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Topics in Anabelian Geometry

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Introduction

In his letter to Faltings [Gro97], Grothendieck proposed a series of conjectures about "anabelian" varieties: it is not entirely clear which varieties should be anabelian, but Grothendieck gave some clues. Their fundamental groups should certainly be highly non abelian, and in dimension one they should coincide with hyperbolic curves, i.e. those with negative Euler characteristic. In higher dimension the picture is not so clear, since Grothendieck gave no precise definition. He said the class of anabelian varieties should contain the so called elementary anabelian varieties, i.e. those ones that can be obtained by subsequent fibrations by hyperbolic curves, and moduli stacks of smooth, hyperbolic curves.

The idea of Grothendieck's anabelian philosophy is that if X is an anabelian variety, and if the base field k is finitely generated over $\mathbb{Q}$, it should be possible to recover the geometry of the variety from the étale fundamental group $\pi_1(X)$ with its projection to the absolute Galois group $\mathrm{Gal}(\bar{k}/k)$. Grothendieck's conjectures predict more precisely how this happens.

Let us fix a base field k finitely generated over $\mathbb{Q}$, and write G_k for $\mathrm{Gal}(\bar{k}/k)$. If X is geometrically connected and x is a geometric point, there is a short exact sequence of étale fundamental groups

$$0 \to \pi_1(\bar{X}, \bar{x}) \to \pi_1(X, x) \to G_k \to 0$$

Let T be another geometrically connected scheme with a geometric point t. Write $\text{Hom-ext}_{G_k}(\pi_1(T,t), \pi_1(X,x))$ for the set of continuous homomorphisms $\pi_1(T,t) \to \pi_1(X,x)$ which commute with the projections to G_k, considered up to conjugation by elements of $\pi_1(\bar{X}, \bar{x})$. There is a natural map

$$\mathrm{Hom}_k(T, X) \to \text{Hom-ext}_{G_k}(\pi_1(T,t), \pi_1(X,x)).$$

For proper varieties, Grothendieck gave two forms of his "main conjecture".

Conjecture (Hom conjecture). If T is smooth and X is smooth, proper and anabelian, then

$$\mathrm{Hom}_k(T, X) \to \text{Hom-ext}_{G_k}(\pi_1(T, t), \pi_1(X, x))$$

is a bijection.

We call this the *hom conjecture*. There is a weaker form of the hom conjecture which restricts the attention to dominant morphisms, and this has been proved by Mochizuki for hyperbolic curves. The second form of the main conjecture is the so called *section conjecture*, which is just the hom conjecture for $T = \operatorname{Spec} k$.

Conjecture (Section conjecture). If X is smooth, proper and anabelian, then

$$X(k) \to \text{Hom-ext}_{G_k}(G_k, \pi_1(X, x))$$

is a bijection.

The present thesis studies various aspects of these conjectures, and is divided in two independent parts. In the first one, contained in chapter 1, we give a form of the section conjecture well suited for Deligne-Mumford stacks and establish various implications between anabelian conjectures. In the second part, contained in chapter 2 and chapter 3, we state a dimensional variant of the section conjecture and make some steps toward proving it for $\mathbb{P}^1$ minus n points, for $n \geq 3$. This dimensional variant relies on the concept of *fce dimension*, which we introduce as a variation of essential dimension. Since fce dimension might be of interest even for mathematicians not interested in anabelian geometry, we have separated its study from the rest of the thesis by reserving chapter 2 for it.

For our work in chapter 1, we need to replace the formalism of étale fundamental groups with the much more convenient one of étale fundamental gerbes, introduced by Niels Borne and Angelo Vistoli in [BV15]. In Appendix A we prove some tools we need which are straightforward generalizations of the work of Borne and Vistoli, and in Appendix B we give a brief comparison between the classical formalism and the one of étale fundamental gerbes, showing how to pass from one to the other.

Here, let us just say that if X is a geometrically connected fibered category over k (it may be a scheme, an algebraic stack or a more general object) there exists a pro-étale gerbe $\Pi_{X/k}$, called the étale fundamental gerbe, with a morphism

$$X \to \Pi_{X/k}$$

characterized by the fact that it is universal among morphisms in finite étale stacks. The étale fundamental gerbe carries the same information as the étale fundamental group together with its projection to $\mathrm{Gal}(k_s/k)$. In particular, if T is a geometrically connected scheme then we may recover $\text{Hom-ext}_{G_k}(\pi_1(T,t), \pi_1(X,x))$ as the set of isomorphism classes of $\Pi_{X/k}(T)$. Moreover, in this formalism the maps of Grothendieck's conjectures are just the one induced by the morphism $X \to \Pi_{X/k}$.

In addition to the new results we prove, our approach with the formalism of fundamental gerbes allows us to reprove easily and in an unified way a lot of classical results that are scattered in the literature: the description of packets of sections, the fact that the section conjecture for proper curves implies the one for open curves, the triviality of centralizers, the going up and going down theorems for étale covers.

Anabelian geometry for DM stacks In chapter 1, we reverse Grothendieck's point of view and define anabelian DM stacks as those satisfying a strong form of the section conjecture: if k is finitely generated over $\mathbb{Q}$, we define a smooth, proper DM stack X as anabelian (resp. fundamentally fully faithful, or fff) if

$$X(k') \to \Pi_{X/k}(k')$$

is an equivalence of categories (resp. fully faithful) for every finitely generated extension k'/k, see Definition 1.2.2.

For schemes, anabelianity amounts to the section conjecture being verified for every finitely generated extension of the base field together with the triviality of centralizers of sections. Since for hyperbolic curves centralizers of sections are already known to be trivial, we get the following.

Corollary 1.2.5. *Let k be finitely generated over $\mathbb{Q}$.*

- *A smooth, proper curve over k is fundamentally fully faithful if and only if its Euler characteristic is less than or equal to 0.*
- *A smooth, proper hyperbolic curve over k is anabelian if and only if it satisfies the section conjecture over every finitely generated extension of k.*

Our definition of anabelianity, which is deeply arithmetic in nature, turns out to be purely geometric in the following sense. Let k'/k be finitely generated and X a smooth, proper DM stack over k. If X is anabelian, then clearly $X_{k'}$ is anabelian, too. The converse is not obvious, but true, see Proposition 1.4.2. Since every DM stack of finite type over $\mathbb{C}$ can be defined over a finitely generated extension of $\mathbb{Q}$, we can define anabelian DM stacks over $\mathbb{C}$ as those ones that are anabelian over every finitely generated extension of $\mathbb{Q}$. Then if $k/\mathbb{Q}$ is finitely generated and $k \subseteq \mathbb{C}$, a smooth and proper DM stack X is anabelian if and only if $X_{\mathbb{C}}$ is anabelian. This is in line with the ideas Grothendieck expressed in his letter to Faltings:

> *Allenfalls soil die anabelsche Eigenschaft eine rein geometrische sein, nämlich sie hängt nur von $\overline{X}$ über dem alg. abg. Körper $\overline{K}$ (oder dem entsprechenden Schema über beliebiger alg. abg. Erweiterung von $\overline{K}$, etwa $\mathbb{C}$) ab.* [1]

Obviously, which DM stacks over $\mathbb{C}$ are anabelian remains a mystery. Still, we think that it is worth observing that this purely arithmetic property depends only on the geometry of the variety, confirming Grothendieck's ideas.

We then give in section 1.5 the first nontrivial example of expected anabelian DM stack, i.e. hyperbolic orbicurves. Orbicurves are essentially curves where we replace some closed points with copies of $B\mu_n$. There is a notion of rational Euler characteristic for orbicurves, i.e. for every orbicurve X there exists a rational number $\chi(X)$ which coincides with the usual Euler characteristic if X is a curve. Hyperbolic orbicurves are those with negative characteristic. It is then natural to conjecture the following:

Conjecture. Smooth, proper, hyperbolic orbicurves over a finitely generated extension of $\mathbb{Q}$ are anabelian.

Niels Borne and Michel Emsalem had already formulated a weaker form of this conjecture (asking only for a bijection and not an equivalence of categories), see [BE14, Conjecture 2]. We then prove that the conjecture for orbicurves is equivalent to the one for curves:

Theorem 1.5.3. *Let k be finitely generated over $\mathbb{Q}$.*

- *A smooth, proper orbicurve X is fundamentally fully faithful if and only if $\chi(X) \leq 0$.*
- *Smooth, proper, hyperbolic orbicurves are anabelian if and only if smooth, proper, hyperbolic curves are anabelian.*

In particular we get that the section conjecture for every hyperbolic curve and every finitely generated extension of $\mathbb{Q}$ implies that the section conjecture holds for orbicurves, too.

[1] In any case, being anabelian is a purely geometric property, that is, one which depends only on $\overline{X}$, defined over the algebraic closure $\overline{K}$ (or the corresponding scheme over an arbitrary algebraically closed extension of K, such as $\mathbb{C}$).

In section 1.6, following an idea of Borne and Emsalem we show (in the case of curves and orbicurves) how using Deligne-Mumford stacks allows to treat the non proper case of the section conjecture as a limit case of the proper case: this greatly clarifies the situation and the meaning of "packets" of sections. As a byproduct, if we merge this picture with Theorem 1.5.3 we obtain a new proof of the fact that the section conjecture for proper curves implies the section conjecture for open curves.

In section 1.7 we prove one of our main results: the fact that the section conjecture implies the hom conjecture. Actually, we show that a stronger form of the section conjecture, i.e. anabelianity, implies a stronger form of the hom conjecture. Grothendieck stated such an implication, but I know of no proof of it in the literature. Our form of the hom conjecture is stronger than Grothendieck's one: we relax the smoothness hypothesis by asking only normality together with a very broad finiteness condition, being *left*, see Definition 1.7.1.

Theorem 1.7.3. *Let X be a DM stack and T an integral, normal left scheme over k. If X is fundamentally fully faithful, then $X(T) \to \Pi_X(T)$ is fully faithful. If X is anabelian, then $X(T) \to \Pi_X(T)$ is an equivalence of categories.*

As a corollary, we prove another fact about anabelian DM stacks: their topological fundamental groups have no finite index abelian subgroups.

Theorem 1.7.6. *Let X be an anabelian DM stack such that $\pi_1(X_{\bar{k}})$ has a finite index abelian subgroup. Then* $\dim X = 0$.

Observe that this result is of the form "anabelian properties" $\Rightarrow$ "fundamental group is far from abelian": conjectures and theorems are usually in the other direction.

Recall that Grothendieck defined a variety as elementary anabelian if it can be obtained by subsequent fibrations in hyperbolic curves. In section 1.8, we enlarge this definition by defining elementary anabelian DM stacks. We prove that elementary anabelian DM stacks are of type $K(G, 1)$ for both the classical and the étale topology, and we show that they are stable under some elementary operations in addition to the ones defining them. Despite the name, elementary anabelian DM stacks are not known to be anabelian.

Theorem 1.8.10. *Let k be finitely generated over $\mathbb{Q}$.*

- *Elementary anabelian DM stacks are fundamentally fully faithful.*
- *Elementary anabelian DM stacks are anabelian if and only if smooth, projective hyperbolic curves are anabelian.*

Again, Grothendieck stated such an implication for elementary anabelian varieties, but I know of no proof of it in the literature.

Fce dimension In chapter 2 we introduce a variant of essential dimension called fce dimension. We introduce and study this variant in order to formulate a dimensional version of the section conjecture, but the concept stands on its own. I have to thank my advisor Angelo Vistoli for the idea of applying essential dimension to the study of the section conjecture.

Essential dimension is a widely studied notion of dimension firstly introduced by Buhler and Reichstein in [BR97]. In its original and most studied form, essential dimension is applied to the space of torsors of a group scheme G: it is the minimum number of parameters needed to define a G-torsor. More precisely, if G is a group scheme over a field k and $T \to \operatorname{Spec} L$ is a torsor with L/k an extension, the essential dimension $\operatorname{ed} T$ of T is the minimum of the transcendence degrees of subextensions over which T is defined. The essential dimension $\operatorname{ed} G$ is the maximum of $\operatorname{ed} T$ where T varies among all G-torsors over all field extensions L/k.

If we want to apply the concept of essential dimension to the section conjecture, we have first to solve a problem: we have to understand the behaviour of essential dimension for pro-étale group schemes, since to the best of our knowledge essential dimension has only been studied for group schemes of finite type, while the étale fundamental group scheme (see Appendix B) is pro-étale.

Unfortunately, it turns out that essential dimension behaves very badly for pro-étale group schemes: it is infinite very often. In fact, we have proved the following.

Proposition 2.1.1. *Let G/k be a proétale group scheme, with $\operatorname{char} k = 0$. Suppose that there exists a prime p and an extension L/k with a nontrivial morphism $G_L \to \mathbb{Z}_p$. Then* $\operatorname{ed} G = \infty$.

Because of Proposition 2.1.1, essential dimension does not give a lot of information about pro-étale group schemes. We address this problem by defining the *fce dimension* as a modification of essential dimension. For group schemes of finite type the fce dimension coincides with essential dimension, but it has a much better behaviour in the pro-étale case, thus we may regard it as a generalization rather than a different version. This is somewhat analogous to what happens when we try to generalize Galois theory from finite to infinite extensions: in order to make the theory interesting, we have to introduce a topology that we could not see in the finite case.

In section 2.4 we describe the class of *very rigid group schemes*: these are pro-étale group schemes having a very peculiar behaviour, in particular they have very strong properties with respect to the fce dimension, properties that are completely false in the algebraic case. In arithmetic, very rigid group schemes are common: over fields finitely generated over $\mathbb{Q}$ some examples are the Tate module of an abelian variety, $\widehat{\mathbb{Z}}(1)$ and $\mathbb{Z}_p(1)$, the étale fundamental group scheme of a smooth curve. In particular, the following facts hold for very rigid group schemes.

Corollary 2.4.16. *If G/k is very rigid, then $\mathrm{H}^1(G, k') \to \mathrm{H}^1(G, k'')$ is injective for every finitely generated extensions $k''/k'/k$.*

Corollary 2.4.19. *Let G be a very rigid group scheme over k, and k'/k a finite extension. Then the fce dimensions of G and $G_{k'}$ are equal.*

Corollary 2.4.21. *Let G be a very rigid group scheme over k, and $H \subseteq G$ a subgroup of finite index. Then the fce dimensions of G and H are equal.*

The dimensional section conjecture We come now to chapter 3, which concerns a dimensional variant of the section conjecture.

We will use the formalism of the étale fundamental group scheme: in characteristic 0, this is just Nori's fundamental group scheme. If X is a geometrically connected variety with a rational base point $x \in X(k)$, the étale fundamental group scheme is a pro-étale group scheme $\underline{\pi}_1(X, x)$ carrying the same information of the étale fundamental group together with its projection to the absolute Galois group. See Appendix B for details.

Using the fce dimension, Grothendieck's section conjecture implies the following, hopefully easier conjecture:

Conjecture 3.0.1. Let X be a smooth, geometrically connected hyperbolic curve over a field k finitely generated over $\mathbb{Q}$, and $x \in X(k)$ a rational point. Then $\underline{\pi}_1(X, x)$ has fce dimension equal to 1.

Just as the injectivity part of the section conjecture is easier, the lower bound of the dimensional version is easier, too, and we do this in Proposition 3.0.3. The whole point of the conjecture is proving the upper bound. Conjecture 3.0.1 could be a first step toward the section conjecture: it would say that there cannot be too many sections, in a dimensional sense.

The dimensional section conjecture seems still very hard. However, something nontrivial can be said about it: we are able to do some explicit computations that might eventually lead to prove it for $\mathbb{P}^1 \setminus \{p_0, \ldots, p_n\}$, with $p_0, \ldots, p_n$ rational points and $n \geq 2$.

Observe moreover the following fact: by taking any rational function and removing ramification points, every curve X has an open set $U \subseteq X$ with a finite extension k'/k and a finite étale cover $U_{k'} \to \mathbb{P}^1_{k'} \setminus \{p_0, \dots, p_n\}$. Since the validity of the dimensional section conjecture is stable both under finite extension of the base field and finite étale cover (see Corollary 3.0.2 and Corollary 3.0.4), proving the conjecture for $\mathbb{P}^1 \setminus \{p_0, \dots, p_n\}$ would tell us that every curve has an open subset for which we know the dimensional section conjecture.

Let us now describe what we can do for $\mathbb{P}^1 \setminus \{p_0, \dots, p_n\}$ for $n \geq 2$. Let $\underline{\pi}_1\left(\mathbb{P}^1 \setminus \{p_0, \dots, p_n\}\right)$ be the étale fundamental group scheme, its abelianization is

$$\underline{\pi}_1\left(\mathbb{P}^1 \setminus \{p_0, \dots, p_n\}\right)^{\mathrm{ab}} = \widehat{\mathbb{Z}}(1)^n = \prod_p \mathbb{Z}_p(1)^n.$$

We have strong evidence that the following consequence of the dimensional section conjecture should hold.

Consequence ★. *The image of the morphism of functors* $\mathrm{Fields}_k \to \mathrm{Set}$

$$\mathrm{H}^1\left(\underline{\pi}_1\left(\mathbb{P}^1 \setminus \{p_0, \dots, p_n\}\right), \_\right) \to \mathrm{H}^1\left(\mathbb{Z}_p(1)^n, \_\right)$$

has fce dimension 1 *for each prime* p *and every* $n \geq 2$.

This would give a strong obstruction: $\mathbb{Z}_p(1)^n$ has fce dimension n, but ★ says that sections coming from $\underline{\pi}_1\left(\mathbb{P}^1 \setminus \{p_0, \dots, p_n\}\right)$ land in a subset of fce dimension 1. The good thing about ★ is that we are able to convert it in a very concrete problem on which we can actually do computations and play with examples. From now on, we focus on this consequence ★ of the conjecture, since there is concrete hope of proving it: actually, we have proved a good part of it.

The computation leading to ★ is based on the 2-nilpotent obstruction in nonabelian cohomology for the group $\underline{\pi}_1(\mathbb{P}^1 \setminus \{p_0, \dots, p_n\})$, an obstruction originally studied by Kirsten Wickelgren in [Wic12] for $n = 2$. As she observed, for the classical section conjecture this obstruction is too coarse. However, it should be fine enough to prove ★. We briefly illustrate the relevant part of Wickelgren's work in section 3.1.

The passage from the first more theoretical part to the explicit computations is quite technical: we do not want to enter here into the details of how this passage is made, but we do want to give an idea of the type of computations we end up with and the results we obtained.

One key tool are valuations in the group $\mathbb{Z}^m$ ordered lexicographically (which we call rank m valuations), and in particular the following statement.

Proposition 3.5.3. *Given an extension L/k and n elements $x_1, \ldots, x_n \in L^*$, then $\mathrm{trdeg}_k k(x_1, \ldots, x_n) \geq m$ if and only if there exists a rank m valuation v trivial on k such that $\det(v(x_1)|\ldots|v(x_n))$ has rank m.*

Recall that if L is a field, $\mathrm{H}^1(\mathbb{Z}_p(1), L) = \varprojlim_n L^*/L^{*p^n}$, we write $\wedge_p L^*$ for this set. In particular, n-uples $(x_1, \ldots, x_n) \in \oplus_n \wedge_p L^*$ define $\mathbb{Z}_p(1)^n$-torsors over L, and it makes sense to compute the fce dimension of a $\mathbb{Z}_p(1)^n$-torsor $(x_1, \ldots, x_n)$. This fce dimension plays the role of the transcendence degree of $k(x_1, \ldots, x_n)/k$ if $x_i \in L^*$. We then conjecture the following analogue of Proposition 3.5.3.

Conjecture 3.3.2. Let p be a prime number, $m \leq n$ positive integers and L/k finitely generated extensions of $\mathbb{Q}$. Consider n elements $x_1, \ldots, x_n \in \wedge_p L^*$. Then $(x_1, \ldots, x_n)$ has fce dimension greater or equal than m if and only if there exists a rank m valuation $v : L^* \to \mathbb{Z}^m$ such that

$$\big(v(x_1)|\ldots|v(x_n)\big) \in M_{n\times m}(\mathbb{Z}_p)$$

has rank m.

We call this the *valuation conjecture*. The valuation conjecture for $m = 2$ implies ★. From now on, we may forget about the dimensional section conjecture and focus on the valuation conjecture, in particular on the case $m = 2$. This is done in sections 3.3 and 3.4.

So, what evidence do we have about the valuation conjecture? While up to now chapter 3 was more expository, at this point it becomes more technical, and the real proofs come in. We have essentially two results.

The first one is that the the valuation conjecture holds in rank 1 in a stronger form, i.e. not at a prime but for the whole $\widehat{L}^* = \varprojlim_n L^*/L^{*n}$.

Theorem 3.3.6. *Let L/k be finitely generated over $\mathbb{Q}$, and consider $x_1, \ldots, x_n$ elements of $\widehat{L}^*$. Then, $(x_1, \ldots, x_n)$ has fce dimension greater or equal than 1 if and only if there exists a valuation $v : L^* \to \mathbb{Z}$ such that $v(x_i) \neq 0 \in \widehat{\mathbb{Z}}$ for some $i = 1, \ldots, n$.*

We stress out a very meaningful fact: in order to prove Theorem 3.3.6, the hypothesis that L, k are finitely generated over $\mathbb{Q}$ is essential, since it allows us to use the Mordell-Weil theorem. This tells us, at the end of this long journey of definitions and conjectures, that a true arithmetic obstruction pops up: we are not just playing with words.

In rank 2 we can divide the conjecture into two very different cases: the one of bounded degree and the one of unbounded degree, here we only study the first one. In bounded degree, we have finished the proof modulo a fact of pure birational geometry, see MathOverflow 306537. This fact is easily seen true for surfaces, hence the proof is complete when $\operatorname{trdeg}_k L = 2$.

Theorem 3.4.3. *Let L/k be fields finitely generated over $\mathbb{Q}$, and $x_1, \dots, x_n \in \wedge_p L^*$ such that $\det(v)(x_i, x_j) = 0$ for every rank 2 valuation v of L/k and every $1 \leq i < j \leq n$. Suppose that $x_1, \dots, x_n$ have bounded degree and $\operatorname{trdeg}_k L = 2$, or that $x_1, \dots, x_n$ have finite support (with no restrictions on $\operatorname{trdeg}_k L$). Then $(x_1, \dots, x_n)$ has fce dimension at most 1.*

A positive answer to MathOverflow 306537 would complete the proof of the rank 2 valuation conjecture in bounded degree.

Like that of Theorem 3.3.6, the proof of Theorem 3.4.3 uses heavily the Mordell-Weil theorem and more generally the arithmetic properties of fields finitely generated over $\mathbb{Q}$. The proof of Theorem 3.4.3 is highly complex, and occupies a good part of chapter 3.

Finally, in the last three sections we have included some technical results in order to unclutter the exposition. In section 3.5 we prove some facts about rank n valuations. In section 3.6 we build a theory of divisors and Picard groups for a finitely generated extension L/k: this is essentially the direct limit of the theory of divisors and Picard groups along projective varieties M with function field L. Finally, in section 3.7 we prove some facts about completions of abelian groups.

Conventions and notations We always work over a field k finitely generated over $\mathbb{Q}$, except in chapter 2 and Appendix A where there are no hypotheses on the base field. Curves and orbicurves will always be smooth, geometrically connected and proper, except if we specify differently.

If X is geometrically connected, we will denote by π_X the structure morphism $X \to \Pi_{X/k}$ of the étale fundamental gerbe. If there is no risk of confusion, we may drop the subscript and write just $\pi : X \to \Pi_{X/k}$.

We write $\pi_1(X, x)$ for classical étale fundamental groups and $\underline{\pi}_1(X, x)$ for étale fundamental group schemes, see Appendix B.

If A is an abelian group and p is a prime, we write $\wedge_p A$ for $\varprojlim_n A/p^n A$ and $\widehat{A}$ for $\varprojlim_n A/nA$. We use the notation $\wedge_p A$ instead of A_p in order to avoid confusion with completions of fields: for instance, $\wedge_p \mathbb{Q}^*$ has nothing to do with $\mathbb{Q}_p$, even if $\wedge_p \mathbb{Z} = \mathbb{Z}_p$.

Chapter 1

Anabelian geometry for DM stacks

Throughout this chapter (except in section 1.6), we restrict our attention to X proper. The reason is that the section conjecture is much easier to handle in the proper case, and if one states the anabelian conjectures for DM stacks rather than schemes then, thanks to an idea of Niels Borne and Michel Emsalem (see [BE14, §2.2.3] and section 1.6), the non-proper case can be seen as a projective limit of the proper case. The upside is that conjectures are much easier to state and manipulate in the proper case, the downside is that one has to embrace the formalism of stacks (which, however, is very appropriate for the anabelian world).

1.1 Stacky going up and going down theorems

To understand precisely how anabelian geometry for DM stacks should look like, the single most important fact to understand is how the section conjecture behaves along finite étale morphism. In a classical context, i.e. for schemes, this situation is well understood and packed in the so called "going up" and "going down" theorems, see [Sti13, Propositions 110, 111]. The formalism of étale fundamental gerbes is particularly well suited for the study of this situation: in fact, if $Y \to X$ is a finite étale morphism, the natural diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & \Pi_{Y/k} \arrow[d] \\ X \arrow[r] & \Pi_{X/k} \end{tikzcd}$$

is 2-cartesian, see Proposition A.3.2. This fact makes the study of finite étale morphism with respect to the section conjecture particularly easy, even for stacks.

Remark 1.1.1. In [Sti13, Propositions 110, 111], there are hypotheses on the so called centralizers of sections. If $s \in \Pi_{X/k}(k)$ corresponds to a section $s : \mathrm{Gal}(\bar{k}/k) \to \pi_1(X, \bar{x})$, the centralizer of s is the subgroup of elements of $\pi_1(X_{\bar{k}}, \bar{x})$ centralizing the image of s. We don't need them, since the notion of centralizer of a section (see [Sti13, §3.3]) fits nicely in our point of view without any additional work.

In fact, if we think of $\pi_1(X_{\bar{k}}, \bar{x})$ as

$$\mathrm{Aut}_{\Pi_{X/k}}(\pi_X(\bar{x})) = \underline{\mathrm{Aut}}_{\Pi_{X/k}}(\pi_X(\bar{x}))(\bar{k}) \simeq \underline{\mathrm{Aut}}_{\Pi_{X/k}}(s)(\bar{k}),$$

then the centralizer of s is the group of elements of $\underline{\mathrm{Aut}}_{\Pi_{X/k}}(s)(\bar{k})$ which satisfy Galois descent, i.e. rational points of the group scheme $\underline{\mathrm{Aut}}_{\Pi_{X/k}}(s)$. Hence saying that a section $s \in \Pi_{X/k}(k)$ has trivial centralizer simply means that $\underline{\mathrm{Aut}}_{\Pi_{X/k}}(s)$ has no rational points apart from the identity.

Proposition 1.1.2 (Going up)**.** *Let X, Y be geometrically connected fibered categories and $f : Y \to X$ a representable, finite étale morphism. The following are true:*

(i) If $X(k) \to \Pi_{X/k}(k)$ is fully faithful, then $Y(k) \to \Pi_{Y/k}(k)$ is fully faithful, too.

(ii) If $X(k) \to \Pi_{X/k}(k)$ is an equivalence, then $Y(k) \to \Pi_{Y/k}(k)$ is an equivalence, too.

Proof. By Proposition A.3.2, the 2-commutative diagram

$$\begin{tikzcd} Y \arrow[r, "\pi_Y"] \arrow[d, "f"] & \Pi_{Y/k} \arrow[d, "\Pi_f"] \\ X \arrow[r, "\pi_X"] & \Pi_{X/k} \end{tikzcd}$$

is 2-cartesian.

(i) Since $X(k) \to \Pi_{X/k}(k)$ is fully faithful, its base change

$$Y(k) = X(k) \times_{\Pi_{X/k}(k)} \Pi_{Y/k}(k) \to \Pi_{Y/k}(k)$$

is fully faithful too.

(ii) Since $X(k) \to \Pi_{X/k}(k)$ is an equivalence, its base change

$$Y(k) = X(k) \times_{\Pi_{X/k}(k)} \Pi_{Y/k}(k) \to \Pi_{Y/k}(k)$$

is an equivalence too.

$\square$

Definition 1.1.3. Let $\mathcal{C}, \mathcal{D}$ be categories and $f : \mathcal{C} \to \mathcal{D}$ a functor, $p \in \mathcal{C}$ an object. We say f is fully faithful at p if $\mathrm{Aut}_{\mathcal{C}}(p) \to \mathrm{Aut}_{\mathcal{D}}(f(p))$ is bijective.

Remark 1.1.4. Suppose that $\mathcal{C}, \mathcal{D}$ are small categories in which all morphisms are isomorphisms. For example, $X(S)$ has this form for every stack X and every scheme S. A functor $f : \mathcal{C} \to \mathcal{D}$ is fully faithful if and only if it is fully faithful at every point and is injective on isomorphism classes.

Lemma 1.1.5 (Extension of the base field)**.** *Let $f : A \to B$ be a morphism of fibered categories over k which are stacks in the étale topology, and L/k a finite Galois extension. Then the following are true.*

(i) Let $a \in A(k)$ be a rational point, $a_L \in A(L)$ the pullback of a. If $A(L) \to B(L)$ is fully faithful at a_L over L, then $A(k) \to B(k)$ is fully faithful at a.

(ii) If $A(L) \to B(L)$ is fully faithful, then $A(k) \to B(k)$ is fully faithful, too.

(iii) Let $b \in B(k)$ be a rational point, and suppose that $A(L) \to B(L)$ is fully faithful. Then b is in the essential image of $A(k) \to B(k)$ if and only if b_L is in the essential image of $A(L) \to B(L)$.

(iv) If $A(L) \to B(L)$ is an equivalence, then $A(k) \to B(k)$ is an equivalence, too.

Proof. (i) We have a commutative diagram

$$\begin{tikzcd}
\mathrm{Aut}_A(a) \arrow[r] \arrow[d, hook] & \mathrm{Aut}_B(f(a)) \arrow[d, hook] \\
\mathrm{Aut}_A(a_L) \arrow[r, "\sim"] & \mathrm{Aut}_B(f(a_L))
\end{tikzcd}$$

where the vertical arrows are injective, and the lower arrow is bijective by hypothesis. Both A and B are stacks in the étale topology, hence the $\underline{\mathrm{Isom}}$ functors are sheaves and satisfy Galois descent. This means that the groups in the upper row are just the $\mathrm{Gal}(L/k)$-invariant elements of the groups in the lower row. Since the lower horizontal arrow is clearly equivariant, we get that the upper horizontal row is bijective, too.

(ii) Thanks to point (i), $A(k) \to B(k)$ is fully faithful at every point. If $a, a' \in A(k)$ are such that $f(a) \simeq f(a')$, then $f(a_L) \simeq f(a'_L)$ and hence $a_L \simeq a'_L$. Since we have already identified automorphisms groups, $a_L \simeq a'_L$ descends to $a \simeq a'$, hence $A(k) \to B(k)$ is fully faithful.

(iii) The "only if" part is obvious. Now suppose that $b_L \simeq f(a')$ is in the essential image of $A(L) \to B(L)$. For every $\sigma \in \mathrm{Gal}(L/k)$, we have an isomorphism

$$\varphi_\sigma : \sigma^* f(a') \simeq \sigma^* b_L = b_L \simeq f(a')$$

which corresponds to an isomorphism $\psi_\sigma : \sigma^*(a') \simeq a'$ since $A(L) \to B(L)$ is fully faithful by hypothesis.

Now, we have $\varphi_{\sigma\rho} = \varphi_\sigma \circ \sigma^* \varphi_\rho$ by direct computation. Since $A(L) \to B(L)$ is fully faithful, this means that we also have $\psi_{\sigma\rho} = \psi_\sigma \circ \sigma^* \psi_\rho$ and hence by Galois descent there exists $a \in A(k)$ such that $a_L \simeq a'$. Let us check that $f(a) \simeq b$.

We have a chain of isomorphisms

$$f(a)_L = f(a_L) \simeq f(a') \simeq b_L,$$

we have to check that this is Galois invariant. This amounts to the fact that, by definition, $f(\psi_\sigma) = \varphi_\sigma$.

(iv) This is a direct consequence points (ii) and (iii).

□

In the following, we will use without mention the fact that, if X is a geometrically connected fibered category and L/k is a finite, separable extension, the natural morphism $\Pi_{X_L/L} \to \Pi_{X/k} \times_k L$ is an isomorphism (see Proposition A.2.11).

Proposition 1.1.6 (Going down)**.** *Let X and Y be geometrically connected fibered categories which are stacks in the étale topology, and $f : Y \to X$ a representable, finite étale morphism. The following are true:*

(i) If $Y_L(L) \to \Pi_{Y_L}(L)$ is fully faithful for every finite, separable extension L/k, then $X(k) \to \Pi_{X/k}(k)$ is fully faithful.

(ii) If $Y_L(L) \to \Pi_{Y_L}(L)$ is an equivalence for every finite, separable extension L/k, then $X(k) \to \Pi_{X/k}(k)$ is an equivalence.

Proof. As in Proposition 1.1.2, we are going to use the fact that the 2-commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\pi_Y} & \Pi_{Y/k} \\ \downarrow f & & \downarrow \Pi_f \\ X & \xrightarrow{\pi_X} & \Pi_{X/k} \end{array}$$

is 2-cartesian, see Proposition A.3.2. However, the proofs will be much more complex: the main problem is that, while a k-rational point of Y defines a k-rational point of X, the converse is not true, hence we will need to enlarge the base field and then use Galois descent to get back to k.

(i) First, let us check that $X(k) \to \Pi_{X/k}(k)$ is fully faithful at every point, next we will show that it is injective on isomorphism classes. Choose $p \in X(k)$, since $Y \to X$ is finite étale there exists a finite Galois extension L and a point $p' \in Y_L(L)$ such that $f(p') \simeq p_L$. Thanks to Lemma 1.1.5.(i), we may suppose $L = k$, $f(p') \simeq p$.

Now, we have an isomorphism

$$\mathrm{Aut}_Y(p') \simeq \mathrm{Aut}_X(p) \times_{\mathrm{Aut}_{\Pi_{X/k}}(\pi_X(p))} \mathrm{Aut}_{\Pi_{Y/k}}(\pi_Y(p')),$$

and we also know that

$$\mathrm{Aut}_Y(p') \simeq \mathrm{Aut}_{\Pi_{Y/k}}(\pi_Y(p'))$$

by hypothesis. In particular, $\mathrm{Aut}_X(p) \to \mathrm{Aut}_{\Pi_{X/k}}(\pi_X(p))$ is injective: let us show that it is surjective, too.

Suppose that $g \in \mathrm{Aut}_{\Pi_{X/k}}(\pi_X(p))$ does not come from an element of $\mathrm{Aut}_X(p)$, thanks to Proposition A.3.2 the triple

$$(p, g, \pi_Y(p'))$$

gives us a point $p'' \in Y(k)$ such that $\pi_Y(p'') \simeq \pi_Y(p')$ and $f(p'') \simeq p$. Now the fact that g does not come from $\mathrm{Aut}_X(p)$ means exactly that the isomorphism $f(p'') \simeq p \simeq f(p')$ does not lift to an isomorphism $p'' \simeq p'$, but we have that $\pi_Y(p'') \simeq \pi_Y(p')$ is a lifting of $\pi_X(f(p'')) \simeq \pi_X(f(p'))$, which is absurd since $Y(k) \to \Pi_{Y/k}(k)$ is fully faithful by hypothesis.

Let us check now that $X(k) \to \Pi_{X/k}(k)$ is injective on isomorphism classes. Suppose that we have an isomorphism $\alpha : \pi_X(p) \simeq \pi_X(q)$ for some $p, q \in X(k)$. As before, if we can find a finite Galois extension

L and an isomorphism $\varphi : p_L \simeq q_L$ such that $\pi_{X_L}(\varphi) = \alpha_L$, then φ descends to an isomorphism $p \simeq q$. In fact, if $\sigma \in \mathrm{Gal}(L/k)$,

$$\pi_{X_L}(\sigma^* \varphi) = \sigma^* \alpha_L = \alpha_L = \pi_{X_L}(\varphi),$$

and this implies that $\pi_{X_L}(\sigma^*\varphi \circ \varphi^{-1}) = \mathrm{id}_{\pi_{X_L}(p_L)}$ and hence $\sigma^*\varphi \circ \varphi^{-1} = \mathrm{id}_{p_L}$ because we already know that π_{X_L} is fully faithful at every point.

Hence, up to a finite Galois extension we may suppose that there exists $p' \in Y(k)$ with $f(p') = p$. Since

$$\Pi_f(\pi_Y(p')) = \pi_X(f(p')) = \pi_X(p) \simeq \pi_X(q)$$

and thanks to Proposition A.3.2, there exists a point $q' \in Y(k)$ such that $\pi_Y(q') \simeq \pi_Y(p')$ and $f(q') \simeq q$. Now since $Y(k) \to \Pi_{Y/k}(k)$ is fully faithful by hypothesis and $\pi_Y(p') \simeq \pi_Y(q')$, we get an isomorphism $p' \simeq q'$ which induces an isomorphism $p \simeq q$ as desired.

(ii) This is a direct application of point (i) and Lemma 1.1.5.(iii), together with the observation that every section $\mathrm{Spec}\, k \to \Pi_{X/k}$ lifts to a section of $\Pi_{Y/k}$ up to a finite, separable field extension: in fact, $\mathrm{Spec}\, k \times_{\Pi_{X/k}} \Pi_{Y/k}$ is a finite étale scheme. To check that $\mathrm{Spec}\, k \times_{\Pi_{X/k}} \Pi_{Y/k}$ is finite étale, observe that up to an extension k'/k we have

$$\mathrm{Spec}\, k' \times_{\Pi_{X/k}} \Pi_{Y/k} \simeq \mathrm{Spec}\, k' \times_X Y$$

for some point $\mathrm{Spec}\, k' \to X$, since $\Pi_{X/k}$ is a gerbe and hence all points are fpqc locally isomorphic.

□

1.2 Anabelian DM stacks

Now that we have established what happens along finite, étale covers, we want to understand what the section conjecture for DM stacks should look like. Clearly, one can just directly translate Grothendieck's section conjecture to DM stacks. Here we hope to show that the right thing to conjecture in general is slightly stronger (but equivalent in the case of hyperbolic curves).

Proposition 1.2.1. *Let X be a proper, smooth, geometrically connected Deligne-Mumford stack over k. The following are equivalent:*

1. *for every finitely generated extension k'/k and for every finite étale connected cover $Y \to X_{k'}$,*
$$Y(k') \to \text{Hom-ext}_{G_{k'}}(G_{k'}, \pi_1(Y))$$
is bijective (resp. injective) on isomorphism classes,

2. *the natural map*
$$X(k') \to \Pi_{X/k}(k')$$
is an equivalence of categories (resp. fully faithful) for every finitely generated extension k'/k.

