SCUOLA NORMALE SUPERIORE DI PISA


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# Complete constant scalar curvature metrics with funnel-like ends and relativistic initial data foliated by constant mean curvature tori.

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# Abstract

By the Teichmüller Theory it is known that every complete non-compact hyperbolic surface with finite volume presents one or more ends called cusps. These surfaces can always be seen as Gromov-Hausdorff limits of families of complete, noncompact, hyperbolic surfaces whit funnel-like ends. In this thesis we consider a Riemannian manifold $(M, g)$ of dimension $n \geq 2$ with constant negative scalar curvature equal to $-n(n-1)$ whose ends are cusps and we prove the existence of a family of complete constant scalar curvature metrics $(g_\varepsilon)_{\varepsilon>0}$ converging to $g$ smoothly on compact subset of $M$, as $\varepsilon \to 0$. The cusps of $g$ are replaced with funnel-like ends in each metric of the family. An important feature of these ends is that they can be foliated by hypersurfaces whose mean curvature is constant and ranging from $-m(\varepsilon)$ to $n-1$, where $m(\varepsilon) > 0$ and $\lim_{\varepsilon \to 0^+} m(\varepsilon) = n-1$.
As a byproduct of our construction, we are able to select totally umbilical solutions to the Einstein constraint equations with apparent horizons. We recall that a triple $(M, g, K)$, where $(M, g)$ is a Riemannian manifold and $K$ is a symmetric $(0, 2)$-tensor on $M$, is said to be a solution of the Einstein constraint equations if the following system is satisfied

$$\begin{cases} \mathrm{Scal}_g - |K|_g^2 + (\mathrm{tr}_g K)^2 = 2\Lambda \\ \mathrm{div}_g K = d\mathrm{tr} K. \end{cases} \tag{1}$$

In this text the cosmological constant $\Lambda$ is assumed to lie in the interval $[-n(n-1)/2, n^2(n-1)(n-2)/2)$. Whenever it is nonempty, the boundary $\partial M$ of $M$ is called an apparent horizon if its outward null expansion $\theta_+$ is vanishing. It is worth recalling that the outward null expansion of $\partial M$ is defined as

$$\theta_+ = -\mathrm{tr}_g K + K(\nu, \nu) + (n-1)H,$$

where $\nu$ is the exterior unit normal of $\partial M$ view as an hypersurface in $M$, and $H$ is the mean curvature of $\partial M$ computed with respect to $\nu$.
In the setting described above, the triple $(M, g_\varepsilon, \lambda g_\varepsilon)$, with $\lambda = \sqrt{1 + \frac{2\Lambda}{n(n-1)}}$, automatically provides a solution to the Einstein constraint equations. Moreover it is possible to truncate each end of $g_\varepsilon$ along the leaf of the CMC foliation with mean curvature $H = \lambda \in [0, n-1)$, producing an hyperboloidal

solution with apparent horizons. Finally, we prove that these relativistic initial data $(M, g_\varepsilon, \lambda g_\varepsilon)$ verify the Riemannian Penrose inequality conjecture.

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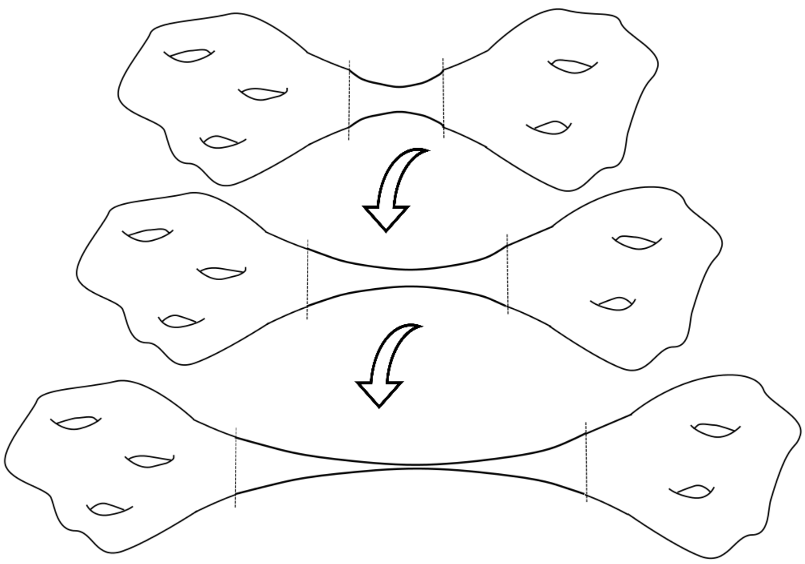
# Chapter 1

# Introduction

Initially, the idea was to generalize in arbitrary dimension a famous fact descending from the Teichmüller theory (see [83]). Briefly, it is known that every complete hyperbolic metric with finite volume defined on a non-compact surface, whose ends are *cusps*, can be obtained as Gromov-Hausdorff limit of complete hyperbolic metrics with funnels (see [43], [86]). Both cusps and funnels are ends diffeomorphic to semicylinders, but geometrically the circular slices of the first ones shrink toward the infinity whereas the circular slices of the last ones initially shrink but then, after reaching a minimal geodesic, they begin to expand toward the infinity. This fact is not the most famous fact of the Teichmüller theory, which studies the moduli space of complete hyperbolic metrics with finite volume existing on a given surface (usually but not necessarily oriented and of finite type) up to isometries (called *Teichmüller space*), but rather it is a tool used in this theory. Since the Teichmüller theory considers metrics with finite-volume and funnels do not have finite volume, usually the previous fact is presented in a different way. They are considered surfaces of finite-type with boundary and the complete hyperbolic metrics on it must have finite volume and geodesic boundary. However every geodesic boundary can be uniquely extended to a funnel and viceversa every funnel can be cut along its minimal geodesic to give raise to a surface with geodesic boundary. In any case, our initial goal was to extend the fact that every cusp can be seen as limit of funnels in higher dimension and not just for hyperbolic surface. The first problem was then to define *cusps* and *funnels* in general dimension. We also had to decide what kind of requirement use in dimension $n \geq 3$ to replace the hypothesis of hyperbolic metrics for surfaces, since in dimension $n = 2$ there is essentially a unique concept of curvature. If on one hand it was not a problem to extend the definition of *cusps* for any dimension considering manifolds with constant sectional curvature(this definition already exists in literature, see [59] for instance, although our definition is a little more general since we will not set conditions on the curvature), on the other hand (cfr Sec 2.1) the right setting for extending the definition of *funnels* turned out to be the one of *Riemannian manifolds with constant negative scalar curvature.* The result we could prove was then the following: *Given a Riemannian manifold* $(M, g)$ *with negative constant scalar curvature and cusps, it is possible to find a family of metrics with negative constant scalar curvature* $(g_\varepsilon)$ *that replace the cusps of* $g$ *by ends asymptotic to funnels (or more precisely, to funnel-like ends. See Definition 2.1) and such that* $g_\varepsilon \to g$ *as* $\varepsilon \to 0$ *smoothly on compact subsets of* $M$*. Moreover every end of* $g_\varepsilon$ *contains a unique minimal graph-hypersurface.* The proof of this result was achieved with a *Yamabe problem* approach (see [55] or Section 5.7 for a quick introduction), working with weighted functional spaces (originally introduced in [40] and used in a setting close to our by [71] and [41]). The existence of the metrics $g_\varepsilon$ could be deduced by [10], however if we would apply their result we would not be able to deduce the convergence of $g_\varepsilon$ to $g$, so we had to choose a different way to face the problem (see Section 3.5).

With similar arguments of the proof, we think it is simple to generalize a second famous fact of the Teichmüller theory, which actually follows from the one above. In fact if one consider a simple loop geodesic of a complete hyperbolic metric (a point in the Teichmüller space) and if one shrink the length of this geodesic describing a curve in the Teichmüller space, then concretely the surface develops a neck which becomes longer and finer as the geodesic shrinks as in Figure 1. The limit case when the geodesic collapse to a point (namely when its length is zero) splits the neck in two different ends (cusps) and is not a point of the Teichmüller space (it changed the topology of the surface!). In particular we can consider two different complete hyperbolic surfaces with finite volume and we can glue them through their cusps (in couples, not necessarily all the cusps) simply reverting the previous discussion. In higher dimension we think that it is possible to glue Riemannian manifold with constant negative scalar curvature whose ends are cusps just through their cusps (see Section 4.4), using the techniques of our main result with some simple adjustments.

Let us come back to the construction of the metric $g_\varepsilon$ approximating $g$ described some lines above. We asked if the minimal hypersurface included in the ends of $g_\varepsilon$ is actually a leaf of a CMC foliation, as it happens in the 2-dimensional case. The presence of a CMC foliation for the ends of manifolds would have been very interesting. In fact there are physical consequence related to this problem. For instance one can try to use it to define a mass (even more a center of mass) in the spirit of the milestone [46]. This problem was overstudied for asymptotically flat manifolds, but in our case the funnel-like ends turned out to be asymptotically hyperbolic. Although some authors like [66] and [5] studied the problem of the existence of a CMC foliation for some classes of asymptotically hyperbolic manifolds, the funnel-like end did not fall inside this classes of manifolds. However we could adapt the arguments of [5] to prove what we expected: *the ends of*



$g_\varepsilon$ *could be foliated by CMC hypersurfaces.* Let us explain the physical consequences emerging from this result one step at a time. First, both $(M, g)$ and $(M, g_\varepsilon)$ can be regarded as totally umbilical solutions of the Einstein's constraint equations. This is not yet a consequence of the CMC foliation but is simply related to the constant scalar curvature. More precisely (see Subsection 4.1.2) this means that there are Lorentzian manifolds satisfying the Einstein's field equations that contains $(M, g)$ or $(M, g_\varepsilon)$ as isometrically embedded totally umbilical (that is with extrinsic curvature proportional to the induced metric) space-like hypersurface (see [17] or Chapter 4 for an introduction). Due to the presence of a CMC foliation, we are also able to find *apparent horizons* (see Section 4.2) in $(M, g_\varepsilon)$. Without discussing now this meaning, it is now sufficient to know that this implies the presence of black holes in the spacetime associated to $(M, g_\varepsilon)$. Here they are important two observations. One is that our black holes are *exotic* in the sense that their gravitational ends do not propagate spherically (as when one consider usual spherical black holes), but are rather toroidal (the literature is less rich in this setting, however toroidal black holes have been considered in [1]). The other is that since $(M, g_\varepsilon)$ tends to $(M, g)$ as $\varepsilon \to 0$, then the same happens to their respective associated spacetimes. The black holes of the spacetime relative to $(M, g_\varepsilon)$ run away toward the infinity as $\varepsilon \to 0$, and that is why the spacetime relative to $(M, g)$ do not present black holes and $(M, g)$ has not apparent horizons.

As we mentioned before, another consequence of the existence of a CMC foliation of our asymptotically hyperbolic manifolds is that we can consider a positive mass $m_\varepsilon$ of $(M, g_\varepsilon)$, despite the most celebrated Positive Mass Theorem considers asymptotically flat manifolds. Actually there exists a Positive

Mass Theorem defined for asymptotically hyperbolic manifolds whose ends are topologically semicylinders with spherical base, but we mentioned that our ends are semicylinders with toroidal base and the literature is not very deep in this direction. Therefore the mass was defined following the analogies with the works of [5] and [24]. Once defined a mass, at least when $n = 3$, the presence of a CMC foliation (including an apparent horizon) for $(M, g_\varepsilon)$ introduces to a well-known problem, called *Riemannian Penrose inequality* conjecture. For an introduction to this problem, we recommend [24]. Most of the results about the Penrose inequality concern asymptotically flat manifolds. In general, and particularly in our setting, the Penrose inequality is still a conjecture. For the manifolds $(M, g_\varepsilon)$, assuming for simplicity the existence of a unique end, it is equivalent to show that

$$\left(\frac{A_\varepsilon}{16\pi}\right)^{1/2} + 4\left(\frac{A_\varepsilon}{16\pi}\right)^{3/2} \leq m_\varepsilon,$$

where $A_\varepsilon$ is the area of the apparent horizon of $(M, g_\varepsilon)$. Notice that the constants $16\pi$ do not have a physical meaning (for spherical slices, they simplify some computation) since the mass $m_\varepsilon$ is definite up to multiply by numerical constants. Precisely, the right value for $m_\varepsilon$ (or equivalently of the constants that can replace $16\pi$) is chosen in such a way that there is a model satisfying the Penrose inequality as an equality. In fact, the Penrose inequality often contains a rigidity statement: if it holds as an equality, the considered manifold is isometric to a model case. We were able to show that *the Riemannian manifolds* $(M, g_\varepsilon)$ *strictly verify the Penrose inequality.* If the Penrose inequality conjecture was true in general, then this heuristically means that our spacetime do not present singularities with exception of the Big Bang and of singularities within event horizons (therefore not observable from outside). This is known as the *Cosmic censorship hypothesis* (see [44] or [79] for an introduction).

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The Chapter 2 gives the definitions of *cusps* and *funnel-like ends* (a subclass of asymptotically hyperbolic ends), which are the possible ends we will consider in this text. The Chapter 3 is so organized. In Section 3.1 we state the main result. Namely, we consider a Riemannian manifold $(M, g)$ with cusps and constant negative scalar curvature, and we build a family of metrics $(g_\varepsilon)_\varepsilon$ on $M$ which is in some sense close to $g$ for small $\varepsilon$ and preserve the constant scalar curvature. Every metric $g_\varepsilon$ has asymptotically hyperbolic ends which can be foliated by CMC hypersurfaces, one of the leaves is minimal. The proof of the main result begins in Section 3.2, where we build approximate solutions $h_\varepsilon$, which can not be chosen as exact solutions $g_\varepsilon$ since they lack a constant scalar curvature in a compact subset of the ends of $M$. The exact solutions will be obtained via a conformal transformation of $h_\varepsilon$, this reduce to a non-compact Yamabe problem. This is solved by a fixed-point approach in Subsection 3.5, after introducing a suitable weighted space of functions where to work in Section 3.3 and proving an a priori uniform estimate for the linear operator associated to our problem in Section 3.4. After that, we have to check the existence of a CMC foliation for the ends of $g_\varepsilon$. In Section 3.6 we prove that such a foliation exists on a compact subset of the ends, then in Section 3.7 we extend it to the whole ends. Due to the existence and the properties of the found CMC foliation, we can apply the main result to the General Relativity. In Chapter 4 we will see in what sense the manifolds involved in our main result can be seen as solutions of the Einstein's constraint equations with apparent horizons. We are also able to check the validity of the Riemannian Penrose Inequality for our manifolds. This inequality is still a conjecture for general asymptotically hyperbolic manifolds. Finally we included an appendix containing useful result which are applied in Chapter 3.

# Chapter 2

# Special ends of Riemannian manifolds

This chapter treats particular types of ends which can be encountered when considering non-compact ends equipped with a warped product metric with constant negative scalar curvature. These ends will be the main characters of this text and are called *cusps* and *funnel-like ends* since their definitions, which is given in the next sections, extend the 2-dimensional notion of ″cusps‶and ″funnels‶respectively. Actually we will give the two mentioned definitions without assuming any hypothesis on the scalar curvature, however we have been inspired by the following observation. Notice that by Proposition 5.1, a warped product which can be written as $dr^2 + \psi(r)^{4/n}\bar{g}$, where $(\Sigma, \bar{g})$ is a scalar flat manifold of dimension $n-1$, have constant negative curvature equal to $-n(n-1)$ if and only if it holds $\ddot{\psi} = \frac{n^2}{4}\psi$. So we can take $\psi(r) = a\mathrm{e}^{nr/2} + b\mathrm{e}^{-nr/2}$ for some $a, b \in \mathbb{R}$ such that $\psi > 0$. In practise the definition of funnel-like end comes from the case $a \neq 0$ and $b \neq 0$, up to a translation of $r$, whereas the definition of cusp comes from the case $b = 0$ (or $a = 0$, up to replace $r$ by $-r$).

## 2.1 Funnel-like ends

**Definition 2.1.** Let $(M, g)$ be a complete Riemannian manifold of dimension $n \geq 2$. A ***funnel-like end*** is an open subset of $M$ diffeomorphic to $(r_0, +\infty) \times \Sigma$ for some $r_0 \leq 0$ and some compact Riemannian manifold $(\Sigma, \bar{g})$ of dimension $n-1$, such that

$$g|_{(s_0,+\infty)\times\Sigma} = dr^2 + \varepsilon^2 \left(2\cosh\left(\frac{nr}{2}\right)\right)^{4/n} \bar{g}, \tag{2.1}$$

with $\varepsilon > 0$.

The origin of the name descends from the fact that the areas of the slices $\{r\} \times \Sigma$ increase as $r \to +\infty$ so that they recall a funnel. By Proposition 5.1 a funnel-like end has constant negative scalar curvature if and only if $\mathrm{R}_{\bar{g}} = 0$, and it is never hyperbolic nor Einstein (except for $n = 2$). In dimension $n = 2$ we get precisely the usual concept of funnel. By Lemma 5.4 the slices $\{r\} \times \Sigma$ have second fundamental form $\mathrm{II} = \varepsilon^2 \left(2\cosh\left(\frac{nr}{2}\right)\right)^{4/n} \tanh(nr/2)\bar{g}$ and therefore constant mean curvature $H = (n-1)\tanh(nr/2)$ computed with respect to the unit normal vector field $\partial_r$. It is possible to prove that the funnel-like ends are conformally compact, more precisely they are asymptotically hyperbolic. We recall that a Riemannian manifold $(Z, \gamma)$ is called *conformally compact* if $Z$ is the interior of a compact Riemannian manifold $(\hat{Z}, \hat{\gamma})$ with non-empty boundary and if there exists a smooth non-negative function $\omega\colon \hat{Z} \to \mathbb{R}$ (called *boundary defining function*) such that $\{\omega = 0\} = \partial\hat{Z}$, $d\omega \neq 0$ on $\partial\hat{Z}$ and $\gamma = \omega^{-2}\hat{\gamma}$ on $Z$. If moreover $|d\log\omega|_\gamma \to 1$ as $\omega \to 0$, then $(Z, \gamma)$ is called *asymptotically hyperbolic*. The reason of this name descends from the fact that on a conformally compact manifold the curvature tensor R of $\gamma$ takes the form

$$\mathrm{R}_{ijkl} = -|d\omega|^2_{\hat{\gamma}}(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}) + O(\omega), \quad \text{as } \omega \to 0,$$

thus it holds $\sec(\gamma) \to -1$ as $\omega \to 0$ if and only if $(Z, \gamma)$ is asymptotically hyperbolic. Let us check that a funnel-like end is asymptotically hyperbolic. First set $t = r + \log\varepsilon$ so that

$$g = dt^2 + \mathrm{e}^{2t}\left(1 + \varepsilon^n \mathrm{e}^{-nt}\right)^{4/n} \bar{g}.$$

From this expression we see that $g$ is asymptotic to $dt^2 + \mathrm{e}^{2t}\bar{g}$ independently from $\varepsilon$, so we will check the thesis for the latter metric. We chose $\omega(t) := \mathrm{e}^{-t}$ and we claim that it it the wanted boundary defining function. The function $\omega$ is positive, $\omega \to 0$ as $t \to +\infty$ and $|d\log\omega|_{dt^2+\mathrm{e}^{2t}\bar{g}} = 1$, moreover $\omega^2(dt^2 + \mathrm{e}^{2t}\bar{g}) = d\omega^2 + \bar{g}$ can be extended as a smooth cylindrical metric up to the boundary $\{\omega = 0\}$, corresponding to $t \to +\infty$. This proves the claim.

## 2.2 Cusps

**Definition 2.2.** Let $(M, g)$ be a complete Riemannian manifold of dimension $n \geq 2$. A ***cusp*** is an open subset of $M$ diffeomorphic to $(s_0, +\infty) \times \Sigma$ for some $s_0 \in \mathbb{R}$ and some compact Riemannian manifold $(\Sigma, \bar{g})$ of dimension $n - 1$, such that

$$g|_{(s_0,+\infty)\times\Sigma} = ds^2 + \mathrm{e}^{-2s}\bar{g}. \tag{2.2}$$

The origin of the name descends from the fact that the areas of the slices $\{s\} \times \Sigma$ decrease exponentially as $s \to +\infty$. By Proposition 5.1 a cusp has constant negative scalar curvature if and only if $\mathrm{R}_{\bar{g}} = 0$. Similarly it can be shown that a cusp is Einstein ($\mathrm{Ric}_g = -(n-1)g$) if and only if $\bar{g}$ is Einstein ($\mathrm{Ric}_{\bar{g}} = 0$) and that a cusp is hyperbolic if and only if $\bar{g}$ is flat. In dimension $n = 2$ we get the usual definition of cusp. By

Lemma 5.4 the slices $\{s\} \times \Sigma$ have second fundamental form $\mathrm{II} = -\mathrm{e}^{-2s}\bar{g}$ and therefore constant mean curvature $H = -(n-1)$ computed with respect to the unit normal vector field $\partial_s$. A cusp is not conformally compact, in fact it can be easily shown using Gauss Lemma that in suitable coordinates a conformally compact metric can be written in the form $d\sigma^2 + \omega^{-2}(\sigma)G_\sigma$. Here $G_\sigma$ is a family of metrics on $\Sigma$ tending to a limit metric $G_0$ on $\Sigma$, as $\sigma \to +\infty$, and $\omega(\sigma) \to 0$, as $\sigma \to +\infty$. In particular the sections $\{\sigma = \text{const.}\}$ become "larger" when approaching the infinity, whereas in the cusps they shrink. We already discussed in what sense every complete hyperbolic surface with finite volume presents a cusp in a neighbourhood of every puncture. This result can be actually extended on higher dimension by the Margulis thick-thin decomposition [19, pag. 133]. Namely, every complete hyperbolic manifold with finite volume contains a compact part (*thick part*) whose complementary set (*thin part*) is the finite disjoint union of open subsets each one isometric to $\big((s_0, +\infty) \times \Sigma, ds^2 + \mathrm{e}^{-2s}\bar{g}\big)$, where $(\Sigma, \bar{g})$ is some flat manifold. Therefore the ends of a complete hyperbolic metric with finite volume are all examples of cusps.

