in \mathbb{R}^2 : 0 < x < \pi/2, |y| < e^{-\tan x}\}$. Then (3.0.5) fails at the cuspidal point $(1, 0)$: define $z_\varepsilon = (\pi/2 - \varepsilon, 0)$ for any $0 < \varepsilon < \pi/4$, then for any Lipschitz curve γ starting from z_ε parametrized by arc-length and with length larger than ε,

$$\frac{d(\gamma(\varepsilon), \mathbb{R}^2\backslash\Omega)}{\varepsilon} \leq \frac{d(z_{2\varepsilon}, \mathbb{R}^2\backslash\Omega)}{\varepsilon} = \frac{e^{-\tan(\pi/2-2\varepsilon)}}{\varepsilon} \xrightarrow{\varepsilon\to 0} 0.$$

Patching procedure

Let us present the patching procedure [GS05, Mi09] that we shall apply to get Theorem 3.0.1. Recall that μ is the Borel measure absolutely continuous with respect to $\mathfrak{m}$ with density $w_o = V(o, \mathrm{d}(o, \cdot))^{p/(N-p)}\mathrm{d}(o, \cdot)^{-Np/(N-p)}$. For a given set $\{\cdot\}$, we denote by $Card\{\cdot\}$ its cardinality. Let $A \subset A^\# \subset X$ be two Borel sets such that $0 < \mathfrak{m}(A) \leq \mathfrak{m}(A^\#) < +\infty$.

Definition 3.0.6. A countable family $(U_i, U_i^*, U_i^\#)_{i\in I}$ of Borel subsets of X with finite $\mathfrak{m}$-measure is called a good covering of $(A, A^\#)$ with respect to $(\mu, \mathfrak{m})$ if:

1. for every $i \in I$, $U_i \subset U_i^* \subset U_i^\#$;
2. there exists a $\mathfrak{m}$-negligible Borel set $E \subset A^\#$ such that $X\backslash A^\# \subset \bigcup_i U_i$;
3. (**overlapping condition at level 3**)
   there exists $Q_1 > 0$ such that for every $i_0 \in I$, $Card(\{i \in I : U_{i_0}^\# \cap U_i^\# \neq \emptyset\}) \leq Q_1$;
4. (**embracing condition between level 1 and 2**)
   for every $(i, j) \in I^2$ such that $\overline{U_i} \cap \overline{U_j} \neq \emptyset$, there exists $k(i, j) \in I$ such that $U_i \cup U_j \subset U_{k(i,j)}^*$;
5. (**measure control of the embracing condition**)
   there exists $Q_2 > 0$ such that for every $i, j \in I$, if $\overline{U_i} \cap \overline{U_j} \neq \emptyset$,

$$\begin{array}{ll} (i) & \mu(U_{k(i,j)}^*) \leq Q_2 \min(\mu(U_i), \mu(U_j)); \\ (ii) & \mathfrak{m}(U_{k(i,j)}^*) \leq Q_2 \min(\mathfrak{m}(U_i), \mathfrak{m}(U_j)). \end{array}$$

Assume that $(U_i, U_i^*, U_i^\#)_{i\in I}$ is a good covering of $(A, A^\#)$ with respect to $(\mu, \mathfrak{m})$. Let us explain how to define out of $(U_i, U_i^*, U_i^\#)_{i\in I}$ a canonical weighted graph $(\mathcal{V}, \mathcal{E}, \mu)$, where $\mathcal{V}$ is the set of vertices of the graph, $\mathcal{E}$ is the set of edges, and μ is a weight on the graph (i.e. a function $\mu : \mathcal{V} \sqcup \mathcal{E} \to \mathbb{R}$). We define $\mathcal{V}$ by associating to each U_i a vertex i (informally, we put a point i on each U_i). Then we define $\mathcal{E}$ as

$$\mathcal{E} := \{(i, j) \in \mathcal{V} \times \mathcal{V} : i \neq j \text{ and } \overline{U_i} \cap \overline{U_j} \neq \emptyset\}.$$

We will write $i \sim j$ whenever $(i, j) \in \mathcal{E}$. Note that two vertices are linked if the associated pieces of the covering intersect. But in practice, we will always consider good coverings such that $\mathring{U}_i \cap \mathring{U}_j = \emptyset$ for every $i \neq j$, so roughly speaking, we are just linking two vertices i and j if they correspond to adjacent pieces U_i and U_j. Afterwards we weight the vertices of the graph setting $\mu(i) := \mu(U_i)$ for every $i \in \mathcal{V}$ (the repeated use of the letter "μ" won't cause any trouble), and the edges setting $\mu(i,j) := \max(\mu(i), \mu(j))$ for every $(i,j) \in \mathcal{E}$.

The patching theorem (Theorem 3.0.11) states that if some local inequalities are true on the pieces of the good covering, and if a discrete inequality holds on the associated canonical weighted graph, then the local inequalities can be patched into a global one. Let us give the precise definitions.

Definition 3.0.7. We say that the good covering $(U_i, U_i^*, U_i^\#)_{i\in I}$ satisfies local continuous L^p Sobolev-Neumann inequalities if there exists a constant $S_c > 0$ such that for every $i \in I$,

1. **(levels 1-2)** for any measurable function $u : U_i^* \to \mathbb{R}$ and any upper gradient $g \in L^p(U_i^*, \mathfrak{m})$ of u,
$$\left(\int_{U_i} |u - \langle u\rangle_{U_i}|^{p^*} \,\mathrm{d}\mu\right)^{\frac{1}{p^*}} \leq S_c \left(\int_{U_i^*} g^p \,\mathrm{d}\mathfrak{m}\right)^{\frac{1}{p}};$$

2. **(levels 2-3)** for any measurable function $u : U_i^\# \to \mathbb{R}$ and any upper gradient $g \in L^p(U_i^\#, \mathfrak{m})$ of u,
$$\left(\int_{U_i^*} |u - \langle u\rangle_{U_i^*}|^{p^*} \,\mathrm{d}\mu\right)^{\frac{1}{p^*}} \leq S_c \left(\int_{U_i^\#} g^p \,\mathrm{d}\mathfrak{m}\right)^{\frac{1}{p}}.$$

Definition 3.0.8. We say that the weighted graph $(\mathcal{V}, \mathcal{E}, \mu)$ satisfies a discrete (p,p) Poincaré inequality if there exists a constant $S_d > 0$ such that for every $f \in L^p(\mathcal{V}, \mu)$,

$$\left(\sum_{i\in\mathcal{V}} |f(i)|^p \mu(i)\right)^{\frac{1}{p}} \leq S_d \left(\sum_{\{i,j\}\in\mathcal{E}} |f(i) - f(j)|^p \mu(i,j)\right)^{\frac{1}{p}}.$$

Remark 3.0.9. Note that here we differ from Minerbe's terminology, which call the above inequality a discrete L^p Sobolev-Dirichlet inequality of order ∞. More generally, we say that a discrete L^p Sobolev-Dirichlet inequality of order k holds if there exists a constant S_d such that for every $f \in L^p(\mathcal{V}, \mu)$,

$$\left(\sum_{i\in\mathcal{V}} |f(i)|^{\frac{pk}{k-p}} \mu(i)\right)^{\frac{k-p}{pk}} \leq S_d \left(\sum_{\{i,j\}\in\mathcal{E}} |f(i) - f(j)|^p \mu(i,j)\right)^{\frac{1}{p}}.$$

As we don't need this general definition, we have chosen the terminology "Poincaré" which seems more natural.

In the following statements, we consider $1 \leq q < +\infty$.

Definition 3.0.10. A good covering $(U_i, U_i^*, U_i^\#)_{i\in I}$ of $(A, A^\#)$ is called a (p,q) patchwork if it satisfies the local continuous L^p Sobolev-Neumann inequalities and if the associated weighted graph $(\mathcal{V}, \mathcal{E}, m)$ satisfies the discrete (q,q) Poincaré inequality.

We are now in a position to state the patching theorem.

Theorem 3.0.11. *Assume that* $(A, A^{\#})$ *admits a* (p, q) *patchwork. Then there exists a constant* $C = C(p, q, Q_1, Q_2, S_c, S_d) > 0$ *such that for any Borel function* $u : X \to \mathbb{R}$ *admitting an upper gradient* $g \in L^p(X, \mathfrak{m})$*,*

$$\left(\int_A |u|^q \, \mathrm{d}\mu\right)^{\frac{1}{q}} \leq C \left(\int_{A^{\#}} g^p \, \mathrm{d}\mathfrak{m}\right)^{\frac{1}{p}}.$$

The proof of Theorem 3.0.11 can be copied *verbatim* from [Mi09, Th. 1.8], replacing norm of gradients by upper gradients, thus we omit it. Nonetheless, let us stress that this proof does not require any extra assumption on $(X, \mathrm{d}, \mathfrak{m})$.

A similar statement holds if we replace the discrete (q, q) Poincaré inequality by a discrete (q, q) Poincaré-Neumann inequality, namely there exists a constant $S_d > 0$ such that for every $f : \mathcal{V} \to \mathbb{R}$ with finite support,

$$\left(\sum_{i \in \mathcal{V}} |f(i) - \mathfrak{m}(f)|^p \mu(i)\right)^{\frac{1}{p}} \leq S_d \left(\sum_{\{i,j\} \in \mathcal{E}} |f(i) - f(j)|^p \mu(i, j)\right)^{\frac{1}{p}},$$

where $\mathfrak{m}(f) = \left(\sum_{i\,:\,f(i) \neq 0} \mathfrak{m}(i)\right)^{-1} \sum_i f(i)\mathfrak{m}(i)$. Note that the terminology "Poincaré-Neumann" used here comes from the mean-value in the left-hand side which draws an analogy with the local Poincaré inequality applied in the study of the eigenvalues of the Laplacian with Neumann boundary conditions on bounded Euclidean domains, see [Sa02, Sect. 1.5.2].

Theorem 3.0.12. *Assume that* $(A, A^{\#})$ *admits a finite good covering* $(U_i, U_i^*, U_i^{\#})_{i \in I}$ *satisfying the local continuous* L^p *Sobolev-Neumann inequalities and whose associated weighted graph* $(\mathcal{V}, \mathcal{E}, m)$ *satisfies the above discrete* (q, q) *Poincaré-Neumann inequality. Then there exists a constant* $C = C(p, q, Q_1, Q_2, S_c, S_d) > 0$ *such that for any* $u \in L^1(X, \mu)$ *admitting an upper gradient* $g \in L^p(X, \mathfrak{m})$*,*

$$\left(\int_A |u - \langle u \rangle_A|^q \, \mathrm{d}\mu\right)^{\frac{1}{q}} \leq C \left(\int_{A^{\#}} g^p \, \mathrm{d}\mathfrak{m}\right)^{\frac{1}{p}}.$$

See [Mi09, Th. 1.10] for the proof.

Proof of Theorem 3.0.1

Let $(X, \mathrm{d}, \mathfrak{m})$ be a non-compact $\mathrm{CD}(0, N)$ space with $N \geq 3$. Take $1 \leq p < N$ and $p < \eta < N$, and assume that the growth condition (3.0.1) holds. Let us recall that $p^* = pN/(N - p)$ and that μ is the measure absolutely continuous with respect to $\mathfrak{m}$ with density $w_o = V(o, \mathrm{d}(o, \cdot))^{p/(N-p)} \mathrm{d}(o, \cdot)^{-Np/(N-p)}$.

As pointed out by Minerbe [Mi09], on Riemannian manifolds, the local continuous L^2 Sobolev-Neumann inequalities can be derived from the doubling condition and a uniform local $(2, 2)$ strong Poincaré inequality, both implied by the non-negativity of the Ricci curvature. However, the discrete $(2^*, 2^*)$ Poincaré inequality requires the addition of a reverse doubling condition (3.0.6), which is an immediate consequence of the growth condition (3.0.1).

Lemma 3.0.13. *There exist* $A > 0$ *and* $C_{RD} > 0$ *such that*

$$\frac{V(o, R)}{V(o, r)} \geq C_{RD} \left(\frac{R}{r}\right)^{\eta} \qquad \forall\, A < r \leq R. \tag{3.0.6}$$

Proof. The growth condition (3.0.1) implies the existence of $A > 0$ such that for any $r \geq A$, $\Theta_{inf}/2 \leq r^{-\eta} V(o,r) \leq 2\Theta_{sup}$. Take $R \geq r$. Then $R^{-\eta}V(o,R) \geq \Theta_{inf}/2$, whence the result with $C_{RD} = \Theta_{inf}/(4\Theta_{sup})$. □

Remark 3.0.14. With no loss of generality, we can (and will) assume that $A = 1$.

We shall need the following result, namely a local L^p-Sobolev inequality, which is a well-known consequence of the doubling and Poincaré properties of $(X, \mathrm{d}, \mathfrak{m})$.

Proposition 3.0.15. *There exists a constant $C = C(N,p) > 0$ such that for any function $u \in L^1_{loc}(X, \mathfrak{m})$, any upper gradient g of u, and any ball $B = B_R(x) \subset X$,*

$$\left(\fint_B |u - u_B|^{p^*} \,\mathrm{d}\mathfrak{m} \right)^{1/p^*} \leq CR \left(\fint_B g^p \,\mathrm{d}\mathfrak{m} \right)^{1/p} \tag{3.0.7}$$

or, equivalently,

$$\left(\int_B |u - u_B|^{p^*} \,\mathrm{d}\mathfrak{m} \right)^{1/p^*} \leq C \frac{R}{V(x,R)^{1/N}} \left(\int_B g^p \,\mathrm{d}\mathfrak{m} \right)^{1/p}. \tag{3.0.8}$$

Proof. Assume that $u : X \to \mathbb{R}$ is a Borel function admitting an upper gradient $g \in L^p(X, \mathfrak{m})$. Assume that B is a ball of X with radius R. Using Hölder's inequality, the Poincaré inequality (Theorem 2.1.16) gives

$$\fint_B |u - u_B| \,\mathrm{d}\mathfrak{m} \leq 2^{N+2} r \left(\fint_{2B} g^p \,\mathrm{d}\mu \right)^{1/p}.$$

As $\eta > p$ we are in a position to apply 1. of [HK00, Th. 5.1], which implies

$$\left(\fint_B |u - u_B|^{p^*} \,\mathrm{d}\mathfrak{m} \right)^{1/p^*} \leq Cr \left(\fint_{10B} g^p \,\mathrm{d}\mathfrak{m} \right)^{1/p}.$$

To turn this weak inequality into a strong one, let us apply [HK00, Th. 9.7] to the ball B. As $(X, \mathrm{d}, \mathfrak{m})$ is a $CD(0,N)$ space, the metric structure (X, d) is proper and geodesic, then all the balls of X are John domain with a common constant [HK00, Cor. 9.5]. The fact that there exists a constant $C > 0$ such that for every ball $B(x,\rho) \subset B$ with $\rho < 2r$,

$$\mathfrak{m}(B(x,r)) \geq C \left(\frac{\rho}{2r} \right)^{\eta} \mathfrak{m}(B),$$

is easily verified using the doubling condition. Then [HK00, Th. 9.7] applies and gives the result. □

Remark 3.0.16. Theorem 9.7 of [HK00] is stated for *weak* John domains, a generalization of the notion of John domain to structures without enough rectifiable curves (especially fractals, see [HK00, p.39] for details). However being a John domain implies being a weak John domain, and $CD(0,N)$ spaces are geodesics so they contain enough rectifiable curves.

Finally, let us state a result whose proof can be taken from [Mi09, Prop. 2.8], replacing smooth functions by measurable ones, norm of gradients by upper gradients, and the strong local $(2,2)$ Poincaré inequality used there by Rajala's Poincaré inequality (Theorem 2.1.16). Notice that even if Theorem 2.1.16 provides only a weak inequality, one can harmlessly substitute it to the strong one used in the smooth case, because it is applied to a function f which is Lipschitz on a ball B and extended by 0 outside of B.

Proposition 3.0.17. *There exists $\kappa_0 = \kappa_0(N, \eta, p) > 1$ such that for every $R > 0$, for any couple of points x, y in the geodesic sphere $S(o, R)$, there exists a rectifiable curve from x to y that remains inside $B(o, R)\backslash B(o, \kappa_0^{-1} R)$.*

Remark 3.0.18. It is worth pointing out that the conclusion of Proposition 3.0.17 can be understood as a connectedness property, as it implies that any annulus $B(o, \kappa_o^{i+2})\backslash B(o, \kappa_o^{i-1})$ must be connected. Moreover, the proof can be carried out with only the doubling and Poincaré properties, thus the conclusion holds for any PI doubling space.

Let us prove now Theorem 3.0.1.

<u>STEP 1:</u> The good covering.

Let us give explain in a few words on how to construct a good covering on $(X, \mathrm{d}, \mathfrak{m})$. We refer to [Mi09, Section 2.3.1] for the details. Define κ as the square-root of the constant κ_0 given by Proposition 3.0.17. Then for any $R > 0$, two connected components of $B(o, \kappa R)\backslash B(o, R)$ are always contained in one component of $B(o, \kappa R)\backslash B(o, \kappa^{-1} R)$. Let us write $A_i = B(o, \kappa^i)\backslash B(o, \kappa^{i-1})$ for any $i \in \mathbb{N}$.

Let γ be a line starting at o, i.e. a continuous function $\gamma : [0, +\infty) \to X$ such that $\gamma(o) = 0$ and $\mathrm{d}(\gamma(t), \gamma(s)) = |t - s|$ for any $s, t \geq 0$. Such a line can be obtained as follows. For $x_1 \in S(o, 1)$, let $\gamma_1 : [0, 1] \to X$ be a geodesic between o and x_1. Define then recursively $x_n := \arg\min\{\mathrm{d}(x_{n-1}, x) \ : \ x \in S(o, n)\}$ and γ_n geodesic between x_{n-1} and x_n for any $n \geq 1$. The concatenation of all the γ_n provides the desired γ.

Then for any integer i, denote by $(U'_{i,a})_{0 \leq a \leq h'_i}$ the connected components of A_i, $U'_{i,0}$ being the one which intersects γ. Let us prove that the numbers h'_i are uniformly bounded. This was stated without proof in [Mi09].

Lemma 3.0.19. *There exists a constant $h = h(N, \kappa) < \infty$ such that $\sup_i h'_i \leq h$.*

Proof. Take $i \in \mathbb{N}$. For every $0 \leq a \leq h'_i$, pick x_a in $U_{i,a} \cap S(o, (\kappa^i + \kappa^{i-1})/2)$. As the balls $V(x_a, (\kappa^i - \kappa^{i-1})/4)$, $0 \leq a \leq h'_i$, are disjoints and all included in $V(o, \kappa_i)$,

$$h'_i \min_{0 \leq a \leq h'_i} V(x_a, (\kappa^i - \kappa^{i-1})/4) \leq \sum_{0 \leq a \leq h'_i} V(x_a, (\kappa^i - \kappa^{i-1})/4) \leq V(o, \kappa^i).$$

Assume for simplicity that $\min\limits_{0 \leq a \leq h'_i} V(x_a, (\kappa^i - \kappa^{i-1})/4) = V(x_0, (\kappa^i - \kappa^{i-1})/4)$. Notice that $d(o, x_0) \leq \kappa_i$. Then

$$h'_i \leq \frac{V(o, \kappa^i)}{V(x_0, (\kappa^i - \kappa^{i-1})/4)} \leq \frac{V(x_0, \kappa^i + \mathrm{d}(o, x_0))}{V(x_0, (\kappa^i - \kappa^{i-1})/4)} \leq \left(\frac{8\kappa^i}{\kappa^i - \kappa^{i-1}}\right)^N$$

by the doubling condition. Whence the result with $h = \left(\frac{8\kappa}{\kappa - 1}\right)^N$. □

Define then the covering $(U'_{i,a}, U'^*_{i,a}, U'^\#_{i,a})_{i \in \mathbb{N}, 0 \leq a \leq h'_i}$ where $U'^*_{i,a}$ is by definition the union of the sets $U'_{j,b}$ such that $\overline{U'_{j,b}} \cap \overline{U'_{i,a}} \neq \emptyset$, and $U'^\#_{i,a}$ is by definition the union of the sets $U'^*_{j,b}$ such that $\overline{U'^*_{j,b}} \cap \overline{U'^*_{i,a}} \neq \emptyset$. Note that $(U'_{i,a}, U'^*_{i,a}, U'^\#_{i,a})_{i \in \mathbb{N}, 0 \leq a \leq h'_i}$ is not necessarily a good covering, as there is no reason a priori that it satisfies the measure control of the overlapping condition: the pieces $U'_{i,a}$ may be arbitrary small compared to their neighbors. Thus whenever $\overline{U'_{i,a}} \cap S(o, \kappa^i) = \emptyset$, we define $U_{i-1,a} = U'_{i,a} \cup U'_{i-1,a'}$ where a' is such that $\overline{U'_{i,a}} \cap \overline{U_{i-1,a'}} \neq \emptyset$; otherwise we define $U_{i,a} = U'_{i,a}$. In other words, we incorporate small pieces $U'_{i,a}$ into the adjacent piece $U'_{i-1,a'}$. Then we define $U^*_{i,a}$ and $U^\#_{i,a}$ in a similar way than $U'^*_{i,a}$ and $U'^\#_{i,a}$. Using the doubling condition, one can show that $(U_{i,a}, U^*_{i,a}, U^\#_{i,a})_{i\in\mathbb{N},0\le a\le h_i}$ is a good covering on (X, d) with respect to $(\mu, \mathfrak{m})$, with constants Q_1 and Q_2 depending only on N.

STEP 2: The discrete (p^*, p^*) Poincaré inequality.

Let us denote by $(\mathcal{V}, \mathcal{E}, \mu)$ the weighted graph obtained from the good covering $(U_{i,a}, U^*_{i,a}, U^\#_{i,a})_{i\in\mathbb{N},0\le a\le h_i}$. Define the degree $\deg(i)$ of a vertex i as the number of vertices j such that $i \sim j$. As a consequence of Lemma 3.0.19, $\sup \deg(i) : i \in \mathcal{V} \le 2h$. Moreover, the doubling condition implies easily the existence of a number $C \ge 1$ such that for every $i, j \in E$, $C^{-1}\mathfrak{m}(i) \le \mathfrak{m}(j) \le C\mathfrak{m}(j)$. Thus by [Mi09, Prop. 1.12], the discrete $(1, 1)$ Poincaré inequality implies the (q, q) one for every $q \ge 1$. But the discrete $(1, 1)$ Poincaré inequality is equivalent to the isoperimetric inequality ([Mi09, Prop. 1.14]): there exists a constant $\mathcal{I} > 0$ such that for any $\Omega \subset \mathcal{V}$ with finite measure,

$$\frac{\mu(\Omega)}{\mu(\partial\Omega)} \le \mathcal{I}$$

where $\partial\Omega := \{(i, j) \in \mathcal{E} : i \in \Omega, j \notin \Omega\}$. The only ingredients to prove this isoperimetric inequality are the doubling and reverse doubling conditions, see Section 2.3.3 in [Mi09]. Then the discrete (q, q) Poincaré inequality holds for any $q \ge 1$, with a constant S_d depending only on q, η, Θ_{inf}, Θ_{sup} and on the doubling and Poincaré constants of $(X, \mathrm{d}, \mathfrak{m})$, i.e. on N.

STEP 3: The local continuous L^p Sobolev-Neumann inequalities.

Let us explain how to get the local continuous L^p Sobolev-Neumann inequalities. We start by deriving from the local L^p-Sobolev inequality (3.0.7) a crucial technical result, namely a L^p-Sobolev-type inequality on connected Borel subsets of annuli.

Lemma 3.0.20. *Let $R > 0$ and $\alpha > 1$. Let A be a connected Borel subset of $B(o, \alpha R)\backslash B(o, R)$. For $0 < \delta < 1$, denote by $(A)_\delta$ the δ-neighborhood of A, i.e. $(A)_\delta = \bigcup_{x\in A} B_\delta(x)$. Then there exists a constant $C = C(N, \delta, \alpha, p) > 0$ such that for any measurable function $u : (A)_\delta \to [-\infty, +\infty]$ and any upper gradient $g \in L^p((A)_\delta, \mathfrak{m})$,*

$$\left(\int_A |u - u_A|^{p^*} \,\mathrm{d}\mathfrak{m}\right)^{1/p^*} \le C \frac{R^p}{V(o, R)^{p/N}} \left(\int_{(A)_\delta} g^p \,\mathrm{d}\mathfrak{m}\right)^{1/p}.$$

Proof. Define $s = \delta R$ and choose $(x_j)_{j\in J}$ an s-lattice of A (a maximal set of points whose distance between two of them is at least s). Set $V_i = B(x_i, s)$ and $V^*_i = V^\#_i = B(x_i, 3s)$. Using the doubling condition, there is no difficulty in proving that $(V_i, V^*_i, V^\#_i)$ is a good covering of (X, d) with respect to $(\mathfrak{m}, \mathfrak{m})$. A discrete (p^*, p^*) Poincaré-Neumann inequality holds on the associated weighted graph, as one can easily check following the lines of [Mi09, Lem. 2.10]. The local continuous L^p Sobolev-Neumann inequalities stem from the proof of [Mi09, Lem. 2.11], where we replace (14) there by Proposition 3.0.15. Then Theorem 3.0.12 gives the result. □

Let us prove that Lemma 3.0.20 implies the local continuous L^p Sobolev-Neumann inequalities with a constant S_c depending only on N, η and p. Take a piece of the good covering $U_{i,a}$. Choose $\delta = (1 - \kappa^{-1})/2$ so that $(U_{i,a})_\delta \subset U^*_{i,a}$. Take a measurable function

$u : U^*_{i,a} \to [-\infty, +\infty]$ and an upper gradient g of u. By the triangle inequality and the elementary fact $|x+y|^{p^*} \le 2^{p^*-1}(|x|+|y|)$ holding for any $x, y \in \mathbb{R}$,

$$\int_{U_{i,a}} |u - \langle u \rangle_{U_{i,a}}|^{p^*} \, \mathrm{d}\mu \le 2^{p^*} \inf_{c\in\mathbb{R}} \int_{U_{i,a}} |u - c|^{p^*} \, \mathrm{d}\mu \le 2^{p^*} \int_{U_{i,a}} |u - u_{U_{i,a}}|^{p^*} w_o \, \mathrm{d}\mathfrak{m}.$$

As w_o is a radial function, let us define $\bar{w}_o(r) = w_o(x)$ for $r = \mathrm{d}(o,x)$. Note that by Bishop-Gromov theorem, $\bar{w}_o$ is a decreasing function, so

$$\int_{U_{i,a}} |u - \langle u \rangle_{U_{i,a}}|^{p^*} \, \mathrm{d}\mu \le 2^{p^*} \bar{w}_o(\kappa^{i-1}) \int_{U_{i,a}} |u - u_{U_{i,a}}|^{p^*} \, \mathrm{d}\mathfrak{m}.$$

Applying Lemma 3.0.20 with $A = U_{i,a}$, $R = \kappa^{i-1}$ and $\alpha = \kappa^2$, we get

$$\begin{aligned} \int_{U_{i,a}} |u - \langle u \rangle_{U_{i,a}}|^{p^*} \, \mathrm{d}\mu \le & C^{p^*} 2^{p^*} \frac{\kappa^{p^*(i-1)}}{V(o,\kappa^{i-1})^{p^*/N}} \bar{w}_o(\kappa^{i-1}) \left(\int_{U^*_{i,a}} g^p \, \mathrm{d}\mathfrak{m} \right)^{p^*/p} \\ \le & \, C \left(\int_{U^*_{i,a}} g^p \, \mathrm{d}\mathfrak{m} \right)^{p^*/p} \end{aligned}$$

where we used the same letter C to denote different constants depending only on N, p, and κ. As κ depends only on N, η and p, we get the result.

An analogous argument implies the inequalities between levels 2 and 3.

STEP 4: Conclusion.

Apply Theorem 3.0.11 to get the result.

Weighted Nash inequality

Let us now prove Theorem 3.0.2. To this purpose, we need a standard lemma which states that the relaxation procedure defining Ch can be achieved with slopes of Lipschitz functions *with bounded support*. We give a proof for convenience.

Lemma 3.0.21. *Let $u \in H^{1,2}(X, \mathrm{d}, \mathfrak{m})$. Then*

$$\mathrm{Ch}(u) = \inf \left\{ \liminf_{n\to\infty} \int_X |\nabla u_n|^2 \, \mathrm{d}\mathfrak{m} : (u_n)_n \subset \mathrm{Lip}_{bs}(X), \|u_n - u\|_{L^2(X,\mathfrak{m})} \to 0 \right\}.$$

In particular, for any $u \in H^{1,2}(X, \mathrm{d}, \mathfrak{m})$, there exists a sequence $(u_n)_n \subset \mathrm{Lip}_{bs}(X)$ such that $\|u - u_n\|_{L^2(X,\mathfrak{m})} \to 0$ and $\||\nabla u_n|\|_{L^2(X,\mathfrak{m})} \to \mathrm{Ch}(u)$ when $n \to +\infty$.

Proof. Choose a point $o \in X$ and for every $n \in \mathbb{N}^*$, let χ_n be a Lipschitz function constant equal to 1 on $B(o,n)$, to 0 on $X \backslash B(o, n+1)$ and such that $|\nabla \chi_n| \le 2$. Take $f \in \mathrm{Lip}(X) \cap L^2(X,\mathfrak{m})$ and define, for every $n \in \mathbb{N}$, $f_n = f\chi_n$. Using the chain rule and Young's inequality for some $\varepsilon > 0$, denoting by $\mathrm{Lip}(\chi_n)(\le 2)$ the Lipschitz constant of χ_n, we get

$$\begin{aligned} |\nabla f_n|^2 & \le \left(\chi_n |\nabla f| + f \, \mathrm{Lip}(\chi_n) 1_{B(0,n+1)\backslash B(0,n)} \right)^2 \\ & \le (1+\varepsilon)|\nabla f|^2 + 4(1/(1+\varepsilon)) f^2 1_{B(0,n+1)\backslash B(0,n)}. \end{aligned}$$

Integrating over X and taking the limit superior, it implies

$$\limsup_{n\to\infty} \int_X |\nabla f_n|^2 \, \mathrm{d}\mathfrak{m} \le (1+\varepsilon) \int_X |\nabla f|^2 \, \mathrm{d}\mathfrak{m},$$

and letting ε go to 0 leads to

$$\limsup_{n\to\infty}\int_X |\nabla f_n|^2\,\mathrm{d}\mathfrak{m} \leq \int_X |\nabla f|^2\,\mathrm{d}\mathfrak{m}.$$

Then for $u \in H^{1,2}(X,\mathrm{d},\mathfrak{m})$, for any sequence $(u_k)_k \subset \mathrm{Lip}(X)\cap L^2(X,\mathfrak{m})$ $L^2(X,\mathfrak{m})$-converging to u, considering for any $k\in\mathbb{N}$ a sequence $(v_{k,n})_n \subset \mathrm{Lip_b}(X)$ built as above, a diagonal argument provides a sequence $(v_{k,n(k)})_k$ such that

$$\liminf_{k\to\infty}\int_X |\nabla v_{k,n(k)}|^2\,\mathrm{d}\mathfrak{m} \leq \liminf_{k\to\infty}\int_X |\nabla u_k|^2\,\mathrm{d}\mathfrak{m}.$$

Taking the infimum among all sequences $(u_k)_k$ L^2-converging to u leads to the result. □

We can now prove Theorem 3.0.2. The proof presented here is the standard way to deduce a Nash inequality from a Sobolev inequality, see for instance [BBGL12].

Proof. By the previous lemma it is sufficient to prove the result for $u \in \mathrm{Lip}_{bs}(X)\cap L^1(X,\mu)$. By Hölder's inequality,

$$\|u\|_{L^2(X,\mu)} \leq \|u\|^{\theta}_{L^1(X,\mu)}\|u\|^{1-\theta}_{L^{2^*}(X,\mu)}$$

where $\frac{1}{2} = \frac{\theta}{1} + \frac{1-\theta}{2^*}$ i.e. $\theta = \frac{2}{N+2}$. Then by Theorem 3.0.1 applied for $p = 2 < \eta$,

$$\|u\|_{L^2(X,\mu)} \leq C\|u\|^{\frac{2}{N+2}}_{L^1(X,\mu)}\||\nabla u|\|^{\frac{N}{N+2}}_{L^2(X,\mathfrak{m})}.$$

The result follows from the previous inequality raised to the power $2(N+2)/N$. □

Bound on the corresponding heat kernel

We consider now a $\mathrm{RCD}(0,N)$ space $(X,\mathrm{d},\mathfrak{m})$ satisfying (3.0.1) for some $\eta > 2$ and such that there exists $C_o > 1$ and $r_o > 0$ such that

$$C_o^{-1} \leq \frac{\mathfrak{m}(B_r(x))}{r^N} \leq C_o \qquad \forall x\in X,\ \ \forall\, 0<r<r_o. \tag{3.0.9}$$

Let us explain which weighted heat kernel we are dealing with. We consider $w_o = V(o,\mathrm{d}(o,\cdot))^{2/(N-2)}\mathrm{d}(o,\cdot)^{-2N/(N-2)}$, i.e. the case $p = 2$. Recall that $\mu = w_o\mathfrak{m}$, and note that $L^2(X,\mathfrak{m}) \subset L^2(X,\mu)$ as w_o is a bounded function (this follows from Bishop-Gromov's theorem and (3.0.10)).

Recall that by definition, $H^{1,2}_{loc}(X,\mathrm{d},\mathfrak{m}) = \{f \in L^2_{loc}(X,\mathfrak{m}) : \varphi f \in H^{1,2}(X,\mathrm{d},\mathfrak{m})\ \ \forall \varphi \in \mathrm{Lip}_c(X)\}$, and that as an immediate consequence of the boundedness of w_o, we have $f \in L^2_{loc}(X,\mathfrak{m})$ if and only if $f \in L^2_{loc}(X,\mu)$.

Define the Dirichlet form Q on $L^2(X,\mu)$ as follows:

$$Q(f) = \begin{cases}\int_X |\nabla f|_*^2\,\mathrm{d}\mathfrak{m} & \text{if } f\in H^{1,2}_{loc}(X,\mathrm{d},\mathfrak{m}) \text{ with } |\nabla f|_* \in L^2(X,\mathfrak{m})\\ +\infty & \text{otherwise.}\end{cases}$$

Q is easily seen to be convex. Moreover, since convergence in $L^2_{loc}(X,\mathfrak{m})$ and in $L^2_{loc}(X,\mu)$ are equivalent, Q is a $L^2(X,\mu)$-lower semicontinuous functional on $L^2(X,\mu)$, so we can apply the general theory of gradient flow to define the semi-group $(h^\mu_t)_{t>0}$ associated to Q which is characterized by the property that for any $f \in L^2(X,\mu)$, $t \to h^\mu_t f$ is locally absolutely continuous on $(0,+\infty)$ with values in $L^2(X,\mu)$, and

$$\frac{\mathrm{d}}{\mathrm{d}t}h^\mu_t f = -Ah^\mu_t f \quad \text{for } \mathscr{L}^1\text{-a.e. } t\in(0,+\infty),$$

where the self-adjoint operator $-A$ associated to Q is defined on a dense subset $\mathcal{D}(A)$ of $\mathcal{D}(Q) = \{Q < +\infty\}$ and characterized by:

$$Q(f,g) = \int_X (Af)g \, \mathrm{d}\mu \qquad \forall f \in \mathcal{D}(A), \, \forall g \in \mathcal{D}(Q).$$

Note that by the Markov property, each h_t^μ can be uniquely extended from $L^2(X,\mu) \cap L^1(X,\mu)$ to a contraction from $L^1(X,\mu)$ to itself.

We start with a preliminary lemma stating that the Nash inequality also holds for the Dirichlet form Q. For convenience, from now on we write $L^p(\mathfrak{m})$, $L^p(\mu)$, etc... instead of $L^p(X,\mathfrak{m})$, $L^p(X,\mu)$, etc.