Proof. Suppose that $X(k') \to \Pi_{X/k}(k')$ is an equivalence (resp. fully faithful). Then by Proposition A.2.11 $X_{k'}(k') \to \Pi_{X_{k'}/k'}(k')$ is an equivalence (resp. fully faithful), too, and hence $Y(k') \to \text{Hom-ext}_{G_{k'}}(G_{k'}, \pi_1(Y))$ is bijective (resp. injective) thanks to the going up theorem Proposition 1.1.2.

Suppose now that (1) holds, let k'/k be a finitely generated extension and $x \in X(k')$ a point and $\pi(x) \in \Pi_{X/k}(k')$. Since by hypothesis $X(k') \to \Pi_{X/k}(k')$ is bijective (resp. injective) on isomorphism classes, then we only have to show that
$$\underline{\text{Aut}}_X(x) \to \underline{\text{Aut}}_{\Pi_{X/k}}(\pi(x))$$
induces a bijection on k'-rational points. Thanks to Proposition A.2.11, we may suppose $k' = k$.

Suppose that $\varphi \in \underline{\text{Aut}}_{\Pi_{X/k}}(\pi(x))(k)$ is not in the image. Then, since $\Pi_{X/k} = B\underline{\text{Aut}}_{\Pi_{X/k}}(\pi(x))$ is profinite, there exists a finite index subgroup $H \subseteq \underline{\text{Aut}}_{\Pi_{X/k}}(\pi(x))$ such that $\varphi \notin H$. Consider the fiber product

$$\begin{tikzcd}
Y \arrow[rr] \arrow[d] & & BH \arrow[d] \\
X \arrow[r] & B\underline{\text{Aut}}_{\Pi_{X/k}}(\pi(p)) = \Pi_{X/k} &
\end{tikzcd}$$

where BH identifies naturally with $\Pi_{Y/k}$. In fact, the universal property of $\Pi_{Y/k}$ gives us a natural map $\Pi_{Y/k} \to BH$, and thanks to Proposition A.3.2 $\Pi_{Y/k}, BH$ are both subgerbes of $\Pi_{X/k}$ with the same finite index, hence they coincide. Then id, φ define two non isomorphic rational points $q, q' \in Y(k)$ over $p \in X(k)$ with the same image in $BH(k) = \Pi_{Y/k}(k)$, but this is absurd since by hypothesis $Y(k') \to \text{Hom-ext}_{G_{k'}}(G_{k'}, \pi_1(Y))$ is injective.

We want now to prove that $\underline{\text{Aut}}_X(x) \to \underline{\text{Aut}}_{\Pi_{X/k}}(\pi(x))$ is injective. Since $\underline{\text{Aut}}_X(x)$ is finite étale, up to enlarging the base field we may suppose that $\underline{\mathrm{Aut}}_X(x)$ is discrete, and we can consider a finite index subgroup $H \subseteq \underline{\mathrm{Aut}}_{\Pi_{X/k}}(\pi(x))$ such that

$$H \cap \mathrm{im}\left(\underline{\mathrm{Aut}}_X(x) \to \underline{\mathrm{Aut}}_{\Pi_{X/k}}(\pi(x))\right) = \{\mathrm{id}\}.$$

Pass to the fiber product $Y = X \times_{\Pi_{X/k}} BH$ as above, $x \in X(k)$ and the distinguished point $\operatorname{Spec} k \to BH$ define a rational point $y \in Y(k)$. We have that $\underline{\mathrm{Aut}}_Y(y) \subseteq \underline{\mathrm{Aut}}_X(x)$ is the kernel of $\pi_X : \underline{\mathrm{Aut}}_X(x) \to \underline{\mathrm{Aut}}_{\Pi_{X/k}}(\pi_X(x))$: in fact,

$$\underline{\mathrm{Aut}}_Y(y) = \pi_X^{-1}(H) = \pi_X^{-1}(\mathrm{id}) \subseteq \underline{\mathrm{Aut}}_X(x).$$

Hence, we want to prove that $\underline{\mathrm{Aut}}_Y(y)$ is trivial.

Now, since

$$\pi_Y : Y(k') \to \Pi_{Y/k}(k')$$

is by hypothesis injective on isomorphism classes for every finitely generated extension k'/k and $\underline{\mathrm{Aut}}_Y(y) \to \underline{\mathrm{Aut}}_{\Pi_{Y/k}}(\pi_Y(y))$ factorizes through the identity, we get that

$$B\underline{\mathrm{Aut}}_Y(y)(k') \subseteq \pi_Y^{-1}(\pi_Y(y))(k') = \{y\}$$

has only one isomorphism class for every finitely generated k'/k, i.e. the group scheme $\underline{\mathrm{Aut}}_Y(y)$ is special. But an étale special group is trivial, as desired. □

We define now *anabelian DM stacks* as those DM stacks satisfying the equivalent conditions of Proposition 1.2.1.

Definition 1.2.2. Let X be a smooth, proper, geometrically connected Deligne-Mumford stack. We say that X is *anabelian* (resp. *fundamentally fully faithful, or fff*) if the natural morphism

$$X(k') \to \Pi_{X/k}(k')$$

is an equivalence of categories (resp. fully faithful) for every finitely generated extension k'/k.

Remark 1.2.3. There are some remarks to be made about the definition of anabelian DM stack.

- As we will see later, even if this definition seems deeply arithmetic in nature, it is actually purely geometric: if $k \subseteq \mathbb{C}$, the anabelianity of X depends only on $X_{\mathbb{C}}$, see Remark 1.4.3.

- Extending the definition to Deligne-Mumford stacks seems natural for at least two reasons. One is that moduli stacks of curves are expected to be anabelian, the second is that hyperbolic orbicurves are anabelian if and only if hyperbolic curves are anabelian, see Theorem 1.5.3. We address the question "why not Artin stacks?" in section 1.3.

- Classical conjectures and theorems in anabelian geometry are stated in terms of isomorphisms classes, rather than equivalence of categories. However, both the points of a Deligne-Mumford stack and the étale fundamental gerbe have a natural structure of a category whose morphisms are invertible rather than that of a set, hence asking an equivalence of categories seems more natural, particularly in view of Proposition 1.2.1.

In the following, we show what it means for a scheme to be anabelian in the classical terms of the section conjecture and of centralizers of sections, see [Sti13, §3.3].

Lemma 1.2.4. *Let X be a smooth, proper, geometrically connected scheme. Then X is anabelian (resp. fff) if and only if*

- *$X_{k'}$ satisfies the section conjecture (resp. the injectivity part of the section conjecture) for every finitely generated extension k'/k, and*

- *for every $x \in X(k')$, the associated section in $\mathrm{Hom\text{-}ext}_{G_{k'}}(G_{k'}, \pi_1(X))$ has trivial centralizer.*

Proof. As we have shown in Remark 1.1.1, the automorphism groups of the points of the fundamental gerbe correspond to centralizers of sections of the étale fundamental group, hence if X is a scheme asking an equivalence of categories corresponds to asking a bijection on isomorphism classes together with the triviality of centralizers. □

Corollary 1.2.5. *Let k be finitely generated over $\mathbb{Q}$.*

- *Smooth proper curves over k are fundamentally fully faithful if and only if they have positive genus.*

- *Hyperbolic curves over k are anabelian if and only if they satisfy the section conjecture over every finitely generated extension of the base field.*

Proof. For smooth, proper curves with Euler characteristc less than or equal to 0, centralizers of sections coming from rational points are trivial, thanks either to [Sti13, Proposition 36, Proposition 104] or to the full faithfulness part of Proposition 1.2.1. □

Proposition 1.2.6. *Let Y, X be smooth, proper, geometrically connected DM stacks, and $Y \to X$ a finite étale covering. Then Y is anabelian (resp. fff) if and only if X is anabelian (resp. fff).*

Proof. This is a straightforward application of the going up and down theorems Proposition 1.1.2, Proposition 1.1.6. □

1.3 Artin stacks vs Deligne-Mumford stacks

One may wonder: why DM stacks and not Artin stacks? The answer is based on one's taste. DM stacks seem more natural, since $\Pi_{X/k}$ is pro-étale and Proposition 1.2.1 fails for Artin stacks. For example, if G is a connected algebraic group, then condition (1) of Proposition 1.2.1 holds for BG if an only if G is special, while condition (2) if and only if G is trivial. Hence, it makes a difference if we choose condition (1) or (2) as definition of anabelianity for Artin stacks.

If we choose (1), we should for instance consider $B\,\mathrm{GL}_n$ as anabelian even if $B\,\mathrm{GL}_n \to \Pi_{B\,\mathrm{GL}_n} = \operatorname{Spec} k$ is not an equivalence of categories on rational points, and this seems not very pleasant. On the other hand, if we choose (2), the following proposition shows that we get back to DM stack.

Proposition 1.3.1. *Let X be a geometrically connected Artin stack. Suppose that*

$$X(k') \to \Pi_{X/k}(k')$$

is fully faithful for every finitely generated extension k'/k. Then X is a Deligne-Mumford stack.

Proof. Let $x : \operatorname{Spec}\Omega \to X$ be any geometric point, we want to show that $\underline{\mathrm{Aut}}_X(x)$ is finite étale. Since X is of finite type, we may suppose that x is defined over a finitely generated extension k'/k. Thanks to Proposition A.2.11, we may suppose $k' = k$, i.e. $x \in X(k)$ is a rational point.

Let $\pi(x) \in \Pi_{X/k}(k)$ be the image of x, we have an homomorphism of group schemes

$$\underline{\mathrm{Aut}}_X(x) \xrightarrow{\pi} \underline{\mathrm{Aut}}_{\Pi_{X/k}}(\pi(x)).$$

This homomorphism is injective: in fact, if the kernel $\ker(\pi) \subseteq \underline{\mathrm{Aut}}_X(x)$ is nontrivial, since it is of finite type up to enlarging the base field we may suppose that there exists a rational point $\varphi \in \ker(\pi)(k)$ different from the identity. But $\underline{\mathrm{Aut}}_X(x) \to \underline{\mathrm{Aut}}_{\Pi_{X/k}}(\pi(x))$ is injective on rational points by hypothesis, hence $\ker(\pi)$ is trivial and

$$\underline{\mathrm{Aut}}_X(x) \subseteq \underline{\mathrm{Aut}}_{\Pi_{X/k}}(\pi(x))$$

is a subgroup scheme. Now, $\underline{\mathrm{Aut}}_X(x)$ is of finite type and $\underline{\mathrm{Aut}}_{\Pi_{X/k}}(\pi(x))$ is pro-étale, hence $\underline{\mathrm{Aut}}_{\Pi_{X/k}}(\pi(x))$ is finite étale, as desired. □

1.4 Étale covers by algebraic spaces

It turns out that anabelian DM stacks (actually, fff is enough) must have a non obvious topological feature, i.e. they have a finite étale cover by an algebraic space.

Proposition 1.4.1. *Let X be a geometrically connected DM stack locally of finite type over k, and suppose that the natural morphism*

$$X(k') \to \Pi_{X/k}(k')$$

is fully faithful for every finitely generated k'/k.

Then $X \to \Pi_{X/k}$ is representable by algebraic spaces and there exists a profinite étale cover $\widetilde{X} \to X$ with $\widetilde{X}$ an algebraic space.

If moreover X is of finite type and separated, there exists a finite gerbe Φ with a representable morphism $X \to \Phi$, and a finite étale cover $E \to X$ with E an algebraic space.

Proof. Let $\operatorname{Spec}\Omega \to X$ be any geometric point. Since X is locally of finite type, x is defined over a finitely generated extension k'/k, $x \in X(k')$. Let $\pi(x) \in \Pi_{X/k}(k')$ its image. Up to extending k', we may suppose that the finite étale group scheme $\underline{\mathrm{Aut}}_X(x)$ over k' is discrete. Since $X(k') \to \Pi_{X/k}(k')$ is fully faithful, the map

$$\underline{\mathrm{Aut}}_X(x)(k') \to \underline{\mathrm{Aut}}_{\Pi_{X/k}}(\pi(x))(k')$$

is injective. In particular, since $\underline{\mathrm{Aut}}_X(x)$ is discrete, the homomorphism of group schemes

$$\underline{\mathrm{Aut}}_X(x) \to \underline{\mathrm{Aut}}_{\Pi_{X/k}}(x)$$

is injective, hence $X \to \Pi_{X/k}$ is representable.

We want now to show that X has a profinite étale cover by an algebraic space. Since X is locally of finite type over k, there exists a finite extension k'/k and a point $x_0 \in X(k')$, let $\pi(x_0) \in \Pi_{X/k}(k')$ be its image. Then just take the fiber product

$$\begin{array}{ccc} \widetilde{X} & \longrightarrow & \operatorname{Spec} k' \\ \downarrow & & \downarrow \\ X & \longrightarrow & \Pi_{X/k} \end{array}$$

Since $X \to \Pi_{X/k}$ is representable, $\widetilde{X}$ is an algebraic space.

Suppose now that X is of finite type and separated. Let $\xi_1, \dots, \xi_n$ be the generic points of the irreducible components of X. Since $\underline{\mathrm{Aut}}_X(\xi_i)$ is finite for every i, thanks to the hypothesis there exists a finite gerbe Φ_1 with a morphism $X \to \Phi_1$ such that $B\underline{\mathrm{Aut}}_X(\xi_i) \to \Phi_1$ is representable for every i. Hence, there exists a dense open subset $U_1 \subseteq X$ such that $U_1 \to \Phi_1$ is representable: U_1 is open since it is the locus where the relative inertia $I_{X/\Phi_1} \to X$ is an isomorphism. Now take the generic points of the irreducible components of $X \setminus U_1$, and repeat the argument in order to find $X \to \Phi_2 \to \Phi_1$ and $U_2 \supseteq U_1$ with $U_2 \to \Phi_2$ representable. Since X is of finite type, the process ends.

In order to find E, since Φ is finite there exists a finite, separable extension k'/k and a section $\operatorname{Spec} k' \to \Phi$. Take $E = \operatorname{Spec} k' \times_\Phi X$. □

A priori, our definition of anabelian DM stack depends on the base field k. Thanks to the existence of finite étale covers by algebraic spaces we have proved in Proposition 1.4.1 we can show that anabelianity does not depend on the base field.

Proposition 1.4.2. *Let k'/k be finitely generated extensions of $\mathbb{Q}$, and X a smooth, proper, geometrically connected DM stack over k. Then X is anabelian (resp. fff) if and only if $X_{k'}$ is anabelian (resp. fff).*

Proof. We only do this for anabelianity, the argument for fff is strictly contained.

If X is anabelian, $X_{k'}$ is anabelian by definition since $\Pi_{X_{k'}/k'} = \Pi_{X/k} \times_k k'$ thanks to Proposition A.2.11.

On the other hand, suppose that $X_{k'}$ is anabelian. If k'/k is finite, up to a finite extension we may suppose that it is Galois, too. Then this is the content of Lemma 1.1.5.

Now let k'/k be any finitely generated extension. We want to reduce ourselves to the case in which X is an algebraic space. Observe that since we already know the case in which k'/k is finite, we may replace k with any finite extension. Thanks to Proposition 1.4.1 there exists a finite gerbe Φ' over k' and a fully faithful morphism $X_{k'} \to \Phi'$. But since Φ' is finite, up to finite extensions of both k' and k we may suppose that $X_{k'} \to \Phi'$ is the base change of some faithful morphism $X \to \Phi$, with Φ finite over k. This last fact essentially reduces to the fact that étale covers are defined over a finite extension of the base field, which in turn is equivalent to the invariance of the étale fundamental group along algebraically closed extensions. Up to another finite extension of k, we may suppose that we have a rational point $x \in X(k)$, and we may also replace $X \to \Phi$ with a Nori-reduced morphism, see [BV15, Lemma 5.12]. The fact that $X \to \Phi$ is Nori-reduced essentially means that the fiber product $X \times_\Phi \operatorname{Spec} k$ (where $\operatorname{Spec} k \to \Phi$ is the image of $x \in X(k)$) is geometrically connected. Thanks to Proposition 1.2.6, X is anabelian if and only if $X \times_\Phi \operatorname{Spec} k$ is anabelian, and $X \times_\Phi \operatorname{Spec} k$ is an algebraic space.

Hence, we have reduced ourselves to the case in which X is an algebraic space. Let L/k be a finitely generated extension, we want to show that $X(L) \to \Pi_{X/k}(L)$ is an equivalence. There exists a finitely generated extension L' of k' containing L, up to extensions we may suppose $L = k$ and $L' = k'$.

First, we must show that $\pi_X : X(k) \to \Pi_{X/k}(k)$ is fully faithful. Since X is an algebraic space, this amounts to showing injectivity on isomorphism classes together with the fact that for every $x \in X(k)$, $\underline{\mathrm{Aut}}_{\Pi_{X/k}}(\pi_X(x))(k)$ is trivial. But these are direct consequences of the analogous facts for $X(k') = X_{k'}(k') \to \Pi_{X/k}(k') = \Pi_{X_{k'}/k'}(k')$, which are true by hypothesis.

Finally, we have to show essential surjectivity of $\pi_X : X(k) \to \Pi_{X/k}(k)$. Choose $s \in \Pi_{X/k}(k)$, by hypothesis $s_{k'} = \pi_X(x')$ for some $x' \in X(k')$. Consider now the residue field $k(x')$ of $x' \in X$.

If $k(x') \neq k$, then up to enlarging k' we may find an automorphism σ of k'/k such that $\sigma^* x' \neq x' \in X(k')$. Now, x' and $\sigma^* x'$ both have image $s_{k'} \in \Pi_{X/k}(k')$ since $s_{k'}$ is defined over k and thus $\sigma^* s_{k'} = s_{k'}$, but $X(k') \to \Pi_{X/k}(k')$ is fully faithful by hypothesis, thus we have an absurd.

Hence $k(x') = k$, and $x' = x_{k'}$ for some rational point $x \in X(k)$. We want to show that $\pi_X(x) = s$ using the fact that

$$\pi_X(x)_{k'} = \pi_X(x_{k'}) = \pi_X(x') = s_{k'}.$$

This is a consequence of the fact that s is a $\underline{\pi}_1(X, x) = \underline{\mathrm{Aut}}_{\Pi_{X/k}}(x)$-torsor, $\pi_X(x)$ is the trivial $\underline{\pi}_1(X, x)$-torsor and $\underline{\pi}_1(X, x)$ is what we call a *rigid* group scheme, see section 2.4, Lemma 2.4.9, Lemma 2.4.14. In particular, we are using the fact that a torsor under a rigid group scheme which is trivialized by a finitely generated extension was already trivial on the base field, see Lemma 2.4.14. □

Remark 1.4.3. Thanks to Proposition 1.4.2, we can see anabelianity as a geometric property, rather than an arithmetic one, and this is coherent with Grothendieck's ideas. In fact, if X is a smooth, proper DM stack of over $\mathbb{C}$, since X is of finite type we have some subfield $k \subseteq \mathbb{C}$ finitely generated over $\mathbb{Q}$ and a DM stack X' over k with an isomorphism $X \simeq X'_{\mathbb{C}}$. We can then define X to be anabelian if X' is anabelian: thanks to Proposition 1.4.2, this definition does not depend on the choice of X'.

Hence, if k is finitely generated over $\mathbb{Q}$ and $k \subseteq \mathbb{C}$ is any embedding of k in $\mathbb{C}$, X is anabelian if and only if $X_{\mathbb{C}}$ is anabelian, i.e. anabelianity is a geometric notion. Clearly this is a tautology, we are not really able to describe in purely geometrical terms which DM stacks over $\mathbb{C}$ are anabelian: still we think it is worth observing that this arithmetic property depends only on the geometry of the variety.

1.5 Orbicurves

The first non trivial example of expected anabelian DM stacks are hyperbolic orbicurves.

Consider X a smooth, connected curve over k, $(D_i, r_i)_{i=1,\ldots,n}$ a finite family of reduced, effective Cartier divisors D_i together with a positive integer r_i. We can define the associated root stack $\mathfrak{X}$, and will call such a stack simply an *orbicurve*. It is a Deligne-Mumford stack of finite type $\mathfrak{X}$ with a morphism $f : \mathfrak{X} \to X$ such that f^*D_i has an r_i-th root. Moreover, $\mathfrak{X} \to X$ is universal among algebraic stacks $\mathfrak{Y}$ with morphisms $\mathfrak{Y} \to X$ with this property.

Essentially, we are putting an orbifold structure of ramification r_i on the divisor D_i: for example, if $D_i = p$ is a rational point, we are replacing p with a copy of $B\mu_{r_i}$. Outside of the divisors D_i, $\mathfrak{X} \to X$ is an isomorphism. For a precise definition see [AGV08, Appendix B.2]. In order to be clear, we will use Fraktur letters for orbicurves and normal ones for schemes.

If $\bar{X}$ is the smooth compactification of X and $D_\infty = \bar{X} \setminus X$, then the Euler-Poincaré characteristic of $\mathfrak{X}$ is

$$\chi(\mathfrak{X}) = 2 - 2g - \deg D_\infty - \sum_i \frac{r_i - 1}{r_i} \deg D_i.$$

If $\mathfrak{Y} \to \mathfrak{X}$ is a finite étale cover of degree d, the Riemann-Hurwitz formula implies that

$$\chi(\mathfrak{Y}) = d\chi(\mathfrak{X}).$$

The orbicurve $\mathfrak{X}$ is hyperbolic if $\chi(\mathfrak{X}) < 0$, elliptic if $\chi(\mathfrak{X}) = 0$ and parabolic if $\chi(\mathfrak{X}) > 0$, except one case: if $g = 0$, $\deg D_\infty = 2$ and there is no ramification, then we say that $\mathfrak{X}$ is parabolic even if it has characteristic 0. At the end of §1.6 we explain why we had to make this distinction. Observe that this is coherent with our intuition from complex geometry, since the universal covering of $\mathbb{P}^1_{\mathbb{C}}$ minus two points is the complex plane and not the unit disc: parabolic curves are exactly those covered by the complex plane and $\mathbb{P}^1_{\mathbb{C}}$, while elliptic and hyperbolic ones are covered by the unit disc.

The main fact that allows us to compare curves and orbicurves is that almost every orbicurve has a finite étale covering which is a curve. In fact we can reduce to the complex case, and in turn to a topological problem about surfaces using the Riemann existence theorem.

For surfaces, this problem has been solved by Bundgaard, Nielsen and Fox with a mistake later corrected by Chau (see [Nie48], [BN51], [Fox52] and [Cha83] for the original papers and [Nam87, Theorem 1.2.15] for a more comprehensive treatment). There are some parabolic orbisurfaces supported on the sphere which obviously can't be covered by ordinary surfaces because they have a finite universal covering which is not a surface, but that's all, in all other cases it is possible.

Proposition 1.5.1. *Let k be a field of characteristic 0, and $\mathfrak{X}/k$ an orbicurve defined over a smooth, connected curve X with smooth compactification $\bar{X}$ by ramification data $(D_i, r_i)_{i=1,\dots,n}$ with $1 < r_1 < \cdots < r_n$. Set $D_\infty = \bar{X} \setminus X$. Suppose that we are not in one of the following cases:*

- $D_\infty = \emptyset$, $g(\bar{X}) = 0$, $n = 1$, $\deg D_1 = 1$;
- $D_\infty = \emptyset$, $g(\bar{X}) = 0$, $n = 2$, $\deg D_1 = \deg D_2 = 1$.

Then there exists a finite extension k'/k and a smooth connected curve Y defined over k' with a finite étale cover $Y \to \mathfrak{X}_{k'}$.

Proof. Since everything is of finite type, from standard arguments we can obtain the general case once we know the theorem is true for k finitely generated over $\mathbb{Q}$. Suppose then that k is finitely generated over $\mathbb{Q}$ and fix an immersion of $k \subseteq \mathbb{C}$.

Consider the curve X on which $\mathfrak{X}$ is supported, the topological set $X^{\mathrm{an}}_{\mathbb{C}}$ is a compact oriented surface while $\mathfrak{X}^{\mathrm{an}}_{\mathbb{C}}$ is a compact orbifold supported on $X^{\mathrm{an}}_{\mathbb{C}}$. We can regard unramified covers $Y \to \mathfrak{X}^{\mathrm{an}}_{\mathbb{C}}$ with Y a compact surface as ramified covers $Y \to X^{\mathrm{an}}_{\mathbb{C}}$ such that all the points over $D_{i,\mathbb{C}}$ have ramification r_i. By [Nam87, Theorem 1.2.15], such a cover exists for almost all ramification data on oriented surfaces, the only exceptions being the sphere with exactly one critical value and the sphere with two critical values with different ramification.

Hence we have a topological unramified orbifold covering $Y \to \mathfrak{X}^{\mathrm{an}}_{\mathbb{C}}$. By applying the Riemann existence theorem to $Y \to X^{\mathrm{an}}_{\mathbb{C}}$, we can regard Y as a smooth, proper curve over $\mathbb{C}$ with a morphism $Y \to X_{\mathbb{C}}$. Consider the closed subset $R = \bigcup_i D_i \subseteq X$, its base change $R_{\mathbb{C}} \subseteq X_{\mathbb{C}}$ is the ramification locus of $Y \to X_{\mathbb{C}}$. By Lemma 1.5.2, there exists a finite extension $k \subseteq k' \subseteq \mathbb{C}$ and a morphism of curves $Y' \to X_{k'}$ whose base change to $\mathbb{C}$ is isomorphic to $Y \to X_{\mathbb{C}}$. By the universal property of the root stack $\mathfrak{X}_{k'}$ this gives a finite étale covering $Y' \to \mathfrak{X}_{k'}$, as desired. □

In the proof of Proposition 1.5.1 we have used the following lemma, which is widely known (when $k = \mathbb{Q}$ and $X = \mathbb{P}^1$ it is the easy implication of Belyi's theorem), but for which we could not find a reference.

Lemma 1.5.2. *Let $k \subseteq K$ be fields of characteristic 0, with $K = \bar{K}$. Let X/k and Y/K be smooth, projective curves with a branched covering*

$$f : Y \to X_K$$

such that all the ramification values are defined over a finite extension of k.

Then there exists a finite extension $k \subseteq k' \subseteq K$ and a branched covering $f' : Z \to X_{k'}$ whose base change to K is isomorphic to $Y \to X_K$.

Proof. Since everything is of finite type, it is enough to find such a covering $Z \to X_{k'}$ for $k' = \bar{k} \subseteq K$. By hypothesis, there exists an open subset $U \subseteq X$ such that $Y|_{U_K} \to U_K$ is unramified.

Since $\bar{k}$ and K are algebraically closed of characteristic 0, $\pi_1(\mathcal{U}_K) = \pi_1(\mathcal{U}_{\bar{k}})$ and hence there exists a finite étale morphism $g : V \to U_{\bar{k}}$ whose base change to K is $Y|_{U_K} \to U_K$. Let Z be a smooth completion of V, g extends to a finite morphism $Z \to X_{\bar{k}}$. It is now obvious that the base change of $Z \to X_{\bar{k}}$ is isomorphic to $Y \to X_K$. □

Now that we know that we can cover every hyperbolic orbicurve with an hyperbolic curve, we get the following.

Theorem 1.5.3. *Let k be finitely generated over $\mathbb{Q}$.*

- *A smooth, proper orbicurve X is fundamentally fully faithful if and only if $\chi(X) \leq 0$.*
- *Smooth, proper, hyperbolic orbicurves are anabelian if and only if smooth, proper, hyperbolic curves are anabelian.*

Proof. Thanks to Proposition 1.2.6, Proposition 1.4.2 and Proposition 1.5.1, we may reduce to one of the following cases: X is either a curve or a simply connected orbicurve. Both these cases are obvious. □

1.6 Open case

There is a version of the section conjecture for open curves. If X is a smooth geometrically connected curve with smooth completion $\bar{X}$, every "missing" rational point $x \in \bar{X} \setminus X(k)$ defines a so called packet of cuspidal sections $\mathcal{P}_x \subseteq \Pi_{X/k}(k)$, see for example [EH08]. The section conjecture for open curves says that if k is finitely generated over $\mathbb{Q}$ and X has negative Euler characteristic, every section $s \in \Pi_{X/k}(k)$ comes either from a rational point of x or from a packet of cuspidal sections.

As showed by Niels Borne and Michel Emsalem in [BE14, §2.2.3], the section conjecture for orbicurves implies easily the section conjecture for open curves. If we put together their observation and Theorem 1.5.3, we obtain a new proof of the following classical result (see [Sti13, Proposition 103]).

Theorem 1.6.1. *The section conjecture for proper curves implies the section conjecture for open curves.* □

Let us show how the ideas of Borne and Emsalem fit nicely in our formalism, giving a clear picture of packets of tangential points and of the section conjecture for open curves. Let X be a smooth connected curve over a field k of characteristic 0 with smooth compactification $\bar{X}$, set $D = \bar{X} \setminus X$. Let X_n be the orbicurve supported over $\bar{X}$ with ramification of degree n along the divisor D, and

$$\widehat{X} = \varprojlim_n X_n$$

their projective limit: it is an fpqc stack with natural morphisms $X \hookrightarrow \widehat{X}$ and $\widehat{X} \twoheadrightarrow \bar{X}$.

Remark 1.6.2. The proalgebraic stack $\widehat{X}$ is the *infinite root stack* associated to the logarithmic structure given by D on $\bar{X}$, see [TV18].

Moreover, the natural morphism

$$\Pi_{\widehat{X}} \to \varprojlim_n \Pi_{X_n}$$

is an isomorphism: in fact, if Φ is a finite étale stack, thanks to [BV15, Proposition 3.8] we have equivalences

$$\mathrm{Hom}(\varprojlim_n \Pi_{X_n}, \Phi) \simeq \varinjlim_n (\Pi_{X_n}, \Phi) \simeq \varinjlim_n \mathrm{Hom}(X_n, \Phi) \simeq \mathrm{Hom}(\widehat{X}, \Phi).$$

Let $\eta_n : X_n \to \bar{X}$, $\eta : \widehat{X} \to \bar{X}$ be the natural morphisms. If $p \in \bar{X} \setminus X(k)$ is a rational point, the fiber $\eta_n^{-1}(p)$ over p is non canonically isomorphic to $B\mu_n$, hence $\eta_n^{-1}(p)(k) \simeq k^*/k^{*n}$ and this implies that $X_n \to \bar{X}$ is surjective on rational points. By taking a coherent sequence of points in $\eta_n^{-1}(p)$ for every n we get an isomorphism $\eta^{-1}(p) \simeq B\widehat{\mathbb{Z}}(1)$ and hence $\widehat{X} \to \bar{X}$ is surjective on rational points too. The *packet* of tangential points at p is

$$\eta^{-1}(p)(k) \simeq B\widehat{\mathbb{Z}}(1)(k) = \varprojlim_n k^*/k^{*n} = \widehat{k}^*.$$

Since we are in characteristic 0, Abhyankar's lemma implies that the natural map

$$\Pi_{X/k} \to \Pi_{\widehat{X}/k}$$

is an isomorphism. In fact, checking that $\Pi_{X/k} \to \Pi_{\widehat{X}/k}$ is an isomorphism is equivalent to checking that it induces an isomorphism of $\underline{\mathrm{Isom}}$ sheaves, and this in turn means asking an isomorphism of étale fundamental groups. For this, see [Bor09, Proposition 3.2.2].

A simple way of seeing this is observing that is enough to prove the isomorphism over an algebraically closed field, over which we can use the standard presentation of the fundamental group. In fact, if we remove a point from a curve, we are adding a generator with infinite order to the presentation, while if we replace it with $B\mu_n$ we are adding a generator of order n: it is then clear that for $n \to \infty$ we get the desired convergence.

Hence, the section conjecture for an hyperbolic open curve X can be reinterpreted by asking that, if $k/\mathbb{Q}$ is finitely generated,

$$\widehat{X}(k) \to \Pi_{\widehat{X}/k}(k) = \Pi_{X/k}(k)$$

is a bijection (or an equivalence of categories). If X is hyperbolic, $\chi(X) < 0$, and hence $\chi(X_n) < 0$ for n big enough. If we know that the section conjecture for proper curves is true, then it is true for proper orbicurves too thanks to Theorem 1.5.3, hence

$$X_n(k) \to \Pi_{X_n/k}(k)$$

is an equivalence of categories for n big enough. Passing to the limit, the same is true for $\widehat{X}$, and we get Theorem 1.6.1.

At this point, it is natural to see what happens for open orbicurves. If $\mathfrak{X}$ is an open orbicurve we can define $\mathfrak{X}_n$ and $\widehat{\mathfrak{X}}$ as above. If $\chi(\mathfrak{X}) < 0$ and $k/\mathbb{Q}$ is finitely generated, the section conjecture for $\mathfrak{X}$ says that

$$\widehat{\mathfrak{X}}(k) \to \Pi_{\widehat{\mathfrak{X}}/k}(k) = \Pi_{\mathfrak{X}/k}(k)$$

is an equivalence of categories. The same argument as above shows that this is equivalent to the classical section conjecture for proper curves.

Finally, let us look more closely at the injectivity part of the section conjecture. Recall that an open orbicurve $\mathfrak{X}$ is hyperbolic if $\chi(\mathfrak{X}) < 0$, elliptic if $\chi(\mathfrak{X}) = 0$ and parabolic if $\chi(\mathfrak{X}) > 0$, with one exception: if $\mathfrak{X} = X$ is a curve of genus 0 and $\chi(X) = 0$ (i.e. $\deg(\bar{X} \setminus X) = 2$), X is parabolic even if $\chi(X) = 0$. It is clear now why we had to make this distinction: with this definition, an open orbicurve $\mathfrak{X}$ is hyperbolic (resp. elliptic, parabolic) if and only if $\widehat{\mathfrak{X}}$ can be expressed as a projective limit of hyperbolic (resp. elliptic, parabolic) proper orbicurves.

In fact, $\chi(\mathfrak{X}_n) \geq \chi(\mathfrak{X})$ converges to $\chi(\mathfrak{X})$ from above, and if $\mathfrak{X}$ is not proper we have a strict inequality $\chi(\mathfrak{X}_n) > \chi(\mathfrak{X})$. Hence, if $\chi(\mathfrak{X}) = 0$ and $\mathfrak{X}$ is not proper, $\chi(\mathfrak{X}_n) > 0$ for every n. It is immediate to check that this happens only for $\mathfrak{X}$ a curve of genus 0 without a divisor of degree 2.

By a direct application of Theorem 1.5.3, we thus get that for an orbicurve $\mathfrak{X}$ the section map is fully faithful for every finitely generate extension k'/k if and only if $\mathfrak{X}$ is elliptic or hyperbolic.

1.7 From the section conjecture to the hom conjecture

If X is anabelian, we expect the functor

$$X(T) \to \Pi_{X/k}(T)$$

to be an equivalence for a much larger class than finitely generated extensions of k. At least, we should have smooth schemes: we actually show that normality together with a finiteness condition on local rings is enough.

Recall that a k-algebra is essentially of finite type if it is the localization of a k-algebra of finite type.

Definition 1.7.1. Let k be a field. A k-scheme T is *left* over k (short for *locally essentially of finite type*) if $\mathcal{O}_{T,p}$ is essentially of finite type over k for every $p \in T$.

Remark 1.7.2. This condition on local rings may seem strange at first glance, but it is really everything that we need: there is no need of conditions on open neighbourhoods. Observe that this definition is somewhat similar to Mochizuki's smooth pro-varieties [Moc99, Definition 16.4]. Imposing that the local rings are essentially of finite type ensures both the fact that residue fields are finitely generated over k and that local rings are noetherian. Being left is a quite general finiteness condition: it contains schemes locally of finite type, finitely generated extensions of k, curves with an arbitrary set of closed points removed. For example,

$$\varprojlim_n \mathbb{A}^1_k \setminus \{1, \ldots, n\} = \operatorname{Spec} k \left[x, \frac{1}{x+1}, \frac{1}{x+2}, \ldots\right]$$

is left.

Theorem 1.7.3. *Let X be a smooth, proper DM stack and T an integral, normal left scheme over k. If X is fff, then $X(T) \to \Pi_X(T)$ is fully faithful. If X is anabelian, then $X(T) \to \Pi_X(T)$ is an equivalence of categories.*

Proof. **Full faithfulness.** Let $t_1, t_2 : T \to X$ be two morphisms, $\pi(t_1), \pi(t_2)$ their images in $\Pi_{X/k}(T)$ and $(t_1, t_2) \in X \times X(T)$. Then $\underline{\mathrm{Isom}}_X(t_1, t_2)$ is proper, unramified and hence finite over T (because X is separated and DM, hence it has proper and unramified diagonal), while $\underline{\mathrm{Isom}}_{\Pi_{X/k}}(\pi(t_1), \pi(t_2))$ is pro-étale over T (because $\Pi_{X/k}$ is a pro-étale gerbe, hence it has pro-étale diagonal).

Since $\underline{\mathrm{Isom}}_X(t_1, t_2) \to T$ is finite, $\underline{\mathrm{Isom}}_{\Pi_{X/k}}(\pi(t_1), \pi(t_2)) \to T$ is pro-étale and T is normal, we have that

$$\underline{\mathrm{Isom}}_X(t_1, t_2)(T) = \underline{\mathrm{Isom}}_X(t_1, t_2)(k(T)),$$

$$\underline{\mathrm{Isom}}_{\Pi_{X/k}}(\pi(t_1), \pi(t_2))(T) = \underline{\mathrm{Isom}}_{\Pi_{X/k}}(\pi(t_1), \pi(t_2))(k(T)),$$

and hence

$$\underline{\mathrm{Isom}}_X(t_1, t_2)(T) \xrightarrow{\sim} \underline{\mathrm{Isom}}_{\Pi_{X/k}}(\pi(t_1), \pi(t_2))(T)$$

because by hypothesis

$$\underline{\mathrm{Isom}}_X(t_1, t_2)(k(T)) \xrightarrow{\sim} \underline{\mathrm{Isom}}_{\Pi_{X/k}}(\pi(t_1), \pi(t_2))(k(T)).$$

Essential surjectivity. Let T be an integral, normal left scheme over k with a morphism $T \to \Pi_{X/k}$. Thanks to Proposition 1.4.1, there exists a finite gerbe Φ and a representable morphism $X \to \Phi$. Hence we have an induced morphism $T \to \Pi_{X/k} \to \Phi$, and by hypothesis we have a generic section $\operatorname{Spec} k(T) \to X$ which induces a section $\operatorname{Spec} k(T) \to X' = X \times_\Phi T$. Since $X \to \Phi$ is representable, X' is an algebraic space, call $Z \subseteq X'$ the closure of $\operatorname{Spec} k(T) \to X'$. Finally, let $\widetilde{X}$ be $X \times_{\Pi_{X/k}} T$, we also have a generic section $\operatorname{Spec} k(T) \to \widetilde{X}$. The situation is illustrated in the following diagram.

$$\begin{tikzcd}
\widetilde{X} = X \times_{\Pi_{X/k}} T \arrow[rr, bend left] & Z \arrow[r, hook] \arrow[dr, "\pi"] & X' = X \times_{\Phi} T \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{Spec} k(T) \arrow[u, bend left] \arrow[ur] \arrow[rr] & & T \arrow[r] \arrow[ul, dotted, bend left, "\exists ?"] & \Phi
\end{tikzcd}$$

If we can show that $Z \to T$ is an isomorphism, we have a section $T \to X'$ which by composition gives us a section $T \to X$ generically isomorphic to the morphism $\operatorname{Spec} k(T) \to X$ we started with. As we have shown in the preceding part about full faithfulness, this implies that $T \to X$ lifts the initial morphism $T \to \Pi_{X/k}$.