# Chapter 3

# Complete hyperbolic metrics with funnel-like ends

## 3.1 Statement of the main result

**Theorem 3.1** (Main result). *Let $(M, g)$ be a $n$-dimensional Riemannian manifold ($n \geq 2$) with constant scalar curvature equal to $-n(n-1)$, consisting of a compact core and $k$ cusps (see Definition 2.2). For $i = 1, \dots, k$ fix a positive parameter $\varepsilon_i$, one for each cusp. If $\varepsilon = (\varepsilon_1, \dots, \varepsilon_k)$ is small enough then there exists a metric $g_\varepsilon$ on $M$ such that:*

(1) *The metric $g_\varepsilon$ has constant scalar curvature $-n(n-1)$;*

(2) *The metric $g_\varepsilon$ replaces every one of the $k$ cusps of $g$ with an asymptotically hyperbolic end. More precisely each of these ends is conformal and asymptotic to a funnel-like end (see Definition 2.1);*

(3) *The metric $g_\varepsilon$ smoothly converges to $g$ as $\varepsilon \to 0$ on compact subsets of $M$;*

(4) *Every asymptotically hyperbolic end of $g_\varepsilon$ is foliated by a family of weakly stable CMC hypersurfaces whose mean curvatures span $(-m, n-1) \subset \mathbb{R}$, for some positive $m = m(\varepsilon) \in \mathbb{R}$ which can be different for each end and satisfies $\lim_{\varepsilon \to 0} m = n - 1$. Moreover the area $\mathrm{Area}_{g_\varepsilon}(\Sigma_{\min})$ of the minimal hypersurface $\Sigma_{\min} \subset M$ belonging to an asymptotically hyperbolic end of $g_\varepsilon$ is comparable to $\varepsilon_i^{n-1}$, where $\varepsilon_i$ is the parameter relative to the involved end. This means that there exists a constant $c > 1$ independent of $\varepsilon$ such that $c^{-1}\varepsilon_i^{n-1} \leq \mathrm{Area}_{g_\varepsilon}(\Sigma_{\min}) \leq c\varepsilon_i^{n-1}$.*

## 3.2 Approximate solutions

The metric $g_\varepsilon$ that we are going to build for small $\varepsilon$ in order to prove Theorem 3.1 will turn out to be a small perturbation of an approximate solution $h_\varepsilon$. In this subsection we define the metric $h_\varepsilon$ and we study its main properties. The metric $h_\varepsilon$ is an approximate solution meaning that it fulfills the requirements of Theorem 3.1 but the first point (its negative scalar curvature is not constant in the whole manifold $M$).

Consider $(M, g)$ as in the hypothesis of Theorem 3.1. For simplicity we will suppose that $(M, g)$ is a Riemannian manifold of dimension $n \geq 2$ with constant scalar curvature $\mathrm{R}_g = -n(n-1)$ and with a *unique* cusp. By definition this means that there exists a subset $U \subset M$ such that $M \setminus U$ is relatively compact and $U$ is diffeomorphic to $[s_0, +\infty) \times \Sigma$ with

$$g|_U = ds^2 + \mathrm{e}^{-2s}\bar{g}. \tag{3.1}$$

As a consequence of Proposition 5.1, necessarily $(\Sigma, \bar{g})$ is a compact manifold of dimension $n-1$ with zero scalar curvature. Without loss of generality we can assume $s_0 = 0$. Introduce now a smooth cut-off function $\eta \colon M \to [0, 1]$ vanishing outside the cusp $U$ and coinciding with 1 in $\{s \geq 1\} \subset U$, then define for $\varepsilon > 0$ the metric

$$h_\varepsilon := \begin{cases} g, & \text{in } M \setminus U \\ ds^2 + \mathrm{e}^{-2s}\left(1 + \eta(s)\varepsilon^n \mathrm{e}^{ns}\right)^{4/n} \bar{g} & \text{in } U. \end{cases} \tag{3.2}$$

The metric $h_\varepsilon$, which will be called *approximate solution*, has an asymptotically hyperbolic end isometric to a funnel-like end. To see this, set $r = s + \log \varepsilon$, then in $\{s \geq 1\}$ it holds

$$h_\varepsilon = dr^2 + \varepsilon^2 \left(2\cosh\left(\frac{nr}{2}\right)\right)^{4/n} \bar{g}. \tag{3.3}$$

By construction $\mathrm{R}_{h_\varepsilon} + n(n-1)$ vanishes everywhere outside the region $\{0 < s < 1\}$. We have the following estimate:

**Lemma 3.2.** *The following inequality holds for the scalar curvature of the approximate solutions $h_\varepsilon$ defined above. For every $k \in \mathbb{N}$ there exists $C > 0$ such that for small $\varepsilon > 0$, one has*

$$\|R_{h_\varepsilon} + n(n-1)\|_{\mathscr{C}^k(M)} \leq C\varepsilon^n, \tag{3.4}$$

*where $\mathscr{C}^k(M)$ can be equivalently computed with respect to $h_\varepsilon$ or $g$. The constant $C$ only depends on $\eta$, $k$ and $n$.*

*Proof.* We restrict ourselves to the case where $0 < s < 1$. For bounded $s$ the metrics $g$ and $h_\varepsilon$ are comparable and so are their induced norms. By Proposition 5.1 it follows that that in $\{0 \leq s \leq 1\}$ one has

$$\mathrm{R}_{h_\varepsilon} + n(n-1) = -4\frac{n-1}{n}\varepsilon^n \mathrm{e}^{ns} \frac{\ddot{\eta}(s) + n\dot{\eta}(s)}{1 + \eta(s)\varepsilon^n \mathrm{e}^{ns}},$$

so it is sufficient to prove that

$$\sup_{s\in[0,1]}\left|\frac{d^k}{ds^k}\left(\mathrm{e}^{ns}\frac{\ddot{\eta}(s)+n\dot{\eta}(s)}{1+\eta(s)\varepsilon^n\mathrm{e}^{ns}}\right)\right|\le C_1$$

for some constant $C_1 = C_1(n,k,\eta)$. That is true since we are bounding a ratio whose denominator is greater than 1 and whose numerator consists on products of derivatives of $\mathrm{e}^{ns}$, $\ddot{\eta}(s)+n\dot{\eta}(s)$ and $1+\eta(s)\varepsilon^n\mathrm{e}^{ns}$, all bounded in terms of $n$, $k$ and $\|\eta\|_{\mathscr{C}^{k+2}}$. □

We conclude this subsection analysing three remarkable behaviours of $h_\varepsilon$ as the parameter $\varepsilon$ tends to zero depending on the subset of $M$ where one wants to consider this limit. The first lemma involves the compact core of $M$.

**Lemma 3.3.** *The metrics $h_\varepsilon$ in* (3.2) *converge to $g$ as $\varepsilon\to 0$ in $\mathscr{C}^\infty$-norm on compact subsets of $M$.*

*Proof.* Let $\Omega$ be a compact subset of $M$, we can suppose without loss of generality that $\Omega = M\setminus\{s>C\}$ for some large $C$. Since $h_\varepsilon$ coincides with $g$ outside the cusp, the lemma holds away from $\{0\le s\le C\}$. By the local expression of $g$ and $h_\varepsilon$ in this region, it is sufficient to prove that

$$\lim_{\varepsilon\to 0}\sup_{s\in[0,C]}\left|\frac{d^k}{ds^k}\left(\eta(s)\varepsilon^n\mathrm{e}^{ns}\right)\right| = 0$$

for every $k\in\mathbb{N}$. This follows from the boundedness of $\mathrm{e}^{ns}$ and of the derivatives of the cut-off function $\eta$, and we are done. □

The second lemma describes the behaviour of $h_\varepsilon$ along the end of $M$.

**Lemma 3.4.** *Let us consider the coordinate $t = s+2\log\varepsilon$. Then the metrics $h_\varepsilon$ defined in* (3.2) *converge to the metric $dt^2+\mathrm{e}^{2t}\bar{g}$ with all their derivatives uniformly on subsets of the form $\{t\ge a\}$, $a\in\mathbb{R}$, as $\varepsilon\to 0$.*

*Proof.* It is sufficient to observe that

$$h_\varepsilon = dt^2+\mathrm{e}^{2t}\left(1+\varepsilon^n\mathrm{e}^{-nt}\right)^{4/n}\bar{g} \tag{3.5}$$

on $\{s\ge 1\} = \{t\ge 1+2\log\varepsilon\}$, then the thesis follows with a symmetric argument as in the previous Lemma 3.3. □

Finally, we notice that by (3.3) the metric $h_\varepsilon$ collapses to $dr^2$ as $\varepsilon\to 0$ in a neighbourhood of $\{r=0\}$, which is an bottleneck region between the compact core of $M$ and its end. However, the next lemma shows that it is possible to introduce a conformal factor which not only prevents the collapse of $h_\varepsilon$ but also makes it to converge to a cylinder.

**Lemma 3.5.** *Let us consider the coordinate $r = s + \log\varepsilon$. Then for every $a \in \mathbb{R}$ the Riemannian manifold $\{|r| < -\log\varepsilon + a\}$ equipped with $\varepsilon^{-2}\left(2\cosh\left(\frac{nr}{2}\right)\right)^{-4/n} h_\varepsilon$ defined in (3.2) converge in the Gromov-Hausdorff sense to the cylinder $\mathbb{R}\times\Sigma$ as $\varepsilon \to 0$.*

*Proof.* Let $\rho = \rho(r)$ be the solution of the 1st-order ODE

$$\dot\rho(r) = \varepsilon^{-1}\left(2\cosh\left(\frac{nr}{2}\right)\right)^{-2/n}, \qquad \rho(0) = 0. \tag{3.6}$$

Since $\dot\rho > 0$, this function is actually a new coordinate on $\{|r| < -\log\varepsilon + a\}$. A direct computation from (3.3) gives

$$\varepsilon^{-2}\left(2\cosh\left(\frac{nr}{2}\right)\right)^{-4/n} h_\varepsilon = d\rho^2 + \bar g.$$

Thus the manifold $\{|r| < -\log\varepsilon + a\}$ equipped with the metric $\varepsilon^{-2}\left(2\cosh\left(\frac{nr}{2}\right)\right)^{-4/n} h_\varepsilon$ is isometric to the limited cylinder $\left(\{|\rho| < \rho_0\}, d\rho^2 + \bar g\right)$ for some $\rho_0 = \rho_0(a,\varepsilon)$. The positive constant $\rho_0$ is given by

$$\rho_0 = \frac{1}{\varepsilon}\int_0^{-\log\varepsilon + a} \frac{dr}{(\cosh(nr/2))^{2/n}}$$

and tends to $+\infty$ as $\varepsilon \to 0$, for every fixed $a \in \mathbb{R}$. This implies the thesis. □

As first goal, we want to perturb the approximate solution $h_\varepsilon$ with a conformal factor in order to get a metric $g_\varepsilon$ with constant negative scalar curvature on $M$.

## 3.3 Weighted functional spaces

### The Yamabe Problem

Consider a perturbation of the approximate solution $h_\varepsilon$ in the form

$$g_\varepsilon := (1 + v_\varepsilon)^{\frac{4}{n-2}} h_\varepsilon, \quad \text{if } n \ge 3 \qquad \text{and} \qquad g_\varepsilon = e^{2v_\varepsilon} h_\varepsilon, \quad \text{if } n = 2. \tag{3.7}$$

Here $v_\varepsilon$ is smooth on $M$ and such that the conformal factor is positive. If we want $g_\varepsilon$ to satisfy $\mathrm{R}_{g_\varepsilon} = -n(n-1)$, then by Proposition 4.5 this is equivalent to solve the well-known *Yamabe equation*

$$\begin{cases} (\Delta_{h_\varepsilon} - 2)\, v_\varepsilon = \frac{1}{2}\mathrm{R}_{h_\varepsilon} + 1 + Q(v_\varepsilon) & \text{if } n = 2, \\ \\ (\Delta_{h_\varepsilon} - n)\, v_\varepsilon = \frac{n-2}{4(n-1)}\left(\mathrm{R}_{h_\varepsilon} + n(n-1)\right)(1 + v_\varepsilon) + Q(v_\varepsilon) & \text{if } n \ge 3, \end{cases} \tag{3.8}$$

where $Q$ is a quadratic remainder defined as

$$Q(v) := \begin{cases} e^{2v} - 2v - 1 & \text{if } n = 2, \\ \frac{n(n-2)}{4}(1+v)^{\frac{n+2}{n-2}} - \frac{n(n+2)}{4}v - \frac{n(n-2)}{4} & \text{if } n \ge 3. \end{cases} \tag{3.9}$$

A remarkable property of the quadratic remainder is that its coefficients do not depend on the parameter $\varepsilon > 0$.

*REMARK* 3.1. It is very important to point out that the original part of this text is *not* to prove the existence of a smooth solution $v_\varepsilon$ of the Yamabe equation (3.8). In fact the existence and uniqueness of such a solution in this context is guaranteed (at least for $n \geq 3$) by classical works like [71], [12] and [13]. For what concerns the case $n = 2$, analogous results can be found in [86] and [87]. It is also possible to use the literature (cfr. Theorem 1.2 of [9]) to show that $v_\varepsilon$ tends to zero along the ends of $M$. In short the points (1) and (2) of Theorem 3.1 can be easily deduced by well-known results, but this does not hold for points (3) and (4). In fact the originality of this text involves an estimate for $v_\varepsilon$ which will look like

$$\|v_\varepsilon\|_{\mathscr{C}^{k,\alpha}_\delta(M,h_\varepsilon)} \leq C\varepsilon^n. \tag{3.10}$$

The precise definition of this norm is treated in the next subsection, basically it gives us two information. Firstly it implies that $v_\varepsilon$ and its derivatives tends to zero as $\varepsilon \to 0$. This will allow us to deduce point (3) and to prove that there is a CMC foliation on a compact subset of each end of $(M, g_\varepsilon)$, if $\varepsilon$ is small enough. Secondly it implies that for fixed $\varepsilon$ the function $v_\varepsilon$ (and derivatives) decrease exponentially as it approaches the infinity. This will allow us to extend the compact CMC foliation to the whole end of $(M, g_\varepsilon)$. For these reasons we decided to infer the existence of a solution $v_\varepsilon$ of (3.8) (which is a posteriori the same solution provided by the literature) with a different method (fixed-point), that also gives us the estimates (3.10).

The proof of the existence of a smooth solution $v_\varepsilon$ of the Yamabe equation satisfying (3.10) is so organized:

- We conclude this Section introducing the space of functions $\mathscr{C}^{k,\alpha}_\delta(M, h_\varepsilon)$ where to face the problem and we motivate the decision for this choice.
- In Sections 3.4 we prove some a priori estimates (Proposition 3.7) for the operator $\Delta_{h_\varepsilon} - n$. This requires some preliminary lemmas.
- In Section 3.5 we apply the fixed-point method under the guise of a contraction theorem, getting the desired result.

### Weighted Hölder spaces

The usual Riemannian functional spaces of $(M, h_\varepsilon)$ are not the right choice to solve the Yamabe equation (3.8). The reason is hidden in the behaviour of $h_\varepsilon$ far away from the compact core of $M$ as $\varepsilon \to 0$. We recall that $h_\varepsilon$ is close to $g$ for small $\varepsilon > 0$ on compact subsets of the form $M \setminus \{s > \text{const.}\}$ by Lemma 3.3, it collapses around $\{r = 0\}$ as $\varepsilon \to 0$ but a suitable conformal rescaling makes it to be close to a cylindrical metric by Lemma 3.5, whereas it is close to the asymptotically hyperbolic metric $dt^2 + \mathrm{e}^{2t}\bar{g}$ on subsets of the form $\{t > \text{const.}\}$, with $t = s + 2\log\varepsilon$, by Lemma 3.4. These three distinct behaviours of $h_\varepsilon$ suggest the next definition.

**Definition 3.1.** The manifold $M$ is the union of $\mathcal{K}$, $\mathcal{N}$ and $\mathcal{E}$, with the following definitions:

- The ***compact core*** $\mathcal{K} := M \setminus \{s \geq 3\}$ is a relatively compact subset, which we improperly denote as $\mathcal{K} = \{s < 3\}$. It contains a neighbourhood of the cut-off area of $\eta$, see (3.2);
- The ***neck region*** $\mathcal{N} := \{2 < s < -2\log\varepsilon - 2\}$ is a neighbourhood of $\{s = -\log\varepsilon\}$. In this region is useful to work with the variable $r = s + \log\varepsilon$ so that $\mathcal{N} = \{|r| < -\log\varepsilon - 2\}$ is centred at $\{r = 0\}$. It is also convenient to define a new variable $\rho$ by

$$\dot{\rho}(r) = \varepsilon^{-1}\left(2\cosh\left(\frac{nr}{2}\right)\right)^{-2/n}, \qquad \rho(0) = 0; \tag{3.11}$$

- The ***funnel-like end*** $\mathcal{E} := \{s > -2\log\varepsilon - 3\}$ is the end of $(M, g_\varepsilon)$. In this region is useful to work with the variable $t = s + 2\log\varepsilon$ so that $\mathcal{E} = \{t > -3\}$.

A qualitative picture of this partition of $M$ is represented in Figure 3.3. We can finally define the announced space of functions on $(M, h_\varepsilon)$.

**Definition 3.2.** Fix $k \in \mathbb{N}$, $\alpha \in (0,1)$ and $\delta \in \mathbb{R}$. Given an open subset $U \subseteq M$ and a function $f \in \mathscr{C}^{k,\alpha}(U)$, we define

$$\|f\|_{\mathscr{C}^{k,\alpha}_\delta(U,h_\varepsilon)} := \|f\|_{\mathscr{C}^{k,\alpha}(\mathcal{K}\cap U,g)} + \|f\|_{\mathscr{C}^{k,\alpha}(\mathcal{N}\cap U,g_{\mathrm{cyl}})} + \|f\|_{\mathscr{C}^{k,\alpha}_\delta(\mathcal{E}\cap U,g_{\mathrm{ah}})}, \tag{3.12}$$

where $g_{\mathrm{cyl}} = d\rho^2 + \bar{g}$ and $g_{\mathrm{ah}} = dt^2 + \mathrm{e}^{2t}\bar{g}$. In the equation (3.12) above, we have set

$$\|f\|_{\mathscr{C}^{k,\alpha}(\mathcal{K}\cap U,g)} = \max_{j\leq k}\sup_{\mathcal{K}\cap U}|\nabla^{(j)}_g f| + \sup_{p\neq q\in\mathcal{K}\cap U}\frac{|\nabla^{(k)}_g f(p) - \nabla^{(k)}_g f(q)|_g}{d_g(p,q)^\alpha},$$

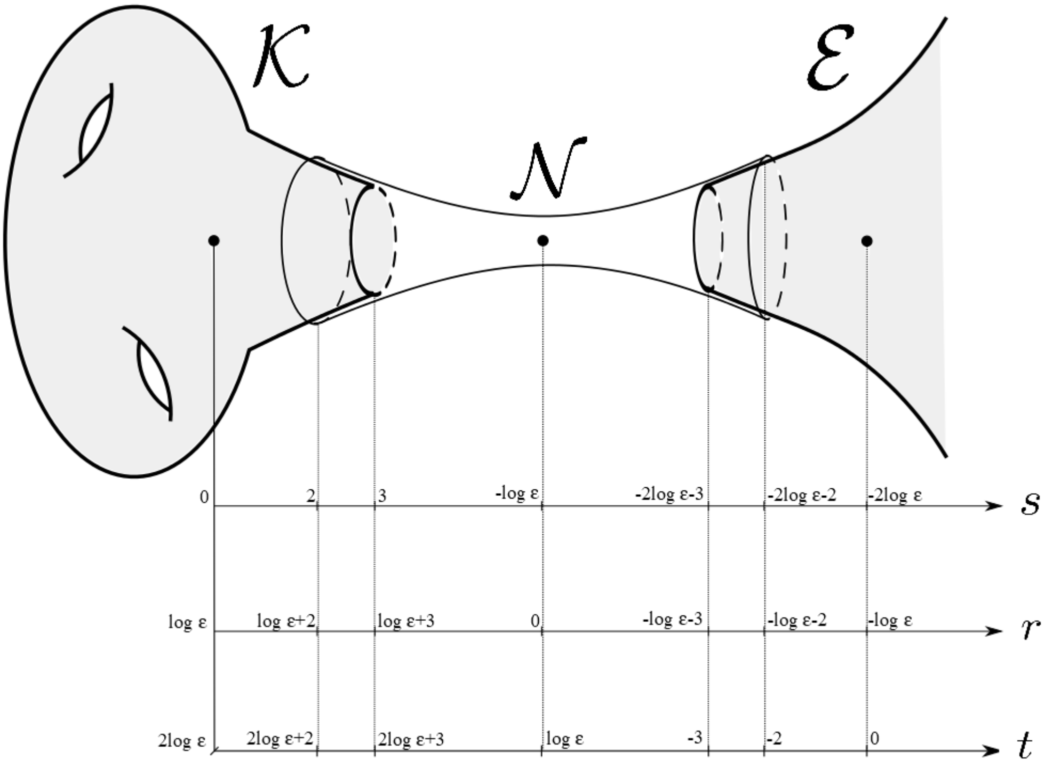
we have also set

$$\|f\|_{\mathscr{C}^{k,\alpha}(\mathcal{N}\cap U,g_{\mathrm{cyl}})} = \max_{j\leq k}\sup_{\mathcal{N}\cap U}|\nabla^{(j)}_{g_{\mathrm{cyl}}} f|_{g_{\mathrm{cyl}}} + \sup_{p\neq q\in\mathcal{N}\cap U}\frac{|\nabla^{(k)}_{g_{\mathrm{cyl}}} f(p) - \nabla^{(k)}_{g_{\mathrm{cyl}}} f(q)|_{g_{\mathrm{cyl}}}}{d_{g_{\mathrm{cyl}}}(p,q)^\alpha}$$

and finally we have set

$$\|f\|_{\mathscr{C}^{k,\alpha}_\delta(\mathcal{E}\cap U,g_{\mathrm{ah}})} = \max_{j\leq k}\sup_{\mathcal{E}\cap U}\mathrm{e}^{\delta t}|\nabla^{(j)}_{g_{\mathrm{ah}}} f|_{g_{\mathrm{ah}}} + \sup_{p\neq q\in\mathcal{E}\cap U}\mathrm{e}^{\delta t(p)}\frac{|\nabla^{(k)}_{g_{\mathrm{ah}}} f(p) - \nabla^{(k)}_{g_{\mathrm{ah}}} f(q)|_{g_{\mathrm{ah}}}}{d_{g_{\mathrm{ah}}}(p,q)^\alpha}.$$

The Banach space containing the functions that satisfy $\|f\|_{\mathscr{C}^{k,\alpha}_\delta(U,h_\varepsilon)} < +\infty$ is then denoted $\mathscr{C}^{k,\alpha}_\delta(U, h_\varepsilon)$ and can be regarded as a *weighted Hölder space* (see [41] [9]).