Lemma 3.0.22. *Let $(X, \mathrm{d}, \mathfrak{m})$ be a $\mathrm{RCD}(0,N)$ space with $N \geq 3$ satisfying (3.0.1) for some $\eta > 2$ and such that there exists $C_o > 1$ and $r_o > 0$ such that*

$$C_o^{-1} \leq \frac{\mathfrak{m}(B_r(x))}{r^N} \leq C_o \qquad \forall x \in X, \;\; \forall \, 0 < r < r_o, \tag{3.0.10}$$

and Q as above. Then there exists a constant $C = C(N, \eta, \Theta_{inf}, \Theta_{sup}) > 0$ such that for any $u \in L^1(\mu) \cap \mathcal{D}(Q)$,

$$\|u\|_{L^2(\mu)}^{2+\frac{4}{N}} \leq C \|u\|_{L^1(\mu)} Q(u).$$

Proof. Let $u \in L^1(\mu) \cap \mathcal{D}(Q)$. Then $u \in L^2_{loc}(\mathfrak{m})$, $\varphi u \in H^{1,2}(X, \mathrm{d}, \mathfrak{m})$ for any $\varphi \in \mathrm{Lip}_c(X)$ and $|\nabla u|_* \in L^2(\mathfrak{m})$. In particular, if we take $(\chi_n)_n$ as in the proof of Lemma 3.0.21, we get that $\chi_n u \in H^{1,2}(X, \mathrm{d}, \mathfrak{m})$ for any $n \in \mathbb{N}$. Then there exists a sequence $(u_{n,k})_k \subset \mathrm{Lip}_{bs}(X)$ such that $u_{n,k} \to \chi_n u$ in $L^2(\mathfrak{m})$ and $\int_X |\nabla u_{n,k}|^2 \, \mathrm{d}\mathfrak{m} \to \int_X |\nabla(\chi_n u)|_*^2 \, \mathrm{d}\mathfrak{m}$ for any $n \in \mathbb{N}$. For any given $n \in \mathbb{N}$, apply Theorem 3.0.2 to the functions $u_{n,k}$ to get

$$\|u_{n,k}\|_{L^2(\mu)}^{2+\frac{4}{N}} \leq C \|u_{n,k}\|_{L^1(\mu)}^{\frac{4}{N}} \int_X |\nabla u_{n,k}|^2 \, \mathrm{d}\mathfrak{m}$$

for any $k \in \mathbb{N}$. As the $u_{n,k}$ and $\chi_n u$ have bounded support, the $L^2(\mathfrak{m})$ convergence $u_{n,k} \to \chi_n u$ is equivalent to the $L^2_{loc}(\mathfrak{m})$, $L^2_{loc}(\mu)$, $L^2(\mu)$ and $L^1(\mu)$ convergences. Therefore, passing to the limit $k \to +\infty$, we get

$$\|\chi_n u\|_{L^2(\mu)}^{2+\frac{4}{N}} \leq C \|\chi_n u\|_{L^1(\mu)}^{\frac{4}{N}} \int_X |\nabla(\chi_n u)|_*^2 \, \mathrm{d}\mathfrak{m}.$$

By an argument similar to the proof of Lemma 3.0.21, we can show that

$$\limsup_{n \to +\infty} \int_X |\nabla(\chi_n u)|_*^2 \, \mathrm{d}\mathfrak{m} \leq \int_X |\nabla u|_*^2 \, \mathrm{d}\mathfrak{m}.$$

And monotone convergence ensures that $\|\chi_n u\|_{L^2(\mu)} \to \|u\|_{L^2(\mu)}$ and $\|\chi_n u\|_{L^1(\mu)} \to \|u\|_{L^1(\mu)}$. Whence the result. ☐

Theorem 3.0.23 (Bound of the weighted heat kernel)**.** *Assume that $N \geq 3$. Let $(X, \mathrm{d}, \mathfrak{m})$ be a $\mathrm{RCD}(0,N)$ space satisfying the growth condition (3.0.1) for some $\eta > 2$ and the regularity condition (3.0.10) for some $C_o > 1$ and $r_o > 0$. Then there exists $C = C(N, \eta, \Theta_{inf}, \Theta_{sup}) > 0$ such that*

$$\|h_t^\mu\|_{L^1(X,\mu) \to L^\infty(X,\mu)} \leq \frac{C}{t^{N/2}}, \qquad \forall t > 0,$$

or equivalently, for any $t > 0$, h_t^μ admits a kernel p_t^μ with respect to μ such that for every $x, y \in X$,

$$p_t^\mu(x,y) \leq \frac{C}{t^{N/2}}.$$

To prove this theorem we follow closely the lines of [Sa02, Th. 4.1.1]. The constant C may differ from line to line, note however that it will always depend only on N, Θ_{inf} and Θ_{sup}. For better readability, we will write $L^p(\mu)$ and $L^p(\mathfrak{m})$ instead of $L^p(X,\mu)$ and $L^p(X,\mathfrak{m})$ respectively.

Proof. Let $u \in L^1(X,\mu)$ be such that $\|u\|_{L^1(X,\mu)} = 1$. We claim that for any chosen $t > 0$, we have $\|h_t^\mu u\|_{L^2(\mu)} \le \frac{C}{t^{N/4}}$. First of all, by density of $\mathrm{Lip}_{bs}(X)$ in $L^1(X,\mu)$, we can assume $u \in \mathrm{Lip}_{bs}(X)$ with $\|u\|_{L^1(X,\mu)} = 1$. Furthermore, since by the Markov property the operator $h_t^\mu : L^1(X,\mu) \cap L^2(X,\mu) \to \mathcal{D}(Q)$ extends uniquely to a contraction operator from $L^1(X,\mu)$ to itself, we can assume $u \in L^1(X,\mu) \cap L^2(X,\mu)$, so that $h_t^\mu u \in L^1(X,\mu) \cap \mathcal{D}(Q)$ and $\|h_t^\mu u\|_{L^1(X,\mu)} \le 1$. Therefore, we can apply Lemma 3.0.22 to get

$$\|h_t^\mu u\|_{L^2(X,\mu)}^{2+\frac{4}{N}} \le CQ(h_t^\mu u).$$

As $\displaystyle\int_X |\nabla h_t^\mu u|_*^2 \,\mathrm{d}\mathfrak{m} = \int_X (A h_t^\mu u) h_t^\mu u \,\mathrm{d}\mathfrak{m} = -\int_X \left(\frac{\mathrm{d}}{\mathrm{d}t} h_t^\mu u\right) h_t^\mu u \,\mathrm{d}\mathfrak{m} = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|h_t^\mu u\|_{L^2(X,\mu)}^2$, we finally end up with the following differential inequality:

$$\|h_t^\mu u\|_{L^2(\mu)}^{2+4/N} \le -\frac{C}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|h_t^\mu u\|_{L^2(\mu)}^2 \qquad \forall t > 0.$$

Writing $\varphi(t) = \|h_t^\mu u\|_{L^2(\mu)}^2$ and $\psi(t) = \frac{N}{2}\varphi(t)^{-2/N}$ for any $t > 0$, we get $\frac{2}{C} \le -\psi'(t)$ and thus $\frac{2}{C}t \le \psi(t) - \psi(0)$. As $\psi(0) = \frac{N}{2}\|h_t^\mu u\|_{L^2(\mu)}^{-4/N} \ge 0$, we obtain $\frac{2}{C}t \le \psi(t)$, leading to

$$\|h_t^\mu u\|_{L^2(\mu)} \le \frac{C}{t^{N/4}},$$

and therefore $\|h_t^\mu\|_{L^1(\mu)\to L^2(\mu)} \le \frac{C}{t^{N/4}}$. Using the self-adjointness of $h_t f$, one deduces by duality $\|h_t^\mu u\|_{L^2(\mu)\to L^\infty(\mu)} \le \frac{C}{t^{N/4}}$. Finally the semi-group property

$$\|h_t^\mu\|_{L^1(\mu)\to L^\infty(\mu)} \le \|h_{t/2}^\mu\|_{L^1(\mu)\to L^2(\mu)}\|h_{t/2}^\mu\|_{L^2(\mu)\to L^\infty(\mu)}$$

implies the result.

□

Adimensional weighted Sobolev inequalities in PI doubling spaces

As the patching procedure does not use any special properties of $\mathrm{CD}(0,N)$ spaces apart from the doubling condition and Poincaré inequality, the proof of Theorem 3.0.1 carries over verbatim on geodesic PI doubling spaces (note that contrary to the weak local $(1,p)$-Poincaré inequality of Definition 2.2.17, here we assume that a strong local (p,p)-Poincaré inequality holds, namely there exist $R > 0$ and $C_P = C_P(p,R) > 0$ such that for all $f \in L^0(X,\mathfrak{m})$ and $g \in \mathrm{UG}^p(f)$,

$$\left(\fint_{B_r} |f - f_{B_r}|^p \,\mathrm{d}\mathfrak{m}\right)^{1/p} \le C_P r \left(\fint_{B_r} g^p \,\mathrm{d}\mathfrak{m}\right)^{1/p}$$

holds for all ball B_r with radius $0 < r < R$). Nevertheless, the constant N appearing in (3.0.2) is only an upper bound on the dimension, so some information related to the dimension of the spaces might get lost. Therefore, following a suggestion of T. Coulhon, we provide in [T17b] the following family of weighted Sobolev inequalities, calling it "adimensional" because the dimension does not appear directly.

Theorem 3.0.24. *Let* $(X, \mathrm{d}, \mathfrak{m})$ *be a PI doubling space with constants* $C_D > 0$, $1 \le p < +\infty$ *and* $C_P > 0$ *such that* (3.0.1) *holds for some* $o \in X$ *and* $p < \eta < \log_2(C_D)$. *Then for any* $f \in L^0(X, \mathfrak{m})$,

$$\left(\int_X f^q w_{p',q}^q \, \mathrm{d}\mathfrak{m}\right)^{1/q} \le C \left(\int_X g^{p'} \, \mathrm{d}\mathfrak{m}\right)^{1/p'} \tag{3.0.11}$$

holds for any $1 \le p' < \eta$, *any* $p \le q \le p^\star := \frac{p \log_2 C_D}{\log_2 C_D - p}$ *and any* $g \in UG^{p'}(f)$. *Here the weight* $w_{p,q}$ *is defined by:*

$$w_{p,q}(x) = \frac{V(o, \mathrm{d}(o,x))^{\frac{1}{p} - \frac{1}{q}}}{\mathrm{d}(o,x)}, \quad \forall x \in X.$$

Chapter 4

Weyl's law on RCD*(K,N) spaces

This chapter presents the main results of [AHT18], which are:

- the pointwise convergence of heat kernels for a convergent sequence of $\mathrm{RCD}^*(K,N)$ spaces (Theorem 4.0.6) which is a generalization of Ding's Riemannian results [D02],

- a sharp criterion for the validity of Weyl's law on compact $\mathrm{RCD}^*(K,N)$ spaces (Theorem 4.0.9). Let us point out that it is not known yet whether there exist $\mathrm{RCD}^*(K,N)$ spaces which do not satisfy this criterion, since all known examples satisfy it.

We conclude with a proof of the expansions of the heat kernel (4.0.21) and (4.0.22) on a compact $\mathrm{RCD}^*(K,N)$ space using eigenvalues and eigenfunctions. This proof is taken from [AHPT17, App. A], but the expansion (4.0.21) already played a major role in [AHT18] in which it was given without sufficiently many details, so we prefer to include it to this chapter.

A brief account on classical Weyl's law type results

Named after H. Weyl [We11] who established it in dimension 2 and 3 in connection with the black-body radiation experiment (see [ANPS09] for a nice historical account), the classical Weyl's law describes the asymptotic behavior of the eigenvalues of the Laplacian on bounded domains of $\mathbb{R}^n$. More precisely, if $\Omega \subset \mathbb{R}^n$ is a bounded domain, standard arguments from the theory of compact operators and elliptic regularity ensure that the spectrum of (minus) the Dirichlet Laplacian on Ω is a discrete sequence of positive numbers $(\lambda_i)_{i\in\mathbb{N}}$ which can be ordered, counting multiplicity, as $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots$ and such that $\lambda_i \to +\infty$ when $i \to +\infty$. Weyl's law states that

$$\lim_{\lambda\to+\infty} \frac{N(\lambda)}{\lambda^{n/2}} = \frac{\omega_n}{(2\pi)^n}\mathcal{L}^n(\Omega)$$

where $N(\lambda) = \sharp\{i \in \mathbb{N} : \ \lambda_i \le \lambda\}$, ω_n is the volume of the n-dimensional Euclidean unit ball, and $\mathcal{L}^n(\Omega)$ is the n-dimensional Lebesgue measure of Ω.

Among the possible generalizations of Weyl's law, one can replace the bounded domain $\Omega \subset \mathbb{R}^n$ by a n-dimensional closed (i.e. compact without boundary) manifold. The Laplacian is then replaced by the Laplace-Beltrami operator of the manifold, and the term $\mathcal{L}^n(\Omega)$ is replaced by $\mathcal{H}^n(M)$, where $\mathcal{H}^n$ denotes the n-dimensional Hausdorff measure. It has been proved by B. Levitan in [Le52] that Weyl's law is still true in that case.

Another generalization concerns compact Riemannian manifolds (M,g) equipped with the distance d induced by the metric g and a measure with positive smooth density e^{-f}

with respect to the volume measure $\mathcal{H}^n$. For such spaces $(M, \mathrm{d}, e^{-f}\mathcal{H}^n)$, called weighted Riemannian manifolds, one has

$$\lim_{\lambda \to +\infty} \frac{N_{(M,\mathrm{d},e^{-f}\mathcal{H}^n)}(\lambda)}{\lambda^{n/2}} = \frac{\omega_n}{(2\pi)^n}\mathcal{H}^n(M), \tag{4.0.1}$$

where $N_{(M,\mathrm{d},e^{-f}\mathcal{H}^n)}(\lambda)$ denotes the counting function of the (weighted) Laplacian $\Delta^f := \Delta - \langle \nabla f, \nabla \cdot \rangle$ of $(M, \mathrm{d}, e^{-f}\mathcal{H}^n)$. This result is a consequence of [Hö68]. We stress that in the asymptotic behavior (4.0.1) the information of the weight, e^{-f}, disappears (as we obtain by different means in Example 4.0.15). This sounds a bit surprising because the Hausdorff dimension is a purely metric notion, whereas the Laplace-Beltrami operator on weighted Riemannian manifolds and more generally the Laplacian on $\mathrm{RCD}^*(K,N)$ spaces does depend on the reference measure.

Eigenvalues and eigenfunctions of compact $\mathrm{RCD}^*(K,N)$ spaces

Thanks to the Cheeger energy Ch, the sequence of eigenvalues can be defined on any metric measure space $(X, \mathrm{d}, \mathfrak{m})$ via Courant's min-max procedure:

$$\lambda_i := \min\left\{ \max_{f \in S,\, \|f\|_{L^2}=1} \mathrm{Ch}(f) : \ S \subset H^{1,2}(X,\mathrm{d},\mathfrak{m}), \ \dim(S) = i \right\} \qquad i \geq 1. \tag{4.0.2}$$

We then define

$$N_{(X,\mathrm{d},\mathfrak{m})}(\lambda) := \#\{i \geq 1 : \ \lambda_i \leq \lambda\}$$

as the "inverse" function of $i \mapsto \lambda_i$. Notice that the formula makes sense even though Ch is not quadratic or equivalently even though Δ is not a linear operator. Moreover, if $\tilde{\mathrm{d}}$ is a distance bi-Lipschitz equivalent to d, meaning that $c^{-1}\tilde{\mathrm{d}} \leq \mathrm{d} \leq c\tilde{\mathrm{d}}$ for some $c > 0$, then $\mathrm{Lip}(X,\mathrm{d}) = \mathrm{Lip}(X,\tilde{\mathrm{d}})$, for any $f \in \mathrm{Lip}(X,\mathrm{d})$ one has $c^{-1}|\nabla f|_{*,\tilde{\mathrm{d}}} \leq |\nabla f|_{*,\mathrm{d}} \leq c|\nabla f|_{*,\tilde{\mathrm{d}}}$, and consequently $c^{-1}\tilde{\mathrm{Ch}} \leq \mathrm{Ch} \leq c\tilde{\mathrm{Ch}}$ where $\tilde{\mathrm{Ch}}$ is the Cheeger energy of $(X, \tilde{\mathrm{d}}, \mathfrak{m})$. Then (4.0.2) shows that the growth rate of $N_{(X,\mathrm{d},\mathfrak{m})}$ does not change if we replace the distance d by $\tilde{\mathrm{d}}$. This observation also holds if we perturb the measure $\mathfrak{m}$ by a factor uniformly bounded away from 0 and $+\infty$. Notice also that if (X,d) is doubling we can always find a Dirichlet form $\mathcal{E}$ with $C^{-1}\mathcal{E} \leq \mathrm{Ch} \leq C\mathcal{E}$, with C depending only on the metric doubling constant, see [ACDM15] (a result previously proved in [Ch99] for PI doubling metric measure spaces). Thus, the replacement of Ch with $\mathcal{E}$ makes the standard tools of Linear Algebra applicable. However, in the case of $\mathrm{CD}(K,\infty)$ spaces with non-linear Laplacian, Weyl's law is still an open question. Note that in this context, the stability of the Krasnoselskii spectrum of the Laplace operator with respect to measured Gromov-Hausdorff convergence has been established by L. Ambrosio, S. Honda and J. Portegies in [AHP18] under a suitable compactness assumption.

When $(X, \mathrm{d}, \mathfrak{m})$ is a compact $\mathrm{RCD}^*(K,N)$ space, the operator Δ is linear and the space $H^{1,2}(X,\mathrm{d},\mathfrak{m})$ is Hilbert. As Rellich-Kondrachov theorem [HK00, Thm. 8.1] implies that the injection $H^{1,2}(X,\mathrm{d},\mathfrak{m}) \hookrightarrow L^2(X,\mathfrak{m})$ is compact, we can apply standard arguments of spectral theory [Bé86] to show the existence of an orthonormal basis of $L^2(X,\mathfrak{m})$ made of eigenfunctions of Δ, namely functions φ_i such that $\Delta\varphi_i = \lambda_i\varphi_i$, with $\varphi_0 \equiv 1/\sqrt{\mathfrak{m}(X)}$ corresponding to $\lambda_0 = 0$. As in the Riemannian case for (minus) the Dirichlet Laplacian, the sequence $(\lambda_i)_i$ can be ordered in increasing order and is such that $\lambda_i \to +\infty$ when $i \to +\infty$.

Technical preliminaries

Before going further, let us recall some technical results. We start with basic differentiation properties of measures.

Proposition 4.0.1. *If μ is a locally finite and nonnegative Borel measure in X and $S \subset X$ is a Borel set, one has*

$$\mu(S) = 0 \quad \Longrightarrow \quad \mu(B_r(x)) = o(r^k) \ \textit{for } \mathcal{H}^k\textit{-a.e. } x \in S. \tag{4.0.3}$$

In addition,

$$\mu(S) = 0, \ S \subset \{x : \limsup_{r \to 0^+} \frac{\mu(B_r(x))}{r^k} > 0\} \quad \Longrightarrow \quad \mathcal{H}^k(S) = 0. \tag{4.0.4}$$

Finally, if $\mu = f\mathcal{H}^k \llcorner S$ with S countably k-rectifiable, one has

$$\lim_{r \to 0^+} \frac{\mu(B_r(x))}{\omega_k r^k} = f(x) \qquad \textit{for } \mathcal{H}^k\textit{-a.e. } x \in S. \tag{4.0.5}$$

Proof. The proof of (4.0.3) and (4.0.4) can be found for instance in [F69, 2.10.19] in a much more general context. See also [AT03, Theorem 2.4.3] for more specific statements and proofs. The proof of (4.0.5) is given in [K94] when $\mu = \mathcal{H}^k \llcorner S$, with S countably k-rectifiable and having locally finite $\mathcal{H}^k$-measure (the proof uses the fact that for any $\varepsilon > 0$ we can cover $\mathcal{H}^k$-almost all of S by sets S_i which are biLipschitz deformations, with biLipschitz constants smaller than $1 + \varepsilon$, of $(\mathbb{R}^i, \|\cdot\|_i)$, for suitable norms $\|\cdot\|_i$). In the general case a simple comparison argument gives the result. □

We shall also need the two next auxiliary results.

Lemma 4.0.2. *Let f_i, g_i, f, $g \in L^1(X, \mathfrak{m})$. Assume that f_i, $g_i \to f$, g $\mathfrak{m}$-a.e. respectively, that $|f_i| \le g_i$ $\mathfrak{m}$-a.e., and that $\lim_{i\to\infty} \|g_i\|_{L^1} = \|g\|_{L^1}$. Then $f_i \to f$ in $L^1(X, \mathfrak{m})$.*

Proof. Obviously $|f| \le g$ $\mathfrak{m}$-a.e. Applying Fatou's lemma for $h_i := g_i + g - |f_i - f| \ge 0$ yields

$$\int_X \liminf_{i\to\infty} h_i \mathrm{d}\mathfrak{m} \le \liminf_{i\to\infty} \int_X h_i \mathrm{d}\mathfrak{m}.$$

Then by assumption the left hand side is equal to $2\|g\|_{L^1}$, and the right hand side is equal to $2\|g\|_{L^1} - \limsup_i \|f_i - f\|_{L^1}$. It follows that $\limsup_i \|f_i - f\|_{L^1} = 0$, which completes the proof. □

The proof of the next classical result can be found, for instance, in [F71, Sec. XIII.5, Theorem 2].

Theorem 4.0.3 (Karamata's Tauberian theorem)**.** *Let ν be a nonnegative and locally finite measure in $[0, +\infty)$ and set*

$$\hat{\nu}(t) := \int_{[0,+\infty)} e^{-\lambda t} \mathrm{d}\nu(\lambda) \qquad t > 0.$$

Then, for all $\gamma > 0$ and $a \in [0, +\infty)$ one has

$$\lim_{t \to 0^+} t^\gamma \hat{\nu}(t) = a \qquad \Longleftrightarrow \qquad \lim_{\lambda \to +\infty} \frac{\nu([0, \lambda])}{\lambda^\gamma} = \frac{a}{\Gamma(\gamma + 1)}.$$

In particular, if $\gamma = k/2$ with k integer, the limit in the right hand side can be written as $a\omega_k/\pi^{k/2}$.

Remark 4.0.4. Following the proofs of Theorems 10.2 and 10.3 in [S79], we prove in [AHT18, Sect. 5] the so-called Abelian one-sided implications and inequalities:

$$\liminf_{t\to 0^+} t^\gamma \hat{\nu}(t) \ge \Gamma(\gamma+1) \liminf_{\lambda\to+\infty} \frac{\nu([0,\lambda])}{\lambda^\gamma}, \tag{4.0.6}$$

$$\limsup_{\lambda\to+\infty} \frac{\nu([0,\lambda])}{\lambda^\gamma} < +\infty \quad\Longrightarrow\quad \limsup_{t\to 0^+} t^\gamma \hat{\nu}(t) \le \Gamma(\gamma+1) \limsup_{\lambda\to+\infty} \frac{\nu([0,\lambda])}{\lambda^\gamma} \tag{4.0.7}$$

as well as the so-called Tauberian one-sided implications and inequalities:

$$\limsup_{\lambda\to+\infty} \frac{\nu([0,\lambda])}{\lambda^\gamma} \le e \limsup_{t\to 0^+} t^\gamma \hat{\nu}(t), \tag{4.0.8}$$

$$\liminf_{t\to 0^+} t^\gamma \hat{\nu}(t) > 0,\ \limsup_{t\to 0^+} t^\gamma \hat{\nu}(t) < +\infty \quad\Longrightarrow\quad \liminf_{\lambda\to+\infty} \frac{\nu([0,\lambda])}{\lambda^\gamma} > 0. \tag{4.0.9}$$

When $(X,\mathrm{d},\mathfrak{m})$ is a $\mathrm{RCD}^*(K,N)$ space, we can apply Karamata's Theorem 4.0.3 to the measure $\mu = \sum_i \delta_{\lambda_i}$ to relate the growth rate of $N_{(X,\mathrm{d},\mathfrak{m})}(\lambda) = \nu([0,\lambda])$ with the behavior of $\hat{\nu}(t) = \sum_i e^{-\lambda_i t}$ when $t \downarrow 0$. Assuming compactness of (X,d), the expansion (4.0.21) implies that

$$\hat{\nu}(t) = \int_X p(x,x,t)\,\mathrm{d}\mathfrak{m} \qquad \forall t > 0,$$

reducing the focus to the short-time behavior of the above right-hand side.

Pointwise convergence of heat kernels

Our approach is based on a blow-up procedure. To explain it, let us fix a pointed measured Gromov-Hausdorff convergent sequence $(X_i,\mathrm{d}_i,\mathfrak{m}_i,x_i) \stackrel{mGH}{\to} (X,\mathrm{d},\mathfrak{m},x)$ of $\mathrm{RCD}^*(K,N)$-spaces. Recall that by [GMS15, Prop. 3.30], we can adopt the extrinsic point of view on this convergence, embedding all spaces into a doubling and complete metric space (Y,d_Y) by isometric embeddings $\psi_i : X_i \hookrightarrow Y$, $\psi : X \hookrightarrow Y$ such that $\mathrm{d}_Y(\psi_i(x_i),\psi(x)) \to 0$ and $(\psi_i)_\sharp \mathfrak{m}_i \stackrel{C_{\mathrm{bs}}(Y)}{\rightharpoonup} (\psi)_\sharp \mathfrak{m}$ as $i \to \infty$. Without any loss of generality, we can assume that $\mathrm{supp}(\mathfrak{m}_i) = X_i$ for any i, and identify the spaces $(X_i,\mathrm{d}_i,\mathfrak{m}_i,x_i)$ with their image by φ_i, namely $(\varphi_i(X_i), \mathrm{d}_{|\varphi_i(X_i)}, (\varphi_i)_\#\mathfrak{m}_i, \varphi_i(x_i))$. For simplicity, we can also assume that the doubling and complete space (Y,d_Y) is (X,d), in which case each space X_i is a subset of X supporting the measure $\mathfrak{m}_i$, and we have $\mathfrak{m}_i \stackrel{C_{\mathrm{bs}}(X)}{\rightharpoonup} \mathfrak{m}$. We shall denote $y_i \stackrel{GH}{\longrightarrow} y$ whenever a sequence $y_i \in X_i$ is such that $\mathrm{d}(y_i,y) \to 0$.

Let us start with a crucial technical proposition which allows to turn L^2-weak/strong convergence into pointwise convergence.

Proposition 4.0.5. *Let $f_i \in C(X_i)$ and $f \in C(X)$. Assume (X,d) proper and*

$$\sup_i \sup_{X_i \cap B_R(x_i)} |f_i| < +\infty \qquad \forall R > 0.$$

Assume moreover that $\{f_i\}_i$ is locally equi-continuous, i.e. for any $\epsilon > 0$ and any $R > 0$ there exists $\delta > 0$ independent of i such that

$$(y,z) \in (X_i \cap B_R(x_i))^2 \ \ \mathrm{d}(y,z) < \delta \quad\Longrightarrow\quad |f_i(y) - f_i(z)| < \epsilon. \tag{4.0.10}$$

Then the following are equivalent:

(1) $\lim_{k\to\infty} f_{i(k)}(y_{i(k)}) = f(y)$ *whenever* $y \in \mathrm{supp}\,\mathfrak{m}$, $i(k) \to \infty$ *and* $X_{i(k)} \ni y_{i(k)} \stackrel{GH}{\longrightarrow} y$,

(2) f_i *$L^2_{\rm loc}$-weakly converge to* f*,*

(3) f_i *$L^2_{\rm loc}$-strongly converge to* f*.*

Proof. We prove the implication from (1) to (3) and from (2) to (1), since the implication from (3) to (2) is trivial.

Assume that (2) holds, let $\epsilon > 0$ and let $y_i \to y$. Take ζ nonnegative, with support contained in $B_\delta(y)$ and with $\int \zeta d\mathfrak{m} = 1$. Thanks to (4.0.10) and the continuity of f, for δ sufficiently small we have

$$(f_i(y_i) - \epsilon) \int \zeta d\mathfrak{m}_i \leq \int \zeta f_i d\mathfrak{m}_i \leq (f_i(y_i) + \varepsilon) \int \zeta d\mathfrak{m}_i \qquad f(y) - \epsilon \leq \int \zeta f d\mathfrak{m} \leq f(y) + \varepsilon$$

Since $\int \zeta f_i d\mathfrak{m}_i \to \int \zeta f d\mathfrak{m}$ and $\int \zeta d\mathfrak{m}_i \to \int \zeta d\mathfrak{m} = 1$, from the arbitrariness of ε we obtain that $f_i(y_i) \to f(y)$. A similar argument, for arbitrary subsequences, gives (1).

In order to prove the implication from (1) to (3) we prove the implication from (1) to (2). Assuming with no loss of generality that f_i and f are nonnegative, for any $\zeta \in C_{\rm bs}(X)$ nonnegative, (1) and the compactness of the support of ζ give that for any $\varepsilon > 0$ and any $s > 0$ the set $X_i \cap \{f_i\zeta > s\}$ is contained in the ϵ-neighbourhood of $\{f\zeta > s\}$ for i large enough, so that

$$\limsup_{i \to \infty} \mathfrak{m}_i(\{f_i\zeta > s\}) \leq \mathfrak{m}(\{f\zeta \geq s\}).$$

Analogously, any open set $A \Subset \{f\zeta > s\}$ is contained for i large enough in the set $\{f_i\zeta > s\} \cup (X \setminus X_i)$, so that

$$\liminf_{i \to \infty} \mathfrak{m}_i(\{f_i\zeta > s\}) \geq \mathfrak{m}(\{f\zeta > s\}).$$

Combining these two informations, Cavalieri's formula and the dominated convergence theorem provide $\int_X f_i\zeta d\mathfrak{m}_i \to \int_X f\zeta d\mathfrak{m}$ and then, since ζ is arbitrary, (2).

Now we can prove the implication from (1) to (3). Thanks to the equiboundedness assumption, the sequence $g_i := f_i^2$ is locally equi-continuous as well and g_i pointwise converge to $g := |f|^2$ in the sense of (1), applying the implication from (1) to (2) for g_i gives

$$\lim_{i \to \infty} \int_{X_i} \zeta^2 f_i^2 d\mathfrak{m}_i = \int_X \zeta^2 f^2 d\mathfrak{m} \qquad \forall \zeta \in C_{\rm bs}(X),$$

which yields (3). □

Here is our generalization/refinement of Ding's results [D02, Theorems 2.6, 5.54 and 5.58] from the Ricci limit setting to our setting, via a different approach.

Theorem 4.0.6 (Pointwise convergence of heat kernels)**.** *The heat kernels* p_i *of* $(X_i, {\rm d}_i, \mathfrak{m}_i)$ *satisfy*

$$\lim_{i \to \infty} p_i(x_i, y_i, t_i) = p(x, y, t)$$

whenever $(x_i, y_i, t_i) \in X_i \times X_i \times (0, +\infty) \to (x, y, t) \in \operatorname{supp} \mathfrak{m} \times \operatorname{supp} \mathfrak{m} \times (0, +\infty)$*.*

Proof. By rescaling ${\rm d} \to (t/t_i)^{1/2}{\rm d}$, without any loss of generality we can assume that $t_i \equiv t$. Let $f \in C_{\rm bs}(X)$. Recall that, viewing f as an element of $L^2 \cap L^\infty(X_i, \mathfrak{m}_i)$, Theorem 2.4.20 provides L^2-strong convergerce of $h_t^i f$ to $h_t f$. By the estimate [AGS14b, Theorem 6.5] valid in all ${\rm RCD}(K, \infty)$ spaces, defining $I_0(t) := t$ and $I_S(t) := (e^{St} - 1)/S$ for $S \neq 0$, we have

$$\sqrt{2I_{2K}(t)} {\rm Lip}(h_t^i f, \operatorname{supp} \mathfrak{m}) \leq \|f\|_{L^\infty(X, \mathfrak{m})},$$

so the functions $h_t^i f$ are equi-Lipschitz on X. Applying Proposition 4.0.5 yields then $h_t^i f(y_i) \to h_t f(y)$ for any $y_i \xrightarrow{GH} y$.

On the other hand, the Gaussian estimate (2.3.6) shows that $\sup_i \|p_i(\cdot, y_i, t)\|_{L^\infty} < \infty$. By definition, since

$$h_t f(y_i) = \int_{X_i} p_i(z, y_i, t) f(z) \mathrm{d}\mathfrak{m}_i(z), \qquad h_t f(y) = \int_X p(z, y, t) f(z) \mathrm{d}\mathfrak{m}(z),$$

we see that $p_i(\cdot, y_i, t)$ L^2_{loc}-weakly converge to $p(\cdot, y, t)$. Moreover, since thanks to (2.3.9) the functions $p_i(\cdot, y_i, t)$ are locally equi-Lipschitz continuous, choosing any continuous extension of $p(\cdot, y, t)$ to the whole of X and applying Proposition 4.0.5 once more to $p_i(\cdot, y_i, t)$ we obtain that $p_i(x_i, y_i, t)$ converge to $p(x, y, t)$ for any $x_i \xrightarrow{GH} x$, which completes the proof. □

We deduce an important corollary concerning the heat kernel of a fixed $\mathrm{RCD}^*(K, N)$ space $(X, \mathrm{d}, \mathfrak{m})$. Recall Definition 2.3.13 in which regular sets of $(X, \mathrm{d}, \mathfrak{m})$ were introduced.

Corollary 4.0.7 (Short time diagonal behavior of heat kernel on the regular set)**.** *Let $(X, \mathrm{d}, \mathfrak{m})$ be a $\mathrm{RCD}^*(K, N)$ space with $K \in \mathbb{R}$ and $N \in (1, +\infty)$. Set $n = \dim_{\mathrm{d},\mathfrak{m}}(X)$. Then*

$$\lim_{t \to 0^+} \mathfrak{m}(B_{t^{1/2}}(x)) p(x, x, t) = \frac{\omega_n}{(4\pi)^{n/2}} \tag{4.0.11}$$

for any n-dimensional regular point x of $(X, \mathrm{d}, \mathfrak{m})$.

Proof. Let us recall that for any $r > 0$ and any $C > 0$ the heat kernel $\hat{p}(x, y, t)$ of the rescaled $\mathrm{RCD}^*(r^2 K, N)$ space $(X, r^{-1}\mathrm{d}, C\mathfrak{m})$ is given by $\hat{p}(x, y, t) = C^{-1} p(x, y, r^2 t)$. Applying this for $r := t^{1/2}$, $C := \frac{1}{\mathfrak{m}(B_t(x))}$ with Theorem 4.0.6 shows

$$\lim_{t \to 0^+} \mathfrak{m}(B_{t^{1/2}}(x)) p(x, x, t) = \lim_{t \to 0^+} p^t(x, x, 1) = p_{\mathbb{R}^n}(0_n, 0_n, 1) = \frac{\omega_n}{(4\pi)^{n/2}},$$

where p^t, $p_{\mathbb{R}^n}$ denote the heat kernels of $(X, t^{-1/2}\mathrm{d}, \frac{\mathfrak{m}}{\mathfrak{m}(B_{t^{1/2}}(x))})$, $\left(\mathbb{R}^n, \mathrm{d}_{\mathbb{R}^n}, \frac{\mathcal{H}^n}{\omega_n}\right)$, respectively. □

Weyl's law

Let us discuss now Weyl's law on compact $\mathrm{RCD}^*(K, N)$ spaces $(X, \mathrm{d}, \mathfrak{m})$. As the example of weighted Riemannian manifolds suggests, the asymptotic behavior of the eigenvalues of the Laplacian is related to the behavior of the Hausdorff measure on regular sets with respect to the restriction of the reference measure. Recall that by Theorem (2.3.16), denoting by n the dimension of $(X, \mathrm{d}, \mathfrak{m})$, we have $\mathfrak{m} \llcorner \mathcal{R}_n \ll \mathscr{H}^n$, and the set $\mathcal{R}_n^* \subset \mathcal{R}_n$ made of those points at which the density θ of $\mathfrak{m}$ w.r.t. $\mathscr{H}^n$ is finite and non-zero is such that $\mathfrak{m}(\mathcal{R}_n \setminus \mathcal{R}_n^*) = 0$.

We are now in a position to introduce a first criterion. We always have $\mathcal{H}^n(\mathcal{R}_n^*) > 0$ and, if an assumption slightly stronger than the finiteness of n-dimensional Hausdorff measure holds, we obtain Weyl's law in the weak asymptotic form. For simplicity we use the following notation: $f(\lambda) \sim g(\lambda)$ if there exists $C > 1$ satisfying $C^{-1} f(\lambda) \le g(\lambda) \le C f(\lambda)$ for sufficiently large λ.