Since $X' \to T$ is proper, $\pi : Z \to T$ is surjective (Z is closed and its image contains the generic point of T), we want to show that it is injective too.

Let $z \in Z$ be a point, with image $\pi(z) : \operatorname{Spec} k(z) \to T$ in T. By hypothesis, we have a unique lifting $\operatorname{Spec} k(z) \to \widetilde{X}$ of $\pi(z)$, call z' the composition $\operatorname{Spec} k(z) \to \widetilde{X} \to X'$. We will prove that $z = z'$ by induction on $\operatorname{ht}_Z(z)$. Observe that T, and hence Z, may happen to be not locally of finite type over k, still our hypothesis that T is left is enough to show that the height is finite. In fact, in order to compute the height, we may localize everything to $\pi(z) \in T$: $\mathcal{O}_{T,\pi(z)}$ is essentially of finite type and hence noetherian by hypothesis.

If $\operatorname{ht}_Z(z) = 0$, then z is the generic point of Z, i.e. the image of the section $\operatorname{Spec} k(T) \to Z$. But then $z = z'$ by definition of $\operatorname{Spec} k(T) \to X'$.

If $\operatorname{ht}_Z(z) > 0$, there exists a germ of a non constant curve on Z passing through z. More precisely, there exists a DVR R and a morphism $\operatorname{Spec} R \to Z$ such that the closed point maps to z and the open point maps to a point $z_0 \neq z$. In fact, up to an étale cover Z is a scheme near z, thus we may take as R the normalization of a dimension 1 integral quotient of $\mathcal{O}_{Z,z}$. Moreover, $\mathcal{O}_{Z,z}$ is the localization of a $\mathcal{O}_{T,\pi(z)}$ algebra of finite type, and hence R is essentially of finite type too. Since $\operatorname{ht}_Z(z_0) < \operatorname{ht}_Z(z)$, by induction hypothesis $z_0 = z_0'$, i.e. z_0 is the image of the unique lifting $\operatorname{Spec} k(R) \to \widetilde{X}$ of $\pi(z_0) : \operatorname{Spec} k(R) \to T$.

Thanks to the valuative criterion, we may lift $\operatorname{Spec} R \to T$ to a morphism $\operatorname{Spec} R \to \widetilde{X}$. Here we are using the valuative criterion of universal closedness [Stacks, Tag 0A3X]: in order to use it, we don't need finite type hypotheses, but just the fact that $\widetilde{X} \to T$ is universally closed and separated. This is true, since $\widetilde{X} = X \times_{\Pi_{X/k}} T \to X \times T$ is representable by integral morphisms of schemes (it is obtained by base change from the diagonal of $\Pi_{X/k}$), and hence both separated and universally closed, while $X \times T \to T$ is proper since it is the base change of $X \to \operatorname{Spec} k$. If one wants to avoid this general valuative criterion, we can also use the fact that $\widetilde{X}$ is a projective limit of algebraic spaces of finite type.

Hence, by composing with $\widetilde{X} \to X'$ we obtain another morphism $\operatorname{Spec} R \to X'$. We have thus two morphisms $\operatorname{Spec} R \to X'$ sending the open point to z_0, one of them sends the closed point to z and the other one to z', and their compositions $\operatorname{Spec} R \to T$ are equal by construction. But $X' \to T$ is separated, and hence $z = z'$.

Since every point $z \in Z$ is uniquely determined by its image $\pi(z) \in T$, we have that $Z \to T$ is injective too. Hence, we know that $Z \to T$ is a $1:1$ proper map. Since Z is an algebraic space quasi-finite over a scheme, Z is a scheme, too. Moreover, Z is integral by construction (it is the closure of $\operatorname{Spec} k(T) \to X'$), and by Zariski's main theorem [Stacks, Tag 05K0] $Z \to T$ is a $1:1$ birational finite morphism. Since T is normal, we get that $Z \to T$ is an isomorphism too.

Actually, we have cheated, since in order to apply Zariski's main theorem we need T to be quasi-compact and quasi-separated and this is not a consequence of our hypotheses, but this is easily fixed. Cover T by open affine schemes T_i, for each i the argument above works since T_i is quasi-compact and quasi-separated, hence we have a section $T_i \to X$ of $T_i \to T \to \Pi_{X/k}$. We already know the fact that $X(T_i \cap T_j) \to \Pi_{X/K}(T_i \cap T_j)$ is an equivalence, hence the gluing data on $T_i \cap T_k \rightrightarrows \Pi_{X/k}$ gives us gluing data on $T_i \cap T_j \rightrightarrows X$, and thus finally we get a global section $T \to X$. □

Corollary 1.7.4. *If hyperbolic curves satisfy the section conjecture, then they satisfy the hom conjecture.*

Proof. If hyperbolic curves satisfy the section conjecture, then they are anabelian thanks to Corollary 1.2.5. Hence, they satisfy the hom conjecture thanks to Theorem 1.7.3. □

Thanks to Corollary 1.7.4, we can also see the anabelian conjecture proved by Mochizuki as a particular case of the section conjecture, rather than a different one.

Corollary 1.7.5. *For hyperbolic curves, the section conjecture implies the dominant version of the anabelian conjecture (i.e. the restriction of Mochizuki's theorem [Moc99, Theorem A] to fields finitely generated over $\mathbb{Q}$).* □

Theorem 1.7.3 allows us to prove easily that the topological fundamental group of an anabelian DM stack has no abelian finite index subgroup. We know no other result of the form "if a variety shows anabelian behaviour, then its fundamental group is far from being abelian": conjectures and theorems are always in the other direction.

Theorem 1.7.6. *Let X be an anabelian DM stack such that $\pi_1(X_{\bar{k}})$ has a finite index abelian subgroup. Then* $\dim X = 0$.

Proof. Thanks to Proposition 1.4.1 and Proposition 1.2.6, up to a finite étale covering we may suppose that X is an algebraic space. Up to another finite étale covering and a finite extension of the base field, we may suppose that $\pi_1(X_{\bar{k}})$ is abelian and X has a rational point $x_0 \in X(k)$. Let $\mathrm{Sm_k}$ be the category of smooth varieties over k. Since X is anabelian, thanks to Theorem 1.7.3 X and $\Pi_{X/k}$ define two naturally equivalent functors $\mathrm{Sm_k^{op}} \to \mathrm{Set}$ (by taking equivalence classes of $\Pi_{X/k}(T)$ for every $T \in \mathrm{Sm_k}$). The fact that the fundamental group of $X_{\bar{k}}$ is abelian implies that the gerbe $\Pi_{X/k}$ is abelian and hence its functor is enriched in groups with identity $\pi(x_0) \in \Pi_{X/k}(x_0)$, thus the same is true for the functor defined by X and x_0.

Now take an étale cover $U \to X$ with U a scheme, and let $R = U \times_X U$. Then, since U and R are smooth varieties, $X(U)$ and $X(R)$ are groups with the structure inherited from $\Pi_{X/k}(U)$ and $\Pi_{X/k}(R)$, this allows us to construct the usual maps $m : X \times X \to X$, $i : X \to X$ giving the group structure to X. Hence, the functor of points of X is enriched in groups over the whole category of schemes over k and not just the smooth ones. This implies that X is not only an algebraic space but also a scheme: the rough idea is that there exists a nonempty open subset which is a scheme, and then we can move it around with the group structure. For an actual proof, see [Art69, Theorem 4.1].

Hence, X is actually a proper group scheme, i.e. an abelian variety. But it is well known that an abelian variety of positive dimension is not anabelian, see for instance MathOverflow 92927 where a proof is given for elliptic curves (the proof actually works without modifications for positive dimensional abelian varieties). □

1.8 Elementary anabelian DM stacks

Recall that a proper, geometrically connected variety X is elementary anabelian if there exists a chain of smooth, proper morphisms

$$X = X_0 \to X_1 \to \cdots \to X_n = \operatorname{Spec} k$$

with $X_i \to X_{i+1}$ either a finite étale morphism or a fibration whose fibers are geometrically connected hyperbolic curves. We want to extend this definition to *elementary anabelian DM stacks*.

Definition 1.8.1. Let $Y \to X$ be a smooth, proper morphism representable morphism of codimension 1 with geometrically connected fibers of algebraic stacks. Let $D_1, \ldots, D_n \subseteq Y$ be distinct, reduced effective Cartier divisors étale over X and $d_1, \ldots, d_n$ positive integers. Write $\mathbf{D} = (D_1, \ldots, D_n)$, $\mathbf{r} = (r_1, \ldots, r_n)$. As described in [AGV08, Appendix B.2], we can construct the root stack

$$\sqrt[\mathbf{r}]{\mathbf{D}/Y}$$

We call a morphism of the form $\sqrt[\mathbf{r}]{\mathbf{D}/Y} \to X$ a *family of orbicurves*.

Let $\sqrt[\mathbf{r}]{\mathbf{D}/Y} \to X$ be a family of orbicurves, and suppose that X is connected. Let g be the genus of the fibers of $Y \to X$, and d_i be the degree of $D_i \to X$. Then the fibers of the family are orbicurves of rational Euler characteristic

$$2 - 2g - \sum_i \frac{r_i - 1}{r_i} d_i.$$

The fibers of the family are resp. parabolic, elliptic or hyperbolic if the Euler characteristic is resp. positive, zero or negative.

Definition 1.8.2. Elementary anabelian DM stacks over k are DM stacks defined by recursion in the following way.

1. $\operatorname{Spec} k$ is elementary anabelian.

2. If $Y \to X$ is a family of hyperbolic orbicurves and X is elementary anabelian, then Y is elementary anabelian.

3. If $Y \to X$ is finite, representable and étale, then X is elementary anabelian if and only if Y is elementary anabelian.

4. If k'/k is a finitely generated extension, then X is elementary anabelian over k if and only if $X_{k'}$ is elementary anabelian over k'.

Remark 1.8.3. Despite the name, it is obviously not known that elementary anabelian DM stacks are anabelian (with respect to our definition): this is equivalent to the section conjecture for hyperbolic curves, see Theorem 1.8.10.

In the analytic context, hyperbolic orbicurves are $K(G, 1)$ spaces (they have a covering by an hyperbolic curve, see Proposition 1.5.1), and these are $K(G, 1)$). By using the long exact sequence of a fibration, it is then immediate to check that elementary anabelian DM stacks are $K(G, 1)$ spaces, too. We want to show that this is true also for étale homotopy in the sense of Artin and Mazur, i.e. that the higher étale homotopy groups of elementary anabelian DM stacks are trivial. By [AM69, Theorem 6.7], it is enough to check that the topological fundamental group of elementary anabelian DM stacks is *good* in the sense of Serre, see [Ser65, §I.2.6].

Recall that a discrete group G is good if the natural homomorphism

$$\mathrm{H}^q(\widehat{G}, M) \to \mathrm{H}^q(G, M)$$

is an isomorphism for every finite G-module M, where $\widehat{G}$ is the profinite completion of G. We recall some facts about good groups.

Facts 1.8.4. [Ser65, §I.2.6 Exercises 1, 2]

1. Finite groups and finitely generated free groups are good.
2. If we have an exact sequence

   $$1 \to N \to E \to G \to 1$$

   with G good and N finitely generated, then

   $$1 \to \widehat{N} \to \widehat{E} \to \widehat{G} \to 1$$

   is exact.
3. In the situation of the preceding point, if we assume that N is good and $\mathrm{H}^q(N, M)$ is finite for every finite E-module M, then E is good too.
4. If M is finite and N is either finite or finitely generated and free, then $H^q(N, M)$ is finite. If N is obtained by successive extension starting from finite groups and finitely generated free groups, by taking the long exact sequence in cohomology we see that $\mathrm{H}^q(N, M)$ is still finite. Hence, thanks to the preceding point, all groups obtained by successive extensions starting from finite groups and finitely generated free groups are good.

Remark 1.8.5. In the following, we will need the long exact sequence of étale homotopy groups of a fibration. The standard reference for this is Friedlander's paper [Fri73, Corollary 4.8], but unfortunately it covers only fibrations of schemes, not DM stacks. Since this is not the place to generalize Friedlander's theorem, we use Facts 1.8.4.2 as a workaround: over $\mathbb{C}$ we can pass to the associated topological orbifold, take exact sequences in topology and then pass to profinite completions using Facts 1.8.4.2, since our DM stacks have no higher homotopy groups.

Lemma 1.8.6. *Fix an embedding of k in $\mathbb{C}$. If X is an elementary anabelian DM stack over k, then $X^{\mathrm{an}}_{\mathbb{C}}$ is of type $K(G,1)$ and its topological fundamental group is good in the sense of Serre.*

Proof. Hyperbolic curves are $K(G,1)$, and their topological fundamental group is obtained by successive extensions from free groups, hence it is good. Thanks to Proposition 1.5.1, every hyperbolic orbicurve has a finite étale cover which is a curve, hence we get the result for orbicurves too. We conclude by induction on dimension by taking the long exact sequence of a fibration along families of hyperbolic orbicurves. □

Corollary 1.8.7. *The étale homotopy type of an elementary anabelian DM stack is of type $K(G,1)$.*

Proof. Just apply [AM69, Theorem 6.7] and Lemma 1.8.6. □

Lemma 1.8.8. *If X is an elementary anabelian DM stack and $f : Y \to X$ is a finite, étale gerbe, then Y is an elementary anabelian DM stack.*

Proof. Fix an embedding of k in $\mathbb{C}$, since the definition of elementary anabelian DM stack is invariant under base field extension we may suppose $k = \mathbb{C}$. In fact, we will obtain a chain of elementary operations as in the definition of elementary anabelian DM stacks ending in $Y_{\mathbb{C}}$. Since everything is of finite type over k, these operations will then be defined over a finitely generated extension k' of k, and hence we will know that $Y_{k'}$ is elementary anabelian. But then Y is elementary anabelian too, by definition.

Consider a geometric point $y \in Y(\mathbb{C})$ and its image $x \in X(\mathbb{C})$. The fiber Y_x is a finite étale gerbe of the form BG for some finite group G. Passing to the associated topological orbifolds, we may consider the topological homotopy exact sequence

$$1 \to G \to \pi_1^{\mathrm{top}}(Y) \to \pi_1^{\mathrm{top}}(X) \to 1,$$

where $\pi_2^{\mathrm{top}}(X)$ is 0 by Lemma 1.8.6. Since G is finite and $\pi_1^{\mathrm{top}}(X)$ is good, we can pass to profinite completions

$$1 \to G \to \widehat{\pi_1^{\mathrm{top}}(Y)} = \pi_1(Y) \to \widehat{\pi_1^{\mathrm{top}}(X)} = \pi_1(X) \to 1.$$

Since G is finite, there exists a connected, finite étale cover $Y' \to Y$ such that $\pi_1(Y') \cap G = \{1\} \subseteq \pi_1(Y)$. Consider now the composition $Y' \to Y \to X$: a priori, it is proper étale, but since $\pi_1(Y') \to \pi_1(X)$ is injective then we conclude that it is representable too. Hence, we have two finite etale covers $Y' \to Y$ and $Y' \to X$: since X is and elementary anabelian DM stack, Y' and Y are elementary anabelian DM stacks too. □

Corollary 1.8.9. *If X is an elementary anabelian DM stack and $f : Y \to X$ is a proper, étale morphism, then Y is an elementary anabelian DM stack.*

Proof. We work directly on $\mathbb{C}$ as we have done in Lemma 1.8.8. In order to reduce to Lemma 1.8.8, consider the Stein factorization

$$Y \to \operatorname{Spec} f_* \mathcal{O}_Y \to X.$$

We want to show that $\operatorname{Spec} f_* \mathcal{O}_Y \to X$ is finite étale and $Y \to \operatorname{Spec} f_* \mathcal{O}_Y$ is a finite étale gerbe.

Up to taking a smooth covering of X (f_* commutes with flat base change), we may suppose that X is a scheme of finite type over $\mathbb{C}$. Since f is proper and X is locally of finite type, pushforward of coherent sheaves is coherent, see [Fal03], and hence $\operatorname{Spec} f_* \mathcal{O}_Y \to X$ is a finite morphism. Moreover, by hypothesis now the automorphism groups of geometric points of Y are finite étale, hence Y is a DM stack. Let $Y \to M$ be the coarse moduli space of Y, we have a natural morphism $M \to \operatorname{Spec} f_* \mathcal{O}_Y$ since X, and thus $\operatorname{Spec} f_* \mathcal{O}_Y$, is a scheme. On the other hand, $M \to X$ is proper and quasi-finite, hence affine, and this gives us a natural morphism in the other direction $\operatorname{Spec} f_* \mathcal{O}_Y \to M$. These are easily checked to be inverses. In particular, we get that $Y \to \operatorname{Spec} f_* O_Y = M$ is an homeomorphism on points.

Now take a surjective étale cover $U \to Y$ with U a scheme, the composition $U \to X$ is étale. By looking at the composition

$$U \to \operatorname{Spec} f_* \mathcal{O}_Y = M \to X,$$

since $Y \to \operatorname{Spec} f_* \mathcal{O}_Y$ is surjective we get that $\operatorname{Spec} f_* \mathcal{O}_Y \to X$ is étale.

Finally, we have to show that since Y is étale over its coarse moduli space M, then $Y \to M$ is a gerbe. Hence, take a scheme S with a morphism $S \to M$ and two sections $S \rightrightarrows Y$. We have a diagram

$$\begin{tikzcd} & Y \arrow[d, "\Delta"] \\ S \arrow[r] \arrow[ur, dotted] & Y \times_M Y \end{tikzcd}$$

and we want to find the dotted arrow, étale locally on S. But since $Y \to M$ is an étale coarse moduli space, $Y \to Y \times_M Y$ is a surjective étale morphism, hence we can find sections étale locally as desired. □

Theorem 1.8.10. *Elementary anabelian DM stacks are fundamentally fully faithful. If proper, hyperbolic curves satisfy the section conjecture, elementary anabelian DM stacks are anabelian too.*

Proof. We do this by induction checking that full faithfulness and anabelianity are preserved along the elementary operations that define elementary anabelian DM stacks.

Obviously, $\operatorname{Spec} k$ is anabelian since $\Pi_{\operatorname{Spec} k/k} = \operatorname{Spec} k$. If $Y \to X$ is finite étale, then by Proposition 1.2.6 Y is anabelian (resp. fff) if and only if X is anabelian (resp. fff). Both properties are also preserved along finitely generated extensions of the base field thanks to Proposition 1.4.2. We only have to check that full faithfulness and anabelianity are preserved along families of hyperbolic orbicurves.

Let $Y \to X$ be a family of hyperbolic orbicurves. Call $\Pi_{Y/X}$ the fiber product $X \times_{\Pi_{X/k}} \Pi_{Y/k}$, we have a natural 2-commutative diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[dr] & \Pi_{Y/X} \arrow[r] \arrow[d] & \Pi_{Y/k} \arrow[d] \\ & X \arrow[r] & \Pi_{X/k} \end{tikzcd}$$

Fix a point $x \in X$, and consider the fiber

$$\Pi_{Y/X,x} = \Pi_{Y/X} \times_X \operatorname{Spec} k(x) = \Pi_{Y/k} \times_{\Pi_{X/k}} \operatorname{Spec} k(x).$$

There is a natural map $Y_x \to \Pi_{Y/X,x}$.

Claim: $Y_x \to \Pi_{Y/X,x}$ is the étale fundamental gerbe of Y_x. Thanks to Proposition A.2.11, we may assume $k(x) = k = \bar{k}$ is algebraically closed. Fix a base point $y \in Y_x$. Then, since X has trivial topological second homotopy group, there is an exact sequence of étale fundamental groups

$$0 \to \pi_1^{\text{top}}(Y_x, y) \to \pi_1^{\text{top}}(Y, y) \to \pi_1(X, x)^{\text{top}} \to 0.$$

Since $\pi_1^{\text{top}}(X, x)$ is good in the sense of Serre thanks to Lemma 1.8.6, thanks to what we have said in Facts 1.8.4 about good groups we may pass to profinite completions, i.e. étale fundamental groups:

$$0 \to \pi_1(Y_x, y) \to \pi_1(Y, y) \to \pi_1(X, x) \to 0.$$

Since $\Pi_{Y/X,x} = \Pi_{Y/k} \times_{\Pi_{X/k}} \operatorname{Spec} k(x)$, there is also a short exact sequence

$$0 \to \mathrm{Aut}_{\Pi_{Y/X,x}}(y) \to \mathrm{Aut}_{\Pi_{Y/k}}(y) \to \mathrm{Aut}_{\Pi_{X/k}}(x) \to 0,$$

and there are natural identifications

$$\pi_1(Y_x, y) = \mathrm{Aut}_{\Pi_{Y_x/k}}(y),\ \pi_1(Y, y) = \mathrm{Aut}_{\Pi_{Y/k}}(y),\ \pi_1(X, x) = \mathrm{Aut}_{\Pi_{X/k}}(x).$$

These fit in a commutative diagram of short exact sequences, identifying $Y_x \to \Pi_{Y/X,x}$ with the étale fundamental gerbe $Y_x \to \Pi_{Y_x/k}$.

We can make another induction on dimension, hence $X(k') \to \Pi_{X/k}(k')$ is fully faithful and an equivalence if proper hyperbolic orbicurves satisfy the section conjecture, and the same holds for its base change $\Pi_{Y/X} \to \Pi_{Y/k}$. These holds for $Y \to \Pi_{Y/X}$ too, since we can work fiberwise on $Y_x \to \Pi_{Y_x/k}$: in fact, thanks to Theorem 1.5.3, since Y_x is an hyperbolic orbicurve we have that $Y_x(k') \to \Pi_{Y_x/k}(k')$ is fully faithful and an equivalence if proper, hyperbolic curves satisfy the section conjecture.

Finally, by composition these holds for

$$Y \to \Pi_{Y/X} \to \Pi_{Y/k}.$$

□

Chapter 2

Fce dimension

In this chapter we introduce a variant of essential dimension called fce dimension. Our main driving reason is the formulation of a dimensional version of the section conjecture: we do this in chapter 3. However, we think that the theory of fce dimension has its own interest independently of the section conjecture, so we dedicate to it a separate chapter.

The starting point is essential dimension, as defined by Buhler and Reichstein and later generalized by Merkurjev. Let us recall how it is defined.

Fix a base field k, and consider the category Fields_k of field extensions of k. Let $F : \mathrm{Fields}_k \to \mathrm{Set}$ be a functor: the two main examples to keep in mind are the functor of points of a scheme and the functor of torsors of a group scheme. We want to define the essential dimension of F: in a sentence, this is the minimum number of parameters needed to define an object of F.

More precisely, consider an extension L/k and an object $\alpha \in F(L)$. For every subextension $L/E/k$, we may ask ourselves if α is defined over E, i.e. if there exists an object $\beta \in F(E)$ such that $\beta_L = \alpha$. If one is lucky, it may exist the minimal field of definition: for example, if F is the functor of points of a scheme, then the minimal field of definition is just the residue field. The transcendence degree over k of the minimal field of definition measures the "complexity" of the object α.

In general, the minimal field of definition will not exists. Still, it exists the minimum of transcendence degrees of fields over which α is defined: this is the essential dimension of α. In formulas,

$$\operatorname{ed} \alpha = \min\{\operatorname{trdeg}_k E \mid \alpha \text{ is defined over } E\}.$$

Then the essential dimension $\operatorname{ed} F$ of F is the maximum of $\operatorname{ed} \alpha$ where α varies among all objects of $F(L)$ over all extensions L/k: every object of F

can be defined using $\operatorname{ed} F$ parameters, and $\operatorname{ed} F$ is minimal with respect to this property.

If F is the functor of points of a scheme X, by looking at the generic points of X we find that the essential dimension of F is just the dimension of X.

Suppose now that F is the functor of torsors of a group scheme G. In many cases, G-torsors classify interesting objects: for example, $O(n)$-torsors correspond to rank n quadratic forms, while S_n-torsors classify étale algebras of degree n. Hence, for example, $\operatorname{ed} O(n)$ is the minimum number of parameters needed in order to define a generic rank n quadratic form, and $\operatorname{ed} S_n$ does the same for étale algebras. Essential dimension gives a unified environment to study all of these problems. In the context of the section conjecture, $\underline{\pi}_1(X,x)$-torsors classify points of the space of sections ($\underline{\pi}_1(X,x)$ is the *étale fundamental group scheme*, see Appendix B for a brief introduction).

It turns out that essential dimension behaves badly for pro-étale group schemes: it is infinite very often, and hence it doesn't give much information. We show this in section 2.1, and then we propose a way to fix this problem by defining the fce dimension as a variation of essential dimension in sections 2.2 and 2.3. Since fce dimension coincides with essential dimension for algebraic group schemes and has a better behaviour in the proalgebraic case, one may think as fce dimension as a generalization of essential dimension, rather than as an alternative. Finally, in section 2.4 we define and study the class of *very rigid* group schemes, which are pro-étale group schemes with a very peculiar behaviour with respect to the fce dimension.

2.1 Essential dimension of pro-étale groups

Let us work in characteristic 0. In this section, we are going to prove the following.

Proposition 2.1.1. *Let G/k be a proétale group scheme. Suppose that there exists an extension k'/k and a prime p with a nontrivial morphism $G_{k'} \to \mathbb{Z}_p$. Then* $\operatorname{ed} G = \infty$.

Proposition 2.1.1 is a consequence of the following more general result.

Proposition 2.1.2. *Let G/k be a group scheme. Suppose that there exists an extension k'/k and a prime p with morphisms $\mathbb{Z}_p \to G_{k'}$ and $G_{k'} \to \mathbb{Z}_p$ whose composition $\mathbb{Z}_p \to \mathbb{Z}_p$ is the identity. Then* $\operatorname{ed} G = \infty$.

Proof of 2.1.1. Let G be pro-étale as in 2.1.1. Since $\operatorname{ed} G_{k'} \leq \operatorname{ed} G$, we may suppose $k' = k = \bar{k}$ and hence think of $G \to \mathbb{Z}_p$ simply as a continuous map of profinite topological groups. The image of $G \to \mathbb{Z}_p$ is a nontrivial closed subgroup, and since these are are all of the form $\mathbb{Z}_p = p^n\mathbb{Z}_p \subseteq \mathbb{Z}_p$ up to changing the morphism $G \to \mathbb{Z}_p$ we may suppose it is surjective. Let $g \in G$ be an element mapping to $1 \in \mathbb{Z}_p$. Recall that $G = \varprojlim_{i\in I} G_i$ is profinite, and write $g_i \in G_i$ for the image of g in G_i. Since G_i is finite, g_i has some finite order n_i and if $j \geq i$ then $n_i | n_j$. Hence we can define morphisms $\widehat{\mathbb{Z}} \to G_i$ lifting to a morphism $\widehat{\mathbb{Z}} \to G$.

The composition $\widehat{\mathbb{Z}} \to G \to \mathbb{Z}_p$ is by construction the natural projection $\widehat{\mathbb{Z}} \to \mathbb{Z}_p$, hence if we compose with the embedding $\mathbb{Z}_p \subseteq \widehat{\mathbb{Z}}$ we have that $\mathbb{Z}_p \to G \to \mathbb{Z}_p$ is the identity, and we can apply Proposition 2.1.2. □

Let us now prove Proposition 2.1.2. Since $\operatorname{ed} G_{k'} \leq \operatorname{ed} G$ and under the hypothesis of 2.1.2 we have $\operatorname{ed} \mathbb{Z}_p \leq \operatorname{ed} G_{k'}$, it is enough to prove that $\operatorname{ed} \mathbb{Z}_p = \operatorname{ed} \mathbb{Z}_p(1) = \infty$. We are actually going to work with $\mathbb{Z}_p(1)$ rather than with $\mathbb{Z}_p$, but since essential dimension can only decrease along extensions of the base field and $\mathbb{Z}_p$, $\mathbb{Z}_p(1)$ are isomorphic over an algebraically closed field, working with $\mathbb{Z}_p(1)$ is sufficient.

Rank n valuations In order to prove that $\operatorname{ed} \mathbb{Z}_p(1) = \infty$ we need the theory of rank n valuations, i.e. valuations in the group $\mathbb{Z}^n$ ordered lexicographically. This is developed properly in section 3.3: here we just say some facts.

If M is a normal variety over k and $V \subseteq M$ is a codimension one subvariety, the local ring of the generic point of V is a DVR, hence V defines a discrete valuation $k(M)^* \to \mathbb{Z}$. This generalizes to rank n valuations: if we have a sequence $V_0, \dots, V_n$ with $V_0 = M$ and V_{i+1} a codimension 1 subvariety in the normalization of V_i, then we can define a rank n valuation $v : k(M)^* \to \mathbb{Z}^n$ associated to this sequence.

Let $A \subseteq k(M)$ be the valuation ring of v: if M is proper, the valuative criterion of properness gives us a morphism $\operatorname{Spec} A \to M$. The image of the unique closed point of $\operatorname{Spec} A$ is called the *center* of v, and if the valuation is constructed as above it coincides with the image of the generic point of V_n along the composition $V_n \to V_{n-1} \to \cdots \to V_0 = M$.

Rank n valuations are tightly connected with the concept of algebraic independence. The simplest incarnation of this connection is the fact that a rational function f over a smooth, projective variety M is algebraic over the base field if and only if its divisor is 0, i.e. if $v(f) = 0$ for every discrete valuation trivial over k. Rank n valuations allow us to generalize this fact.

In particular, the following is for us the single most important fact about rank n valuations.

Proposition 3.5.3. *Given an extension L/k and n elements $x_1, \dots, x_n \in L^*$, they are algebraically independent over k if and only if there exists a rank n valuation v trivial on k such that* $\det v(x_1, \dots, x_n) \neq 0$.

Most of the time, we are going to use Proposition 3.5.3. Sometimes, however, it is useful to have some kind of control over the valuations we use.

For example, consider the rational function x in the function field $k(x,y)$ of $\mathbb{P}^2$. Since x is transcendental, Proposition 3.5.3 ensures the existence of a discrete valuation $v : k(x,y)^* \to \mathbb{Z}$ such that $v(x) \neq 0$. There are two obvious valuation that one immediately sees: the one associated with the line $\{x = 0\} \subseteq \mathbb{P}^2$, for which $v(x) = 1$, and the one associated with the line at infinity, for which $v(x) = -1$.

However, there are infinite other valuations nontrivial on x: for instance, we may blow up the origin of $\mathbb{P}^2$ and consider the valuation associated to the exceptional divisor. Since x has order 1 on the exceptional divisor E, it is possible that Proposition 3.5.3 gives us the valuation associated to E as output, which we don't see on $\mathbb{P}^2$: we want to avoid this, we want to "see" our valuation on our fixed variety.

In rank 1 this type of control is easy, since a rational function on a smooth, projective variety is algebraic over k if and only if it has 0 divisor. In higher rank we have the following lemma.

Lemma 3.5.5. *Let k be a field of characteristic* 0 *and M a smooth variety over k of dimension n. Let $k(M)/L/k$ be a subextension of transcendence degree $m \leq n$. Then there exist a transcendence basis $x_1, \dots, x_m \in L$ and a rank m valuation $v : k(M)^* \to \mathbb{Z}^m$ such that*

- $\det(v)(x_1, \dots, x_m) \neq 0$,
- *the center of v is the generic point of a codimension m subvariety of M.*

Recall now that the group of $\mathbb{Z}_p(1)$-torsors over a field k is

$$\mathrm{H}^1(\mathbb{Z}_p(1), k) = \varprojlim_n k^*/k^{*p^n},$$

we call this group $\wedge_p k^*$. We are going to show that over every field of characteristic 0 and for any integer n we can find a $\mathbb{Z}_p(1)$-torsor of essential dimension n: this implies immediately that $\operatorname{ed} \mathbb{Z}_p(1) = \operatorname{ed} \mathbb{Z}_p = \infty$.

One of the two implications of Proposition 3.5.3 works for elements of $\wedge_p L^*$, too: if we have n elements $x_1, \ldots, x_n \in \wedge_p L^*$ they define a $\mathbb{Z}_p(1)^n$-torsor, hence we can substitute the concept of algebraic independence with "essential dimension at least n". Observe moreover that if v is a rank n valuation on L then $(v(x_1)|\ldots|v(x_n))$ is a $n \times n$ matrix with coefficients in $\mathbb{Z}_p$, hence it makes sense to define $\det(v)(x_1, \ldots, x_n) \in \mathbb{Z}_p$.

Lemma 2.1.3. *Consider $x_1, \ldots, x_n \in \wedge_p L^*$ and suppose that for some rank n valuation $v : L^* \to \mathbb{Z}^n$ we have*

$$\det(v)(x_1, \ldots, x_n) \neq 0 \in \mathbb{Z}_p.$$

Then $\mathrm{ed}(x_1, \ldots, x_n) \geq n$.

Proof. The reduction modulo p^s of $\det(v)(x_1, \ldots, x_n)$ is nonzero for some s large enough. This implies that the image of $(x_1, \ldots, x_n)$ in $(L^*/L^{*p^s})^n = \mathrm{H}^1(\mu_{p^s}^n, L)$ has essential dimension n.

In fact, for any choice of $x_{1,s}, \ldots, x_{n,s} \in L^*$ such that $x_i \cong x_{i,s} \pmod{L^{p^s}}$ we have that $x_{1,s}, \ldots, x_{n,s}$ are algebraically independent thanks to Proposition 3.5.3. □

We can now prove the following lemma, which directly implies that $\mathrm{ed}_k \mathbb{Z}_p(1) = \infty$ and hence Proposition 2.1.2.

Lemma 2.1.4. *Let $\alpha_1, \ldots, \alpha_n \in \mathbb{Z}_p$ be linearly independent over $\mathbb{Q}$. Then*

$$\prod_{i=1}^{n} t_i^{\alpha_i} \in \wedge_p k(t_1, \ldots, t_n)^* = \mathrm{H}^1(\mathbb{Z}_p(1), k(t_1, \ldots, t_n))$$

has essential dimension n.

Proof. Suppose that $\mathrm{ed}\, t_1^{\alpha_1} \cdots t_n^{\alpha_n} < n$, this means that there exists a subfield $k' \subseteq k(t_1, \ldots, t_n)$ of transcendence degree $n-1$ such that $t_1^{\alpha_1} \cdots t_n^{\alpha_n}$ is in the image of $\wedge_p k'^* \to \wedge_p k(t_1, \ldots, t_n)^*$.

Identify $k(t_1, \ldots, t_n)$ with the function field of $\mathbb{P}^n$, and choose a transcendence basis $x_2, \ldots, x_n$ of k' as in Lemma 3.5.5. We have then a rank $n-1$ valuation v' whose center is the generic point of an irreducible curve $C \subseteq \mathbb{P}^n$ and such that $\det(v')(x_2, \ldots, x_n) \neq 0$.

There is at least one of the coordinate hyperplanes not containing C, say $H_1 = \{t_1 = 0\}$. Choose a point p in the normalization $\overline{C}$ of C mapping to a point of $C \cap H_1$. Then we may use p to extend v' to a rank n valuation $v : k(t_1, \ldots, t_n) \to \mathbb{Z}^n$ whose first $n-1$ coordinates are just v' (for details on how to construct this extension, see Lemma 3.5.2).

Write

$$v(t_1^{\alpha_1}\cdots t_n^{\alpha_n}) = \left(\sum_{i=1}^{n} r_{j,i}\alpha_i\right)_{j=1,\ldots,n}$$

for some $r_{j,i} \in \mathbb{Q}$. By construction, $r_{j,1} = 0$ for $j = 1, \ldots, n-1$ and $r_{n,1} \neq 0$ since $C \not\subseteq H_1$ but $p \in \overline{C}$ maps to a point of H_1. Hence, the determinant of

$$\left(v(t_1^{\alpha_1}\cdots t_n^{\alpha_n})|v(x_2)|\ldots|v(x_n)\right)$$

has the form

$$r_{n,1}\cdot \det(v')(x_2,\ldots,x_n)\cdot\alpha_1 + \sum_{j=2}^{n} s_j\cdot\alpha_j$$

for some $s_j \in \mathbb{Q}$, $j = 2, \ldots, n$. Since $\det(v')(x_2,\ldots,x_n) \neq 0$ is rational and $\alpha_1, \ldots, \alpha_n$ are linearly independent over $\mathbb{Q}$, this determinant is different from 0.

Thanks to Lemma 2.1.3, this implies that

$$\mathrm{ed}(t_1^{\alpha_1}\cdots t_n^{\alpha_n}, x_2, \ldots, x_n) = n.$$

On the other hand, both $t_1^{\alpha_1}\cdots t_n^{\alpha_n}$ and $x_2, \ldots, x_n$ are defined on k', which has transcendence degree $n-1$ over k, hence we have an absurd.

□

In order to fix the fact that essential dimension is infinite very often for pro-étale group schemes, we are going to define two new variants of essential dimension, i.e. finite type essential dimension and continuous essential dimension. With respect to classical essential dimension, finite type essential dimension introduces a modification in the definition at the level of the whole functor, while continuous essential dimension operates on single objects. Fce dimension is what we end up with if we merge these two variants in single definition.

2.2 Finite type essential dimension

As we have said above, finite type essential dimension is defined only at the level of functors. This means that for single objects we still use classical essential dimension, but we change the definition of the dimension of the functor.

Observe that if G is an affine group scheme of finite type over k and L/k is an extension, every G-torsor $T \to \operatorname{Spec} L$ is defined over a finitely generated extension of k, for example thanks to the existence of versal torsors (which are defined over a finitely generated extension). Hence, in order to compute the essential dimension of G, we may forget about the existence of torsors defined over extensions of k which are not finitely generated. This is true for points of algebraic stacks of finite type, too.

If G is not of finite type, extensions which are not finitely generated make a difference. In particular, étale fundamental group schemes are usually not of finite type, and the section conjecture has hope to be true only for finitely generated extensions of $\mathbb{Q}$. Hence we give the following definition.

Definition 2.2.1. Let $\mathrm{F} : \mathrm{Fields_k} \to \mathrm{Set}$ be a functor from the category of extensions of k to Set. The *finite type essential dimension* $\operatorname{fed}\mathrm{F}$ is the supremum of the essential dimensions $\operatorname{ed}(\alpha)$ where α varies among objects $\alpha \in \mathrm{F}(\mathrm{K})$ with K a finitely generated extension of k.