*REMARK* 3.2. Apparently the parameter $\varepsilon$ is not involved in the right-hand side of (3.12). This optical illusion comes from the fact that both $g_{\mathrm{cyl}}$ and $g_{\mathrm{ah}}$ are written in terms of variables $\rho$ and $t$, which depend on $\varepsilon$.

We observe that the three different norms used to define $\|f\|_{\mathscr{C}^{k,\alpha}_{\delta}(U,h_\varepsilon)}$ are equivalent in the overlapping subsets of $M$, uniformly in $\varepsilon$, as showed in the next lemma.

**Lemma 3.6.** *In the setting of Definition 3.2 there exists $C > 0$ independent of $\varepsilon > 0$ such that*

$$\begin{cases} C^{-1}\|f\|_{\mathscr{C}^{k,\alpha}(\mathcal{K}\cap\mathcal{N},g)} \leq \|f\|_{\mathscr{C}^{k,\alpha}(\mathcal{N}\cap\mathcal{K},g_{\mathrm{cyl}})} \leq C\|f\|_{\mathscr{C}^{k,\alpha}(\mathcal{K}\cap\mathcal{N},g)} \\ C^{-1}\|f\|_{\mathscr{C}^{k,\alpha}_{\delta}(\mathcal{E}\cap\mathcal{N},g_{\mathrm{ah}})} \leq \|f\|_{\mathscr{C}^{k,\alpha}(\mathcal{N}\cap\mathcal{E},g_{\mathrm{cyl}})} \leq C\|f\|_{\mathscr{C}^{k,\alpha}_{\delta}(\mathcal{E}\cap\mathcal{N},g_{\mathrm{ah}})} \end{cases} \tag{3.13}$$

*for every $f \in \mathscr{C}^{k,\alpha}(M)$.*

*Proof.* For the first inequality, we notice that on $\mathcal{K}\cap\mathcal{N} = \{2 < s < 3\}$ the norm induced by the metric $g = ds^2 + \mathrm{e}^{-2s}\bar{g}$ is equivalent to the norm induced by the metric $ds^2+\bar{g}$. This is true because $\mathrm{e}^{-2s}$ is uniformly bounded above and below by positive constants for $2 < s < 3$, and the same holds for the absolute value of its derivatives. On the other hand, by (3.11) and $r = s + \log \varepsilon$, it can be computed that

$$g_{\mathrm{cyl}} = \mathrm{e}^{2s} \left(1 + \varepsilon^n \mathrm{e}^{-ns}\right)^{-4/n} ds^2 + \bar{g}.$$

Since $\mathrm{e}^{2s} \left(1 + \varepsilon^n \mathrm{e}^{-ns}\right)^{-4/n}$ can be uniformly bounded above and below by positive constants for $2 < s < 3$, and the same for the absolute value of its derivatives, also the norm induced by $g_{\mathrm{cyl}}$ is equivalent to the norm induced by $ds^2 + \bar{g}$ on $\mathcal{K} \cap \mathcal{N}$. This shows the first inequality. The second one can be proved similarly on $\mathcal{E} \cap \mathcal{N} = \{-3 < t < -2\}$, but in this case one has also to notice that for $-3 < t < 2$ the weight function $\mathrm{e}^{\delta t}$ is uniformly bounded above and below by positive constants. □

We conclude this subsection motivating why we considered the definition (3.12) to study elliptic problems on $(M, h_\varepsilon)$.

### The norm on $\mathcal{K}$

In the compact part $\mathcal{K}$ of $M$ the norm $\|\cdot\|_{\mathscr{C}^{k,\alpha}(\mathcal{K}\cap U, g)}$ is essentially the unique reasonable choice. One could equivalently replace this norm by norm

$$\|f\|_{\mathscr{C}^{k,\alpha}(\mathcal{K}\cap U, h_\varepsilon)} = \max_{j \le k} \sup_{\mathcal{K}\cap U} |\nabla^{(j)}_{h_\varepsilon} f| + \sup_{p \ne q \in \mathcal{K}\cap U} \frac{|\nabla^{(k)}_{h_\varepsilon} f(p) - \nabla^{(k)}_{h_\varepsilon} f(q)|_{h_\varepsilon}}{d_{h_\varepsilon}(p,q)^\alpha}.$$

induced by $h_\varepsilon$. The fact that those norms are equivalent descends from Lemma 3.3, namely there exists $C > 0$ independent of small $\varepsilon > 0$ such that

$$C^{-1} \|\cdot\|_{\mathscr{C}^{k,\alpha}(\mathcal{K}\cap U, g)} \le \|\cdot\|_{\mathscr{C}^{k,\alpha}(\mathcal{K}\cap U, h_\varepsilon)} \le C \|\cdot\|_{\mathscr{C}^{k,\alpha}(\mathcal{K}\cap U, g)}$$

for every $f \in \mathscr{C}^{k,\alpha}(M)$. However, the choice of $\|\cdot\|_{\mathscr{C}^{k,\alpha}(\mathcal{K}\cap U, g)}$ has the advantage of being independent of $\varepsilon$.

### The cylindrical norm on $\mathcal{N}$

As mentioned many times, the usual norm of $h_\varepsilon$ degenerates as $\varepsilon \to 0$ on the neck region $\mathcal{N}$ of $M$ and therefore the same would happen to its induced metric. For this reason, in order to study $(\mathcal{N}, h_\varepsilon)$ for small $\varepsilon$ it is useful to consider a sort of blow-up. Precisely, we observe that by Lemma 3.5 the metric $\varepsilon^{-2} \left(2\cosh\left(\frac{nr}{2}\right)\right)^{-4/n} h_\varepsilon$ coincides with $g_{\mathrm{cyl}}$ on $\mathcal{N}$, which does not collapse as $\varepsilon$. Replacing $h_\varepsilon$ by $g_{\mathrm{cyl}}$ has then some advantages, the most essential are the Schauder estimates (Lemma 3.11) uniformly in small $\varepsilon$, which differently fail in a collapsing setting such as $(\mathcal{N}, h_\varepsilon)$ when $\varepsilon \to 0$. Notice that this blow-up of the neck region becomes negligible near $\partial\mathcal{N}$, meaning that $g_{\mathrm{cyl}}$ is comparable with $g$ on $\mathcal{K} \cap \mathcal{N}$ and is comparable with $h_\varepsilon$ on $\mathcal{N} \cap \mathcal{E}$ in the sense of Lemma 3.6.

### The weighted norm on $\mathcal{E}$

Here is where the word "weighted" assumes its full significance. The norm $\|\cdot\|_{\mathscr{C}^{k,\alpha}_\delta(\mathcal{E}\cap U, g_{\mathrm{ah}})}$ in the funnel-like end is actually coherent with the classical *weighted Hölder norm* defined for asymptotically hyperbolic manifold or, even more generally, for conformally compact manifold [40, pag. 53], [41, pag. 206], [9, Definition 2.2]. In fact we recall that $g_{\mathrm{ah}}$, as well as $h_\varepsilon$, is conformally compact with boundary defining function $\mathrm{e}^{-t}$. These weighted norms provide a good environment where to develop the theory of particular elliptic operators, such as $\Delta_{h_\varepsilon} - n$ (cfr.[41, section 3]). Indeed the non-compactness of the manifold is in some sense balanced by the introduction of a weight function along the end and some classical results that holds on compact manifold can be traduced in this non-compact case, such as the Schauder estimates ([41], Proposition 3.4) and results about isomorphisms ([41], Theorem 3.10). With such kind of results we will be able to invert $\Delta_{h_\varepsilon} - n$ and transform the Yamabe equation (3.8) in a fixed-point problem. Notice that by Lemma 3.4 one could equivalently replace $g_{\mathrm{ah}}$ by $h_\varepsilon$ in the definition of $\|\cdot\|_{\mathscr{C}^{k,\alpha}_\delta(\mathcal{E}\cap U, g_{\mathrm{ah}})}$ and all the what follows would hold as well.

## 3.4 Linearization of the problem

In this section we analyse the equation $(\Delta_{h_\varepsilon} - n)u = f$ and, with the help of the functional setting introduced in the previous subsection, we prove the following proposition.

**Proposition 3.7.** *Let $(M, h_\varepsilon)$ be the Riemannian manifold defined in (3.2) and fix $\alpha \in (0,1)$ and $(n-1)/2 < \delta < n$. Then for every $f \in \mathscr{C}^{0,\alpha}_\delta(M, h_\varepsilon)$ there exist a unique $u \in \mathscr{C}^{2,\alpha}_\delta(M, h_\varepsilon)$ such that $(\Delta_{h_\varepsilon} - n)u = f$. Moreover there exists $\varepsilon_0 > 0$ and $C > 0$ such that*

$$\|u\|_{\mathscr{C}^{2,\alpha}_\delta(M,h_\varepsilon)} \leq C\,\|f\|_{\mathscr{C}^{0,\alpha}_\delta(M,h_\varepsilon)}\,, \tag{3.14}$$

*where the constant $C > 0$ is independent of $u$, $f$ and $\varepsilon \in (0, \varepsilon_0)$.*

The proof of the proposition above will be given after a short list of lemmas. We need a couple of lemmas that are Liouville-type results, in the sense that we want to conclude $u = 0$ from the assumption that $u$ solves the condition $(\Delta - n)u = 0$ and has a certain decay along the ends of particular manifolds. We also need some lemmas concerning the Interior Schauder estimates for the case of non-compact manifolds such as cylinders and conformally compact manifolds.

### 3.4.1 Preliminary Lemmas

#### Liouville-type results

We present two results about the injectivity of $\Delta - n$ in the cuspidal and the asymptotically hyperbolic case.

**Lemma 3.8.** *Let $(M, g)$ be a complete Riemannian manifold of dimension $n \geq 2$ containing a compact subset $U$ such that $M \setminus U$ is diffeomorphic to $(0, +\infty) \times \Sigma$, where $(\Sigma, \bar{g})$ is a compact Riemannian manifold, and such that $g|_{M\setminus U} = ds^2 + \mathrm{e}^{-2s}\bar{g}$ (namely $(M, g)$ is the manifold with a single cusp introduced at the beginning of Section 3.2). Suppose that $u$ is a $\mathscr{C}^2$-function on $(M, g)$ satisfying $\Delta_g u = nu$. If $|u|, |\partial_s u| < C$ for some $C > 0$, then $u = 0$.*

*Proof.* Let $\Omega =: M \setminus \{s > s_0\}$ denote a compact subset with smooth boundary. Multiplying $\Delta_g u = nu$ by $u$ and integrating by parts on $\Omega$ we get

$$\int_\Omega u\Delta_g u + |\nabla u|_g^2 = \int_{\partial\Omega} u\frac{\partial u}{\partial s}.$$

The right-hand side is bounded by $C^2\mathrm{Area}(\{s = s_0\})$, therefore $\int_\Omega nu^2 + |\nabla u|_g^2 \leq C_1\mathrm{e}^{-(n-1)s_0}$ for some constant $C_1$ independent of $s_0$. Taking the limit for $s_0 \to +\infty$, we get $u = 0$. □

**Lemma 3.9.** *Suppose that $u$ is a $\mathscr{C}^2$-function on $(\mathbb{R} \times \Sigma, dt^2 + \mathrm{e}^{2t}\bar{g})$, with $(\Sigma, \bar{g})$ a compact Riemannian manifold of dimension $n - 1$. Suppose $\Delta u = nu$. If $|u|, |\partial_t u| < C \min\{1, \mathrm{e}^{-\delta t}\}$ for some $C > 0$ and $(n - 1)/2 < \delta$, then $u = 0$.*

*Proof.* Similarly to the previous lemma, we integrate $\Delta u = nu$ by parts on $\Omega = \{-t_0 \leq t \leq t_0\}$, for some large $t_0 \in \mathbb{R}$. Then

$$\int_\Omega nu^2 + |\nabla u|_g^2 \leq \int_{\{t=-t_0\}} \left|u\frac{\partial u}{\partial t}\right| + \int_{\{t=t_0\}} \left|u\frac{\partial u}{\partial t}\right| \leq C_1(\mathrm{e}^{-(n-1)t_0} + \mathrm{e}^{-2\delta t_0}\mathrm{e}^{(n-1)t_0}),$$

for some constant $C_1 > 0$ independent of $t_0$. Taking the limit for $t_0 \to +\infty$ we get the result for $(n - 1)/2 < \delta$. □

*REMARK* 3.3. The two lemmas above are stated in the precise way that they will be used. We are not saying that these results are sharp for what concerns the bound of the function $u$ along the ends or the restriction of $\delta$ in $((n - 1)/2, n)$. About the last argument, we recall that the operator $\Delta_{h_\varepsilon} - n$ studied in Proposition 3.7 is an isomorphism for $\delta \in (0, n)$ but we will get the uniform bound (3.14) for a smaller range of $\delta$ exactly because of the restriction done in Lemma 3.9. We emphasize here that this will *not* imply a loss of information about the control of the solution $v_\varepsilon$ of (3.8), in fact if $\delta > \delta'$ then $\mathcal{C}^{k,\alpha}_\delta(M, h_\varepsilon) \subset \mathcal{C}^{k,\alpha}_{\delta'}(M, h_\varepsilon)$, thus the best control for $v_\varepsilon$ is got when $\delta$ is close to $n$.

### Interior Schauder Estimates

We are going to state *ad hoc* Interior Schauder Estimates for particular Riemannian manifolds. Despite these results can be extended in more general settings, we tried to keep the background as conformal as possible to the proof of Proposition 3.7, where they are applied.

The first lemma we want to prove is a simple translation of the classical Schauder estimates in the setting of our Riemannian manifold with a cusp.

**Lemma 3.10.** *Let $(M,g)$ be a complete Riemannian manifold of dimension $n \geq 2$ containing a compact subset $U$ such that $M \setminus U$ is diffeomorphic to $(0,+\infty) \times \Sigma$, where $(\Sigma, \bar{g})$ is a compact Riemannian manifold, and such that $g|_{M\setminus U} = ds^2 + \mathrm{e}^{-2s}\bar{g}$ (namely $(M,g)$ is the manifold with a single cusp introduced at the beginning of Section 3.2). They hold the Schauder estimates for the second order elliptic operator $\Delta - n$ in $\mathcal{K} := \{s < 3\} \subset \mathcal{K}' := \{s < 4\}$, that is for every $k \in \mathbb{N}$ and $\alpha \in (0,1)$ there exists $C > 0$ such that*

$$\|u\|_{\mathscr{C}^{k+2,\alpha}(\mathcal{K},g)} \leq C\left(\|(\Delta - n)u\|_{\mathscr{C}^{k,\alpha}(\mathcal{K}',g)} + \|u\|_{\mathscr{C}^{0}(\mathcal{K}',g)}\right)$$

*for every $u \in \mathscr{C}^{k+2,\alpha}(\mathcal{K}', g)$.*

*Proof.* This result will follow applying the classical Schauder estimates (cfr. [40], Corollary 6.3) on a finite covering of $\mathcal{K}$ by local patches. Since $\mathcal{K}'$ is bounded, then $R := \min\{1, \mathrm{injrad}(\mathcal{K}', g)\} > 0$ and we can assume

$$\|\cdot\|_{\mathscr{C}^{k,\alpha}(U,g)} = \max_{j\leq k} \sup_{U} |\nabla_g^{(j)} \cdot |_g + \sup_{0<d(p,q)<R/2} \frac{|\nabla_g^{(k)} \cdot (p) - \nabla_g^{(k)} \cdot (q)|_g}{d_g(p,q)^\alpha}$$

for every open subset $U \subseteq \mathcal{K}'$. By compactness of $\overline{\mathcal{K}}$ it is possible to find a finite number of balls $\{V_\beta\}_\beta$ centred at some $p \in \mathcal{K}$ with radius $R/4$ covering $\mathcal{K}$. Denoting by $U_\beta$ the ball concentric to $V_\beta$ with radius $R/2$, we have by construction

$$\mathcal{K} \subseteq \bigcup_\beta V_\beta \subset \bigcup_\beta U_\beta \subseteq \mathcal{K}'.$$

The classical Schauder estimates then assert that

$$\|u\|_{\mathscr{C}^{k+2,\alpha}(V_\beta,g)} \leq C_\beta\left(\|(\Delta - n)u\|_{\mathscr{C}^{k,\alpha}(U_\beta,g)} + \|u\|_{\mathscr{C}^{0}(U_\beta,g)}\right)$$

with $C_\beta > 0$ independent of $u$. Since $U_\beta \subset \mathcal{K}'$, then

$$\max_\beta \|u\|_{\mathscr{C}^{k+2,\alpha}(V_\beta,g)} \leq \max_\beta C_\beta\left(\|(\Delta - n)u\|_{\mathscr{C}^{k,\alpha}(\mathcal{K}',g)} + \|u\|_{\mathscr{C}^{0}(\mathcal{K}',g)}\right).$$

We got the thesis if we prove that there exists $c > 0$ independent of $u$ such that

$$\|u\|_{\mathscr{C}^{k+2,\alpha}(\mathcal{K},g)} \leq c \max_\beta \|u\|_{\mathscr{C}^{k+2,\alpha}(V_\beta,g)}.$$

By construction $\mathcal{K} \subseteq \bigcup_\beta V_\beta$, it follows that it is enough to check the estimate above for the Hölder coefficient of $\|u\|_{\mathscr{C}^{k+2,\alpha}(\mathcal{K},g)}$. Namely, we have to check that for every $p, q \in \mathcal{K}$ such that $0 < d(p,q) < R/2$, there exists $\beta$ such that

$$\frac{|\nabla_g^{(k)} u(p) - \nabla_g^{(k)} u(q)|_g}{d_g(p,q)^\alpha} \le c\|u\|_{\mathscr{C}^{k+2,\alpha}(V_\beta,g)}.$$

The proof of this fact is standard. Choose $\beta$ such that $p \in V_\beta$. Then two cases may occur. If $q \in V_\beta$ then the thesis follows with $c = 1$. If $q \notin V_\beta$ then $d(p,q) \ge R/4$ and therefore

$$\frac{|\nabla^{(k)} u(p) - \nabla^{(k)} u(q)|}{d(p,q)^\alpha} \le (R/4)^{-\alpha}(|\nabla_g^{(k)} u(p)|_g + |\nabla_g^{(k)} u(q)|_g) \le c\|u\|_{\mathscr{C}^{k+2,\alpha}(V_\beta,g)}$$

with $c = 2(R/4)^{-\alpha}$. □

The second lemma we want to prove concerns the cylinder. We want to show that an analogue of the previous proposition can be applied to the cylinder without involving the diameter of the larger open subset.

**Lemma 3.11.** *Consider the cylinder $(\mathbb{R} \times \Sigma, g_{\mathrm{cyl}} = d\rho^2 + \bar{g})$ with compact fiber $(\Sigma, \bar{g})$ and let $\mathcal{L}$ be a second order uniformly elliptic operator on $\mathbb{R} \times M$ of the form*

$$\mathcal{L} = \partial_\rho^2 + \Delta_{\bar{g}} + a(\rho)\partial_\rho + b(\rho), \tag{3.15}$$

*for some smooth function $a(\rho)$, $b(\rho)$. Fix $D > 0$ and $D' := D + d$, for some $0 < d < D$, and consider the following encapsulated open subsets $\Omega := \{|\rho| < D\}$ and $\Omega' := \{|\rho| < D'\}$. There exists a constant $C > 0$ such that if it holds $Lu = f$ for some $f \in \mathscr{C}^{k,\alpha}(\Omega', g_{\mathrm{cyl}})$ and some $u \in \mathscr{C}^{k+2,\alpha}(\Omega', g_{\mathrm{cyl}})$ then one has the following estimate*

$$\|u\|_{\mathscr{C}^{k+2,\alpha}(\Omega,g_{\mathrm{cyl}})} \le C\left(\|f\|_{\mathscr{C}^{k,\alpha}(\Omega',g_{\mathrm{cyl}})} + \|u\|_{\mathscr{C}^0(\Omega',g_{\mathrm{cyl}})}\right). \tag{3.16}$$

*The constant $C$ only depends on $(\Sigma, \bar{g})$, $\alpha \in (0,1)$, $k \in \mathbb{N}$, a lower and a upper bound for $d$ and an upper bound for $\|a\|_{\mathscr{C}^{k,\alpha}(\mathbb{R})}$ and $\|b\|_{\mathscr{C}^{k,\alpha}(\mathbb{R})}$.*

*Proof.* It is not restrictive to suppose that

$$\|f\|_{\mathscr{C}^{k,\alpha}(U,g_{\mathrm{cyl}})} = \max_{j\le k} \sup_U |\nabla_{g_{\mathrm{cyl}}}^{(j)} f| + \sup_{0<d_{g_{\mathrm{cyl}}}(p,q)<R/2} \frac{|\nabla_{g_{\mathrm{cyl}}}^{(k)} f(p) - \nabla_{g_{\mathrm{cyl}}}^{(k)} f(q)|_{g_{\mathrm{cyl}}}}{d_{g_{\mathrm{cyl}}}(p,q)^\alpha}$$

with $R = \mathrm{injrad}\,(\mathbb{R} \times \Sigma, g_{\mathrm{cyl}})$. Notice that $R > 0$ by compactness of $\Sigma$. The idea is to cover $\Omega$ and $\Omega'$ by smaller compact cylinder in order to apply the classical Schauder estimates. For $\rho_0 \in (-D_1 + d, D_1 - d)$, set

$$V_{\rho_0} := (\rho_0 - d, \rho_0 + d) \times \Sigma \quad \text{and} \quad U_{\rho_0} := (\rho_0 - 2d, \rho_0 + 2d) \times \Sigma.$$

By construction

$$\bigcup_{\rho_0} V_{\rho_0} = \Omega \quad \text{and} \quad \bigcup_{\rho_0} U_{\rho_0} = \Omega'.$$

Up to covering $\Sigma$ by a finite number of coordinate patches as in the proof of the previous lemma, it is then possible to apply the classical Schauder estimates to prove that

$$\|u\|_{\mathscr{C}^{k+2,\alpha}(V_{\rho_0},g_{\mathrm{cyl}})} \leq C_1 \left( \|f\|_{\mathscr{C}^{k,\alpha}(U_{\rho_0},g_{\mathrm{cyl}})} + \|u\|_{\mathscr{C}^0(U_{\rho_0},g_{\mathrm{cyl}})} \right),$$

where the constant $C_1$ only depends on $n$, $\alpha$, $k$, the ellipticity constant of $\mathcal{L}$, a positive lower bound for $d(V_{\rho_0}, \partial U_{\rho_0})$, an upper bound for the $\mathscr{C}^{k,\alpha}(U_{\rho_0})$-norms of $a$ and $b$ and an upper bound for the diameter of $U_{\rho_0}$. All those quantities can be controlled by $(\Sigma, \bar{g})$, $k$, $\alpha$, a lower and upper bound for $d$ and an upper bound for $\|a\|_{\mathscr{C}^{k,\alpha}(\mathbb{R})}$ and $\|b\|_{\mathscr{C}^{k,\alpha}(\mathbb{R})}$. Since $U_{\rho_0}$ is contained in $\Omega'$, then

$$\|u\|_{\mathscr{C}^{k+2,\alpha}(V_{\beta,\rho_0},g_{\mathrm{cyl}})} \leq C_1 \left( \|f\|_{\mathscr{C}^{k,\alpha}(\Omega',g_{\mathrm{cyl}})} + \|u\|_{\mathscr{C}^0(\Omega',g_{\mathrm{cyl}})} \right),$$

thus the thesis would follow if we show that

$$\|u\|_{\mathscr{C}^{k+2,\alpha}(\Omega,g_{\mathrm{cyl}})} \leq C_2 \sup_{\rho_0} \|u\|_{\mathscr{C}^{k+2,\alpha}(V_{\rho_0},g_{\mathrm{cyl}})}$$

for some $C_2 > 0$ depending only on the same factors as $C_1$. Since $\{V_{\rho_0}\}_{\rho_0}$ is an open covering for $\Omega$, the non trivial inequality only concerns the Hölder part of $\|u\|_{\mathscr{C}^{2,\alpha}(\Omega,g_{\mathrm{cyl}})}$, namely we have to check that for $p \neq q$ in $\Omega$ with $d_{g_{\mathrm{cyl}}}(p,q) < R/2$, one has

$$\frac{|\nabla^{(k+2)}_{g_{\mathrm{cyl}}} u(p) - \nabla^{(k+2)}_{g_{\mathrm{cyl}}} u(q)|_{g_{\mathrm{cyl}}}}{d_{g_{\mathrm{cyl}}}(p,q)^\alpha} \leq C_2 \sup_{\rho_0} \|u\|_{\mathscr{C}^{k+2,\alpha}(V_{\rho_0},g_{\mathrm{cyl}})}. \tag{3.17}$$

So we fix $p \in \Omega$, which belongs to $V_{\rho_0}$ for some $\rho_0$. Then two cases may occur. If $q \in V_{\beta,\rho_0}$ too, then (3.17) is trivial with $C_2 = 1$. If $q \notin V_{\beta,\rho_0}$ then $d_{g_{\mathrm{cyl}}}(p,q) \geq d$, therefore

$$\frac{|\nabla^{(k+2)}_{g_{\mathrm{cyl}}} u(p) - \nabla^{(k+2)}_{g_{\mathrm{cyl}}} u(q)|_{g_{\mathrm{cyl}}}}{d_{g_{\mathrm{cyl}}}(p,q)^\alpha} \leq 2d^{-\alpha} \sup_{V_{\rho_0}} |\nabla^{(k+2)}_{g_{\mathrm{cyl}}} u|_{g_{\mathrm{cyl}}}$$

and we got the thesis with $C_2 = 2d^{-\alpha}$. □

The analogous interior Schauder estimates for unbounded domains in the context of conformally compact manifold has been obtained by Graham and Lee [41, Proposition 3.4], even in a more general setting involving *elliptic uniformly degenerate operators* and *weighted tensorial spaces*. For our purpose, it is enough to have the following result.