Theorem 4.0.8. *Let $(X, \mathrm{d}, \mathfrak{m})$ be a compact $\mathrm{RCD}^*(K, N)$ space with $K \in \mathbb{R}$ and $N \in (1, +\infty)$, let $n = \dim_{\mathrm{d},\mathfrak{m}}(X)$ and let $\mathcal{R}_n^*$ be as in (2.3.16) of Theorem 2.3.16. Then we have*

$$\liminf_{t \to 0^+} \left(t^{n/2} \sum_i e^{-\lambda_i t} \right) \ge \frac{1}{(4\pi)^{n/2}} \mathcal{H}^n(\mathcal{R}_n^*) > 0. \tag{4.0.12}$$

In particular, if $N_{(X,\mathsf{d},\mathfrak{m})}(\lambda) \sim \lambda^i$ *as* $\lambda \to +\infty$ *for some* i*, then Remark 4.0.4 gives* $i \geq n/2$*. In addition*

$$\limsup_{s\to 0^+} \int_X \frac{s^n}{\mathfrak{m}(B_n(x))} \mathrm{d}\mathfrak{m}(x) < +\infty \quad \iff \quad N_{(X,\mathsf{d},\mathfrak{m})}(\lambda) \sim \lambda^{n/2} \ (\lambda \to +\infty). \tag{4.0.13}$$

Proof. In order to prove (4.0.12) we first notice that the combination of (4.0.11) and (2.3.18) gives

$$\lim_{t\to 0^+} t^{n/2} p(x,x,t) = \frac{1}{(4\pi)^{n/2}} \chi_{\mathcal{R}_n^*}(x) \frac{\mathrm{d}\mathcal{H}^n \llcorner \mathcal{R}_n^*}{\mathrm{d}\mathfrak{m} \llcorner \mathcal{R}_n^*}(x) \quad \text{for } \mathfrak{m}\text{-a.e. } x \in X.$$

Using the identity $t^{n/2} \sum_i e^{-\lambda_i t} = \int_X t^{n/2} p(x,x,t) \mathrm{d}\mathfrak{m}(x)$ and Fatou's lemma we obtain

$$\liminf_{t\to 0} \left(t^{n/2} \sum_i e^{-\lambda_i t} \right) \geq \frac{1}{(4\pi)^{n/2}} \int_{\mathcal{R}_n^*} \frac{\mathrm{d}\mathcal{H}^n \llcorner \mathcal{R}_n^*}{\mathrm{d}\mathfrak{m} \llcorner \mathcal{R}_n^*} \mathrm{d}\mathfrak{m} = \frac{1}{(4\pi)^{n/2}} \mathcal{H}^n(\mathcal{R}_n^*).$$

The heat kernel estimate (2.3.6) shows

$$C^{-1} \frac{t^{n/2}}{\mathfrak{m}(B_{t^{1/2}}(x))} \leq t^{n/2} p(x,x,t) \leq C \frac{t^{n/2}}{\mathfrak{m}(B_{t^{1/2}}(x))} \tag{4.0.14}$$

for some $C > 1$, which is independent of t and x. Thus the upper bound on p gives

$$\limsup_{t\to 0^+} t^{n/2} \int_X p(x,x,t) \mathrm{d}\mathfrak{m}(x) \leq C \limsup_{s\to 0^+} \int_X \frac{s^n}{\mathfrak{m}(B_s(x))} \mathrm{d}\mathfrak{m}(x) < +\infty.$$

We can now invoke Remark 4.0.4 to obtain the implication $\Rightarrow$ in (4.0.13). The proof of the converse implication is similar and uses the lower bound in (4.0.14). □

Under the stronger assumption (4.0.15) (notice that both the finiteness of the limit and the equality of the integrals are part of the assumption) we can recover Weyl's law in the stronger form.

Theorem 4.0.9. *Let* $(X,\mathsf{d},\mathfrak{m})$ *be a compact* $\mathrm{RCD}^*(K,N)$ *space with* $K \in \mathbb{R}$ *and* $N \in (1,+\infty)$*, and let* $n = \dim_{\mathsf{d},\mathfrak{m}}(X)$*. Then*

$$\lim_{s\to 0^+} \int_X \frac{s^n}{\mathfrak{m}(B_s(x))} \mathrm{d}\mathfrak{m}(x) = \int_X \lim_{s\to 0^+} \frac{s^n}{\mathfrak{m}(B_s(x))} \mathrm{d}\mathfrak{m}(x) < +\infty \tag{4.0.15}$$

if and only if

$$\lim_{\lambda\to+\infty} \frac{N_{(X,\mathsf{d},\mathfrak{m})}(\lambda)}{\lambda^{n/2}} = \frac{\omega_n}{(2\pi)^n} \mathcal{H}^n(\mathcal{R}_n^*) < +\infty. \tag{4.0.16}$$

Proof. We first assume that (4.0.15) holds. Taking (2.3.18) and (4.0.14) into account, we can apply Lemma 4.0.2 with $f_t(x) = t^{n/2} p(x,x,t)$ and $g_t(x) = Ct^{n/2}/\mathfrak{m}(B_{t^{1/2}}(x))$ to get

$$\begin{aligned} \lim_{t\to 0^+} t^{n/2} \int_X p(x,x,t) \mathrm{d}\mathfrak{m}(x) &= \int_X \lim_{t\to 0^+} t^{n/2} p(x,x,t) \mathrm{d}\mathfrak{m}(x) \\ &= \int_{\mathcal{R}_n^*} \frac{1}{(4\pi)^{n/2}} \frac{\mathrm{d}\mathcal{H}^n \llcorner \mathcal{R}_n^*}{\mathrm{d}\mathfrak{m} \llcorner \mathcal{R}_n^*} \mathrm{d}\mathfrak{m} \\ &= \frac{1}{(4\pi)^{n/2}} \mathcal{H}^n(\mathcal{R}_n^*) \end{aligned}$$

which shows (4.0.16) by Karamata's Tauberian theorem.

Next we assume that (4.0.16) holds. Then by (2.3.18) and Karamata's Tauberian theorem again, (4.0.16) is equivalent to

$$\lim_{t\to 0^+} t^{n/2}\int_X p(x,x,t)\mathrm{d}\mathfrak{m}(x) = \int_X \lim_{t\to 0^+} t^{n/2}p(x,x,t)\mathrm{d}\mathfrak{m}(x) < +\infty. \tag{4.0.17}$$

Let $f_t(x) := t^{n/2}/\mathfrak{m}(B_{t^{1/2}}(x))$. Then the heat kernel estimate (4.0.14) shows that we can apply Lemma 4.0.2 with $g_t(x) = Ct^{n/2}p(x,x,t)$ to get (4.0.15). □

By the stability of RCD conditions with respect to mGH-convergence and [CC97, Theorem 5.1], noncollapsed Ricci limit spaces give typical examples of $\mathrm{RCD}^*(K,N)$ spaces $(X,\mathrm{d},\mathfrak{m})$ with $\dim_{\mathrm{d},\mathfrak{m}}X = N$. For such metric measure spaces Weyl's law was proven in [D02] by Ding. Thus the following corollary also recovers his result.

Corollary 4.0.10. *Let $(X,\mathrm{d},\mathfrak{m})$ be a compact $\mathrm{RCD}^*(K,N)$ space with $K\in\mathbb{R}$ and $N\in(1,+\infty)\cap\mathbb{N}$, and assume that $N=\dim_{\mathrm{d},\mathfrak{m}}X$. Then (4.0.16) holds.*

Proof. The existence of a functions $g\in L^1(\mathcal{R}^*_N,\mathcal{H}^N)$ such that

$$g(x,t) := \frac{t^N}{\mathfrak{m}(B_t(x))}\frac{\mathrm{d}\mathfrak{m}\llcorner\mathcal{R}^*_N}{\mathrm{d}\mathcal{H}^N\llcorner\mathcal{R}^*_N}(x) \le g(x) \qquad \forall t\in(0,1)$$

for $\mathcal{H}^N$-a.e. $x\in\mathcal{R}^*_N$ follows directly from the Bishop-Gromov inequality, since $\mathfrak{m}(B_r(x))/r^k$ is bounded from below by a positive constant. Then the proof follows by the dominated convergence theorem in conjunction with Theorem 4.0.9. □

Applications

Let us provide a serie of examples to which Theorem 4.0.9 can be applied.

Example 4.0.11. Let us consider the following $\mathrm{RCD}^*(N-1,N)$ space:

$$(X,\mathrm{d},\mathfrak{m}) := \left([0,\pi],\mathrm{d}_{[0,\pi]},\sin^{N-1}t\mathrm{d}t\right)$$

for $N\in(1,\infty)$ (note that this is a Ricci limit space if N is an integer, see for instance [AH17a]). Then we can apply Theorem 4.0.9 with $k=1$ and $\mathcal{R}^*_1=\mathcal{R}_1=(0,\pi)$, because of $\sup_{t<1}\|g(\cdot,t)\|_{L^\infty}<\infty$, where g is as in Corollary 4.0.10. Thus we have Weyl's law:

$$\lim_{\lambda\to+\infty}\frac{N_{(X,\mathrm{d},\mathfrak{m})}(\lambda)}{\lambda^{1/2}} = \frac{\omega_1}{2\pi}\mathcal{H}^1((0,\pi)) = 1.$$

Example 4.0.12 (Iterated suspensions)**.** Let us apply now Theorem 4.0.9 to iterated suspensions of $(X,\mathrm{d},\mathfrak{m})$ as in Example 4.0.11:

$$\begin{cases}(X_1,\mathrm{d}_1,\mathfrak{m}_1) := ([0,\pi],\mathrm{d}_{[0,\pi]},\sin^{N-1}t\mathrm{d}t),\\ (X_{n+1},\mathrm{d}_{n+1},\mathfrak{m}_{n+1}) := ([0,\pi],\mathrm{d}_{[0,\pi]},\sin t\mathrm{d}t)\times^1(X_n,\mathrm{d}_n,\mathfrak{m}_n).\end{cases}$$

Recall that the spherical suspension $([0,\pi],\mathrm{d}_{[0,\pi]},\sin t\mathrm{d}t)\times^1(X,\mathrm{d},\mathfrak{m})$ of a metric measure space $(X,\mathrm{d},\mathfrak{m})$ is the quotient of the product $[0,\pi]\times X$ by the identification of every point of $\{0\}\times X$ and $\{\pi\}\times X$ into two distinct points, equipped with the product measure $\mathrm{d}\mu := \sin t\mathrm{d}t\times\mathfrak{m}$ and with the distance $\mathrm{d}_{\mathrm{susp}}$ defined by

$$\cos\mathrm{d}_{\mathrm{susp}}((t,x),(s,y)) = \cos t\cos s + \sin t\sin s\cos(\min\{\mathrm{d}(x,y),\pi\}).$$

Note that $(X_n, \mathrm{d}_n, \mathfrak{m}_n)$ is a $\mathrm{RCD}^*(N+n-2, N+n-1)$ space (see [K15b]) and that (X_n, d_n) are isometric to a hemisphere of the n-dimensional unit sphere $\mathbb{S}^n$ as metric spaces.

Then we can apply Theorem 4.0.9 because an elementary calculation similar to the one of Example 4.0.11 shows that $\sup_{t<1} \|g_n(\cdot, t)\|_{L^\infty} < \infty$. Thus Weyl's law follows:

$$\lim_{\lambda \to +\infty} \frac{N_{(X_n \mathrm{d}_n, \mathfrak{m}_n)}(\lambda)}{\lambda^{n/2}} = \frac{\omega_n}{(2\pi)^n} \mathcal{H}^n(X_n) = \frac{\omega_n}{(2\pi)^n} \frac{\mathcal{H}^n(\mathbb{S}^n)}{2}.$$

Example 4.0.13 (Gaussian spaces)**.** For noncompact $\mathrm{RCD}(K, \infty)$ spaces the behavior of the spectrum is different, and requires a more delicate analysis. For instance (see [Mil15, (2.2)]) the n-dimensional Gaussian space $(X, \mathrm{d}, \mathfrak{m}) := (\mathbb{R}^n, \mathrm{d}_{\mathbb{R}^n}, \gamma_n)$ satisfies

$$\lim_{\lambda \to +\infty} \frac{N_{(X,\mathrm{d},\mathfrak{m})}(\lambda)}{\lambda^n} = \frac{1}{\Gamma(n+1)}.$$

Theorem 4.0.9 implies the following corollary for Ahlfors regular $\mathrm{RCD}^*(K, N)$ spaces.

Corollary 4.0.14 (Weyl's law on compact Ahlfors regular $\mathrm{RCD}^*(K, N)$ spaces - especially Alexandrov spaces)**.** *Let* $(X, \mathrm{d}, \mathfrak{m})$ *be a compact* $\mathrm{RCD}^*(K, N)$ *space with* $K \in \mathbb{R}$ *and* $N \in (1, +\infty)$. *Assume that* $(X, \mathrm{d}, \mathfrak{m})$ *is Ahlfors n-regular for some* $n \in \mathbb{N}$, *i.e. there exists* $C > 1$ *such that*

$$C^{-1} r^n \le \mathfrak{m}(B_r(x)) \le C r^n \qquad \forall x \in X,\ r \in (0, 1).$$

Then we have Weyl's law:

$$\lim_{\lambda \to +\infty} \frac{N_{(X,\mathrm{d},\mathfrak{m})}(\lambda)}{\lambda^{n/2}} = \frac{\omega_n}{(2\pi)^n} \mathcal{H}^n(X). \tag{4.0.18}$$

In particular this holds if $(X, \mathrm{d}, \mathfrak{m})$ *is an n-dimensional compact Alexandrov space.*

Proof. Note that by the Ahlfors n-regularity of $(X, \mathrm{d}, \mathfrak{m})$, any tangent cone at x also satisfies the Ahlfors n-regularity, which implies that $\mathcal{R}_i = \emptyset$ for any $i \neq n$. In particular since $\mathcal{H}^n \ll \mathfrak{m} \ll \mathcal{H}^n$, we have

$$\mathfrak{m}(X \setminus \mathcal{R}_n) = \mathcal{H}^n(X \setminus \mathcal{R}_n) = 0. \tag{4.0.19}$$

Then Theorem 4.0.9 can be applied with $g_k \equiv c$ for some $c > 0$, which proves (4.0.18) by (4.0.19). The final statement follows from the compatibility between Alexandrov spaces and RCD spaces [Pet11, ZZ10]. □

Example 4.0.15. Let us discuss the simplest case we can apply Corollary 4.0.14; let M be a compact n-dimensional manifold and let $f \in C^2(M)$. Then, thanks to (2.3.15), for any $N \in (n, \infty)$ there exists $K \in \mathbb{R}$ such that $(M, \mathrm{d}, e^{-f}\mathcal{H}^n)$ is a $\mathrm{RCD}^*(K, N)$ space. Moreover since $\left(M, \mathrm{d}, e^{-f}\mathcal{H}^n\right)$ is Ahlfors n-regular, Corollary 4.0.14 yields Weyl's law:

$$\lim_{\lambda \to +\infty} \frac{N_{(M,\mathrm{d},e^{-f}\mathcal{H}^n)}(\lambda)}{\lambda^{n/2}} = \frac{\omega_n}{(2\pi)^n} \mathcal{H}^n(M).$$

In order to give another application of Weyl's law on compact finite dimensional Alexandrov spaces, let us recall that two compact finite dimensional Alexandrov spaces are said to be *isospectral* if the spectrums of their Laplacians coincide. See for instance [S85, EW13] for constructions of isospectral manifolds and of isospectral Alexandrov spaces (see also [KMS01] for analysis on Alexandrov spaces).

It is also well-known as a direct consequence of Perelman's stability theorem [Per91] (see also [Kap07]) that for fixed $n \in \mathbb{N}$, $K \in \mathbb{R}$ and d, $v > 0$ the isometry class of n-dimensional compact Alexandrov spaces X of sectional curvature bounded below by K with $\operatorname{diam} X \le d$ and $\mathcal{H}^n(X) \ge v$ has only finitely many topological types. By using this and Weyl's law, we can prove the following result which is a generalization of topological finiteness results for isospectral spaces proven in [BPP92, Stan05, Har16] to Alexandrov spaces.

Corollary 4.0.16 (Topological finiteness theorem for isospectral Alexandrov spaces)**.** *Let $\chi := \{(X_u, \mathrm{d}_u, \mathcal{H}^{n_u})\}_{u \in U}$ be a class of compact finite dimensional Alexandrov spaces with a uniform sectional curvature bound from below. Assume that there exists $C > 1$ such that*

$$\limsup_{\lambda \to +\infty} \frac{N_{(X_u, \mathrm{d}_u, \mathcal{H}^{n_u})}(\lambda)}{N_{(X_v, \mathrm{d}_v, \mathcal{H}^{n_v})}(\lambda)} \le C \tag{4.0.20}$$

for all u, $v \in U$. Then χ has only finitely many topological types.

In particular, any class of isospectral compact finite dimensional Alexandrov spaces with a uniform sectional curvature bound from below has only finitely many members up to homeomorphism.

Proof. By an argument similar to the proof of [BPP92, Corollary 1.2] (or [Stan05, Proposition 7.4]) with [VR04, Corollary 1] there exists $d > 0$ such that $\operatorname{diam} X_u \le d$ for any $u \in U$. Since Weyl's law (4.0.18) with (4.0.20) implies that there exist $n \in \mathbb{N}$ and $v > 0$ such that $\dim X_u \equiv n$ and $\mathcal{H}^n(X_u) \ge v$ for any $u \in U$, the topological finiteness result stated above completes the proof. □

Expansions of the heat kernel

Throughout this paragraph we assume that $(X, \mathrm{d}, \mathfrak{m})$ is a metric measure space with (X, d) compact, $\mathfrak{m}(X) = 1$ (this is not restrictive, up to a normalization) and $\operatorname{supp} \mathfrak{m} = X$. We denote by D the diameter of (X, d).

The main aim of this paragraph is to provide a complete proof of the expansions

$$p(x, y, t) = \sum_{i \ge 0} e^{-\lambda_i t} \varphi_i(x) \varphi_i(y) \qquad \text{in } C(X \times X) \tag{4.0.21}$$

for any $t > 0$ and

$$p(\cdot, y, t) = \sum_{i \ge 0} e^{-\lambda_i t} \varphi_i(y) \varphi_i \qquad \text{in } H^{1,2}(X, \mathrm{d}, \mathfrak{m}) \tag{4.0.22}$$

for any $y \in X$ and $t > 0$, where p denotes the locally Hölder representative of the heat kernel in the case when, in addition, $(X, \mathrm{d}, \mathfrak{m})$ is $\mathrm{RCD}^*(K, N)$ space. Our goal is to justify the convergence of the series in (4.0.21) and (4.0.22): as soon as this is secured, a standard argument shows that they provide good representatives of the heat kernel. Here and in the sequel $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \to +\infty$ are the eigenvalues of $-\Delta$, and $\varphi_0, \varphi_1, \varphi_2, \ldots$ are corresponding eigenfunctions forming an orthonormal basis of $L^2(X, \mathfrak{m})$, with $\varphi_0 \equiv 1$.

The following proposition is a consequence of [HK00, Th. 5.1 and Th. 9.7].

Proposition 4.0.17. *Assume, in addition, that $(X, \mathrm{d}, \mathfrak{m})$ is a PI space, with doubling constant $C_D \le 2^N$ for some $N > 2$ and Poincaré constant C_P. Then there exists a constant $C_S = C_S(N, C_P, D) > 0$ such that*

$$\left(\int_X |f - f_X|^{2N/(N-2)} \, \mathrm{d}\mathfrak{m} \right)^{(N-2)/2N} \le C_S \left(\int_X |\nabla f|^2 \, \mathrm{d}\mathfrak{m} \right)^{1/2}$$

for any $f \in H^{1,2}(X, \mathrm{d}, \mathfrak{m})$, where f_X denotes the mean-value of f over X.

The following result, well-known for compact Riemannian manifolds, provides by Moser's iteration technique a polynomial lower bound for the eigenvalues of $-\Delta$. The estimate we provide is not sharp, but sufficient for our purposes.

Proposition 4.0.18. *Assuming that $(X, \mathrm{d}, \mathfrak{m})$ is a $\mathrm{RCD}^*(K, N)$ space, there exists a constant $C_0 = C_0(D, K, N) > 0$ such that*

$$\lambda_i \geq C_0 i^{1/N} \qquad \forall i \geq 1.$$

Proof. Take $i \geq 1$, write $E_i = \mathrm{Span}(\varphi_1, \ldots, \varphi_i)$ and recall that, under our assumptions, we can and will use the continuous version of the φ_i, which are even Lipschitz [J16]. We claim that there exists $f_o \in E_i$ such that $\sup f_o^2 \geq i$ and $\|f_o\|_2 = 1$. Let us define the continuous function $F = \sum_{j=1}^{i} \varphi_j^2$ and let $p \in X$ be a maximum point of F. Then

$$f_o(x) := \frac{1}{\sqrt{F(p)}} \sum_{j=1}^{i} \varphi_j(p)\varphi_j(x)$$

satisfies $\|f_o\|_2 = 1$ and $f_o(p) = \sqrt{F(p)}$, so that

$$i = \dim E_i = \int_X F \, \mathrm{d}\mathfrak{m} \leq F(p) \leq \sup f_o^2. \tag{4.0.23}$$

We claim now that there exists $C_1 > 0$ depending only on K and N such that

$$\sup |f| \leq C_1 \lambda_i^{N/2} \|f\|_2 \qquad \forall f \in E_i. \tag{4.0.24}$$

Using this claim with $f = f_o$ together with (4.0.23), we obtain the stated lower bound on the λ_i.

As one can easily check, $\Delta f^2 = 2f\Delta f + 2|\nabla f|^2 \geq 2f\Delta f$ for any $f \in E_i$, so that for $k \geq k_0 = 2N/(N-2) > 1$, we estimate

$$\begin{aligned}
\int_X |f|^{k-2} f \Delta f \, \mathrm{d}\mathfrak{m} &\leq \frac{1}{2}\int_X |f|^{k-2}\Delta f^2 \, \mathrm{d}\mathfrak{m} = -\frac{1}{2}\int_X \langle \nabla |f|^{k-2}, \nabla f^2 \rangle \, \mathrm{d}\mathfrak{m} \\
&= -(k-2)\int_X |f|^{k-2} |\nabla |f||^2 \, \mathrm{d}\mathfrak{m} = -\frac{4(k-2)}{k^2}\int_X |\nabla |f|^{k/2}|^2 \, \mathrm{d}\mathfrak{m} \\
&\leq -\frac{4(k-2)}{k^2 C_S}\left(\int_X |f|^{kN/(N-2)} \, \mathrm{d}\mathfrak{m}\right)^{(N-2)/N} \\
&\leq -\frac{1}{kC}\left(\int_X |f|^{kN/(N-2)} \, \mathrm{d}\mathfrak{m}\right)^{(N-2)/N},
\end{aligned}$$

where we used Proposition 4.0.17 (note that $f_X = 0$ for any $f \in E_i$, and that $E_i \subset L^\infty(X, \mathfrak{m})$, so that $|f|^{k/2} \in H^{1,2}(X, \mathrm{d}, \mathfrak{m})$) and C is chosen in such a way that $4(k-2)/(kC_S) \geq 1/C$ for all $k \geq k_0$. Thus, setting $\beta = N/(N-2) > 1$, we get

$$\|f\|_{\beta k}^k \leq kC \int_X |f|^{k-1} |\Delta f| \, \mathrm{d}\mathfrak{m} \leq kC \|f\|_k^{k-1} \|\Delta f\|_{\beta k}, \tag{4.0.25}$$

by Hölder's inequality. A simple reasoning [L12, p. 101] shows that if $h \in E_i$ is such that

$$\frac{\|h\|_{\beta k}}{\|h\|_2} = \max_{f \in E_i \setminus \{0\}} \frac{\|f\|_{\beta k}}{\|f\|_2},$$

then $\|\Delta h\|_{\beta k} \le \lambda_i \|h\|_{\beta k}$, so that (4.0.25) with $f = h$ implies $\|h\|_{\beta k}^{k-1} \le kC\lambda_i \|h\|_k^{k-1}$. Notice that, as soon as C is chosen in such a way that $C \ge C_S$, the inequality holds also in the case $k = 2$. Therefore with $k_j = 2\beta^j$, $j \ge 0$, by induction, we get

$$\max_{f \in E_i \setminus \{0\}} \frac{\|f\|_{k_j}}{\|f\|_2} \le \prod_{\ell=0}^{j-1} (k_\ell C \lambda_i)^{1/(k_\ell - 1)} \qquad \forall j \ge 1.$$

Now, notice that $k_\ell^{1/(k_\ell-1)}$ can be bounded above by a dimensional and that, since $\lambda_1 \ge c(K,N) > 0$, we can choose C so large that $C\lambda_i \ge 1$. Therefore using the inequality

$$\sum_{\ell=0}^{j-1} \frac{1}{k_\ell - 1} \le 2 \sum_{\ell=0}^{\infty} \frac{1}{k_\ell} = \frac{N}{2},$$

letting $j \to \infty$ provides (4.0.24). □

In the following proposition we obtain an explicit estimate on the L^∞ norm and the Lipschitz constant of eigenfunctions of $-\Delta$ in terms of the size of eigenvalues, see also [J16] for related results.

Proposition 4.0.19. *Assuming that $(X, \mathrm{d}, \mathfrak{m})$ is a $\mathrm{RCD}^*(K,N)$ space, whenever $\lambda_i \ge D^{-2}$ one has*

$$\|\varphi_i\|_\infty \le C\lambda_i^{N/4} \|\varphi_i\|_2, \qquad \|\nabla \varphi_i\|_\infty \le C\lambda_i^{(N+2)/4} \|\varphi_i\|_2,$$

with $C = C(K,N,D)$.

Proof. Let us prove only the first inequality, since the proof of the second one goes along similar lines. Throughout this proof, C denotes a generic constant depending on K, N, whose value can also change from line to line. Since φ_i is an eigenfunction with eigenvalue λ_i, for all $t > 0$ one has $P_t \phi_i = e^{-\lambda_i t} \phi_i$, so that

$$\varphi_i(x) = e^{\lambda_i t} \int_X p(x,y,t) \varphi_i(y) \, \mathrm{d}\mathfrak{m}(y) \qquad \forall x \in X.$$

If we use the heat kernel estimates (2.3.6) with $\epsilon = 1$ and the assumption $\mathfrak{m}(X) = 1$ we obtain

$$\begin{aligned} |\varphi_i(x)| &\le e^{\lambda_i t} \int_X p(x,y,t) |\varphi_i(y)| \, \mathrm{d}\mathfrak{m}(y) \le e^{\lambda_i t} \|\varphi_i\|_2 \left(\int_X p(x,y,t)^2 \, \mathrm{d}\mathfrak{m}(y) \right)^{1/2} \\ &\le C \|\varphi_i\|_2 \frac{e^{\lambda_i t + Ct}}{\mathfrak{m}(B_{\sqrt{t}}(x))}. \end{aligned}$$

Now we use the Bishop-Gromov inequality (Theorem 2.1.14) to conclude that, for $t \le D^2$, one has

$$\frac{\mathfrak{m}(B_{\sqrt{t}}(x))}{\mathrm{Vol}_{K,N}(\sqrt{t})} \ge \frac{1}{\mathrm{Vol}_{K,N}(D)}.$$

Therefore,

$$|\varphi_i(x)| \le C \frac{e^{\lambda_i t + Ct}}{\sqrt{\mathrm{Vol}_{K,N}(\sqrt{t})}} \sqrt{\mathrm{Vol}_{K,N}(D)} \|\varphi_i\|_2.$$

Choosing $t = 1/\lambda_i$ the proof is achieved. □

We are now in a position to conclude. The first expansion (4.0.21) is a direct consequence of Proposition 4.0.18 and Proposition 4.0.19. The second expansion (4.0.22) follows, thanks to the simple observation that $\|\nabla \varphi_i\|_2^2 = \lambda_i$.

Chapter 5

Embedding RCD*(K,N) spaces into L^2 via their heat kernel

In the last chapter of this thesis, we present the results of [AHPT17], in which we explain how to extend a Riemannian construction due to P. Bérard, G. Besson and S. Gallot [BBG94, Sect. 3] to the RCD setting.

5.1 Riemannian context

Let us start with a review on the original theorem in the Riemannian context.

A family of smooth embeddings
Let (M, g) be a closed n-dimensional Riemannian manifold equipped with its canonical Riemannian distance d and volume measure vol. For any positive time $t > 0$, the map $\Phi_t : M \to L^2(M)$ is given by

$$\Phi_t(x) = p(x, \cdot, t) \tag{5.1.1}$$

where $p : M \times M \times (0, \infty) \to (0, \infty)$ is the heat kernel on M. The next proposition is similar to [BBG94, Thm. 5] and gives regularity properties of the maps Φ_t.

Proposition 5.1.1. *For any $t > 0$ the map Φ_t is a smooth embedding. Moreover the differential $d_x\Phi_t : T_xM \to L^2(M)$ at $x \in M$ is given by*

$$d_x\Phi_t(v) : y \mapsto g_x(\nabla_x p(x, y, t), v) \qquad \forall v \in T_xM. \tag{5.1.2}$$

In particular

$$\|d_x\Phi_t(v)\|^2_{L^2(M)} = \int_M |g_x(\nabla_x p(x, y, t), v)|^2 \mathrm{dvol}(y) \qquad \forall v \in T_xM.$$

Proof. We first check that Φ_t is a continuous embedding. Continuity is obvious. As (M, d) is compact, it suffices to show that Φ_t is injective. Recall the expression (4.0.21) of the heat kernel, we see that $\Phi_t(x_1) = \Phi_t(x_2)$ yields

$$\sum_i e^{-\lambda_i t}\varphi_i(x_1)\varphi_i(y) = \sum_i e^{-\lambda_i t}\varphi_i(x_2)\varphi_i(y) \quad \text{for vol-a.e. } y \in M. \tag{5.1.3}$$

In particular, multiplying both sides of (5.1.3) by $\varphi_j(y)$ and integrating over M shows that $\varphi_j(x_1) = \varphi_j(x_2)$ holds for all j. Then since $p(x_1, x_1, s) = p(x_1, x_2, s)$ for all $s > 0$ by

(4.0.21), the Gaussian bounds (2.3.6) yield

$$\frac{1}{C_1\mathfrak{m}(B_{s^{1/2}}(x_1))}\exp\left(-C_2 s\right) \le p(x_1,x_1,s) = p(x_1,x_2,s) \le \frac{C_1}{\mathfrak{m}(B_{s^{1/2}}(x_1))}\exp\left(-\frac{\mathrm{d}^2(x_1,x_2)}{5s}+C_2 s\right),$$

i.e. $\exp(-C_2 s) \le C_1^2\exp(-\mathrm{d}^2(x_1,x_2)/(5s)+C_2 s)$. Then letting $s \downarrow 0$ yields $x_1 = x_2$, which shows that Φ_t is injective.

Next we prove the smoothness of Φ_t along with (5.1.2). Let us denote by $q(x,t,v)(\in L^2(M,\mathrm{vol}))$ the right hand side of (5.1.2). Take a smooth curve $c:(-\epsilon,\epsilon)\to M$ with $c(0)=x$ and $c'(0)=v$ and estimate

$$\begin{aligned}
&\left\|\frac{\Phi_t\circ c(h)-\Phi_t\circ c(0)}{h}-q(x,t,v)\right\|_{L^2}^2\\
&=\int_M\left|\frac{p(c(h),y,t)-p(c(0),y,t)}{h}-\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}p(c(s),y,t)\right|^2\mathrm{dvol}(y)\\
&=\int_M\left|\int_0^h\frac{s}{h}\mathrm{Hess}_{p(x,\cdot,t)}\left(c'(s),c'(s)\right)\mathrm{d}s\right|^2\mathrm{dvol}\\
&\le h\int_M\int_0^h\left|\mathrm{Hess}_{p(x,\cdot,t)}\left(c'(s),c'(s)\right)\right|^2\mathrm{d}s\,\mathrm{dvol},
\end{aligned}\tag{5.1.4}$$

where we applied the identity $f(h)=f(0)+f'(0)h-\int_0^h sf''(s)\mathrm{d}s$, valid for any $f\in C^2(-\epsilon,\epsilon)$, to the family of functions $f_y(s):=p(c(s),y,t)$, $y\in M$. Thus, letting $h\to 0$ in (5.1.4) shows that Φ_t is differentiable at $x\in M$ and that (5.1.2) holds. The smoothness of Φ_t follows similarly. □

Pull-back metrics

Viewing $L^2(M)$ as an infinite dimensional manifold whose tangent space at each point is $L^2(M)$ itself, we can see the L^2 scalar product as a "flat" Riemannian metric g_{L^2}. Thanks to Proposition 5.1.1, for any $t>0$ we consider the pull-back metric $\Phi_t^*g_{L^2}$ which writes as follows:

$$[\Phi_t^*g_{L^2}]_x(v,w):=\int_M g_x(\nabla_x p(x,y,t),v)g_x(\nabla_x p(x,y,t),w)\mathrm{dvol}(y),\quad \forall v,w\in T_xM,\forall x\in M.\tag{5.1.5}$$

The asymptotic behavior of $\Phi_t^*g_{L^2}$ was discussed in [BBG94, Thm 5] where one can find the following result.

Theorem 5.1.2. *Denoting by* Ric, Scal *the Ricci and the scalar curvature of* (M,g) *respectively,*

$$c(n)t^{(n+2)/2}\Phi_t^*g_{L^2}=g+\frac{t}{3}\left(\frac{1}{2}\mathrm{Scal}\,g-\mathrm{Ric}\right)+O(t^2),\quad t\downarrow 0,\tag{5.1.6}$$

in the sense of pointwise convergence, where $c(n)$ *is a positive constant depending only on the dimension* n.

The proof of the previous theorem heavily relies on the so-called Minakshisundaram-Pleijel asymptotic formula [MP49]. Recall that for any $x\in M$ the injectivity radius $\mathrm{inj}(x)$ of M at x is set as the supremum of the set of real numbers $r>0$ such that the exponential map $\exp_x$ restricted to $B_r(x)$ is a diffeomorphism onto its image. We write $\mathrm{inj}(M)$ for the injectivity radius of M, which is by definition $\inf\{\mathrm{inj}(x) : x \in M\}$. Note that compactness of M implies that $\mathrm{inj}(M) > 0$. Finally, let us set $\mathrm{InjDiag}(M) := \{(x, y) \in M \times M : |x - y| \leq \mathrm{inj}(M)\}$.

Proposition 5.1.3 (Minakshisundaram-Pleijel asymptotic expansion)**.** *There exists a sequence of smooth functions $u_i : \mathrm{InjDiag}(M) \to \mathbb{R}$, $i \in \mathbb{N}$, such that for any $N \in \mathbb{N}$,*

$$p_t(x, y) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{\mathrm{d}(x,y)}{4t}} \left[\sum_{i=0}^{N} t^i u_i(x, y) + O(t^{N+1}) \right] \qquad t \downarrow 0, \tag{5.1.7}$$

for any $(x, y) \in \mathrm{InjDiag}(M)$. Moreover, (5.1.7) can be differentiated with respect to x, y or t as many time as needed.

A careful study of the first terms u_0, u_1 of the expansion (5.1.7) provides the following extra information.

Proposition 5.1.4.

1. *The first term u_0 in (5.1.7) coincides with $\theta^{-\frac{1}{2}}$ where for any $x \in M$, the function $\theta(x, \cdot)$ defined on $B_{\mathrm{inj}(M)}(x)$ is the density w.r.t. the Lebesgue measure of $(\exp_x^{-1})_{\#}\mathrm{vol} \ll \mathscr{L}^n$.*
2. *The second term u_1 satisfies*

$$u_1(x, x) = \frac{\mathrm{Scal}(x)}{6}, \quad \forall x \in M.$$

Finally, recall that if γ is a length minimizing geodesic on (M, g), one has:

$$\theta(\gamma_s, \gamma_t) \underset{|t-s|\to 0}{=} 1 - \mathrm{Ric}(\dot\gamma_s, \dot\gamma_s)\frac{|t - s|^2}{6} + O(|t - s|^3). \tag{5.1.8}$$

We are now in a position to provide a detailed proof of Theorem 5.1.2.