Clearly, finite type essential dimension makes sense also for functors defined only on the subcategory $\mathrm{FFields}_k \subseteq \mathrm{Fields}_k$ of finitely generated extensions of k. In fact, we can think of fed_k simply as the essential dimension of the restriction of F to $\mathrm{FFields}_k$.

Remark 2.2.2. Since the vast majority of functors for which essential dimension is studied are algebraic stacks of finite type we can think of finite type essential dimension as a generalization of essential dimension rather than as a variation of it.

Example 2.2.3. It is easy to come up with examples of functors for which essential dimension and finite type essential dimension are different. For example, define $F(L) = \{\bullet\}$ if $\operatorname{trdeg}_k L < \infty$, and $F(L) = \{\bullet, *\}$ if $\operatorname{trdeg}_k L = \infty$. Then $\operatorname{fed} F = 0$ and $\operatorname{ed} F = \infty$.

For a less trivial example, let $\underline{\pi}_1(X, x)$ be the étale fundamental group scheme of a smooth, proper hyperbolic curve, with k finitely generated over $\mathbb{Q}$. If Grothendieck's section conjecture is true, then $\operatorname{fed} \underline{\pi}_1(X, x) = 1$, but we have that $\operatorname{ed} \underline{\pi}_1(X, x) = \infty$ since $\underline{\pi}_1(X, x)$ clearly satisfies the hypothesis of Proposition 2.1.1.

With the notion of finite type essential dimension we have solved some problems, because for example over fields finitely generated over $\mathbb{Q}$ it is not so easy to embed $\mathbb{Z}_p(1)$ in a general pro-étale group scheme.

However, the torsors we have given in Lemma 2.1.4 are defined over fields finitely generated over the base field, hence we still have $\operatorname{fed} \mathbb{Z}_p(1) = \operatorname{ed} \mathbb{Z}_p(1) = \infty$, and this is not very pleasant: the essential dimension of μ_{p^n} is 1 for every n, hence we would like $\mathbb{Z}_p(1)$ to have dimension 1.

2.3 Continuous essential dimension

The second variant, continuous essential dimension, is much more subtle. It is not defined at the level of functors but at the level of single objects. We define it only for group schemes, but there are obvious generalizations for other functors (at least gerbes, but not only them). However, the applications we have in mind are not so broad, hence we will not seek generality for generality's sake.

Definition 2.3.1. Let G be a proalgebraic group (every affine group scheme is proalgebraic), and T a G-torsor defined over a field K/k.

The *continuous essential dimension* $\operatorname{ced}_k(T)$ is the supremum of the essential dimensions of $T \times^G H$, where $G \to H$ varies among all morphisms from G to a group of finite type H.

The continuous essential dimension $\operatorname{ced}(G)$ of G is the supremum of $\operatorname{ced}(T)$, where T varies among all G-torsors T over all field extensions K/k.

Remark 2.3.2. As for finite type essential dimension, if G is already an algebraic group scheme, then it is obvious that for every G-torsor T we have $\operatorname{ed}_k T = \operatorname{ced}_k T$, hence $\operatorname{ed}_k G = \operatorname{ced}_k G$. Again, this tells us that we can think of continuous essential dimension as a generalization of essential dimension rather than a variation.

If G is given as a projective limit of groups of finite type $G = \varprojlim_{i \in I} G_i$, since every morphism $G \to H$ to an algebraic group H splits as $G \to G_i \to H$ for some i, then for every G-torsor T we have

$$\operatorname{ced}(T) = \sup_i \operatorname{ed}(T \times^G G_i).$$

Since $\operatorname{ed}(T \times^G G_i) \leq \operatorname{ed}_k G_i$, this also tells us that

$$\operatorname{ced}_k G \leq \liminf_i \operatorname{ed}_k G_i,$$

where we have used the nonstandard, but obvious, notion of lim inf along the projective system I.

The reason why the definition of continuous essential dimension makes sense is the basic fact that if $\Gamma : \mathrm{F} \to \mathrm{G}$ is a functor and $p \in \mathrm{F}(\mathrm{K})$ is an object, $\operatorname{ed}(\Gamma(p)) \leq \operatorname{ed}(p)$. Hence, if T is a G-torsor with $G = \varprojlim_{i \in I} G_i$, the essential dimension of its algebraic approximations $T \times^G G_i$ increases with i and the continuous essential dimension of T is thus the limit of these essential dimensions $\operatorname{ed}_k T \times^G G_i$.

In dimension 0, essential dimension and continuous essential dimension coincide for pro-étale group schemes.

Proposition 2.3.3. *Let G be a pro-étale group scheme over k, and let $T \to \operatorname{Spec} L$ be a G-torsor where L/k is an extension of fields. Then* $\operatorname{ced} T = 0$ *if and only if* $\operatorname{ed} T = 0$*.*

Proof. Since $\operatorname{ced} T \leq \operatorname{ed} T$, one implication is obvious. Let us suppose now that $\operatorname{ced} T = 0$. Up to replacing k with $\bar{k}^L$, we may suppose that k is algebraically closed in L.

Write $G = \varprojlim_i G_i$ with G_i étale group schemes and $G \to G_i$ surjective. By hypothesis, $T_i = T \times^G G_i$ is defined over k, i.e. there exist G_i-torsors $Q_i \to \operatorname{Spec} k$ with isomorphisms $Q_{i,L} \simeq T_i$. We want to show that the Q_i form a projective system whose limit is a G-torsor $Q \to \operatorname{Spec} k$ such that $Q_L \simeq T$.

Let $j \geq i$ in the projective system, and define $Q_{j,i} = Q_j \times^{G_j} G_i$. We want to give G equivariant morphisms $Q_j \to Q_i$ for every $j \geq i$, and this is equivalent to giving G_i equivariant isomorphisms $Q_{j,i} \to Q_i$. Now, $\underline{\operatorname{Isom}}_{G_i}(Q_{j,i}, Q_i)$ is an étale scheme with an L rational point, because we have G equivariant morphisms

$$Q_{j,L} \simeq T_j \to T_i \simeq Q_{i,L}.$$

Since k is algebraically closed in L and $\underline{\operatorname{Isom}}_{G_i}(Q_{j,i}, Q_i)$ is étale, the isomorphism $Q_{j,L} \simeq Q_{i,L}$ given above is defined over k, i.e. it is the base change of an isomorphism $\varphi_{j,i} : Q_j \simeq Q_j$.

These morphisms respect the cocycle condition: if $j \geq i \geq h$, we have $\varphi_{h,i} \circ \varphi_{i,j} = \varphi_{h,j}$. In fact, this equality can be checked after base change to L, and over L it amounts to the commutativity of the following diagram:

$$\begin{tikzcd}
Q_{j,L} \arrow[r, "\sim"] \arrow[dr, "\sim"] & T_j \arrow[r] \arrow[d, equal] & T_i \arrow[r, "\sim"] \arrow[dr, equal] & Q_{i,L} \arrow[d, "\sim"] \\
& T_j \arrow[dr] & & T_i \arrow[d] \\
& & T_h \arrow[r, equal] \arrow[dr, "\sim"] & T_h \arrow[d, "\sim"] \\
& & & Q_{h,L}
\end{tikzcd}$$

which is obvious. Hence $Q = \varprojlim_i Q_i \to \operatorname{Spec} k$ is a G-torsor, and clearly $Q_L \simeq T$. □

Example 2.3.4. Consider $t \in \wedge_p^* k(t)$: thanks to Proposition 2.3.3, we have $\operatorname{ced}(t) = 1$, hence $\operatorname{ced} \mathbb{Z}_p(1) \geq 1$. On the other hand,

$$\operatorname{ced} \mathbb{Z}_p(1) \leq \liminf_n \operatorname{ed} \mu_n = 1,$$

and thus $\operatorname{ced} \mathbb{Z}_p(1) = 1$ as expected.

Finally, we can merge in an obvious way the two definitions above and define the *fce dimension* $\operatorname{fced}_k G$ of a group scheme G.

Definition 2.3.5. If G is an algebraic group scheme, the *fce dimension* $\operatorname{fced}_k G$ of G is the supremum of the continuous essential dimensions $\operatorname{ced}_k T$ where $T \to \operatorname{Spec} K$ is a G-torsor over a field K finitely generated over k.

2.4 Very rigid group schemes

We now introduce and study a particular class of pro-étale group schemes, which we call *very rigid* group schemes. Very rigid group schemes have a very peculiar behaviour with respect to finite type essential dimension (both continuous and not continuous), and as we will see they are not so uncommon in arithmetic. For example, the Mordell-Weil theorem implies that the Tate module of an abelian variety over a field finitely generated over $\mathbb{Q}$ is very rigid. Moreover, étale fundamental group schemes of hyperbolic curves over fields finitely generated over $\mathbb{Q}$ are very rigid, and thus the theory of very rigid group schemes will be useful in the next chapter, where we study a dimensional variant of the section conjecture.

Recall that two group schemes G, H over k are *strong inner forms* of each other if there exists a (G, H)-bitorsor $T \to \operatorname{Spec} k$, i.e. a scheme which is both a left G-torsor and a right H-torsor, and such that the actions commute. See [Bre90] for the theory of bitorsors. Equivalently, G and H are strong inner forms if $BG \simeq BH$, or if $H \simeq \operatorname{Aut}_G(T)$ for some left G-torsor $T \to \operatorname{Spec} k$.

Definition 2.4.1. Let X be a pro-étale scheme over a field k. We say that X is *rigid* if $X(k')$ is empty for every finitely generated extension k'/k. By a small abuse of notation, we call a pro-étale group scheme G rigid if $G \setminus \{\mathrm{id}\}$ is rigid. Moreover, we say that G is *very rigid* if every strong inner form of G over a finitely generated extension of k is rigid.

If the reader is familiar with the language of gerbes, it is clear that being very rigid is a property of pro-étale gerbes.

Remark 2.4.2. If X is pro-étale, all the residue fields of its points are algebraic over k, and hence we can check rigidity only on finite extensions k'/k.

Lemma 2.4.3. *Every rigid abelian group scheme is very rigid.*

Proof. If G is abelian, G is the only strong inner form of G. □

Lemma 2.4.4. *If G is (very) rigid, every subgroup G' is (very) rigid.*

Proof. If G is rigid, it is clear that G' is rigid.

Let k'/k be a finitely generated extensions, and H' a strong inner form of $G'_{k'}$. Then $H' = \underline{\mathrm{Aut}}_{G'}(T')$ for some G'-torsor $T' \to \operatorname{Spec} k'$: we want to show that H' is rigid. Let $T = T' \times^{G'} G$, then it is clear that H' is a subgroup of $H = \underline{\mathrm{Aut}}_G(T)$, which is rigid by hypothesis. □

Lemma 2.4.5. *If k'/k is finitely generated and G is a pro-étale group scheme over k, then G is (very) rigid if and only if $G_{k'}$ is (very) rigid.* □

2.4.1 Examples of very rigid group schemes

Recall that $\mathrm{H}^1(k, \mathbb{Z}_p(1)) = \varprojlim_n k^*/k^{*p^n}$, we write $\wedge_p k^*$ for this group.

Lemma 2.4.6. *The group schemes $\widehat{\mathbb{Z}}(1)$ and $\mathbb{Z}_p(1)$ over a finitely generated extension k of $\mathbb{Q}$ are very rigid. Let L/k be a finitely generated extension, then $k^* \to \wedge_p k^*$ and $\wedge_p k^* \to \wedge_p L^*$ are injective.*

Proof. Since $\widehat{\mathbb{Z}}(1) = \prod_p \mathbb{Z}_p(1)$, it is enough to do this for $\mathbb{Z}_p(1)$.

The fact that the group of units of a number field is finitely generated implies that k^* has finite torsion and 1 is the only p-divisible element for every prime number p. In fact, consider the number field $K = \overline{\mathbb{Q}}^k$ and its integers $\mathcal{O}_K = \overline{\mathbb{Z}}^k$: every torsion element λ of k^* is clearly an element of $\mathcal{O}_K$, and if η is p-divisible then $v(\lambda) = 0$ for every discrete valuation, thus $\eta \in \mathcal{O}_K$ too. The same holds for λ^{-1} and η^{-1}, thus they are units of K.

The kernel of $k^* \to \wedge_p k^*$ are p-divisible elements, thus this is injective. We have moreover $\mathbb{Z}_p(1)(L) = T_p L^* = \{1\}$ for every finitely generated extension L/k, thus $\mathbb{Z}_p(1)$ is rigid and hence very rigid since it is abelian. The fact that $\wedge_p k^* \to \wedge_p L^*$ is injective is a particular case of the following Corollary 2.4.16. □

Lemma 2.4.7. *If A is an abelian variety defined over a finitely generated extension of $\mathbb{Q}$, both its global Tate module TA and its p-adic Tate module T_pA for every prime p are very rigid.*

Proof. This is a direct consequence of the Mordell-Weil theorem. □

Lemma 2.4.8. *Let k be a finitely generated extension of $\mathbb{Q}$ and X is a smooth curve with a rational point $x \in X(k)$. Then $\underline{\pi}_1(X, x)$ is very rigid.*

Proof. We know that the space of sections for the étale fundamental group correspond to the space of $\underline{\pi}_1(X, x)$-torsors, and if s is section corresponding to a torsor T then the centralizers of s correspond to rational points of $\mathrm{Aut}_{\underline{\pi}_1}(T)$. Hence, to say that $\underline{\pi}_1(X, x)$ is very rigid is equivalent to saying that all the sections of the étale fundamental group have trivial centralizers over any finitely generated extension k'/k. This is known, see [Sti13, Proposition 104].

Actually, in [Sti13, Proposition 104] this is done for finite extensions of $\mathbb{Q}_p$. However, from our point of view it is clear that if triviality of centralizers holds for an extension of k then it holds over k (since this is trivially true for very rigid groups), and any finitely generated extension of $\mathbb{Q}$ can be embedded in a finite extension of $\mathbb{Q}_p$. In fact $\mathrm{trdeg}_{\mathbb{Q}} \mathbb{Q}_p = \infty$ because $\mathbb{Q}_p$ is uncountable. □

Lemma 2.4.9. *Let X be an fff DM stack over k (see Definition 1.2.2) and let $x \in X(k)$ be a non-stacky point, meaning that $\underline{\mathrm{Aut}}_X(x)$ is the trivial group scheme. Then $\underline{\pi}_1(X, x)$ is rigid.*

Proof. By definition of fff, $\underline{\mathrm{Aut}}_X(x)(L) \to \underline{\pi}_1(X, x)(L)$ is bijective for every finitely generated extension L/k. □

In the following we give an example of a rigid group which is not very rigid.

Example 2.4.10. Let X be a smooth projective orbicurve over a field k finitely generated over $\mathbb{Q}$ with Euler characteristic less than or equal to 0. Let $x \in X(k)$ be an ordinary point and $y \in X(k)$ a stacky point, meaning that $\underline{\mathrm{Aut}}_X(x) = \{\mathrm{id}\}$ and $\underline{\mathrm{Aut}}_X(y) \neq \{\mathrm{id}\}$.

Then thanks to Theorem 1.5.3 we have that, for every finitely generated extension k'/k, $\underline{\mathrm{Aut}}_X(x)(k') = \underline{\pi}_1(X, x)(k')$ and $\underline{\mathrm{Aut}}_X(y)(k') = \underline{\pi}_1(X, y)(k')$. In particular $\underline{\pi}_1(X, x)$ is rigid but $\underline{\pi}_1(X, y)$ is not, and since they are strong inner forms one of each other then $\underline{\pi}_1(X, x)$ is rigid but not very rigid.

2.4.2 Minimal fields of definition

Lemma 2.4.11. *Let G/k be very rigid, L/k a finite separable extension and $T \to \operatorname{Spec} L$ a G-torsor. Then T has a minimal field of definition, i.e. there exists a subextension $L/E/k$ over which T is defined and such that for every other subextension $L/E'/k$ over which T is defined we have $E \subseteq E'$.*

Proof. Up to enlarging L, we may suppose that L/k is Galois. Let $H \subseteq \mathrm{Gal}(L/k)$ the subgroup of elements σ such that there exists an isomorphism

$\sigma^*T \simeq T$, and let $E = L^H$. Clearly, if T is defined on a subfield $L/E'/k$, then $\mathrm{Gal}(L/E') \subseteq H$ and hence $E \subseteq E'$. Let us show that T is defined over E.

For every $\sigma \in H$, we have an isomorphism $\varphi_\sigma : \sigma^*T \to T$. We have to check the cocycle condition, i.e. that if $\sigma, \tau \in H$ then

$$\varphi_\tau \circ \tau^*\varphi_\sigma \circ \varphi_{\sigma\tau}^{-1} : T \to \tau^*\sigma^*T \to \tau^*T \to T$$

is the identity. But $\varphi_\tau \circ \tau^*\varphi_\sigma \circ \varphi_{\sigma\tau}^{-1}$ is an element of $\underline{\mathrm{Aut}}_G(T)(L)$, which contains only the identity by hypothesis. $\square$

In Lemma 2.4.11, we have shown that torsors for a very rigid group scheme have a minimal field of definition if they are defined over a finite and separable extension. Actually, there are examples where the minimal field of definition exists even for finitely generated transcendental extensions.

Example 2.4.12. Let k be a field of characteristic 0, and consider the field extension $k(t)/k$. Let $x \in \wedge_p k(t)^*$ be defined by the product

$$x = t \cdot (t+1)^p \cdot (t+2)^{p^2} \cdot (t+3)^{p^3} \cdots \in \wedge_p k(t)^*$$

Then x is not defined over any nontrivial subextension $k \subsetneq E \subsetneq k(t)$. Let $k(t)$ be the function field of $\mathbb{P}^1$ and E the function field of a smooth projective curve X with a morphism $\mathbb{P}^1 \to X$. If $\mathbb{P}^1 \to X$ is not an isomorphism then the p-adic divisor $\mathrm{div}(x) \in \wedge_p \mathrm{Div}(\mathbb{P}^1) = \varprojlim_n \mathrm{Div}(\mathbb{P}^1)/p^n \mathrm{Div}(\mathbb{P}^1)$ cannot be the pullback of a p-adic divisor on X, because if we restrict $\mathbb{P}^1 \to X$ to an open subset $U \subseteq \mathbb{P}^1$ it becomes étale, but $\mathrm{div}\, x|_U$ is supported on infinite points each one with different ramification.

It is possible to modify this example in order to construct for every n a $\mathbb{Z}_p(1)$-torsor over $k(t_1, \dots, t_n)$ which is not defined over any subextension, thus giving another proof of the fact that $\mathrm{ed}\, \mathbb{Z}_p(1) = \infty$. In fact, one may build $x_n \in \wedge_p k(t_1, \dots, t_n)^*$ such that its p-adic divisor in $\mathbb{A}^n$ is a "skeleton" obstructing the compression of $\mathbb{A}^n$, meaning that the restriction of x_n to a generic line has the same form of the uncompressible element $x \in \wedge_p k(t)^*$ given above. For example,

$$x_n = t_1 \cdot t_2^p \cdots t_n^{p^{n+1}} \cdot (t_1+1)^{p^{n+2}} \cdots \in \wedge_p k(t_1, \dots, t_n)^*.$$

But then a generic line is uncompressible, and thus the whole torsor is uncompressible.

Minimal fields of definition are common for torsors over very rigid group schemes, but they don't exist in full generality.

Example 2.4.13. Consider $t \in \wedge_p k(t)^*$. Then for every integer n prime with p we have that t is defined over $k(t^n)$: in fact $1/n \in \mathbb{Z}_p$, and hence $t = (t^n)^{1/n}$ is a well defined element of $\wedge_p k(t^n)^*$.

2.4.3 Rigidity properties for torsors

Lemma 2.4.14. *Let G be a rigid group scheme over k. Then for every tower of finitely generated extensions $k''/k'/k$, if $T \to \operatorname{Spec} k'$ is a G-torsor trivialized by k'' then T was already trivial on k'.*

Proof. By hypothesis, we have a section $\operatorname{Spec} k'' \to T \to \operatorname{Spec} k'$ with image $t \in T$. Since G is pro-étale and k''/k' is finitely generated, $k(t)/k'$ is finite. We want to show that, actually, $k(t) = k'$. In fact, if $k(t)$ is strictly greater than k' and L/k' is its Galois closure, then $T \times_{k'} L \simeq G_L$ has more than one rational point, and this is absurd. □

Corollary 2.4.15. *Let G/k be rigid. Then for every finitely generated extension k'/k every nontrivial torsor $T \to \operatorname{Spec} k'$ is rigid.* □

Corollary 2.4.16. *If G/k is very rigid, then $\mathrm{H}^1(G, k') \to \mathrm{H}^1(G, k'')$ is injective for every finitely generated extensions $k''/k'/k$.*

Proof. Let $T_1 \to \operatorname{Spec} k'$, $T_2 \to \operatorname{Spec} k'$ be non isomorphic G-torsors, G acts on the left. The group H of G automorphisms of T_2 is a strong inner form of $G_{k'}$ and hence it is rigid, too. Now, T_2 is a right H-torsor, hence $\underline{\mathrm{Isom}}_G(T_1, T_2)$ is a right H-torsor too. But $\underline{\mathrm{Isom}}_G(T_1, T_2)$ is nontrivial since T_1 and T_2 are non isomorphic and H is rigid, hence $\underline{\mathrm{Isom}}_G(T_1, T_2)$ is rigid too thanks to Corollary 2.4.15. In particular, $T_{1,k''}$ is not isomorphic to $T_{2,k''}$. □

Lemma 2.4.17. *Let G/k be a rigid group scheme, H/k an algebraic group scheme with a morphism $\varphi : G \to H$ and L/k a finitely generated extension.*

i) There exists a factorization $\varphi : G \to G' \to H$ with G' finite étale such that all the points of $G'(L)$ map to the identity of $H(L)$.

ii) Suppose now that we have a finite Galois extension L'/L, and let $G \to G' \to H$ be as above with respect to L'/k, i.e. $G'(L') \to H(L')$ maps every point to the identity. Let $T \to L$ be an G'-torsor over L with $T(L') \neq \emptyset$. Then $T_H = T \times^{G'} H$ is trivial.

Proof. i) Write $G = \varprojlim_i G_i$ with G_i finite étale, for i large enough φ factorizes as $G \to G_i \to H$: we are going to define $G' = G_i$ for some i large enough.

Since $H(L)$ is finite, it is enough to prove that every $h \in H(L)$ different from the identity is not in the image of $G_i(L) \to H(L)$ for i large enough. Hence, consider the projections $\pi_i : G \to G_i$ and $\psi_i : G_i \to H$.

Since G is rigid, it has only one rational point which is the identity, and since the identity doesn't belong to $\varphi^{-1}(h)(L) \subseteq G(L)$ we have $\varphi^{-1}(h)(L) = \varprojlim_i \psi_i^{-1}(h)(L) = \emptyset$. But a projective limit of finite nonempty sets is nonempty, thus $\psi_i^{-1}(h)(L) = \emptyset$ for i large enough, as desired.

ii) Choose any $t \in T(L')$, and let $t_H \in T_H(L')$ its image: we want to show that the residue field of t_H is L. Since $T_{L'} \simeq G'_{L'}$ and $T_{H,L'} = H_{L'}$, by construction $T(L') \to T_H(L')$ maps every L' rational point to t_H. For any $\sigma \in \mathrm{Gal}(L'/L)$, $\sigma^* t \in T(L')$ maps to $\sigma^* t_H$, and hence $\sigma^* t_H = t_H$. Since this is true for any $\sigma \in \mathrm{Gal}(L'/L)$, we have that t_H is L rational, as desired.

□

2.4.4 Rigidity properties for the fce dimension

Proposition 2.4.18. *Let G be a very rigid group scheme over k, L/k a finitely generated extension and $T \to \operatorname{Spec} L$ a G-torsor. Then for every finite separable extension L'/L we have $\operatorname{ed} T = \operatorname{ed} T_{L'}$ and $\operatorname{ced} T = \operatorname{ced} T_{L'}$.*

Proof. The inequalities $\operatorname{ced} T_{L'} \leq \operatorname{ced} T$, $\operatorname{ed} T_{L'} \leq \operatorname{ed} T$ are obvious. Up to extending L', we may suppose that L'/L is Galois.

We have $T_{L'} = Q_{L'}$ for some torsor $Q \to \operatorname{Spec} E'$ over a subextension $L'/E'/k$ with $\operatorname{trdeg}_k E' = n$. Up to substituting E' with $\overline{E'}^{L'}$, we may suppose that E' is algebraically closed in L', and in particular the action of $\mathrm{Gal}(L'/L)$ sends E' to itself. Now define E as the elements of E' fixed by $\mathrm{Gal}(L'/L)$. We have that E'/E is a finite Galois extension, $\mathrm{Gal}(L'/L) \to \mathrm{Gal}(E'/E)$ is surjective and $E \subseteq L$.

Consider $\sigma \in \mathrm{Gal}(L'/L)$, since $T_{L'}$ is defined over L then we have an isomorphism $\varphi(\sigma) : \sigma^* T_{L'} \simeq T_{L'}$ as G-torsors. These $\varphi(\sigma)$ clearly satisfy the cocycle condition $\varphi(\tau) \circ \tau^* \varphi(\sigma) = \varphi(\sigma \circ \tau)$.

Now consider the pro-étale scheme $\underline{\mathrm{Isom}}_G(\sigma^* Q, Q)$, then $\varphi(\sigma)$ defines an L' rational point of $\underline{\mathrm{Isom}}_G(\sigma^* Q, Q)$. Since this scheme is pro-étale and E' is algebraically closed in L', we actually have an E' rational point $\psi(\sigma) \in \underline{\mathrm{Isom}}_G(\sigma^* Q, Q)$, i.e. an isomorphism $\psi(\sigma) : \sigma^* Q \to Q$. The fact that the $\varphi(\sigma)$ respect the cocycle condition directly translates in the same fact for $\psi(\sigma)$, and thus by descent we may define a G-torsor $P \to \operatorname{Spec} E$ such that $P_{E'} \simeq Q$.

Up to now, we have not used the rigidity hypothesis. What makes this lemma fail in general is that there is no reason why we should have

$P_L \simeq T$. But if G is very rigid, then $P_{L'} \simeq T_{L'}$ and hence $P_L \simeq T$ thanks to Corollary 2.4.16.

For continuous essential dimension, the argument is analogous, but subtler. Write $G = \varprojlim_i G_i$, $T_i = T \times^G G_i$ and $H_i = \underline{\mathrm{Aut}}_{G_i}(T_i)$. Fix an index i, we want to show that $\operatorname{ed} T_i \leq \operatorname{ced} T_{L'}$. This is subtle: it may happen that $\operatorname{ed} T_i > \operatorname{ed} T_{i,L'}$, but still we have $\operatorname{ed} T_i \leq \operatorname{ed} T_{j,L'}$ for some $j >> i$.

We have that $H = \varprojlim_i H_i = \underline{\mathrm{Aut}}_G(T)$ is a rigid pro-étale group scheme. Choose $j >> i$ such that $H \to H_j \to H_i$ is as in Lemma 2.4.17.ii with respect to $L'/L/k$.

The same argument as above gives us a torsor $P_j \to \operatorname{Spec} E_j$ such that $P_{j,L'} \simeq T_{j,L'}$, with $\operatorname{trdeg}_k E_j \leq \operatorname{ced} T_{L'}$. Write $P_i = P_j \times^{G_j} G_i$, $E_i = E_j$, we want to show that $P_{i,L} \simeq T_i$.

Consider the étale scheme $I_j = \underline{\mathrm{Isom}}_{G_j}(P_{j,L}, T_j)$: I_j is an H_j-torsor over L, we know it has an L' rational point. We have that

$$I_i = I_j \times^{H_j} H_i = \underline{\mathrm{Isom}}_{G_i}(P_{i,L}, T_i).$$

But then, thanks to Lemma 2.4.17.ii, since $I_j(L')$ is nonempty we get that $I_i \to \operatorname{Spec} L$ is trivial, as desired. □

Corollary 2.4.19. *Let G be a very rigid group scheme over k, and k'/k a finite separable extension. Then* $\operatorname{fed} G = \operatorname{fed} G_{k'}$ *and* $\operatorname{fced} G = \operatorname{fced} G_{k'}$. □

Lemma 2.4.20. *Let G be a very rigid group scheme over k, $H \subseteq G$ a subgroup of finite index and $T \to \operatorname{Spec} L$ an H-torsor with L/k finitely generated. Write $T_G = T \times^H G$. Then* $\operatorname{ed} T_G = \operatorname{ed} T$ *and* $\operatorname{ced} T_G = \operatorname{ced} T$.

Proof. We clearly have $\operatorname{ed} T_G \leq \operatorname{ed} T$ and $\operatorname{ced} T_G \leq \operatorname{ced} T$.

Thanks to Proposition 2.4.18 up to a finite separable extension of L we may suppose that the étale scheme T_G/H is discrete, i.e. all of its points are L rational.

Suppose that $T_G \simeq Q_L$ for some G-torsor $Q \to \operatorname{Spec} E$ with $L/E/k$ a subextension. Up to a finite extension of E, we may suppose that it is algebraically closed in L.

Consider now the finite étale scheme Q/H with the projection $\pi : Q \to Q/H$. For every rational point $e \in Q/H(E)$ the fiber $\pi^{-1}(e)$ is an H-torsor lifting Q to H. Every other lifting of Q to H has this form, and thus we have a map $Q/H(E) \to \mathrm{H}^1(H, E)$ whose image are all the liftings of Q to H. This is a particularly rough form of nonabelian long exact sequence in cohomology, since $H \subseteq G$ is not even normal.

Since

$$(Q/H)_L = T_G/H$$

is discrete, i.e. all of its points are rational, and $E \subseteq L$ is algebraically closed then Q/H is discrete too. Consider now the following commutative diagram:

$$\begin{tikzcd} Q/H(E) \arrow[r] \arrow[d, "\sim"] & \mathrm{H}^1(H,E) \arrow[d] \\ T_G/H(L) \arrow[r] & \mathrm{H}^1(H,L) \end{tikzcd}$$

since the left vertical arrow is bijective and $T \in \mathrm{H}^1(H,L)$ is in the image of $T_G/H(L) \to \mathrm{H}^1(H,L)$, it is in the image of $\mathrm{H}^1(H,E) \to \mathrm{H}^1(G,E)$ too. This gives us an H-torsor $P \to \operatorname{Spec} E$ such that $P_L \simeq T$, and hence $\operatorname{ed} T \leq \operatorname{ed} T_G$.

Write now $G = \varprojlim_{i \in I} G_i$ with $G \to G_i$ surjective, $H_i = \operatorname{im}(H \to G_i)$. We have $H = \varprojlim_{i \in I} H_i$ and for i large enough $G_i/H_i = G/H$. Write $T_i = T \times^H H_i$.

Since $T_{i,G_i}/H_i = T_G/H$ is discrete for i large enough, we may repeat the argument above and hence $\operatorname{ed} T_i = \operatorname{ed} T_{i,G_i}$, which implies $\operatorname{ced} T = \operatorname{ced} T_G$. $\square$

Corollary 2.4.21. *Let G be a very rigid group scheme over k, and $H \subseteq G$ a subgroup of finite index. Then* $\operatorname{fed} G = \operatorname{fed} H$ *and* $\operatorname{fced} G = \operatorname{fced} H$*.*

Proof. Thanks to Lemma 2.4.20, we have $\operatorname{fed} H \leq \operatorname{fed} G$ and $\operatorname{fced} H \leq \operatorname{fced} G$.

Let T be a G-torsor over L, with L finitely generated over k. There exists a finite separable extension L'/L such that $T/H(L')$ is nonempty, i.e. $T_{L'} \simeq P_G$ for some H-torsor $P \to \operatorname{Spec} L'$. But then $\operatorname{ed} T = \operatorname{ed} T_{L'} = \operatorname{ed} P$ and $\operatorname{ced} T = \operatorname{ced} T_{L'} = \operatorname{ced} P$ thanks to Lemma 2.4.20 and Proposition 2.4.18. Hence $\operatorname{fed} H \geq \operatorname{fed} G$ and $\operatorname{fced} H \geq \operatorname{fced} G$.

$\square$

Chapter 3

The dimensional section conjecture

In this chapter we are going to study a dimensional version of Grothendieck's section conjecture, and make some steps toward proving it for $\mathbb{P}^1 \setminus \{p_0, \dots, p_n\}$ for $n \geq 2$.

For $\mathbb{P}^1 \setminus \{p_0, \dots, p_n\}$ we use the 2 nilpotent obstruction, originally studied by Kirsten Wickelgren in [Wic12], in order to reduce ourselves to a very explicit problem about valuations in the group $\mathbb{Z}^n$ ordered lexicographically. We call this problem the *valuation conjecture*. In sections 3.3 and 3.4 we introduce many techniques for the study of the valuation conjecture, and prove many subcases of it. In particular, we have proved the rank 1 case and a good part of the rank 2 case, and the rank 2 case is the one we need for the dimensional section conjecture.

This chapter relies on the concept of fce dimension, which we have introduced and studied in chapter 2. If the reader is only interested in the applications of fce dimension to the section conjecture, he may skip sections 2.1 and 2.4, while 2.2 and 2.3 are necessary since they contain the basic definitions.

Let X be a smooth, geometrically connected hyperbolic curve. Since $\dim X = 1$ and packets of cuspidal sections are in natural bijection with $\mathrm{H}^1(k, \widehat{\mathbb{Z}}(1)) = \widehat{k}^*$ which has fce dimension 1, Grothendieck's section conjecture implies the following, which is hopefully easier.

Conjecture 3.0.1. Let X be a smooth, geometrically connected hyperbolic curve over a field k finitely generated over $\mathbb{Q}$, and $x_0 \in X(k)$ a rational point. Then $\mathrm{fced}_k\, \underline{\pi}_1(X, x_0) = 1$.

If X is a complete curve, since there are no packets Grothendieck's section conjecture implies more strongly that $\mathrm{fed}_k\, \underline{\pi}_1(X, x_0) = 1$, not only that

$\operatorname{fced}_k \underline{\pi}_1(X, x_0) = 1$. However, we think that in the profinite case continuity problems should still be taken into account for theoretical reasons, even in the complete case.

In Conjecture 3.0.1 we require the existence of a rational point, but this is not a problem: the validity of the dimensional section conjecture is invariant along finite extensions of the base field.

Corollary 3.0.2. *If X is a smooth hyperbolic curve over a field k finitely generated over $\mathbb{Q}$ and k'/k is a finite extension, then the dimensional section conjecture holds for X if and only if it holds for $X_{k'}$.*

Proof. Apply Lemma 2.4.8 and Corollary 2.4.19. □

The dimensional section conjecture is quite different from usual essential dimension problems: here the lower bound is easy and the upper bound is hard, usually the converse is true.

Proposition 3.0.3. *Let X be a smooth, elliptic or hyperbolic curve over a field k finitely generated over $\mathbb{Q}$ with a rational point $x \in X(k)$. Then $\operatorname{fced}_k \underline{\pi}_1(X, x) \geq 1$.*

Proof. Let $\xi \in X$ be the generic point, and $T \to \operatorname{Spec} k(X)$ its associated $\underline{\pi}_1(X, x)$-torsor. We want to show that $\operatorname{ced} T = 1$, and this is equivalent to $\operatorname{ed} T = 1$ thanks to Proposition 2.3.3 and the fact that $\operatorname{trdeg}_k k(X) = 1$. Actually, we are going to show more: $k(X)$ is the minimal field of definition of T.

Suppose that there exists a nontrivial subextension $k \subsetneq E \subsetneq k(X)$ such that T is defined over E. Then there exists a finite extension $L/k(X)$ and an automorphism σ of L fixing E but not fixing $k(X)$. Hence, we have $\sigma^* T_L \simeq T_L$, but $\sigma^* \xi_L \neq \xi_L$. This is absurd, because $L/\mathbb{Q}$ is finitely generated and X is either elliptic or hyperbolic, and thus $X(L) \to B\underline{\pi}_1(X, x)(L)$ is injective on isomorphism classes. □

Just as Grothendieck's section conjecture, the dimensional section conjecture is stable under finite étale covers.

Corollary 3.0.4. *Let $\pi : Y \to X$ be a finite étale cover of geometrically connected smooth curves over a field k finitely generated over $\mathbb{Q}$, and $y \in Y(k)$ a rational point with image $x \in X(k)$. Then $\operatorname{fed} \underline{\pi}_1(Y, y) = \operatorname{fed} \underline{\pi}_1(X, x)$ and $\operatorname{fced} \underline{\pi}_1(Y, y) = \operatorname{fced} \underline{\pi}_1(X, x)$.*

Proof. Apply Lemma 2.4.8 and Corollary 2.4.21. □

Proving Conjecture 3.0.1 could be a step toward Grothendieck's conjecture: it would say that "there cannot be too many sections" in a dimensional sense.

3.1 The 2 nilpotent obstruction for $\mathbb{P}^1 \setminus \{0,1,\infty\}$

In this brief section, we describe the basic obstruction which we are going to use to study the dimensional section conjecture in the case of $\mathbb{P}^1 \setminus \{0,1,\infty\}$, or more generally $\mathbb{P}^1 \setminus \{p_0, \dots, p_n\}$ for $n \geq 2$. We do not give proofs, these can be found in [Wic12].

Fix any rational base point x_0 of $\mathbb{P}^1 \setminus \{0,1,\infty\}$ and write $\pi = \underline{\pi}_1(\mathbb{P}^1 \setminus \{0,1,\infty\}, x_0)$, we want to study its fce dimension: if the section conjecture is true, it is 1. In order to do this, it turns out to be useful to study the 2-nilpotent quotient of π. Call $[\pi]_1 = [\pi,\pi]$ the commutator subgroup, and denote by $[\pi]_{n+1} = [\pi, [\pi]_n]$ the lower central series. We have a central extension

$$0 \to [\pi]_1/[\pi]_2 \to \pi/[\pi]_2 \to \pi/[\pi]_1 \to 0$$

with isomorphisms $\pi/[\pi]_1 = \pi^{ab} \simeq \widehat{\mathbb{Z}}(1)^2$, $[\pi]_1/[\pi]_2 \simeq \widehat{\mathbb{Z}}(2)$. For every finitely generated extension L/k, this gives an obstruction

$$\widehat{\delta} : \mathrm{H}^1(L, \widehat{\mathbb{Z}}(1)^2) = \Pi^{ab}(L) \to \mathrm{H}^2(L, \widehat{\mathbb{Z}}(2))$$

which has been studied by Kirsten Wickelgren, along with higher order obstructions, in [Wic12]. In particular, she proved in [Wic12, Proposition 7] that $\widehat{\delta}$ is simply the cup product, and that this obstruction is not sufficient to prove the section conjecture: there are elements of $\ker \widehat{\delta}$ not coming from rational points of $\mathbb{P}^1 \setminus \{0,1,\infty\}$ or from packets.