**Lemma 3.12.** *In $(\mathbb{R}\times\Sigma, g_{\mathrm{ah}} := dt^2 + \mathrm{e}^{2t}\bar{g})$, where $(\Sigma,\bar{g})$ is a compact Riemannian manifold, they hold the weighted Schauder estimates for the second order elliptic operator $\Delta - n$ in $\mathcal{E} := \{t > -3\} \subset \mathcal{E}' := \{t > -4\}$, that is for every $k \in \mathbb{N}$, $\delta \in \mathbb{R}$, $\alpha \in (0,1)$ there exists $C > 0$ such that*

$$\|u\|_{\mathscr{C}^{k+2,\alpha}_\delta(\mathcal{E},g_{\mathrm{ah}})} \leq C\left(\|(\Delta - n)u\|_{\mathscr{C}^{k,\alpha}_\delta(\mathcal{E}',g_{\mathrm{ah}})} + \|u\|_{\mathscr{C}^{0}_\delta(\mathcal{E}',g_{\mathrm{ah}})}\right)$$

*for every $u \in \mathscr{C}^{k+2,\alpha}_\delta(\mathbb{R}\times\Sigma)$.*

*Proof.* As mentioned above, this lemma is an application of [41, Proposition 3.4], which is in turn a consequence of the analysis of elliptic operators in [40]. To see that, we first notice that the only involved end is the neighbourhood of $\{t = +\infty\}$. Thus we can assume that $(\mathbb{R}\times\Sigma, g_{\mathrm{ah}})$ is replaced by the Riemannian manifold with boundary $([-5,+\infty), g_{\mathrm{ah}})$, which has a unique end. We report here the statement of [41, Proposition 3.4], followed by some comments and explanations of their terms.
"*Let $k \in \mathbb{N}$, $0 < \alpha < 1$ and $(M,g)$ a conformally compact manifold (eventually with boundary). Let $\Omega \subset \Omega'$ be two subsets of $M$ such that $d(\partial\Omega',\Omega) \geq 1/2$ and let $P$ be an elliptic degenerate operator. If $u \in \mathscr{C}^2(\Omega') \cap \Lambda^\delta_{0,0}(\Omega')$* and $Pu \in \Lambda^\delta_{k,\alpha}(\Omega')$, then $u \in \Lambda^\delta_{k+2,\alpha}(\Omega)$ and

$$\|u\|_{\Lambda^\delta_{k+2,\alpha}(\Omega)} \leq C\left(\|Pu\|_{\Lambda^\delta_{k,\alpha}(\Omega')} + \|u\|_{\Lambda^\delta_{0,0}(\Omega')}\right)$$

for some constant $C$ independent of $u$, $\Omega$ and $\Omega'$. "
We observed in Section 2.1 that $g_{\mathrm{ah}}$ is asymptotically hyperbolic (hence conformally compact) for $t \to +\infty$. Therefore $(M,g) := ([-5,+\infty), g_{\mathrm{ah}})$ is conformally compact (with boundary). We can therefore consider on $(M,g)$ the weighted Banach spaces $\Lambda^\delta_{k,\alpha}(U)$ introduced by Graham and Lee for every open subset $U \subset M$. As remarked in [9, Definition 2.2 and following remarks], these spaces correspond to $\mathscr{C}^{k,\alpha}_\delta(U, g_{\mathrm{ah}})$ of this text and the respective norms are equivalent. With the choice $\Omega = \mathcal{E}$ and $\Omega' = \mathcal{E}'$, the proof of the lemma would descend from the result of Graham and Lee reported above if the operator $P = \Delta - n$ falls inside their definition of *elliptic degenerate operators.* Those are operators whose coefficients obey to some growing condition depending on the boundary defining function, the precise definition can be found in [41, pag. 209]. The same authors (see [41, pag. 212]) prove that the operator $\Delta - n$ is elliptic degenerate, so the result follows. □

### 3.4.2 A priori estimates

We are ready to prove Proposition 3.7. Actually the only point to be shown is the uniform estimate (3.14), in fact the involved operator is an isomorphism by [41, Corollary 3.11] or [9, Lemma 4.2 with $\xi = n$] for $0 < \delta < n$. We will prove (3.14) by contradiction. Suppose by contradiction that for every $j \in \mathbb{N}$ there exists $(\varepsilon_j, u_j, f_j) \in \mathbb{R}^+ \times \mathscr{C}^{2,\alpha}_\delta(M, h_{\varepsilon_j}) \times \mathscr{C}^{0,\alpha}_\delta(M, h_{\varepsilon_j})$ such that

(*i*) $\varepsilon_j \to 0$, as $j \to +\infty$;

(*ii*) $(\Delta_{h_{\varepsilon_j}} - n)u_j = f_j$, $\forall j \in \mathbb{N}$;

(*iii*) $\|u_j\|_{\mathscr{C}^{2,\alpha}_\delta(M,h_{\varepsilon_j})} = 1$, $\forall j \in \mathbb{N}$;

(*iv*) $\|f_j\|_{\mathscr{C}^{0,\alpha}_\delta(M,h_{\varepsilon_j})} \to 0$, as $j \to +\infty$.

The proof then proceeds as follows.

Step 1: We show that up to subsequence $\|u_j\|_{\mathscr{C}^{2,\alpha}(\mathcal{K},g)} \to 0$ as $j \to +\infty$ with the help of the Liouville-type result obtained Lemma 3.8 and the interior Schauder estimates (Lemma 3.10);

Step 2: We show that up to subsequence $\|u_j\|_{\mathscr{C}^{2,\alpha}_\delta(\mathcal{E},g_{\mathrm{ah}})} \to 0$ as $j \to +\infty$ with the help of the Liouville-type result obtained in Lemma 3.9 and the weighted Schauder estimates (Lemma 3.12);

Step 3: Finally we use the previous two steps and the Schauder estimates for the cylinder (Lemma 3.11) to show that also $\|u_j\|_{\mathscr{C}^{2,\alpha}(\mathcal{N},g_{\mathrm{cyl}})} \to 0$ as $j \to +\infty$ (up to subsequence) and we get a contradiction with (*iii*).

It is useful to introduce the following open subsets of $M$

$$\mathcal{K}' = \{s < 4\}, \qquad \mathcal{E}' = \{t > -4\}, \qquad \mathcal{N}' = \{|r| < -\log\varepsilon - 1\}.$$

These subsets extend (respectively) $\mathcal{K}$, $\mathcal{E}$ and $\mathcal{N}$ by a unitary length and will be used to apply the interior estimates. Clearly all the preliminary observations, such as Lemma 3.6, obtained in the previous section for $\mathcal{K}$, $\mathcal{E}$ and $\mathcal{N}$ adapts to $\mathcal{K}'$, $\mathcal{E}'$ and $\mathcal{N}'$ as well.

*Step 1:* We are going to show that up to subsequence

$$\lim_{j\to+\infty} \|u_j\|_{\mathscr{C}^{2,\alpha}(\mathcal{K},g)} = 0. \tag{3.18}$$

Consider the compact subset $\{s \leq 1\}$. From (*iii*) it follows that

$$\|u_j\|_{\mathscr{C}^{2,\alpha}(\{s\leq 1\},g)} \leq 2$$

for $j$ large enough. By Ascoli-Arzelà, there is a subsequence of $(u_j)$ converging to some $u^{(1)}$ in $\mathscr{C}^2(\{s \leq 1\}, g)$. Now look at this convergent subsequence in the larger compact subset $\{s \leq 2\}$. Since $\|u_j\|_{\mathscr{C}^{2,\alpha}(\{s\leq 2\},g)}$ is still bounded independently of large $j$'s, there is a further subsequence converging to some $u^{(2)}$ in $\mathscr{C}^2(\{s \leq 2\}, g)$ and necessarily $u^{(2)}$ extends $u^{(1)}$. We can iterate this argument for

$$\{s \leq 1\} \subset \{s \leq 2\} \subset \{s \leq 3\} \subset \dots$$

and so on. This produces a function $u^{(\infty)}$ defined on $M$ such that for every compact subset $\{s \leq c\} \subset M$ there exists a subsequence of the $u_j$'s converging to $u^{(\infty)}$ in the $\mathscr{C}^2(\{s \leq c\}, g)$-norm. Moreover, since $|u_j|, |\partial_s u_j| < 2$ in every compact subset $\{s \leq c\}$ for $j$ large enough, then $|u^{(\infty)}|, |\partial_s u^{(\infty)}| < 2$ on $M$. We can also conclude that $(\Delta_g - n)u^{(\infty)} = 0$, in fact this holds in the strong sense on every compact subset of the form $\{s \leq c\}$ passing to the limit from $(\Delta_{h_{\varepsilon_j}} - n)u_j = f_j$ and applying both *(iv)* and Lemma 3.3. Then we can apply Lemma 3.8 and we get $u^{(\infty)} = 0$. We have in particular proved the existence of a subsequence $(u_j)$ converging to zero in $\mathscr{C}^2(\mathcal{K}', g)$. We apply Lemma 3.10 to this subsequence in $\mathcal{K} \subset \mathcal{K}'$, getting

$$\|u_j\|_{\mathscr{C}^{2,\alpha}(\mathcal{K},g)} \leq C_1 \left( \|(\Delta_g - n)u_j\|_{\mathscr{C}^{0,\alpha}(\mathcal{K}',g)} + \|u_j\|_{\mathscr{C}^0(\mathcal{K}')} \right) \tag{3.19}$$

for some $C_1 > 0$ independent of $j$. We know that $\|u_j\|_{\mathscr{C}^0(\mathcal{K}')} \to 0$ as $j \to +\infty$, on the other hand from Lemma 3.3 we also know that $\|(\Delta_g - n)u_j\|_{\mathscr{C}^{0,\alpha}(\mathcal{K}',g)} \leq 2\|f_j\|_{\mathscr{C}^{0,\alpha}_\delta(\mathcal{K}',h_{\varepsilon_j})}$ for large $j$'s, which tends to zero by *(iv)* as $j \to +\infty$. These last two arguments and (3.19) imply (3.18), and we are done.

*Step 2:* We are going to show that up to subsequence

$$\lim_{j \to +\infty} \|u_j\|_{\mathscr{C}^{2,\alpha}_\delta(\mathcal{E},g_{\mathrm{ah}})} = 0. \tag{3.20}$$

This is the analogue of (3.18) in the manifold's end $\mathcal{E}$ and the technique we use is really specular. Consider the subset $\{-1 \leq t\}$. From *(iii)* it follows that $\|u_j\|_{\mathscr{C}^{2,\alpha}_\delta(\{a \leq t\},g_{\mathrm{ah}})} \leq 2$ for $j$ large enough. By Ascoli-Arzelà, there is a subsequence of $(u_j)$ converging to some $u^{(-1)}$ in $\mathscr{C}^2_\delta(\{-1 \leq t\}, g_{\mathrm{ah}})$. Similarly as in the first step, then one extract from this subsequence a further subsequence converging in $\mathscr{C}^2_\delta(\{-2 \leq t\}, g_{\mathrm{ah}})$ and this argument is iterated for

$$\{-1 \leq t\} \subset \{-2 \leq t\} \subset \{-3 \leq t\} \subset \dots$$

and so on. As a result we get a function $u^{(-\infty)}$ defined in $\mathbb{R} \times \Sigma$ such that for every subset $\{-c \leq t\}$ there is a subsequence of the $u_j$'s converging to $u^{(-\infty)}$ in the $\mathscr{C}^2_\delta(\{-c \leq t\}, g_{\mathrm{ah}})$-norm. Moreover, since $|u_j|, |\partial_t u_j| < 2\min\{1, \mathrm{e}^{-\delta t}\}$ in every subset of the form $\{-c \leq t\}$ for $j$ large enough, then $|u^{(-\infty)}|, |\partial_t u^{(-\infty)}| < 2\min\{1, \mathrm{e}^{-\delta t}\}$ on $\mathbb{R} \times \Sigma$. We can also conclude that $(\Delta_{g_{\mathrm{ah}}} - n)u^{(-\infty)} = 0$, in fact this holds in the strong sense on every subset of the form $\{-c \leq t\}$ passing to the limit from $(\Delta_{h_{\varepsilon_j}} - n)u_j = f_j$ and applying both *(iv)* and Lemma 3.4. Then we can apply Lemma 3.9 and we get $u^{(-\infty)} = 0$. We have in particular proved the existence of a subsequence $(u_j)$ converging to zero in $\mathscr{C}^2_\delta(\mathcal{E}', g_{\mathrm{ah}})$. By hypothesis $(n-1)/2 < \delta < n$, so we can apply Lemma 3.12 to this subsequence in $\mathcal{E} \subset \mathcal{E}'$, getting

$$\|u_j\|_{\mathscr{C}^{2,\alpha}_\delta(\mathcal{E},g_{\mathrm{ah}})} \leq C_2 \left( \|(\Delta_{g_{\mathrm{ah}}} - n)u_j\|_{\mathscr{C}^{0,\alpha}_\delta(\mathcal{E}',g_{\mathrm{ah}})} + \|u_j\|_{\mathscr{C}^0(\mathcal{E}')} \right) \tag{3.21}$$

for some $C_2 > 0$ independent of $j$. We know that $\|u_j\|_{\mathscr{C}^0(\mathcal{E}')} \to 0$ as $j \to +\infty$, on the other hand from Lemma 3.4 we also know that $\|(\Delta_{g_{\rm ah}} - n)u_j\|_{\mathscr{C}^{0,\alpha}_\delta(\mathcal{E}', g_{\rm ah})} \leq 2\|f_j\|_{\mathscr{C}^{0,\alpha}_\delta(\mathcal{E}', g_{\rm ah})}$ for large $j$'s, which tends to zero by $(iv)$ as $j \to +\infty$. Using these two considerations in (3.21), we infer (3.20), as wanted.

*Step 3:* We are going to get the contradiction proving that

$$\lim_{j\to+\infty} \|u_j\|_{\mathscr{C}^{2,\alpha}(\mathcal{N}, g_{\rm cyl})} = 0. \tag{3.22}$$

It is not restrictive suppose that the $u_j$'s satisfy (3.18) and (3.20), up to take a subsequence. For every arbitrary constant $\beta > 0$ we can consider the function $\pm u_j - \beta$. By construction $(\Delta_{h_{\varepsilon_j}} - n)(\pm u_j - \beta) = \pm f_j + n\beta$ is non-negative on $\mathcal{N}'$ for $j$ large enough since by $(iv)$ it holds $\sup_{\mathcal{N}'} |f_j| \to 0$ as $j \to +\infty$. On the other hand $\pm u_j - \beta \leq 0$ on $\partial\mathcal{N}'$ by (3.18) and (3.20). We can then apply the maximum principle to the strictly elliptic operator $\Delta_{h_{\varepsilon_j}} - n$ on $\mathcal{N}'$, so that for every $\beta > 0$ there exists $j_0 \in \mathbb{N}$ such that $\pm u_j \leq \beta$ on $\mathcal{N}'$ for every $j > j_0$. This means $\lim_{j\to+\infty} \|u_j\|_{\mathscr{C}^0(\mathcal{N}', g_{\rm cyl})} = 0$. We want to improve this limit to the $\mathscr{C}^{2,\alpha}(\mathcal{N}, g_{\rm cyl})$-norm via Lemma 3.11. Define a second-order elliptic operator $\mathcal{L}_j$ in $\mathcal{N}'$ by

$$\mathcal{L}_j := \phi_j^2(\Delta_{h_{\varepsilon_j}} - n), \quad \phi_j(r) := \varepsilon_j \left(2\cosh\left(\frac{nr}{2}\right)\right)^{2/n}.$$

By construction $\mathcal{L}_j u_j = \phi_j^2 f_j$. Since a direct computation gives

$$\Delta_{h_{\varepsilon_j}} = \partial_r^2 + \frac{\Delta_{\bar g}}{\varepsilon_j^2 \left(2\cosh\left(\frac{nr}{2}\right)\right)^{4/n}} + (n-1)\tanh\left(\frac{nr}{2}\right)\partial_r \tag{3.23}$$

in $\mathcal{N}'$, then with respect to the variable $\rho$ defined in (3.11) by $\dot\rho(r) = 1/\phi_j(r)$ one has

$$\mathcal{L}_j = \partial_\rho^2 + \Delta_{\bar g} + (n-2)\tanh\left(nr(\rho)/2\right)\phi_j(r(\rho))\partial_\rho - n\phi_j(r(\rho))^2. \tag{3.24}$$

The equation (3.24) was derived by (3.23) using that $\partial_r = \dot\rho\partial_\rho$ and $\partial_r^2 = \dot\rho^2\partial_\rho + \ddot\rho\partial_\rho$. We also define $D(j) = \rho(-\log\varepsilon_j - 2)$ and $D'(j) := \rho(-\log\varepsilon_j - 1)$, so that

$$\mathcal{N} = \{|\rho| < D(j)\},\ \mathcal{N}' = \{|\rho| < D'(j)\}.$$

We can apply Lemma 3.11, getting

$$\|u_j\|_{\mathscr{C}^{2,\alpha}(\mathcal{N}, g_{\rm cyl})} \leq C_3\left(\|\phi_j^2 f_j\|_{\mathscr{C}^{0,\alpha}(\mathcal{N}', g_{\rm cyl})} + \|u_j\|_{\mathscr{C}^0(\mathcal{N}', g_{\rm cyl})}\right). \tag{3.25}$$

We claim that $C_3$ does not depend on large $j$'s. In fact by Lemma 3.11 it is sufficient to prove that:

- the quantities $d(j) := D'(j) - D(j)$ can be bounded above and below by positive constants independent of $j$ (large enough);

- the $C^{0,\alpha}$-norms of the coefficients $\tanh\left(nr(\rho)/2\right)\phi_j(r(\rho))$ and $\phi_j(r(\rho))^2$ can be bounded uniformly in $j$.

It is easy to see that for large $j$'s one has $\mathrm{e}^{-2} < \phi_j|_{(-\log\varepsilon_j-2,-\log\varepsilon_j-1)} < 1$. In particular by Lagrange,

$$d(j) = \rho(-\log\varepsilon_j - 1) - \rho(-\log\varepsilon_j - 2) = 1/\phi_j(r^*)$$

for some $r^* \in (-\log\varepsilon_j - 2, -\log\varepsilon_j - 1)$, thus we have the uniform bound for $d(j)$. Notice that on the other hand

$$\lim_{j\to+\infty} D(j) = \lim_{j\to+\infty}\int_0^{-\log\varepsilon_j-2}\frac{dr}{\phi_j(r)} = \int_0^{+\infty}\frac{dr}{(2\cosh(nr/2))^{2/n}}\lim_{j\to+\infty}\frac{1}{\varepsilon_j}$$

is equal to $+\infty$, in particular it follows $d(j) < D(j)$. Finally we have to check that $\phi_j(r(\rho))$ and $\tanh(nr(\rho)/2)$ can be bounded (uniformly for large $j$'s) in $\mathcal{N}'$ together with their first derivative in $\rho$. Since $\partial_\rho = \phi_j\partial_r$, then it is sufficient to check that $\phi_j(r)$ and $\tanh(nr/2)$ can be bounded (uniformly for large $j$'s) in $\{|r| < -\log\varepsilon_j - 1\}$ together with their first derivative in $r$. This is trivial for the hyperbolic tangent, on the other hand one can easily check that $\dot{\phi}_j = \phi_j\tanh(nr/2)$, thus it is sufficient to notice that in $\{|r| < -\log\varepsilon_j - 1\}$ we have $|\phi_j| < 1$. This proves that the constant $C_3$ in (3.25) can be chosen independent of $j$. Using again that $\phi_j$ is bounded in $\mathcal{N}'$ uniformly in $j$ with all its derivatives and using the absurd hypothesis $(iv)$, we have $\|\phi_j^2 f_j\|_{\mathscr{C}^{0,\alpha}(\mathcal{N}',g_{\mathrm{cyl}})} \to 0$ as $j\to+\infty$. Also we have shown that $\|u_j\|_{\mathscr{C}^0(\mathcal{N}')} \to 0$. These two results and (3.25) imply $\|u_j\|_{\mathscr{C}^{2,\alpha}(\mathcal{N},g_{\mathrm{cyl}})} \to 0$ as $j\to+\infty$ and we got the result.