Proof. We will only need the first-order expansion of Minakshisundaram-Pleijel's formula. For convenience, let us write

$$K_t(x, y) = (4\pi t)^{-n/2} e^{-\frac{\mathrm{d}^2(x,y)}{4t}} \quad \text{and} \quad R_t(x, y) = u_0(x, y) + t u_1(x, y) + O(t^2),$$

so that

$$p_t(x, y) = K_t(x, y) R_t(x, y). \tag{5.1.9}$$

Let us fix $x \in M$ and $v \in T_x M$. Then thanks to the expansion (4.0.22),

$$\begin{aligned}
[\Phi_t^* g_{L^2}]_x(v, v) &= \int_M g_x(\nabla_x p(x, y, t), v)^2 \mathrm{dvol}(y) \\
&= \int_M \sum_{i,j} e^{-(\lambda_i + \lambda_j)t} \varphi_i(y)\varphi_j(y) g_x(\nabla_x \varphi_i, v) g_x(\nabla_x \varphi_j, v) \,\mathrm{dvol}(y) \\
&= \sum_i e^{-2\lambda_i t} \,\mathrm{d}_x \varphi_i(v)^2 \\
&= \mathrm{d}_x^2 \,\mathrm{d}_x^1 p_t(v, v).
\end{aligned} \tag{5.1.10}$$

Let us explain the notation "$\mathrm{d}_x^2 \,\mathrm{d}_x^1$". The function $p_t : M \times M \to (0, +\infty)$ depends on two variables, then $\mathrm{d}_x^1 p_t : T_x M \times M \to (0, +\infty)$ is the differential w.r.t. the first variable of p_t at x, and $\mathrm{d}_x^2 \,\mathrm{d}_x^1 p_t : T_x M \times T_x M \to (0, +\infty)$ is the differential with respect to the second variable of $\mathrm{d}_x^1 p_t$ at x. The function $\mathrm{d}_x^2\,\mathrm{d}_x^1 p_t$ is called mixed derivative of p_t at x. The equality (5.1.10) is a direct consequence of the expansion (4.0.22). Let us compute $\mathrm{d}_x^2\,\mathrm{d}_y^1 p_t$ for some $y \in M$, afterwards we will consider only the value for $y = x$.

Thanks to (5.1.9), Leibniz rule and chain rule, for any $y \in M$,

$$\begin{aligned}\mathrm{d}_x^1 p_t(v,y) &= \mathrm{d}_x^1 K_t(v,y)\ R_t(x,y) \ +\ K_t(x,y)\ \mathrm{d}_x^1 R_t(v,y) \\ &= -\frac{1}{4t}(\,\mathrm{d}_x^1[\mathrm{d}^2(\cdot,y)](v))K_t(x,y)R_t(x,y) \\ &\quad + K_t(x,y)\left(\,\mathrm{d}_x^1 u_0(\cdot,y)(v) + t\,\mathrm{d}_x^1[u_1(\cdot,y)](v) + O(t^2)\right).\end{aligned}$$

Then by Leibniz rule, and writing d instead of d for better readability,

$$\begin{aligned}\mathrm{d}_y^2\,\mathrm{d}_x^1 p_t(v,v) = &-\frac{1}{4t}\,\mathrm{d}_y^2 K_t(x,v)\,\mathrm{d}_x^1 d^2(v,y)R_t(x,y) - \frac{1}{4t}K_t(x,y)\,\mathrm{d}_y^2\,\mathrm{d}_x^1 \mathrm{d}^2(v,v)R_t(x,y) \\ &-\frac{1}{4t}K_t(x,y)\,\mathrm{d}_x^1 d^2(v,y)\,\mathrm{d}_y^2 R_t(x,v) + \mathrm{d}_y^2 K_t(x,v)\,\mathrm{d}_x^1 R_t(v,y) \\ &+ K_t(x,y)\,\mathrm{d}_y^2\,\mathrm{d}_x^1 R_t(v,v).\end{aligned}$$

To go on, note that by definition, $\mathrm{d}_x^1 d^2(v,y) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} d^2(\gamma_t,y)$ where γ is a curve such that $\gamma_0 = x$ and $\dot\gamma_0 = v$. If $y = x$, one gets $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} d^2(\gamma_t,x)$ which reads $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} |tW - 0|^2$ in local coordinates, W being some vector of $\mathbb{R}^n$ depending on the system of coordinates. Then $\mathrm{d}_x^1 d^2(v,x) = 0$, so in the above equality, taking $y = x$, the first and third term vanish. One can prove similarly $\mathrm{d}_x^2 d^2(x,v) = 0$. As $\mathrm{d}_x^2 K_t(x,v) = -(4t)^{-1}K_t(x,y)\,\mathrm{d}_y^2 d^2(x,v)$, the fourth term vanish. To deal with the second term, let us compute the mixed second derivative of d^2. Take a curve γ such that $\gamma_0 = x$ and $\dot\gamma_0 = v$. One can choose γ to be a geodesic. Then

$$\mathrm{d}_x^2\,\mathrm{d}_x^1 d^2(v,v) = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} d^2(\gamma_s,\gamma_t) = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} g_x(v,v)|t-s|^2 = -2g_x(v,v).$$

Finally, we get

$$\begin{aligned}\mathrm{d}_x^2\,\mathrm{d}_x^1 p_t(v,v) &= K_t(x,x)\left(\frac{g_x(v,v)}{2t}R_t(x,x) + \mathrm{d}_x^2\,\mathrm{d}_x^1 R_t(v,v)\right) \\ &= (4\pi t)^{-n/2}\left[\frac{g_x(v,v)}{2}\left(\frac{u_0(x,x)}{t} + u_1(x,x)\right) + \mathrm{d}_x^2\,\mathrm{d}_x^1 u_0(v,v) + O(t)\right]\end{aligned}$$

The result follows from Proposition 5.1.4 and the computation $\mathrm{d}_x^2\,\mathrm{d}_x^1 u_0(v,v) = -\frac{1}{3!}\mathrm{Ric}_x(v,v)$ made thereafter. Recall that $u_0 = \theta^{-1/2}$. Let γ be a geodesic such that $\gamma_0 = x$ and $\dot\gamma_0 = v$.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\theta^{-1/2}(\gamma_s,\gamma_t) &= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\left(-\frac{1}{2}\theta^{-3/2}(\gamma_s,\gamma_0)\,\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\theta(\gamma_s,\gamma_t)\right) \\ &= -\frac{1}{2}\left(-\frac{3}{2}\underbrace{\theta^{-5/2}(\gamma_0,\gamma_0)}_{=1}\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\theta(\gamma_s,\gamma_0)\,\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\theta(\gamma_0,\gamma_t)\right. \\ &\qquad\left. + \underbrace{\theta^{-3/2}(\gamma_0,\gamma_0)}_{=1}\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\theta(\gamma_s,\gamma_t)\right)\end{aligned}$$

Thanks to (5.1.8),

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\theta(\gamma_s,\gamma_0) = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\left(1 - \mathrm{Ric}(\dot\gamma_s,\dot\gamma_s)\frac{s^2}{6} + O(s^3)\right) = 0$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\theta(\gamma_s,\gamma_t) &= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\Big(-\frac{1}{3!}\mathrm{Ric}_{\gamma_s}(\dot\gamma_s,\dot\gamma_s)s+O(s^2)\Big)\\ &= -\frac{1}{3!}\mathrm{Ric}_{\gamma_0}(\dot\gamma_0,\dot\gamma_0)\underbrace{\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}s}_{=1}\\ &\quad -\frac{1}{3!}\Big[\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\mathrm{Ric}_{\gamma_s}(\dot\gamma_s,\dot\gamma_s)\Big]\underbrace{s|_{s=0}}_{=0}+\underbrace{O(2s)|_{s=0}}_{0}\\ &= -\frac{1}{3!}\mathrm{Ric}_{\gamma_0}(\dot\gamma_0,\dot\gamma_0).\end{aligned}$$

□

5.2 RCD context

From now on and until the end of this chapter, $K\in\mathbb{R}$ and $N\in[1,+\infty)$ are kept fixed. We move from the Riemannian manifold (M,g) considered in the previous section to a compact $\mathrm{RCD}^*(K,N)$ space $(X,\mathrm{d},\mathfrak{m})$. For any positive time $t>0$, the map $\Phi_t: X\to L^2(X,\mathfrak{m})$ is given by

$$\Phi_t(x)=p(x,\cdot,t) \tag{5.2.1}$$

where $p: X\times X\times(0,\infty)\to(0,\infty)$ is the locally Hölder continuous representative of the heat kernel on $(X,\mathrm{d},\mathfrak{m})$. It is immediate to check that the maps Φ_t are continuous embeddings. Indeed, since (4.0.21) holds true on $(X,\mathrm{d},\mathfrak{m})$, we can carry out the proof of Proposition 5.1.1 to get that Φ_t is an embedding for any $t>0$. Continuity is obvious as we consider the locally Hölder representative of the heat kernel.

First-order differentiation formula

Let us start with an analogue of the differentiation formula (5.1.2) which does not appear in [AHPT17]. Such a precise formula seems hardly reachable on $(X,\mathrm{d},\mathfrak{m})$, as there is no pointwise definition of tangent vectors on $\mathrm{RCD}^*(K,N)$ spaces. Nevertheless, following [G13, Prop. 5.15], we can prove an integrated version of (5.1.2) along Wasserstein geodesics with bounded compression (Proposition 5.2.1), or even along more general curves (Remark 5.2.2). Indeed, for some given $t>0$, let us consider the W_2-continuous curve $(\mu_s:=p(x_s,\cdot,t)\mathfrak{m})_s$, where $(x_s)_{s\in[0,1]}$ is a continuous curve on X. Then for any $s\in[0,1]$ and any $z\in X$, by the Chapman-Kolmogorov property (2.3.3),

$$F_t(s)(z):=\int_X p(x,z,t)\,\mathrm{d}\mu_s(x)=\int_X p(x,z,t)p(x_s,x,t)\,\mathrm{d}\mathfrak{m}(x)=p(x_s,z,2t).$$

If $(X,\mathrm{d},\mathfrak{m})$ is a Riemannian manifold, it follows from Proposition 5.1.1 that $F_t:[0,1]\to L^2(X,\mathfrak{m})$ is continuously differentiable on $[0,1]$, and

$$F_t'(s)=g_{x_s}(\nabla_x p(x_s,\cdot,2t),x_s')\qquad\forall s\in[0,1].$$

From this observation, it appears natural to study the differentiability properties of the $L^2(X,\mathfrak{m})$-valued functions $F_t: s\mapsto\int_X p(x,\cdot,t)\,\mathrm{d}\mu_s$ for suitable W_2-continuous curves $(\mu_s)_s$. In the next proposition we consider W_2-geodesics (μ_s) with bounded compression, meaning that there exists $C>0$ such that $\mu_s\le C\mathfrak{m}$ holds for every $s\in[0,1]$. Let us recall that

Kantorovitch potentials between any given $\mu, \nu \in \mathcal{P}(X)$ are optimizers in Kantorovitch duality formula [Vi03, Th. 1.3]:

$$\frac{1}{2} W_2^2(\mu, \nu) = \sup_{\varphi} \left\{ \int_X \varphi \, \mathrm{d}\mu + \int_X \varphi^c \, \mathrm{d}\nu \right\},$$

where the supremum is taken over all Borel functions $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ such that $\varphi \in L^1(X, \mu)$, and the c-transform φ^c of φ is by definition

$$\varphi^c(y) := \inf_{x \in X} \left\{ \frac{\mathrm{d}^2(x, y)}{2} - \varphi(x) \right\}.$$

On compact spaces, one can always find a locally Lipschitz Kantorovitch potential between two arbitrary probability measures. Let us also recall the Hopf-Lax formula: for any given bounded function $f : X \to \mathbb{R}$, the function $Q_s f : X \to \mathbb{R}$ is defined as

$$Q_s f(x) := \inf_{y \in X} f(y) + \frac{\mathrm{d}^2(x, y)}{2s} \qquad \forall s > 0,$$

and $Q_0 f = f$. The following two simple results are well-known:

- continuity: if f is continuous, then the function $s \mapsto Q_s f(x)$ is continuous on $[0, \infty)$ for any $x \in X$;
- Lipschitz estimate:

$$\mathrm{Lip}(Q_s f) \leq 2 \sqrt{\frac{\sup f - \inf f}{2s}}, \quad \forall s > 0. \tag{5.2.2}$$

Proposition 5.2.1. *For any W_2-geodesic (μ_s) with bounded compression, the $L^2(X, \mathfrak{m})$-valued function F_t defined by:*

$$F_t(s) := \int_X p(x, \cdot, t) \, \mathrm{d}\mu_s(x) \qquad \forall s \in [0, 1], \tag{5.2.3}$$

is continuously differentiable in $[0, 1]$ and

$$F_t'(s) = \int_X \langle \nabla_x p(x, \cdot, t), \nabla [Q_s(-\varphi)](x) \rangle \, \mathrm{d}\mu_s(x) \quad \forall s \in (0, 1] \tag{5.2.4}$$

$$= - \int_X \langle \nabla_x p(x, \cdot, t), \nabla [Q_{1-s}(-\varphi^c)](x) \rangle \, \mathrm{d}\mu_{1-s}(x) \quad \forall s \in [0, 1), \tag{5.2.5}$$

where the function φ is any locally Lipschitz Kantorovitch potential from μ_0 to μ_1.

Proof. From the bounded compression assumption, we know that the W_2-geodesic $(\mu_s)_s$ induces a map $\rho_s : [0, 1] \to L^\infty(X, \mathfrak{m})$ continuous w.r.t. the w^*-$L^\infty(X, \mathfrak{m})$ topology, where ρ_s is the density of μ_s w.r.t. $\mathfrak{m}$. Note also that given two measures μ_0, $\mu_1 \in \mathcal{P}(X)$ and a Kantorovitch potential φ from μ_0 to μ_1, for any $s \in [0, 1]$, the functions $sQ_s(-\varphi)$ and $(1-s)Q_{1-s}(-\varphi^c)$ are Kantorovitch potentials from μ_s to μ_0 and from μ_s to μ_1 respectively. Let us now fix a Lipschitz Kantorovich potential φ from μ_0 to μ_1, whose existence is ensured by the compactness of (X, d). Pick $s_0 \in (0, 1]$. First of all, it is easily checked that $F(s_0) \in L^2(X, \mathfrak{m})$ thanks to the heat kernel upper bound (2.3.6), boundedness of the density ρ_s and compactness of (X, d). Let us prove the convergence

$$\lim_{h \to 0} \frac{F_t(s_0 + h) - F_t(s_0)}{h} = - \int_X \langle \nabla_x p(x, \cdot, t), \nabla Q_{s_0} \varphi(x) \rangle \, \mathrm{d}\mu_{s_0}(x) \quad \text{in } L^2(X, \mathfrak{m}). \tag{5.2.6}$$

As $p(\cdot, y, t) \in H^{1,2}(X, \mathrm{d}, \mathfrak{m}) \cap L^1(X, \mathfrak{m})$ for any $y \in X$, we can apply [G13, Prop. 5.16] to get:

$$\lim_{h\to 0} \int_X p(x,y,t)\,\mathrm{d}\left(\frac{\mu_{s_0+h}-\mu_{s_0}}{h}\right)(x) = -\int_X \langle \nabla_x p(x,y,t), \nabla Q_{s_0}\varphi(x)\rangle\,\mathrm{d}\mu_{s_0}(x), \quad \forall y \in X. \tag{5.2.7}$$

Write $G_h = h^{-1}(F(s_0+h)-F(s_0))$ for any $h > 0$ and $G_0 = -\int_X \langle \nabla_x p(x,\cdot,t), \nabla Q_{s_0}\varphi(x)\rangle\,\mathrm{d}\mu_{s_0}(x)$. Then for any $y \in X$,

$$G_h(y) = \int_X p(x,y,t)\,\mathrm{d}\left(\frac{\mu_{s_0+h}-\mu_{s_0}}{h}\right)(x) = \frac{1}{s_0}\int \frac{p(\gamma_h,y,t)-p(\gamma_0,y,t)}{h}\,\mathrm{d}\pi^-_{s_0}(\gamma)$$

where $\pi^-_{s_0} = (Restr_0^{s_0})_\#\pi$, π is a lifting of the geodesic $(\mu_s)_s$ and $Restr_0^{s_0}$ is the function associating to each curve $\gamma \in C([0,1],X)$ the reparametrization over $[0,1]$ of the restriction to $[0,s_0]$ of γ. By the heat kernel gradient upper bound (2.3.7),

$$|p(\gamma_h,y,t)-p(\gamma_0,y,t)| \le (\sup_{x,y}|\nabla_x p(x,y,t)|)\mathrm{d}(\gamma_h,\gamma_0) \le \frac{C_1 e^{C_2 t}}{\sqrt{t}(\min\limits_{z\in X}\mathfrak{m}(B_{\sqrt{t}}(z)))} h =: C_0(t)h.$$

Here $\inf_z \mathfrak{m}(B_{\sqrt{t}}(z))$ is achieved and positive because of compactness of (X,d) and continuity of the function $z \mapsto \mathfrak{m}(B_{\sqrt{t}}(z))$. This implies that $|G_h(y)| \le C_0(t)/s_0$. Moreover, applying the Lipschitz continuity estimate (5.2.2) and (2.3.7), we get

$$|G_0(y)| \le \int_X |\nabla_x p(x,y,t)||\nabla Q_{s_0}\varphi(x)|\,\mathrm{d}\mu_{s_0}(x) \le 2C_0(t)\sqrt{\frac{\sup\varphi - \inf\varphi}{s_0}}.$$

Hence, by dominated convergence, the pointwise convergence (5.2.7) implies the L^2 convergence (5.2.6) and thus the validity of the formula (5.2.4). The formula for $s_0 \in [0,1)$ is established similarly.

Let us prove now that the function F_t' is continuous in $(0,1]$. Let $s_n, s \in (0,1]$ be such that $s_n \to s$. Writing

$$H_n = \int_X \langle \nabla_x p(x,\cdot,t), \nabla[Q_{s_n}(-\varphi)](x)\rangle\,\mathrm{d}\mu_{s_n}(x),\ \ H = -\int_X \langle \nabla_x p(x,\cdot,t), \nabla[Q_s(-\varphi)](x)\rangle\,\mathrm{d}\mu_s(x),$$

we claim that H_n converge to H pointwise in X. This follows from [G13, Lem. 5.11] applied to the (non relabeled) subsequence of (s_n) such that $\rho_{s_n} \to \rho_{s_\infty}$ $\mathfrak{m}$-a.e., the measures $\mu_n = \mu_{s_n}$, $\mu = \mu_{s_\infty}$ and the functions $g_n = g = p(\cdot,y,t)$ and $f_n = Q_{s_n}\varphi$, $f = Q_{s_\infty}\varphi$. Note that continuity of the function $s \mapsto Q_s\varphi$ and the Lipschitz continuity estimate (5.2.2) imply the required hypotheses on f_n, f. Moreover, assuming w.l.o.g. that $s_n > s/4$ for every n, using again (5.2.2) and (2.3.7) one has

$$|H_n(y)| \le 4C_0(t)\sqrt{\frac{\sup\varphi - \inf\varphi}{s}} \qquad \forall y \in X$$

and similarly one can bound $|H_\infty|$ by $2C_0(t)\sqrt{(\sup\varphi - \inf\varphi)/s}$. Thus, dominated convergence theorem turns the pointwise convergence of H_n into $L^2(X,\mathfrak{m})$ convergence, which implies continuity of F_t' at s. The continuity of F_t' in $[0,1)$ can be proved by similar means using the second formula in (5.2.4) instead of the first one. □

Remark 5.2.2. More generally, it follows from [GH14, Prop. 3.7] that for any 2-absolutely continuous curve $\mu = (\mu_s) \subset \mathcal{P}(X)$ w.r.t. W_2 with bounded compression, for any $t > 0$, the $L^2(X,\mathfrak{m})$-valued function F_t defined by (5.2.3) is absolutely continuous on $[0,1]$, and

$$F_t'(s)(y) = L_s^\mu(p(\cdot,y,t)) \qquad \forall y \in X,$$

for any $s \in [0,1]\backslash\mathcal{N}^\mu$, where $\mathcal{N}^\mu$ is a $\mathscr{L}^1$-negligible subset of $[0,1]$ and $(L_s^\mu)_{s\in[0,1]\backslash\mathcal{N}^\mu}$ is a family of linear maps from $S^2(X,\mathrm{d},\mathfrak{m})$ to $\mathbb{R}$, both $\mathcal{N}^\mu$ and $(L_s^\mu)_{s\in[0,1]\backslash\mathcal{N}^\mu}$ depending only on μ.

Tangent bundle

Before going further, we need to remind the construction of the L^2-tangent bundle $L^2T(X,\mathrm{d},\mathfrak{m})$ introduced by N. Gigli [G18]. Although Gigli's construction can be performed in more general settings, recall that here we are considering a compact RCD$^*(K,N)$ space $(X,\mathrm{d},\mathfrak{m})$ for fixed $K\in\mathbb{R}$ and $N\in[1,+\infty)$. In particular, the infinitesimally Hilbertian condition on $(X,\mathrm{d},\mathfrak{m})$ allows several simplifications, as Gigli's original construction provides first the space $L^2T^*(X,\mathrm{d},\mathfrak{m})$ from which one recovers $L^2T(X,\mathrm{d},\mathfrak{m})$ by duality.

Recall that for any unitary ring $(A,+,\cdot)$, a A-module M is by definition an abelian group $(M,+)$ equipped with an operation $A\times M\to M$ whose properties are those of scalar multiplication for vector spaces, namely bilinearity, associativity, and invariance of any element under the action of the identity element 1_A of A. Such an operation is sometimes called left-multiplication, and one could similarly consider right-multiplications $M\times A\to M$. We won't need these refinements here, and be content with the solely operation $A\times M\to M$ that we simply call multiplication.

Of special interest for us are $L^\infty(X,\mathfrak{m})$-modules. Such modules admit a natural structure of real vector spaces, identifying multiplication by a real number λ with multiplication by the $L^\infty(X,\mathfrak{m})$ function equal $\mathfrak{m}$-a.e. to λ.

The lack of differentiable structure on $(X,\mathrm{d},\mathfrak{m})$ prevents any consistent definition of a tangent space TX in analogy with the tangent bundle TM of a differentiable manifold M. Therefore, to define a first order-calculus on $(X,\mathrm{d},\mathfrak{m})$, a seminal idea due to N. Weaver [W00] in the context of complete separable metric measure spaces is to forget about the tangent space TX in itself, and to look rather for an analogue of the space of L^p sections of this tangent space, for any $1\le p<+\infty$. Indeed, in the differentiable context, such spaces satisfy simple abstract algebraic properties which can be properly turned into definitions.

Slightly modifying the appellation in [G18, Sect. 1.2] for consistency with Weaver's work, we give the following definition.

Definition 5.2.3. (Banach $L^\infty(X,\mathfrak{m})$-module) We say that E is a Banach $L^\infty(X,\mathfrak{m})$-module if it is a $L^\infty(X,\mathfrak{m})$-module equipped with a complete norm $\|\cdot\|_E$ satisfying $\|fv\|_E\le\|f\|_{L^\infty(X,\mathfrak{m})}\|v\|_E$ for any $f\in L^\infty(X,\mathfrak{m})$ and $v\in E$, and if the two following properties hold:

(i) (locality) for any $v\in E$ and any countable family of Borel sets $(A_n)_n$,

$$1_{A_n}v=0\quad\forall n\quad\Rightarrow\qquad 1_{\cup A_n}v=0;$$

(ii) (gluing) for any sequence $(v_n)_n\subset E$ and any countable family of Borel sets $(A_n)_n$ such that $1_{A_n\cap A_m}v_n=1_{A_n\cap A_m}v_m$ for any n,m and $\limsup_{n\to+\infty}\|\sum_{i=1}^n 1_{A_i}v_i\|_E<+\infty$, there exists $v\in E$ gluing all the (v_n,A_n)'s together, in the sense that $1_{A_n}v=1_{A_n}v_n$ for any n and $\|v\|_E\le\liminf_{n\to+\infty}\|\sum_{i=1}^n 1_{A_i}v_i\|_E$.

For instance, the space of smooth vector fields over a n-dimensional differentiable manifold M is a Banach $L^\infty(M,\mathscr{H}^n)$-module.

Let us choose $1\le p<+\infty$. To characterize the space of L^p sections in such an algebraic way, we need a further definition.

Definition 5.2.4. ($L^p(X,\mathfrak{m})$-normed modules) We say that a Banach $L^\infty(X,\mathfrak{m})$-module E is a $L^p(X,\mathfrak{m})$-normed module provided there exists a function $|\cdot|: E \to \{f \in L^p(X,\mathfrak{m}) : f \geq 0\}$, called *local norm*, satisfying:

(a) $|v+v'| \leq |v| + |v'|$ $\mathfrak{m}$-a.e. in X, for all v, $v' \in E$;

(b) $|\chi v| = |\chi||v|$ $\mathfrak{m}$-a.e. in X, for all $v \in E$, $\chi \in L^\infty(X,\mathfrak{m})$;

(c) the function

$$\|v\|_p := \left(\int_X |v(x)|^p \,\mathrm{d}\mathfrak{m}(x)\right)^{1/p} \tag{5.2.8}$$

is a norm in E.

Notice that homogeneity and subadditivity of $\|\cdot\|_p$ are obvious consequences of (a), (b). Note also that the space of L^p vector fields over a n-dimensional differentiable manifold M is a $L^p(M,\mathscr{H}^n)$-normed module.

We arrive now to the core of Gigli's construction. Define the so-called pretangent module $\mathrm{Pre}(X,\mathrm{d},\mathfrak{m})$ as the set of countable families $\{(A_i,f_i)\}_i$ where $(A_i)_i$ is a Borel partition of X and $f_i \in H^{1,2}(X,\mathrm{d},\mathfrak{m})$. Take the quotient of $\mathrm{Pre}(X,\mathrm{d},\mathfrak{m})$ with respect to the equivalence relation

$$\{(A_i,f_i)\}_{i\in I} \sim \{(B_j,g_j)\}_{j\in J} \iff |\nabla(f_i-g_j)|_* = 0 \ \mathfrak{m}\text{-a.e. in } A_i \cap B_j \text{ for any } i,j.$$

Thanks to the locality property of the minimal relaxed slope, it is easily seen that:

- the sum of two families $\{(A_i,f_i)\}_{i\in I}$ and $\{(B_j,g_i)\}_{j\in J}$ defined as $\{A_i \cap B_j, f_i + g_j\}_{(i,j)\in I\times J}$ is well-defined on $\mathrm{Pre}(X,\mathrm{d},\mathfrak{m})/\sim$;
- the multiplication of $\{(A_i,f_i)\}_{i\in I}$ by $\mathfrak{m}$-measurable functions χ taking finitely many values, defined as $\chi\{(A_i,f_i)\} = \{(A_i \cap F_j, z_j f_i)\}_{i,j}$ with $\chi = \sum_{j=1}^N z_j 1_{F_j}$, is also well-defined on $\mathrm{Pre}(X,\mathrm{d},\mathfrak{m})/\sim$;
- the map $|\cdot|_* : \mathrm{Pre}(X,\mathrm{d},\mathfrak{m})/\sim \to L^2(X,\mathfrak{m})$ defined by $|\{(A_i,f_i)\}_{i\in I}|_* := |\nabla f_i|_*$ $\mathfrak{m}$-a.e. on A_i for any i, is well-posed;
- the map $\|\cdot\|_{L^2T} : \mathrm{Pre}(X,\mathrm{d},\mathfrak{m})/\sim \to [0,+\infty)$ defined by $\|\{(A_i,f_i)\}_{i\in I}\|_{L^2T} := \sum_i \int_{A_i} |\nabla f_i|_*^2 \,\mathrm{d}\mathfrak{m}$ defines a norm.

Definition 5.2.5. (The tangent bundle)
The tangent bundle over $(X,\mathrm{d},\mathfrak{m})$ is defined as the completion of the space $\mathrm{Pre}(X,\mathrm{d},\mathfrak{m})/\sim$ w.r.t. the norm $\|\cdot\|_{L^2T}$; it is a $L^2(X,\mathfrak{m})$-normed module denoted by $L^2T(X,\mathrm{d},\mathfrak{m})$.

Note that the terminology "tangent bundle" is a little bit misleading here, for two reasons. First because we only define the analogue of the set of L^2 tangent vector fields, not the whole set of tangent vector fields. Second because $L^2T(X,\mathrm{d},\mathfrak{m})$ is not a bundle in the usual sense. However this terminology is convenient, and it has already appeared many times in the literature, so we stick to it.

In the sequel we shall denote by V, W, etc. the typical elements of $L^2T(X,\mathrm{d},\mathfrak{m})$ and by $|V|$ the local norm. We also start using a more intuitive notation, using ∇f for (the equivalence class of) $\{(X,f)\}$ where $f \in H^{1,2}(X,\mathrm{d},\mathfrak{m})$, and $\sum_i \chi_i \nabla f_i$ for any finite sum $\sum_i \chi_i\{(X,f_i)\}$ where $f_i \in H^{1,2}(X,\mathrm{d},\mathfrak{m})$ and $\chi_i \in L^\infty(X,\mathfrak{m})$ for any i.

The following result is a simple consequence of the definition of $L^2T(X,\mathrm{d},\mathfrak{m})$.

Theorem 5.2.6. *The vector space*

$$\left\{\sum_{i=1}^{n} \chi_i \nabla f_i : \ \chi_i \in L^\infty(X, \mathfrak{m}),\ f_i \in H^{1,2}(X, \mathrm{d}, \mathfrak{m}),\ n \geq 1\right\}$$

is dense in $L^2T(X, \mathrm{d}, \mathfrak{m})$.

More generally, density still holds if the functions χ_i vary in a set $D \subset L^2 \cap L^\infty(X, \mathfrak{m})$ stable under truncations and dense in $L^2(X, \mathfrak{m})$ (as $\mathrm{Lip}_b(X, \mathrm{d}) \cap L^2(X, \mathfrak{m})$).

Let us explain now how to define the cotangent module $L^2T^*(X, \mathrm{d}, \mathfrak{m})$ out of $L^2T(X, \mathrm{d}, \mathfrak{m})$.

Definition 5.2.7. (Dual module of a Banach $L^\infty(X, \mathfrak{m})$-module)
For any Banach $L^\infty(X, \mathfrak{m})$-module $(E, \|\cdot\|_E)$, the space

$$\begin{aligned}\mathrm{Hom}(E, L^1(X, \mathfrak{m})) := \{T : E \to L^1(X, \mathfrak{m}) \text{ linear, bounded as map between Banach spaces}\\ \text{i.e. } \|T\|_{E^*} := \sup\{\|T(v)\|_{L^1(X,\mathfrak{m})} : \|v\|_E = 1\} < +\infty,\\ \text{and satisfying } T(fv) = fT(v) \text{ for any } v \in E \text{ and } f \in L^\infty(X, \mathfrak{m})\}\end{aligned}$$

equipped with $\|\cdot\|_{E^*}$ is a Banach $L^\infty(X, \mathfrak{m})$-module; it is denoted by E^* and called dual module of E.

It turns out that the dual module of a $L^2(X, \mathfrak{m})$-normed module is still a $L^2(X, \mathfrak{m})$-normed module. Whence the following natural definition.

Definition 5.2.8. (The cotangent module)
The dual module of $L^2T(X, \mathrm{d}, \mathfrak{m})$ is denoted by $L^2T^*(X, \mathrm{d}, \mathfrak{m})$ and called cotangent module over $(X, \mathrm{d}, \mathfrak{m})$. Moreover, for any $f \in H^{1,2}(X, \mathrm{d}, \mathfrak{m})$, we shall write $\mathrm{d}f$ for the dual element of ∇f.

The cotangent and tangent bundles over $(X, \mathrm{d}, \mathfrak{m})$ not only have a structure of $L^2(X, \mathfrak{m})$-normed modules: they are also Hilbert $L^\infty(X, \mathfrak{m})$-modules, in the sense of the following definition.

Definition 5.2.9. (Hilbert $L^\infty(X, \mathfrak{m})$-module)
Let $(E, \|\cdot\|_E)$ be a Banach $L^\infty(X, \mathfrak{m})$-module. We call it a Hilbert $L^\infty(X, \mathfrak{m})$-module whenever $\|\cdot\|_E$ is an Hilbertian norm.

Remark 5.2.10. (Hilbert $L^\infty(X, \mathfrak{m})$-modules are $L^2(X, \mathfrak{m})$-normed modules) It is natural to ask whether there exists a relationship between Hilbert $L^\infty(X, \mathfrak{m})$-modules and $L^2(X, \mathfrak{m})$-normed modules. A preliminary result in this direction states that whenever $(E, \|\cdot\|_E)$ is a Hilbert $L^\infty(X, \mathfrak{m})$-module, for any $v \in E$, the map

$$\mu_v : A \mapsto \|\chi_A v\|_E^2$$

defines a non-negative measure on the Borel σ-algebra of (X, d) such that $\mu_v \ll \mathfrak{m}$. Afterwards, it can be shown ([G18, Prop. 1.2.21]) that any Hilbert $L^\infty(X, \mathfrak{m})$-module $(E, \|\cdot\|_E)$ is a $L^2(X, \mathfrak{m})$-normed module whose local norm $|\cdot|_E$ is given by

$$|v|_E := \sqrt{\rho_v} \qquad \forall v \in E$$

where ρ_v is the density of the measure μ_v w.r.t. $\mathfrak{m}$. Moreover, $|\cdot|_E$ satisfies the parallellogram identity: for any $v, v' \in E$,

$$|v + v'|^2 + |v - v'|^2 = 2|v|^2 + 2|v'|^2 \qquad \mathfrak{m}\text{-a.e. on } X.$$

The importance of Hilbert modules lies in the fact that we can define a scalar product on it: if $(E, \|\cdot\|_E)$ is a Hilbert $L^\infty(X,\mathfrak{m})$-module, set

$$\langle v, v'\rangle_E := |v+v'|^2 - |v|^2 - |v'|^2$$

for any $v, v' \in E$. In particular, $L^2T(X,\mathrm{d},\mathfrak{m})$ and $L^2T^*(X,\mathrm{d},\mathfrak{m})$ possess both a scalar product denoted by $\langle\cdot,\cdot\rangle_{L^2T}$ and $\langle\cdot,\cdot\rangle_{L^2T^*}$ respectively in the sequel.

Remark 5.2.11. (Derivations) Another way to define the space $L^2T(X,\mathrm{d},\mathfrak{m})$ is to consider derivations, as in [W00] or [AST17]. Recall that a derivation over a differentiable manifold M is a $\mathbb{R}$-linear map $D : C^\infty(M) \to C^\infty(M)$ satisfying the Leibniz rule, whose pointwise norm is defined by $|D|(x) := \sup\{|D(f)(x)| : f \in C^\infty(M) \text{ with } |\nabla f(x)| = 1\}$ for any $x \in M$. This definition can be extended to the metric measure setting, calling derivation any linear functional $b : \mathrm{Lip}_b(X) \to L^0(X,\mathfrak{m})$ for which there exists some $h \in L^0(X,\mathfrak{m})$ such that $|b(f)| \le h|Df|$ holds $\mathfrak{m}$-a.e. in X, for any $f \in \mathrm{Lip}_b(X)$. The $\mathfrak{m}$-a.e. smallest function h with such property is denoted by $|b|$ and called (local) norm of b. The space of derivations with square integrable local norm coincides with $L^2T(X,\mathrm{d},\mathfrak{m})$: this follows from [G18, Sect. 2.3.1].

Tensors over $(X,\mathrm{d},\mathfrak{m})$ and Hilbert-Schmidt norm

Hilbert $L^\infty(X,\mathfrak{m})$-modules have good tensorization properties. Recall that if M, N are two A-moduli for some commutative ring A, the tensor product $M \otimes N$ of M and N is defined as the quotient C/D where C is the free A-modulus generated by the functions $\{e_{(x,y)} : M \times N \to A\}_{(x,y)\in M\times N}$ defined by

$$e_{(x,y)}(v,v') = \begin{cases} 1_A & \text{if } (v,v') = (x,y), \\ 0 & \text{otherwise.} \end{cases}$$

and D is the submodulus of C generated by the elements

$$\begin{aligned} & e_{(x+u,y)} - e_{(x,y)} - e_{(u,y)}, \\ & e_{(x,y+v)} - e_{(x,y)} - e_{(x,v)}, \\ & e_{(\alpha x,y)} - \alpha e_{(x,y)}, \\ & e_{(x,\alpha y)} - \alpha e_{(x,y)}, \end{aligned}$$

where $x, u \in M$, $y, v \in N$ and $\alpha \in A$. Up to isomorphism of A-modulus, $M \otimes N$ is the unique A-modulus satisfying the following universal property:

there exists a bilinear map $\varphi : M \times N \to M \otimes N$ such that for any A- modulus F and any bilinear map $f : M \times N \to F$, there exists a unique linear map $g : M \otimes N \to F$ such that $f = g \circ \varphi$.

Recall now that the Hilbert tensor product of two Hilbert spaces $(H_1, \langle\cdot,\cdot\rangle_{H_1})$ and $(H_2, \langle\cdot,\cdot\rangle_{H_2})$ is defined as the completion of the algebraic tensor product $H_1 \otimes H_2$ defined above w.r.t. the norm defined out of the following scalar product:

$$\langle x \otimes u, y \otimes v\rangle_{H_1\otimes H_2} := \langle x, y\rangle_{H_1} \langle u, v\rangle_{H_2},$$

for all $x, y \in H_1$ and $u, v \in H_2$. We keep the notation $H_1 \otimes H_2$ to denote this Hilbert space.