Let us look more closely at the map $\widehat{\delta}$. Its domain is

$$\mathrm{H}^1(L, \widehat{\mathbb{Z}}(1))^2 = \left(\varprojlim_n L^*/L^{*n} \right)^2 = \widehat{L}^* \oplus \widehat{L}^*.$$

For the codomain, the Merkurjev-Suslin theorem tells us that the cohomology group $\mathrm{H}^2(L, \widehat{\mathbb{Z}}(2))$ is the completion $\widehat{K}_2^M(L) = \varprojlim_n K_2^M(L)/n$ of the Milnor group

$$K_2^M(L) = L^* \otimes_{\mathbb{Z}} L^* / < (a, 1-a) | a \in L^* > .$$

There is an obvious bilinear structure map $L^* \oplus L^* \to K_2^M(L)$ which induces a map between completions $\widehat{L}^* \oplus \widehat{L}^* \to \widehat{K_2^M}(L)$, and since $\widehat{\delta}$ is the cup product this implies that the following diagram commutes

$$\begin{tikzcd}
\mathrm{H}^1\left(L, \widehat{\mathbb{Z}}(1)^2\right) \arrow[r, "\widehat{\delta}"] \arrow[d, "\sim"] & \mathrm{H}^2\left(L, \widehat{\mathbb{Z}}(2)\right) \\
\widehat{L}^* \oplus \widehat{L}^* \arrow[r] & \widehat{K}_2^M(L) \arrow[u, "\sim"']
\end{tikzcd}$$

since the isomorphism $K_2^M(L)/n \to \mathrm{H}^2(L, \mu_n^{\otimes 2})$ of Merkurjev-Suslin theorem is exactly the one induced by cup product. Hence, we may think of $\widehat{\delta}$ as the completion of the map

$$\delta : L^* \oplus L^* \to K_2^M(L).$$

Wickelgren's work can be easily generalized to $\mathbb{P}^1 \setminus \{p_0, \ldots, p_n\}$ for $n \geq 2$. The topological fundamental group of $\mathbb{P}^1 \setminus \{p_0, \ldots, p_n\}$ is the free group $\mathbb{Z}^{*n}$ on n generators, which we can identify with simple loops around the points $p_1, \ldots, p_n \in \mathbb{P}^1$. The abelianization of the fundamental group is $\mathbb{Z}^n$, while the $[\mathbb{Z}^{*n}]_1/[\mathbb{Z}^{*n}]_2$ is free abelian on $\binom{n}{2}$ generators each corresponding to the commutator of two simple loops, see [War69].

Passing to étale fundamental group schemes, we can merge the topological computation and Wickelgren's work using suitable morphisms

$$\mathbb{P}^1 \setminus \{p_0, \ldots, p_n\} \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$$

$$0 \mapsto 0, \quad p_i \mapsto 1, \quad p_j \mapsto \infty \quad 1 \leq i < j \leq n$$

to show that if π is the étale fundamental group scheme of $\mathbb{P}^1 \setminus \{p_0, \ldots, p_n\}$ then

$$\pi/[\pi]_1 = \widehat{\mathbb{Z}}^n(1),$$
$$[\pi]_1/[\pi]_2 = \widehat{\mathbb{Z}}^{\binom{n}{2}}(2)$$

and the 2 nilpotent obstruction is the completion $\widehat{\omega}_n$ of the map

$$\omega_n : \oplus_n L^* \to \oplus_{\binom{n}{2}} K_2^M(L)$$

$$(x_1, \ldots, x_n) \mapsto \omega_n(x_1, \ldots, x_n) = \Big(\delta(x_i, x_j)\Big)_{i<j}$$

where $\delta : L^* \oplus L^* \to K_2^M(L)$ is the structure map as above. In particular, $\delta = \omega_2$.

As already noted by Wickelgren [Wic12, Proposition 24], this shows that the obstruction given by $\widehat{\delta}$ (or more generally $\widehat{\omega}_n$) is quite coarse. In fact, there are a lot of elements of $\widehat{L}^* \oplus \widehat{L}^*$ which are in $\ker \widehat{\delta}$ but do not come from $\mathbb{P}^1 \setminus \{0, 1, \infty\}(L)$, for example $(x, (1-x)^m)$ or $((-x)^m, x)$ for any $x \in L^*$.

Still, all these examples are at most "one dimensional families", i.e. they have at most essential dimension 1. This is not just an observation: in the following we are going to prove that $\operatorname{ed} \ker \omega_n = 1$.

Obviously, we are interested in the maps $\widehat{\omega}_n$ and $\wedge_p \omega_n$ rather than in ω_n, still this will serve us as a roadmap case.

3.2 Roadmap case

As we have said above, we are going to prove that $\operatorname{ed}\ker\delta = 1$, and thus $\operatorname{ed}\ker\omega_n = 1$. Actually, we have more generally a map $\delta_m : \oplus_m L^* \to K_m^M(L)$ for every m (with $\delta = \delta_2$), and we are going to prove that $\operatorname{ed}\ker\delta_m = m - 1$.

Remark 3.2.1. The notation might be quite confusing, so let us stress this out: δ_m has in general nothing to do with ω_n. For $n = m = 2$ we do have $\delta_2 = \omega_2 = \delta$, but for $m > 3$ the map δ_m is a generalization of $\delta = \delta_2$, while ω_n is a combination of $\binom{n}{2}$ copies of δ. We want to study the essential dimension of the kernel of ω_n for every n, and this reduces to the essential dimension of the kernel of $\delta = \delta_2$. The computation $\operatorname{ed}\ker\delta = 1$ is a particular case of the computation of $\operatorname{ed}\ker\delta_m = m - 1$, so we do the latter.

The proof that $\operatorname{ed}\ker\delta_m = m - 1$ builds on the theory of rank m valuations, i.e. valuations with value group $\mathbb{Z}^m$ ordered lexicographically. Rank m valuations can be seen geometrically in the following way: if M is a smooth projective variety over k, a sequence $(V_0, \dots, V_n)$ where $V_0 = M$ and V_{i+1} is an hypersurface in the normalization of V_i gives a rank m valuation $v : k(M)^* \to \mathbb{Z}^m$. If we allow M to vary in its birational equivalence class, we recover all rank m valuations on $k(M)$.

We develop the theory of rank n valuations in section 3.3, here we just give some examples and results.

Example 3.2.2.

- Let M be a smooth projective variety over k, and $f \in k(M)$ a rational function on M. Then f is algebraic over k if and only if its divisor $\operatorname{div} f$ is trivial. If f is not algebraic over k, there exists an hypersurface $V \subseteq M$ on which f has nonzero order. Let ξ the generic point of V, then $\mathcal{O}_{M,\xi}$ is a DVR defining a valuation $v : k(M) \to \mathbb{Z}$ such that $v(f) \neq 0$.

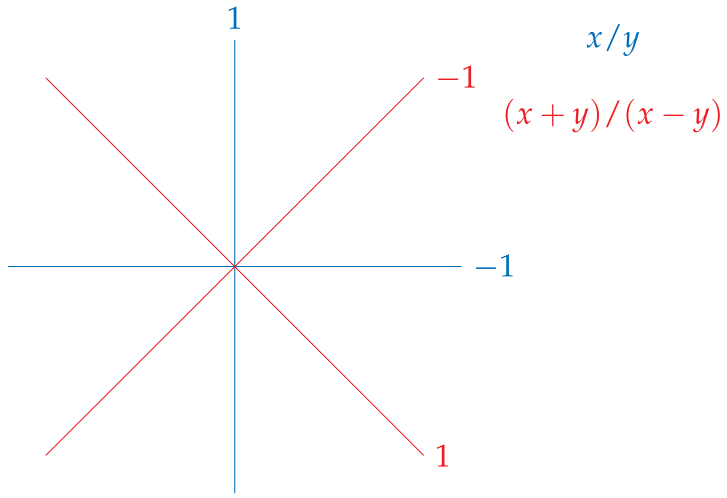
- Let M/k be as above, and consider two nonzero rational functions $f_1, f_2 \in k(M)^*$.

  Suppose that they are algebraically dependent, hence there exists a nonzero polynomial $p \in k[t_1, t_2]$ such that $p(f_1, f_2) = 0$. Since $f_1, f_2 \neq 0$, we may suppose that t_1, t_2 do not divide p. Let $V \subseteq M$ be an hypersurface on which f_1 has nonzero order: up to some elementary operations on f_1 and f_2 like replacing f_2 with $f_2 \cdot f_1^d$ for some d, we may suppose that V is a zero of f_1, but not a zero neither a pole of f_2. Hence $p(0, t_2)$ is a nonzero polynomial telling us that the restriction of f_2 to V is algebraic over k, i.e. constant.

On the other hand, suppose that f_1 and f_2 are algebraically independent, f_2 defines a rational map $M \dashrightarrow \mathbb{P}^1$. Since f_1 and f_2 are algebraically independent, the restriction of f_1 to generic fiber of this map is not constant, hence we may find an hypersurface $V \subseteq M$ which dominates $\mathbb{P}_1$ and which is a zero of f_1. Clearly, V is not a zero neither a pole of f_2, and the restriction of f_2 to V is non constant.

Using rank 2 valuations, the above facts can be packed in the following sentence: two rational functions $f_1, f_2 \in k(M)$ are algebraically independent over k if and only if there exists a rank 2 valuation $v : k(M) \to \mathbb{Z}^2$ such that the 2×2 matrix $(v(f_1)|v(f_2))$ is nonsingular, see Proposition 3.5.3.

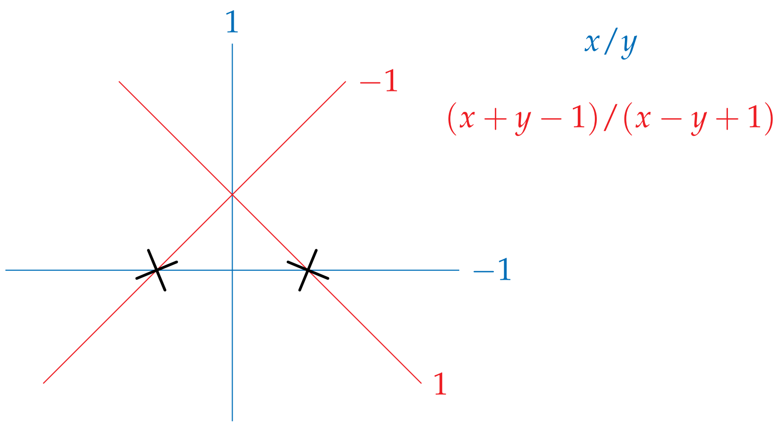
- Let $M = \mathbb{P}^2_k$, $k(M) = k(x, y)$. Then we can say that x/y and $(x+y)/(x-y)$ are algebraically dependent without computations just by looking at a picture.



If we blow up the origin, then the exceptional divisor is not a zero nor a pole of both x/y and $(x+y)/(x-y)$. This can be seen by observing that all the lines are smooth at the origin, hence have multiplicity 1, and thus the two rational functions both have order $1-1=0$ on the exceptional divisor.

Hence, after blowing up the divisors are disjoint, thus there cannot exist any rank 2 valuation with associated nonsingular matrix as above. This implies that x/y and $(x+y)/(x-y)$ are algebraically dependent.

- On the other hand, if we move the divisor of one of the two rational functions above, we get a point (two, actually) where the divisors meet transversally.



  Any of the lines in the support passing through one of these points defines a rank 2 valuation whose associated 2×2 matrix is nonsingular, thus x/y and $(x+y-1)/(x-y+1)$ are algebraically independent.

There are two basic facts about rank m valuation that we need.

- The first fact is a generalization of Example 3.2.2. If L/k is an extension of fields and $x_1, \dots, x_n \in L^*$, then $\operatorname{trdeg}_k k(x_1, \dots, x_n) \geq m$ if and only if there exists a rank m valuation $v : L^* \to \mathbb{Z}^m$ trivial on k such that the $m \times n$ matrix $(v(x_1)| \dots |v(x_n))$ has rank m, see Proposition 3.5.3.

- The second fact is that the tame symbols of discrete valuations can be generalized to tame symbols of rank m valuations. If $v : L^* \to \mathbb{Z}^m$ is a rank m valuation, v defines an homomorphism $\partial_v : K^M_{d+m}(L) \to K^M_d(L_v)$ for every d which is just the usual tame symbol if $m = 1$. If $d = 0$ then ∂_v is just the determinant, i.e. if $x_1, \dots, x_m \in L^*$, then $K^M_0(L_v) = \mathbb{Z}$ and

$$\partial_v(\delta_n(x_1, \dots, x_m)) = \det(v(x_1)| \dots |v(x_m)),$$

  see Proposition 3.5.7.

We need now to define precisely the kernel (and the image) functor, since they are slightly different from the obvious definitions. We are going to say that $(x_1, \dots, x_m) \in \oplus_n L^*$ is an element of $\ker \delta_m(L)$ if there exists an extension L'/L such that $\delta_m(x_1, \dots, x_m)_{L'} = 0 \in K^M_m(L')$. Let us state this more generally.

Definition 3.2.3. Let $F : \mathrm{Fields}_k \to \mathrm{Sets}$ be a functor, and $G \subseteq F$ a subfunctor. We say that $G \subseteq F$ is *geometrically saturated* if for every tower of extension $L/K/k$ and every $a \in F(K)$ such that $a_L \in G(L)$, then $a \in G(K)$.

We can define the *geometric saturation* $\underline{G}_F \subseteq F$ of G in F as the subfunctor of elements $\alpha \in F(K)$ such that $\alpha_L \in G(L)$ for some extension L/K, it is the smallest geometrically saturated subfunctor containing G.

Definition 3.2.4. If $\varphi : G \to F$ is a natural transformation, the *image* $\operatorname{im}\varphi \subseteq F$ is the geometric saturation of the functor $\big(L \mapsto \varphi(L) \subseteq F(L)\big)$. If $0 \in F(k)$ is a distinguished element, the *kernel* $\ker\varphi \subseteq G$ is the geometric saturation of the functor $\Big(L \mapsto \varphi^{-1}(0_L) \subseteq G(L)\Big)$.

Example 3.2.5. To understand why we take geometric saturations (apart from the fact that they make proofs work), think of what should be the image of $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{R}$ when we think of them as functors $\mathrm{Fields}_{\mathbb{R}} \to \mathrm{Sets}$.

It is immediate to check that the essential dimension of a geometrically saturated subfunctor of F is bounded above by the essential dimension of F. Hence, we have $\mathrm{ed}_k \operatorname{im}\varphi \leq \mathrm{ed}_k G$ and $\mathrm{ed}_k \ker\varphi \leq \mathrm{ed}_k F$.

Lemma 3.2.6. *Let ω_n be the natural transformation of functors* $\mathrm{Fields}_k \to \mathrm{Set}$ *associated to the map*

$$\oplus_n L^* \to \oplus_{\binom{n}{2}} K_n^M(L)$$

for every L/k using $\binom{n}{2}$ copies of δ. If $\mathrm{ed}\ker\delta = 1$, *then* $\mathrm{ed}\ker\omega_n = 1$.

Proof. By absurd, suppose that we have $(x_1, \ldots, x_n) \in \ker\omega_n(L)$ with $\mathrm{ed}(x_1, \ldots, x_n) \geq 2$, i.e. $\mathrm{trdeg}_k\, k(x_1, \ldots, x_n) \geq 2$. By definition of $\ker\omega_n$, up to extending L we may suppose that $\omega_n(x_1, \ldots, x_n) = 0$, i.e. $\delta(x_i, x_j) = 0$ for every $i, j = 1, \ldots, n$. Since we know that $\mathrm{ed}\ker\delta = 1$, this means that $\mathrm{trdeg}_k\, k(x_i, x_j) \leq 1$ for every i, j, but this is absurd since we have now that $\mathrm{trdeg}_k\, k(x_1, \ldots, x_n) \geq 2$. □

Proposition 3.2.7. *Let δ_m be the natural transformation of functors* $\mathrm{Fields}_k \to \mathrm{Set}$ *associated to the structure map $\oplus_m L^* \to K_m^M(L)$ for every L/k. Then*

$$\mathrm{ed}_k \ker\delta_m = m - 1.$$

Proof. If $t_1, \ldots, t_{m-1}$ are algebraically independent over k,

$$(t_1, \ldots, t_{m-1}, 1 - t_{m-1}) \in \ker\delta_m(k(t_1, \ldots, t_{m-1}))$$

has essential dimension $m-1$, and hence $\mathrm{ed}_k \ker \delta_m \geq m-1$. Since $\ker \delta_m \subseteq \oplus_m K_1^M$ is geometrically saturated, there are only two possibilities: $\mathrm{ed}_k \ker \delta_m$ is $m-1$ or m. Let us exclude the latter.

Suppose that $(t_1, \ldots, t_m) \in \ker \delta_m(L)$ is such that $\mathrm{ed}_k(t_1, \ldots, t_m) = m$. Since $\ker \delta_m \subseteq \oplus_m K_1^M$ is geometrically saturated, $(t_1, \ldots, t_m)$ has essential dimension m also in $\oplus_m L^*$, i.e. $t_1, \ldots, t_m$ are algebraically independent over k. By definition of $\ker \delta_m$, there exists L'/L such that $\delta_m(t_1, \ldots, t_m) = 0 \in K_m^M(L')$. Up to substituting L with L', we can suppose $\delta_m(t_1, \ldots, t_m) = 0 \in K_m^M(L)$.

Now, since they are algebraically independent, by Proposition 3.5.3 there exists a rank m valuation v with $\det(v)(t_1, \ldots, t_m) \neq 0$. But now, by Proposition 3.5.7

$$0 \neq \det(v)(t_1, \ldots, t_m) = \partial_v \delta_m(t_1, \ldots, t_m) = 0$$

which is absurd. □

3.3 The valuation conjecture

There is strong evidence that the argument in the roadmap case should extend to $\wedge_p \omega_n$ for every prime p, i.e. we expect that

$$\mathrm{fced} \ker \wedge_p \omega_n = 1.$$

Remark 3.3.1. Since we are working with finite type essential dimension, we are considering only finitely generated extensions of k, and the definition of geometric saturation (and hence of $\ker \wedge_p \omega_n$) must be modified accordingly, i.e. we must use geometric saturation "of finite type": just take the same definition restricted to fields finitely generated over k. As before, if a subfunctor $G \subseteq F : \mathrm{Fields}_k \to \mathrm{Set}$ is finite type geometrically saturated, then $\mathrm{fed}\, G \leq \mathrm{fed}\, F$, and $\mathrm{fced}\, G \leq \mathrm{fced}\, F$ if F is the functor of torsors of a proalgebraic group scheme. We are just forgetting the existence of fields not finitely generated over k.

In order to generalize the argument from ω_n to $\wedge_p \omega_n$, the key point is the following conjectural generalization of Proposition 3.5.3.

Conjecture 3.3.2 (Valuation conjecture)**.** Let p be a prime number, $m \leq n$ positive integers and L/k finitely generated extensions of $\mathbb{Q}$. Consider n elements $x_1, \ldots, x_n \in \wedge_p L^*$. Then $\mathrm{ced}(x_1, \ldots, x_n) \geq m$ if and only if there exists a rank m valuation $v : L^* \to \mathbb{Z}^m$ such that $(v(x_1)| \ldots |v(x_n)) \in M_{n \times m}(\mathbb{Z}_p)$ has rank m.

One implication of the conjecture is immediate and true without hypotheses on the fields. In fact, if there exists a rank m valuation v such that

$$\mathrm{rank}(v(x_1)|\ldots|v(x_n)) = m,$$

then there exist $1 \leq i_1 < \cdots < i_m \leq n$ such that

$$\det(v)(x_{i_1}, \ldots, x_{i_m}) \neq 0 \pmod{p^s}$$

for some integer s. This implies that the image of $(x_1, \ldots, x_n) \in \oplus_n \wedge_p L^*$ in $\oplus_n L^*/L^{p^s}$ has essential dimension at least m thanks to Proposition 3.5.3. Hence, the hard part is showing that, if $\det(v)(x_{i_1}, \ldots, x_{i_m}) = 0$ for every rank m valuation and every $1 \leq i_1 < \cdots < i_m \leq n$, then $\mathrm{ced}(x_1, \ldots, x_n) < m$, i.e. for every d we can approximate $(x_1, \ldots, x_n)$ up to p^d with n-uples $(x_{1,d}, \ldots, x_{n,d})$ of elements of L such that $\mathrm{trdeg}_k\, k(x_{1,d}, \ldots, x_{n,d}) \leq m$.

Proposition 3.3.3. *Suppose that Conjecture 3.3.2 holds for $m = 2$, and let $k/\mathbb{Q}$ be finitely generated. Let p be a prime number and ω_n be the morphism of functors* $\mathrm{Fields}_k \to \mathrm{Set}$ *associated to the structure map $\oplus_n \wedge_p L^* \to \oplus_{\binom{n}{2}} \wedge_p K_n^M(L)$ for every L/k. Then*

$$\mathrm{fced}(\ker \wedge_p \omega_n) = 1.$$

Proof. We can just repeat the proof of Proposition 3.2.7, but there is a subtlety. One is tempted to say that, since ω_n is built using $\binom{n}{2}$ copies of $\delta = \omega_2$, the valuation conjecture for $n = m = 2$ is enough. But this is wrong: in the argument of Proposition 3.2.7 we have used the seemingly innocent fact that if $x_1, \ldots, x_n \in \oplus_n L^*$ are such that $\mathrm{ed}(x_1, \ldots, x_n) \geq 2$, then there exist $1 \leq i < j \leq n$ with $\mathrm{ed}(x_i, x_j) = 2$. For elements of L^* this is obvious, while for elements of $\wedge_p L^*$ it is not obvious at all: we need the valuation conjecture, see the following Lemma 3.3.4. □

Lemma 3.3.4. *Suppose that the valuation conjecture holds for some fixed p, n and m, and choose $x_1, \ldots, x_n \in \wedge_p L^*$. If* $\mathrm{ced}(x_1, \ldots, x_n) \geq m$*, then there exist $1 \leq i_1 < \cdots < i_m \leq n$ such that* $\mathrm{ced}(x_{i_1}, \ldots, x_{i_m}) = m$.

Proof. If $x_1, \ldots, x_n$ are elements of L^* this is trivial, but for $\wedge_p L^*$ this is not obvious at all: we need the valuation conjecture. With the valuation conjecture, we find a rank m valuation v such that $(v(x_1)|\ldots|v(x_n))$ has rank m, and hence we can find a non singular square $m \times m$ submatrix. □

Lemma 3.3.4 shows how the valuation conjecture is necessary even for the easiest computations. Thanks to Proposition 3.3.3, for our application to the dimensional section conjecture we need to prove the valuation conjecture for $m = 2$. We have some results in this direction.

In rank 1 the conjecture is proven in a stronger sense (not locally at a prime, but globally), and it a consequence of the Mordell-Weil theorem, see Theorem 3.3.6. We think that this is particularly meaningful: the fact that the Mordell-Weil theorem is necessary tells us that an actual arithmetic obstruction is at work below the surface.

In rank 2 we have proven the conjecture for a large class of elements of $\wedge_p L^*$, see Theorem 3.4.3. Again, the Mordell-Weil theorem is crucial. The general case still eludes us.

3.3.1 Rank 1

Now we want to prove the valuation conjecture in rank 1.

Lemma 3.3.5. *Let L/k be any extension of fields, and take $x_1, \dots, x_n \in \widehat{L}^*$ (resp. $\wedge_p L^*$). Let $k' \subseteq L$ be the algebraic closure of k in L. Then, $\mathrm{ced}_k(x_1, \dots, x_n) = 0$ if and only if $x_1, \dots, x_n$ are in the image of $\widehat{k}'^* \to \widehat{L}^*$ (resp. $\wedge_p k'^* \to \wedge_p L^*$).*

Proof. This is a particular case of Proposition 2.3.3. □

To study the valuation conjecture, it turns out to be useful to have a birational version of the theory of divisors and Picard groups. This is done extensively in appendix B. The following are the useful facts we need to know.

- If L/k is a finitely generated extension, there are abelian groups $\mathrm{Pr}(L) \subseteq \mathrm{Div}(L)$ and $\mathrm{Pic}(L) = \mathrm{Div}(L)/\mathrm{Pr}(L)$ which are the direct limits of $\mathrm{Pr}(M) \subseteq \mathrm{Div}(M)$ and $\mathrm{Pic}(M) = \mathrm{Div}(M)/\mathrm{Pr}(M)$ while M varies among smooth projective models of L/k, with maps in the direct system defined by pullback of divisors.

- For an element $x \in \widehat{L}^*$, there exists a valuation $v : L^* \to \mathbb{Z}$ with $v(x) \neq 0 \in \widehat{\mathbb{Z}}$ if and only if $\mathrm{div}(x) \neq 0 \in \widehat{\mathrm{Div}(L)}$.

- For every smooth projective model M, the natural map $\mathrm{Pic}(M) \to \mathrm{Pic}(L)$ induces an isomorphism on torsion subgroups. As a consequence, if the Mordell-Weil theorem holds over k, then $\mathrm{Pic}(L)$ has finite torsion.

The following theorem shows that Conjecture 3.3.2 holds for $n = 1$ in a strong sense, i.e. not locally at a prime but globally.

Theorem 3.3.6. *Let L/k be finitely generated over $\mathbb{Q}$, and consider $x_1, \dots, x_n$ elements of $\widehat{L}^*$. Then, $\mathrm{ced}_k(x_1, \dots, x_n) \geq 1$ if and only if there exists a valuation $v : L^* \to \mathbb{Z}$ such that $v(x_i) \neq 0 \in \widehat{\mathbb{Z}}$ for some $i = 1, \dots, n$.*

Proof. Thanks to Lemma 3.3.5, $\mathrm{ced}(x_1, \ldots, x_n) \geq 1$ if and only if $\mathrm{ced}(x_i) \geq 1$ for some i, hence it is enough to prove this for $n = 1$, $x_1 = x$.

We already proved one implication in general, i.e. if such a valuation exists then $\mathrm{ced}(x) = 1$.

On the other hand, we have $x \in \widehat{L}^*$ such that $\mathrm{ced}\, x = 1$ and we want to find a valuation $v : L^* \to \mathbb{Z}$ such that $v(x) \neq 0 \in \widehat{\mathbb{Z}}$, i.e. we have to show that the image of x in $\widehat{\mathrm{Div}(L)}$ is $\neq 0$. Let $k' \subseteq L$ be the algebraic closure of k in L. We have two exact sequences:

$$0 \to k'^* \to L^* \to \mathrm{Pr}(L) \to 0$$

$$0 \to \mathrm{Pr}(L) \to \mathrm{Div}(L) \to \mathrm{Pic}(L) \to 0$$

Since $\mathrm{Pr}(L)$ is torsion free and $\mathrm{Pic}(L)$ has finite torsion (see Corollary 3.6.7), thanks to Lemma 3.7.1.(c) we know that also the following sequences are exact:

$$0 \to \widehat{k}'^* \to \widehat{L}^* \to \widehat{\mathrm{Pr}(L)} \to 0$$

$$0 \to \widehat{\mathrm{Pr}(L)} \to \widehat{\mathrm{Div}(L)} \to \widehat{\mathrm{Pic}(L)} \to 0.$$

Since $\mathrm{ced}_k(x) = 1$ thanks to Lemma 3.3.5 we know that x is not in the image of $\widehat{k}'^* \to \widehat{L}^*$, and hence it has nonzero image in $\widehat{\mathrm{Div}(L)}$. □

Example 3.3.7. While the implication $v(x) \neq 0 \Rightarrow \mathrm{ced}(x) = 1$ is clearly true for any extension of fields L/k, for the converse it is crucial that we can use the Mordell-Weil theorem. Here is a counterexample.

Let E be an elliptic curve over $\mathbb{C}$ with identity $e \in E(\mathbb{C})$ and p a prime number. For every $n \in \mathbb{N}$ choose $e_n \in E(\mathbb{C})[p^n]$ such that $pe_{n+1} = e_n$, with $e_0 = e$. This choice is what we cannot do if Mordell-Weil holds. Since $pe_{n+1} = e_n$, the divisor $p[e_{n+1}] - [e_n] - (p-1)[e_0]$ is principal, let f_n be such that $\mathrm{div}\, f_n = p[e_{n+1}] - [e_n] - (p-1)[e_0]$. Define

$$f = \prod_{n=0}^{\infty} f_n^{p^n} = f_0 \cdot f_1^p \cdot f_2^{p^2} \cdots \in \wedge_p \mathbb{C}(E)^*.$$

An easy computation shows that $\mathrm{div}(f) = 0$, i.e. $v(f) = 0 \in \mathbb{Z}_p$ for every discrete valuation $v : \mathbb{C}(E)^* \to \mathbb{Z}$, or equivalently f has order 0 at every closed point of E.

Still, $\mathrm{ced}\, f = 1$, i.e. f is "transcendental" over $\mathbb{C}$. In order to show this, it is enough to show that the image of f in $\mathbb{C}(E)^*/\mathbb{C}(E)^{*p}$ has essential dimension 1. We have that

$$[f] = [f_0] \in \mathbb{C}(E)^*/\mathbb{C}(E)^{*p}.$$

Suppose that $[f_0] \in \mathbb{C}(E)^*/\mathbb{C}(E)^{*p}$ has essential dimension 0. This means that there exists $g \in \mathbb{C}(E)^*$ such that $f_0 \cdot g^p$ is in $\mathbb{C}$. But then $\operatorname{div}(g) = -\operatorname{div}(f_0)/p = [e_0] - [e_1]$ which is absurd, since $[e_0] - [e_1]$ is not principal.

Corollary 3.3.8. *Let $E/L/k$ a tower of finitely generated extensions of $\mathbb{Q}$. If x is an element of $\widehat{L}^*$ and x_E its image in $\widehat{E}^*$, then $\operatorname{ced}_k x = \operatorname{ced}_k x_E$.*

Proof. Since clearly $\operatorname{ced}_k x \geq \operatorname{ced}_k x_E$, the only non trivial case is when $\operatorname{ced}_k(x) = 1$. Then, take a valuation v on L such that $v(x) \neq 0$. We can extend v to a valuation v' on E such that $v'|_L = dv$ for some $d > 0$. But then $v'(x_E) = dv(x) \neq 0$.

This can also be seen as a consequence of Lemma 3.3.5 and Lemma 2.4.6. □

Corollary 3.3.8 is yet another fact that shows how the situation is different from usual essential dimension, since every torsor becomes trivial (and hence of essential dimension 0) after a suitable extension.

Corollary 3.3.9. *Let $x \in \widehat{L}^*$ with L/k finitely generated over $\mathbb{Q}$. Then $\operatorname{ced}(x) = \sup_p \operatorname{ced}(x_p)$, where $x_p \in \wedge_p L^*$ is the image of x through $\widehat{L}^* \to \wedge_p L^*$.*

Proof. If $\operatorname{ced}(x) = 0$, this is obvious. If $\operatorname{ced}(x) = 1$, then there exists some discrete valuation v with $v(x) \neq 0 \in \widehat{\mathbb{Z}}$. Since $\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$, there exists some p such that $v(x_p) = v(x)_p \neq 0 \in \mathbb{Z}_p$, thus $\operatorname{ced}(x_p) = 1$. □

3.4 The valuation conjecture in rank 2. A partial result

Now we want to study the valuation conjecture in rank 2. There are two completely different situations that we are going to specify soon: the one of bounded degree and the one of unbounded degree. In this thesis we only address the case of bounded degree.

In bounded degree, the proof is complete modulo a result of pure birational geometry which at the moment we are able to prove only for surfaces, i.e. the fact that an hypersurface becomes unmovable after enough blow ups along codimension 2 centers contained in it, see MathOverflow 306537. Since this fact is easy to prove for surfaces, if L/k has transcendence degree 2 then the proof is complete.

While the previous sections were more expository, this whole section is devoted to the proof of Theorem 3.4.3 and it is much more technical. For the rest of the section, we fix fields $L/k/\mathbb{Q}$ finitely generated over $\mathbb{Q}$ and $x_1, \ldots, x_n \in \wedge_p L^*$ such that $\det(v(x_i)|v(x_j)) = 0 \in \mathbb{Z}_p$ for every rank 2 valuation v of L/k and every $1 \leq i < j \leq n$. Our target is to show that $\operatorname{ced}(x_1, \ldots, x_n) < 2$, i.e. that for every d we may approximate $x_1, \ldots, x_n$ up to p^d with elements of a subextension $L/E/k$ with $\operatorname{trdeg}_k E \leq 1$.

3.4.1 Bounded degree

As we mentioned above, we are going to work under the additional hypothesis of *bounded degree*. In Remark 3.4.4 we are going to discuss briefly what happens when the degree is unbounded.

Recall that if M is a projective variety with a fixed ample divisor H, we may define the degree of a dimension d cycle Z on M by $\int H^{\cap d} \cap Z \in \mathbb{Z}$.

Lemma 3.4.1. *Let M be a projective variety over k, H an ample divisor and S a set of hypersurfaces. Let $\sim$ be the relation of numerical equivalence between hypersurfaces. Then*

1. *the elements of S have bounded degree with respect to H if and only if $S/\sim$ is finite,*

2. *if M' is another projective variety, $\pi : M' \to M$ a birational morphism and $\widetilde{S}$ the set of proper transforms of hypersurfaces in S, then $S/\sim$ is finite if and only if $\widetilde{S}/\sim$ is finite.*

Proof. 1. We have that $S/\sim$ is a subset of the sharp cone of effective divisors in the Néron-Severi group. Since the Néron-Severi group is finitely generated and the degree is strictly positive on effective nonzero divisors, then it is clear that $S/\sim$ is finite if and only if the degree is bounded on $S/\sim$, and hence on $S/\sim$.

2. Since pushforward respects numerical equivalence, if $\widetilde{S}/\sim$ is finite then $S/\sim$ is finite too. We want to show the converse. Let Exc be the set hypersurfaces in M' contracted by $\pi : M' \to M$. For every $V_i \in S$, we can write

$$\pi^* V_i = \widetilde{V}_i + \sum_{E \in Exc} m_{E,i} E_i$$

for some $m_{E,i}$, where $\widetilde{V}_i$ is the proper transform of V_i.

Claim: there exists an $N > 0$ such that $m_{E,i} < N$ for all i and for all $E \in Exc$. Since the set Exc is finite, we can focus ourselves on a single divisor $E \in Exc$.

If n is the dimension of M' and H' is very ample on M', the intersection of $n-1$ generic elements of $|H'|$ is a curve $C' \subseteq M'$ which has strictly positive intersection with every hypersurface. In particular, we have $d = C \cdot E > 0$. By projection formula,

$$m_{E,i} \cdot d \leq C \cdot \pi^* V_i = \pi_* C \cdot V_i.$$

Since $\pi_* C \cdot V_i$ depends only on the class of numerical equivalence of V_i, its value is bounded with respect to i, hence $m_{E,i}$ is bounded with respect to i too.

Thanks to what we have proved, while i varies there is only a finite number of possibilities for the divisor $\sum_{E \in Exc} m_{E,i} E$. Since, by hypothesis, the classes of numerical equivalence of V_i (and hence of $\pi^* V_i$) are in finite number, we get that there is only a finite number of possible classes of numerical equivalence of $\widetilde{V}_i$.

□

Definition 3.4.2. Let $x \in \wedge_p L^*$ an element. We say that x has *bounded degree* if the hypersurfaces in the support of $\operatorname{div} x$ have bounded degree for some projective model M of L with respect to some ample divisor H on M. Thanks to Lemma 3.4.1, this definition does not depend on M and H. We say that x has *finite support* if there is only a finite number of hypersurfaces in the support of $\operatorname{div} x$.

Let us now state the theorem.

Theorem 3.4.3. *Let L/k be fields finitely generated over $\mathbb{Q}$, and $x_1, \ldots, x_n \in \wedge_p L^*$ such that $\det(v)(x_i, x_j) = 0$ for every rank 2 valuation v of L/k and every $1 \leq i < j \leq n$. Suppose that $x_1, \ldots, x_n$ have bounded degree and $\operatorname{trdeg}_k L = 2$, or that $x_1, \ldots, x_n$ have finite support (with no restrictions on $\operatorname{trdeg}_k L$). Then $\operatorname{ced}_k(x_1, \ldots, x_n) < 2$.*

Remark 3.4.4. To get a feeling of what happens when the degree is unbounded, think of $\mathbb{P}^2$ and $n = 2$. Then $\operatorname{div} x_1$, $\operatorname{div} x_2$ are supported on a infinite number of curves of higher and higher degree. Our hypothesis on rank 2 valuations gives us a piece of information every time two curves of the support meet each other: if the degree is unbounded, we have an abundance of these points. This rigidifies the situation, up to the point that we think the only possibility is that $x_2 = \alpha x_1^\lambda$ with α algebraic over k and $\lambda \in \mathbb{Z}_p$ (or vice versa). If this is true, then it is immediate to check that, even in this case, $\operatorname{ced}_k(x_1, x_2) < 2$.

3.4.2 Label and weight

Let S be the set of discrete valuations $v : L^* \to \mathbb{Z}$ trivial on k such that $v(x_i) \neq 0$ for at least one $i = 1, \ldots, n$. If M is a projective model of M, S_M is the set of hypersurfaces in the support of $\operatorname{div}(x_i)$ for some i: if v is the valuation associated to an hypersurface $V \subseteq M$, then $V \in S_M$ if and only if $v \in S$. Let $\nu_p : \mathbb{Q}^* \to \mathbb{Z}$ the p-adic valuation.

Definition 3.4.5. For each $v \in S$, the *label* $\lambda(v)$ of v is

$$\lambda(v) = [v(x_1) : \cdots : v(x_n)] \in \mathbb{P}^{n-1}(\mathbb{Q}_p).$$

We say that v and v' are friends if they have the same label, they are enemies otherwise.

For every discrete valuation v, even not in the support, the *weight* $\omega(v)$ is the p-adic norm

$$\omega(v) = \left|\left(v(x_1), \ldots, v(x_n)\right)\right|_p = p^{-\min_i\{\nu_p(v(x_i))\}}.$$

We have that $\omega(v) = 0$ if and only if $v \notin S$.

In the following, if M is a model of L/k and $V \subseteq M$ is an hypersurface, we often confuse between V and the discrete valuation associated to it. For example, we will talk about the label and the weight of an hypersurface.

Before going on, we want to give an idea of why we have introduced the definition of label and weight. Suppose $V, W \subseteq M$ are hypersurfaces in the support, and suppose that $C \subseteq V \cap W$ is a codimension 2 subvariety such that V and W are the only hypersurfaces in the support containing C. Let now $\bar{V}$ be the normalization of V, and $\bar{C} \subseteq \bar{V}$ an hypersurface of $\bar{V}$ which dominates $C \subseteq V$. The pair $(V, \bar{C})$ defines a rank 2 valuation $k(M)^* \to \mathbb{Z}^2$. If we apply our hypothesis on rank 2 valuation to v, a simple computation shows that $\lambda(V) = \lambda(W)$.