## 3.5 Exact solutions

Now it is the moment to face the non-linear analysis. We want to prove the existence of a solution $v_\varepsilon$ of the Yamabe equation (3.8) satisfying

$$\|v_\varepsilon\|_{\mathscr{C}^{k,\alpha}_\delta(M,h_\varepsilon)} \le C\varepsilon^n$$

for some $C > 0$ independent of $\varepsilon$. This is done via a contraction theorem.

### 3.5.1 Fixed-point method

The Yamabe equation (3.8) is equivalent to $F_\varepsilon(v_\varepsilon) = v_\varepsilon$, where the operator $F_\varepsilon\colon \mathscr{C}^{k,\alpha}_\delta(M,h_\varepsilon) \to \mathscr{C}^{k+2,\alpha}_\delta(M,h_\varepsilon)$ is defined by

$$F_\varepsilon(v) = \begin{cases} (\Delta_{h_\varepsilon} - n)^{-1}\left(\frac{1}{2}\mathrm{R}_{h_\varepsilon} + 1 + Q(v)\right), & \text{if } n = 2\\ (\Delta_{h_\varepsilon} - n)^{-1}\left(\frac{n-2}{4(n-1)}\left(\mathrm{R}_{h_\varepsilon} + n(n-1)\right)(1+v) + Q(v)\right), & \text{if } n \ge 3. \end{cases} \tag{3.26}$$

We recall that $R_{h_\varepsilon} + n(n-1)$ is compactly supported, so one deduces that the operator $F_\varepsilon$ actually maps $\mathscr{C}^{k+2,\alpha}_\delta(M, h_\varepsilon)$ into $\mathscr{C}^{k,\alpha}_\delta(M, h_\varepsilon)$. The purpose of this section is to prove that $F_\varepsilon$ is a contraction on a ball of $\mathscr{C}^{k,\alpha}_\delta(M, h_\varepsilon)$ centred at 0 with radius proportional to $\varepsilon^n$. By simplicity, we will show this result for $k = 2$, then one get the general result by elliptic regularity and a bootstrap argument. The following lemma is a control of the quadratic term.

**Lemma 3.13.** *Let $C > 0$ be the uniform bound for $(\Delta_{h_\varepsilon} - n)^{-1}$ given by Proposition 3.7. The quadratic term $Q\colon \mathscr{C}^{2,\alpha}_\delta(M, h_\varepsilon) \to \mathscr{C}^{0,\alpha}_\delta(M, h_\varepsilon)$ introduced in* (3.8)*, for $\delta > 0$, satisfies*

$$\|Q(x) - Q(y)\|_{\mathscr{C}^{0,\alpha}_\delta(M,h_\varepsilon)} \le \frac{1}{2C}\|x - y\|_{\mathscr{C}^{2,\alpha}_\delta(M,h_\varepsilon)}, \tag{3.27}$$

*provided that $x, y \in \mathscr{C}^{2,\alpha}_\delta(M, h_\varepsilon)$ are close enough to* 0.

*Proof.* We will prove something slightly stronger, namely that

$$\|Q(x) - Q(y)\|_{\mathscr{C}^{1}_\delta(M,h_\varepsilon)} \le \frac{1}{2C}\|x - y\|_{\mathscr{C}^{2,\alpha}_\delta(M,h_\varepsilon)}.$$

In virtue of the definition of this norm (see Definition 3.2), it will be sufficient to check the validity of both the zero-order estimates

$$\sup_{\mathcal{K}\cup\mathcal{N}} |Q(x) - Q(y)| + \sup_{\mathcal{E}} \mathrm{e}^{\delta t}|Q(x) - Q(y)| \le \frac{1}{4C}\|x - y\|_{\mathscr{C}^{2,\alpha}_\delta(M,h_\varepsilon)} \tag{3.28}$$

and the first-order estimates

$$\begin{aligned}\sup_{\mathcal{K}} |\nabla_g(Q(x) - Q(y))|_g + \sup_{\mathcal{N}} |\nabla_{g_{\mathrm{cyl}}}(Q(x) - Q(y))|_{g_{\mathrm{cyl}}} +\\ + \sup_{\mathcal{E}} \mathrm{e}^{\delta t}|\nabla_{g_{\mathrm{ah}}}(Q(x) - Q(y))|_{g_{\mathrm{ah}}} \le \frac{1}{4C}\|x - y\|_{\mathscr{C}^{2,\alpha}_\delta(M,h_\varepsilon)}.\end{aligned} \tag{3.29}$$

Notice that by construction the operator $Q$ is defined as $Q(u) = q \circ u$, with $q\colon \mathbb{R} \to \mathbb{R}$ given by

$$q(x) = \begin{cases} \mathrm{e}^{2x} - 2x - 1 & \text{in dimension } n = 2,\\ \frac{n(n-2)}{4}(1+x)^{\frac{n+2}{n-2}} - \frac{n(n+2)}{4}x - \frac{n(n-2)}{4} & \text{in dimension } n \ge 3.\end{cases}$$

Notice also that there exist $c_1$ and $c_2$ independent of $\varepsilon$ such that $|q'(x)| < c_1|x|$ and $|q''(x)| < c_2$ if $|x| < 1$. Let us begin with the zero-order estimates. For every $p \in \mathcal{K} \cup \mathcal{N}$ one has

$$\begin{aligned}|Q(x)(p) - Q(y)(p)| &= |q'(x^*)(x(p) - y(p))|\\ &\le c_1|x^*||x(p) - y(p)| \le c_1(|x(p)| + |y(p)|)|x(p) - y(p)|,\end{aligned} \tag{3.30}$$

where by Lagrange $x^*$ lives between $x(p)$ and $y(p)$, so its modulus is less or equal to the sum of $|x(p)|$ and $|y(p)|$. Similarly, if $p \in \mathcal{E}$ one has

$$\mathrm{e}^{\delta t}|Q(x)(p) - Q(y)(p)| \le c_1 c_3 (\mathrm{e}^{\delta t}|x(p)| + \mathrm{e}^{\delta t}|y(p)|)\mathrm{e}^{\delta t}|x(p) - y(p)| \quad (3.31)$$

by the same argument above and the fact that $1 \le c_3 \mathrm{e}^{\delta t}$ in $\mathcal{E}$, with $c_3$ independent of $\varepsilon$. So it follows (3.28) for $\|x\|_{\mathscr{C}^{2,\alpha}_\delta(M,h_\varepsilon)}, \|y\|_{\mathscr{C}^{2,\alpha}_\delta(M,h_\varepsilon)} \le (8Cc_1(1+c_3))^{-1}$. Finally let us face the first-order estimates, separately for $\mathcal{K}$, $\mathcal{N}$ and $\mathcal{E}$. Assume initially $p \in \mathcal{K}$ and let $(x^1, \dots, x^n) = (s, \theta^1, \dots, \theta^{n-1})$ be local coordinates around $p$, where $(\theta^1, \dots, \theta^{n-1})$ are local coordinates for $\Sigma$. With respect to these coordinates, by Lagrange we have

$$\begin{aligned}
&|\partial_i(Q(x))(p) - \partial_i(Q(y))(p)| = |q'(x(p))\partial_i x(p) - q'(y(p))\partial_i y(p)| \\
&\le |q'(x(p))[\partial_i x(p) - \partial_i y(p)]| + |[q'(y(p)) - q'(x(p))]\partial_i y(p)| \\
&\le c_1|x(p)||\partial_i(x-y)(p)| + |q''(x^*)||x(p) - y(p)||\partial_i y(p)| \\
&\le c_1|x(p)||\partial(x-y)(p)| + c_2|x(p) - y(p)||\partial y(p)|.
\end{aligned}$$

Since $\partial_i$ is the derivative which counts in the computation of our weighted Hölder norms $\mathscr{C}^1(\mathcal{K}, g)$, we just proved in $\mathcal{K}$ it holds

$$|\nabla_g(Q(x) - Q(y))|_g \le c_4 \left(\|x\|_{\mathscr{C}^{2,\alpha}_\delta(M,h_\varepsilon)} + \|y\|_{\mathscr{C}^{2,\alpha}_\delta(M,h_\varepsilon)}\right)\|x - y\|_{\mathscr{C}^{2,\alpha}_\delta(M,h_\varepsilon)} \quad (3.32)$$

for some $c_4 > 0$ independent of $\varepsilon$. Assume now that $p \in \mathcal{N}$ and let $(x^1, \dots, x^n) = (\rho, \theta^1, \dots, \theta^{n-1})$ be local coordinates around $p$. With the same arguments of the compact case, we can deduce that in $\mathcal{N}$ it holds

$$|\nabla_{g_{\mathrm{cyl}}}(Q(x) - Q(y))|_{g_{\mathrm{cyl}}} \le c_5 \left(\|x\|_{\mathscr{C}^{2,\alpha}_\delta(M,h_\varepsilon)} + \|y\|_{\mathscr{C}^{2,\alpha}_\delta(M,h_\varepsilon)}\right)\|x - y\|_{\mathscr{C}^{2,\alpha}_\delta(M,h_\varepsilon)} \quad (3.33)$$

for some $c_5 > 0$ independent of $\varepsilon$. Assume finally that $p \in \mathcal{E}$ and let $(x^1, \dots, x^n) = (t, \theta^1, \dots, \theta^{n-1})$ be local coordinates around $p$. Similarly as in the previous cases we have

$$\begin{aligned}
&\mathrm{e}^{\delta t}|\partial_i(Q(x))(p) - \partial_i(Q(y))(p)| = \mathrm{e}^{\delta t}|q'(x(p))\partial_i x(p) - q'(y(p))\partial_i y(p)| \\
&\le \mathrm{e}^{\delta t}|q'(x(p))[\partial_i x(p) - \partial_i y(p)]| + \mathrm{e}^{\delta t}|[q'(y(p)) - q'(x(p))]\partial_i y(p)| \\
&\le c_1\mathrm{e}^{\delta t}|x(p)||\partial_i(x-y)(p)| + |q''(x^*)|\mathrm{e}^{\delta t}|x(p) - y(p)||\partial_i y(p)| \\
&\le c_1c_3\mathrm{e}^{\delta t}|x(p)|\mathrm{e}^{\delta t}|\partial_i(x-y)(p)| + c_2c_3\mathrm{e}^{\delta t}|x(p) - y(p)|\mathrm{e}^{\delta t}|\partial_i y(p)|.
\end{aligned}$$

From this inequality it follows that on $\mathcal{E}$ it holds

$$\mathrm{e}^{\delta t}|\nabla_{g_{\mathrm{ah}}}(Q(x) - Q(y))|_{g_{\mathrm{ah}}} \le c_6 \left(\|x\|_{\mathscr{C}^{2,\alpha}_\delta(M,h_\varepsilon)} + \|y\|_{\mathscr{C}^{2,\alpha}_\delta(M,h_\varepsilon)}\right)\|x - y\|_{\mathscr{C}^{2,\alpha}_\delta(M,h_\varepsilon)} \quad (3.34)$$

for some $c_6 > 0$ independent of $\varepsilon$. Using (3.32), (3.33) and (3.34), we get

$$\sup_{\mathcal{K}} |\nabla_g(Q(x) - Q(y))|_g + \sup_{\mathcal{N}} |\nabla_{g_{\mathrm{cyl}}}(Q(x) - Q(y))|_{g_{\mathrm{cyl}}} + \sup_{\mathcal{E}} \mathrm{e}^{\delta t}|\nabla_{g_{\mathrm{ah}}}(Q(x) - Q(y))|_{g_{\mathrm{ah}}}$$
$$\leq c_7 \left( \|x\|_{\mathscr{C}^{2,\alpha}_\delta(M,h_\varepsilon)} + \|y\|_{\mathscr{C}^{2,\alpha}_\delta(M,h_\varepsilon)} \right) \|x - y\|_{\mathscr{C}^{2,\alpha}_\delta(M,h_\varepsilon)}$$

for some $c_4 > 0$ independent of $\varepsilon$, and (3.29) holds provided to take $\|x\|_{\mathscr{C}^{2,\alpha}_\delta(M,h_\varepsilon)}$ and $\|y\|_{\mathscr{C}^{2,\alpha}_\delta(M,h_\varepsilon)}$ smaller then $(8Cc_7)^{-1}$. □

Finally we can prove the main result of this section, in which it is essential the role of Proposition 3.7.

**Proposition 3.14.** *Fix $(n-1)/2 < \delta < n$ and $\alpha \in (0,1)$. For small $\varepsilon > 0$, the operator $F_\varepsilon \colon \mathscr{C}^{2,\alpha}_\delta(M,h_\varepsilon) \to \mathscr{C}^{2,\alpha}_\delta(M,h_\varepsilon)$ defined in (3.26) is a contraction of a ball of $\mathscr{C}^{2,\alpha}_\delta(M,h_\varepsilon)$ centred at 0 with radius proportional to $\varepsilon^n$.*

*Proof.* We set for convenience

$$a_\varepsilon = \begin{cases} \frac{1}{2}\mathrm{R}_{h_\varepsilon} + 1 & \text{in dimension } n = 2, \\ \frac{n-2}{4(n-1)}\left(\mathrm{R}_{h_\varepsilon} + n(n-1)\right) & \text{in dimension } n \geq 3, \end{cases}$$

and $b_\varepsilon = 0$ if $n = 2$ and $b_\varepsilon = a_\varepsilon$ if $n \geq 3$. Consider the ball

$$B_\rho := \left\{ v \in \mathscr{C}^{2,\alpha}_\delta(M,h_\varepsilon) \quad | \quad \|v\|_{\mathscr{C}^{2,\alpha}_\delta(M,h_\varepsilon)} \leq \rho \right\},$$

for some $\rho > 0$. Due to Proposition 3.7 for every $x \in B_\rho$ one has

$$\begin{aligned} \|F_\varepsilon(x)\|_{\mathscr{C}^{2,\alpha}_\delta(M,h_\varepsilon)} &= \left\| (\Delta_{h_\varepsilon} - n)^{-1} (a_\varepsilon + b_\varepsilon x + Q(x)) \right\|_{\mathscr{C}^{2,\alpha}_\delta(M,h_\varepsilon)} \\ &\leq C \|a_\varepsilon + b_\varepsilon x + Q(x)\|_{\mathscr{C}^{0,\alpha}_\delta(M,h_\varepsilon)} \\ &\leq C \|a_\varepsilon + b_\varepsilon x\|_{\mathscr{C}^{0,\alpha}_\delta(M,h_\varepsilon)} + C \|Q(x)\|_{\mathscr{C}^{0,\alpha}_\delta(M,h_\varepsilon)} . \end{aligned}$$

Assuming $\rho > 0$ small enough, we can apply Lemma 3.13 with $y = 0$, so that $C\|Q(x)\|_{\mathscr{C}^{0,\alpha}_\delta(M,h_\varepsilon)} \leq \rho/2$. Then

$$\|F_\varepsilon(x)\|_{\mathscr{C}^{2,\alpha}_\delta(M,h_\varepsilon)} \leq C_1 \|a_\varepsilon\|_{\mathscr{C}^{0,\alpha}_\delta(M,h_\varepsilon)} + \frac{1}{2}\rho$$

for some $C_1 > 0$ independent of $\varepsilon$. Here we used that $\|a_\varepsilon + b_\varepsilon x\| \leq \|a_\varepsilon\| + \|b_\varepsilon x\|$, the fact that $\|x\|$ is bounded. Since $a_\varepsilon$ is bounded in terms of $\varepsilon^n$ by Lemma 3.2, we got

$$\|F_\varepsilon(x)\|_{\mathscr{C}^{2,\alpha}_\delta(M,h_\varepsilon)} \leq C_2\varepsilon^n + \frac{1}{2}\rho$$

for some $C_2 > 0$ independent of $\varepsilon$ and, as a consequence, $F_\varepsilon(B_\rho) \subseteq B_\rho$ if $\rho \geq 2C_2\varepsilon^n$. So we chose $\rho := 2C_2\varepsilon^n$ and $\varepsilon > 0$ small enough. We claim that $F_\varepsilon$ is a contraction of $B_\rho$. Indeed for $x, y \in B_\rho$ one has

$$\begin{aligned}
\|F_\varepsilon(x) - F_\varepsilon(y)\|_{\mathscr{C}^{2,\alpha}_\delta(M,h_\varepsilon)} &= \left\|(\Delta_{h_\varepsilon} - n)^{-1}\left(b_\varepsilon(x-y) + Q(x) - Q(y)\right)\right\|_{\mathscr{C}^{2,\alpha}_\delta(M,h_\varepsilon)} \\
&\leq C\,\|b_\varepsilon(x-y) + Q(x) - Q(y)\|_{\mathscr{C}^{0,\alpha}_\delta(M,h_\varepsilon)} \\
&\leq C\,\|b_\varepsilon(x-y)\|_{\mathscr{C}^{0,\alpha}_\delta(M,h_\varepsilon)} + C\,\|Q(x) - Q(y)\|_{\mathscr{C}^{0,\alpha}_\delta(M,h_\varepsilon)} \\
&\leq C_3\varepsilon^n\,\|x-y\|_{\mathscr{C}^{0,\alpha}_\delta(M,h_\varepsilon)} + \frac{1}{2}\,\|x-y\|_{\mathscr{C}^{2,\alpha}_\delta(M,h_\varepsilon)}\,.
\end{aligned}$$

In the previous inequality we used again Proposition 3.7, Lemma 3.13 and the bound for $b_\varepsilon$ given by Lemma 3.2. We got the thesis if $\varepsilon > 0$ is small enough. □

### 3.5.2 Estimates of the solution

As a consequence of the previous section, the solution $v_\varepsilon$ of the Yamabe equation (3.8) satisfies

$$\|v_\varepsilon\|_{\mathscr{C}^{2,\alpha}_\delta(M,h_\varepsilon)} \leq C\varepsilon^n \tag{3.35}$$

for some $C > 0$ independent of small $\varepsilon$. More generally, with the same arguments of the previous sections one could prove that for every $k \in \mathbb{N}$ there exists $C > 0$ such that

$$\|v_\varepsilon\|_{\mathscr{C}^{k,\alpha}_\delta(M,h_\varepsilon)} \leq C\varepsilon^n,$$

however the case $k = 2$ is enough to proceed. Actually, it also could be proved that $v_\varepsilon$ is smooth in $\varepsilon$ as well, since the contraction $F_\varepsilon$ depends smoothly on this parameter. In this subsection we want to explicit the meaning of (3.35) focusing on the derivatives up to the second order. As usual we have to distinguish the three distinct regions of $M = \mathcal{K} \cup \mathcal{N} \cup \mathcal{E}$ introduced in Section 3.3.

- For every $p \in \mathcal{K}$, the control (3.35) implies

  $$|v_\varepsilon(p)|, |\nabla_{\bar g} v_\varepsilon(p)|_{\bar g}, |\nabla^2_{\bar g} v_\varepsilon(p)|_{\bar g}, |\partial_s v_\varepsilon(p)|, |\nabla_{\bar g}\partial_s v_\varepsilon(p)|_{\bar g}, |\partial^2_s v_\varepsilon| < C_1\varepsilon^n$$

  for some $C_1 > 0$ independent of $p$ and $\varepsilon$. It was sufficient to apply the definition of the involved norm and the boundedness of $\mathrm{e}^s$ in $\mathcal{K}$.

- For every $p \in \mathcal{N}$, the control (3.35) implies

  $$|v_\varepsilon(p)|, |\nabla_{\bar g} v_\varepsilon(p)|_{\bar g}, |\nabla^2_{\bar g} v_\varepsilon(p)|_{\bar g}, |\partial_\rho v_\varepsilon(p)|, |\nabla_{\bar g}\partial_\rho v_\varepsilon(p)|_{\bar g}, |\partial^2_\rho v_\varepsilon| < C_2\varepsilon^n$$

for some $C_2 > 0$ independent of $p$ and $\varepsilon$. It was sufficient to apply the definition of the involved norm. In terms of the variable $r$, using (3.11), it holds

$$\begin{cases} |v_\varepsilon(p)|, |\nabla_{\bar g} v_\varepsilon(p)|_{\bar g}, |\nabla^2_{\bar g} v_\varepsilon(p)|_{\bar g} < C_3 \varepsilon^n \\ |\partial_r v_\varepsilon(p)|, |\nabla_{\bar g}\partial_r v_\varepsilon(p)|_{\bar g} < C_3\varepsilon^{n-1}\cosh^{-2/n}(nr(p)/2) \\ |\partial_r^2 v_\varepsilon| < C_3 \varepsilon^{n-2}\cosh^{-4/n}(nr(p)/2) \end{cases}$$

for some $C_3 > 0$ independent of $p$ and $\varepsilon$.

- For every $p \in \mathcal{E}$, the control (3.35) implies

$$\begin{cases} |v_\varepsilon(p)|, \mathrm{e}^{-t(p)}|\nabla_{\bar g} v_\varepsilon(p)|_{\bar g}, \mathrm{e}^{-2t(p)}|\nabla^2_{\bar g} v_\varepsilon(p)|_{\bar g} < C_4 \varepsilon^n \mathrm{e}^{-\delta t(p)} \\ |\partial_t v_\varepsilon(p)|, \mathrm{e}^{-t(p)}|\nabla_{\bar g}\partial_t v_\varepsilon(p)|_{\bar g} < C_4\varepsilon^{n}\mathrm{e}^{-\delta t(p)} \\ |\partial_r^2 v_\varepsilon| < C_4 \varepsilon^{n}\mathrm{e}^{-\delta t(p)} \end{cases}$$

for some $C_4 > 0$ independent of $p$ and $\varepsilon$. It was sufficient to apply the definition of the involved norm.