Remark 5.2.12. Note that the tensor product of Hilbert spaces does not satisfy the expected universal property: there exists a bilinear and *continuous* map $\varphi : H_1 \times H_2 \to H_1 \otimes H_2$ such that for any Hilbert space F and any bilinear and *continuous* map $f : H_1 \times H_2 \to F$, there exists a unique linear and *continuous* map $g : H_1 \otimes H_2 \to F$ such that $f = g \circ \varphi$ (see [G18, Rk. 1.5.3] for a counter-example).

Let us consider now two Hilbert $L^\infty(X,\mathfrak{m})$-modules $(H_1, \|\cdot\|_{H_1})$ and $(H_2, \|\cdot\|_{H_2})$ with respective scalar products $\langle\cdot,\cdot\rangle_{H_1}$ and $\langle\cdot,\cdot\rangle_{H_2}$. A priori, there is no reason for the Hilbert tensor product $H_1 \otimes H_2$ to be a Hilbert $L^\infty(X,\mathfrak{m})$-module; actually, the construction of a tensor product in the category of Hilbert $L^\infty(X,\mathfrak{m})$-modules is much more involved. Let us provide the main ideas behind this construction, refering to [G18, Sec. 1.3 and 1.5] for a more complete treatment.

We first define the so-called associated $L^0(X,\mathfrak{m})$-module H_1^0 (and similarly H_2^0) as the completion of H_1^0 w.r.t. the topology induced by the distance

$$\mathrm{d}_{H_1^0}(v,w) := \sum_i \frac{1}{2^i \mathfrak{m}(E_i)} \int_{E_i} \min\{|v-w|, 1\} \,\mathrm{d}\mathfrak{m} \qquad \forall v, w \in H_1^0$$

where $(E_i)_i$ is any countable partition of X with sets of finite and positive measure. Note that although the choice of the partition $(E_i)_i$ might modify the distance $\mathrm{d}_{H_1^0}$, the induced topology is not affected.

Define now the Hilbert $L^\infty(X,\mathfrak{m})$-module tensor product between H_1 and H_2 as the subspace of $H_1^0 \overline{\otimes} H_2^0$ made of those elements A such that

$$\| |A|_{HS} \|_{L^2(X,\mathfrak{m})} < +\infty,$$

the so-called Hilbert-Schmidt local norm $|\cdot|_{HS} : H_1^0 \overline{\otimes} H_2^0 \to L^0(X,\mathfrak{m})$ and the space $H_1^0 \overline{\otimes} H_2^0$ being defined as follows. Set first the product function P as the unique bilinear map $H_1^0 \otimes H_2^0 \to L^0(X,\mathfrak{m})$ such that

$$P(v_1 \otimes v_2, w_1 \otimes w_2) = \langle v_1, w_1 \rangle_{H^1} \langle v_2, w_2 \rangle_{H_2}$$

for all $v_1, w_1 \in H_1^0$ and $v_2, w_2 \in H_1^0$. Afterwards define

$$|A|_{HS} = \sqrt{P(A,A)} \qquad \forall A \in H_1^0 \otimes H_2^0.$$

It follows from the construction that the Hilbert-Schmidt local norm $|\cdot|_{HS}$ satisfies the following natural properties: for all Borel set $E \subset X$, all $A, B \in H_1^0 \overline{\otimes} H_2^0$ and all $f \in L^0(X,\mathfrak{m})$,

$$|A|_{HS} = 0 \quad \mathfrak{m}\text{-a.e. on } E \qquad \Longleftrightarrow \qquad A = 0 \quad \text{on } E,$$

$$|A+B|_{HS} \le |A|_{HS} + |B|_{HS} \quad \mathfrak{m}\text{-a.e. on } X,$$

$$|fA|_{HS} = |f||A|_{HS} \quad \mathfrak{m}\text{-a.e. on } X.$$

Finally, set $H_1^0 \overline{\otimes} H_2^0$ as the completion of $H_1^0 \otimes H_2^0$ w.r.t. the topology $\tau_\otimes$ induced by the distance

$$\mathrm{d}_\otimes(A,B) := \sum_i \frac{1}{2^i \mathfrak{m}(E_i)} \int_{E_i} \min(|A-B|_{HS}, 1) \,\mathrm{d}\mathfrak{m}$$

where $(E_i)_i$ is a countable partition of X with sets of finite and positive measure. Here again the topology $\tau_\otimes$ does not depend on the choice of $(E_i)_i$.

We will especially work with the two following Hilbert $L^\infty(X,\mathfrak{m})$-module tensor products:

$$L^2T(X,\mathrm{d},\mathfrak{m})\otimes L^2T(X,\mathrm{d},\mathfrak{m}) \quad \text{and} \quad L^2T^*(X,\mathrm{d},\mathfrak{m})\otimes L^2T^*(X,\mathrm{d},\mathfrak{m})$$

which are easily shown to be dual one to another; we shall denote by $[\cdot,\cdot] : L^2T(X,\mathrm{d},\mathfrak{m})^{\otimes 2}\times L^2T^*(X,\mathrm{d},\mathfrak{m})^{\otimes 2}\to L^0(X,\mathfrak{m})$ the duality pairing. Following the definition of the class $\mathrm{TestF}(X,\mathrm{d},\mathfrak{m})$ which was defined right after (2.3.12), let us introduce the set

$$\mathrm{Test}_2^0(X,\mathrm{d},\mathfrak{m}) := \left\{\sum_{i=1}^n \chi_i\nabla f_i^1\otimes\nabla f_i^2 : \ \chi_i, f_i^1, f_i^2\in \mathrm{TestF}(X,\mathrm{d},\mathfrak{m}),\ n\geq 1\right\}$$

which shall play an important role later.

RCD **metrics**

Now that the appropriate abstract notions have been introduced, we can provide the definition of RCD metrics on the given compact $\mathrm{RCD}^*(K,N)$ space $(X,\mathrm{d},\mathfrak{m})$. Note that such objects are called “Riemannian metrics” in [AHPT17], but here we prefer to call it RCD metrics, in order to let to the Riemannian world the Riemannian designations. For brevity, we drop “$(X,\mathrm{d},\mathfrak{m})$” from the notation $H^{1,2}(X,\mathrm{d},\mathfrak{m})$, $L^2T(X,\mathrm{d},\mathfrak{m})$, $L^2T(X,\mathrm{d},\mathfrak{m})\otimes L^2T(X,\mathrm{d},\mathfrak{m})$, etc. We will also denote by $L^2\cap L^\infty T$ the subspace of $L^2T(X,\mathrm{d},\mathfrak{m})$ made of those element V such that $|V|\in L^\infty(X,\mathfrak{m})$.

Definition 5.2.13. (RCD metrics) We call RCD metric any symmetric bilinear form $\bar g : L^2T\times L^2T\to L^0$ such that

(i) $\bar g$ is L^∞-linear, meaning that $\bar g(\chi V,W)=\chi\bar g(V,W)$ for any $\chi\in L^\infty$ and $V,W\in L^2T$;

(ii) $\bar g$ is non-degenerate, in the sense that $\bar g(V,V)>0$ $\mathfrak{m}$-a.e. on $\{|V|>0\}$ for any $V\in L^2T$.

The set of RCD metrics supports the following natural partial order:

$$g_1\leq g_2 \iff g_1(V,V)\leq g_2(V,V) \qquad \mathfrak{m}\text{-a.e. in } X,\text{ for all } V\in L^2T. \tag{5.2.9}$$

Moreover, we can single out a canonical RCD metric, as shown in the next proposition.

Proposition 5.2.14. *There exists a unique* RCD *metric* g *such that*

$$g(\nabla f_1,\nabla f_2)=\langle\nabla f_1,\nabla f_2\rangle \qquad \mathfrak{m}\text{-}a.e.$$

for all $f_1,f_2\in H^{1,2}$*. Such a metric is called canonical* RCD *metric of* $(X,\mathrm{d},\mathfrak{m})$*.*

To define the norm of a RCD metric in a natural way, let us recall that the usual norm of a bilinear form b defined over an Euclidean space is given by $|b|=\sup\left\{\frac{b(v_1,v_2)}{\langle v_1,v_2\rangle} : v_1,v_2\in V\backslash\{0\}\right\}$. Thanks to the canonical RCD metric g, we can adapt this definition to our context, setting as “local norm” of any RCD metric $\bar g$ the $\mathfrak{m}$-measurable function defined by

$$|\bar g|(x) := \sup\left\{\frac{\bar g(V_1,V_2)}{\langle V_1,V_2\rangle} : V_1,V_2\in L^2T, V_1(x)\neq 0\neq V_2(x)\right\} \quad \text{for } \mathfrak{m}\text{-a.e. } x\in X.$$

Note that $|\bar g|$ is, up to $\mathfrak{m}$-negligible subsets, the smallest positive $\mathfrak{m}$-a.e. function $s\in L^0$ satisfying $\hat g\leq s(\cdot)g$.

We can finally defined a notion of convergence of RCD metrics. We shall consider only the case when

$$\bar g_i\leq Cg \tag{5.2.10}$$

for some C independent of i.

Definition 5.2.15. (Weak convergence of RCD metrics) We say that a family $(\bar{g}_i)_i$ of RCD metrics weakly converges to $\bar{g}$ if for any $V \in L^2T$, the sequence $\bar{g}_i(V,V)$ weakly converges in L^1 to $\bar{g}(V,V)$.

Note that we don't assume any uniformity w.r.t. $V \in L^2T$ for the L^1-weak convergence $\bar{g}_i(V,V) \to \bar{g}(V,V)$, so weak convergence of RCD metrics must be understood as a pointwise weak convergence.

RCD **metric tensors**

It is not difficult to show that $\mathrm{Bilin}(L^2T \times L^2T; L^0)$, namely the space of $L^\infty(X,\mathfrak{m})$-bilinear maps $L^2T \times L^2T \to L^0$, is a $L^2(X,\mathfrak{m})$-normed module with local norm given by $|F| : x \mapsto \sup\{F(V,V)(x)|V|^{-2}(x) \,:\, V \in L^2T,\ |V|(x) \neq 0\}$ for any $F \in \mathrm{Bilin}(L^2T \times L^2T; L^0)$, and that

$$\mathrm{Bilin}(\mathrm{L^2T \times L^2T; L^0}) \simeq L^2T^* \otimes L^2T^*, \tag{5.2.11}$$

so that we can canonically associate a (0,2) tensor $\bar{\mathbf{g}}$ to any RCD metric $\bar{g}$. Such a tensor is called metric tensor, or lifted metric, of $\bar{g}$. More explicitly, $\bar{\mathbf{g}}$ is the only element in $L^2T^* \otimes L^2T^*$ such that

$$\left[\bar{\mathbf{g}}; \sum_i \chi_i \nabla f_i^1 \otimes \nabla f_i^2\right] = \sum_i \chi_i \bar{g}(\nabla f_i^1, \nabla f_i^2) \quad \mathfrak{m}\text{-a.e.}$$

for any $\sum_i \chi_i \nabla f_i^1 \otimes \nabla f_i^2 \in \mathrm{Test}_2^0$. Note that (5.2.11) provides also a tensorial representation of any linear combination of RCD metrics, and in particular the metric tensor of the difference of two RCD metrics $\bar{g}_1, \bar{g}_2$ is $\bar{\mathbf{g}}_\mathbf{1} - \bar{\mathbf{g}}_\mathbf{2}$.

Definition 5.2.16. (Hilbert-Schmidt norm for RCD metrics) The duality formula (5.2.11) gives rise to a natural dual norm for any RCD metric $\bar{g}$ (more generally, for any symmetric and L^∞-bilinear form $h : L^2T \times L^2T \to L^0$) defined as the smallest $\mathfrak{m}$-measurable function $s : X \to [0,+\infty]$ such that

$$[\bar{\mathbf{g}}, A] \le s|A|_{HS} \qquad \forall A \in \mathrm{Test}_2^0.$$

Such a function is called (dual) Hilbert-Schmidt norm of $\bar{g}$ and denoted by $|\bar{\mathbf{g}}|_{HS}$.

As for any $f_1, f_2 \in H^{1,2}$, we have $\bar{g}(\nabla f_1, \nabla f_2) = [\bar{\mathbf{g}}; \nabla f_1 \otimes \nabla f_2] \le |\bar{\mathbf{g}}|_{HS} |\nabla f_1 \otimes \nabla f_2|_{HS}$ and by definition, $|\nabla f_1 \otimes \nabla f_2|_{HS} = \langle \nabla f_1, \nabla f_2 \rangle = g(\nabla f_1, \nabla f_2)$ $\mathfrak{m}$-a.e., we immediately notice that $|g| \le |\bar{\mathbf{g}}|_{HS}$.

Definition 5.2.17. The space $L^2T_2^0$ (or $L^2T_2^0(X, \mathrm{d}, \mathfrak{m})$) is defined as the completion of Test_2^0 with respect to the norm $\| | \cdot |_{HS} \|_{L^2(X,\mathfrak{m})}$.

Note that if $|\bar{\mathbf{g}}|_{HS} \in L^2$, then $\bar{\mathbf{g}}$ extends uniquely to an element of $L^2T_2^0$, still denoted by $\bar{\mathbf{g}}$.

Following the characterization of strong convergence in Hilbert spaces by the combination of weak convergence and convergence of norms, we can use the Hilbert-Schmidt norm of lifted RCD metrics to provide a notion of strong convergence for RCD metrics.

Definition 5.2.18 (Strong convergence of RCD metrics)**.** We say that RCD metrics $\bar{g}_i$ satisfying (5.2.10) L^2-strongly converge to the Riemannian metric $\bar{g}$ if in addition to L^2-weak convergence, we have $|\bar{\mathbf{g}}_\mathbf{i} - \bar{\mathbf{g}}|_{HS} \to 0$ in $L^2(X,\mathfrak{m})$.

Notice that L^2-strong convergence of $\bar{g}_i$ to $\bar{g}$ implies strong convergence in L^1 of $\bar{g}_i(V,V)$ to $\bar{g}(V,V)$ for all $V \in L^2T$; indeed, because of (5.2.10), by density it suffices to check this for $V \in L^2 \cap L^\infty T$, and for this class of vector fields it follows immediately by

$$|\bar{g}_i(V,V) - \bar{g}(V,V)| \le \|V\|_\infty |\bar{\mathbf{g}}_\mathbf{i} - \bar{\mathbf{g}}|_{HS} |V|,$$

so that by integration the L^1 convergence of $\bar{g}_i(V,V)$ can be obtained.

The following convergence criterion will also be useful.

Proposition 5.2.19. *Let $\bar{g}_i$, $\bar{g}$ be RCD metrics with $|\bar{\mathbf{g}}_\mathbf{i}|_{HS}$, $|\bar{\mathbf{g}}|_{HS} \in L^2(X,\mathfrak{m})$. Then $\bar{g}_i$ L^2-strongly converge to $\bar{g}$ as $i \to \infty$ if and only if*

$$\lim_{i\to\infty} \int_X \bar{g}_i(V,V)\,\mathrm{d}\mathfrak{m} = \int_X \bar{g}(V,V)\,\mathrm{d}\mathfrak{m} \qquad \forall V \in L^2 \cap L^\infty T \tag{5.2.12}$$

and

$$\limsup_{i\to\infty} \int_X |\bar{\mathbf{g}}_\mathbf{i}|^2_{HS}\,\mathrm{d}\mathfrak{m} \le \int_X |\bar{\mathbf{g}}|^2_{HS}\,\mathrm{d}\mathfrak{m}.$$

Proof. One implication is obvious. To prove the converse, by the reflexivity (as Hilbert space) of $L^2T^0_2$ it is sufficient to check the weak convergence of $\bar{\mathbf{g}}_\mathbf{i}$ to $\bar{\mathbf{g}}$. The family of linear continuous functionals

$$\mathbf{g} \mapsto \int_X \mathbf{g}(V,V)\,\mathrm{d}\mathfrak{m} \qquad V \in L^2 \cap L^\infty T$$

separates points in $L^2T^0_2$, therefore weak convergence of $\bar{\mathbf{g}}_\mathbf{i}$ follows by (5.2.12). □

Pull-back metrics

For any $t > 0$, a natural way to define a pull-back Riemannian metric on $(X,\mathrm{d},\mathfrak{m})$ is based on an integral version of (5.1.5), namely $\Phi_t^* g_{L^2}(V_1,V_2)$ satisfies:

$$\int_X \Phi_t^* g_{L^2}(V_1,V_2)(x)\,\mathrm{d}\mathfrak{m}(x) = \int_X \Big(\int_X \langle \nabla_x p(x,y,t), V_1(x)\rangle \langle \nabla_x p(x,y,t), V_2(x)\rangle\,\mathrm{d}\mathfrak{m}(y) \Big)\,\mathrm{d}\mathfrak{m}(x), \qquad \forall V_1,\, V_2 \in L^2T. \tag{5.2.13}$$

To see that this is a good definition, notice that the function $G(x,y)$ in the integral in the right hand side of (5.2.13) is pointwise defined as a map $y \mapsto G(\cdot,y)$ with values in L^2 (L^2 integrability follows by the Gaussian estimate (2.3.7)). By Fubini's theorem also the map $x \mapsto \int G(x,y)\,\mathrm{d}\mathfrak{m}(y)$ is well defined, up to $\mathfrak{m}$-negligible sets, and this provides as with the pointwise definition, up to $\mathfrak{m}$-negligible sets, of $\Phi_t^* g_{L^2}(V_1,V_2)$, namely

$$\Phi_t^* g_{L^2}(V_1,V_2)(x) = \int_X \langle \nabla_x p(x,y,t), V_1(x)\rangle \langle \nabla_x p(x,y,t), V_2(x)\rangle\,\mathrm{d}\mathfrak{m}(y). \tag{5.2.14}$$

As a matter of fact, since many objects of the theory are defined only up to $\mathfrak{m}$-measurable sets, we shall mostly work with the equivalent integral formulation.

It is obvious that (5.2.14) defines a symmetric bilinear form on L^2T with values in L^0 and with the L^∞-linearity property. The next proposition ensures that $g_t = \Phi_t^* g_{L^2}$ is indeed a RCD metric on $(X,\mathrm{d},\mathfrak{m})$, provides an estimate from above in terms of the canonical metric, and the representation of the lifted metric $\mathbf{g}_t$.

Proposition 5.2.20. *Formula (5.2.13) defines a* RCD *metric* g_t *on* L^2T *with*

$$\int_X |\mathbf{g}_t|^2_{HS}\,\mathrm{d}\mathfrak{m} = \sum_i e^{-2\lambda_i t}\int_X g_t(\nabla\varphi_i,\nabla\varphi_i)\,\mathrm{d}\mathfrak{m} = \sum_i e^{-2\lambda_i t}\int_X\int_X |\langle\nabla_x p(x,y,t),\nabla\varphi_i\rangle|^2\,\mathrm{d}\mathfrak{m}(y)\,\mathrm{d}\mathfrak{m}(x), \tag{5.2.15}$$

$$|\mathbf{g}_t|_{HS}(x) = \left|\int_X \nabla_x p(x,y,t)\otimes\nabla_x p(x,y,t)\,\mathrm{d}\mathfrak{m}(y)\right|_{HS} \qquad \text{for } \mathfrak{m}\text{-a.e. } x\in X \tag{5.2.16}$$

and representable as the HS*-convergent series*

$$\mathbf{g}_t = \sum_{i=1}^{\infty} e^{-2\lambda_i t}\mathrm{d}\varphi_i\otimes\mathrm{d}\varphi_i \qquad \text{in } L^2T^0_2. \tag{5.2.17}$$

Moreover, the rescaled metric $t\mathfrak{m}(B_{\sqrt{t}}(\cdot))g_t$ *satisfies*

$$t\mathfrak{m}(B_{\sqrt{t}}(\cdot))g_t \le C(K,N)g \qquad \forall t\in(0,C_4^{-1}), \tag{5.2.18}$$

where C_4 *is the constant in (2.3.7).*

Proof. Let us prove (5.2.18), assuming $0<t<\min\{1,C_4^{-1}\}$. For $V\in L^2T$ and $y\in X$, the Gaussian estimate (2.3.7) with $\epsilon=1$ and the upper bound on t yield

$$\int_X |\langle\nabla_x p(x,y,t),V(x)\rangle|^2\,\mathrm{d}\mathfrak{m}(x) \le \int_X \frac{C_3^2 e^2}{t\mathfrak{m}(B_{\sqrt{t}}(x))^2}\exp\left(\frac{-2\mathrm{d}(x,y)^2}{5t}\right)|V(x)|^2\,\mathrm{d}\mathfrak{m}(x). \tag{5.2.19}$$

By integration with respect to y and taking into account (5.3.7) with $\ell=0$ (applied to the rescaled space $(X,\mathrm{d}_t,\mathfrak{m})$ with $\mathrm{d}_t=\mathrm{d}/\sqrt{t}$, whose constants c_0, c_1, c_2 can be estimated uniformly w.r.t. t, since $(X,\mathrm{d}_t,\mathfrak{m})$ is RCD$^*(t^2K,N)$), we recover (5.2.18).

Let us prove now non-degeneracy of g_t, using the expansion (4.0.22) of $\nabla_x p$. For all $V\in L^2T$ we have

$$\begin{aligned}
&\int_X g_t(V,V)\,\mathrm{d}\mathfrak{m}\\
&= \int_X\int_X \langle\nabla_x p(x,y,t),V(x)\rangle^2\,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}\mathfrak{m}(y)\\
&= \int_X\int_X \left(\sum_i e^{-\lambda_i t}\varphi_i(y)\langle\nabla\varphi_i,V\rangle(x)\right)^2\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}\mathfrak{m}(y)\\
&= \int_X\int_X \sum_{i,j} e^{-(\lambda_i+\lambda_j)t}\varphi_i(y)\varphi_j(y)\langle\nabla\varphi_i,V\rangle(x)\langle\nabla\varphi_j,V\rangle(x)\,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}\mathfrak{m}(y)\\
&= \sum_i e^{-2\lambda_i t}\int_X \langle\nabla\varphi_i,V\rangle^2\,\mathrm{d}\mathfrak{m}.
\end{aligned} \tag{5.2.20}$$

By L^∞-linearity, it suffices to check that $\|g_t(V,V)\|_{L^1}=0$ implies $|V|(x)=0$ for $\mathfrak{m}$-a.e. $x\in X$. Thus assume $\|g_t(V,V)\|_{L^1}=0$. Then (5.2.20) yields that for all i,

$$\langle\nabla\varphi_i,V\rangle(x)=0 \quad \text{for } \mathfrak{m}\text{-a.e. } x\in X. \tag{5.2.21}$$

Since L^2T is generated, in the sense of L^2-modules, by $\{\nabla f:\ f\in H^{1,2}\}$ and since the vector space spanned by φ_i is dense in $H^{1,2}$, it is easily seen that L^2T is generated, in the sense of L^2-modules, also by $\{\nabla\varphi_i:\ i\ge 1\}$. In particular (5.2.21) shows that $V=0$.

In order to prove (5.2.15) and (5.2.17), fix an integer $N \geq 1$ and let

$$\mathbf{g}_t^N := \sum_{i=1}^N e^{-2\lambda_i t} \mathrm{d}\varphi_i \otimes \mathrm{d}\varphi_i.$$

Then

$$\begin{aligned}
\int_X |\mathbf{g}_t^N|_{HS}^2 \,\mathrm{d}\mathfrak{m} &= \sum_{i,j=1}^N e^{-2(\lambda_i+\lambda_j)t} \int_X \langle \nabla\varphi_i, \nabla\varphi_j\rangle^2 \,\mathrm{d}\mathfrak{m} \\
&= \sum_{i=1}^N e^{-2\lambda_i t} \left(\sum_{j=1}^N e^{-2\lambda_j t} \int_X \langle \nabla\varphi_i, \nabla\varphi_j\rangle^2 \,\mathrm{d}\mathfrak{m} \right) \\
&\leq \sum_{i=1}^\infty e^{-2\lambda_i t} \int_X g_t(\nabla\varphi_i, \nabla\varphi_i) \,\mathrm{d}\mathfrak{m} \\
&\leq C \sum_{i=1}^\infty e^{-2\lambda_i t} \int_X |\nabla\varphi_i|^2 \,\mathrm{d}\mathfrak{m} \leq C \sum_{i=1}^\infty e^{-2\lambda_i t} \lambda_i < \infty,
\end{aligned} \tag{5.2.22}$$

where $C = C(K, N, t)$ and we used (5.2.18) and (5.2.20), together with a uniform lower bound on $\mathfrak{m}(B_{\sqrt{t}}(x))$, $x \in \operatorname{supp}\mathfrak{m}$. By Proposition 4.0.18, an analogous computation shows that $\||\mathbf{g}_t^N - \mathbf{g}_t^M|_{HS}\|_2 \to 0$ as N, $M \to \infty$, hence $\mathbf{g}_t^N \to \tilde{\mathbf{g}}_t$.

Passing to the limit in the identity

$$\int_X \langle \mathbf{g}_t^N, \chi^2 \nabla f \otimes \nabla f\rangle \,\mathrm{d}\mathfrak{m} = \sum_{i=1}^N e^{-2\lambda_i t} \int_X \chi^2 \langle \nabla\varphi_i, \nabla f\rangle^2 \,\mathrm{d}\mathfrak{m}$$

with $\chi \in L^\infty$, $f \in \mathrm{TestF}$, we obtain from (5.2.20) with $V = \chi\nabla f$

$$\int_X \langle \tilde{\mathbf{g}}_t, \chi^2 \nabla f \otimes \nabla f\rangle \,\mathrm{d}\mathfrak{m} = \int_X \langle \mathbf{g}_t, \chi^2 \nabla f \otimes \nabla f\rangle \,\mathrm{d}\mathfrak{m}.$$

Hence $\tilde{\mathbf{g}}_t = \mathbf{g}_t$ (in particular $\mathbf{g}_t$ has finite Hilbert-Schmidt norm and it can be extended to $L^2 T_2^0$).

In order to prove (5.2.15) it is sufficient to pass to the limit as $N \to \infty$ in

$$\int_X |\mathbf{g}_t^N|_{HS}^2 \,\mathrm{d}\mathfrak{m} = \int_X \sum_{i,j=1}^N e^{-2(\lambda_i+\lambda_j)t} \langle \nabla\varphi_i, \nabla\varphi_j\rangle^2 \,\mathrm{d}\mathfrak{m}$$

taking (5.2.20) into account.

Finally, (5.2.16) follows by the observation that $\mathbf{g}_t$ is induced by the scalar product, w.r.t. the Hilbert-Schmidt norm, with the vector $\int_X \nabla_x p(x,y,t) \otimes \nabla_x p(x,y,t) \,\mathrm{d}\mathfrak{m}(y)$. □

5.3 Convergence results via blow-up

In this section, we study the L^2-convergence of the rescaled metrics $\mathrm{sc}_t g_t$ as $t \to 0^+$ on a given compact $\mathrm{RCD}^*(K, N)$ space $(X, \mathrm{d}, \mathfrak{m})$. Here the function $\mathrm{sc}_t : X \to \mathbb{R}$ is a suitable scaling function whose expression requires an immediate discussion. In the Riemannian case $(X, \mathrm{d}, \mathfrak{m}) = (M^n, \mathrm{d}_g, \mathrm{vol}_g)$, one knows by (5.1.6) that $\mathrm{sc}_t \equiv c_n t^{(n+2)/2}$ where $c_n > 0$ is a constant depending only on the dimension n. In the RCD setting, we have two choices:

- on one hand, the analogy with the Riemannian setting suggests to take $\mathrm{sc}_t \equiv t^{(n+2)/2}$, where $n = \dim_{\mathrm{d},\mathfrak{m}}(X)$ (recall Theorem 2.3.15);

- on the other hand, since the RCD setting is closer to a weighted Riemannian setting, we can also set $\mathrm{sc}_t = t\mathfrak{m}(B_{\sqrt{t}}(\cdot))$, to take into account the effect of the weight θ, namely the density of $\mathfrak{m}$ w.r.t. $\mathcal{H}^n \llcorner \mathcal{R}_n$.

In both cases, we prove that $\mathrm{sc}_t g_t$ converges to a rescaled version of the canonical Riemannian metric g on $(X, \mathrm{d}, \mathfrak{m})$, where the rescaling reflects the choice of sc_t. To be more precise, we prove in Theorem 5.3.14 that $\hat{g}_t := t\mathfrak{m}(B_{\sqrt{t}}(x))g_t$ converges to $\hat{g} = c_n g$, where c_n is a constant depending only on the dimension (5.3.14). Concerning the other scaling, as $t^{(n+2)/2} = (\sqrt{t}^n/\mathfrak{m}(B_{\sqrt{t}}(x)))t\mathfrak{m}(B_{\sqrt{t}}(x))$, we prove in Theorem 5.3.16 that the limit of $\tilde{g}_t := t^{(n+2)/2} g_t$ is $(\omega_n \theta)^{-1} 1_{\mathcal{R}_n^*} \hat{g}$ (notice that this is a good definition, since θ is well-defined up to $\mathcal{H}^n$-negligible sets and $\mathfrak{m}$ and $\mathcal{H}^n$ are mutually absolutely continuous on $\mathcal{R}_n^*$).

Technical preliminaries

We shall denote by $p_{k,e}$ the Euclidean heat kernel in $\mathbb{R}^k$, given by

$$p_{k,e}(x, y, t) := \frac{1}{(4\pi t)^{k/2}} \exp\left(-\frac{|x-y|^2}{4t}\right) \tag{5.3.1}$$

and recall the classical identity

$$\frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} x^2 \exp\left(-\frac{x^2}{2t}\right) \,\mathrm{d}x = t \tag{5.3.2}$$

for the variance of the centered Gaussian measures. Furthermore, we shall often use the scaling formula

$$\tilde{p}(x, y, s) = b^{-1} p(x, y, a^{-2}s) \qquad \forall x,\, y \in \operatorname{supp}\mathfrak{m},\ \ \forall s > 0 \tag{5.3.3}$$

relating for any a, $b > 0$, the heat kernel $\tilde{p}$ of the rescaled space $(X, a\mathrm{d}, b\mathfrak{m})$ to the heat kernel p of $(X, \mathrm{d}, \mathfrak{m})$.

Because of Bishop-Gromov's inequality (Theorem 2.1.14), the following lemma, whose proof is omitted for brevity, applies to the whole class of RCD$^*(K, N)$ spaces. It is a simple consequence of Cavalieri's formula together with (5.3.5) and its useful corollary:

$$\frac{\mathfrak{m}(B_1(x))}{\mathfrak{m}(B_1(y))} \le c_2 \exp\left(c_1 \mathrm{d}(x, y)\right) \qquad \forall x,\, y \in \operatorname{supp}\mathfrak{m} \tag{5.3.4}$$

with $c_2 = c_0 e^{c_1}$.

Lemma 5.3.1. *Let $(Y, \mathrm{d}_Y, \mathfrak{m}_Y)$ be a metric measure space and let $x \in \operatorname{supp}\mathfrak{m}_Y$ be satisfying*

$$\frac{\mathfrak{m}(B_R(x))}{\mathfrak{m}(B_1(x))} \le c_0 e^{c_1 R} \qquad \forall R \ge 1 \tag{5.3.5}$$

for some constants c_0, $c_1 > 0$. Then:

(1) for any $\delta > 0$ there exists $L_0 = L_0(\delta, c_0, c_1) > 1$ such that

$$\int_{Y \setminus B_{L_0}(x)} \mathfrak{m}_Y(B_1(y)) \exp\left(-\frac{2\mathrm{d}_Y^2(x, y)}{5}\right) \mathrm{d}\mathfrak{m}_Y(y) < \delta(\mathfrak{m}_Y(B_1(x)))^2; \tag{5.3.6}$$

(2) for any $\ell \in \mathbb{Z}$ there exists $C = C(\ell, c_0, c_1) \in [0, \infty)$ such that

$$\int_Y \mathfrak{m}_Y(B_1(y))^\ell \exp\left(-\frac{2\mathrm{d}_Y^2(x, y)}{5}\right) \mathrm{d}\mathfrak{m}_Y(y) \le C(\mathfrak{m}_Y(B_1(x)))^{\ell+1}. \tag{5.3.7}$$

The following result is a consequence of the rectifiability of the set $\mathcal{R}_n$ in Theorem 2.3.15, which provides a canonical isometry between the tangent bundle $L^2T(X, \mathrm{d}, \mathfrak{m})$ as defined later on and the tangent bundle defined via measured Gromov-Hausdorff limits, see [GP16, Thm 5.1] for the proof.

Lemma 5.3.2. *If $(X, \mathrm{d}, \mathfrak{m})$ is $\mathrm{RCD}^*(K, N)$, the canonical metric g of Proposition 5.2.14 satisfies*

$$|\mathbf{g}|_{HS}^2 = n \qquad \mathfrak{m}\text{-a.e. in } X, \text{ with } n = \dim_{\mathrm{d},\mathfrak{m}}(X). \tag{5.3.8}$$

The pointwise convergence of heat kernels for a convergent sequence of $\mathrm{RCD}^*(K, N)$ spaces has been proved in Chapter 4 ([AHT18, Thm. 3.3]); building on this, and using the "tightness" estimate (5.3.9) below, one can actually prove the global $H^{1,2}$-strong convergence.

Theorem 5.3.3 ($H^{1,2}$-strong convergence of heat kernels)**.** *For all convergent sequences $t_i \to t$ in $(0, \infty)$ and $y_i \in X_i \to y \in X$, $p_i(\cdot, y_i, t_i) \in H^{1,2}(X_i, \mathrm{d}_i, \mathfrak{m}_i)$ $H^{1,2}$-strongly converge to $p(\cdot, y, t) \in H^{1,2}(X, \mathrm{d}, \mathfrak{m})$.*

Proof. By a rescaling argument we can assume $t_i = t = 1$. Applying Theorem 2.4.24 for p_i with (2.3.8) yields that $p_i(\cdot, y_i, 1)$ $H^{1,2}_{\mathrm{loc}}$-strongly converge to $p(\cdot, y, 1)$. We claim that for any $\delta > 0$ there exists $L := L(K^-, N, \delta) > 1$ such that for any $\mathrm{RCD}^*(K, N)$-space (Y, d, ν) and any $y \in \operatorname{supp} \nu$ one has (q denoting its heat kernel)

$$\int_{Y \setminus B_L(y)} q^2(z, y, 1) + |\nabla_z q(z, y, 1)|^2 \, \mathrm{d}\nu(z) \le \frac{\delta}{\nu^2(B_1(y))}. \tag{5.3.9}$$

Indeed, let us prove the estimate for q, the proof of the estimate for $|\nabla_z q|$ (based on (2.3.7)) being similar. Combining (5.3.4) with the Gaussian estimate (2.3.6) with $\epsilon = 1$, one obtains

$$\int_{Y \setminus B_L(y)} q^2(z, y, 1) \, \mathrm{d}\nu(z) \le \frac{c_2^2 C_1^2 e^{C_2}}{\nu^2(B_1(y))} \int_{Y \setminus B_L(y)} \exp\Big(-\frac{2}{5}\mathrm{d}^2(z, y) + 2c_1 \mathrm{d}(z, y)\Big) \, \mathrm{d}\nu(z)$$

and then one can use the exponential growth condition on $\nu(B_R(y))$, coming from the Bishop-Gromov estimate (Theorem 2.1.14), to obtain that the left hand side is smaller than $\delta/\nu^2(B_1(y))$ for $L = L(K^-, N, \delta)$ sufficiently large.

Combining (5.3.9) with the $H^{1,2}_{\mathrm{loc}}$-strong convergence of p_i shows that

$$\lim_{i \to \infty} \|p_i(\cdot, y_i, 1)\|_{H^{1,2}(X_i, \mathrm{d}_i, \mathfrak{m}_i)} = \|p(\cdot, y, 1)\|_{H^{1,2}(X, \mathrm{d}, \mathfrak{m})}, \tag{5.3.10}$$

which completes the proof. □

We shall also use the following local compactness theorem under BV bounds, applied to sequences of Sobolev functions.

Theorem 5.3.4. *Assume that $f_i \in H^{1,2}(B_2(x_i))$ satisfy*

$$\sup_i \left(\|f_i\|_{L^\infty(B_2(x_i))} + \int_{B_2(x_i)} |\nabla f_i| \, \mathrm{d}\mathfrak{m}_i \right) < \infty.$$

Then (f_i) has a subsequence L^p-strong convergent on $B_1(x)$ for all $p \in [1, \infty)$.