In general, the situation will not be so simple, for instance there will be a lot (maybe infinite) hypersurfaces in the support containing C. Still, every time two hypersurfaces V, W meet the hypothesis on rank 2 valuation will give us a clue pointing toward the fact that $\lambda(V) = \lambda(W)$. The more an hypersurface has weight, the more it has power to "attract" the hypersurfaces it meets toward its label.

The metric on the space of labels If $[a_1 : \cdots : a_n] \in \mathbb{P}^{n-1}(\mathbb{Q}_p)$, we say that $[a_1 : \cdots : a_n]$ is in *canonical form* if $a_1 \ldots, a_n \in \mathbb{Z}_p \subseteq \mathbb{Q}_p$ and p does not divide at least one of them. The canonical form exists and is unique up to elements of $\mathbb{Z}_p^*$.

The n subsets $U_{i_0} = \{[a_1 : \cdots : a_n] \in \mathbb{P}^{n-1}(\mathbb{Q}_p) | \nu_p(a_{i_0}) \leq \nu_p(a_i) \forall i\}$ of $\mathbb{P}^{n-1}(\mathbb{Q}_p)$ are each canonically isomorphic to $\oplus_{n-1}\mathbb{Z}_p$ via $[a_1 : \cdots : a_n] \mapsto (a_1/a_{i_0}, \ldots, a_n/a_{i_0})$, we call these subsets the canonical charts of $\mathbb{P}^{n-1}(\mathbb{Q}_p)$. The intersection of the $n-1$ canonical charts is $\oplus_{n-1}\mathbb{Z}_p^*$ in all charts.

Lemma 3.4.6. *Let $[a_1 : \cdots : a_n], [b_1 : \cdots : b_n] \in \mathbb{P}^{n-1}(\mathbb{Q}_p)$ be in canonical form. Then*

$$d\left([a_1 : \cdots : a_n], [b_1 : \cdots : b_n]\right) = \max_{i,j} \left| a_i b_j - a_j b_i \right|_p$$

defines an ultrametric distance on $\mathbb{P}^{n-1}(\mathbb{Q}_p)$ with diameter 1 whose restriction to each of the canonical charts is the usual p-adic metric of $\oplus_{n-1}\mathbb{Z}_p$, but the restriction to $\oplus_{n-1}\mathbb{Q}_p \subseteq \mathbb{P}^1(\mathbb{Q}_p)$ is not the p-adic metric of $\oplus_{n-1}\mathbb{Q}_p$.

Proof. Suppose that $[a_1 : \cdots : a_n]$, $[b_1 : \cdots : b_n]$ are in the same chart, say $\nu_p(a_1) \leq \nu_p(a_i)$ and $\nu_p(b_1) \leq \nu_p(b)$. Then, since they are in canonical form $\nu_p(a_1) = \nu_p(b_1) = 0$, and we may suppose that $a_1 = b_1 = 1$. Let i, j be indices such that

$$d\left([1 : a_2 : \cdots : a_n], [1 : b_2 : \cdots : b_n]\right) = \left| a_i b_j - a_j b_i \right|_p.$$

We have

$$\left| a_i b_j - a_j b_i \right|_p = \left| b_j(a_i - a_j) + a_j(b_j - b_i) \right|_p \leq \max\{|a_i - a_j|_p, |b_i - b_j|_p\}$$

and hence the restriction of d to each canonical chart is just the ultrametric distance on $\oplus_{n-1}\mathbb{Z}_p$.

Suppose now that they are not in the same chart, we may suppose for instance $a_1 = 1, b_2 = 1$ and $\nu_p(a_2), \nu_p(b_1) > 0$. Then $|1 - a_2 b_1|_p = 1$ achieves the maximum in the definition of the distance, and thus the distance is 1. From this, it is easy to check the ultrametric inequality by cases, and it is immediate that the diameter is 1 since $\oplus_{n-1}\mathbb{Z}_p$ has diameter 1. The restriction of the metric to $\oplus_{n-1}\mathbb{Q}_p$ is not the usual p-adic metric, for example because the p-adic metric on $\mathbb{Q}_p$ has not finite diameter. $\square$

Observe that, if V and W are two hypersurfaces in the support with associated valuations v, w, by unravelling the definitions we get

$$\omega(V) \cdot \omega(W) \cdot d(\lambda(V), \lambda(W)) = \max_{i,j} |v(x_i)w(x_j) - w(x_i)v(x_j)|_p$$

3.4.3 Domination

Recall that our ground hypothesis is that, for every rank 2 valuation $v : L^* \to \mathbb{Z}^2$, the determinant of $(v(x_i)|v(x_j))$ is 0 for every i, j. How do we use this hypothesis? As we have said in the previous subsection, if two hypersurfaces V, W are the only ones in the support passing through a codimension 2 subvariety C, then $\lambda(V) = \lambda(W)$, but the general situation is much more complicated.

What we are going to do is to define what it means for an hypersurface to *dominate* one of its points: we will show in Proposition 3.4.12 that if an hypersurface V dominates a point q, then for every other hypersurface V' containing q we have $\lambda(V) = \lambda(V')$, and moreover that V dominates an open subset. Once this is done, we will forget the hypothesis about rank 2 valuations, and replace it with the much more tractable concept of domination.

Definition 3.4.7. Let M be a model and V an hypersurface with normalization $\eta : \overline{V} \to V$, and $q \in \overline{V}$ a point of codimension 1. Let V' another surface, and $f \in \mathcal{O}_{M,\eta(q)}$ a local equation for $V' \subseteq M$. Then the *fine intersection multiplicity* $\iota_q(V, V')$ is the valuation of $\eta^\# f \in \mathcal{O}_{\overline{V},q}$, where $\mathcal{O}_{\overline{V},q}$ is a DVR since $\overline{V}$ is normal and p has codimension 1. Since M is smooth, f is well defined up to invertible elements of $\mathcal{O}_{M,\eta(q)}$, hence the definition of fine intersection multiplicity does not depend on the choice of f.

Remark 3.4.8. We have to pay attention: if M is a surface, fine intersection multiplicity is closely related to usual intersection multiplicity, but it's different. For example, it is not symmetric with respect to the curves. If the curve C is regular at a rational point p, then $\iota_q(C, C')$ is indeed usual intersection multiplicity.

In general, on a surface the fine intersection multiplicity is less than or equal to the usual intersection multiplicity: this is because fine intersection multiplicity does not take into account neither the degree of the residue field of the point nor the fact that the usual intersection multiplicity involves several points in the normalization of the curve, not just one.

Lemma 3.4.9. *We have that*

$$\iota_q(V, V') \leq \deg V \cap V'$$

for every V, V', q as above.

Proof. We have a fixed ample divisor H which we use to define degrees, let n be the dimension of M. Since $\overline{\eta(q)} \subseteq V \cap V'$ has codimension 2, by $n - 2$

generic cuts with the ample divisor we reduce to the case in which $\overline{\eta(q)} \cap H^{n-2}$ has dimension 0, i.e. $M \cap H^{n-2}$ is a surface and $V \cap H^{n-2}$, $V' \cap H^{n-2}$ are curves.

Now, fine intersection multiplicity in q is bounded by the usual intersection multiplicity in anyone of the finite points of $\overline{\eta(q)} \cap H^{n-2}$, which in turn is bounded by the degree of the intersection $(V \cap H^{n-2}) \cap (V' \cap H^{n-2})$, and this is exactly $\deg V \cap V'$. □

Definition 3.4.10. Let M be a model. Since by hypothesis the support has bounded degree, let ε_M be the minimum of $|\deg V_1 \cap V_2|_p$ while V_1, V_2 vary among the hypersurfaces in the support, we have $\varepsilon_M > 0$.

Now take V an hypersurface and $q \in V$ a point. We say that V *dominates* q if for every other hypersurface W with normalization $\eta : \overline{W} \to W$ and for every codimension 1 point $q' \in \overline{W}$ such that $q \in \overline{\eta(q')}$, then

$$\varepsilon_M \cdot \omega(V) > \omega(W).$$

Essentially, for V to dominate a point p we ask it to have more weight than all the other hypersurfaces passing through q, with a correction to weights given by the maximum of the degrees of intersections. Thanks to Lemma 3.4.9, this correction will let us manage problems arising from the fact that the hypersurfaces do not intersect transversally.

Lemma 3.4.11. *The set of points of V not dominated by V is closed of pure codimension* 1*.*

Proof. Let W be an hypersurface in the support different from V with a point q of codimension 1 in the normalization of W and such that

$$\omega(W) \geq \varepsilon_M \cdot \omega(V).$$

For every such W and every such q, we get an irreducible component of $V \cap W$ (i.e. $\overline{\eta(q)}$) whose points are not dominated by V, and every point not dominated by V is in such a component for some W and some q. Hence, it is enough to show that there is only a finite number of such hypersurfaces W. But there is only a finite number of hypersurfaces in the support with weight greater than any fixed positive constant, hence we conclude.

□

The following Proposition 3.4.12 is the whole point of the concept of domination: along with Lemma 3.4.11, it shows that an hypersurface V can meet enemy hypersurfaces (i.e. with different label) only in a proper closed subset.

Proposition 3.4.12. *Let M be a model of L/k, V an hypersurface in the support and $q \in V$ a point dominated by V. Then for every hypersurface W in the support containing q we have $\lambda(V) = \lambda(W)$.*

Proof. Observe that it is enough to prove this for $n = 2$, i.e. only two elements $x_1, x_2 \in \wedge_p L^*$: this will let us simplify the proof a lot. Suppose that we have done the case $n = 2$. If we have n elements $x_1, \dots, x_n \in \wedge_p L^*$, there exists some $1 \leq i \leq n$ such that $\omega(V) = |v(x_i)|_p$, where v is the discrete valuation associated to V. Then, if we restrict the definitions of label, weight and domination to the pair (x_i, x_j) for any other index j, we have $\omega_{i,j}(V) = \omega(V)$ and hence V still dominates q. This implies that, if $\lambda_{i,j}$ is the restricted label, $\lambda_{i,j}(V) = \lambda_{i,j}(W)$ for every j such that W is in the support of x_i, x_j, and from this it is immediate to obtain $\lambda(V) = \lambda(W)$.

Hence, let us suppose now that $n = 2$. Since the intersection of two hypersurfaces has pure codimension 2 in M, we may suppose that q is a point of height 2 in M, and height 1 in W. Choose any point $\bar{q} \in \overline{W}$ over q, where $\overline{W}$ is the normalization of W. The pair $(\overline{W}, \bar{q})$ defines a rank 2 valuation $\mathfrak{w} : L^* \to \mathbb{Z}^2$. We want to use our hypothesis on the vanishing of the determinant of this valuation.

Let $\{W_i\}_i$ be the set of hypersurfaces different from V and W containing q. Call v, w, w_i the respective discrete valuations. Then the matrix which, by hypothesis, has vanishing determinant is the transpose of:

$$\begin{bmatrix} w(x_1) & \iota_{\bar{q}}(W,V)v(x_1) + \sum_i \iota_{\bar{q}}(W,W_i)w_i(x_1) \\ & \\ w(x_2) & \iota_{\bar{q}}(W,V)v(x_2) + \sum_i \iota_{\bar{q}}(W,W_i)w_i(x_2) \end{bmatrix}$$

i.e. we have

$$\iota_{\bar{q}}(W,V)\left(w(x_1)v(x_2) - v(x_1)w(x_2)\right) =$$
$$= -\sum_i \iota_{\bar{q}}(W,W_i)\left(w(x_1)w_i(x_2) - w_i(x_1)w(x_2)\right)$$

Recall that out final target is to show that $\lambda(V) = \lambda(W)$: we want to use the equality above to show that the distance between $\lambda(V)$ and $\lambda(W)$ is "little". By taking p-adic norms in the equality above, we get

$$\left|\iota_{\bar{q}}(W,V) \cdot \left(w(x_1)v(x_2) - v(x_1)w(x_2)\right)\right|_p =$$

$$= \left|\iota_{\bar{q}}(W,V)\right|_p \cdot \omega(V) \cdot \omega(W) \cdot d(\lambda(V), \lambda(W))$$

and

$$\left| -\sum_i \iota_{\bar{q}}(W, W_i) \left(w(x_1)w_i(x_2) - w_i(x_1)w(x_2) \right) \right|_p \leq$$

$$\leq \omega(W) \cdot \max_i \left(\left| \iota_{\bar{q}}(W, W_i) \right|_p \cdot \omega(W_i) \cdot d(\lambda(W_i), \lambda(W)) \right) \leq$$

$$\leq \omega(W) \cdot \max_i \omega(W_i) \cdot \max_i d(\lambda(W_i), \lambda(W)) <$$

$$< \omega(W) \cdot \varepsilon_M \cdot \omega(V) \cdot \max_i d(\lambda(W_i), \lambda(W)),$$

where in the last inequality we have used the hypothesis that V dominates q and hence $\omega(W_i) < \varepsilon_M \cdot \omega(V)$. To impose the strict inequality, we are also making the assumption that $\max_i d(\lambda(W_i), \lambda(W)) \neq 0$: this is safe, because if it is 0 then also $d(\lambda(V), \lambda(W)) = 0$ by the inequality above and hence $\lambda(V) = \lambda(W)$ as desired.

Anyway, putting all the information together and simplifying $\omega(V)$ and $\omega(W)$, we get

$$\left| \iota_{\bar{q}}(W, V) \right|_p \cdot d(\lambda(V), \lambda(W)) < \varepsilon_M \cdot \max_i d(\lambda(W_i), \lambda(W)),$$

and, using that $\left| \iota_{\bar{q}}(W, V) \right|_p \geq \varepsilon_M$ thanks to Lemma 3.4.9 and the definition of ε_M,

$$d(\lambda(V), \lambda(W)) < \max_i d(\lambda(W_i), \lambda(W)).$$

The distance d has diameter 1 and takes values in the set $\{1, p^{-1}, p^{-2}, \ldots, 0\}$. In particular, we get

$$d(\lambda(V), \lambda(W)) < 1,$$

which is equivalent to

$$d(\lambda(V), \lambda(W)) \leq p^{-1}.$$

But we can now repeat the argument replacing W with W_i for every i, and hence $d(\lambda(V), \lambda(W_i)) \leq p^{-1}$ too. By ultrametric inequality, this tells us that $d(\lambda(W_i), \lambda(W)) \leq p^{-1}$. Now we apply again the inequality above, and we get

$$d(\lambda(V), \lambda(W)) < \max_i d(\lambda(W_i), \lambda(W)) \leq p^{-1},$$

which is equivalent to

$$d(\lambda(V), \lambda(W)) \leq p^{-2}.$$

We can go on by induction, and thus we get that

$$d(\lambda(V), \lambda(W)) \leq p^{-n}$$

for every $n > 0$, i.e. $d(\lambda(V), \lambda(W)) = 0$ as desired.

□

Recall that by definition two hypersurfaces are friends if they have the same label, and enemies otherwise.

Corollary 3.4.13. *Every hypersurface in the support meets enemies only in a finite number of codimension* 1 *subvarieties.*

Proof. Just put together Lemma 3.4.11 and Proposition 3.4.12. □

3.4.4 Separating labels

Definition 3.4.14. An hypersurface V of a model M is *bad* if there exists an enemy hypersurface W with $V \cap W \neq 0$.

In this subsection we prove the following proposition.

Proposition 3.4.15. *Suppose that x_1 and x_2 have finite support, or that they have bounded degree and* $\operatorname{trdeg}_k L = 2$*. Then there exists a model M without bad hypersurfaces.*

Lemma 3.4.16. *Let M be a model. Then there exists a finite set $\mathcal{Q}$ of codimension* 2 *subvarieties of M such that every bad hypersurface contains one element of $\mathcal{Q}$.*

Proof. Observe that the whole point of this lemma is that $\mathcal{Q}$ is finite: while there may be an infinite number of bad hypersurfaces, they all pass through a finite number of codimension 2 subvarieties of M.

Since we are in bounded the degree, there is only a finite number of algebraic equivalence classes of hypersurfaces in the support, hence we can focus on a single class $[V]$ where V is a bad hypersurface. Let W be an enemy hypersurface such that $V \cap W \neq \emptyset$. Thanks to Corollary 3.4.13, there is only a finite number of hypersurfaces Q of W where W can meet enemies, put all of them in $\mathcal{Q}$.

Now, if every bad hypersurface algebraically equivalent to V is an enemy of W, we have finished, since they all have nonempty intersection with W, hence they all contain an element of $\mathcal{Q}$. If $V' \sim V$ is bad and friend with W, just choose W' enemy of V' with $V' \cap W' \neq \emptyset$ and put in $\mathcal{Q}$ the finite set of hypersurfaces Q' of W' where W can meet enemies. Now W and W' are enemies and they have nonempty intersection with every element of $[V]$, hence we conclude. □

Lemma 3.4.17. *Let* $\operatorname{trdeg}_k L = 2$*. Then there exists a model* M *with a finite number of bad curves.*

Proof. Let M_0 be any smooth, projective model of L/k, it is a surface. Thanks to Lemma 3.4.16, there exists a finite number of points of M such that every bad curve contains at least one of them: let $M_1 \to M_0$ be the blow up of all of them. Now the possible bad curves of M_1 are either exceptional divisors or proper transform of bad curves of M_0. In the first case, they have negative self intersection. In the second case, the self intersection of the proper transform is strictly less than the self intersection of the curve on M_0, since every bad curve of M_0 passed through one of the points we have blown up.

Hence, after a finite number of blow ups, since we are in bounded degree and hence the self intersection is bounded too, we get that every bad curve has negative self intersection. But a curve with negative self intersection is unique in its numerical equivalence class, and by hypothesis x_1 and x_2 are supported on a finite number of numerical equivalence classes, hence we get the thesis. □

Remark 3.4.18. It should be possible to generalize Lemma 3.4.17 to dimension higher than 2. For example, such a generalization follows from a positive answer to MathOverflow 306537. We stress out that this missing part is purely a problem of birational geometry, it has nothing to do with all the machinery introduced here.

Thanks to Lemma 3.4.17, we can complete the proof of Proposition 3.4.15 using resolution of singularities.

In fact, let $V_1, \dots, V_r \subseteq M$ the bad hypersurfaces in the model. If $\operatorname{trdeg}_k L = 2$ there is a finite number of them thanks to Lemma 3.4.17, otherwise by hypothesis. Apply embedded resolution of singularities to $V_1 \cup \cdots \cup V_r$: we get a birational projective morphism $M' \to M$ such that the pullback of $V_1 \cup \cdots \cup V_r$ is normal crossing. In particular, we may suppose that all the bad hypersurfaces are normal crossing: but in this situation there cannot be any bad hypersurfaces.

In fact, let V, V' be two enemy hypersurfaces with nonempty intersection, and let $C \subseteq V \cap V'$ be an irreducible component. Since bad hypersurfaces are normal crossing and V, V' are enemies, no other hypersurface in the support can contain C (if W is such an hypersurface, then W is enemy of at least one between V and V', and then W is bad, which is contrast with the fact that bad hypersurfaces are normal crossing). But then if we apply directly our hypothesis on rank 2 valuations to the rank 2 valuation associated to the pair (V, C), we get that $\lambda(V) = \lambda(V')$, which is absurd since we are supposing that V and V' are enemies. $\square$

3.4.5 Approximating p-adic principal divisors

Fix a positive integer d. We want to find $x_{1,d}, \ldots, x_{n,d} \in L^*$ such that $x_{i,d} \cong x_i \pmod{L^{*p^d}}$ and $\operatorname{trdeg}_k k(x_{1,d}, \ldots, x_{n,d}) \leq 1$. Thanks to Proposition 3.4.15, there exists a model M such that, if V and V' are two hypersurfaces in the support with nonempty intersection, then $\lambda(V) = \lambda(V')$. Recall that S_M is the union of the supports of $\operatorname{div}(x_1), \ldots, \operatorname{div}(x_n)$.

Before starting, we want to make a simplification: later, it will be a problem if $[0 : a_2 : \cdots : a_n] \in \mathbb{P}^{n-1}(\mathbb{Q}_p)$ is actually a label for some hypersurface in the support and some $a_2, \ldots, a_n \in \mathbb{Q}_p$, we want to show that up to a change of coordinates we can rule out this possibility. Observe that $\mathbb{Z}_p$ is uncountable, while the number of hypersurfaces is countable (because M is of finite type over a field which is finitely generated over $\mathbb{Q}$), hence there exists an hyperplane of $\mathbb{P}^{n-1}(\mathbb{Q}_p)$ not containing any label. In formulas, there exist $\eta_2, \ldots, \eta_n \in \mathbb{Z}_p$ such that the support of

$$\operatorname{div}(x_1 \cdot x_2^{\eta_2} \cdots x_n^{\eta_n})$$

contains the supports of $\operatorname{div}(x_1), \ldots, \operatorname{div}(x_n)$, which is equivalent to saying that up to replacing x_1 with $x_1 \cdot x_2^{\eta_2} \cdots x_n^{\eta_n}$ the first coordinate of $\lambda(V)$ is nonzero for every hypersurface V in the support. Observe that hypotheses and thesis are equivalent after the replacement, hence we can safely do this, and suppose that the support of $(x_1, \ldots, x_n)$ is equal to the support of x_1.

Let $\operatorname{Div}(S_M) \subseteq \operatorname{Div}(M)$ be the subgroup of divisors supported on S_M and $\operatorname{Pic}(S_M)$ its image in $\operatorname{Pic}(M)$. The kernel $\operatorname{Pr}(S_M)$ of $\operatorname{Div}(S_M) \to \operatorname{Pic}(S_M)$ is the group of principal divisors supported on S_M. We have a commutative diagram with exact rows and injective vertical arrows:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Pr}(S_M) \arrow[r] \arrow[d, hook] & \operatorname{Div}(S_M) \arrow[r] \arrow[d, hook] & \operatorname{Pic}(S_M) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \operatorname{Pr}(M) \arrow[r] & \operatorname{Div}(M) \arrow[r] & \operatorname{Pic}(M) \arrow[r] & 0
\end{tikzcd}$$

We have that $\operatorname{Pic}(S_M)$, $\operatorname{Pic}(M)$ and $\operatorname{Pic}(M)/\operatorname{Pic}(S_M)$ are finitely generated and $\operatorname{Div}(M)/\operatorname{Div}(S_M)$ is free, hence by applying the functor $\wedge_p$ thanks to Lemma 3.7.1.c we still have a diagram with exact rows and injective vertical arrows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \wedge_p \operatorname{Pr}(S_M) & \longrightarrow & \wedge_p \operatorname{Div}(S_M) & \longrightarrow & \wedge_p \operatorname{Pic}(S_M) & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & \wedge_p \operatorname{Pr}(M) & \longrightarrow & \wedge_p \operatorname{Div}(M) & \longrightarrow & \wedge_p \operatorname{Pic}(M) & \longrightarrow & 0 \end{array}$$

Clearly, $\operatorname{div}(x_i) \in \wedge_p \operatorname{Div}(S_M)$ for $i = 1, \dots, n$ and it maps to 0 in $\wedge_p \operatorname{Pic}(S_M)$ and hence $\operatorname{div}(x_i) \in \wedge_p \operatorname{Pr}(S_M)$. This means that we can approximate $\operatorname{div}(x_i)$ with principal divisors supported on S_M.

3.4.6 Approximating $\operatorname{div}(x_1)$

We want now to use the exact sequences above in order to approximate $\operatorname{div}(x_1)$ up to p^{d+*}, where $*$ is a sum of certain positive constants that will be useful later in the proof. The reader may skip for a moment how these constants are chosen, and come back when the constants come into play.

Let $\{S_i\}_i$ be the set of connected components of S_M. There is only a finite number of them, say $S_1, \dots, S_r$, such that in S_i there is an hypersurface V of weight $\omega(V)$ greater or equal to p^{-d}.

Choosing a Since we have reduced ourselves to the case in which S_M is equal to the support of $\operatorname{div}(x_1)$, for every $i = 1, \dots, r$ and every $j = 2, \dots, n$ there exists $\lambda_{i,j} \in \mathbb{Q}_p$ such that

$$\operatorname{div}(x_j)|_{S_i} = \lambda_{i,j} \operatorname{div}(x_1)|_{S_i}.$$

Choose a such that $|\lambda_{i,j}|_p \le p^a$ for every $i = 1, \dots, r$ and $j = 2, \dots, n$.

Choosing b Choose C any smooth curve in M such that C is not contained in any of the S_i but it intersects all of them. This can be done by taking a suitable power of an ample divisor. For $i = 1, \dots, r$, let $b_i \ge 0$ be the p-adic valuation of

$$\deg(\operatorname{div}(x_1)|_{S_i} \cap C).$$

Thanks to the choice of C the degree above is different from 0, and hence $b_i \ne +\infty$ for every i. Then choose b such that $b \ge b_i$ for every $i = 1, \dots, r$.

Choosing c **and** c' Consider $\operatorname{Pic}^0(M)$. Thanks to Faltings' proof of the Tate conjecture, there is only a finite number of abelian varieties isogenous to $\operatorname{Pic}^0(M)$. From this, it is easy to prove that there is only a finite number of abelian varieties A with a surjective homomorphism $\mathrm{Pic}^0(M) \to A$. These abelian varieties A have finite torsion thanks to Mordell-Weil. Let c be such that $p^c\alpha = 0$ for every abelian variety A which is a quotient of $\mathrm{Pic}^0(M)$ and every torsion element $\alpha \in A(k)$ whose order is a power of p.

For c', let it be a positive integer such that $p^{c'}\beta = 0$ for every torsion element $\beta \in \mathrm{NS}(M)$ in the Néron-Severi group whose order is a power of p.

The idea of the proof is the following: we are going to start by approximating $\mathrm{div}(x_1)$ with a principal divisor $D_{1,d}$ up to p^{d+a+b}, and then define a candidate divisor $D_{j,d}$ for the approximation of $\mathrm{div}(x_j)$ such that $D_{1,d}$ and $D_{j,d}$ respect the criterion on rank 2 valuations. Once this is done, we will have a problem: $D_{j,d}$ is not principal. We will then proceed to modify $D_{j,d}$ one step at time, and at each of these steps we will lose some precision in the approximation of $\mathrm{div}(x_j)$.

In its last incarnation, the divisor $D_{j,d}$ will still satisfy the criterion on rank 2 valuations together with $D_{1,d}$, it will be principal, and it will approximate $\mathrm{div}(x_j)$ up to $p^{d-c-c'}$. Since the constants c, c' do not depend on d, we have that $d - c - c'$ goes to infinity with d, and then the approximation works. This passage is very delicate: the constants a, b do depend on d, if this was true even for c, c' the proof would have failed.

Since we have shown above that we can approximate $\mathrm{div}(x_1)$ with principal divisors supported on S_M, let $D_{1,d} \in \mathrm{Div}(S_M)$ be a principal divisor such that

$$D_{1,d} \cong \mathrm{div}(x_1) \pmod{p^{d+a+b}}.$$

Now choose $f \in L^*$ any rational function such that $\mathrm{div}\, f = D_{1,d}$. Up to enlarging M, f defines a morphism $f : M \to \mathbb{P}^1$. Observe that:

- if we enlarge M, it remains true that the new model separates labels;
- the set $\{\lambda_{i,j}\}_{i,j}$ does not change, hence the constant a remains the same;
- thanks to projection formula, if we replace C with its proper transform then the degrees of $\mathrm{div}(x_1)|_{S_i} \cap C$ and $D_{1,d}|_{S_i} \cap C$ do not change, hence the constant b remains the same;
- $\mathrm{Pic}^0(M)$ is a birational invariant, hence the constant c does not change;

- the torsion of the Néron-Severi group is a birational invariant, hence c' does not change, too.

By definition, $D_{1,d} = f^*[0] - f^*[\infty]$. Let $g : M \to X$ be the Stein factorization of f, where X is a smooth curve with a finite morphism $X \to \mathbb{P}^1$. By pulling back the divisor $[0] - [\infty]$ along $X \to \mathbb{P}^1$, we get that $D_{1,d} = g^* E_{1,d}$ for some principal divisor $E_{1,d}$ on X. Write

$$E_{1,d} = \sum_i e_i \cdot [p_i]$$

for some $e_i \in \mathbb{Z}$, $e_i \neq 0$, and p_i closed points in X, hence

$$D_{1,d} = \sum_i e_i \cdot g^*[p_i].$$

By construction, each $g^*[p_i] = [g^{-1}(p_i)]$ is connected and supported on S_M, hence it is contained in a connected component S_i of S_M. For $i = 1, \ldots, r$ and $j = 2, \ldots, n$, choose $\lambda_{i,j,d} \in \mathbb{Z}[1/p]$ a rational number with a denominator which is a power of p such that $|\lambda_{i,j,d} - \lambda_{i,j}|_p < p^{d+a+b}$. For $i \neq 1, \ldots, r$, just define $\lambda_{i,j,d} = 0$ for every $j = 2, \ldots, n$.

3.4.7 First approximation of $\operatorname{div}(x_j)$, $j \geq 2$. The constant a.

From now on, $D_{1,d}$, the approximation of $\operatorname{div}(x_1)$, will be fixed. We are going to define a first approximation $D_{j,d}$ of $\operatorname{div}(x_j)$ for every $j = 2, \ldots, n$, and then we are going to make subsequent modifications in order to make it principal. The argument works in parallel for every $j = 2, \ldots, n$, the reader may think of a fixed j, for example $j = 2$, for the rest of the proof.

Define

$$D_{j,d} = \sum_i \lambda_{i,j,d} \cdot e_i \cdot g^*[p_i]$$

for every $j = 2, \ldots, n$.

Since $\lambda_{i,j,d} \in \mathbb{Z}[1/p]$ is not an integer, a priori $D_{j,d}$ is an element of $\operatorname{Div}(M) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$: actually, $D_{j,d}$ is an honest divisor, i.e. an element of $\operatorname{Div}(M)$. The reason is that $D_{j,d}$ approximates $\operatorname{div}(x_j)$, which is a integer p-adic divisor. In fact, if $i \neq 1, \ldots, r$, $\lambda_{i,j,d} = 0$, hence there is no problem. If $i = 1, \ldots, r$, we have

$$D_{j,d} - \operatorname{div}(x_j)|_{S_i} = \lambda_{i,j,d} \cdot e_i \cdot g^*[p_i] - \lambda_i \operatorname{div}(x_1)|_{S_i} =$$

$$= (\lambda_{i,j,d} - \lambda_{i,j}) \cdot e_i \cdot g^*[p_i] + \lambda_{i,j} \cdot (e_i \cdot g^*[p_i] - \operatorname{div}(x_1)|_{S_i})$$

and if we take p-adic norms we have

$$\left|\left(D_{j,d} - \mathrm{div}(x_j)\right)|_{S_i}\right|_p \leq$$

$$\leq \max\left\{\left|(\lambda_{i,j,d} - \lambda_i)\cdot e_i \cdot g^*[p_i]\right|_p, \left|\lambda_i \cdot (e_i \cdot g^*[p_i] - \mathrm{div}(x_1)|_{S_i})\right|_p\right\} \leq$$

$$\leq \max\left\{p^{-d-a-b}, \left|\lambda_{i,j}\right|_p \cdot \left|e_i \cdot g^*[p_i] - \mathrm{div}(x_1)|_{S_i}\right|_p\right\} \leq$$

$$\leq \max\left\{p^{-d-a-b}, p^a \cdot \left|e_i \cdot g^*[p_i] - \mathrm{div}(x_1)|_{S_i}\right|_p\right\}.$$

By construction,

$$\left|e_i \cdot g^*[p_i] - \mathrm{div}(x_1)|_{S_i}\right|_p \leq \left|D_{1,d} - \mathrm{div}(x_1)\right|_p \leq p^{-d-a-b}$$

and hence we have

$$\left|\left(D_{j,d} - \mathrm{div}(x_j)\right)|_{S_i}\right|_p \leq p^{-d-b}$$

and thus

$$|D_{j,d} - \mathrm{div}(x_j)|_p \leq p^{-d}$$

since $D_{j,d}$ approximates $\mathrm{div}(x_j)$ up to p^{d+b} on $S_1, \ldots, S_r$ and up to p^d on the other connected components of the support. We have "spent" the constant a in order to manage the denominators of $\lambda_{i,j,d}$.

Now, if $D_{j,d}$ was not integral, we would have $|D_{j,d} - \mathrm{div}(x_j)|_p > 1$, since $\mathrm{div}(x_j) \in \wedge_p \mathrm{Div}(M)$, and this is absurd.

3.4.8 Second approximation of $\mathrm{div}(x_j)$, $j \geq 2$. The constant b.

Hence, we have constructed a principal divisor $D_{1,d}$ which approximates $\mathrm{div}(x_1)$ up to p^{d+a+b}, and $D_{j,d}$ a divisor which approximates $\mathrm{div}(x_j)$ up to p^{d+b} on $S_1, \ldots, S_r$ and up to p^d on the other components of the support. Observe moreover that $D_{1,d}$ and $D_{j,d}$ satisfy automatically the hypothesis on rank 2 valuations, i.e. they are "algebraically dependent": this can be seen either because the divisors are one multiple of the other on each connected component or because they both are pullback of divisors defined on the curve X.

If $D_{j,d}$ was principal, we would have almost finished. We are going to modify $D_{j,d}$ to make it principal, in such a way that it still approximates $\mathrm{div}(x_j)$ and it still comes from a divisor on X.

The first step is to "spend" the constant b in order to find a divisor $R_{j,d}$ on X such that

$$\deg(D_{j,d} + p^d g^* R_{j,d}) \cap C = 0.$$

Recall that

$$|\deg(\mathrm{div}(x_1)|_{S_i} \cap C)|_p = p^{-b_i}$$

Since $|\mathrm{div}(x_1) - D_{1,d}|_p \le p^{-d-a-b}$ we have that

$$|\deg(D_{1,d}|_{S_i} \cap C)|_p = p^{-b_i},$$

too, because $d + a + b > b_i$. Write $\deg(D_{1,d}|_{S_i} \cap C) = q_i p^{b_i}$, with q_i prime with p. Since we have shown that $D_{j,d}$ is integral and $\deg(D_{j,d}|_{S_i} \cap C) = \lambda_{i,j,d} q_i p^{b_i}$, we have that $p^{b_i}\lambda_{i,j,d}$ is integral too.

Clearly, $\deg(\mathrm{div}(x_j) \cap C) = 0$, and $|D_{j,d} - \mathrm{div}(x_j)|_p \le p^{-d-b}$, hence we have that

$$\deg(D_{j,d} \cap C) = \sum_{i=1}^{r} q_i p^{b_i} \lambda_{i,j,d} = q'_j \cdot p^{d+b}$$

for some integer q'_j. Write now

$$R_{j,d} = \sum_{i=1}^{r} \gamma_{i,j} e_i [p_i] \in \mathrm{Div}(X),$$

for some integers $\gamma_{i,j}$. We want to choose these $\gamma_{i,j}$ such that

$$0 = \deg\Big((D_{j,d} + p^d g^* R_{j,d}) \cap C\Big) =$$

$$= q'_j \cdot p^{d+b} + p^d \sum_i \gamma_{i,j} \deg(D_{1,d} \cap C) =$$

$$= q'_j \cdot p^{d+b} + p^d \sum_{i=1}^{r} \gamma_{i,j} q_i p^{b_i},$$

which is equivalent to

$$q'_j p^b = -\sum_{i=1}^{r} \gamma_{i,j} q_i p^{b_i}$$

We can find $\gamma_{1,j}, \ldots, \gamma_{r,j}$ if and only if $\gcd(q_1 p^{b_1}, \ldots, q_r p^{b_r})$ divides $q'_j p^b$. But q_i is prime with p for every i, and $b \ge b_i$ for every i, thus it is sufficient that $\gcd(q_1, \ldots, q_r)$ divides q'_j. But this is true, since

$$\sum_{i=1}^{r} q_i p^{b_i} \lambda_{i,j,d} = q'_j \cdot p^{d+b},$$

q_i is prime with p and $p^{b_i}\lambda_{i,j,d}$ is integral.

3.4.9 Last approximation of $\mathrm{div}(x_j)$, $j \geq 2$. The constants c, c'.

Hence, up to replacing $D_{j,d}$ with $D_{j,d} + p^d g^* R_{j,d}$, we may suppose that $\deg(D_{j,d} \cap C) = 0$, and

$$|D_{j,d} - \mathrm{div}(x_j)|_p \leq p^{-d}.$$

Observe that, up to now, we could not hope to approximate $\mathrm{div}(x_j)$ better than this: $D_{j,d}$ is supported on $S_1, \ldots, S_r$ which are the components with hypersurfaces of weight at most p^{-d}. To get a better approximation, we need to consider other components of the support, hence consider a larger d and thus larger a and b, but luckily not larger c, c', since these do not depend on d.

Let us then take the last step: we need to find a divisor $R'_{j,d}$ on X such that $D_{j,d} - p^{d-c-c'} \cdot g^* R'_{j,d}$ is principal. This will be a principal divisor approximating $\mathrm{div}(x_j)$ up to $p^{d-c-c'}$: since the constants c, c' do not depend on d, $p^{d-c-c'}$ goes to infinity with d, hence the approximation works.

Since $|D_{j,d} - \mathrm{div}(x_j)|_p \leq p^{-d}$, we have that $[D_{j,d}] = p^d Q_j$ for some $Q_j \in \wedge_p \mathrm{Pic}(M)$. But since $\mathrm{Pic}(M)$ is finitely generated, by direct computation $\wedge_p \mathrm{Pic}(M)/\mathrm{Pic}(M)$ has no p-torsion, $[D_{j,d}] \in \mathrm{Pic}(M)$ and hence $Q_j = [D_{j,d}]/p^d$ is actually an element of $\mathrm{Pic}(M)$. We want to show that $p^{c+c'} Q_j$ is the pullback of a divisor $R'_{j,d}$ on X.

Observe now that $p^a \lambda_{i,j,d}$ is integer, by definition of a. Hence

$$p^a D_{j,d} = g^* \sum_i p^a \lambda_{i,j,d} e_i [p_i]$$

is the pullback of a divisor defined on X. I claim now that $\sum_i p^a \lambda_{i,j,d} e_i [p_i]$ defines an element of $\mathrm{Pic}^0(X)$, i.e. it has degree 0. In fact, consider the composition

$$C \to M \to X,$$

by construction it is a dominant morphism of curves. Hence, the degree of $\sum_i p^a \lambda_{i,j,d} e_i [p_i]$ is 0 if and only if its pullback on C has degree 0. But its pullback to C is exactly $p^a D_{j,d} \cap C$, which has degree 0. In particular, $p^a D_{j,d} = p^{d+a} Q_j$ is in $\mathrm{Pic}^0(M)$ and it comes from an element of $\mathrm{Pic}^0(X)$.