The estimates above will play a central role in the next section, where we will construct a CMC foliation for the ends of $(M, g_\varepsilon)$. More generally, the arguments above can be used to show that for $k_1, k_2 \in \mathbb{N}$ it holds

$$\begin{cases} \sup_{\mathcal{K}} \left|\nabla^{(k_1)}_{\bar g}\partial_s^{k_2} v_\varepsilon\right| = O(\varepsilon^n) \\ \sup_{\mathcal{N}} \left|\cosh^{2k_2/n}(nr/2)\nabla^{(k_1)}_{\bar g}\partial_s^{k_2} v_\varepsilon\right| = O(\varepsilon^{n-k_2}) \\ \sup_{\mathcal{E}} \left|\mathrm{e}^{(\delta-k_1)t}\nabla^{(k_1)}_{\bar g}\partial_s^{k_2} v_\varepsilon\right| = O(\varepsilon^n). \end{cases}$$

## 3.6 Local CMC foliations

The aim of this section is to prove Theorem 3.1-(4). In the previous pages we built on $(M, g)$ a family of metrics $\{g_\varepsilon\}_\varepsilon$, defined for small $\varepsilon > 0$, with constant negative scalar curvature. Each $g_\varepsilon$ is a conformal perturbation of the approximate solution $h_\varepsilon$ defined in (3.2) and we were able to show the points (1), (2) and (3) of Theorem 3.1. Now we want to show that every end of $g_\varepsilon$ is foliated by a family of weakly stable CMC hypersurfaces (the definition of *weakly stable* hypersurface is at the end of this introduction), analysing the possible values of the mean curvatures as well as the area of the minimal leaf belonging to this foliation.

Since this problem only involves the ends of $(M, g_\varepsilon)$, we will assume for simplicity that through this section

$$M = (\log\varepsilon + 2, +\infty) \times \Sigma.$$

We recall that

$$g_\varepsilon = \mathrm{e}^{2u_\varepsilon} h_\varepsilon, \tag{3.36}$$

where $h_\varepsilon$ is the *model metric* defined by

$$h_\varepsilon = dr^2 + \varepsilon^2\phi^2(r)\bar{g}, \quad \text{with} \quad \phi(r) := \left(2\cosh\left(\frac{nr}{2}\right)\right)^{2/n}, \tag{3.37}$$

and $(\Sigma, \bar{g})$ is a compact manifold with vanishing scalar curvature. We defined the function $u_\varepsilon$ by

$$\mathrm{e}^{2u_\varepsilon} := (1+v_\varepsilon)^{\frac{4}{n-2}}, \quad \text{if } n \geq 3 \qquad \text{and} \qquad u_\varepsilon := v_\varepsilon, \quad \text{if } n = 2. \tag{3.38}$$

Here $v_\varepsilon$ is the function constructed in the previous sections, which is smooth and, in virtue of what we observed in Subsection 3.5.2, satisfies

$$\|v_\varepsilon\|_{\mathscr{C}^{2,\alpha}_\delta(M,h_\varepsilon)} \leq C\varepsilon^n, \qquad \text{for some } \frac{n-1}{2} < \delta < n.$$

In particular the above estimate implies that, in terms of $u_\varepsilon$, for every real number $\delta \in ((n-1)/2, n)$ there exist $C_1, C_2 > 0$ independent of $\varepsilon$ such that

$$\sup_{(\log\varepsilon+2, -\log\varepsilon-2)\times\Sigma} |u_\varepsilon| + |\nabla_{g_{\mathrm{cyl}}} u_\varepsilon|_{g_{\mathrm{cyl}}} + |\nabla^2_{g_{\mathrm{cyl}}} u_\varepsilon|_{g_{\mathrm{cyl}}} \leq C_1\varepsilon^n, \tag{3.39}$$

where $g_{\mathrm{cyl}} := \varepsilon^{-2}\phi^{-2}(r)h_\varepsilon$ is the cylindrical metric already introduced in (3.11), and

$$\sup_{(-\log\varepsilon-3, +\infty)\times\Sigma} \mathrm{e}^{\delta(r+\log\varepsilon)}\left(|u_\varepsilon| + |\nabla_{h_\varepsilon} u_\varepsilon|_{h_\varepsilon} + |\nabla^2_{h_\varepsilon} u_\varepsilon|_{h_\varepsilon}\right) \leq C_2\varepsilon^n. \tag{3.40}$$

The main result of this section is actually stronger then Theorem 3.1-(4). In fact we are going to prove the following Theorem 3.15. We recall that the mean curvatures are computed with respect to the unit normal vector pointing toward the infinity.

**Theorem 3.15.** *Let $(\Sigma, \bar{g})$ be a compact Riemannian manifold of dimension $n-1$, $n \geq 2$. Let $(M, g_\varepsilon) = ((\log\varepsilon + 2, +\infty) \times \Sigma, g_\varepsilon)$ be defined by* (3.36)-(3.37) *and assume it holds* (3.39)-(3.40). *Then for every $R > 0$ there exist $\varepsilon_R > 0$ and $C_R > 0$ with the following property:*

*(i) For every $\varepsilon \in (0, \varepsilon_R)$ and for every $\rho \in [\log\varepsilon + 3, -\log\varepsilon + R]$ there exists a function $\psi(\cdot\,; \varepsilon, \rho) \in \mathscr{C}^{2,\alpha}(\Sigma, \bar{g})$ such that*

$$S(\varepsilon, \rho) := \{(\rho + \psi(\theta; \varepsilon, \rho), \theta) \mid \theta \in \Sigma\}$$

*is an hypersurface with constant mean curvature equal to $(n-1)\tanh(n\rho/2)$ sitting inside $(M, g_\varepsilon)$;*

*(ii) The hypersurface $S(\varepsilon, \rho)$ belongs to a small neighbourhood of $\{r = \rho\}$, since it holds*

$$\|\psi(\cdot\,; \varepsilon, \rho)\|_{\mathscr{C}^2(\Sigma,\bar{g})} \leq C_R\varepsilon^{n-1}; \tag{3.41}$$

(*iii*) *The family* $\{S(\varepsilon,\rho)\}_{\rho\in[\log\varepsilon+3,-\log\varepsilon+R]}$ *provide a weakly stable CMC foliation of a compact subset of* $M$*;*

(*iv*) *The area of the minimal hypersurface* $S(\varepsilon,0)$ *is comparable to* $\varepsilon^{n-1}$*, meaning that there exists* $c>1$ *depending only on* $n$ *and* $\mathrm{Area}_{\bar g}(\Sigma)$ *such that*
$$c^{-1}\varepsilon^{n-1}\le \mathrm{Area}_{g_\varepsilon}S(\varepsilon,0)\le c\varepsilon^{n-1}.$$

*Moreover, there exists* $R_0>0$ *depending only on* $(\Sigma,\bar g)$ *and the constant* $C_2$ *of* (3.40) *such that if* $R>R_0$ *then:*

(*v*) *The foliation above can be uniquely extended to a weakly stable CMC foliation* $\{S(\varepsilon,\rho)\}_{\rho\ge\log\varepsilon+3}$ *of the whole end, whose leaves' constant mean curvatures increase towards the infinity and tend to* $n-1$*.*

*REMARK* 3.4. Notice that in Theorem 3.15 all the hypersurfaces $S(\varepsilon,\rho)$ are automatically included in $(\log\varepsilon+2,+\infty)\times\Sigma$ if it holds (3.41), provided that $\varepsilon_R$ is small enough. Notice also that the mean curvature of $S(\varepsilon,\rho)$ for $\rho=\log\varepsilon+3$ tends to $-(n-1)$ as $\varepsilon\to 0$ in according to Theorem 3.1-(4).

*REMARK* 3.5. We emphasize that in the theorem above we do not require $g_\varepsilon$ to have constant scalar curvature. In this sense, this theorem is more general then the setting of Theorem 3.1, where $g_\varepsilon$ is the metric obtained via a conformal transformation of $h_\varepsilon$ imposing that the scalar curvature equal to $-n(n-1)$. In fact it is sufficient to consider any conformal perturbation of $h_\varepsilon$ satisfying (3.39)-(3.40) in order to guarantee the existence of a CMC foliation of the ends. Differently, it will be necessary to assume $\mathrm{R}_{g_\varepsilon}=-n(n-1)$ when we will prove the Riemannian Penrose inequality (Section 4.3).

Before proving the theorem above, we recall the notion of *weakly stable* foliation for the reader's convenience.

## Weakly stable hypersurfaces

An hypersurface $S$ embedded in a Riemannian manifold is called *weakly stable* if the second variation of the area of $S$ under volume preserving variations is non-negative. Equivalently, we give this analytic definition:

**Definition 3.3.** Assume that $S$ is an oriented hypersurface in a Riemannian manifold $(M,g)$ such that $M\setminus S$ has a unique unbounded connected component. Let $\nu$ be the unit normal vector pointing toward the infinity and $\mathrm{II}(X,Y)=-\langle\nabla_XY,\nu\rangle$ be the second fundamental form of $S\subset M$. The operator
$$L_S:=\Delta_S+\mathrm{Ric}(\nu,\nu)+|\mathrm{II}|^2$$
is called *Jacobi operator* of $S$. A constant mean curvature hypersurface $S$ is called ***weakly stable*** iff $\int_S L_S(f)f dS\le 0$ for every $f\in\mathscr{C}^\infty(S)$ with $\int_S f dS=0$.

### 3.6.1 The model case

In this subsection we show that Theorem 3.15 easily holds replacing $g_\varepsilon$ by the model metric $h_\varepsilon$, which is the case $u_\varepsilon = 0$. Consider the hypersurface $S = \{r = \rho\}$ in the model manifold

$$(M, h_\varepsilon) := (\mathbb{R} \times \Sigma, dr^2 + \varepsilon^2 \phi^2(r)\bar{g}),$$

where $(\Sigma, \bar{g})$ is compact manifold of dimension $n-1$ and $\phi(r) := (2\cosh(nr/2))^{2/n}$. As $\rho$ varies in $\mathbb{R}$ the hypersurfaces $\{r = \rho\}$ provide a foliation and by Lemma 5.4 the mean curvature of $S$ is constant and equal to $(n-1)\phi^{-1}(\rho)\dot{\phi}(\rho) = (n-1)\tanh(n\rho/2)$, computed with respect to the unit vector field $\nu = \partial_r$. In particular there is a unique minimal leaf (for $\rho = 0$) and it is easy to compute its area, which is equal to $2^{\frac{2(n-1)}{n}}\mathrm{Area}_{\bar{g}}(\Sigma)\varepsilon^{n-1}$. Let us check that $S$ is weakly stable. On one hand, it is easy to compute the Laplacian $\Delta_S = \varepsilon^{-2}\phi^{-2}(\rho)\Delta_{\bar{g}}$, on the other hand by Proposition 5.1

$$\mathrm{Ric}(\nu, \nu) = \frac{n(n-1)}{2}\left(\frac{n-2}{n}\tanh^2\left(\frac{n\rho}{2}\right) - 1\right)$$

while by Lemma 5.4 the second fundamental form of $S$ is $\mathrm{II} = \varepsilon^2\phi(\rho)\dot{\phi}(\rho)\bar{g}$ and therefore

$$|\mathrm{II}|^2 = (n-1)\tanh^2\left(\frac{n\rho}{2}\right).$$

By summing the last results we get the Jacobi operator of $S$ given by

$$\begin{aligned} L_S &= \varepsilon^{-2}\phi^{-2}(\rho)\Delta_{\bar{g}} - \frac{n(n-1)}{2}\frac{1}{\cosh^2(n\rho/2)} \\ &= \varepsilon^{-2}\phi^{-2}(\rho)\Delta_{\bar{g}} - 2n(n-1)\phi^{-n}(\rho). \end{aligned} \tag{3.42}$$

For every $f \in \mathscr{C}^\infty(\Sigma)$ with vanishing mean value

$$\int_S L_S(f)f = -\int_S \varepsilon^{-2}\phi^{-2}(\rho)|\nabla f|^2 - 2n(n-1)\phi^{-n}(\rho)f^2 \leq 0$$

simply integrating by parts, therefore $S$ is weakly stable.

The idea of the proof of Theorem 3.15 is to prove an analogue of the model case above for the conformal perturbation $g_\varepsilon$. More precisely, the proof will be divided into two parts: we need to consider separately the construction of the CMC foliation in a compact part of the manifold's end (which is faced in this section), namely points $(i)$-$(iv)$ of Theorem 3.15, and the construction on the infinity of the end (which is faced in the next), namely point $(v)$ of Theorem 3.15. The necessity of dividing the proof in these two cases has already been discussed by Ambrozio in a similar setting (cfr. [5]), where he constructed a weakly stable CMC foliation for perturbations of asymptotically anti-de Sitter spaces.

Here is how the next subsections are organized in order to prove Theorem 3.15, except for the last point which is obtained in the next section.

- In Subsection 3.6.2 we follow the idea of Ambrozio [5] to product CMC hypersurfaces with the Implicit Function Theorem. This approach has the advantage of being short and does not require too computation, it has also the advantage of simplifying the proof that the these hypersurfaces provide a foliation as well as that each leaf is weakly stable. However, with this approach we are not able to conclude the estimate (3.41) about the location of the leaf with prescribed mean curvature. This estimate describes how well the CMC foliation of $g_\varepsilon$ approximate the CMC foliation of the model metric $h_\varepsilon$ for small $\varepsilon > 0$.

- For this reason we will build the same CMC foliation with a different method. Precisely we reduce the problem to solving a fixed-point problem. In Subsection 3.6.3 the construction of the leaf of mean curvature equal to $(n-1)\tanh(n\rho/2)$ is reduced to prove the convergence of an iterative scheme of the form

$$L(\varepsilon,\rho)[x_{j+1}] = \varepsilon^2\phi^2(\rho)Q(x_j;\varepsilon,\rho) + E(x_j;\varepsilon,\rho),$$

  where $L(\varepsilon,\rho)\colon \mathscr{C}^{2,\alpha}(\Sigma,\bar g) \to \mathscr{C}^{0,\alpha}(\Sigma,\bar g)$ is a linear elliptic operator inducing an isomorphism between Hölder spaces, $Q(\,\cdot\,;\varepsilon,\rho)\colon \mathscr{C}^{2,\alpha}(\Sigma,\bar g) \to \mathscr{C}^{0,\alpha}(\Sigma,\bar g)$ is a quadratic term and $E(\,\cdot\,;\varepsilon,\rho)\colon \mathscr{C}^{2,\alpha}(\Sigma,\bar g) \to \mathscr{C}^{0,\alpha}(\Sigma,\bar g)$ is an *error term* depending on the conformal perturbation $u_\varepsilon$;

- The Subsections 3.6.4, 3.6.5 and 3.6.6 contain some preliminary results about the operators $E$, $L$ and $Q$ respectively. Those results trigger the iterative scheme, which is proved to converge in Subsection 3.6.7. In this way we are able to prove the points (*i*) and (*ii*) of Theorem 3.15.

- In Subsection 3.6.8 we deal the weak stability of the leaves, proving the point (*iii*) of Theorem 3.15. Finally in Subsection 3.6.9 we compute the volume of the minimal leaf, proving the point (*iv*) of Theorem 3.15.

Throughout these subsections we will make large use of (3.39), while (3.40) will be crucial in the next section, concerning the unique extension of the foliation to the infinity.

### 3.6.2 Local existence

In this subsection we follow the implicit function approach of Ambrozio [5] to show the existence of a weakly stable CMC foliation on a compact subset of the end of the Riemannian manifold $(M, g_\varepsilon)$. With this method it is easy to prove the weak stability, however a precise information about the position of the single leaf is missing. Such a control, as mentioned before, will be achieved in the next subsections.

Let $I$ be a compact neighbourhood of $[\log\varepsilon+3, -\log\varepsilon+R]$, for instance we can chose

$$I = [\log\varepsilon + 3 - 1/2, -\log\varepsilon + R + 1/2].$$

For every $u \in \mathscr{C}^{2,\alpha}_\delta(M, h_\varepsilon)$, for every $\sigma \in I$ and for every $x \in \mathscr{C}^{2,\alpha}(\Sigma)$, let us denote by $H(\sigma, u, x) \in \mathscr{C}^{0,\alpha}(\Sigma)$ the mean curvature of the hypersurface $\{r = \sigma + x\}$ with respect to the metric $\mathrm{e}^{2u}h_\varepsilon$ and the unit normal vector pointing toward $r = +\infty$. By Lemma 5.4 one has

$$e^{u(\sigma+x,\cdot)}H(\sigma,u,x) = -\,\phi^{-1}(\sigma+x)\mathrm{div}\left(\frac{\nabla x}{\sqrt{\phi^2(\sigma+x)+|\nabla x|^2}}\right) + \frac{(n-1)\dot{\phi}(\sigma+x)}{\sqrt{\phi^2(\sigma+x)+|\nabla x|^2}} + (n-1)\frac{\partial_r u(\sigma+x,\cdot) - \phi^{-2}(\sigma+x)\langle\nabla u(\sigma+x,\cdot),\nabla x\rangle}{\sqrt{1+\phi^{-2}(\sigma+x)|\nabla x|^2}} \tag{3.43}$$

as equality on $\Sigma$, where all the geometric objects refers to $\bar{g}$. Consider now the Banach spaces of functions

$$A := \left\{x \in \mathscr{C}^{2,\alpha}(\Sigma) \quad \text{s.t.} \quad \fint_\Sigma x = 0\right\}$$

and

$$B := \left\{y \in \mathscr{C}^{0,\alpha}(\Sigma) \quad \text{s.t.} \quad \fint_\Sigma y = 0\right\},$$

and define the $C^1$-operator

$$\Phi\colon I \times \mathscr{C}^{2,\alpha}_\delta(M,h_\varepsilon) \times A \to B \tag{3.44}$$

by $\Phi(\sigma,u,x) = H(\sigma,u,x) - \fint_\Sigma H(\sigma,u,x)$. Observe that $\Phi(\sigma,u,x) = 0$ if and only if $\{r = \sigma + x\}$ is a CMC hypersurface with respect to $\mathrm{e}^{2u}h_\varepsilon$. In particular $\Phi(\sigma,0,0) = 0$, as we noticed in Subsection 3.6.1. Let us verify that $D_x\Phi_{(\sigma,0,0)}\colon A \to B$ is an isomorphism for every $\sigma \in I$, so that it is possible to apply the Implicit Function Theorem. Since $v$ has mean value equal to zero, a direct computation gives

$$D_x\Phi_{(\sigma,0,0)}[v] = \lim_{t\to 0}\frac{1}{t}(\Phi(\sigma,0,tv) - \Phi(\sigma,0,0)) = \varepsilon^{-2}\phi^{-2}(\sigma)\Delta_{\bar{g}}v - 2n(n-1)\phi^{-n}(\sigma)v,$$

which is an isomorphism from $A$ to $B$ as mentioned. By the Implicit Function Theorem there is a $C^1$-application $x(\sigma,u) \in A$ uniquely determined by the equation $\Phi(\sigma,u,x(\sigma,u)) = 0$ in a neighbourhood of $(\sigma,0,0)$. By compactness of $I$, we can assume that $x$ is defined in $I \times U$, where $U$ is a neighbourhood of $0 \in \mathscr{C}^{2,\alpha}_\delta(M,h_\varepsilon)$. Here is the crucial point where the following argument can not be applied to the whole end of $M$ but we have to consider a parameter $\rho$ bounded above. By construction, the hypersurface

$$\hat{S}(u,\sigma) := \{r = \sigma + x(\sigma,u)\}$$

has constant mean curvature with respect to $e^{2u}h_\varepsilon$, but we do not know its value and in general it may not coincide with $(n-1)\tanh(n\sigma/2)$, which is the mean curvature of $\{r=\sigma\}$ in $(M,h_\varepsilon)$. Now we choose $u=u_\varepsilon$ as in the hypothesis of Theorem 3.15. Due to (3.39) and (3.40), for small $\varepsilon$, depending on $I$ thus on $R$, the set $I\times\{u_\varepsilon\}$ belongs to the domain of $x$. We claim that the hypersurfaces $\hat{S}(u_\varepsilon,\sigma)$ provide a weakly stable foliation of a compact subset of $M$ as $\sigma$ varies in $I$. In order to prove that it is a foliation, it is sufficient to check that for small $\varepsilon$ one has

$$\partial_\sigma(\sigma+x(\sigma,u_\varepsilon))>0. \tag{3.45}$$

By continuity, this hold if $\partial_\sigma(\sigma+x(\sigma,0))>0$, which is trivial since $x(\sigma,0)=0$. Similarly, we can conclude that $\hat{S}(u_\varepsilon,\sigma)$ is weakly stable up to restrict $\varepsilon$ again. In fact in Subsection 3.6.1 we proved that

$$\int_{\hat{S}(\sigma,0)} L_{\hat{S}(\sigma,0)}(f)f\,d\hat{S}\le -c\|f\|^2_{L^2(\hat{S}(\sigma,0))}$$

for every $f\in\mathscr{C}^\infty(\hat{S}(\sigma,0))$ with zero average and some constant $c=c(n,R)>0$. By continuity it holds

$$\int_{\hat{S}(\sigma,u_\varepsilon)} L_{\hat{S}(\sigma,u_\varepsilon)}(f)f\,d\hat{S}\le 0$$

for small $\varepsilon$, and the claim follows. Up to now, we constructed a local weakly stable CMC foliation but the leaf $\hat{S}(u_\varepsilon,\sigma)$ may not correspond to the hypersurface $S(\varepsilon,\sigma)$ with constant mean curvature equal to $(n-1)\tanh(n\sigma/2)$ announced in Theorem 3.15. In the next pages we will build the CMC hypersurfaces $\{S(\varepsilon,\rho)\}_{\rho\in[\log\varepsilon+3,-\log\varepsilon+R]}$ and we will use this subsection to prove that they are a weakly stable foliation.