Proof. The proof of the compactness w.r.t. L^1-strong convergence can be obtained arguing as in [AH17a, Prop. 7.5] (where the result is stated in global form, for normalized metric measure spaces, even in the BV setting), using good cut-off functions, see also [H15, Prop. 3.39] where a uniform L^p bound on gradients, for some $p > 1$ is assumed. Then, because of the uniform L^∞ bound, the convergence is L^p-strong for any $p \in [1, \infty)$, see [AH17a, Prop. 1.3.3(e)]. □

Harmonic points

We now introduce another technical concept, namely harmonic points of vector fields. Those are points at which a vector field infinitesimally (meaning after blow-up of the metric measure space) looks like the gradient of a harmonic function.

Let us first recall the definition of Lebesgue point.

Definition 5.3.5 (Lebesgue point)**.** Let $f \in L^p_{loc}(X, \mathfrak{m})$ with $p \in [1, \infty)$. We say that $x \in X$ is a p-Lebesgue point of f if there exists $a \in \mathbb{R}$ such that

$$\lim_{r \to 0} \fint_{B_r(x)} |f(y) - a|^p \, \mathrm{d}\mathfrak{m}(y) = 0.$$

The real number a is uniquely determined by this condition and denoted by $f^*(x)$. The set of p-Lebesgue points of f is Borel and denoted by $\mathrm{Leb}_p(f)$.

Note that the property of being a p-Lebesgue point and $f^*(x)$ do not depend on the choice of the representative in the equivalence class, and that $x \in \mathrm{Leb}_p(f)$ implies $\fint_{B_r(x)} |f(y)|^p \, \mathrm{d}\mathfrak{m} \to |f^*(x)|^p$ as $r \downarrow 0$. It is well-known (see e.g. [Hein01]) that the doubling property ensures that $\mathfrak{m}(X \setminus \mathrm{Leb}_p(f)) = 0$, and that the set $\{x \in \mathrm{Leb}_p(f) : \ f^*(x) = f(x)\}$ (which does depend on the choice of representative in the equivalence class) has full measure in X. When we apply these properties to a characteristic function $f = 1_A$ we obtain that $\mathfrak{m}$-a.e. $x \in A$ is a point of density 1 for A and $\mathfrak{m}$-a.e. $x \in X \setminus A$ is a point of density 0 for A.

Definition 5.3.6 (Harmonic point of a function)**.** Let $x \in X$, $z \in B_R(x) \cap \operatorname{supp} \mathfrak{m}$ and let $f \in H^{1,2}(B_R(x), \mathrm{d}, \mathfrak{m})$. We say that z *is a harmonic point of* f if $z \in \mathrm{Leb}_2(|\nabla f|)$ and for any $(Y, \mathrm{d}_Y, \mathfrak{m}_Y, y) \in \mathrm{Tan}(X, \mathrm{d}, \mathfrak{m}, z)$, mGH limit of $(X, t_i^{-1}\mathrm{d}, \mathfrak{m}(B_{t_i}(z))^{-1}\mathfrak{m}, z)$, where $t_i \to 0^+$, there exist a subsequence $(t_{i(j)})$ of (t_i) and $\hat{f} \in \mathrm{Lip}(Y, \mathrm{d}_Y) \cap \mathrm{Harm}(Y, \mathrm{d}_Y, \mathfrak{m}_Y)$ such that the rescaled functions

$$f_{j,z} := \frac{1}{t_{i(j)}} \left(f - \fint_{B_{t_{i(j)}}(z)} f \, \mathrm{d}\mathfrak{m} \right)$$

in the spaces $(X, t_{i(j)}^{-1}\mathrm{d}, \mathfrak{m}(B_{t_{i(j)}}(z))^{-1}\mathfrak{m}, z)$, $H^{1,2}_{\mathrm{loc}}$-strongly converge to $\hat{f}$ as $j \to \infty$. We denote by $H(f)$ the set of harmonic points of f.

Note that being an harmonic point also does not depend on the choice of versions of f and $|\nabla f|$ and that this notion is closely related to the differentiability of f at x. For instance when $(X, \mathrm{d}, \mathfrak{m}) = (M, g, \mathrm{vol})$ is a smooth Riemannian manifold and $f \in C^1(M)$, every point $x \in M$ is a harmonic point of f, and the function $\hat{f}$ appearing by blow-up is unique and equals the differential of f at x. On the other hand if $f(x) = |x|$ on $\mathbb{R}^n$, then 0_n is not an harmonic point of f.

Another way to understand Definition 5.3.6 is the following. The rescaled functions $f_{j,z}$ tells us how the function f differs from satisfying the mean-value property at the scale $B_{t_{i(j)}}(z)$.

The definition of harmonic point can be extended to vector fields as follows.

Definition 5.3.7 (Harmonic point of L^2-vector fields)**.** Let $V \in L^2T(X, \mathrm{d}, \mathfrak{m})$ and let $z \in \operatorname{supp}\mathfrak{m}$. We say that z *is a harmonic point of* V if there exists $f \in H^{1,2}(X, \mathrm{d}, \mathfrak{m})$ such that $z \in H(f)$ and

$$\lim_{r\downarrow 0} \fint_{B_r(z)} |V - \nabla f|^2 \, \mathrm{d}\mathfrak{m} = 0. \tag{5.3.11}$$

We denote by $H(V)$ the set of harmonic points of V.

Obviously, if $V = \nabla f$ for some $f \in H^{1,2}(X, \mathrm{d}, \mathfrak{m})$, then Definition 5.3.7 is compatible with Definition 5.3.6. Notice also that, as a consequence of (5.3.11) and the condition $z \in \mathrm{Leb}_2(|\nabla f|)$, $\fint_{B_r(z)} |V|^2 \, \mathrm{d}\mathfrak{m}$ converge as $r \to 0^+$ to $(|\nabla f|^*)^2(z)$ and we shall denote this precise value by $|V|^{2*}(z)$. By Lebesgue theorem, this limit coincides for $\mathfrak{m}$-a.e. $z \in H(V)$ with $|V|^2(z)$. The statement and proof of the following result are very closely related to Cheeger's Theorem 2.2.18; we simply adapt the proof and the statement to our needs.

Theorem 5.3.8. *For all* $V \in L^2T(X, \mathrm{d}, \mathfrak{m})$ *one has* $\mathfrak{m}(X \setminus H(V)) = 0$.

Proof. Step 1: the case of gradient vector fields $V = \nabla f$. Recall that $\mathrm{RCD}^*(K, N)$ spaces are doubling and satisfy a local Poincaré inequality, see (2.1.16). We fix $z \in \mathrm{Leb}_2(|\nabla f|)$ where (2.2.9) of Theorem 2.2.18 holds and we prove that $z \in H(f)$. Let (t_i) and $(Y, \mathrm{d}_Y, y, \mathfrak{m}_Y)$, $f_{t_i,z}$ be as in Definition 5.3.6. Take $R > 1$, set $\mathrm{d}_i = t_i^{-1}\mathrm{d}$, $\mathfrak{m}_i = \mathfrak{m}/\mathfrak{m}(B_{t_i}(x))$ and write $H^{1,2}_i, L^2_i$ and Ch_i for $H^{1,2}(B^{\mathrm{d}_i}_R(z), \mathrm{d}_i, \mathfrak{m}_i), L^2(B^{\mathrm{d}_i}_R(z), \mathfrak{m}_i)$ and $\mathrm{Ch}_{(B^{\mathrm{d}_i}_R(z), \mathrm{d}_i, \mathfrak{m}_i)}$ respectively. Along with the existence in $[0, \infty)$ of the limit $(|\nabla f|^*(x))^2$ of $\fint_{B_r(z)} |\nabla f|^2 \, \mathrm{d}\mathfrak{m}$ as $r \downarrow 0$, this provides for i large enough a uniform control of the $H^{1,2}_i$-norms of $f_{t_i,z}$: on $B^{\mathrm{d}_i}_R(z)$,

$$\begin{aligned}\|f_{t_i,z}\|^2_{H^{1,2}_i} = \|f_{t_i,z}\|^2_{L^2_i} + \mathrm{Ch}_i(f_{t_i,z}) = t_i^{-2}\|f - \fint_{B^{\mathrm{d}_i}_1} f \, \mathrm{d}\mathfrak{m}\|^2_{L^2_i} + \frac{\mathfrak{m}(B^{\mathrm{d}_i}_R(z))}{\mathfrak{m}(B^{\mathrm{d}_i}_1(z))} \fint_{B^{\mathrm{d}}_{t_iR}(z)} |\nabla f|^2 \, \mathrm{d}\mathfrak{m} \\ \le C(K, N, R)((|\nabla f|^*(x))^2 + 1),\end{aligned}$$

where $C(K, N, R) > 0$ depends on the doubling and Poincaré constants. Thus, since $R > 1$ is arbitrary, by Theorem 2.4.23 and a diagonal argument there exist a subsequence (s_i) of (t_i) and $\hat{f} \in H^{1,2}_{\mathrm{loc}}(Y, \mathrm{d}_Y, \mathfrak{m}_Y)$ such that $f_{s_i,z}$ $H^{1,2}_{\mathrm{loc}}$-weakly converge to $\hat{f}$.

Let us prove that $f_{s_i,z}$ is a $H^{1,2}_{\mathrm{loc}}$-strong convergent sequence. Let $R > 0$ where (2.4.4) holds and let $h_{i,R}$ be the harmonic replacement of $f_{s_i,z}$ on $B^{\mathrm{d}_i}_R(z)$ (recall $\mathrm{d}_i = s_i^{-1}\mathrm{d}$). Then applying Proposition 2.4.26 yields that $h_{i,R}$ $H^{1,2}$-weakly converge to the harmonic replacement h_R of $\hat{f}$ on $B_R(y)$. Since $h_{i,R}$ are harmonic, by Theorem 2.4.24, $h_{i,r}$ $H^{1,2}$-strongly converge to h_R on $B_r(z)$ for any $r < R$.

Note that Proposition 2.2.23 yields

$$\begin{aligned}\int_{B^{\mathrm{d}_i}_R(z)} |\nabla(f_{s_i,z} - h_{i,R})|^2 \, \mathrm{d}\mathfrak{m}_i &= \int_{B^{\mathrm{d}_i}_R(z)} |\nabla f_{s_i,z}|^2 \, \mathrm{d}\mathfrak{m}_i - \int_{B^{\mathrm{d}_i}_R(z)} |\nabla h_{i,R}|^2 \, \mathrm{d}\mathfrak{m}_i \\ &= \fint_{B_{Rs_i}(z)} |\nabla f|^2 \, \mathrm{d}\mathfrak{m} - \inf_{\varphi \in H^{1,2}_0(B_{Rs_i}(z), \mathrm{d}, \mathfrak{m})} \fint_{B_{Rs_i}(z)} |\nabla(f + \varphi)|^2 \, \mathrm{d}\mathfrak{m}.\end{aligned} \tag{5.3.12}$$

Thus, since by our choice of z the right hand side of (5.3.12) goes to 0 as $i \to \infty$, the Poincaré inequality gives $\|f_{s_i,z} - h_{i,R}\|_{L^2(B^{\mathrm{d}_i}_R(z))} \to 0$, hence $f_{s_i,z}$ $H^{1,2}$-weakly converge to h_R on $B_R(y)$, so that $\hat{f} = h_R$ on $B_R(y)$. In addition, the $H^{1,2}$-strong convergence on balls

$B_r(z)$, $r < R$, of the functions $h_{i,R}$ shows that $f_{s_i,z}$ $H^{1,2}$-strongly converge to $\hat{f}$ on $B_r(z)$ for any $r < R$. Since R has been chosen subject to the only condition (2.4.4), which holds with at most countably many exceptions, we see that $\hat{f} \in \mathrm{Harm}(Y, \mathrm{d}_Y, \mathfrak{m}_Y)$ and that $f_{s_i,z}$ $H^{1,2}_{\mathrm{loc}}$-strongly converge to $\hat{f}$.

Finally, let us show that $\hat{f}$ has a Lipschitz representative. It is easy to check that the condition $z \in \mathrm{Leb}_2(|\nabla f|)$, namely

$$\lim_{r\downarrow 0} \fint_{B_r(z)} \left| |\nabla f| - |\nabla f|^*(z) \right|^2 \mathrm{d}\mathfrak{m} = 0$$

with the $H^{1,2}_{\mathrm{loc}}$-strong convergence of $f_{s_i,z}$ yield $|\nabla \hat{f}|(w) = |\nabla f|^*(z)$ for $\mathfrak{m}_Y$-a.e. $w \in Y$. Thus the Sobolev-Lipschitz property shows that $\hat{f}$ has a Lipschitz representative.
Step 2: the general case when $V \in L^2(T(X, \mathrm{d}, \mathfrak{m}))$. Let C, M, k, F_i be given by Theorem 2.2.18. It is sufficient to prove existence of f as in Definition 5.3.7 for $\mathfrak{m}$-a.e. $x \in C$. Since $\int_{B_r(x)\setminus C} |V|^2 \,\mathrm{d}\mathfrak{m} = o(\mathfrak{m}(B_r(x)))$ for $\mathfrak{m}$-a.e. $x \in C$, we can assume with no loss of generality, possibly replacing V by $1_{X\setminus C}V$, that $V = 0$ on $X \setminus C$. As illustrated in [G18, Cor. 2.5.2] (by approximation of the χ_i by simple functions) the expansion (2.2.10) gives also

$$1_C \left(\nabla f - \sum_{i=1}^{k} \alpha_i \nabla F_i \right) = 0$$

for all $f \in \mathrm{Lip}(X, \mathrm{d}) \cap H^{1,2}$, with $\sum_i \alpha_i^2 \le M|\nabla f|$ $\mathfrak{m}$-a.e. on C. By the approximation in Lusin's sense of Sobolev by Lipschitz functions and the locality of the pointwise norm, the same is true for Sobolev functions f. Eventually, by linearity and density of gradients, we obtain the representation

$$V = \sum_{i=1}^{k} \alpha_i \nabla F_i$$

for suitable coefficients $\alpha_i \in L^2(X, \mathfrak{m})$, null on $X \setminus C$. It is now easily seen that if x is an harmonic point for all F_i and a 2-Lebesgue point of all α_i, then $x \in H(V)$ with

$$f(y) := \sum_{i=1}^{k} \alpha_i^*(x) F_i(y).$$

□

The behavior of $t\mathfrak{m}(B_{\sqrt{t}}(x))g_t$ as $t \downarrow 0$

The main purpose of this paragraph is to prove Theorem 5.3.14, i.e. the L^2-strong convergence of the metrics

$$\hat{g}_t := t\mathfrak{m}(B_{\sqrt{t}}(\cdot))g_t \xrightarrow{t\downarrow 0} \hat{g}, \tag{5.3.13}$$

where $\hat{g}$ is the normalized Riemannian metric on $(X, \mathrm{d}, \mathfrak{m})$ defined by $c_n g$, where $n = \dim_{\mathrm{d},\mathfrak{m}}(X)$, the dimensional constant c_n is given by

$$c_n := \frac{\omega_n}{(4\pi)^n} \int_{\mathbb{R}^n} \left| \partial_{x_1} (e^{-|x|^2/4}) \right|^2 \mathrm{d}x = \frac{\omega_n}{4\sqrt{2\pi}^n}, \tag{5.3.14}$$

and we used (5.3.2) for the explicit computation of the integral.

Here is an important proposition whose proof contains the main technical ingredients that shall be used in the sequel.

Proposition 5.3.9. *("pointwise" convergence) Let $V \in L^2(T(X,\mathrm{d},\mathfrak{m}))$ and $y \in \mathcal{R}_n \cap H(V)$. Then*

$$\lim_{t\downarrow 0}\int_X t\mathfrak{m}(B_{\sqrt{t}}(x))|\langle\nabla_x p(x,y,t),V(x)\rangle|^2\,\mathrm{d}\mathfrak{m}(x) = c_n|V|^{2*}(y). \tag{5.3.15}$$

Proof. As $y \in H(V)$, there exists $f \in H^{1,2}$ such that $⨍_{B_r(x)}|V-\nabla f|^2\,\mathrm{d}\mathfrak{m} \to 0$ as $r\downarrow 0$. With $W = V - \nabla f$, let us first prove that

$$\lim_{t\downarrow 0}\int_X t\mathfrak{m}(B_{\sqrt{t}}(x))|\langle\nabla_x p(x,y,t),W(x)\rangle|^2\,\mathrm{d}\mathfrak{m}(x) = 0. \tag{5.3.16}$$

Using the heat kernel estimate (2.3.7) with $\epsilon = 1$ we need to estimate, for $0 < t < C_4^{-1}$,

$$\int_X \frac{1}{\mathfrak{m}(B_{\sqrt{t}}(x))}\exp\left(-\frac{2\mathrm{d}^2(x,y)}{5t}\right)|W(x)|^2\,\mathrm{d}\mathfrak{m}(x)$$

and use (5.3.4) to reduce the proof to the estimate of

$$\frac{1}{\mathfrak{m}(B_{\sqrt{t}}(y))}\int_X \exp\left(-\frac{2\mathrm{d}^2(x,y)}{5t}+c_1\frac{\mathrm{d}(x,y)}{\sqrt{t}}\right)|W(x)|^2\,\mathrm{d}\mathfrak{m}(x).$$

Using the identity $\int f(\mathrm{d}(\cdot,y))\,\mathrm{d}\mu = -\int_0^\infty \mu(B_r(y))f'(r)\,\mathrm{d}r$ with $\mu_y := \exp(c_1\mathrm{d}(\cdot,y)/\sqrt{t})|W|^2\mathfrak{m}$ and $f_y(r) = \exp(-2r^2/(5t))$, we need to estimate

$$-\frac{1}{\mathfrak{m}(B_{\sqrt{t}}(y))}\int_0^\infty \mu(B_r(x))f_y'(r)\,\mathrm{d}r.$$

Now, write $\mu_y(B_r(y)) \le \omega(r)\exp(c_2 r/\sqrt{t})\mathfrak{m}(B_r(y))$ with ω bounded and infinitesimal as $r\downarrow 0$ and use the change of variables $r = s\sqrt{t}$ to see that it suffices to estimate

$$\frac{4}{5}\int_0^\infty\left(\omega(s\sqrt{t})\frac{\mathfrak{m}(B_{s\sqrt{t}}(y))}{\mathfrak{m}(B_{\sqrt{t}}(y))}\right)\exp\left(c_1 s - \frac{2s^2}{5}\right)s\,\mathrm{d}s.$$

Now we can split outer the integration in $(0,1)$ and in $(1,\infty)$; the former obviously gives an infinitesimal contribution as $t\downarrow 0$; the latter can be estimated with the exponential growth condition (5.3.5) on $\mathfrak{m}(B_r(y))$ and gives an infinitesimal contribution as well. This proves (5.3.16).

Now, setting $c_n(L) = \omega_n/(4\pi)^n\int_{B_L(0)}|\partial_{x_1}(e^{-|x|^2/4})|^2\mathrm{d}x \uparrow c_n$ as $L\uparrow\infty$, we shall first prove that

$$\lim_{t\downarrow 0}\int_{B_{L\sqrt{t}}(y)} t\mathfrak{m}(B_{\sqrt{t}}(x))|\langle\nabla_x p(x,y,t),V(x)\rangle|^2\,\mathrm{d}\mathfrak{m}(x) = c_n(L)|V|^{2*}(y) \tag{5.3.17}$$

for any $L < \infty$. Taking (5.3.16) into account, it suffices to prove that

$$\lim_{t\downarrow 0}\int_{B_{L\sqrt{t}}(y)} t\mathfrak{m}(B_{\sqrt{t}}(x))|\langle\nabla_x p(x,y,t),\nabla f(x)\rangle|^2\,\mathrm{d}\mathfrak{m}(x) = c_n(L)(|\nabla f|^*)^2(y) \quad \forall L\in[0,\infty). \tag{5.3.18}$$

In order to prove (5.3.18), for $t > 0$ let us consider the rescaling $\mathrm{d} \mapsto \mathrm{d}_t := t^{-1/2}\mathrm{d}$, $\mathfrak{m} \mapsto \mathfrak{m}_t := \mathfrak{m}(B_{\sqrt{t}}(y))^{-1}\mathfrak{m}$. We denote by p_t the heat kernel on the rescaled space $(X,\mathrm{d}_t,\mathfrak{m}_t)$. Applying (5.3.3) with $a := \sqrt{t}^{-1}$, $b := \frac{1}{\mathfrak{m}(B_{\sqrt{t}}(y))}$ and $s := t$ yields (notice that the factor $t = a^{-2}$ disappears by the scaling term in the definition of $f_{\sqrt{t},y}$ and the scaling of gradients)

$$\int_{B_{L\sqrt{t}}(y)} t\mathfrak{m}(B_{\sqrt{t}}(x))|\langle \nabla_x p(x,y,t), \nabla f(x)\rangle|^2 \,\mathrm{d}\mathfrak{m}(x)$$
$$= \int_{B_L^{\mathrm{d}_t}(y)} \mathfrak{m}_t(B_1^{\mathrm{d}_t}(x))|\langle \nabla_x p_t(x,y,1), \nabla f_{\sqrt{t},y}(x)\rangle|^2 \,\mathrm{d}\mathfrak{m}_t(x). \tag{5.3.19}$$

Take a sequence $t_i \downarrow 0$, let (s_i) be a subsequence of (t_i) and $\hat{f}$ be a Lipschitz and harmonic function on $\mathbb{R}^n$ as in Definition 5.3.6 (i.e. $\hat{f}$ is the limit of $f_{\sqrt{s_i},y}$). Note that $\hat{f}$ has necessarily linear growth. Since linear growth harmonic functions on Euclidean spaces are actually linear or constant functions, we see that $\nabla \hat{f} = \sum_j a_j \frac{\partial}{\partial x_j}$ for some $a_j \in \mathbb{R}$. Then, letting $i \to \infty$ in the right hand side of (5.3.19) shows

$$\lim_{i\to\infty} \int_{B_L^{\mathrm{d}_{s_i}}(y)} \mathfrak{m}_{s_i}(B_1^{\mathrm{d}_{s_i}}(x))|\langle \nabla_x p_{s_i}(x,y,1), \nabla f_{\sqrt{s_i},y}(x)\rangle|^2 \,\mathrm{d}\mathfrak{m}_{s_i}(x)$$
$$= \int_{B_L(0_n)} \hat{\mathcal{H}}^n(B_1(x))|\langle \nabla_x q_n(x,0_n,1), \nabla \hat{f}(x)\rangle|^2 \,\mathrm{d}\hat{\mathcal{H}}^n(x), \tag{5.3.20}$$

where $\hat{\mathcal{H}}^n = \mathcal{H}^n/\omega_n$ (hence $\hat{\mathcal{H}}^n(B_1(x)) \equiv 1$) and q_n denotes the heat kernel on $(\mathbb{R}^n, \mathrm{d}_{\mathbb{R}^n}, \hat{\mathcal{H}}^n)$. Since (5.3.1) and (5.3.3) give

$$q_n(x, 0_n, 1) \equiv \frac{\omega_n}{(4\pi)^{n/2}} e^{-|x|^2/4},$$

a simple computation shows that the right hand side of (5.3.20) is equal to $c_n(L)(\sum_j |a_j|^2)$. Finally, from

$$(|\nabla f|^*(z))^2 = \lim_{r\downarrow 0}\left(\frac{1}{\mathfrak{m}(B_r(y))}\int_{B_r(y)} |\nabla f|^2 \,\mathrm{d}\mathfrak{m}\right) = \lim_{i\to\infty}\int_{B_1^{\mathrm{d}_{s_i}}(y)} |\nabla f_{\sqrt{s_i},y}|^2 \,\mathrm{d}\mathfrak{m}_{s_i}$$
$$= \int_{B_1(0_n)} |\nabla \hat{f}|^2 \,\mathrm{d}\hat{\mathcal{H}}^n = \sum_j |a_j|^2, \tag{5.3.21}$$

we have (5.3.17) because (t_i) is arbitrary.

In order to obtain (5.3.15) it is sufficient to let $L \to \infty$ in (5.3.17), taking into account that $c_n(L) \uparrow c_n$ as $L \uparrow \infty$ and that, arguing as for (5.3.16), one can prove that

$$\lim_{L\to\infty} \sup_{0<t<C_4^{-1}} \int_{X\setminus B_{L\sqrt{t}}(y)} |\langle \nabla_x p(x,y,t), W(y)\rangle|^2 \,\mathrm{d}\mathfrak{m}(x) = 0.$$

□

Corollary 5.3.10. *Let A be a Borel subset of X. Then for any $V \in L^2(T(X,\mathrm{d},\mathfrak{m}))$ and $y \in H(V) \cap \mathcal{R}_n$, one has*

(1) if $\int_{B_r(y)\cap A} |V|^2 \,\mathrm{d}\mathfrak{m} = o(\mathfrak{m}(B_r(y)))$ as $r \downarrow 0$, we have

$$\lim_{t\downarrow 0} \int_A t\mathfrak{m}(B_{\sqrt{t}}(x))|\langle \nabla_x p(x,y,t), V(x)\rangle|^2 \,\mathrm{d}\mathfrak{m}(x) = 0; \tag{5.3.22}$$

(2) if $\int_{B_r(y)\setminus A} |V|^2 \,\mathrm{d}\mathfrak{m} = o(\mathfrak{m}(B_r(y)))$ as $r \downarrow 0$, we have

$$\lim_{t\downarrow 0} \int_A t\mathfrak{m}(B_{\sqrt{t}}(x))|\langle \nabla_x p(x,y,t), V(x)\rangle|^2 \,\mathrm{d}\mathfrak{m}(x) = c_n|V|^{2*}(y). \tag{5.3.23}$$

In particular, if $V \in L^p(T(X, \mathrm{d}, \mathfrak{m}))$ *for some* $p > 2$, (5.3.22) *holds if* A *has density* 0 *at* y, *and* (5.3.23) *holds if* A *has density* 1 *at* y.

Proof. (1) Let $W = 1_A V$ and notice that our assumption gives that $y \in H(W)$, with $f \equiv 0$, so that $|W|^{2*}(y) = 0$. Therefore (5.3.22) follows by applying Proposition 5.3.9 to W. The proof of (5.3.23) is analogous. □

Remark 5.3.11. Thanks to the estimate (5.3.4), a similar argument provides also the following results for all $y \in H(V) \cap \mathcal{R}_n$:

(1) if $\int_{B_r(y)\cap A} |V|^2 \,\mathrm{d}\mathfrak{m} = o(\mathfrak{m}(B_r(y)))$ as $r \downarrow 0$,

$$\lim_{t\downarrow 0} \int_A t\mathfrak{m}(B_{\sqrt{t}}(y))|\langle \nabla_x p(x, y, t), V(x)\rangle|^2 \,\mathrm{d}\mathfrak{m}(x) = 0; \tag{5.3.24}$$

(2) if $\int_{B_r(y)\setminus A} |V|^2 \,\mathrm{d}\mathfrak{m} = o(\mathfrak{m}(B_r(y)))$ as $r \downarrow 0$,

$$\lim_{t\downarrow 0} \int_A t\mathfrak{m}(B_{\sqrt{t}}(y))|\langle \nabla_x p(x, y, t), V(x)\rangle|^2 \,\mathrm{d}\mathfrak{m}(x) = c_n |V|^{2*}(y). \tag{5.3.25}$$

Theorem 5.3.12. *Let* $V \in L^2(T(X, \mathrm{d}, \mathfrak{m}))$. *Then for any Borel subsets* A_1, A_2 *of* X *we have*

$$\lim_{t\downarrow 0} \int_{A_1} \left(\int_{A_2} t\mathfrak{m}(B_{\sqrt{t}}(x))|\langle \nabla_x p(x, y, t), V(x)\rangle|^2 \,\mathrm{d}\mathfrak{m}(x) \right) \mathrm{d}\mathfrak{m}(y) = \int_{A_1\cap A_2} \hat{g}(V, V) \,\mathrm{d}\mathfrak{m}. \tag{5.3.26}$$

Proof. Taking the uniform L^∞ estimate (5.2.18) into account, it is enough to prove the result for $V \in L^\infty T$, since this space is dense in $L^2 T$. Take $y \in X$. By (5.2.19), for $0 < t < C_4^{-1}$, we get

$$\int_X t\mathfrak{m}(B_{\sqrt{t}}(x))|\langle \nabla_x p(x, y, t), V(x)\rangle|^2 \,\mathrm{d}\mathfrak{m}(x) \le \int_X \frac{C_3^2 e^2 \|V\|_\infty^2}{\mathfrak{m}(B_{\sqrt{t}}(x))} \exp\left(\frac{-2\mathrm{d}(x, y)^2}{5t} \right) \mathrm{d}\mathfrak{m}(x) \tag{5.3.27}$$

and, by applying (5.3.7) to the rescaled space $\mathrm{d}_t := \sqrt{t}^{-1}\mathrm{d}$, we obtain that the left hand side in (5.3.27) is uniformly bounded as function of y.

Thus, denoting by A_2^* the set of points of density 1 of A_2 and by A_2^{**} the set of points of density 0 of A_2 (so that $\mathfrak{m}(X \setminus (A_2^* \cup A_2^{**})) = 0$), the dominated convergence theorem, Corollary 5.3.10 and the definition of $\hat{g}$ imply

$$\begin{aligned}
&\int_{A_1} \left(\int_{A_2} t\mathfrak{m}(B_{\sqrt{t}}(x))|\langle \nabla_x p(x, y, t), V(x)\rangle|^2 \,\mathrm{d}\mathfrak{m}(x) \right) \mathrm{d}\mathfrak{m}(y) \\
&= \int_{\mathcal{R}_n\cap A_1\cap A_2^*} \left(\int_{A_2} t\mathfrak{m}(B_{\sqrt{t}}(x))|\langle \nabla_x p(x, y, t), V(x)\rangle|^2 \,\mathrm{d}\mathfrak{m}(x) \right) \mathrm{d}\mathfrak{m}(y) \\
&+ \int_{\mathcal{R}_n\cap A_1\cap A_2^{**}} \left(\int_{B_2} t\mathfrak{m}(B_{\sqrt{t}}(x))|\langle \nabla_x p(x, y, t), V(x)\rangle|^2 \,\mathrm{d}\mathfrak{m}(x) \right) \mathrm{d}\mathfrak{m}(y) \\
&\to \int_{\mathcal{R}_n\cap A_1\cap A_2^*} c_n |V|^{2*}(y) \,\mathrm{d}\mathfrak{m}(y) = \int_{A_1\cap A_2} \hat{g}(V, V) \,\mathrm{d}\mathfrak{m}.
\end{aligned} \tag{5.3.28}$$

□

Remark 5.3.13. Building on Remark 5.3.11, one can prove by a similar argument

$$\lim_{t\downarrow 0}\int_{A_1}\left(\int_{A_2} t\mathfrak{m}(B_{\sqrt{t}}(y))|\langle\nabla_x p(x,y,t),V(x)\rangle|^2\,\mathrm{d}\mathfrak{m}(x)\right)\mathrm{d}\mathfrak{m}(y)=\int_{A_1\cap A_2}\hat{g}(V,V)\,\mathrm{d}\mathfrak{m}. \tag{5.3.29}$$

In order to improve the convergence of the $\hat{g}_t$ from weak to strong, a classical Hilbertian strategy is to prove convergence of the Hilbert norms. In our case, at the level of $\hat{\mathbf{g}}_\mathbf{t}$ (and taking (5.3.8) and (5.2.16) into account), this translates into

$$\limsup_{t\downarrow 0}\int_X\left(t\mathfrak{m}(B_{\sqrt{t}}(x))\right)^2\left|\int_X\nabla_x p(x,y,t)\otimes\nabla_x p(x,y,t)\,\mathrm{d}\mathfrak{m}(y)\right|^2_{HS}\mathrm{d}\mathfrak{m}(x)\le nc_n^2\mathfrak{m}(X). \tag{5.3.30}$$

The proof of this estimate requires a more delicate blow-up procedure, and to its proof we devoted Appendix B. Notice that, by using the (non-sharp) estimate of the left hand side in (5.3.30) with $\int\left[t\mathfrak{m}(B_{\sqrt{t}}(\cdot))\int|\nabla_x p|^2\,\mathrm{d}\mathfrak{m}\right]^2\mathrm{d}\mathfrak{m}$ one obtains $n^2c_n^2\mathfrak{m}(X)$, but this upper bound is not sufficient to obtain the convergence of the Hilbert-Schmidt norms.

We are now in a position to prove the main theorem of this paragraph. Let us recall the Dunford-Pettis theorem which states that a family $(f_i)_i\subset L^1(X,\mathfrak{m})$ is relatively compact w.r.t. the weak topology of $L^1(X,\mathfrak{m})$ if and only if it is equi-integrable, and the Vitali-Hahn-Saks theorem which implies that whenever a family of absolutely continuous measures $(\mu_i=f_i\mathfrak{m})_i$ is such that $\mu_i(A)\to\mu(A)$ for any Borel set $A\subset X$, with μ being a Borel measure, then $(f_i)_i$ is an equi-integrable family.

Theorem 5.3.14. *The family of RCD metrics $\hat{g}_t$ in (5.3.13) L^2-strongly converges to $\hat{g}$ as $t\downarrow 0$ according to Definition 5.2.18. In particular one has L^1-strong convergence of $\hat{g}_t(V,V)$ to $\hat{g}(V,V)$ as $t\downarrow 0$ for all $V\in L^2T$.*

Proof. For all $V\in L^2T$, the L^1-weak convergence of $\hat{g}_t(V,V)$ to $\hat{g}(V,V)$ follows easily from Theorem 5.3.12: indeed, choosing $A_1=X$, we obtain that $\int_{A_2}\hat{g}_t(V,V)\,\mathrm{d}\mathfrak{m}$ converge as $t\downarrow 0$ to $\int_{A_2}\hat{g}(V,V)\,\mathrm{d}\mathfrak{m}$ for any Borel set $A_2\subset X$. The Vitali-Hahn-Saks and Dunford-Pettis theorems then grants convergence in the weak topology of L^1.

By combining (5.2.16), (5.3.30) and (5.3.8) we have

$$\begin{aligned}&\limsup_{t\downarrow 0}\int_X|\hat{\mathbf{g}}_t|^2_{HS}\,\mathrm{d}\mathfrak{m}\\ =\;&\limsup_{t\downarrow 0}\int_X\left(t\mathfrak{m}(B_{\sqrt{t}}(x))\right)^2\left|\int_X\nabla_x p(x,y,t)\otimes\nabla_x p(x,y,t)\,\mathrm{d}\mathfrak{m}(y)\right|^2_{HS}\mathrm{d}\mathfrak{m}(x)\\ =\;&nc_n^2\mathfrak{m}(\mathcal{R}_n)=\int_X|\hat{\mathbf{g}}|^2_{HS}\,\mathrm{d}\mathfrak{m}.\end{aligned}\tag{5.3.31}$$

The L^2-strong convergence now comes from Proposition 5.2.19. □

The behavior of $t^{(n+2)/2}g_t$ as $t\downarrow 0$

Let us now consider the convergence result

$$\tilde{g}_t:=t^{(n+2)/2}g_t\to\tilde{g},$$

where $n=\dim_{\mathrm{d},\mathfrak{m}}(X)$ and, with our notation $\mathfrak{m}=\theta\mathcal{H}^n$, the normalized metric $\tilde{g}$ is defined by

$$\tilde{g}=\frac{c_n}{\omega_n\theta}1_{\mathcal{R}_n^*}g.$$

Let us start with the analog of Theorem 5.3.12 in this setting.