This means that $[Q_j] \in \mathrm{NS}(M)$ is a torsion element whose order is a power of p, and hence by definition of c' we have that $p^{c'} Q_j \in \mathrm{Pic}^0(M)$. Since $p^{d+a} Q_j$ comes from $\mathrm{Pic}^0(X)$, we have that $p^{c'} Q_j$ defines a torsion element of $\operatorname{Pic}^0(M)/\operatorname{Pic}^0(X)$. By definition of c, we have that $p^{c+c'}Q_j$ is 0 in $\operatorname{Pic}^0(M)/\operatorname{Pic}^0(X)$, hence $p^{c+c'}Q_j$ comes from $\operatorname{Pic}^0(X)$. Say we have $p^{c+c'}Q_j = g^*R'_{j,d}$, with $R'_{j,d} \in \operatorname{Pic}^0(X)$. Then

$$p^{a+c+c'}[D_{j,d}] = p^{d+a+c+c'}Q_j = p^{d+a}g^*R'_{j,d},$$

i.e.

$$p^{c+c'}[D_{j,d}] - p^d g^* Q'_j = p^d(p^{c'}Q_j - g^*R'_{j,d}) \in \operatorname{Pic}^0(M)[p^a].$$

But then $p^{c'}Q_j - g^*R'_{j,d}$ is a torsion element of $\operatorname{Pic}^0(M)$ whose order is a power of p, and hence by definition of c we have that $p^c(p^{c'}Q_j - g^*R'_{j,d}) = 0$. In particular, since c, c' do not depende on d we may suppose $d \geq 2c + c'$ and hence

$$[D_{j,d}] - p^{d-c-c'}g^*R'_{j,d} = p^{d-2c-c'} \cdot p^c \cdot (p^{c'}Q_j - g^*R'_{j,d}) = 0 \in \operatorname{Pic}^0(M),$$

as desired.

3.4.10 From divisors to rational functions

Write $d' = d - c - c'$. Let $k' \subseteq L$ be the algebraic closure of k in L. We have an exact sequence

$$0 \to k'^* \to L^* \to \operatorname{Div}(M) \to 0,$$

and since $\operatorname{Div}(M)$ is torsion free this induces an exact sequence

$$0 \to k'^*/k'^{*p^{d'}} \to L^*/L^{*p^{d'}} \to \operatorname{Div}(M)/p^{d'}\operatorname{Div}(M) \to 0.$$

Let $f_i \in L^*$ for $i = 1, \dots, n$ be any rational function such that $\operatorname{div}(f_i) = D_{i,d}$. By construction, $\operatorname{div}(f_i)$ and $\operatorname{div}(x_i)$ have the same image in the group $\operatorname{Div}(M)/p^{d'}\operatorname{Div}(M)$. Hence, up to replacing f_i with $f_i \cdot g_i$ for some $g_i \in k'^*$, we may suppose

$$x_i \cong f_i \pmod{L^{*p^{d'}}}.$$

Now, f_1 and f_j are algebraically dependent for every $j = 2, \dots, n$: we can check this on divisors, and $\operatorname{div}(f_1) = D_{1,d}$, $\operatorname{div}(f_j) = D_{j,d}$ are constructed precisely in order to make the criterion of rank 2 valuations work. If $D_{1,d} = 0$, then by construction $D_{i,d} = 0$ for every i, and hence $\operatorname{trdeg}_k k(f_1, \dots, f_n) = 0$. If $D_{1,d} \neq 0$, f_1 is not algebraic over k, and this implies that $\operatorname{trdeg}_k k(f_1, \dots, f_n) = 1$.

By repeating this for every d, we get that

$$\operatorname{ced}_k(x_1, \dots, x_n) \leq 1.$$

□

3.5 Higher valuations

In this section we present the theory of valuation in the group $\mathbb{Z}^n$ ordered lexicographically, which we call rank n valuations. Like discrete valuations, rank n valuation carry some geometric content, since they can be "built" using discrete valuations.

Definition 3.5.1. Fix a field L and a valuation $v : L^* \to \mathbb{Z}^n$ where the value group $\mathbb{Z}^n$ has the lexicographic order (we want v to be surjective, $\mathbb{Z}^n$ must really be the value group). We are going to call such a valuation simply a rank n valuation. This is not entirely correct, other ordered groups with rank n exist, but there is no ambiguity since we are not going to use any other value group.

If v is a rank n valuation on L, its determinant $\det(v)$ is the composition

$$\det(v) : \oplus_n L^* \xrightarrow{v^{\oplus n}} \mathbb{Z}^{n^2} = M_n(\mathbb{Z}) \xrightarrow{\det} \mathbb{Z}$$

All rank n valuations we consider will be implicitly assumed trivial on the base field k. Recall that a lower triangular matrix with only ones on the diagonal is called a lower unitriangular matrix, and these are precisely the automorphisms of $\mathbb{Z}^n$ as an ordered group.

Lemma 3.5.2.

1. *Let v be a rank n valuation. Then there exists a canonical way to construct n discrete valuations*
$$v_1 : L^* \to \mathbb{Z}$$
$$v_2 : L^*_{v_1} \to \mathbb{Z}$$
$$\vdots$$
$$v_n : L^*_{v_1,\ldots,v_{n-1}} \to \mathbb{Z}$$
associated to v.

2. *On the converse, given n discrete valuations $v_1, \ldots, v_n$ as above, it is possible to construct a rank n valuation v such that $v_1, \ldots, v_n$ are the valuations associated to v.*

3. *Let v, v' be two rank n valuations $L^* \to \mathbb{Z}^n$. The following are equivalent:*
   - *v, v' are isomorphic valuations,*
   - *v, v' differ by a lower unitriangular matrix,*

- *$v_i = v'_i$ for every $i = 1, \dots, n$,*
- $\det(v) = \det(v')$.

Proof. 1. If v is a rank n valuation, construct $v_1, \dots, v_n$ in the following way. Since $\mathbb{Z}$ has the lexicographical order, the first coordinate of v (the "most important one") is itself a valuation with value group $\mathbb{Z}$, call it $v_1 : L^* \to \mathbb{Z}$.

Now take $a, a' \in L^*$ with $v_1(a) = v_1(a') = 0$. If a and a' map to the same element of $L^*_{v_1}$, we have that $a' - a$ maps to 0 and hence $v_1(a' - a) > 0$ and $v(a' - a) > v(a)$.

This tells us that $v(a') = v((a' - a) + a) = v(a)$, i.e. v defines a map $L^*_{v_1} \to \mathbb{Z}^n$. Since the first coordinate is 0, we may ignore it, thus getting a map $L^*_{v_1} \to \mathbb{Z}^{n-1}$. It can be checked that this is a rank $n-1$ valuation, thus we conclude by induction.

2. Let $v_1, \dots, v_n$ be as above, we want to construct v. By induction, we have a rank $n-1$ valuation $w : L^*_{v_1} \to \mathbb{Z}^{n-1}$: we want to put together v_1 and w to construct v. Fix $\pi \in L^*$ an uniformizing parameter for v_1. Now for any $a \in L^*$ define

$$v(a) = \left(v_1(a), w\left(a \cdot \pi^{-v_1(a)}\right)\right) \in \mathbb{Z} \oplus \mathbb{Z}^{n-1} = \mathbb{Z}^n.$$

It can be easily checked that v satisfies the properties of a rank n valuation, and that its associated discrete valuations are $v_1, \dots, v_n$. Observe that the construction of v depends on the choices of the uniformizing parameter π and of w.

3. Since the ordered automorphisms of $\mathbb{Z}^n$ are given by lower unitriangular matrices, v and v' are isomorphic as abstract valuations if and only if they differ by such a matrix.

If two rank n valuations v, v' differ by a lower unitriangular matrix, it is obvious that $\det(v) = \det(v')$ and that they have the same associated discrete valuations.

On the other hand, suppose that v and v' have the same associated discrete valuations $v_1, \dots, v_n$. Let $\pi_i \in L^*$ be such that $v_j(\pi_i) = 0$ for every $j < i$, and $v_i(\pi_i) = 1$. Let c_i, c'_i be the coordinates of v and v': these are in general different from v_i, v'_i, but still we have $c_j(\pi_i) = c'_j(\pi_i) = 0$ for $j < i$ and $c_i(\pi_i) = c'_i(\pi_i) = 1$ (see the construction of point (1)).

This tells us that the square matrices

$$\big(v(\pi_1)|\dots|v(\pi_n)\big)\,,\ (v'(\pi_1)|\dots|v'(\pi_n))$$

are both lower unitriangular. Hence, up to multiplying v' by a lower unitriangular matrix we may suppose that $v(\pi_i) = v'(\pi_i)$ for every $i = 1, \dots, n$. But now, given any $a \in L^*$, it is easy to write by recursion

$$a = a_1 = \pi_1^{v_1(a_1)} \cdot a_2 = \pi_1^{v_1(a_1)} \cdot \pi_2^{v_2(a_2)} \cdot a_3 = \cdots = \prod_i \pi^{v_i(a_i)} \cdot a_{n+1}$$

with $v_i(a_j) = 0$ for $i < j$. In particular, $v(a_{n+1}) = v'(a_{n+1}) = 0$, and

$$v(a) = \sum_i v_i(a_i) \cdot v(\pi_i) = \sum_i v_i(a_i) \cdot v'(\pi_i) = v'(a).$$

Finally, suppose that $\det(v) = \det(v')$. Suppose by absurd that $v_1 \neq v_1'$. In particular, there exists $a_1 \in L^*$ such that $v_1(a_1) = 0$ and $v_1'(a_1) = d > 0$. We may find $b_2, \dots, b_n$ such that

$$\big(v'(a_1)|v'(b_2)|\dots|v'(b_n)\big)$$

has nonzero determinant and $v_1'(b_i) = 0$ for every i. By hypothesis, the corresponding matrix with v instead of v' has nonzero determinant. Since $v_1(a_1) = 0$, this tells us that there exists at least one i, say $i = 2$, with $v_1(b_2) \neq 0$. Up to replacing b_2 with b_2^{-1}, we may suppose $v_1(b_2) > 0$.

Since $v_1(a_1) = 0$, $v_1'(a_1) > 0$, $v_1(b_2) > 0$ and $v_1'(b_2) = 0$ we have that $v(a_1) < v(b_2)$ and $v'(a_1) > v'(b_2)$, hence

$$v(b_2 + a_1) = v(a_1),$$

$$v'(b_2 + a_1) = v'(b_2).$$

Thus,

$$\det\big(v(a_1)|v(b_2+a_1)|v(b_3)|\dots|v(b_n)\big) =$$
$$= \det\big(v(a_1)|v(a_1)|v(b_3)|\dots|v(b_n)\big) = 0$$

while

$$\det\big(v'(a_1)|v'(b_2+a_1)|v'(b_3)|\dots|v'(b_n)\big) =$$
$$= \det\big(v'(a_1)|v'(b_2)|v'(b_3)|\dots|v'(b_n)\big) \neq 0,$$

which is absurd since by hypothesis $\det(v) = \det(v')$. Hence, $v_1 = v_1'$.

Once we know that $v_1 = v'_1$, we may use v and v' to construct rank $n-1$ valuations $w, w' : L^*_{v_1} \to \mathbb{Z}$ as we have done in point (1). Since $\det(v) = \det(v')$ and $v_1 = v'_1$, we get that $\det(w) = \det(w')$, hence by induction we conclude that $v_i = v'_i$ for every $i > 1$, too.

□

Thanks to Lemma 3.5.2, we may think of rank n valuations geometrically in the following way. If L is finitely generated over k, take M a normal, projective variety such that $L = k(M)$. Then to construct a rank n valuation $v : L^* \to \mathbb{Z}^n$ trivial on k we may choose an hypersurface V_1 in M, then an hypersurface V_2 in the normalization of V_1, an hypersurface V_3 in the normalization of V_2 and so on. If we allow M to vary among smooth, projective varieties with function field L, then this construction recovers all rank n valuations. The first non obvious case is on the function field of a surface: here a rank 2 valuation corresponds to an irreducible curve on the surface together with a closed point on the normalization of the curve.

3.5.1 Algebraic dependence and higher valuations

Proposition 3.5.3. *Given an extension L/k and n elements $x_1, \dots, x_n \in L^*$, they are algebraically independent over k if and only if there exists a rank n valuation v trivial on k such that* $\det v(x_1, \dots, x_n) \neq 0$.

Proof. One implication, i.e. $\det(v)(x_1, \dots, x_n) \neq 0$ implies algebraic independence, is classical, see for example [ZS60, ch.VI, §10, rmk.B].

The other implication can be done by induction. For $n = 0$, the empty set is algebraically independent and the empty matrix has determinant 1, hence the unique 0-valuation works.

Let now $n > 0$ be a positive integer, and suppose we have proven the lemma for $n-1$. Choose $x_1, \dots, x_n \in L^*$ which are algebraically independent. Consider the discrete valuation $v : k(x_1, \dots, x_n) \to \mathbb{Z}$ such that $v(x_n) = 1$ and $v(p) = 0$ if $p \in k[x_1, \dots, x_n]$ is prime with x_n. We can extend v to a valuation $v' : L \to \mathbb{Z}$ in the sense that, if $t \in k(x_1, \dots, x_n)$, $v(t) > 0$ if and only if $v'(t) > 0$. Since the restriction of v' to $k(x_1, \dots, x_{n-1})$ is trivial, we have an immersion

$$k(x_1, \dots, x_{n-1}) \hookrightarrow L_{v'}$$

and hence $x_1, \dots, x_{n-1}$ are algebraically independent also in $L_{v'}$. By induction hypothesis, there exists a rank $n-1$ valuation $u : L_{v'} \to \mathbb{Z}^{n-1}$ such that $\det(u)(x_1, \dots, x_{n-1}) \neq 0$. Now, combining u with v', we obtain a rank

n valuation $v : L^* \to \mathbb{Z}^n$ such that

$$\det(v)(x_1, \dots, x_n) = v'(x_n) \det(u)(x_1, \dots, x_{n-1})$$

since v' is zero when restricted to $k(x_1, \dots, x_{n-1})$. Now,

$$\det(u)(x_1, \dots, x_{n-1}) \neq 0$$

by inductive hypothesis and $v'(x_n) \neq 0$ because $v(x_n) = 1$, hence we have $\det(v)(x_1, \dots, x_n) \neq 0$ too. □

Corollary 3.5.4. *Given n elements $x_1, \dots, x_n \in L^*$, the transcendence degree $\operatorname{trdeg}_k k(x_1, \dots, x_n)$ is the maximum rank of the $d \times n$ matrices $v(x_1, \dots, x_n)$, where $v : L^* \to \mathbb{Z}^d$ is a valuation.*

Proof. Let $t = \operatorname{trdeg}_k k(x_1, \dots, x_n)$, up to reordering we may suppose $t = \operatorname{trdeg}_k k(x_1, \dots, x_t)$. Proposition 3.5.3 gives us a valuation $v : L^* \to \mathbb{Z}^t$ such that $v(x_1, \dots, x_t)$ has nonzero determinant. Then clearly $\operatorname{rk} v(x_1, \dots, x_n) = t$.

On the other hand, take a valuation $v : L^* \to \mathbb{Z}^d$. The rational rank of the restriction of v to $k(x_1, \dots, x_n)$ is at most $\operatorname{trdeg}_k k(x_1, \dots, x_n)$, see [ZS60, ch.VI, §10, rmk.B]. This means that $\operatorname{rk} v(x_1, \dots, x_n) \leq \operatorname{trdeg}_k k(x_1, \dots, x_n)$. □

If M is a smooth, projective model of L and $x \in L$ is transcendental over k, we do not only know that $v(x) \neq 0$ for some discrete valuation: we actually know that this holds for a valuation associated to an hypersurface of M. In some sense, this remains true for rank n valuations.

Recall that if we have a valuation of a finitely generated extension L/k in some ordered group and M is a proper model of L/k, we can define the *center* of the valuation: if A is the value ring with fraction field L, then the valuative criterion of properness gives us a morphism $\operatorname{Spec} A \to M$ and the center of the valuation is the image of the closed point. If A is a DVR and the valuation is associated to an hypersurface $V \subseteq M$, then the center of the valuation is the generic point of V.

Lemma 3.5.5. *Let k be a field of characteristic 0 and M a smooth variety over k of dimension n. Let $k(M)/L/k$ be a subextension of transcendence degree $m \leq n$. Then there exist a transcendence basis $x_1, \dots, x_m \in L$ and a rank m valuation $v : k(M)^* \to \mathbb{Z}^m$ such that*

- $\det(v)(x_1, \dots, x_m) \neq 0$,
- *the center of v is the generic point of a codimension m subvariety of M.*

Proof. Observe that we are not asking that M is proper.

Choose any $x_1, \dots, x_m \in L$ transcendental over k. Thanks to generic smoothness there exists an open subset $U \subseteq M$ over which $(x_1, \dots, x_m)$ defines a smooth dominant morphism $(x_1, \dots, x_m) : U \to \mathbb{A}^m$. In particular $U \to \mathbb{A}^m$ is open, and since every open subset of $\mathbb{A}^m$ contains a rational point up to a translation we may suppose that the origin of $\mathbb{A}^m$ is in the image. Then the closed subsets $V_i = \{x_i = 0\} \subseteq U$ meet transversally, and this allows us to define a rank m valuation v using the subsequent closed subsets $V_1 \supseteq V_1 \cap V_2 \supseteq \cdots \supseteq V_1 \cap \cdots \cap V_m$. The center of V is the generic point of $V_1 \cap \cdots \cap V_m$ which has codimension m as desired. $\square$

3.5.2 Higher tame symbols

Definition 3.5.6. Let v be a rank n valuation, call $v_1, \dots, v_n$ its associated discrete valuations as above. For every $d \geq 0$, the composition

$$K^M_{d+n}(L) \xrightarrow{\partial_{v_1}} K^M_{d+n-1}(L_{v_1}) \xrightarrow{\partial_{v_2}} \dots \xrightarrow{\partial_{v_n}} K^M_d(L_{v_1,\dots,v_n}) = K^M_d(L_v)$$

is the *tame symbol* ∂_v of v, where the single maps are the tame symbols of the discrete valuations v_i (see [GS06, Proposition 7.1.4]).

Proposition 3.5.7. *Let v be a rank n valuation on L. Then the composition*

$$\varphi_v : \oplus_n L^* \to K^M_n(L) \xrightarrow{\partial_v} K^M_0(L_v) = \mathbb{Z}$$

is equal to $\det(v)$.

Proof.
- If $y_n \in L^*$ is such that $v(y_n) = 0$, then $\varphi_v(x_1, \dots, x_{n-1}, y_n) = 0$ for every $x_1, \dots, x_{n-1} \in L^*$. Let us do this by induction. If $n = 1$, it is obvious. Suppose we have proved it for $n - 1$, and call v' the rank $n - 1$ valuation on L_{v_1} induced by $v_2, \dots, v_n$. Observe that y_n can be seen as an element of all the subsequent residue fields, and that $v_i(y_n) = 0$ for every i.

  If $\pi, u_2, \dots, u_n \in L^*$ are such that $v_1(\pi) = 1$ and $v_1(u_i) = 0$ for every $i = 2, \dots, n$, then

  $$\partial_{v_1}([\pi, u_2, \dots, u_n]) = [u_2, \dots, u_n]$$

  by definition of ∂_{v_1}.

  Now, we can write $x_i = x_i \cdot \pi^{-v_1(x_i)} \cdot \pi^{v_1(x_i)}$ and use this to expand $\varphi_v(x_1, \dots, x_{n-1}, y_n)$ using multilinearity and alternance to obtain a sum of terms of the form $\varphi_v(\pi, u_2, \ldots, u_n)$ for some $\pi, u_2, \ldots, u_{n-1}$ as above, and $u_n = y_n$. Now, every element of this form is 0. In fact, we have by definition

$$\varphi_v(\pi, u_2, \ldots, u_{n-1}, y_n) = \varphi_{v'}(u_2, \ldots, u_{n-1}, y_n)$$

which is 0 by induction hypothesis.

- $\varphi_v(x_1, \ldots, x_n)$ depends only on the matrix $(v(x_1)| \ldots |v(x_n))$: in fact if $y_1, \ldots, y_n$ are such that $v(y_i) = 0$ for every i, by multilinearity

  $$\varphi_v(y_1 x_1, \ldots, y_n x_n) = \varphi_v(x_1, \ldots, x_n) + T$$

  where T is a sum of elements of the form

  $$\varphi_v(\ldots, y_i, \ldots).$$

  But then $T = 0$ for what we have shown above. Hence, φ_v defines a multilinear, alternating map $\psi_v : M_n(\mathbb{Z}) \to \mathbb{Z}$.

- To conclude, we only need to show that ψ_v is the determinant, and to do this we only need to check that it is 1 on the identity. Hence, let $\pi_1, \ldots, \pi_n \in L^*$ be such that $v(\pi_1, \ldots, \pi_n)$ is the identity (they exist because, by hypothesis, $v : L^* \to \mathbb{Z}^n$ is surjective). Now, $v(\pi_1, \ldots, \pi_n) = \mathrm{Id}$ implies that $v_i(\pi_j) = 0$ for $i < j$ and $v_i(\pi_i) = 1$. We can do this by induction: for $n = 1$ it is obvious, suppose we have done it for $n - 1$ and call v' the induced $n - 1$-valuation on L_{v_1}. We have

  $$\psi_v(\mathrm{Id}) = \varphi_v(\pi_1, \ldots, \pi_n) = \varphi_{v'}(\pi_2, \ldots, \pi_n) = \psi_{v'}(\mathrm{Id}) = 1.$$

□

3.6 Divisors and Picard group of a function field

Let L/k be a finitely generated extension, G an ordered group, $v : L^* \to G$ a valuation trivial on k and M a model of L, i.e. a smooth, projective variety over k with a fixed identification $L = k(M)$ of extension of k. There is a natural way of extending v to divisors, i.e. to define an homomorphism

$$\mathrm{Div}(M) \to G$$

which we will call v again such that

$$v(\mathrm{div}(f)) = v(f)$$

for every $f \in L^*$.

Let $R \subseteq K(M)$ be the valuation ring of v. By the valuative criterion of properness, there exists a unique map $\operatorname{Spec} R \to M$ extending $\operatorname{Spec} k(M) \to M$, call p the image of the unique closed point of $\operatorname{Spec} R$. If $D \in \operatorname{Div}(M)$ is a divisor, choose an open neighborhood U of p such that there exists $\alpha \in L^*$ with $\operatorname{div}(\alpha)|_U = D|_U$. Define

$$v(D) = v(\alpha) \in G.$$

- This definition does not depend on the choice of U and α. If U', α' is another choice, choose an open affine neighborhood $\operatorname{Spec} A$ of p contained in $U \cap U'$. Since $\operatorname{div}(\alpha/\alpha')|_{\operatorname{Spec} A} = 0$ and A is normal, then α/α' is invertible in $A \subseteq R \subseteq K(M)$, and hence $v(\alpha) = v(\alpha')$.
- On principal divisors $\operatorname{div}(f)$, this definition obviously coincides with $v(f)$.
- The resulting map $\operatorname{Div}(M) \to G$ is an homomorphism: if $D, D' \in \operatorname{Div}(M)$ are represented locally by α, α', then $D + D'$ is represented locally by $\alpha\alpha'$.

If M' is another model of L and $\pi : M' \to M$ is a birational map respecting the identifications $k(M) = L = k(M')$, we have a pullback map $\pi^* : \operatorname{Div}(M) \to \operatorname{Div}(M')$ such that $\pi^* \operatorname{div}_M(f) = \operatorname{div}_{M'}(f)$, and $v(\pi^*(D)) = v(D)$ for every $D \in \operatorname{Div}(M)$.

Remark 3.6.1. Since the divisor of a rational function is constructed using only discrete valuations, the fact that $v(\operatorname{div} f) = v(f)$ for every valuation $v : k(M) \to G$ supports the philosophy that we should only consider discrete valuations in algebraic geometry. This seems not a very smart thing to say in a thesis using heavily other types of valuation, but in our defense we think of valuations with group $\mathbb{Z}^n$ only as a nice packaging of n discrete valuations as in Lemma 3.5.2.

Definition 3.6.2. Let V be the set of all discrete valuations on L which are trivial on k. We call $\operatorname{Div}(L) \subseteq \prod_{v \in V} \mathbb{Z}$ the subgroup of the elements which are in the image of $\operatorname{Div}(M) \to \prod_{v \in V} \mathbb{Z}$ for some model M of L. An element of $\operatorname{Div}(L)$ is a divisor of L/k.

If M', M are models of L/k and $\pi : M' \to M$ is birational, we have homomorphisms

$$\operatorname{div} : L^* \to \operatorname{Div}(L)$$

$$\operatorname{Div}(M) \to \operatorname{Div}(L),\ \operatorname{Div}(M') \to \operatorname{Div}(L)$$

satisfying obvious commutative diagrams along with the divisor map $\mathrm{div}_M : L^* \to \mathrm{Div}(M)$ and the pullback $\pi^* : \mathrm{Div}(M) \to \mathrm{Div}(M')$. Pay attention: $\mathrm{Div}(M) \to \mathrm{Div}(L)$ and $\mathrm{Div}(M') \to \mathrm{Div}(L)$ do not commute with pushforward $\pi_* : \mathrm{Div}(M') \to \mathrm{Div}(M)$.

Lemma 3.6.3. *Let L/k be a finitely generated extension, and consider the two direct systems of groups $\mathrm{Div}(M)$, $\mathrm{Pic}(M)$ where M varies among smooth, projective models M of L/k, and transition maps are given by pullback. Then the natural maps*

$$\varinjlim_M \mathrm{Div}(M) \to \mathrm{Div}(L),$$
$$\varinjlim_M \mathrm{Pic}(M) \to \mathrm{Pic}(L)$$

are isomorphisms.

Proof. Let k' be the algebraic closure of k in L, then if M' and M are smooth projective models with a birational morphism $\pi : M' \to M$ we have exact sequences

$$\begin{tikzcd}
0 \arrow[r] & L^*/k'^* \arrow[r] \arrow[d, equal] & \mathrm{Div}(M) \arrow[r] \arrow[d, hook, "\pi^*"] & \mathrm{Pic}(M) \arrow[r] \arrow[d, hook, "\pi^*"] & 0 \\
0 \arrow[r] & L^*/k'^* \arrow[r] & \mathrm{Div}(M') \arrow[r] & \mathrm{Pic}(M') \arrow[r] & 0
\end{tikzcd}$$

hence it is enough to show the thesis for $\mathrm{Div}(L)$. But this is obvious, since $\mathrm{Div}(M) \to \mathrm{Div}(L)$ is injective for every M and $\varinjlim_M \mathrm{Div}(M) \to \mathrm{Div}(L)$ is surjective by definition of $\mathrm{Div}(L)$. $\square$

Lemma 3.6.4. *Let L/k be an extension of fields, and x an element of $\widehat{L}^*$ (resp. $\wedge_p L^*$). Then there exists a valuation v on L such that $v(x) \neq 0 \in \widehat{\mathbb{Z}}$ (resp. $\mathbb{Z}_p$) if and only if the image of x in $\widehat{\mathrm{Div}(L)}$ (resp. $\wedge_p \mathrm{Div}(L)$) is nonzero.*

Proof. First, let us prove that $\prod_{v\in V} \mathbb{Z}/\mathrm{Div}(L)$ is torsion free. Let D be an element of $\prod_{v\in V} \mathbb{Z}$ such that nD is the image of a divisor $C \in \mathrm{Div}(X)$ for some integer n and some model X of L. It is obvious that $C = nC' \in \mathrm{Div}(X)$ is a multiple of a divisor C': since nC' maps to nD, then C' maps to D because $\prod_{v\in V} \mathbb{Z}$ is torsion free.

Since $\prod_{v\in V} \mathbb{Z}/\mathrm{Div}(L)$ is torsion free, thanks to Lemma 3.7.1.(c) the homomorphism

$$\widehat{\mathrm{Div}(L)} \to \widehat{\prod_{v\in V} \mathbb{Z}} = \prod_{v\in V} \widehat{\mathbb{Z}}$$

is injective. $\square$

Definition 3.6.5. Let $\mathrm{Pr}(L) \subseteq \mathrm{Div}(L)$ be the subgroup of principal divisors $\mathrm{div}(f)$ for some $f \in L^*$. The Picard group of L is $\mathrm{Pic}(L) = \mathrm{Div}(L)/\mathrm{Pr}(L)$.

Proposition 3.6.6. *For every normal, projective model M of L,* $\mathrm{Div}(M) \to \mathrm{Div}(L)$ *induces an injective homomorphism* $\mathrm{Pic}(M) \to \mathrm{Pic}(L)$ *which restricts to an isomorphism on torsion subgroups.*

Proof. Clearly, the map $\mathrm{Div}(M) \to \mathrm{Div}(L)$ passes to the quotient $\mathrm{Pic}(M) \to \mathrm{Pic}(L)$. The fact that it is injective comes from the fact that $\mathrm{Div}(M) \to \mathrm{Div}(L)$ is injective, together with the fact that if D maps to $\mathrm{div}_L(f) \in \mathrm{Div}(L)$, we clearly have $D = \mathrm{div}_M(f)$. Let us prove that torsion elements of $\mathrm{Pic}(L)$ are in the image.

Call V_M the set of valuations which correspond to codimension 1 irreducible subvarieties of M. If $D \in \mathrm{Div}(L)$, the support $\operatorname{supp} D$ is the set of valuations v such that $v(D) \neq 0$. Since D comes from some model M' of L, and since $V_M \setminus V_{M'}$ is finite (this is obvious if we have a birational morphism $M \to M'$, and we can reduce to this case), then $V_M \cap \operatorname{supp} D$ is finite.

Now suppose that $D \in \mathrm{Div}(L)$ is such that $nD = \mathrm{div}_L(f)$ for some $f \in L^*$. We can define a divisor D' on M by the formula

$$D' = \sum_{v \in V_M \cap \operatorname{supp} D} v(D)[v].$$

We claim that $D' \in \mathrm{Div}(M)$ maps to $D \in \mathrm{Div}(L)$. In fact, since $\mathrm{Div}(L)$ is torsion free, it is enough to show that nD' maps to $nD = \mathrm{div}_L(f)$. But now clearly

$$nD' = \sum_{v \in V_M \cap \operatorname{supp} D} v(nD)[v] = \sum_{v \in V_M \cap \operatorname{supp} D} v(f)[v] = \mathrm{div}_M(f)$$

and we know that the map $\mathrm{Div}(M) \to \mathrm{Div}(L)$ behaves well on principal divisors. □

Corollary 3.6.7. *Let k be field over which the Mordell-Weil theorem holds, and L/k a finitely generated extension. Then* $\mathrm{Pic}(L)$ *has finite torsion.*

Proof. Choose M a smooth, projective model of L, it is enough to show that $\mathrm{Pic}(M)$ has finite torsion. But this is true, since both $\mathrm{Pic}^0(M) = \underline{\mathrm{Pic}}^0(M)(k)$ and $\mathrm{NS}(M)$ are finitely generated. □

3.7 Some facts about completion

If $p \in \mathbb{Z}$ is a prime, write $p^{-\infty}\mathbb{Z} \subseteq \mathbb{Q}$ for the subgroup of rational numbers of the form $p^{-n}a$ for some $a \in \mathbb{Z}$, $n \geq 0$. If A is abelian group A, write $T_pA = \mathrm{Hom}(p^{-\infty}\mathbb{Z}/\mathbb{Z}, A)$ for the usual p-adic Tate module, and $TA = \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, A)$ for the global Tate module of A. We have

$$TA = \prod_p T_pA$$

and hence

$$\mathbb{Q}/\mathbb{Z} = \bigotimes_p p^{-\infty}\mathbb{Z}/\mathbb{Z}.$$

Write also $\wedge_p A$ for the p-completion $\varprojlim_n A/p^nA$, and $\widehat{A}$ for the global completion $\varprojlim_n A/nA$, we have a natural identification

$$\widehat{A} = \prod_p \wedge_p A.$$

We write the elements of $\wedge_p A$ either as $\sum_{n=0}^{\infty} a_n p^n$ with $a_i \in A$ or as $(a'_n)_n$ with $a'_{n+1} = a'_n + p^n \alpha_n \in A$, depending on which representation is more convenient.

We say that A has *limited p-torsion* if there exists a positive integer n_0 such that, for $n \geq n_0$, $A[p^n] = A[p^{n+1}]$. If A has limited p-torsion, in particular $T_pA = 0$, but the converse is false. We say that A has *limited torsion* if it has limited p-torsion for every p.

Lemma 3.7.1. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of abelian groups, and p a prime. There is a <u>non</u> exact complex*

$$0 \to T_pA \to T_pB \to T_pC \to \wedge_p A \to \wedge_p B \to \wedge_p C \to 0$$

such that

$$\wedge_p B \to \wedge_p C \to 0,$$
$$0 \to T_pA \to T_pB \to T_pC$$

are exact. If we impose hypotheses on p-torsion of A, B and C, we get higher levels of exactness.

(a) If A has limited p-torsion, then $T_pA = 0$ and

$$0 \to T_pB \to T_pC \to \ker\left(\wedge_p A \to \wedge_p B\right)$$

is exact.

(b) If B has limited p-torsion, then $T_pA = T_pB = 0$ and

$$0 \to T_pC \to \wedge_p A \to \ker\left(\wedge_p B \to \wedge_p C\right)$$

is exact.

(c) If C has limited p-torsion, then $T_pC = 0$, $T_pA = T_pB$ and

$$0 \to \wedge_p A \to \wedge_p B \to \wedge_p C \to 0$$

is exact.

Analogous statements hold for global completions and global Tate modules.

Remark 3.7.2. In general, one may compute the cohomology of the complex by taking the spectral sequence associated to the limit of the exact complexes

$$0 \to A[p^n] \to B[p^n] \to C[p^n] \to A/p^nA \to B/p^nB \to C/p^nC \to 0.$$

It turns out then that the cohomology we want to compute is equal to a shift of the cohomology of the complex

$$0 \to \varprojlim_n{}^1 A[p^n] \to \varprojlim_n{}^1 B[p^n] \to \varprojlim_n{}^1 C[p^n] \to 0.$$

Asking for A to have limited p-torsion implies the Mittag-Leffler condition for the projective system $(A[p^n])_n$, but the converse is not true: for example, we could have $T_pA \neq 0$ even if the condition is satisfied. Hence we could prove a more general version of points (a) and (b) only asking the Mittag-Leffler condition, which ensures the vanishing of $\varprojlim^1$. However, for point (c) (which is the most useful one for our applications) it is not clear to us if the Mittag-Leffler condition is enough: in fact, point (c) does not only prove that $\varprojlim_n^1 C[p^n] = 0$, but that the whole sequence is exact, hence that $\varprojlim_n^1 A[p^n] = \varprojlim_n^1 B[p^n]$ too.

Proof. Since $\widehat{A} = \prod_p \wedge_p A$ and $TA = \prod_p T_pA$, it is clearly enough to prove the p-adic statements.

- Let us define the map $T_pC \to \wedge_p A$. Let $c_n \in C[p^n]$ be such that $pc_{n+1} = c_n$. Choose $b_n \in B$ mapping to c_n with $b_0 = 0$, $pb_{n+1} - b_n$

maps to 0 in C and hence it is an element of A. Then the image $(c_n)_n \in T_pC$ in $\wedge_p A$ is

$$\sum_{n=0}^{\infty} p^n(pb_{n+1} - b_n).$$

The fact that this is well defined, it is an homomorphism and that the resulting sequence

$$0 \to T_pA \to T_pB \to T_pC \to \wedge_p A \to \wedge_p B \to \wedge_p C \to 0$$

is a complex is straightforward.

- $\wedge_p B \to \wedge_p C$ is surjective. Consider $\sum_n p^n c_n \in \wedge_p C$ and choose $b_n \in B$ mapping to c_n, then $\sum_n p^n b_n$ maps to $\sum_n p^n c_n$.

- The sequence $0 \to T_pA \to T_pB \to T_pC$ is exact. This comes directly from the fact that the functor T_p is equal to $\mathrm{Hom}(p^{-\infty}\mathbb{Z}/\mathbb{Z}, -)$, and hence left exact.

- Suppose now that A has limited p-torsion. We have to check is that if $(c_n)_n \in T_pC$ maps to 0 in $\wedge_p A$, then it comes from an element of T_pB. By definition, $(c_n)_n$ maps to $\sum_n p^n(pb_{n+1} - b_n)$ with $b_0 = 0$ and $b_n \in B$ mapping to $_n \in C$. This means that for every i

  $$\sum_{n=0}^{i-1} p^n(pb_{n+1} - b_n) = p^i b_i \in p^i A$$

  and hence, up to replacing b_n with $b_n - a_n$ for a suitable a_n, we may suppose $p^n b_n = 0$ for every n. This implies that $pb_{n+1} - b_n \in A$ is an element of p^n torsion. Recall now that A has limited p-torsion, hence there exists n_0 such that for $n \geq n_0$ we have $p^{n_0}(pb_{n+1} - b_n) = 0$. This means that $(p^{n_0} b_{n+n_0})_n$ defines an element of T_pB mapping to $(c_n)_n \in T_pC$.

- Suppose now that B has limited p-torsion. In particular, A has limited p-torsion too, and $T_pC \to \wedge_p A$ is injective thanks to point (a). Hence, to prove point (b) we only need to check that every $(a_n)_n \in \wedge_p A$ mapping to 0 in $\wedge_p B$ comes from T_pC.

  The fact that $(a_n)_n$ maps to 0 in $\wedge_p B$ means that for every n there exists $\beta_n \in B_n$ with $a_n = p^n \beta_n$. By hypothesis, B has limited p-torsion, i.e. there exists n_0 such that $B[p^n] = B[p^{n_0}]$ for every $n \geq n_0$. Let $c_n \in C$ be the image of $p^{n_0}\beta_{n+n_0}$. It is straightforward to check that $(c_n)_n$ defines an element of T_pC mapping to $(a_n)_n \in \wedge_p A$.

- Finally, suppose that C has limited p-torsion, and fix n_0 such that $C[p^n] = C[p^{n_0}]$ for every $n \geq n_0$. Clearly, $T_pC = 0$ and $T_pA = T_pB$, we need to check that $\wedge_p A \to \wedge_p B$ is injective and that an element of $\wedge_p B$ mapping to $0 \in \wedge_p C$ comes from $\wedge_p A$.