### 3.6.3 Iterative approach

Conformally to the assumption of this section, we consider the manifold $M=(2+\log\varepsilon,+\infty)\times\Sigma$ and the metric $g_\varepsilon$ defined by (3.36). Let $\psi\colon\Sigma\to\mathbb{R}$ be a regular function and assume that the hypersurface $\{r=\rho+\psi\}$ is included in $(M,g_\varepsilon)$. Then by Lemma 5.4 the mean curvature $H_{g_\varepsilon}$ of this hypersurface, which clearly depends on $\rho$ and $\psi$, is given by

$$\begin{aligned}H_{g_\varepsilon}&=e^{-u_\varepsilon(\rho+\psi,\cdot)}H_{h_\varepsilon}\\&\quad+(n-1)e^{-u_\varepsilon(\rho+\psi,\cdot)}\frac{\partial_r u_\varepsilon(\rho+\psi,\cdot)-\varepsilon^{-2}\phi^{-2}(\rho+\psi)\langle\nabla\psi,\nabla u_\varepsilon(\rho+\psi,\cdot)\rangle}{\sqrt{1+\varepsilon^{-2}\phi^{-2}(\rho+\psi)|\nabla\psi|^2}},\end{aligned} \tag{3.46}$$

where

$$\begin{aligned} H_{h_\varepsilon} &= -\varepsilon^{-1}\phi^{-1}(\rho+\psi)\ \mathrm{div}\left(\frac{\nabla\psi}{\sqrt{\varepsilon^2\phi^2(\rho+\psi)+|\nabla\psi|^2}}\right) \\ &\quad + (n-1)\frac{(\dot\phi/\phi)(\rho+\psi)}{\sqrt{1+\varepsilon^{-2}\phi^{-2}(\rho+\psi)|\nabla\psi|^2}} \end{aligned} \tag{3.47}$$

is the mean curvature of $\{r = \rho + \psi\}$ with respect to $(M, h_\varepsilon)$. As usual the geometric quantities are computed with respect to $\bar g$. We recall that all the mean curvatures are computed with respect to the unit normal vector pointing toward $r = +\infty$. In view of (3.46) and (3.47), the hypersurface $\{r = \rho+\psi\}$ has constant mean curvature equal to $H(\rho) := (n-1)(\dot\phi/\phi)(\rho) = (n-1)\tanh(n\rho/2)$ if and only if $F(\psi;\varepsilon,\rho) = 0$, where

$$F(\psi;\varepsilon,\rho) = -\mathrm{e}^{u_\varepsilon(\rho+\psi,\cdot)}\sqrt{1+\varepsilon^{-2}\phi^{-2}(\rho+\psi)|\nabla\psi|^2}\,(H_{g_\varepsilon} - H(\rho))\,.$$

Notice that $H(\rho)$ is precisely the mean curvature of $\{r = \rho\}$ in the model manifold $(M, h_\varepsilon)$. In what follows we explicit the equation $F(\psi;\varepsilon,\rho) = 0$ and we rewrite it in a smarter way with the purpose of applying a fixed-point method to solve it. After using (3.46), we explicit $H_{h_\varepsilon}$ in

$$\begin{aligned} F(\psi;\varepsilon,\rho) &= -\sqrt{1+\varepsilon^{-2}\phi^{-2}(\rho+\psi)|\nabla\psi|^2}H_{h_\varepsilon} \\ &\quad - (n-1)\left(\partial_r u_\varepsilon(\rho+\psi,\cdot) - \varepsilon^{-2}\phi^{-2}(\rho+\psi)\langle\nabla\psi, \nabla u_\varepsilon(\rho+\psi,\cdot)\rangle\right) \\ &\quad + \mathrm{e}^{u_\varepsilon(\rho+\psi,\cdot)}\sqrt{1+\varepsilon^{-2}\phi^{-2}(\rho+\psi)|\nabla\psi|^2}H(\rho) \end{aligned}$$

expending the divergence term. This gives

$$\begin{aligned} F(\psi;\varepsilon,\rho) &= \varepsilon^{-2}\phi^{-2}(\rho+\psi)\Delta\psi \\ &\quad - \varepsilon^{-4}\phi^{-4}(\rho+\psi)\frac{\mathrm{Hess}\psi(\nabla\psi,\nabla\psi)}{1+\varepsilon^{-2}\phi^{-2}(\rho+\psi)|\nabla\psi|^2} \\ &\quad - \varepsilon^{-2}\phi^{-2}(\rho+\psi)\frac{(\dot\phi/\phi)(\rho+\psi)|\nabla\psi|^2}{1+\varepsilon^{-2}\phi^{-2}(\rho+\psi)|\nabla\psi|^2} \\ &\quad - (n-1)(\dot\phi/\phi)(\rho+\psi) \\ &\quad - (n-1)\left(\partial_r u_\varepsilon(\rho+\psi,\cdot) - \varepsilon^{-2}\phi^{-2}(\rho+\psi)\langle\nabla\psi, \nabla u_\varepsilon(\rho+\psi,\cdot)\rangle\right) \\ &\quad + \mathrm{e}^{u_\varepsilon(\rho+\psi,\cdot)}\sqrt{1+\varepsilon^{-2}\phi^{-2}(\rho+\psi)|\nabla\psi|^2}H(\rho). \end{aligned}$$

We want to linearise each line of this expression at $\psi = 0$ up to the first order, except for the two last lines which involve $u_\varepsilon$ and will be treated later. This makes sense since $\psi = 0$ provides a solution in the model case $u_\varepsilon = 0$.

Notice that the second and third line are quadratic in $\psi$. So we get

$$
\begin{aligned}
F(\psi;\varepsilon,\rho) = {} & \varepsilon^{-2}\phi^{-2}(\rho)\Delta\psi + \varepsilon^{-2}\left(\phi^{-2}(\rho+\psi) - \phi^{-2}(\rho)\right)\Delta\psi \\
& - \varepsilon^{-4}\phi^{-4}(\rho+\psi)\frac{\mathrm{Hess}\psi(\nabla\psi,\nabla\psi)}{1+\varepsilon^{-2}\phi^{-2}(\rho+\psi)|\nabla\psi|^2} \\
& - \varepsilon^{-2}\phi^{-2}(\rho+\psi)\frac{(\dot\phi/\phi)(\rho+\psi)|\nabla\psi|^2}{1+\varepsilon^{-2}\phi^{-2}(\rho+\psi)|\nabla\psi|^2} \\
& - (n-1)(\dot\phi/\phi)(\rho+\psi) + H(\rho) + 2n(n-1)\phi^{-n}(\rho)\psi + H(\rho) - 2n(n-1)\phi^{n}(\rho)\psi \\
& - (n-1)\left(\partial_r u_\varepsilon(\rho+\psi,\cdot) - \varepsilon^{-2}\phi^{-2}(\rho+\psi)\langle\nabla\psi,\nabla u_\varepsilon(\rho+\psi,\cdot)\rangle\right) \\
& + \mathrm{e}^{u_\varepsilon(\rho+\psi,\cdot)}\sqrt{1+\varepsilon^{-2}\phi^{-2}(\rho+\psi)|\nabla\psi|^2}H(\rho).
\end{aligned}
$$

which is reorganized as

$$
\begin{aligned}
F(\psi;\varepsilon,\rho) = {} & \varepsilon^{-2}\phi^{-2}(\rho)\Delta\psi - 2n(n-1)\phi^{n}(\rho)\psi \\
& + \varepsilon^{-2}\left(\phi^{-2}(\rho+\psi) - \phi^{-2}(\rho)\right)\Delta\psi \\
& - \varepsilon^{-4}\phi^{-4}(\rho+\psi)\frac{\mathrm{Hess}\psi(\nabla\psi,\nabla\psi)}{1+\varepsilon^{-2}\phi^{-2}(\rho+\psi)|\nabla\psi|^2} \\
& - \varepsilon^{-2}\phi^{-2}(\rho+\psi)\frac{(\dot\phi/\phi)(\rho+\psi)|\nabla\psi|^2}{1+\varepsilon^{-2}\phi^{-2}(\rho+\psi)|\nabla\psi|^2} \\
& - (n-1)(\dot\phi/\phi)(\rho+\psi) + H(\rho) + 2n(n-1)\phi^{-n}(\rho)\psi \\
& + (n-1)\varepsilon^{-2}\phi^{-2}(\rho+\psi)\langle\nabla\psi,\nabla u_\varepsilon(\rho+\psi,\cdot)\rangle \\
& - (n-1)\partial_r u_\varepsilon(\rho+\psi,\cdot) \\
& + \mathrm{e}^{u_\varepsilon(\rho+\psi,\cdot)}\sqrt{1+\varepsilon^{-2}\phi^{-2}(\rho+\psi)|\nabla\psi|^2}H(\rho) - H(\rho).
\end{aligned}
$$

In this way the first line is linear in $\psi$, from the second to the sixth line we have terms quadratic in $\psi$, while the last two lines are not quadratic in $\psi$ but can be rearranged so that they are negligible for $u_\varepsilon$ close to zero. Indeed we can write

$$
\begin{aligned}
F(\psi;\varepsilon,\rho) = & \Big(\varepsilon^{-2}\phi^{-2}(\rho)\Delta - 2n(n-1)\phi^{n}(\rho)\Big)\psi \\
& + \Big(\varepsilon^{-2}\left(\phi^{-2}(\rho+\psi) - \phi^{-2}(\rho)\right)\Delta\psi \\
& - \varepsilon^{-4}\phi^{-4}(\rho+\psi)\frac{\mathrm{Hess}\psi(\nabla\psi,\nabla\psi)}{1+\varepsilon^{-2}\phi^{-2}(\rho+\psi)|\nabla\psi|^2} \\
& - \varepsilon^{-2}\phi^{-2}(\rho+\psi)\frac{(\dot\phi/\phi)(\rho+\psi)|\nabla\psi|^2}{1+\varepsilon^{-2}\phi^{-2}(\rho+\psi)|\nabla\psi|^2} \\
& - (n-1)(\dot\phi/\phi)(\rho+\psi) + H(\rho) + 2n(n-1)\phi^{-n}(\rho)\psi \\
& + (n-1)\varepsilon^{-2}\phi^{-2}(\rho+\psi)\langle\nabla\psi,\nabla u_\varepsilon(\rho+\psi,\cdot)\rangle \\
& + \mathrm{e}^{u_\varepsilon(\rho+\psi,\cdot)}H(\rho)(\sqrt{1+\varepsilon^{-2}\phi^{-2}(\rho+\psi)|\nabla\psi|^2}-1)\Big) \\
& + \Big((\mathrm{e}^{u_\varepsilon(\rho+\psi,\cdot)}-1)H(\rho) - (n-1)\partial_r u_\varepsilon(\rho+\psi,\cdot)\Big).
\end{aligned}
$$

The advantage of the formula above is that now $F(\psi;\varepsilon,\rho) = 0$ can be equivalently written as

$$
\varepsilon^{-2}\phi^{-2}(\rho)L(\varepsilon,\rho)[\psi] = Q(\psi;\varepsilon,\rho) + \varepsilon^{-2}\phi^{-2}(\rho)E(\psi;\varepsilon,\rho), \tag{3.48}
$$

where:

- We introduced the linear operator $L(\varepsilon,\rho)$ of the second order defined in $(\Sigma,\bar g)$ by
$$
L(\varepsilon,\rho) := \Delta - 2n(n-1)\varepsilon^2\phi^{2-n}(\rho). \tag{3.49}
$$
Notice that $\varepsilon^{-2}\phi^{-2}(\rho)L(\varepsilon,\rho)$ is precisely the Jacobi operator computed in Example **??**;

- We introduced the operator
$$
\begin{aligned}
Q(\psi;\varepsilon,\rho) : = & -\varepsilon^{-2}\left(\phi^{-2}(\rho+\psi) - \phi^{-2}(\rho)\right)\Delta\psi \\
& + \varepsilon^{-4}\phi^{-4}(\rho+\psi)\frac{\mathrm{Hess}\psi(\nabla\psi,\nabla\psi)}{1+\varepsilon^{-2}\phi^{-2}(\rho+\psi)|\nabla\psi|^2} \\
& + \varepsilon^{-2}\phi^{-2}(\rho+\psi)\frac{(\dot\phi/\phi)(\rho+\psi)|\nabla\psi|^2}{1+\varepsilon^{-2}\phi^{-2}(\rho+\psi)|\nabla\psi|^2} \\
& + (n-1)(\dot\phi/\phi)(\rho+\psi) - H(\rho) - 2n(n-1)\phi^{-n}(\rho)\psi \\
& - (n-1)\varepsilon^{-2}\phi^{-2}(\rho+\psi)\langle\nabla\psi,\nabla u_\varepsilon(\rho+\psi,\cdot)\rangle \\
& - \mathrm{e}^{u_\varepsilon(\rho+\psi,\cdot)}H(\rho)(\sqrt{1+\varepsilon^{-2}\phi^{-2}(\rho+\psi)|\nabla\psi|^2}-1)
\end{aligned} \tag{3.50}
$$
which is quadratic in $\psi$;

- We introduced a last term $E(\psi;\varepsilon,\rho)$ defined by

$$E(\psi;\varepsilon,\rho) := \varepsilon^2\phi^2(\rho)\Big(-(\mathrm{e}^{u_\varepsilon(\rho+\psi,\cdot)}-1)H(\rho)+(n-1)\partial_r u_\varepsilon(\rho+\psi,\cdot)\Big) \tag{3.51}$$

which depends on $\psi$ but is not quadratic in $\psi$. It has the advantage of being small if $u_\varepsilon$ is close to zero. Notice that $F(0;\varepsilon,\rho)$ is equal to $\varepsilon^{-2}\phi^{-2}(\rho)E(0;\varepsilon,\rho)$, therefore we will usual refer to $E(\psi;\varepsilon,\rho)$ as the *error term.*

We want to build a solution $\psi(\cdot\,;\varepsilon,\rho)$ of $F(\psi;\varepsilon,\rho)=0$ through the iterative scheme

$$\begin{cases} x_0 = 0, \\ L(\varepsilon,\rho)[x_{j+1}] = \varepsilon^2\phi^2(\rho)Q(x_j;\varepsilon,\rho)+E(x_j;\varepsilon,\rho). \end{cases} \tag{3.52}$$

To do that, we need some preliminary analytical estimates about the *error term* $E(\cdot\,;\varepsilon,\rho)$, about the linear operator $L(\varepsilon,\rho)$ and about the quadratic remainder $Q(\cdot\,;\varepsilon,\rho)$. The constant appearing in the estimates for $Q$ and $E$, as we will see in the next, can not be found independently of $R$. Therefore, as well as in the approach of Ambrozio, also our method does not work for constructing a CMC foliation of the whole end at once, but we need another argument to extend it to the infinity.

### 3.6.4 Control of $E$

The operator

$$E(\psi;\varepsilon,\rho) = (n-1)\varepsilon^2\phi^2(\rho)\partial_r u_\varepsilon(\rho+\psi,\cdot) - H(\rho)\varepsilon^2\phi^2(\rho)\left(\mathrm{e}^{u_\varepsilon(\rho+\psi,\cdot)}-1\right) \tag{3.53}$$

involves the function $u_\varepsilon$ defined in (3.38) and satisfying (3.39)-(3.40). In particular for every $(n-1)/2<\delta<n$ there exists $C>0$ independent of small $\varepsilon>0$ such that

$$\sup_{r\in(\log\varepsilon+2,-\log\varepsilon-2)}\ \sup_{\Sigma}|u_\varepsilon|+|\nabla u_\varepsilon|+|\nabla^2 u_\varepsilon|<C\varepsilon^n, \tag{3.54}$$

where the geometric quantities are computed with respect to $g_{\mathrm{cyl}}=\varepsilon^{-2}\phi^{-2}(r)h_\varepsilon$, and

$$\sup_{t\in(-3,+\infty)}\ \sup_{\Sigma}\mathrm{e}^{\delta t}\left(|u_\varepsilon|+|\nabla u_\varepsilon|+|\nabla^2 u_\varepsilon|\right)<C\varepsilon^n, \tag{3.55}$$

where the geometric quantities are computed with respect to $dt^2+\mathrm{e}^{2t}\bar{g}$, $t=r+\log\varepsilon$. The following lemma contains two property of $u_\varepsilon$ descending from (3.54) and (3.55), stated in the setting as they will be applied.

**Lemma 3.16.** *Let $(\Sigma,\bar{g})$ be a compact Riemannian manifold of dimension $n-1$, $n\geq 2$. For every $R>0$ there exist $\varepsilon_R>0$ and $C>0$ with the following* *property. For every* $\varepsilon \in (0, \varepsilon_R)$ *and for every function* $u_\varepsilon \colon (\log\varepsilon + 2, +\infty) \to \mathbb{R}$ *of class* $\mathscr{C}^2$ *satisfying* (3.54) *and* (3.55)*, one has*

$$\|u_\varepsilon(r,\cdot)\|_{\mathscr{C}^2(\Sigma)} < C\varepsilon^n \quad \textit{and} \quad \|\partial_r u_\varepsilon(r,\cdot)\|_{\mathscr{C}^1(\Sigma)} < C\varepsilon^{n-1} \tag{3.56}$$

*for every* $r \in (\log\varepsilon + 2, -\log\varepsilon + R + 1)$.

*Proof.* In this proof $(\theta^1, \dots, \theta^{n-1})$ will denote local coordinates for $\Sigma$ and $\frac{\partial}{\partial\theta^i}$ will be denoted by $\partial_i$. It is useful to separate the proof for $r \in (\log\varepsilon + 2, -\log\varepsilon - 2)$ and then for $r \in [-\log\varepsilon - 2, -\log\varepsilon + R + 1)$. Let us begin assuming $r \in (\log\varepsilon + 2, -\log\varepsilon - 2)$. In this case we introduce a new coordinate $\rho = \rho(r)$ by $\rho(0) = 0$ and $\varepsilon^{-1}\phi^{-1}(r)dr = d\rho$. With this notation we have $g_{\mathrm{cyl}} = d\rho^2 + \bar{g}$ and (3.54) implies

$$|u_\varepsilon|, |\partial_i u_\varepsilon|, |\partial_i\partial_j u_\varepsilon|, |\partial_\rho u_\varepsilon|, |\partial_i\partial_\rho u_\varepsilon| < C_1\varepsilon^n$$

for some $C_1$ independent of $\varepsilon$ and $r$. Since $\partial_r = \varepsilon^{-1}\phi^{-1}\partial_\rho$ and since $\phi^{-1}$ is uniformly bounded, we got (3.56). It remains to consider the case $r \in [-\log\varepsilon - 2, -\log\varepsilon + R + 1)$, namely $t \in [-2, R+1)$. Since in this case $\mathrm{e}^{\delta t}$ can be uniformly bounded in terms of $R$, then (3.55) implies

$$|u_\varepsilon|, |\partial_i u_\varepsilon|, |\partial_i\partial_j u_\varepsilon|, |\partial_t u_\varepsilon|, |\partial_i\partial_t u_\varepsilon| < C_2\varepsilon^n$$

for some $C_1$ independent of $\varepsilon$ and $r$. Since $\partial_r = \partial_t$, we got (3.56). □

As a corollary, we have the following result.

**Lemma 3.17.** *Let* $(\Sigma, \bar{g})$ *be a compact Riemannian manifold of dimension* $n-1$, $n \geq 2$. *For every* $R > 0$ *there exist* $\varepsilon_R > 0$ *and* $C_R > 0$ *with the following property. For every* $\varepsilon \in (0, \varepsilon_R)$*, for every* $\rho \in [\log\varepsilon + 3, -\log\varepsilon + R]$ *and for every* $\psi \in \mathscr{C}^{2,\alpha}(\Sigma)$ *satisfying* $\|\psi\|_{\mathscr{C}^{2,\alpha}(\Sigma)} < 1$*, one has*

$$\|E(\psi;\varepsilon,\rho)\|_{\mathscr{C}^1(\Sigma)} \leq C_R\varepsilon^{n+1},$$

*where* $E(\psi;\varepsilon,\rho)$ *is the operator defined by* (3.53) *with respect to a function* $u_\varepsilon$ *satisfying* (3.54) *and* (3.55).

*Proof.* Since by definition

$$E(\psi;\varepsilon,\rho) := (n-1)\varepsilon^2\phi^2(\rho)\partial_r u_\varepsilon(\rho+\psi,\cdot) - H(\rho)\varepsilon^2\phi^2(\rho)\left(\mathrm{e}^{u_\varepsilon(\rho+\psi,\cdot)} - 1\right),$$

it is sufficient to apply the previous lemma noticing that $\rho + \psi$ lies between $\log\varepsilon + 2$ and $-\log\varepsilon + R + 1$. □

*REMARK* 3.6. We recall that the next goal is to build a solution $\psi(\cdot\,;\varepsilon,\rho)$ of Theorem 3.15 via the iterative scheme (3.52). Due to the previous lemma, one can heuristically guess why estimate (3.41) should hold. In fact $\psi$ has to solve $F(\psi;\varepsilon,\rho) = 0$ and by the estimates of this subsection it is possible to show $F(-c\varepsilon^{n-1};\varepsilon,\rho) < 0$ and $F(c\varepsilon^{n-1};\varepsilon,\rho) > 0$ for some $c > 0$ independent of $\rho$ and $\varepsilon$. In other words, the constants $-c\varepsilon^n$ and $c\varepsilon^n$ provide respectively a sub-solution and a super-solution for our problem. This seems to justify the control of the exact solution in (3.41). However, we will obtain this estimate with the iterative scheme.

### 3.6.5 Control of $L$

In this subsection we study the linear operator $L(\varepsilon,\rho)$ defined in (3.49). It is easy to see that in general it is not possible to have $C > 0$ independent of $\varepsilon$ such that

$$\|x\|_{\mathscr{C}^{2,\alpha}} \le C\|y\|_{\mathscr{C}^{0,\alpha}},$$

whenever $L(\varepsilon,\rho)[x] = y$. To see that, it is sufficient to take $x = 1$ and $\varepsilon \to 0$. However, a similar bound can be obtained replacing $x$ by $x - ⨍_\Sigma x$.