Theorem 5.3.15. *Let $V \in L^\infty(T(X, \mathrm{d}, \mathfrak{m}))$ and $A_1 \subset \mathcal{R}_n^*$ Borel. If*

$$\lim_{r\downarrow 0} \int_{A_1} \frac{r^n}{\mathfrak{m}(B_r(y))} \,\mathrm{d}\mathfrak{m}(y) = \int_{A_1} \lim_{r\downarrow 0} \frac{r^n}{\mathfrak{m}(B_r(y))} \,\mathrm{d}\mathfrak{m}(y) < \infty, \tag{5.3.32}$$

then for any Borel set $A_2 \subset X$ one has

$$\lim_{t\downarrow 0} \int_{A_1} \left(\int_{A_2} t^{(n+2)/2} |\langle \nabla_x p(x,y,t), V(x)\rangle|^2 \,\mathrm{d}\mathfrak{m}(x) \right) \mathrm{d}\mathfrak{m}(y) = \frac{c_n}{\omega_n} \int_{A_1\cap A_2} |V|^2 \,\mathrm{d}\mathcal{H}^n. \tag{5.3.33}$$

Proof. Recall that (2.3.18) of Theorem 2.3.16 gives that $r^n/\mathfrak{m}(B_r(y))$ converges as $r \to 0$ to $1/(\omega_n \theta(y))$ for $\mathfrak{m}$-a.e. $y \in A_1$, where θ is the density of $\mathfrak{m}$ w.r.t. $\mathcal{H}^n$. By an argument similar to the proof of Theorem 5.3.14, using also with (5.3.5) we obtain that for any $y \in X$ and any $t < 1/C_4$ one has

$$\varphi_t(y) := t\mathfrak{m}(B_{\sqrt{t}}(y))) \int_{A_2} |\langle \nabla_x p(x,y,t), V(x)\rangle|^2 \,\mathrm{d}\mathfrak{m}(x) \le C(K,N)\|V\|_\infty^2. \tag{5.3.34}$$

Let

$$f_t(y) := \frac{\sqrt{t}^n}{\mathfrak{m}(B_{\sqrt{t}}(y))} 1_{A_1}(y)\varphi_t(y), \qquad g_t(y) := C(K,N)\|V\|_\infty^2 1_{A_1}(y) \frac{\sqrt{t}^n}{\mathfrak{m}(B_{\sqrt{t}}(y))}, \tag{5.3.35}$$

so that (5.3.34) gives $f_t(y) \le g_t(y)$. Note that (5.3.24) and (5.3.25) yield

$$\lim_{t\downarrow 0} f_t(y) = \frac{c_n}{\omega_n} 1_{A_2}(y) \frac{1}{\theta(y)} |V|^2(y) \quad \text{for } \mathfrak{m}\text{-a.e. } y \in A_1. \tag{5.3.36}$$

Applying Lemma 4.0.2 with $g(y) = C(K,N)\|V\|_\infty^2 1_{A_1}(y)/(\omega_n\theta(y))$ and taking (5.3.32) into account we get

$$\lim_{t\downarrow 0} \int_X f_t \,\mathrm{d}\mathfrak{m} = \int_X \lim_{t\downarrow 0} f_t \,\mathrm{d}\mathfrak{m} = \frac{c_n}{\omega_n} \int_{A_1\cap A_2} |V|^2 \,\mathrm{d}\mathcal{H}^n, \tag{5.3.37}$$

which proves (5.3.33). □

We are now in a position to prove the main result of this paragraph.

Theorem 5.3.16. *Assume that*

$$\lim_{r\downarrow 0} \int_{\mathcal{R}_n^*} \frac{r^n}{\mathfrak{m}(B_r(y))} \,\mathrm{d}\mathfrak{m}(y) = \int_{\mathcal{R}_n^*} \lim_{r\downarrow 0} \frac{r^n}{\mathfrak{m}(B_r(y))} \,\mathrm{d}\mathfrak{m}(y) < +\infty. \tag{5.3.38}$$

Then $\tilde{g}_t$ L^2-strongly converge to $\tilde{g}$ as $t \downarrow 0$.

Proof. Let $A_2 \subset X$ be a Borel set and $V \in L^\infty T$. Then Fubini's theorem leads to

$$\int_{A_2} \tilde{g}_t(V,V) \,\mathrm{d}\mathfrak{m} = \int_X \int_{\mathcal{R}_n^*\cap A_2} \langle \nabla_x p(x,y,t), V(x)\rangle \,\mathrm{d}\mathfrak{m}(x) \,\mathrm{d}\mathfrak{m}(y)$$

Then, we can apply Theorem 5.3.15 to get

$$\int_{A_2} \tilde{g}_t(V,V) \,\mathrm{d}\mathfrak{m} \to \frac{c_n}{\omega_n} \int_{\mathcal{R}_n^*\cap A_2} |V|^2 \,\mathrm{d}\mathcal{H}^n = \int_{A_2} \tilde{g}(V,V) \,\mathrm{d}\mathfrak{m}.$$

This implies the convergence of $\tilde{g}_t(V,V)$ to $\tilde{g}(V,V)$ in the weak topology of $L^1(X,\mathfrak{m})$ by the Vitali-Hahn-Saks theorem. Let us prove now the L^2-strong convergence of $\tilde{\mathbf{g}}_t$ to $\tilde{\mathbf{g}}$ as $t \downarrow 0$ using Proposition 5.2.19. Since the scaling factors depend only on t and x, it is immediately seen that

$$|\tilde{\mathbf{g}}|_{HS} = \frac{c_n}{\omega_n \theta} 1_{\mathcal{R}_n^*} |\mathbf{g}|_{HS}, \qquad |\tilde{\mathbf{g}}_t|_{HS} = t^{(n+2)/2} |\mathbf{g}_t|_{HS}.$$

Let us write for clarity $F(x,t) = \left|\int_X \nabla_x p(x,y,t) \otimes \nabla_x p(x,y,t)\, \mathrm{d}\mathfrak{m}(y)\right|_{HS}$. Applying (5.2.16), (5.3.30) and (5.3.8) we get

$$\begin{aligned}
\limsup_{t\downarrow 0} \int_X |\tilde{\mathbf{g}}_t|_{HS}^2 \,\mathrm{d}\mathfrak{m} &= \limsup_{t\downarrow 0} \int_{\mathcal{R}_n} t^{n+2} |\mathbf{g}_t|_{HS}^2 \,\mathrm{d}\mathfrak{m} \\
&= \limsup_{t\downarrow 0} \int_{\mathcal{R}_n} t^{n+2} F^2(x,t)\,\mathrm{d}\mathfrak{m}(x) \\
&\le \int_{\mathcal{R}_n} \limsup_{t\downarrow 0} \left(\frac{t^{(n+2)/2}}{t\mathfrak{m}(B_{\sqrt{t}}(x))} \right)^2 t^2 \mathfrak{m}(B_{\sqrt{t}}(x))^2 F^2(x,t)\,\mathrm{d}\mathfrak{m}(x) \\
&= \int_{\mathcal{R}_n} \frac{1}{\omega_n^2 \theta^2} n c_n^2 \,\mathrm{d}\mathfrak{m}(x) = \int_X |\tilde{\mathbf{g}}|_{HS}^2 \,\mathrm{d}\mathfrak{m}.
\end{aligned}$$

Notice that we are enabled to pass to the limit under the integral sign thanks to (5.3.38) and Lemma 4.0.2, since the convergence in (5.3.30) is dominated. □

We obtain in particular the following corollary when the metric measure space $(X, \mathrm{d}, \mathfrak{m})$ is Ahlfors n-regular: indeed, in this case obviously one has $n = \dim_{\mathrm{d},\mathfrak{m}}(X)$, $\mathfrak{m}$ and $\mathcal{H}^n$ are mutually absolutely continuous and the existence of the limits in (5.3.38), as well as the validity of the equality, are granted by the rectifiability of $\mathcal{R}_n$ and by the dominated convergence theorem.

Corollary 5.3.17. *Assume that $\mathfrak{m}$ is Ahlfors n-regular, i.e. there exists $C \ge 1$ such that*

$$C^{-1} \le \frac{\mathfrak{m}(B_r(x))}{r^n} \le C \qquad \text{for all } r \in (0,1) \text{ and all } x \in X.$$

Then $t^{(n+2)/2} g_t$ L^2-strongly converge to $c_n(\omega_n\theta)^{-1} g$ as $t \downarrow 0$.

Behavior with respect to the mGH-convergence

Let us fix a mGH-convergent sequence of compact RCD$^*(K,N)$-spaces:

$$(X_j, \mathrm{d}_j, \mathfrak{m}_j) \overset{mGH}{\to} (X, \mathrm{d}, \mathfrak{m}).$$

In this section we can adopt the extrinsic point of view of Section 2.3, viewing when necessary all metric measure spaces as isometric subsets of a compact metric space (Y, d), with X_j convergent to X w.r.t. the Hausdorff distance and $\mathfrak{m}_j$ weakly convergent to $\mathfrak{m}$.

Let us denote by $\lambda_{i,j}$, λ_i, $\varphi_{i,j}$, φ_i the corresponding eigenvalues and eigenfunctions of $-\Delta_j$, $-\Delta$, respectively, listed taking into account their multiplicity (we will also use a similar notation below), recall that $\{\varphi_{i,j}\}_{i\ge 0}$ are orthonormal bases of $L^2(X_j, \mathfrak{m}_j)$ and that, according to [GMS15], for any i one has $\lambda_{i,j} \to \lambda_i$ as $j \to \infty$, so called spectral convergence. In addition, by the uniform bound on the diameters of the spaces, we know from Proposition 4.0.19 (see also [J16]) that uniform Lipschitz continuity of eigenfunctions holds, i.e.

$$\sup_j \|\nabla \varphi_{i,j}\|_{L^\infty} < \infty \qquad \forall i \ge 0. \tag{5.3.39}$$

With no loss of generality, we can also assume that the $\varphi_{i,j}$ are restrictions of Lipschitz functions defined on Y, with Lipschitz constant equal to $\|\nabla\varphi_{i,j}\|_{L^\infty(X_j,\mathfrak{m}_j)}$.

Although the following lemma was already discussed in the proof of [GMS15, Thm. 7.8], we give the proof for the reader's convenience.

Lemma 5.3.18. *Under the same setting as above, there exist $j(k)$ and an L^2-orthonomal basis $\{\psi_i\}_{i\geq 0}$ of $L^2(X,\mathfrak{m})$ such that $\varphi_{i,j(k)}$ $H^{1,2}$-strongly converge to ψ_i for all i, with uniform convergence.*

Proof. Since $\||\nabla\varphi_{i,j}|\|_{L^2}^2 = \lambda_{i,j}$, by Theorem 2.4.24 and a diagonal argument there exist a subsequence $j(k)$ and $\psi_i \in L^2(X,\mathfrak{m})$ such that $\varphi_{i,j(k)}$ $H^{1,2}$-strongly converge as $k\to\infty$ to ψ_i for all $i\geq 0$, with L^2-weak convergence of $\Delta_{j(k)}\varphi_{i,j(k)}$ to $\Delta\psi_i$. In particular we obtain that $\Delta\psi_i = \lambda_i\psi_i$ for all i and that

$$\int_X \psi_\ell\psi_m \,\mathrm{d}\mathfrak{m} = \lim_{k\to\infty}\int_{X_{j(k)}} \varphi_{\ell,j(k)}\varphi_{m,j(k)}\,\mathrm{d}\mathfrak{m}_{j(k)} = \delta_{\ell m}.$$

Thus, as written above, $\{\psi_i\}_{i\geq 0}$ is an L^2-orthonormal basis of $L^2(X,\mathfrak{m})$. □

Taking Lemma 5.3.18 into account, with no loss of generality in the sequel we can assume that $\varphi_{i,j}$ $H^{1,2}$-strongly converge to φ_i for all $i\geq 0$, in addition with uniform convergence in Y.

Definition 5.3.19. We say that RCD metrics $h_j \in L^2(T_2^0(X_j,\mathrm{d}_j,\mathfrak{m}_j))$ L^2-weakly converge to $h\in L^2(T_2^0(X,\mathrm{d},\mathfrak{m}))$ if $\sup_j\int_{X_j}|\mathbf{h}_j|_{HS}^2\,\mathrm{d}\mathfrak{m}_j<\infty$ and $h_j(\nabla g_j,\nabla g_j)$ L^2-weakly converge to $h(\nabla g,\nabla g)$ whenever g_j $H^{1,2}$-strongly converge to g with $\sup_j\||\nabla g_j|\|_{L^\infty}<\infty$. L^2-strong convergence is defined by requiring, in addition, that $\lim_j\int_{X_j}|\mathbf{h}_j|_{HS}^2\,\mathrm{d}\mathfrak{m}_j = \int_X|\mathbf{h}|_{HS}^2\,\mathrm{d}\mathfrak{m}$.

It is not difficult to show several fundamental properties of L^2-strong/weak convergence of metrics, including L^2-weak compactness and lower semicontinuity of L^2-norms with respect to L^2-weak convergence as previously discussed in the case of metrics on a fixed space; in particular, the convergence can be improved from weak to strong if and only if

$$\limsup_j \int_{X_j}|\mathbf{h}_j|_{HS}^2\,\mathrm{d}\mathfrak{m}_j \leq \int_X |\mathbf{h}|_{HS}^2\,\mathrm{d}\mathfrak{m}.$$

Theorem 5.3.20. *Let $t_j\to t\in(0,\infty)$, let $\Phi_{t_j}^j : X_j\to L^2(X_j,\mathfrak{m}_j)$ be the corresponding embeddings and let $g_{t_j}^{X_j}$ be the corresponding pull-back metrics in $(X_j,\mathrm{d}_j,\mathfrak{m}_j)$. Then $g_{t_j}^{X_j}$ L^2-strongly converge to g_t^X and $\Phi_t^j(X_j)$, endowed with the $L^2(X_j,\mathfrak{m}_j)$ distance, GH-converge to $\Phi_t(X)$ endowed with the $L^2(X,\mathfrak{m})$ distance.*

Proof. By rescaling with no loss of generality we can assume that $t_j\equiv t=1$.

Let us prove first the convergence of metrics.

For all $N\geq 1$, recalling the representation formula (5.2.17) for the metrics, we define

$$\mathbf{G}_j^N := \sum_{i\geq N}^{\infty} e^{-2\lambda_{i,j}}\mathrm{d}\varphi_{i,j}\otimes\mathrm{d}\varphi_{i,j}\left(=\mathbf{g}_{1,j}-\mathbf{g}_{1,j}^{N-1}\right)$$

and define $\mathbf{G}^N$ analogously. Note that as the $\varphi_{i,j}$'s are orthogonal in $L^2(X_j,\mathfrak{m}_j)$, one can show that $\mathrm{d}\varphi_{i,j}\otimes\mathrm{d}\varphi_{i,j}$ are orthogonal in $L^2T(X_j,\mathrm{d}_j,\mathfrak{m}_j)^{\otimes^2}$, and therefore $|\mathbf{g_{1,j}}|_{HS}^2 = |\mathbf{g}_{1,j}^{N-1}|_{HS}^2 + |\mathbf{G}_j^N|_{HS}^2$. Then, arguing as in (5.2.22), we get

$$\int_{X_j}|\mathbf{G}_j^N|_{HS}^2\,\mathrm{d}\mathfrak{m}_j = \sum_{\ell,\,m\geq N}^{\infty} e^{-2(\lambda_{\ell,j}+\lambda_{m,j})}\int_{X_j}\langle\nabla\varphi_{\ell,j},\nabla\varphi_{m,j}\rangle^2\,\mathrm{d}\mathfrak{m}_j \leq C\sum_{\ell\geq N}^{\infty}\lambda_{\ell,j}e^{-2\lambda_{\ell,j}} \tag{5.3.40}$$

with $C = C(K,N)$, and a similar estimate holds for $\int_X |\mathbf{G}^N|^2_{HS}\,\mathrm{d}\mathfrak{m}$. On the other hand, since

$$\int_{X_j}\int_{X_j} |\nabla_x p_j(x,y,1)|^2\,\mathrm{d}\mathfrak{m}_j(x)\,\mathrm{d}\mathfrak{m}_j(y) = \sum_{\ell=1}^{\infty} \lambda_{\ell,j} e^{-2\lambda_{\ell,j}}$$

and

$$\int_{X_j}\int_{X_j} |\nabla_x p_j(x,y,1)|^2\,\mathrm{d}\mathfrak{m}_j(x)\,\mathrm{d}\mathfrak{m}_j(y) \to \int_X\int_X |\nabla_x p(x,y,1)|^2\,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}\mathfrak{m}(y),$$

taking also the spectral convergence into account we get

$$\sum_{\ell\geq N}^{\infty} \lambda_{\ell,j} e^{-2\lambda_{\ell,j}} \to \sum_{\ell\geq N}^{\infty} \lambda_\ell e^{-2\lambda_\ell} \qquad \forall N. \tag{5.3.41}$$

In particular for any $\epsilon > 0$ there exists N such that for all sufficiently large j

$$\sum_{\ell\geq N}^{\infty} \lambda_{\ell,j} e^{-2\lambda_{\ell,j}} + \sum_{\ell\geq N}^{\infty} \lambda_\ell e^{-2\lambda_\ell} < \epsilon.$$

Thus, for sufficiently large j one has

$$\int_{X_j} |\mathbf{G}_j^N|^2_{HS}\,\mathrm{d}\mathfrak{m}_j + \int_X |\mathbf{G}^N|^2_{HS}\,\mathrm{d}\mathfrak{m} < 2C\epsilon. \tag{5.3.42}$$

On the other hand, since $\varphi_{\ell,j}$ $H^{1,2}$-strongly converge to φ_ℓ, (5.3.39) yields that $\langle\nabla\varphi_{\ell,j},\nabla\varphi_{m,j}\rangle$ L^p-strongly converge to $\langle\nabla\varphi_\ell,\nabla\varphi_m\rangle$ for all $p\in[1,\infty)$. In particular, as $j\to\infty$ we get

$$\begin{aligned}\int_{X_j} |\mathbf{g}_{1,j}^N|^2_{HS}\,\mathrm{d}\mathfrak{m}_j &= \sum_{\ell,m=1}^{N} e^{-2(\lambda_{\ell,j}+\lambda_{m,j})}\int_{X_j}\langle\nabla\varphi_{\ell,j},\nabla\varphi_{m,j}\rangle^2\,\mathrm{d}\mathfrak{m}_j\\ &\to \sum_{\ell,m=1}^{N} e^{-2(\lambda_\ell+\lambda_m)}\int_X\langle\nabla\varphi_\ell,\nabla\varphi_m\rangle^2\,\mathrm{d}\mathfrak{m} = \int_X |\mathbf{g}_1^N|^2_{HS}\,\mathrm{d}\mathfrak{m}.\end{aligned} \tag{5.3.43}$$

Since ϵ is arbitrary, combining (5.3.42) with (5.3.43) yields

$$\int_{X_j} |\mathbf{g}_{1,j}|^2_{HS}\,\mathrm{d}\mathfrak{m}_j \to \int_X |\mathbf{g}_1|^2_{HS}\,\mathrm{d}\mathfrak{m}. \tag{5.3.44}$$

Since it is easy to check that Lemma 5.3.18 yields that $\mathbf{g}_{1,j}^{N-1}$ L^2-weakly converge to $\mathbf{g}_1^{N-1}$, combining (5.3.42) with (5.3.44) completes the proof of the L^2-strong convergence of metrics.

Now we prove the second part of the statement. Using the eigenfunctions $\phi_{i,j}$ we can embed isometrically all $\Phi_t(X_j)\subset L^2(X_j,\mathfrak{m}_j)$ into ℓ_2, and then we need only to prove the Hausdorff convergence inside ℓ_2 of the sets W_j to W, where

$$W_j = \left\{\left(e^{-\lambda_{i,j}}\phi_{i,j}(x)\right)_{i\geq 1}:\ x\in X_j\right\},\qquad W = \left\{\left(e^{-\lambda_i}\phi_i(x)\right)_{i\geq 1}:\ x\in X\right\}.$$

By Proposition 4.0.18 and 4.0.19 in the next section, for all $\epsilon>0$ there exists $N\in\mathbb{N}$ such that for all j

$$\sum_{i\geq N+1} e^{-2\lambda_{i,j}}\|\varphi_{i,j}\|^2_{L^\infty} < \epsilon^2 \qquad \sum_{i\geq N+1} e^{-2\lambda_i}\|\varphi_i\|^2_{L^\infty} < \epsilon^2.$$

Denoting $\pi^N : \ell_2 \to \ell_2$ the projection defined by $\pi^N((x)_i) := (x_1, \dots, x_N, 0, \dots)$, from this it is easy to get

$$\mathrm{d}_H^{\ell_2}(W_j, W_j^N) < \epsilon, \qquad \mathrm{d}_H^{\ell_2}(W, W^N) < \epsilon,$$

where $W_j^N := \pi^N(W_i), W^N := \pi^N(W)$. Hence, by the triangle inequality, it suffices to check that $\mathrm{d}_H^{\ell_2}(W_j^N, W^N) \to 0$ for N fixed. Since

$$W_j^N = \left\{ (e^{-\lambda_{1,j}}\phi_{1,j}(x), e^{-\lambda_{2,j}}\phi_{2,j}(x), \dots, e^{-\lambda_{N,j}}\phi_{N,j}(x), 0, 0, \dots) : \ x \in X_j \right\},$$

and an analogous formula holds for W^N, from the uniform convergence of the $\varphi_{i,j}$ to φ_i we immediately get that $\mathrm{d}_H^{\ell_2}(W_j^N, W^N) \to 0$. □

Remark 5.3.21. The canonical RCD metrics g^{X_j} L^2-weakly converge to g^X, which is a direct consequence of [AH17a, Thm. 5.7]. In particular the lower semicontinuity of the L^2-norms of g^{X_j}:

$$\liminf_{j\to\infty} \int_{X_j} |\mathbf{g}^{X_j}|_j^2 \, \mathrm{d}\mathfrak{m}_j \ge \int_X |\mathbf{g}^X|^2 \, \mathrm{d}\mathfrak{m} \tag{5.3.45}$$

yields

$$\liminf_{j\to\infty} \dim_{\mathrm{d}_j,\mathfrak{m}_j}(X_j) \ge \dim_{\mathrm{d},\mathfrak{m}}(X) \tag{5.3.46}$$

because Lemma 5.3.2 shows that $\int_{X_j} |\mathbf{g}^{X_j}|^2 \, \mathrm{d}\mathfrak{m} = \dim_{\mathrm{d}_j,\mathfrak{m}_j}(X_j)\mathfrak{m}_j(X_j), \int_X |\mathbf{g}^X|^2 \, \mathrm{d}\mathfrak{m} = \dim_{\mathrm{d},\mathfrak{m}}(X)\mathfrak{m}(X)$.

This allows us to define the notion that $\{(X_j, \mathrm{d}_j, \mathfrak{m}_j)\}_j$ *is a noncollapsed convergent sequence to* $(X, \mathrm{d}, \mathfrak{m})$ if the condition $\lim_{j\to\infty} \dim_{\mathrm{d}_j,\mathfrak{m}_j}(X_j) = \dim_{\mathrm{d},\mathfrak{m}}(X)$ holds (see also [K17]). Moreover it is noncollapsed sequence if and only if

$$\lim_{j\to\infty} \int_{X_j} |\mathbf{g}^{X_j}|_j^2 \, \mathrm{d}\mathfrak{m}_j = \lim_{j\to\infty} \dim_{\mathrm{d}_j,\mathfrak{m}_j}(X_j)\mathfrak{m}_j(X_j) = \dim_{\mathrm{d},\mathfrak{m}}(X)\mathfrak{m}(X) = \int_X |\mathbf{g}^X|^2 \, \mathrm{d}\mathfrak{m}$$

that is, g^{X_j} L^2-strongly converge to g^X. (these observation are justified even for noncompact case if we replace X_j, X by $B_1(x_j), B_1(x)$, where $x_j \to x$). One of the important points in Theorem 5.3.20 is that the RCD metrics $g_{t_j}^{X_j}$ are L^2-strongly convergent even without the noncollapsed assumption.

5.4 Proof of the limsup estimate

In this section, we will prove the estimate (5.3.30) which implies the strong L^2 convergence of RCD metrics $\hat{g}_t \to \hat{g}$ and $\tilde{g}_t \to \tilde{g}$ when $t \downarrow 0$.

To this purpose, we will notably need the local notion of Hessian developed within the framework of N. Gigli's theory [G18] and defined as symmetric bilinear form on $L^2(T(X, \mathrm{d}, \mathfrak{m}))$. In particular we will use the fact that this Hessian is defined for all $f \in D(\Delta)$, with an integral estimate coming from Bochner's inequality [G18, Cor. 3.3.9]

$$\int_X |\mathrm{Hess}_f|^2 \mathrm{d}\mathfrak{m} \le \int_X \left(|\Delta f|^2 - K|\nabla f|^2 \right) \mathrm{d}\mathfrak{m} \qquad \forall f \in D(\Delta). \tag{5.4.1}$$

In addition, we shall use the property [G18, Prop. 3.3.22] that, for all f, $g \in D(\Delta)$ with $|\nabla f|$, $|\nabla g| \in L^\infty(X, \mathfrak{m})$, one has $\langle \nabla f, \nabla g \rangle \in H^{1,2}(X, \mathrm{d}, \mathfrak{m})$, with

$$\nabla \langle \nabla f, \nabla g \rangle = \mathrm{Hess}_f(\nabla g, \cdot) + \mathrm{Hess}_g(\nabla f, \cdot) \qquad \mathfrak{m}\text{-a.e. in } X. \tag{5.4.2}$$

We set

$$F(x,t) := \left(t\mathfrak{m}(B_{\sqrt{t}}(x))\right)^2 \left|\int_X \nabla_x p(x,y,t) \otimes \nabla_x p(x,y,t)\, \mathrm{d}\mathfrak{m}(y)\right|^2_{HS}.$$

and we notice that the Gaussian estimate (2.3.7) provides a uniform upper bound on the L^∞ norm of $F(\cdot,t)$, for $0 < t \le 1$. Now, we claim that (5.3.30) follows by Proposition 5.4.1 below; indeed, by integration of both sides we get

$$\lim_{t\downarrow 0} \int_X \frac{1}{\mathfrak{m}(B_{\sqrt{t}}(\bar{x}))} \int_{B_{\sqrt{t}}(\bar{x})} F(x,t)\, \mathrm{d}\mathfrak{m}(x)\, \mathrm{d}\mathfrak{m}(\bar{x}) = nc_n^2 \mathfrak{m}(X)$$

and, thanks to Fubini's theorem, the right hand side can be represented as

$$\lim_{t\downarrow 0} \int_X F(x,t) \Big(\int_{B_{\sqrt{t}}(x)} \frac{1}{\mathfrak{m}(B_{\sqrt{t}}(\bar{x}))}\, \mathrm{d}\mathfrak{m}(\bar{x})\Big).\, \mathrm{d}\mathfrak{m}(x)$$

Since it is easily seen that $\int_{B_{\sqrt{t}}(x)} \frac{1}{\mathfrak{m}(B_{\sqrt{t}}(\bar{x}))}\, \mathrm{d}\mathfrak{m}(\bar{x})$ are uniformly bounded and converge to 1 as $t \downarrow 0$ for all $x \in \mathcal{R}_n$ (in particular $\mathfrak{m}$-a.e. x), from the dominated convergence theorem we obtain (5.3.30).

Hence, we devote the rest of the appendix to the proof of the proposition.

Proposition 5.4.1. *For all $\bar{x} \in \mathcal{R}_n$ one has*

$$\lim_{t\downarrow 0} \frac{1}{\mathfrak{m}(B_{\sqrt{t}}(\bar{x}))} \int_{B_{\sqrt{t}}(\bar{x})} F(x,t)\, \mathrm{d}\mathfrak{m}(x) = nc_n^2, \tag{5.4.3}$$

with c_n defined as in (5.3.14).

Proof. Let us fix $t_j \downarrow 0$ and consider the mGH convergent sequence

$$(X, \mathrm{d}_j, x, \mathfrak{m}_j) := \left(X, \sqrt{t_j}^{-1}\mathrm{d}, \bar{x}, \frac{\mathfrak{m}}{\mathfrak{m}(B_{\sqrt{t_j}}(\bar{x}))}\right) \overset{mGH}{\to} \left(\mathbb{R}^n, \mathrm{d}_{\mathbb{R}^n}, 0_n, \tilde{\mathcal{L}}^n\right), \tag{5.4.4}$$

where $\tilde{\mathcal{H}}^n := \mathcal{H}^n/\omega_n$.

Setting

$$F_j(x) := (t_j\mathfrak{m}(B_{\sqrt{t_j}}(\bar{x})))^2 \left|\int_X \nabla_x p(x,y,t_j) \otimes \nabla_x p(x,y,t_j)\, \mathrm{d}\mathfrak{m}(y)\right|^2_{HS}.$$

we claim that, in order to get (5.4.3), it is sufficient to prove that

$$\lim_{j\to\infty} \frac{1}{\mathfrak{m}(B_{\sqrt{t_j}}(\bar{x}))} \int_{B_{\sqrt{t_j}}(\bar{x})} F_j(x)\, \mathrm{d}\mathfrak{m}(x) = nc_n^2. \tag{5.4.5}$$

Indeed, letting

$$H_j(x) := \left|\int_X \nabla_x p(x,y,t_j) \otimes \nabla_x p(x,y,t_j)\, \mathrm{d}\mathfrak{m}(y)\right|^2_{HS}, \tag{5.4.6}$$

so that $F_j(x) = \left(t_j\mathfrak{m}(B_{\sqrt{t_j}}(\bar{x}))\right)^2 H_j(x)$, one has

$$\begin{aligned}
&\frac{1}{\mathfrak{m}(B_{\sqrt{t_j}}(\bar{x}))}\int_{B_{\sqrt{t_j}}(\bar{x})}\left|\left(t_j\mathfrak{m}(B_{\sqrt{t_j}}(\bar{x}))\right)^2 H_j(x) - \left(t_j\mathfrak{m}(B_{\sqrt{t_j}}(x))\right)^2 H_j(x)\right| \mathrm{d}\mathfrak{m}(x)\\
&= \int_{B_1^{\mathrm{d}_j}(\bar{x})}\left|1-(\mathfrak{m}_j(B_1^{\mathrm{d}_j}(x)))^2\right|\left|\int_X \nabla_x p_j(x,y,1)\otimes\nabla_x p_j(x,y,1)\,\mathrm{d}\mathfrak{m}_j(y)\right|_{HS}^2 \mathrm{d}\mathfrak{m}_j(x)\\
&\le C\int_{B_1^{\mathrm{d}_j}(\bar{x})}\left|1-(\mathfrak{m}_j(B_1^{\mathrm{d}_j}(x)))^2\right| \mathrm{d}\mathfrak{m}_j(x) \to C\int_{B_1(0_n)}\left|1-(\tilde{\mathcal{H}}^n(B_1(x)))^2\right| \mathrm{d}\tilde{\mathcal{H}}^n(x) = 0,
\end{aligned}$$

where C comes from a Gaussian bound.

Applying Proposition 2.4.26 (more precisely [AH17b, Cor. 4.12]) for the standard coordinate functions $h_i : \mathbb{R}^n \to \mathbb{R}$ yields that (possibly extracting a subsequence) the existence of Lipschitz functions $h_{i,j} \in D(\Delta^j)$, harmonic in $B_3^{\mathrm{d}_j}(\bar{x})$, such that $h_{i,j}$ $H^{1,2}$-strongly converge to h_i on $B_3(0_n)$ with respect to the convergence (5.4.4). Here and in the sequel we are denoting Δ^j the Laplacian of $(X, \mathrm{d}_j, \mathfrak{m}_j)$. Note that gradient estimates for solutions of Poisson's equations given in [J16] show

$$C := \sup_{i,j} \||\nabla h_{i,j}|_j\|_{L^\infty(B_2^{\mathrm{d}_j}(\bar{x}))} < \infty, \tag{5.4.7}$$

where $|\cdot|_j$ denotes the modulus of gradient in the rescaled space.

On the other hand Bochner's inequality (we use here and in the sequel the notation Hess^j for the Hessian in the rescaled space) shows

$$\frac{1}{2}\int_X \Delta^j\varphi|\nabla h_{i,j}|_j^2\,\mathrm{d}\mathfrak{m}_j \ge \int_X \varphi\left(|\mathrm{Hess}^j_{h_{i,j}}|^2 + t_jK|\nabla h_{i,j}|_j^2\right)\mathrm{d}\mathfrak{m}_j \tag{5.4.8}$$

for all $\varphi \in D(\Delta^j)$ with $\Delta^j\varphi \in L^\infty(X, \mathfrak{m}_j)$ and $\operatorname{supp}\varphi \subset B_3^{\mathrm{d}_j}(\bar{x})$. In particular, taking as $\varphi = \varphi_j$ the good cut-off functions constructed in [MN14] we obtain

$$\lim_{j\to\infty}\int_{B_2^{\mathrm{d}_j}(\bar{x})} |\mathrm{Hess}^j_{h_{i,j}}|^2\,\mathrm{d}\mathfrak{m}_j = 0. \tag{5.4.9}$$

Let us define functions $a_j^{\ell,m} : B_2^{\mathrm{d}_j}(\bar{x}) \to \mathbb{R}$, $a^{\ell,m} : B_2(0_n) \to \mathbb{R}$ by

$$a_j^{\ell,m}(x) := \int_X \langle\nabla_x p_j(x,y,1), \nabla h_{\ell,j}(x)\rangle_j\langle\nabla_x p_j(x,y,1), \nabla h_{m,j}(x)\rangle_j\,\mathrm{d}\mathfrak{m}_j(y),$$

$$a^{\ell,m}(x) := \int_{\mathbb{R}^n} \langle\nabla_x q(x,y,1), \nabla h_\ell(x)\rangle\langle\nabla_x q(x,y,1), \nabla h_m(x)\rangle\,\mathrm{d}\tilde{\mathcal{H}}^n(y),$$

respectively, where $p_j(x,y,t)$ is the heat kernel of $(X, \mathrm{d}_j, \mathfrak{m}_j)$ and $q(x,y,t)$ is the heat kernel of $(\mathbb{R}^n, \mathrm{d}_{\mathbb{R}^n}, \tilde{\mathcal{H}}^n)$ (we also use the $\langle\cdot,\cdot\rangle_j$ notation to emphasize the dependence of these objects on the rescaled metric). Notice that the explicit expression (5.3.3) of $q(x,y,t)$ provides the identity $a^{\ell,m} = c_n^2\delta_{\ell,m}$.

Now let us prove that $a_j^{\ell,m}$ L^p-strongly converge to $a^{\ell,m}$ on $B_1(0_n)$ for all $p \in [1,\infty)$. It is easy to check the uniform L^∞ boundedness by the Gaussian estimate (2.3.7) and (5.4.7), and the L^p-weak convergence by Theorem 5.3.3. To prove improve the convergence from weak to strong, thanks to the compactness result stated in Theorem 5.3.4, it suffices to prove that $a_j^{\ell,m} \in H^{1,2}(B_2^{\mathrm{d}_j}(\bar{x}), \mathrm{d}_j, \mathfrak{m}_j)$ for all j, and that

$$\sup_j \int_{B_2^{\mathrm{d}_j}(\bar{x})} |\nabla a_j^{\ell,m}|_j\,\mathrm{d}\mathfrak{m}_j < \infty. \tag{5.4.10}$$

Thus, let us check that (5.4.10) holds as follows. For any $y \in X$, the Leibniz rule and (5.4.2) give

$$\begin{aligned} & \nabla\left(\langle \nabla_x p_j(x,y,1), \nabla h_{\ell,j}(x)\rangle_j \langle \nabla_x p_j(x,y,1), \nabla h_{m,j}(x)\rangle_j\right) \qquad (5.4.11) \\ = \;& \langle \nabla_x p_j(x,y,1), \nabla h_{\ell,j}(x)\rangle_j \left(\mathrm{Hess}^j_{p_j(\cdot,y,1)}(\nabla h_{m,j},\cdot) + \mathrm{Hess}^j_{h_{m,j}}(\nabla p_j(\cdot,y,1),\cdot)\right) \\ + \;& \langle \nabla_x p_j(x,y,1), \nabla h_{m,j}(x)\rangle_j \left(\mathrm{Hess}^j_{p_j(\cdot,y,1)}(\nabla h_{\ell,j},\cdot) + \mathrm{Hess}^j_{h_{\ell,j}}(\nabla p_j(\cdot,y,1),\cdot)\right). \end{aligned}$$

Now, recalling that $(X, \mathrm{d}_j, \mathfrak{m}_j)$ arises from the rescaling of a fixed compact space, the Gaussian estimate (2.3.7) yields that $\langle \nabla_x p_j(x,y,1), \nabla h_{\ell,j}(x)\rangle_j \langle \nabla_x p_j(z,y,1), \nabla h_{m,j}(x)\rangle_j$ belong to $H^{1,2}(B_2^{\mathrm{d}_j}(\bar{x}), \mathrm{d}_j, \mathfrak{m}_j)$, with norm for j fixed uniformly bounded w.r.t. y. Hence, we can commute differentiation w.r.t. x and integration w.r.t. y to obtain that $a_j^{\ell,m} \in H^{1,2}(B_2^{\mathrm{d}_j}(\bar{x}), \mathrm{d}_j, \mathfrak{m}_j)$ with

$$\nabla a_j^{\ell,m} = \int_X \nabla\left(\langle \nabla_x p_j(x,y,1), \nabla h_{\ell,j}(x)\rangle_j \langle \nabla_x p_j(x,y,1), \nabla h_{m,j}(x)\rangle_j\right) \mathrm{d}\mathfrak{m}_j(y) \qquad (5.4.12)$$

$\mathfrak{m}_j$-a.e. in $B_2^{\mathrm{d}_j}(\bar{x})$. From (5.4.11) we then get

$$\begin{aligned} |\nabla a_j^{\ell,m}| \;\le\;& C\left(|\mathrm{Hess}^j_{p_j(\cdot,y,1)}(\nabla h_{\ell,j},\cdot)| + |\mathrm{Hess}^j_{p_j(\cdot,y,1)}(\nabla h_{m,j},\cdot)|\right)|\nabla p_j(\cdot,y,1)|_j \\ &+ C\left(|\mathrm{Hess}^j_{h_{\ell,j}}(\nabla p(\cdot,y,1),\cdot)| + |\mathrm{Hess}^j_{h_{m,j}}(\nabla p(\cdot,y,1),\cdot)|\right)|\nabla p_j(\cdot,y,1)|_j \end{aligned}$$

where C is the constant in (5.4.7), so that using (5.4.7) once more we get

$$\begin{aligned} &\|\nabla a_j^{\ell,m}\|_{L^1(B_2^{\mathrm{d}_j}(\bar{x}))} \\ &\le \tilde{C}\left(\int_X\int_{B_2^{\mathrm{d}_j}(\bar{x})} |\mathrm{Hess}^j_{p_j(\cdot,y,1)}|^2 \,\mathrm{d}\mathfrak{m}_j(x)\,\mathrm{d}\mathfrak{m}_j(y)\right)^{1/2} \left(\int_X\int_{B_2^{\mathrm{d}_j}(\bar{x})} |\nabla_x p_j(x,y,1))|_j^2 \,\mathrm{d}\mathfrak{m}_j(x)\,\mathrm{d}\mathfrak{m}_j(y)\right)^{1/2} \\ &+ \tilde{C}\int_X\int_{B_2^{\mathrm{d}_j}(\bar{x})} \left(|\mathrm{Hess}^j_{h_{\ell,j}}|(x)|\nabla_x p_j(x,y,1)|_j^2 + |\mathrm{Hess}^j_{h_{m,j}}|(x)|\nabla_x p_j(x,y,1)|_j^2\right) \mathrm{d}\mathfrak{m}_j(x)\,\mathrm{d}\mathfrak{m}_j(y) \end{aligned} \qquad (5.4.13)$$

for some positive constant $\tilde{C}$ (recall that the Hessian norm is the Hilbert-Schmidt norm). Note that the second term of the right hand side of (5.4.13) is uniformly bounded with respect to j because of the Gaussian estimate (2.3.7) and (5.4.9).