  If $(a_n)_n \in \wedge_p A$ maps to $0 \in \wedge_p B$, as in the preceding point $a_n = p^n \beta_n$ for some $\beta_n \in B$. Let $\gamma_n \in C$ be the image of β_n, it is an element of $C[p^n]$. Since C has limited p-torsion, $p^{n_0}\gamma_{n+n_0} = 0$ for every n, i.e. $p^{n_0}\beta_{n+n_0} = \alpha_n$ for some $\alpha_n \in A$. But then $p^n\alpha_n = a_{n+n_0}$, and $a_{n+n_0} = p^n\delta_n + a_n$ for some δ_n, hence $(a_n)_n = (p^n(\alpha_n - \delta_n))_n$ is $0 \in \wedge_p A$.

  If $(b_n)_n \in \wedge_p B$ maps to $0 \in \wedge_p C$, up to changing b_n by a multiple of p^n we may suppose that b_n maps to 0 in C, hence it comes from an element $a_n \in A$. It is straightforward to check that $(p^{n_0}a_{n+n_0})_n$ defines an element of $\wedge_p A$ mapping to $(b_n)_n = (p^{n_0}b_{n+n_0})_n \in \wedge_p B$.

□

Corollary 3.7.3. *Let $\varphi : V \to W$ be an homomorphism of abelian groups, and $K \subseteq V$ the kernel of V. If W and W/V have limited p-torsion, then*

$$0 \to \wedge_p K \to \wedge_p V \to \wedge_p W \to \wedge_p W/V \to 0$$

is exact.

Proof. This is a direct consequence of applying Lemma 3.7.1.c to

$$0 \to K \to V \to V/K \to 0$$

$$0 \to V/K \to W \to W/K \to 0.$$

□

Appendix A

Étale fundamental gerbes

Almost everything in this brief appendix is already known to the mathematical community, we claim no originality. In particular, most of the ideas and results are already implicit in [BV15] and in the original paper by Deligne [Del89]. Anyway, we could not find a satisfying reference, since [BV15] is mostly concerned with the Nori fundamental gerbe rather than the étale one, and hence the theorems regarding the étale fundamental gerbe are not expressed in the right generality. In particular, they always work with inflexible fibered categories, while geometrically connected is the right hypothesis. See also [TZ17, §2,3,4], where part of what is contained in this appendix is done under minor additional hypotheses. To our knowledge, the only thing that is new is the proof of Proposition A.3.2 (a similar theorem for the Nori fundamental gerbe is in [ABETZ17, Theorem III]).

We want to stress out that our effort to state results in maximal generality is not for its own sake: it just happens to work with rather nasty objects that are not even algebraic stacks, like the infinite root stacks of section 1.6. Since the theory works for raw fibered categories without any additional hypothesis, we want to give statements in this generality.

A.1 Geometrically connected fibered categories

Definition A.1.1. [TZ17, Definition 2.5] A fibered category X over k is connected if $\mathrm{H}^0(\mathcal{O}_X, X)$ has no nontrivial idempotents.

If S is a scheme and X is a fibered category, we say that a morphism $X \to S$ is surjective if for every point $s \in S$ there exist a field Ω and a morphism $\operatorname{Spec} \Omega \to X$ with image s in S.

Lemma A.1.2. *A fibered category X/k is not connected if and only there exists a surjective morphism $X \to \operatorname{Spec} k \sqcup \operatorname{Spec} k$.*

Proof. If $X \to \operatorname{Spec} k \sqcup \operatorname{Spec} k = \operatorname{Spec} k \times k$ is surjective, the pullback of 1×0 is a nontrivial idempotent. On the other hand, if $e \in \mathrm{H}^0(\mathcal{O}_X, X)$ is a nontrivial idempotent and $S \to X$ is a morphism, we can define a morphism $S \to \operatorname{Spec} k \sqcup \operatorname{Spec} k$ by sending $S_{e=0}$ to one point and $S_{e=1}$ to the other one. This defines a morphism $X \to \operatorname{Spec} k \sqcup \operatorname{Spec} k$. Since e is nontrivial, then for some schemes S, S' with morphisms $S, S' \to X$ we have $S_{e=0} \neq \emptyset$ and $S'_{e=1} \neq \emptyset$, i.e. $X \to \operatorname{Spec} k \sqcup \operatorname{Spec} k$ is surjective. □

Let X_1, X_2 be two fibered categories over k. It is possible to define the disjoint union $X_1 \sqcup X_2$: is S is a scheme, a morphism $S \to X_1 \sqcup X_2$ is a decomposition of $S = S_1 \sqcup S_2$ with S_1, S_2 open and closed together with a pair of morphisms $s_i : S_i \to X_i$.

Definition A.1.3. We define the *clopen* topology on the category of schemes as the Grothendieck topology for which a cover $\{U_i \to U\}_i$ is a jointly surjective set of morphisms $U_i \to U$ which are both closed and open immersions.

The clopen topology is very coarse, in particular is coarser than the Zariski topology.

Lemma A.1.4. *If X is a connected fibered category over k and $X \simeq X_1 \sqcup X_2$, then either X_1 or X_2 is empty. If X is a stack in the clopen topology the converse hold, i.e. we can write it as a non trivial disjoint union if and only if it is disconnected.*

Proof. If X_1 and X_2 are both non empty, 1×0 and 0×1 in $\mathrm{H}^0(\mathcal{O}_X, X) = \mathrm{H}^0(\mathcal{O}_{X_1}, X_1) \times \mathrm{H}^0(\mathcal{O}_{X_2}, X_2)$ are nontrivial idempotents.

Let now $e \in \mathrm{H}^0(\mathcal{O}_X, X)$ be a nontrivial idempotent. For every scheme S define

$$X_1(S) = \{s \in X(S) | s^*e = 1 \in \mathrm{H}^0(\mathcal{O}_S, S)\},$$

$$X_2(S) = \{s \in X(S) | s^*e = 0 \in \mathrm{H}^0(\mathcal{O}_S, S)\}.$$

We have a natural morphism $X \to X_1 \sqcup X_2$ sending a morphism $s : S \to X$ to the pair s_1, s_2 where s_1 is the restriction of s to $S_{e=1}$ and s_2 is the restriction of s to $S_{e=0}$. Since $S_{e=1}$ and $S_{e=0}$ are open subsets of S such that $S_{e=0} \sqcup S_{e=1} = S$, if X is a stack in the clopen topology we get that $X \to X_1 \sqcup X_2$ is an equivalence. □

Remark A.1.5. If X is an algebraic stack, this is equivalent to asking that the underlying topological space $|X|$ (see [Stacks, Tag 04XE]) is connected.

On one hand, if $X = X_1 \sqcup X_2$, then $|X| = |X_1| \sqcup |X_2|$. On the other hand, if $|X| = U_1 \sqcup U_2$ is disconnected, the fact that for every scheme S the natural morphism $|S| \to |X|$ is continuous allows us to define two fibered categories X_1, X_2 such that $|X_i| = U_i$ and $X = X_1 \sqcup X_2$.

If k'/k is a finite extension of fields, the Weil restriction along k'/k is the right adjoint to the functor of base change along $\operatorname{Spec} k' \to \operatorname{Spec} k$. More concretely, if X is a fibered category over k and Y is a fibered category over k', the Weil restriction $\mathrm{R}_{k'/k} Y$ is a fibered category over k with an equivalence of categories

$$\mathrm{Hom}_k(X, \mathrm{R}_{k'/k} Y) \simeq \mathrm{Hom}_{k'}(X_{k'}, Y)$$

functorial in X and Y. We can construct $\mathrm{R}_{k'/k} Y$ as the fibered product $\mathrm{Aff}/k \times_{\mathrm{Aff}/k'} Y$. When Y is represented by a scheme, $\mathrm{R}_{k'/k} Y$ is represented by its Weil restriction which is a scheme, too. If Y is represented by a finite stack and k'/k is separable, then $\mathrm{R}_{k'/k} Y$ is represented by a finite stack too, see [BV15, Lemma 6.2].

Lemma A.1.6. *Let k'/k be a finite, separable extension, and Y a finite étale stack over k'. Then $\mathrm{R}_{k'/k} Y$ is a finite étale stack over k, too.*

Proof. In the proof of [BV15, Lemma 6.2], from a finite groupoid presentation $R \rightrightarrows U$ of Y they construct a finite groupoid presentation $R' \rightrightarrows U'$ of $\mathrm{R}_{k'/k} Y$. Following their construction, it is immediate to check that if $R \rightrightarrows U$ is étale, $R' \rightrightarrows U'$ is étale too. □

Recall that a fibered category is *concentrated* [BV16, Definition 4.1] if there exists an affine scheme U and a representable, quasi separated, quasi compact and faithfully flat morphism $U \to X$.

If X is concentrated and $u : U \to X$ is as above, set $R = U \times_X U$, we obtain an fpqc groupoid $(r_1, r_2) : R \rightrightarrows U$ in algebraic spaces. From standard arguments in descent theory we get an exact sequence

$$0 \to \mathrm{H}^0(X, \mathcal{O}_X) \xrightarrow{u^*} \mathrm{H}^0(U, \mathcal{O}_U) \xrightarrow{r_1^* - r_2^*} \mathrm{H}^0(R, \mathcal{O}_R)$$

and hence it follows easily that for any field extension k'/k,

$$\mathrm{H}^0(X_{k'}, \mathcal{O}_{X_{k'}}) = \mathrm{H}^0(X, \mathcal{O}_X) \otimes_k k'.$$

Lemma A.1.7. *Let X be a category fibered over k, and k_s/k a separable closure. Consider the following:*

(i) $X_{k'}/k'$ is connected for every extension k'/k,

(ii) X_{k_s}/k_s *is connected,*

(iii) $X_{k'}/k'$ *is connected for every finite, separable extension* k'/k*,*

(iv) k *is the only étale subalgebra of* $\mathrm{H}^0(X, \mathcal{O}_X)$*,*

(v) $\operatorname{Spec} \mathrm{H}^0(X, \mathcal{O}_X)$ *is geometrically connected.*

In general, we have implications $(i) \Leftrightarrow (ii) \Rightarrow (iii) \Leftrightarrow (iv) \Leftrightarrow (v)$*. If* X *is an algebraic space or it is concentrated, then* $(iii) \Rightarrow (ii)$ *holds, too.*

Proof. $(i) \Rightarrow (ii)$ Obvious.

$(ii) \Rightarrow (i)$ Suppose that $X_{k'} \to \operatorname{Spec} k' \sqcup \operatorname{Spec} k'$ is a surjective morphism. Up to enlarging k', we may suppose that $k_s \subseteq k'$. Let S be a scheme over k_s, and $S \to X_{k_s}$ a morphism. By [Stacks, Tag 0363] and [Stacks, Tag 0383], $S_{k'} \to S$ is open and induces a bijection of connected components.

In particular, we can write $S = S_1 \sqcup S_2$ such that $S_{i,k'} \to S_{k'} \to \operatorname{Spec} k' \sqcup \operatorname{Spec} k'$ maps to the i-th point, for $i = 1, 2$. This allows us to define a morphism $S \to \operatorname{Spec} k_s \sqcup \operatorname{Spec} k_s$ whose base change is $S_{k'} \to \operatorname{Spec} k' \sqcup \operatorname{Spec} k'$, and thus a morphism $X_{k_s} \to \operatorname{Spec} k_s \sqcup \operatorname{Spec} k_s$ whose base change is $X_{k'} \to \operatorname{Spec} k' \sqcup \operatorname{Spec} k'$. The morphism $X_{k_s} \to \operatorname{Spec} k_s \sqcup \operatorname{Spec} k_s$ is clearly surjective, and this is absurd.

$(ii) \Rightarrow (iii)$ If $X_{k'} \to \operatorname{Spec} k' \sqcup \operatorname{Spec} k'$ is surjective, then $X_{k_s} \to \operatorname{Spec} k_s \sqcup \operatorname{Spec} k_s$ is surjective too.

$(iii) \Rightarrow (iv)$ Suppose that $A \subseteq \mathrm{H}^0(X, \mathcal{O}_X)$ is a nontrivial finite étale subalgebra of degree $d > 1$, there exists a scheme S with a morphism $S \to X$ such that the composition $S \to X \to \operatorname{Spec} A$ is dominant. Now choose k'/k a finite separable extension which splits A. The base change

$$X_{k'} \to \operatorname{Spec} A_{k'} = \operatorname{Spec} k'^d$$

is surjective because $S_{k'} \to X_{k'} \to \operatorname{Spec} k'^d$ is surjective. But this is absurd, since $d > 1$ and $X_{k'}$ is connected.

$(iv) \Rightarrow (iii)$ Suppose that k'/k is a finite separable extension and that we have a surjective morphism $X_{k'} \to \operatorname{Spec} k' \sqcup \operatorname{Spec} k'$, this induces a morphism $X \to \mathrm{R}_{k'/k}(\operatorname{Spec} k' \sqcup \operatorname{Spec} k')$. Since $\mathrm{R}_{k'/k}(\operatorname{Spec} k' \sqcup \operatorname{Spec} k')$ is a finite étale scheme, by hypothesis we have a factorization

$$X \to \operatorname{Spec} k \to \mathrm{R}_{k'/k}(\operatorname{Spec} k' \sqcup \operatorname{Spec} k').$$

But this gives a factorization

$$X_{k'} \to \operatorname{Spec} k' \to \operatorname{Spec} k' \sqcup \operatorname{Spec} k'$$

which is absurd.

$(iv) \Leftrightarrow (v)$ This is well known.

For the implication $(iii) \Rightarrow (ii)$, if X is concentrated we have

$$\mathrm{H}^0(X_{k'}, \mathcal{O}_{X_{k'}}) = \mathrm{H}^0(X, \mathcal{O}_X) \otimes_k k'$$

for every extension k'/k, hence we can reduce to the case of affine schemes which is well known. If X is an algebraic space, this is [Stacks, Tag 0A17]. □

Definition A.1.8. Let X be a fibered category. We say that X is *geometrically connected* if the equivalent conditions (iii), (iv) and (v) of Lemma A.1.7 hold for X.

A.2 Existence and base change

Definition A.2.1. An fpqc stack Γ over a field k is *pro-étale* if it is the limit of a projective system of finite, étale stacks over k, in the sense of [BV15, Definition 3.5].

Remark A.2.2. In [BV15, Definition 3.5] they define the limit of a projective system $(\Gamma_i)_i$ of affine fpqc gerbes as a category fibered in groupoids which turns out to be an fpqc stack. Actually, it is straightforward to check that the definition works without any modification for a projective system (Γ_i) of categories fibered in groupoids, and if Γ_i is an fpqc stack for every i then also the limit is an fpqc stack. Moreover, if Γ_i is an affine fpqc gerbe for every i and the limit is not empty, then the limit is an fpqc gerbe too, see [BV15, Proposition 3.7].

Definition A.2.3. Let X be a fibered category over k, and Π a pro-étale gerbe with a morphism $X \to \Pi$. Then $X \to \Pi$ is an *étale fundamental gerbe* if, for every finite, étale stack Φ, the functor

$$\operatorname{Hom}(\Pi, \Phi) \to \operatorname{Hom}(X, \Phi)$$

is an equivalence of categories.

Lemma A.2.4. *Let X be a fibered category with an étale fundamental gerbe $X \to \Pi$, and Φ a pro-étale stack. Then*

$$\mathrm{Hom}(\Pi, \Phi) \to \mathrm{Hom}(X, \Phi)$$

is an equivalence of categories. In particular, the étale fundamental gerbe is unique up to a canonical equivalence.

Proof. This is a straightforward application of the definition of the étale fundamental gerbe and of pro-étale stacks. □

The following simple lemma is rather enlightening in the sense that it draws the line between the étale setting and the Nori setting: its failure for finite stacks is what makes Nori's fundamental gerbe subtler than the étale one.

Lemma A.2.5. *Let Φ be a finite étale stack. Then the natural morphism*

$$\Phi \to \operatorname{Spec} \mathrm{H}^0(\Phi, \mathcal{O}_\Phi)$$

is a gerbe.

Proof. We give an elementary proof. See also [TZ17, Proposition 3.2] for a more technical proof for finite, reduced stacks.

If k'/k is an extension, it is easy to check that $\Phi \to \operatorname{Spec} \mathrm{H}^0(\Phi, \mathcal{O}_\Phi)$ is a gerbe if and only if $\Phi_{k'} \to \operatorname{Spec} \mathrm{H}^0(\Phi_{k'}, \mathcal{O}_{\Phi_{k'}})$ is a gerbe. Hence, we may suppose $k = \bar{k}$.

Choose now a finite étale groupoid $R \rightrightarrows U$ giving a presentation of Φ. Since $k = \bar{k}$ and R, U are finite étale, they are simply finite disjoint unions of points. Hence we can write

$$\Phi = \sqcup_i BG_i$$

where G_i are finite discrete groups. Now it is obvious that

$$\Phi = \sqcup_i BG_i \to \sqcup_i \operatorname{Spec} k$$

is a gerbe. □

Corollary A.2.6. *Let X be a fibered category. Then X is geometrically connected if and only if every morphism $X \to \Gamma$ where Γ is a finite étale stack has a factorization*

$$X \to \Gamma' \to \Gamma$$

where Γ' is a finite étale gerbe.

Proof. Suppose that X is geometrically connected. Consider the composition

$$X \to \Gamma \to \operatorname{Spec} \mathrm{H}^0(\Gamma, \mathcal{O}_\Gamma).$$

Since X is geometrically connected and $\mathrm{H}^0(\Gamma, \mathcal{O}_\Gamma)$ is finite étale, we have a factorization

$$X \to \operatorname{Spec} k \to \operatorname{Spec} \mathrm{H}^0(\Gamma, \mathcal{O}_\Gamma).$$

Set $\Gamma' = \operatorname{Spec} k \times_{\operatorname{Spec} \mathrm{H}^0(\Gamma, \mathcal{O}_\Gamma)} \Gamma$, we have a factorization

$$X \to \Gamma' \to \Gamma$$

and Γ' is a gerbe over $\operatorname{Spec} k$ thanks to Lemma A.2.5.

On the other hand, if $A \subseteq \mathrm{H}^0(X, \mathcal{O}_X)$ is a nontrivial étale subalgebra, the natural morphism $X \to \operatorname{Spec} A$ cannot factorize through any finite gerbe. □

Theorem A.2.7. *Let X be a fibered category over k. Then X has an étale fundamental gerbe if and only if it is geometrically connected.*

Proof. The proof is completely analogous to the proof of [BV15, Theorem 5.7], so we don't repeat it. The reason why everything works is Corollary A.2.6, which shows that geometrically connected fibered categories and finite étale stacks satisfy the same formal property of inflexible fibered categories and finite stacks. See also [TZ17, Proposition 4.3] for a proof under some minor additional hypotheses. □

Proposition A.2.8. *Let k'/k be an algebraic and separable extension, X a geometrically connected fibered category over k. Suppose that either*

(a) k' is finite over k, or

(b) X is concentrated.

Then $X_{k'}$ is geometrically connected over k' and $\Pi_{X_{k'}/k'} = \operatorname{Spec} k' \times \Pi_{X/k}$.

Proof. Again, the proof is completely analogous to the one of [BV15, Proposition 6.1]. In the proof we replace [BV15, Lemma 6.2], i.e. the fact that the Weil restriction of finite stacks is finite, with Lemma A.1.6, i.e. the fact that the Weil restriction of finite étale stacks is finite étale. □

Suppose now that we are in characteristic 0. Following Borne and Vistoli, we have shown that the étale fundamental gerbe behaves well under algebraic field extensions: we want to show that, actually, it behaves well with respect to any field extension. The idea is to rephrase the theorem in terms of étale fundamental groups, and then use the fact that in characteristic 0 the étale fundamental group is invariant along extensions of algebraically closed fields, see [SGA1, Proposition 4.6].

Lemma A.2.9. *If G, H are pro-étale groups and k'/k is an extension of algebraically closed fields, then the natural functor*

$$\mathrm{Hom}_k(B_k G, B_k H) \to \mathrm{Hom}_{k'}(B_{k'} G, B_{k'} H)$$

is an equivalence.

Proof. Both categories have the same description in purely group theoretic terms. Let us explain this.

We can think of G and H as topological groups, and BG, BH as categories with only one object. Consider now the category $\mathrm{Hom}_{\mathrm{top}}(BG, BH)$ of functors $BG \to BH$: its objects are just continuous homomorphisms $G \to H$, and every $h \in H$ defines an arrow $\varphi \to h^{-1}\varphi h$ where $\varphi : G \to H$ is a continuous homomorphism.

Since k, k' are algebraically closed and G, H are pro-étale, homomorphisms of group schemes $G \to H$ correspond to continuous homomorphisms of the associated topological groups, and the same is true for $G_{k'}, H_{k'}$. We have a natural morphism $\mathrm{Hom}_{\mathrm{top}}(BG, BH) \to \mathrm{Hom}(B_k G, B_k H)$ which is an equivalence of categories, and the same is true for k'. □

For the following Lemma A.2.10, I would like to thank Marc Hoyois who suggested the use of noetherian approximation in order to reach full generality, see MathOverflow 294847.

Lemma A.2.10. *Let k'/k be an extension of algebraically closed fields. Consider X a concentrated fibered category over k, and Φ a finite étale stack over k. Then the natural functor*

$$\mathrm{Hom}_k(X, \Phi) \to \mathrm{Hom}_{k'}(X_{k'}, \Phi_{k'})$$

is an equivalence of categories.

Proof. Let us prove this firstly under the additional hypothesis that X is a scheme of finite type over k. Under this hypothesis, connected components are open, hence we may suppose that X is connected and Φ is of the form BG for some finite group G. Fix any point $x \in X(k)$. Thanks to [SGA1, Exposé XIII, Proposition 4.6], $\pi_1(X, x) = \pi_1(X_{k'}, x_{k'})$.

We have thus

$$\mathrm{Hom}_k(X, B_k G) = \mathrm{Hom}_k(B_k \pi_1(X, x), B_k G) =$$

$$= \mathrm{Hom}_{k'}(B_{k'}\pi_1(X_{k'}, s_{k'}), B_{k'}G) = \mathrm{Hom}_{k'}(X_{k'}, B_{k'}G).$$

Let us now generalize to X quasi compact, quasi separated scheme. By noetherian approximation [TT90, Theorem C.9], we can write X as an inverse limit $\varprojlim_i X_i$, with X_i of finite type over k. Since Φ is finite,

$$\mathrm{Hom}_k(X, \Phi) = \varinjlim_i \mathrm{Hom}_k(X_i, \Phi) =$$

$$= \varinjlim_i \mathrm{Hom}_{k'}(X_{i,k'}, \Phi_{k'}) = \mathrm{Hom}_{k'}(X_{k'}, \Phi_{k'}).$$

Finally, if X is a concentrated fibered category, let U be a quasi compact and quasi separated scheme with a representable, quasi separated, quasi compact and faithfully flat morphism $U \to X$. Set $R = U \times_X U$, R is again quasi compact and quasi separated. Let $\mathrm{Hom}(R \rightrightarrows U, \Phi)$ be the category of morphism $U \to \Phi$ satisfying the usual cocycle condition on R. Descent theory tells us that $\mathrm{Hom}(R \rightrightarrows U, \Phi)$ is naturally equivalent to $\mathrm{Hom}(X, \Phi)$, even if X is not a stack and hence $X \neq [U/R]$. Since U and R are quasi-compact and quasi separated, by the preceding case we conclude that

$$\mathrm{Hom}_{k'}(X_{k'}, \Phi_{k'}) = \mathrm{Hom}_{k'}(R_{k'} \rightrightarrows U_{k'}, \Phi_{k'}) =$$
$$= \mathrm{Hom}_k(R \rightrightarrows U, \Phi) = \mathrm{Hom}_k(X, \Phi).$$

□

Proposition A.2.11. *Let k be a field of characteristic 0. If X is a geometrically connected, concentrated fibered category over k, then the natural map $\Pi_{X_{k'}/k'} \to \Pi_{X/k} \times_k k'$ is an isomorphism for every field extension k'/k.*

Proof. Thanks to Proposition A.2.8, it is immediate to reduce to the case in which k and k' are both algebraically closed. We have to show that $\Pi_{X/k} \times_k k'$ has the universal property of the étale fundamental gerbe of X.

Since k' is algebraically closed, every finite étale stack over k' has the form $\sqcup_i B_{k'}G_i$ for some finite number of finite groups G_i. In particular, every finite étale stack over k' is isomorphic to $\Phi_{k'}$ for some finite étale stack Φ over k, hence it is enough to show that $\Pi_{X/k} \times_k k'$ has the universal property with respect to stacks of the form $\Phi_{k'}$ with Φ finite étale over k.

Now observe that $\Pi_{X/k}$, being a gerbe over $\mathrm{Spec}\, k$, is concentrated: in fact, any morphism $\mathrm{Spec}\, L \to \Pi_{X/k}$ with L a field is representable, quasi compact, quasi separated and faithfully flat. Hence both X and $\Pi_{X/k}$ are concentrated and thanks to Lemma A.2.10 we have

$$\mathrm{Hom}_{k'}(X_{k'}, \Phi_{k'}) = \mathrm{Hom}_k(X, \Phi) =$$

$$= \mathrm{Hom}_k(\Pi_{X/k}, \Phi) = \mathrm{Hom}_{k'}(\Pi_{X/k} \times_k k', \Phi_{k'}).$$

□

A.3 Étale coverings of fibered categories

Lemma A.3.1. *Let $f : Y \to X$ be a representable, finite étale morphism of fibered categories. If X is connected then f has a constant degree, i.e. there exists an integer d such that for every scheme S and every morphism $s : S \to X$ the étale covering $S \times_X Y \to S$ has constant degree d.*

Proof. If S is a scheme, $s \in X(S)$ an object and $d \geq 0$ an integer, the locus $S_{=d}$ of points p of S such that $Y \times_X S \to S$ has degree d over p is an open and closed subscheme of S, set $S_{\neq d} = S \setminus S_{=d}$. This allows to define a morphism $X \to \operatorname{Spec} k \sqcup \operatorname{Spec} k$ sending $S_{=d}$ to the first point and $S_{\neq d}$ to the second point, and if there exist morphisms $S, S' \to X$ such that $S_{=d}$ and $S'_{\neq d}$ are both nonempty then X is not connected, and this is absurd.

There exists some d_0 and a morphism $S \to X$ such that $S_{=d_0}$ is nonempty, hence for every morphism $S' \to X$ we have $S'_{=d_0} = S'$, i.e. $Y \to X$ has constant degree d_0. □

Proposition A.3.2. *Let $Y \to X$ be a representable, finite étale morphism of geometrically connected fibered categories. Then the natural 2-commutative diagram*

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & \Pi_{Y/k} \arrow[d] \\ X \arrow[r] & \Pi_{X/k} \end{tikzcd}$$

is 2-cartesian.

Proof. Thanks to Lemma A.3.1, $Y \to X$ is a finite cover of fixed degree d. Let $d \times X$ be the disjoint union of d copies of X, we have a finite cover $d \times X \to X$ of degree d. The group S_d acts on the fibered category $Z = \underline{\mathrm{Isom}}_X(d \times X, Y)$ by automorphisms of $d \times X$ making it into an S_d-torsor over X. If S is a scheme with a morphism $S \to Z$, we have a trivialization $d \times S \simeq Y \times_X S$. The first copy of $d \times S$ gives us a morphism $S \to Y$, and thus by Yoneda's lemma we have a S_{d-1} invariant morphism $Z \to Y$ which is actually a S_{d-1}-torsor.

All of this can be packed by saying that we have a morphism $X \to BS_d$ with identifications $Z = X \times_{BS_d} \operatorname{Spec} k$ and $Y = X \times_{BS_d} BS_{d-1}$. Moreover, define $\Pi = \Pi_{X/k} \times_{BS_d} BS_{d-1}$ and $\Lambda = \operatorname{Spec} k \times_{BS_d} \Pi_{X/k}$. We have a 2-cartesian diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & \Lambda \arrow[r] \arrow[d] & \operatorname{Spec} k \arrow[d] \\ Y \arrow[r] \arrow[d] & \Pi \arrow[r] \arrow[d] & BS_{d-1} \arrow[d] \\ X \arrow[r] & \Pi_{X/k} \arrow[r] & BS_d \end{tikzcd}$$

Since Π is pro-étale, if we show that it satisfies the universal property of $\Pi_{Y/k}$ then we have that $\Pi = \Pi_{Y/k}$ thanks to Lemma A.2.4, and hence the thesis.

Consider now a finite étale stack Φ: we want to show that

$$\operatorname{Hom}_k(\Pi, \Phi) \to \operatorname{Hom}_k(Y, \Phi)$$

is an equivalence of categories. Let $\rho : Z \times S_d \to Z$ be the action. If $Y \to \Phi$ is a morphism, consider the composition

$$\rho_\Phi : Z \times S_d \xrightarrow{\rho} Z \to Y \to \Phi.$$

For every $g \in S_d$, this defines a morphism $\rho_\Phi(\cdot, g) : Z \to \Phi$. If $h \in S_{d-1} \subseteq S_d$, since $Z \to Y$ is S_{d-1} invariant we get that $\rho_\Phi(\cdot, g) = \rho_\Phi(\cdot, gh) : Z \to \Phi$, hence $\rho_\Phi(\cdot, [g])$ is well defined for $[g] \in S_d/S_{d-1}$. This gives us an S_d-equivariant morphism

$$Z \to \Phi^{S_d/S_{d-1}}$$

where S_d acts on $\Phi^{S_d/S_{d-1}}$ via left multiplication on S_d/S_{d-1}.

On the other hand, if we have an S_d-equivariant morphism $Z \to \Phi^{S_d/S_{d-1}}$, it is S_{d-1}-invariant since S_{d-1} acts trivially on S_d/S_{d-1}. Hence we have an induced morphism

$$Y \to \Phi^{S_d/S_{d-1}},$$

which, composed with the projection $\Phi^{S_d/S_{d-1}} \to \Phi$ on the identity component, gives a morphism $Y \to \Phi$. It is easy to check that these constructions are inverses and give an equivalence of categories

$$\operatorname{Hom}(Y, \Phi) \xrightarrow{\sim} \operatorname{Hom}^{S_d}(Z, \Phi^{S_d/S_{d-1}}).$$

Since $Z \to X$ is an S_d-torsor, we also have an equivalence

$$\operatorname{Hom}^{S_d}(Z, \Phi^{S_d/S_{d-1}}) \xrightarrow{\sim} \operatorname{Hom}_{BS_d}(X, [\Phi^{S_d/S_{d-1}}/S_d])$$

and their composition

$$\mathrm{Hom}(Y, \Phi) \xrightarrow{\sim} \mathrm{Hom}_{BS_d}(X, [\Phi^{S_d/S_{d-1}}/S_d]).$$

We can repeat the same argument with $\Pi_{X/k}$, Π and Λ instead of X, Y and Z, finding an equivalence

$$\mathrm{Hom}(\Pi, \Phi) \xrightarrow{\sim} \mathrm{Hom}_{BS_d}(\Pi_{X/k}, [\Phi^{S_d/S_{d-1}}/S_d]).$$

But since $[\Phi^{S_d/S_{d-1}}/S_d]$ is a finite étale stack there is another equivalence

$$\mathrm{Hom}_{BS_d}(X, [\Phi^{S_d/S_{d-1}}/S_d]) \xrightarrow{\sim} \mathrm{Hom}_{BS_d}(\Pi_{X/k}, [\Phi^{S_d/S_{d-1}}/S_d]).$$

Composing these three, we obtain the desired equivalence

$$\mathrm{Hom}(Y, \Phi) \xrightarrow{\sim} \mathrm{Hom}(\Pi, \Phi).$$

□

Appendix B

Étale fundamental groups vs gerbes. A dictionary

We give here a brief comparison between the formalism of étale fundamental groups and étale fundamental gerbes, showing how to pass from one to the other.

There is an intermediate step between the two which is much easier to understand: the étale fundamental group scheme. On one hand, the étale fundamental group scheme already has many of the technical advantages of étale fundamental gerbes, on the other hand it requires much less machinery, hence it is a good compromise.

If the base field k is separably closed, the étale fundamental group scheme is just the étale fundamental group with a "pro-discrete" scheme structure. In general, let X be a geometrically connected scheme (or algebraic stack) and $x \in X(k)$ a <u>rational</u> point. Write $\pi_1(X, x)$ for the usual fundamental group scheme and $\pi_1(X_{k_s}, x) \subseteq \pi_1(X, x)$ for the geometric part, where k_s is the separable closure of k. Then there exists a pro-étale group scheme $\underline{\pi}_1(X, x)$ such that, if k_s is the separable closure of k, then

$$\pi_1(X_{k_s}, x) = \underline{\pi}_1(X, x)(k_s),$$

i.e. the group of geometric points of the étale fundamental group scheme of X is the classical étale fundamental group of X_{k_s}.

One may recover not only the geometric part but the full étale fundamental group. We have a natural action of $\mathrm{Gal}(k_s/k)$ by group automorphisms on the group of geometric points $\underline{\pi}_1(X, x)(k_s)$, and classical étale fundamental group $\pi_1(X, x)$ is just the semi-direct product:

$$\pi_1(X, x) = \underline{\pi}_1(X, x)(k_s) \rtimes \mathrm{Gal}(k_s/k).$$

One may also go in the other direction, i.e. from the classical étale fundamental group to the étale fundamental group scheme, but this is less obvious since it requires descent theory. If the base point x is rational, $\mathrm{Gal}(k_s/k)$ acts on the pair (X_{k_s}, x_{k_s}) and hence on $\pi_1(X_{k_s}, x_{k_s})$. By fpqc descent along $\operatorname{Spec} k_s \to \operatorname{Spec} k$, this gives us a pro-étale group scheme over k which is exactly the étale fundamental group scheme.

The difference between the two formalisms is that, while the étale fundamental group encodes the information about the base field in the short exact sequence

$$0 \to \pi_1(X_{k_s}, x) \to \pi_1(X, x) \to \mathrm{Gal}(k_s/k) \to 0,$$

the étale fundamental group scheme encodes it in its nontrivial scheme structure. The étale fundamental group scheme behaves well with respect to extensions of the base field: if k'/k is algebraic and separable, then

$$\underline{\pi}_1(X_{k'}, x) = \underline{\pi}_1(X, x) \times_{\operatorname{Spec} k} \operatorname{Spec} k'.$$

If the characteristic of k is 0, then the formula above holds for *every* field extension k'/k.

Let us recall briefly how the étale fundamental group scheme is constructed. There are several points of view, all equivalent.

The easiest way is to mimic Nori's construction contained in [Nor82]. In fact, the étale fundamental group scheme is the quotient of Nori's fundamental group scheme by the connected component of the identity. In particular, in characteristic 0 the connected component of the identity is trivial, hence they coincide. Let us recall Nori's construction (in the étale case): consider the category $\mathcal{C}$ of triples (G, T, t) where G is a finite étale group scheme, $T \to X$ is a G-torsor and $t : \operatorname{Spec} k \to T$ is a rational point over the base point $x \in X(k)$. There is an obvious notion of morphism between triples. Clearly, this category may not have an initial object, since one can take increasingly large torsors. We then introduce by force an initial object by considering not only finite étale groups but also pro-étale groups. Then in the pro-category of $\mathcal{C}$ there exists an initial triple $(\underline{\pi}_1(X, x), U, u)$ with $\underline{\pi}_1(X, x)$ a pro-étale group scheme and U an universal torsor.

If the reader enjoys Galois categories, the following may be enlightening. Consider the category Fet_X of finite étale covers of X. Since our base point x is rational, we may define a fibre functor $\omega : \mathrm{Fet}_X \to \mathrm{Ét}_k$ in the category $\mathrm{Ét}_k$ of étale algebras over k. Then we can take the sheaf of automorphisms of the fibre functor $\underline{\mathrm{Aut}}(\omega)$: this is represented by a pro-étale group scheme, i.e. $\underline{\pi}_1(X, x)$.

The étale fundamental group scheme can be constructed via Tannaka duality, too, see [BV15, §11]. In characteristic 0, it is the Tannaka dual of the Tannakian category of finite connections.

The space of sections of the classical étale fundamental group corresponds to the space of torsors of the étale fundamental group scheme. The map assigning to a rational point of X a $\underline{\pi}_1(X,x)$-torsor is particularly simple: recall that by Nori's construction we have a universal $\underline{\pi}_1(X,x)$-torsor $U \to X$, to a rational point $y \in X(k)$ we simply associate the restriction $U_y \to \operatorname{Spec} k$ of the universal torsor. Using fibre functors in étale algebras, if ω_x is the fibre functor associated to x and ω_y is the fibre functor associated to y, then to y we associate the $\underline{\pi}_1(X,x)$-torsor $\underline{\mathrm{Isom}}(\omega_x, \omega_y)$.

The main problem of the étale fundamental group scheme is that it needs the existence of a rational base point: étale fundamental gerbes solve this problem, giving a base point free theory.

If X is geometrically connected, the étale fundamental gerbe is a pro-étale gerbe $\Pi_{X/k}$ with a structural morphism

$$X \to \Pi_{X/k}$$

which is universal among morphism $X \to \Phi$, where Φ is a finite étale stack. The étale fundamental gerbe coincides with the quotient stack of Deligne's fundamental groupoid, see [Del89, pp. 10.17-18] and [BV15, Theorem 8.3]. If X has a rational point $x \in X(k)$, then

$$\Pi_{X/k} = B\underline{\pi}_1(X,x).$$

If we have a rational base point, we have said above that the space of sections of $\pi_1(X,x) \to \mathrm{Gal}(k_s/k)$ corresponds to the space of $\underline{\pi}_1(X,x)$-torsors, which in turn is simply the set of isomorphism classes of $B\underline{\pi}_1(X,x)(k)$. Using étale fundamental gerbes, this is true even if we have no rational base point: the space of sections corresponds to isomorphism classes of $\Pi_{X/k}(k)$, see [BV15, Proposition 9.3].

Actually, we may replace $\operatorname{Spec} k$ with any geometrically connected scheme T:

Proposition B.0.1. *Let X/k be a quasi-compact, quasi-separated and geometrically connected algebraic stack with a geometric point $\bar{x} : \operatorname{Spec}\Omega \to X$, and T any geometrically connected scheme with a geometric point $\bar{t} : \operatorname{Spec}\Omega \to X$. There is a (non canonical) equivalence of categories*

$$\Pi_{X/k}(T) \to \text{Hom-ext}_{G_k}(\pi_1(T,\bar{t}), \pi_1(X,\bar{x}))$$

that composed with the canonical functor $X(T) \to \Pi_{X/k}(T)$ is a lifting of the natural map

$$\mathrm{Hom}_k(T,X) \to \text{Hom-ext}_{G_k}(\pi_1(T,\bar{t}), \pi_1(X,\bar{x})).$$

This is a straightforward generalization of [BV15, Proposition 9.3], we don't repeat the proof. Proposition B.0.1 gives a very natural environment for the anabelian conjectures:

$$\left\{ \begin{array}{c} \text{morphisms in the} \\ \text{"classifying space" of } \pi_1 \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{hom. of fundamental} \\ \text{groups as extensions of } G_k \end{array} \right\} / \sim$$

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