Note that (2.3.8) and (2.3.7) with Lemma 5.3.1 show

$$\sup_j\left(\int_X\int_{B_2^{\mathrm{d}_j}(\bar{x})} |\Delta_x^j p_j(x,y,1)|^2 \,\mathrm{d}\mathfrak{m}_j(x)\,\mathrm{d}\mathfrak{m}_j(y) + \int_X\int_{B_2^{\mathrm{d}_j}(\bar{x})} |\nabla_x p_j(x,y,1))|_j^2 \,\mathrm{d}\mathfrak{m}_j(x)\,\mathrm{d}\mathfrak{m}_j(y)\right) < \infty.$$

In particular by applying (5.4.1) to the scaled spaces, with a sequence of good cut-off functions constructed in [MN14], we obtain

$$\sup_j \int_X\int_{B_2^{\mathrm{d}_j}(\bar{x})} |\mathrm{Hess}^j_{p_j(\cdot,y,1)}|^2 \,\mathrm{d}\mathfrak{m}_j(x)\,\mathrm{d}\mathfrak{m}_j(y) < \infty.$$

Thus (5.4.13) yields (5.4.10), which completes the proof of the L^p-strong convergence of $a_j^{\ell,m}$ to $a^{\ell,m}$ for all $p \in [1,\infty)$.

Then, since $a^{\ell,m} = c_n^2\delta_{\ell m}$ we get

$$\lim_{j\to\infty}\int_{B_1^{\mathrm{d}_j}(\bar{x})} \sum_{\ell,m} |a_j^{\ell,m}|^2 \,\mathrm{d}\mathfrak{m}_j = \int_{B_1(0_n)} \sum_{\ell,m} |a^{\ell,m}|^2 \,\mathrm{d}\tilde{\mathcal{H}}^n = nc_n^2. \qquad (5.4.14)$$

Hence, to finish the proof of (5.4.5), and then of the proposition, it suffices to check that

$$\fint_{B_{\sqrt{t_j}}(\bar{x})} \left| \sum_{\ell,m} |a_j^{\ell,m}|^2 - \big(t_j \mathfrak{m}(B_{\sqrt{t_j}}(\bar{x}))\big)^2 \left| \int_X \nabla_x p(x,y,t_j) \otimes \nabla_x p(x,y,t_j)\, \mathrm{d}\mathfrak{m}(y) \right|_{HS}^2 \right| \mathrm{d}\mathfrak{m}(x) \tag{5.4.15}$$

is infinitesimal as $j \to \infty$.

To prove this fact, we first state an elementary property of Hilbert spaces whose proof is quite standard, and therefore omitted: for any r-dimensional Hilbert space $(V, \langle \cdot, \cdot \rangle)$, $\epsilon > 0$, $\{e_i\}_{i=1}^r \subset V$ one has the implication

$$|\langle e_i, e_j \rangle - \delta_{ij}| < \epsilon \quad \forall i, j \quad \Rightarrow \quad \left| |v|^2 - \sum_{i=1}^r |\langle v, e_i \rangle|^2 \right| \leq C(r)\epsilon^2 |v|^2 \quad \forall v \in V. \tag{5.4.16}$$

Note that the scaling property (5.3.3) of the heat kernel gives

$$\begin{aligned} &\fint_{B_{\sqrt{t_j}}(\bar{x})} \left| \sum_{\ell,m} |a_j^{\ell,m}|^2 - \big(t_j \mathfrak{m}(B_{\sqrt{t_j}}(\bar{x}))\big)^2 \left| \int_X \nabla_x p(x,y,t_j) \otimes \nabla_x p(x,y,t_j)\, \mathrm{d}\mathfrak{m}(y) \right|_{HS}^2 \right| \mathrm{d}\mathfrak{m}(x) \\ &= \int_{B_1^{\mathrm{d}_j}(\bar{x})} \left| \sum_{\ell,m} |a_j^{\ell,m}|^2 - \left| \int_X \nabla_x p_j(x,y,1) \otimes \nabla_x p_j(x,y,1)\, \mathrm{d}\mathfrak{m}_j(y) \right|_{HS}^2 \right| \mathrm{d}\mathfrak{m}_j(x) \\ &= \int_{B_1^{\mathrm{d}_j}(\bar{x})} \left| \sum_{\ell,m} |a_j^{\ell,m}|^2 - G_j \right| \mathrm{d}\mathfrak{m}_j, \end{aligned} \tag{5.4.17}$$

where

$$G_j(x) := \left| \int_X \nabla_x p_j(x,y,1) \otimes \nabla_x p_j(x,y,1)\, \mathrm{d}\mathfrak{m}_j(y) \right|_{HS}^2.$$

Let

$$\epsilon_j := \max_{\ell,m} \int_{B_1^{\mathrm{d}_j}(\bar{x})} |\langle \nabla h_{\ell,j}, \nabla h_{m,j} \rangle_j - \delta_{\ell m}|\, \mathrm{d}\mathfrak{m}_j.$$

Then notice that for all ℓ, m one has

$$\int_{B_1^{\mathrm{d}_j}(\bar{x})} |\langle \nabla h_{\ell,j}, \nabla h_{m,j} \rangle_j - \delta_{\ell m}|\, \mathrm{d}\mathfrak{m}_j \to \int_{B_1(0_n)} |\langle \nabla h_\ell, \nabla h_m \rangle - \delta_{\ell m}|\, \mathrm{d}\tilde{\mathcal{H}}^n = 0$$

as $j \to \infty$. In particular $\epsilon_j \to 0$.

Let

$$K_j^{\ell,m} := \left\{ w \in B_1^{\mathrm{d}_j}(\bar{x}) : \ |\langle \nabla h_{\ell,j}, \nabla h_{m,j} \rangle_j(w) - \delta_{lm}| > \sqrt{\epsilon_j} \right\}.$$

Then the Markov inequality and the definition of ϵ_j give $\mathfrak{m}_j(K_j^{\ell,m}) \leq \sqrt{\epsilon_j}$, so that $K_j := \bigcup_{\ell,m} K_j^{\ell,m}$ satisfy $\mathfrak{m}_j(K_j) \to 0$ as $j \to \infty$.

On the other hand, (5.4.16) with $r = n^2$ yields

$$\int_{B_1^{\mathrm{d}_j}(\bar{x}) \setminus K_j} \left| \sum_{\ell,m} |a_j^{\ell,m}|^2 - G_j \right| \mathrm{d}\mathfrak{m}_j \leq C(n^2)\epsilon_j \int_{B_1^{\mathrm{d}_j}(\bar{x})} |G_j|^2\, \mathrm{d}\mathfrak{m}_j \to 0, \tag{5.4.18}$$

where we used $\sup_j \|G_j\|_{L^\infty(B_1^{\mathrm{d}_j}(\bar{x}))} < \infty$, as a consequence of the Gaussian estimate (2.3.7).

Then since

$$\int_{K_j} \left| \sum_{\ell,m} |a_j^{\ell,m}|^2 - G_j \right| \mathrm{d}\mathfrak{m}_j \leq \sqrt{\mathfrak{m}_j(K_j)} \left(\int_{B_1^{\mathrm{d}_j}(\bar{x})} \left| \sum_{\ell,m} |a_j^{\ell,m}|^2 - G_j \right|^2 \mathrm{d}\mathfrak{m}_j \right)^{1/2} \to 0,$$

where we used the uniform L^p-bounds on $a_j^{\ell,m}$ for all p, we have

$$(5.4.17) = \int_{K_j} \left| \sum_{\ell,m} |a_j^{\ell,m}|^2 - G_j \right| \mathrm{d}\mathfrak{m}_j + \int_{B_1^{\mathrm{d}_j}(\bar{x}) \setminus K_j} \left| \sum_{\ell,m} |a_j^{\ell,m}|^2 - G_j \right| \mathrm{d}\mathfrak{m}_j \to 0.$$

Thus we have that the expression in (5.4.15) is infinitesimal as $j \to \infty$, which completes the proof of Proposition 5.4.1. □

Bibliography

[Ale51] A.D. Alexandrov: *A theorem on triangles in a metric space and some of its applications*, Trudy Mat. Inst. Steklov **38** (1951), 5–23.

[Ale57] A.D. Alexandrov: *Uber eine Verallgemeinerung der Riemannschen Geometrie*, Schr. Forschungsinst. Math. **1** (1957), 33–84.

[ACDM15] L. Ambrosio, M. Colombo, S. Di Marino: *Sobolev spaces in metric measure spaces: reflexivity and lower semicontinuity of slope*, Adv. Studies in Pure Math. **67** (2015), 1–58.

[AG13] L. Ambrosio, N. Gigli: *A User's Guide to Optimal Transport,* in: Modelling and Optimisation of Flows on Networks, Lecture Notes in Mathematics **2062**, Springer, Berlin, Heidelberg, 2013.

[AGS08] L. Ambrosio, N. Gigli, G. Savaré: *Gradient flows in metric spaces and in the space of probability measures*, Lectures in Mathematics, ETH Zürich, Birkhäuser, 2008.

[AGS13] L. Ambrosio, N. Gigli, G. Savaré: *Density of Lipschitz functions and equivalence of weak gradients in metric measure spaces*, Rev. Mat. Iberoam. **29** (2013), 969–996.

[AGS14a] L. Ambrosio, N. Gigli, G. Savaré: *Calculus and heat flow in metric measure spaces and applications to spaces with Ricci bounds from below*, Invent. Math. **195** (2014), 289–391.

[AGS14b] L. Ambrosio, N. Gigli, G. Savaré: *Metric measure spaces with Riemannian Ricci curvature bounded from below*, Duke Math. J. **163** (2014), 1405–1490.

[AGS15] L. Ambrosio, N. Gigli, G. Savaré: *Bakry-Émery curvature-dimension condition and Riemannian Ricci curvature bounds*, Annals of Probability, **43** (2015), 339–404.

[AH17a] L. Ambrosio, S. Honda: *New stability results for sequences of metric measure spaces with uniform Ricci bounds from below*, in: Measure Theory in Non-Smooth Spaces, edited by Nicola Gigli, De Gruyter Press, 2017, 1–51.

[AH17b] L. Ambrosio, S. Honda: *Local spectral convergence in* $\mathrm{RCD}^*(K,N)$ *spaces*, ArXiv preprint 1703.04939 (2017).

[AHPT17] L. Ambrosio, S. Honda, J. Portegies, D. Tewodrose: *Embedding of* $\mathrm{RCD}^*(K,N)$ *spaces in* L^2 *via eigenfunctions*, in preparation.

[AHP18] L. Ambrosio, S. Honda, J. Portegies: *Continuity of nonlinear eigenvalues in* $\mathrm{CD}(K,\infty)$ *spaces with respect to measured Gromov-Hausdorff convergence.* Calc. Var. **34** (2018), 1–23.

[AHT18] L. Ambrosio, S. Honda, D. Tewodrose: *Short-time behavior of the heat kernel and Weyl's law on* $\mathrm{RCD}^*(K,N)$*-spaces.* Annals of Global Analysis and Geometry **53** (2018), 1–23.

[AST17] L. Ambrosio, F. Stra, D. Trevisan: *Weak and strong convergence of derivations and stability of flows with respect to MGH convergence.* J. Funct. Anal. **272**(3) (2017), 1182–1229.

[AT03] L. Ambrosio, P. Tilli: *Topics on Analysis in Metric Spaces*, Oxford Lecture Series in Mathematics and Its Applications, OUP Oxford, 2003.

[ANPS09] W. Arendt, R. Nittka, W. Peter, F. Steiner: *Weyl's law: Spectral properties of the Laplacian in Mathematics and Physics*, in: Mathematical Analysis of Evolution, Information, and Complexity, edited by W. Arendt and W. P. Schleich, 2009, 1–72.

[BS10] K. Bacher, K.-T. Sturm: *Localization and tensorization properties of the curvature-dimension condition for metric measure spaces*, J. Funct. Anal. **259** (2010), 28–56.

[BS14] K. Bacher, K.-T. Sturm: *Ricci bounds for Euclidean and Spherical Cones*, in: Griebel M. (eds) Singular Phenomena and Scaling in Mathematical Models, Springer, Cham, 2014.

[Ba91] D. Bakry: *Inégalités de Sobolev faibles: un critère* $\Gamma 2$, Séminaire de Probabilités, XXV, 234–261, in: Lecture Notes in Math. **1485**, Springer, Berlin, 1991.

[Ba94] D. Bakry: *L'hypercontractivité et son utilisation en théorie des semi-groupes*, in: Lecture on probability theory: Ecole d'été de Saint-Flour XXII-1992, Lecture Notes in Math. **1581**, Springer Verlag, Berlin, 1994.

[BBGL12] D. Bakry, F. Bolley, I. Gentil, P. Maheux: *Weighted Nash inequalities*, Revista Matematica Iberoamericana **28**(3) (2012), 879–906.

[BE85] D. Bakry, M. Émery: *Diffusion hypercontractives* Séminaire de probabilités, XIX, in: Lecture Notes in Math. **1123** (1985), 177–206.

[BGL14] D. Bakry, I. Gentil, M. Ledoux: *Analysis and Geometry of Markov Diffusion operators.* Grundlehren der mathematisches Wissenschaften **348**, Springer, 2014.

[Bé86] P. Bérard: *Spectral geometry: direct and inverse problems*, with an Appendix by G. Besson, Lecture Notes in Mathematics **1207**, Springer-Verlag, 1986.

[BBG94] P. Bérard, G. Besson, S. Gallot: *Embedding Riemannian manifolds by their heat kernel*, Geometric and Functional Analysis **4**(4) (1994), 373–398.

[BKMR18] J. Bertrand, C. Ketterer, I. Mondello, T. Richard: *Stratified spaces and synthetic Ricci curvature bounds*, ArXiV preprint 1804.08870 (2018).

[BD59] A. Beurling, J. Deny: *Dirichlet spaces*, Proc. Natl. Acad. Sci. USA **45** (1959), 208–215.

[Bis63] R. Bishop: *A relation between volume, mean curvature, and diameter*, Amer. Math. Soc. Not. **10** (1963), 364.

[BH91] N. Bouleau, F. Hirsch: *Dirichlet Forms and Analysis on Wiener Spaces*, De Gruyter Studies in Mathematics **14**, De Gruyter, Berlin, 1991.

[Bre87] Y. Brenier: *Décomposition polaire et réarrangement monotone des champs de vecteurs*, C. R. Acad. Sci. Paris Sér. I Math. **305** (1987), 805–808.

[Bre91] Y. Brenier: *Polar factorization and monotone rearrangement of vector-valued functions*, Comm. Pure Appl. Math. **44** (1991), 375–417.

[BPP92] R. Brooks, P. Petersen, P. Perry: *Compactness and finiteness theorems for isospectral manifolds*, J. Reine Angew Math. **426** (1992), 67–90.

[BS18] E. Brué, D. Semola: *Constancy of the dimension for* $\mathrm{RCD}(K,N)$ *spaces via regularity of Lagrangian flows*, ArXiV preprint 1804.07128v1 (2018).

[BGP92] Y. Burago, M. Gromov, G. Perelman: *Aleksandrov spaces with curvatures bounded below*, Uspekhi Mat. Nauk. **47** (1992), 3–51.

[Bus82] P. Buser: *A note on the isoperimetric constant*, Ann. Sci. École Norm. Sup. **4** (1982), 213–230.

[CS86] E. Carlen, D. Stroock: *An application of the Bakry-Emery criterion to infinite dimensional diffusions*, Sém. de Probabilités XX, Azéma J. and Yor. M. (eds) LNM **1204** (1986), 341–348.

[CMi16] F. Cavalletti, E. Milman: *The globalization theorem for the curvature-dimension condition,* ArXiv Preprint 1612.07623.

[Ch99] J. Cheeger: *Differentiability of Lipschitz functions on metric measure spaces*, Geom. Funct. Anal. **9** (1999), 428–517.

[CC96] J. Cheeger, T. H. Colding: *Lower bounds on Ricci curvature and the almost rigidity of warped products,* Ann. of Math. **144**(2) (1996), 189–237.

[CC97] J. Cheeger, T. H. Colding: *On the structure of spaces with Ricci curvature bounded below. I.*, J. Differential Geom. **46**(3) (1997), 406–480.

[CC00a] J. Cheeger, T. H. Colding: *On the structure of spaces with Ricci curvature bounded below. II.*, J. Differential Geom. **54** (2000), 13–35.

[CC00b] J. Cheeger, T. H. Colding: *On the structure of spaces with Ricci curvature bounded below. III.*, J. Differential Geom. **54** (2000), 37–74.

[CG71] J. Cheeger, D. Gromoll: *The splitting theorem for manifolds of non-negative Ricci curvature*, J. Differential Geom. **6** (1971), 119–128.

[CW77] R.R. Coifman, G. Weiss: *Extensions of Hardy spaces and their use in analysis,* Bull. Amer. Math. Soc. **83**(4) (1977), 569–645.

[CN12] T.H. Colding, A. Naber: *Sharp Hölder continuity of tangent cones for spaces with a lower Ricci curvature bound and applications,* Annals of Math. **176** (2012), 1173–1229.

[CMS01] D. Cordero-Erausquin, R. McCann, M. Schmuckenschläger: *A Riemannian interpolation inequality à la Borell, Brascamp and Lieb,* Invent. Math. **146** (2001), 219–257.

[D97] E. B. Davies: *Non-Gaussian aspects of heat kernel behaviour*, J. London Math. Soc. **55**(2) (1997), 105-125.

[DePhMR16] G. De Philippis, A. Marchese, F. Rindler: *On a conjecture of Cheeger,* in: Measure theory in non-smooth spaces, De Gruyter Open, Warsaw, 2017, 145-155.

[D02] Y. Ding: *Heat kernels and Green's functions on limit spaces,* Comm. Anal. Geom. **10** (2002), 475-514.

[E15] A. Einstein: *Zur allgemeinen Relativitätstheorie,* Sitzungsberichte der Preuss. Akad. der Wissensch. (1915), 101–134.

[EW13] A. Engel, M. Weiandt: *Isospectral Alexandrov spaces,* Ann. Glob. Anal. Geom. **44** (2013), 501–515.

[EKS16] M. Erbar, N. Juillet: *Smoothing and non-smoothing via a flow tangent to the Ricci flow,* to appear in Journal des Mathématiques Pures et Appliquées, Arxiv preprint: 1603.00280 (2016).

[EKS15] M. Erbar, K. Kuwada, K.-T. Sturm: *On the equivalence of the entropic curvature-dimension condition and Bochner's inequality on metric measure spaces,* Invent. Math. **201** (2015), 993–1071.

[F69] H. Federer: *Geometric Measure Theory,* Springer, 1969.

[F71] W. Feller: *An Introduction to Probability Theory and its Applications,* Vol.II, 2nd edn. New York: Wiley, 1971.

[Fu57] B. Fuglede: *Extremal length and functional completion*, Acta Math. **98** (1957), 171–219.

[F87] K. Fukaya: *Collapsing of Riemannian manifolds and eigenvalues of the Laplace operator*, Invent. Math. **87** (1987), 517–547.

[FOT10] M. Fukushima, Y. Oshima, M. Takeda: *Dirichlet Forms and Symmetric Markov Processes*, De Gruyter Studies in Mathematics **19**, 2010.

[GHL04] S. Gallot, D. Hulin, J. Lafontaine: *Riemannian Geometry,* Third edition, Universitext, Springer-Verlag, Berlin Heidelberg, 2004.

[GM14] N. Garofalo, A. Mondino: *Li-Yau and Harnack type inequalities in* $\mathrm{RCD}^*(K,N)$ *metric measure spaces,* Nonlinear Anal. **95** (2014), 721-734.

[G13] N. Gigli: *The splitting theorem in non-smooth context*, ArXiv preprint: 1302.5555 (2013).

[G15] N. Gigli: *On the differential structure of metric measure spaces and applications,* Mem. Am. Math. Soc. **236** (2015), vi-91.

[G18] N. Gigli: *Nonsmooth differential geometry – An approach tailored for spaces with Ricci curvature bounded from below*, Mem. Amer. Math. Soc. **251** (2018), v-161.

[GH14] N. Gigli, B. Han: *The continuity equation on metric measure spaces,* Calc. Var. **53**(1-2) (2015), 149–177.

[GKO13] N. Gigli, K. Kuwada, S. Ohta: *Heat flow on Alexandrov spaces,* Comm. Pure and Appl. Math. **66**(3) (2013), 307–331.

[GM14] N. Gigli, C. Mantegazza: *A flow tangent to the Ricci flow via Heat Kernels and Mass Transport,* Advances in Math. **250** (2014), 74-104.

[GMS15] N. Gigli, A. Mondino, G. Savaré: *Convergence of pointed non-compact metric measure spaces and stability of Ricci curvature bounds and heat flows*, Proceedings of the London Mathematical Society **111** (2015), 1071–1129.

[GP16] N. Gigli, E. Pasqualetto: *Behaviour of the reference measure on* RCD *spaces under charts,* ArXiV preprint: 1607.05188v2 (2016).

[Gr92] A. Grigor'yan: *Heat equation on a non-compact Riemannian manifold*, Matem. Sbornik **182** (1992), 55–87, Engl. Transl. in Math. USSR Sb 72, 47–77.

[GS05] A. Grigor'yan, L. Saloff-Coste: *Stability results for Harnack inequalities*, Ann. Inst. Fourier **55**(3) (2005), 825–890.

[Gro07] M. Gromov: *Metric structures for Riemannian and non-Riemannian spaces,* Modern Birkhäuser Classics, Birkhäuser, Boston, MA, english ed., 2007.

[Gro81] M. Gromov: *Groups of polynomial growth and expanding maps*, Publ. Math. de l'IHES **53** (1981), 53–78.

[Gro91] M. Gromov: *Sign and geometric meaning of curvature*, Rondi. Sem. Mat. Fis. Milano **61** (1991), 9–123.

[Ha95] P. Hajlasz: *Sobolev spaces on an arbitrary metric space*, J. Potential Anal. **5** (1995), 403–415.

[HK95] P. Hajlasz, P. Koskela: *Sobolev meets Poincaré*, C.R. Acad. Sci. Paris **320** (1995), 1211–1215.

[HK00] P. Hajlasz, P. Koskela: *Sobolev met Poincaré*, Me. Am. Math. Soc. **145** (2000), 1–101.

[Ha17] B.-X. Han: *Ricci tensor on* $\mathrm{RCD}^*(K,N)$ *spaces*, ArXiv preprint 1412.0441v3 (2017).

[Har16] J. Harvey: *Equivariant Alexandrov Geometry and Orbifold Finitenes,* J. Geom. Anal. **26** (2016), 1925–1945.

[He11] H.-J. Hein: *Weighted Sobolev inequalities under lower Ricci curvature bounds,* Proceedings of the American Mathematical Society **139** (2011), 2943–2955.

[Hein01] J. Heinonen: *Lectures on analysis on metric spaces*, Springer, New York, 2001.

[HeK98] J. Heinonen, P. Koskela: *Quasiconformal maps in metric spaces with controlled geometry*, Acta Math. **181** (1998), 1–101.

[H15] S. Honda: *Ricci curvature and L^p-convergence*, J. Reine Angew Math. **705** (2015), 85–154.

[H16] S. Honda: *Ricci curvature and Orientability,* ArXiv preprint: 1610.02932.

[Hö68] L. Hörmander: *The spectral function of an elliptic operator*, Acta Mathematica **121** (1968), 193–218.

[Ivr80] V. Ivrii: *Second term of the spectral asymptotic expansion for the Laplace-Beltrami operator on manifold with boundary,* Funct. Anal. Appl. **14**(2) (1980), 98–106.

[J16] R. Jiang: *The Li-Yau Inequality and Heat Kernels on Metric Measure Spaces,* J. Math. Pures Appl. **104**(1) (2015), 29-57.

[JLZ16] R. Jiang, H. Li, and H.-C. Zhang: *Heat Kernel Bounds on Metric Measure Spaces and Some Applications,* Potent. Anal. **44** (2016), 601–627.

[JK098] R. Jordan, D. Kinderlehrer, F. Otto: *The variational formulation of the Fokker-Planck equation,* SIAM J. Math. Anal. **29**(1) (1998), 1–17.

[Ju09] N. Juillet: *Geometric Inequalities and Generalized Ricci Bounds in the Heisenberg Group*, Intern. Math. Res. Notes **2009**(13) (2009), 2347–2373.

[Kap07] V. Kapovitch: *Perelman's Stability Theorem*, Surveys in Differential geometry, Metric and Comparison geometry, vol XI , International press (2007), 103–136.

[Kap05] V. Kapovitch: *Restrictions on collapsing with a lower sectional curvature bound,* Math. Zeit. **249**(3) (2005), 519–539.

[KM17] M. Kell and A. Mondino: *On the volume measure of non-smooth spaces with Ricci curvature bounded below,* To appear in Annali SNS, ArXiV preprint: 1607.02036v2 (2017).

[K13a] C. Ketterer: *Ricci curvature bounds for warped products,* J. of Funct. Anal. **265**(2) (2013), 266–299.

[K15b] C. Ketterer: *Cones over metric measure spaces and the maximal diameter theorem,* J. Math. Pures Appl. **103**(5) (2015), 1228–1275.

[KM96] J. Kinnunen, O. Martio: *The Sobolev capacity on metric spaces,* Ann. Acad. Scien. Fenn. Math. **21** (1996), 367–382.

[K94] B.Kirchheim: *Rectifiable metric spaces: local structure and regularity of the Hausdorff measure,* Proc. AMS **121** (1994), 113–123.

[K17] Y. Kitabeppu: *A sufficient condition to a regular set of positive measure on RCD spaces*, ArXiv preprint: 1708.04309 (2017).

[KS93] J.N. Korevaar, R.M. Schoen: *Sobolev spaces and harmonic maps for metric space targets,* Comm. Anal. Geom. **1** (1993), 561–659.

[KM98] P. Koskela, P. MacManus: *Quasiconformal mappings and Sobolev spaces*, Studia Mathematica **131**(1) (1998), 1–17.

[KMS01] K. Kuwae, Y. Machigashira, T. Shioya: *Sobolev spaces, Laplacian, and heat kernel on Alexandrov spaces*, Math. Z. **238** (2001), 269—316.

[KS11] K. Kuwae, T. Shioya: *A topological splitting theorem for weighted Alexandrov spaces*, Tohoku Math. J. **63** (2011), 59—76.

[Le00] M. Ledoux: *The geometry of Markov diffusion generators*, Annales de la faculté des sciences de Toulouse **9**(2) (2000), 305–366.

[Le01] B. Levi: *Sul principio di Dirichlet*, Rend. Circ. Mat. Palermo **22** (1901), 293–359.

[Le52] B. M. Levitan: *On the asymptotic behavior of the spectral function of the second order elliptic equation*, in Russian, Izv. AN SSSR Ser. Mat. **16(1)** (1952), 325–352.

[L12] P. Li: *Geometric Analysis,* Cambridge studies in advanced mathematics **134**, Cambridge University Press, 2012.

[LY86] P. Li, S.-T. Yau: *On the parabolic kernel of the Schrödinger operator,* Acta Math. **156** (1986), 153-201.

[LV09] J. Lott, C. Villani: *Ricci curvature for metric-measure spaces via optimal transport*, Ann. of Math. **169** (2009), 903–991.

[Mc97] R. McCann: *A convexity principle for interacting gases,* Adv. Math. **128** (1997), 153–179.

[Mil15] E. Milman: *Spectral Estimates, Contractions and Hypercontractivity,* ArXiv preprint: 1508.00606v3.

[MP49] S. Minakshisundaram, A. Pleijel: *Some properties of the eigenfunctions of the Laplace-operator on Riemannian manifolds*, Canadian Journ. of Math. **1** (1949), 242–256.

[Mi09] V. Minerbe: *Weighted Sobolev inequalities and Ricci flat manifolds*, G.A.F.A. **18**(5) (2009), 1696–1749.

[MN14] A. Mondino, A. Naber: *Structure Theory of Metric-Measure Spaces with Lower Ricci Curvature Bounds I*, to appear in Journ. European Math. Soc., ArXiv preprint:1405.2222.

[Mo69] U. Mosco: *Convergence of convex sets and of solutions of variational inequalities,* Advances in Math. **3** (1969), 510–595.

[Mo64] J. Moser: *A Harnack inequality for parabolic differential equations,* Communications on Pure and Applied Mathematics **17**(1) (1964), 101–134.

[Oh09] S.-I. Ohta *Finsler interpolation inequalities,* Calc. Var. Partial Differential Equations **36** (2009), 211–249.

[OS09] S.-I. Ohta, K.-T. Sturm *Heat flow on Finsler manifolds,* Comm. Pure Appl. Math. **62**(10) (2009), 1386–1433.

[Ot01] F. Otto *The geometry of dissipative evolution equations: the porous medium equation,* Comm. Part. Diff. Eq. **26** (2001) 1-2, 101–174.

[OV00] F. Otto, C. Villani: *Generalization of an Inequality by Talagrand and Links with the Logarithmic Sobolev Inequality,* J. of Funct. Anal. **173**(2) (2000), 361–400.

[Per91] G. Perelman: *Alexandrov spaces with curvatures bounded from below II.,* preprint (1991).

[Pet11] A. Petrunin: *Alexandrov meets Lott-Villani-Sturm,* Münster J. Math. **4** (2011), 53–64.

[Raj12] T. Rajala: *Local Poincaré inequalities from stable curvature conditions on metric spaces*, Calc. Var. **44**(3) (2012), 477–494.

[Raj16] T. Rajala: *Failure of the local-to-global property for* $\mathrm{CD}(K,N)$ *spaces*, Ann. Sc. Norm. Super. Pisa Cl. Sci. **16** (2016), 45–68.

[RS14] T. Rajala, K.-T. Sturm: *Non-branching geodesics and optimal maps in strong* $\mathrm{CD}(K,\infty)$ *spaces*, Calc. Var. **50** (2014), 831–846.

[Ric02] G. Ricci-Curbastro: *Formule fondamentali nella teoria generale delle varietà e della loro curvatura*, Rend. Acc. Lincei **XI**, 1902.

[RL01] G. Ricci-Curbastro, T. Levi-Civita: *Méthodes de Calcul différentiel absolu et leurs applications*, Math. Ann. **B. 54** (1901), Préface.

[VR04] M.-K. von Renesse: *Heat Kernel Comparison on Alexandrov Spaces with Curvature Bounded Below*, Potential Anal. **21**(2) (2004), 151–176.

[RS05] M.-K. von Renesse, K.-T. Sturm: *Transport inequalities, gradient estimates, entropy, and Ricci curvature*, Comm. Pure Appl. Math. **58** (2005), 923–940.

[Sa92] L. Saloff-Coste: *A note on Poincaré, Sobolev, and Harnack inequalities*, Intern. Math. Res. Not. **1992**(2) (1992), 27–38.

[Sa02] L. Saloff-Coste: *Aspect of Sobolev-type inequalities*, London Mathematical Society Lecture Note Series **289**, Cambridge University Press, 2002.

[S14] G. Savaré: *Self-improvement of the Bakry-Émery condition and Wasserstein contraction of the heat flow in* $RCD(K,\infty)$ *metric measure spaces,* Discrete Contin. Dyn. Syst. **34** (2014), 1641–1661.

[Se96] S. Semmes: *Finding curves on General Spaces through Quantitative Topology, with Applications to Sobolev and Poincaré Inequalities*, Selecta Math. **2**(2) (1996), 155–295.

[Sh00] N. Shanmugalingam: *Newtonian spaces: an extension of Sobolev spaces to metric measure spaces*, Rev. Mat. Iberoamericana, **16** (2000), 243–279.

[S79] B. Simon: *Functional Integration and Quantum Physics,* Academic Press, 1979.

[Stan05] E. Stanhope: *Spectral Bounds on Orbifold Isotropy,* Ann. Glob. Anal. Geom. **27** (2005), 355–375.

[St94] K.-T. Sturm: *Analysis on local Dirichlet spaces. I. Recurrence, conservativeness and L^p-Liouville properties*, J. Reine Angew. Math. **456** (1994), 173-196.

[St95] K.-T. Sturm: *Analysis on local Dirichlet spaces. II. Upper Gaussian estimates for the fundamental solutions of parabolic equations*, Osaka J. Math. **32**(2) (1995), 275-312.

[St96] K.-T. Sturm: *Analysis on local Dirichlet spaces. III. The parabolic Harnack inequality*, J. Math. Pures Appl. **75**(3) (1996), 273-297.

[St98] K.-T. Sturm: *Diffusion processes and heat kernels on metric spaces*, Ann. Probability **26**(1) (1998), 1–55.

[St06a] K.-T. Sturm: *On the geometry of metric measure spaces, I.*, Acta Math. **196** (2006), 65–131.

[St06b] K.-T. Sturm: *On the geometry of metric measure spaces, II.*, Acta Math. **196** (2006), 133–177.

[S85] T. Sunada: *Riemannian coverings and isospectral manifolds*, Ann. of Math. **121**(1), 169–186 (1985).

[T17a] D. Tewodrose: *Weighted Sobolev inequalities on CD(0,N) spaces*, submitted paper, HAL Id : hal-01477215v2.

[T17b] D. Tewodrose: *Adimensional weighted Sobolev inequalities in PI doubling spaces*, in preparation.

[To54] A. Tonolo: *Commemorazione di Gregorio Ricci-Curbastro nel primo centenario della nascita,* Rendiconti del Seminario Matematico della Università di Padova **23** (1954), 1–24.

[To64] V. A. Topogonov: *Riemanian spaces which contains straight lines*, Amer. Math. Soc. Transl. **2** (1964), 287–290.

[Vi03] C. Villani: *Topics in Optimal Transpotation*, Graduate Studies in Mathematics **58**, Amer. Math. Soc., Providence, RI, 2003.

[Vi09] C. Villani: *Optimal transport. Old and new*, Grundlehren der Mathematischen Wissenschaften **338**, Springer-Verlag, Berlin, 2009.

[W00] N. Weaver: *Lipschitz algebras and derivations. II. Exterior differentiation*, J. Funct. Anal. **178** (2000), 64–112.

[We11] H. Weyl: *Über die asymptotische Verteilung der Eigenwerte*, Nachrichten der Königlichen Geselischaft der Wissenschaften zu Göttingen (1911), 110–117.

[ZZ10] H.-C. Zhang and X.-P. Zhu: *Ricci curvature on Alexandrov spaces and rigidity theorems*, Commun. Anal. Geom. **18** (2010), 503–553.