**Lemma 3.18.** *Let $(\Sigma,\bar{g})$ be a compact Riemannian manifold of dimension $n-1$, $n \ge 2$. Then there exists $C > 0$ such that for every $\rho \in \mathbb{R}$ and $0 < \varepsilon < 1$ the second-order elliptic operator $L(\varepsilon,\rho)\colon \mathscr{C}^{2,\alpha}(\Sigma) \to \mathscr{C}^{0,\alpha}(\Sigma)$ defined by (3.49) satisfies the following property. If $x \in \mathscr{C}^{2,\alpha}(\Sigma)$ and $y \in \mathscr{C}^{0,\alpha}(\Sigma)$ verify $L(\varepsilon,\rho)[x] = y$ then*

$$\left\| x - ⨍_\Sigma x \right\|_{\mathscr{C}^{2,\alpha}(\Sigma)} \le C\|y\|_{\mathscr{C}^{0,\alpha}(\Sigma)}.$$

*Proof.* Assume to prove the next Lemma 3.19, then the thesis will follow with

$$\kappa = 2n(n-1)\varepsilon^2\phi^{2-n}(\rho),$$

noticing that $\phi^{2-n}$ is uniformly bounded and so for $\varepsilon < 1$ we have $\kappa < \kappa_0$ for some constant $\kappa_0$ depending only on $n$. ☐

**Lemma 3.19.** *Fix $\kappa_0 > 0$. Let $(\Sigma,\bar{g})$ be a compact Riemannian manifold and consider the elliptic operator $L_\kappa := \Delta - \kappa$, for some constant $0 < \kappa < \kappa_0$. Then there exists $C > 0$ such that for every $x \in \mathscr{C}^{2,\alpha}(\Sigma)$ and $y \in \mathscr{C}^{0,\alpha}(\Sigma)$ satisfying $L_\kappa x = y$, one has*

$$\left\| x - ⨍_\Sigma x \right\|_{\mathscr{C}^{2,\alpha}(\Sigma)} \le C\|y\|_{\mathscr{C}^{0,\alpha}(\Sigma)}.$$

*The constant $C = C(\kappa_0)$ does not depend on $x$, $y$ and $\kappa$.*

*Proof.* First, we recall that by compactness of $\Sigma$ and by the fact that $\kappa > 0$, the operator $\Delta - \kappa$ induces an isomorphism from

$$\left\{ x \in \mathscr{C}^{2,\alpha}(\Sigma) \quad \text{s.t.} \quad ⨍_\Sigma x = 0 \right\}$$

to
$$\left\{ y \in \mathscr{C}^{0,\alpha}(\Sigma) \quad \text{s.t.} \quad \fint_\Sigma x = 0 \right\}.$$
Therefore it holds
$$\left\| x - \fint_\Sigma x \right\|_{\mathscr{C}^{2,\alpha}(\Sigma)} \leq C_1 \left\| y - \fint_\Sigma y \right\|_{\mathscr{C}^{0,\alpha}(\Sigma)} \leq C_2 \|y\|_{\mathscr{C}^{0,\alpha}(\Sigma)}$$
for some constant $C_1, C_2 > 0$ independent of $x$ and $y$, but that a priori can depend on $\kappa$. We can assume w.l.o.g. that $\operatorname{Area}_{\bar{g}} \Sigma = 1$. Suppose by contradiction that $\kappa_j \to 0$, $L_{\kappa_j} x_j = y_j$, $\|x_j - \fint_\Sigma x_j\|_{\mathscr{C}^{2,\alpha}(\Sigma)} = 1$ and $\|y_j\|_{\mathscr{C}^{0,\alpha}(\Sigma)} \to 0$. Up to subsequence $x_j - \fint_\Sigma x_j \to v$ in $\mathscr{C}^2$. Now notice that
$$\begin{aligned} &(\Delta - \kappa_j)\left( x_j - \fint_\Sigma x_j \right) = \\ &L_{\kappa_j}\left( x_j - \fint_\Sigma x_j \right) = y_j + \kappa_j \fint_\Sigma x_j \end{aligned}$$
and taking the mean integral
$$\kappa_j \fint_\Sigma x_j = -\fint_\Sigma y_j,$$
so
$$\left| \kappa_j \fint_\Sigma x_j \right| \leq \fint_\Sigma |y_j| \leq \|y_j\|_{\mathscr{C}^{0,\alpha}(\Sigma)} \to 0.$$
On the other hand, taking the pointwise limit of $L_{\kappa_j} x_j = y_j$, we get $\Delta v = 0$. By compactness of $\Sigma$, the function $v$ is constant. Moreover
$$\fint_\Sigma v = \lim_j \fint_\Sigma (x_j - \fint_\Sigma x_j) = 0,$$
so $x_j - \fint_\Sigma x_j \to 0$ in $\mathscr{C}^2(\Sigma)$. Now let $\{\Omega_k\}_k$ be a finite covering of open balls of $\Sigma$ with fixed radius equal to $(\operatorname{injrad}\Sigma)/4$ and let $\{\Omega'_k\}_k$ the covering of $\Sigma$ such that $\Omega'_k$ is the ball concentric with $\Omega_k$ but with radius $(\operatorname{injrad}\Sigma)/2$. By the Schauder Interior Estimates there exists $C > 0$ depending only on $(\Sigma, \bar{g})$ such that
$$\begin{aligned} &\left\| x_j - \fint_\Sigma x_j \right\|_{\mathscr{C}^{2,\alpha}(\Omega_k)} \\ &\leq C \left( \left\| y_j + \kappa_j \fint_\Sigma x_j \right\|_{\mathscr{C}^{0,\alpha}(\Omega'_k)} + \left\| x_j - \fint_\Sigma x_j \right\|_{\mathscr{C}^{0}(\Omega'_k)} \right). \end{aligned}$$
Taking the supremum over the covering, we get
$$\begin{aligned} 1 = &\sup_{\{\Omega_k\}} \left\| x_j - \fint_\Sigma x_j \right\|_{\mathscr{C}^{2,\alpha}(\Omega_k)} \\ &\leq C \left( \|y_j\|_{\mathscr{C}^{0,\alpha}(\Sigma)} + \left| \kappa_j \fint_\Sigma x_j \right| + \left\| x_j - \fint_\Sigma x_j \right\|_{\mathscr{C}^{0}(\Sigma)} \right) \end{aligned}$$

and we got a contradiction since all the terms on the right hand side tend to zero as $j \to +\infty$. $\square$

### 3.6.6 Control of $Q$

In this subsection we prove an estimate for $Q(\psi;\varepsilon,\rho)$. The following lemma guarantees that if $\psi \in \mathscr{C}^{2,\alpha}(\Sigma)$ satisfies $\|\psi\|_{\mathscr{C}^{2,\alpha}(\Sigma)} = O(\varepsilon^{n-1})$ and $\|\psi - \fint_\Sigma \psi\|_{\mathscr{C}^{2,\alpha}(\Sigma)} = O(\varepsilon^{n+1})$ as $\varepsilon \to 0$, then one has $\|Q(\psi;\varepsilon,\rho)\|_{\mathscr{C}^{0,\alpha}(\Sigma)} = o(\varepsilon^{n-1})$ as $\varepsilon \to 0$. Due to that we will be able to show in the next subsection that the iterative scheme (3.52) converges.

**Lemma 3.20.** *Let $(\Sigma,\bar g)$ be a compact Riemannian manifold of dimension $n-1$, $n \geq 2$. For every $R > 0$ there exist $\varepsilon_R > 0$ and $C_R > 0$ with the following property. For every $\varepsilon \in (0,\varepsilon_R)$, for every $\rho \in [\log\varepsilon + 3, -\log\varepsilon + R]$ and for every $\psi \in \mathscr{C}^{2,\alpha}(\Sigma)$ satisfying $\|\psi\|_{\mathscr{C}^{2,\alpha}(\Sigma)} < 1$ and $\|\psi - \fint_\Sigma \psi\|_{\mathscr{C}^{2,\alpha}(\Sigma)} < \varepsilon^2$, one has*

$$\|Q(\psi;\varepsilon,\rho)\|_{\mathscr{C}^{0,\alpha}(\Sigma)} \leq C_R\bigg(\varepsilon^{-2}\,\|\psi\|_{\mathscr{C}^{2,\alpha}(\Sigma)}\left\|\psi - \fint_\Sigma \psi\right\|_{\mathscr{C}^{2,\alpha}(\Sigma)} + \left\|\psi - \fint_\Sigma \psi\right\|_{\mathscr{C}^{2,\alpha}(\Sigma)} + \|\psi\|^2_{\mathscr{C}^{2,\alpha}(\Sigma)}\bigg),$$

*where $Q(\psi;\varepsilon,\rho)$ is the operator defined by (3.50).*

*Proof.* In this proof we assume that $(\theta^1,\ldots,\theta^{n-1})$ are local coordinates in $\Sigma$ and we denote $\frac{\partial}{\partial\theta^i}$ by $\partial_i$. Also, the constants $C_1, C_2, \ldots$ appearing in this proof are independent of $\psi$, $\varepsilon$ and $\rho$, but may depend on $R$. In order to simplify the computation, we decompose $Q(\cdot\,;\varepsilon,m) = Q_1 + Q_2 + \cdots + Q_6$ with

$$Q_1(\psi) = -\varepsilon^{-2}\left(\phi^{-2}(\rho+\psi) - \phi^{-2}(\rho)\right)\Delta\psi,$$

$$Q_2(\psi) = \varepsilon^{-4}\phi^{-4}(\rho+\psi)\frac{\mathrm{Hess}\psi(\nabla\psi,\nabla\psi)}{1+\varepsilon^{-2}\phi^{-2}(\rho+\psi)|\nabla\psi|^2},$$

$$Q_3(\psi) = \varepsilon^{-2}\phi^{-2}(\rho+\psi)\frac{(\dot\phi/\phi)(\rho+\psi)|\nabla\psi|^2}{1+\varepsilon^{-2}\phi^{-2}(\rho+\psi)|\nabla\psi|^2},$$

$$Q_4(\psi) = (n-1)\left((\dot\phi/\phi)(\rho+\psi) - (\dot\phi/\phi)(\rho) - 2n\phi^{-n}(\rho)\psi\right),$$

$$Q_5(\psi) = -(n-1)\varepsilon^{-2}\phi^{-2}(\rho+\psi)\bar g^{ij}\partial_i\psi\partial_j u_\varepsilon(\rho+\psi,\cdot)$$

and

$$Q_6(\psi) = -(n-1)(\dot{\phi/\phi})(\rho)\left(\sqrt{1+\varepsilon^{-2}\phi^{-2}(\rho+\psi)|\nabla\psi|^2} - 1\right)e^{u_\varepsilon(\rho+\psi,\cdot)},$$

with $\phi(r) = (2\cosh(nr/2))^{2/n}$. Therefore it is enough to check the required estimate separately for $Q_1,\ldots,Q_6$.

*1)* The $\mathscr{C}^{0,\alpha}$-norm of $Q_1(\psi)$ can be uniformly bounded in terms of

$$\varepsilon^{-2}\|\phi^{-2}(\rho+\psi)-\phi^{-2}(\rho)\|_{\mathscr{C}^1(\Sigma)}\|\Delta\psi\|_{\mathscr{C}^{0,\alpha}(\Sigma)}.$$

By Lagrange

$$|\phi^{-2}(\rho+\psi)-\phi^{-2}(\rho)|=2|\phi^{-3}(\psi^*)\dot{\phi}(\psi^*)\psi|$$

for some $\psi^*$ pointwise lying between $\rho+\psi$ and $\rho$, whereas

$$|\partial_i(\phi^{-2}(\psi+\rho)-\phi^{-2}(\rho))|=2|\phi^{-3}(\psi)\dot{\phi}(\psi)\partial_i\psi|.$$

As a consequence

$$\begin{aligned}\|Q_1(\psi)\|_{\mathscr{C}^{0,\alpha}(\Sigma)} &\leq C_1\varepsilon^{-2}\|\psi\|_{\mathscr{C}^1(\Sigma)}\|\Delta\psi\|_{\mathscr{C}^{0,\alpha}(\Sigma)}\\ &\leq C_2\varepsilon^{-2}\|\psi\|_{\mathscr{C}^{2,\alpha}(\Sigma)}\left\|\psi-\fint_\Sigma\psi\right\|_{\mathscr{C}^{2,\alpha}(\Sigma)};\end{aligned}$$

*2)* The $\mathscr{C}^{0,\alpha}$-norm of $Q_2(\psi)$ can be uniformly bounded in terms of

$$\varepsilon^{-4}\|\phi^{-4}(\rho{+}\psi)\|_{\mathscr{C}^1(\Sigma)}\left\|\nabla^2\psi\right\|_{\mathscr{C}^{0,\alpha}(\Sigma)}\|\nabla\psi\|^2_{\mathscr{C}^{0,\alpha}(\Sigma)}\left\|(1+\varepsilon^{-2}\phi^{-2}(\rho+\psi)|\nabla\psi|^2)^{-1}\right\|_{\mathscr{C}^1(\Sigma)},$$

which can be bounded in turn by

$$\varepsilon^{-4}\|\phi^{-4}(\rho{+}\psi)\|_{\mathscr{C}^1(\Sigma)}\left\|\psi-\fint_\Sigma\psi\right\|^3_{\mathscr{C}^{0,\alpha}(\Sigma)}\left\|(1+\varepsilon^{-2}\phi^{-2}(\rho+\psi)|\nabla\psi|^2)^{-1}\right\|_{\mathscr{C}^1(\Sigma)}.$$

We get the thesis for $Q_2$ if we show that

$$\|\phi^{-4}(\rho{+}\psi)\|_{\mathscr{C}^1(\Sigma)}\leq C_3 \quad\text{and}\quad \left\|(1+\varepsilon^{-2}\phi^{-2}(\rho+\psi)|\nabla\psi|^2)^{-1}\right\|_{\mathscr{C}^1(\Sigma)}\leq C_4.$$

The first inequality follows from the fact that both

$$|\phi^{-4}(\rho+\phi)|$$

and

$$|\partial_i(\phi^{-4}(\rho+\phi))|=4|\phi^{-4}(\rho+\phi)\tanh(n(\rho+\psi)/2)\partial_i\psi|$$

are uniformly bounded. Here we used $\|\psi\|_{\mathscr{C}^{2,\alpha}(\Sigma)}<1$. The second inequality follows from the fact that both

$$|(1+\varepsilon^{-2}\phi^{-2}(\psi)|\nabla\psi|^2)^{-1}|$$

and

$$\begin{aligned}\left|\partial_i(\varepsilon^{-2}\phi^{-2}(\psi)|\nabla\psi|^2)\right| \leq & 2\left|\phi^{-2}(\rho+\psi)\tanh(n(\rho+\psi)/2)\right|\varepsilon^{-2}|\nabla\psi|^2\\ &+\phi^{-2}(\rho+\psi)\varepsilon^{-2}|\partial_i\bar{g}^{jk}\partial_j\psi\partial_k\psi|\\ &+2\phi^{-2}(\rho+\psi)\varepsilon^{-2}|\bar{g}^{jk}\partial_{ij}\psi\partial_k\psi|\end{aligned}$$

are uniformly bounded. Here we used $(1+\varepsilon^{-2}\phi^{-2}(\psi)|\nabla\psi|^2)^{-1}\leq 1$, the uniform boundedness of the functions $\phi^{-2}(r)$ and $(\dot{\phi}/\phi)(r)=(n-1)\tanh(nr/2)$ and the hypothesis $\|\psi-\fint_\Sigma\psi\|_{\mathscr{C}^{2,\alpha}(\Sigma)}<\varepsilon^2$;

*3)* Using as in the previous points the boundedness of the functions $\phi^{-2}(r)$ and $(\dot{\phi}/\phi)(r) = \tanh(nr/2)$ (and derivatives) as well as the uniform boundedness of $\left\|(1+\varepsilon^{-2}\phi^{-2}(\psi)|\nabla\psi|^2)^{-1}\right\|_{\mathscr{C}^1(\Sigma)}$, one concludes that the $\mathscr{C}^{0,\alpha}$-norm of $Q_3(\psi)$ can be uniformly bounded in terms of $\varepsilon^{-2}\|\nabla\phi\|^2_{\mathscr{C}^{0,\alpha}(\Sigma)}$. The hypothesis on $\psi$ then implies

$$\|Q_3(\psi)\|_{\mathscr{C}^{0,\alpha}(\Sigma)} \leq C_5 \left\|\psi - \fint_\Sigma \psi\right\|_{\mathscr{C}^{2,\alpha}(\Sigma)};$$

*4)* If one set $f(r) := (n-1)\tanh(nr/2)$, then a simple computation shows that

$$Q_4(\psi) = f(\rho+\psi) - f(\rho) - f'(\rho)\psi.$$

This means that $Q_4$ is nothing but the Taylor expansion of $f(\rho+\psi)$ truncated to the first order. As a consequence, by Lagrange there exists $\psi^*$ pointwise lying between $\rho$ and $\rho+\psi$ such that

$$\begin{aligned} Q_4(\psi) &= (n-1)f''(\psi^*)\psi^2 \\ &= -2n^2(n-1)\phi^{-n-1}(\psi^*)\dot{\phi}(\psi^*)\psi^2. \end{aligned}$$

Reasoning as in the previous points, we obtain

$$\|Q_4(\psi)\|_{\mathscr{C}^{0,\alpha}(\Sigma)} \leq C_6\|\psi\|^2_{\mathscr{C}^{2,\alpha}(\Sigma)};$$

*5)* The $\mathscr{C}^{0,\alpha}$-norm of $Q_5(\psi)$ can be uniformly bounded in terms of

$$\varepsilon^{-2}\|\phi^{-2}(\rho+\psi)\|_{\mathscr{C}^{0,\alpha}(\Sigma)}\|\nabla\psi\|_{\mathscr{C}^{0,\alpha}(\Sigma)}\|\nabla u_\varepsilon(\rho+\psi,\cdot)\|_{\mathscr{C}^{0,\alpha}(\Sigma)}.$$

By uniform boundedness of $\phi^{-2}(r)$ and derivatives we deduce

$$\|Q_5(\psi)\|_{\mathscr{C}^{0,\alpha}(\Sigma)} \leq C_7\varepsilon^{-2}\left\|\psi - \fint_\Sigma \psi\right\|_{\mathscr{C}^{2,\alpha}(\Sigma)} \sup_{\theta\in\Sigma} \|u_\varepsilon(r,\cdot)\|_{\mathscr{C}^2(\Sigma)}|_{r=\rho+\psi(\theta)}$$

Since $\rho+\psi$ lies in $(\log\varepsilon+2, -\log\varepsilon+R+1)$, we can apply Lemma 3.16. As a consequence $\|Q_4(\psi)\|_{\mathscr{C}^{0,\alpha}(\Sigma)}$ is bounded in terms of $\left\|\psi - \fint_\Sigma\psi\right\|_{\mathscr{C}^{2,\alpha}(\Sigma)}$;

*6)* Since $u_\varepsilon$ is bounded, since it holds $(n-1)|\tanh(n\rho/2)| < n-1$ and since $\left(\sqrt{1+x}-1\right) < x$ for $x > 0$, then for sure

$$|Q_6(\psi)| \leq C_8\varepsilon^{-2}\phi^{-2}(\rho+\psi)|\nabla\psi|^2,$$

which is bounded by $\left\|\psi - \fint_\Sigma\psi\right\|_{\mathscr{C}^{2,\alpha}(\Sigma)}$ with the same arguments of the previous points. Dealing with the derivatives, a direct computation shows that

$$\begin{aligned}
\partial_i Q_6(\psi) &= \varepsilon^{-2}\frac{-2\phi^{-3}(\rho+\psi)\dot{\phi}(\rho+\psi)\partial_i\psi|\nabla\psi|^2}{\sqrt{1+\varepsilon^{-2}\phi^{-2}(\rho+\psi)|\nabla\psi|^2}}e^{u_\varepsilon(\rho+\psi,\cdot)} \\
&+\varepsilon^{-2}\frac{2\phi^{-2}(\rho+\psi)g^{kl}\partial_{il}\psi\partial_k\psi}{\sqrt{1+\varepsilon^{-2}\phi^{-2}(\rho+\psi)|\nabla\psi|^2}}e^{u_\varepsilon(\rho+\psi,\cdot)} \\
&+\left(\sqrt{1+\varepsilon^{-2}\phi^{-2}(\rho+\psi)|\nabla\psi|^2}-1\right)e^{u_\varepsilon(\rho+\psi,\cdot)}\partial_r u_\varepsilon(\rho+\psi,\cdot)\partial_i\psi \\
&+\left(\sqrt{1+\varepsilon^{-2}\phi^{-2}(\rho+\psi)|\nabla\psi|^2}-1\right)e^{u_\varepsilon(\rho+\psi,\cdot)}\partial_i u_\varepsilon(\rho+\psi,\cdot).
\end{aligned}$$

This implies

$$\begin{aligned}
\|Q_6(\psi)\|_{\mathscr{C}^{0,\alpha}(\Sigma)} &\leq C_9\left\|\psi-\fint_\Sigma\psi\right\|_{\mathscr{C}^{2,\alpha}(\Sigma)} \\
&+C_{10}\varepsilon^{-2}\|\nabla\psi\|^3_{\mathscr{C}^0(\Sigma)} \\
&+C_{11}\varepsilon^{-2}\|\nabla^2\psi\|_{\mathscr{C}^0(\Sigma)}\|\nabla\psi\|_{\mathscr{C}^0(\Sigma)} \\
&+C_{12}\varepsilon^{-2}\|\nabla\psi\|^3_{\mathscr{C}^0(\Sigma)}\sup_{\theta\in\Sigma}\|\partial_r u_\varepsilon(r,\cdot)\|_{\mathscr{C}^0(\Sigma)}|_{r=\rho+\psi(\theta)} \\
&+C_{13}\varepsilon^{-2}\|\nabla\psi\|^2_{\mathscr{C}^0(\Sigma)}\sup_{\theta\in\Sigma}\|u_\varepsilon(r,\cdot)\|_{\mathscr{C}^1(\Sigma)}|_{r=\rho+\psi(\theta)} \\
&\leq C_{14}\left\|\psi-\fint_\Sigma\psi\right\|_{\mathscr{C}^{2,\alpha}(\Sigma)}.
\end{aligned}$$

The last inequality follows from the same arguments used in the previous points and Lemma 3.16.

□

### 3.6.7 Convergence of the Newton scheme

Now we are able to prove that the iterative scheme (3.52), that for simplicity we report below in (3.57), converge (up to subsequence) to a solution $\psi(\cdot\,;\varepsilon,\rho)$ of Theorem 3.15.

**Proposition 3.21.** *Let $(\Sigma,\bar{g})$ be a compact Riemannian manifold of dimension $n-1$, $n\geq 2$. For every $R>0$ there exist $\varepsilon_R>0$ and $c>0$ satisfying the following property. For every $\varepsilon\in(0,\varepsilon_R)$ and for every $\rho\in[\log\varepsilon+3,-\log\varepsilon+R]$ the functions $x_j$'s, $j\in\mathbb{N}$, given by the iterative scheme*

$$\begin{cases} x_0=0, \\ L(\varepsilon,\rho)[x_{j+1}]=\varepsilon^2\phi^2(\rho)Q(x_j;\varepsilon,\rho)+E(x_j;\varepsilon,\rho), \end{cases} \tag{3.57}$$

*satisfy*

$$\left\|x_j-\fint_\Sigma x_j\,dV_{\bar{g}}\right\|_{\mathscr{C}^{2,\alpha}(\Sigma)}\leq c\varepsilon^{n+1}\quad\text{and}\quad\left|\fint_\Sigma x_j\,dV_{\bar{g}}\right|\leq c\varepsilon^{n-1}. \tag{3.58}$$

*Therefore, up to subsequence, $x_j$ tends to some function $\psi(\cdot\,;\varepsilon,\rho)\in\mathscr{C}^2(\Sigma)$ in $\mathscr{C}^2(\Sigma)$-norm, as $j\to+\infty$. Moreover it holds*

$$\left\|\psi-\fint_\Sigma \psi\, dV_{\bar g}\right\|_{\mathscr{C}^2(\Sigma)}\le c\varepsilon^{n+1}\quad \textit{and}\quad \left|\fint_\Sigma \psi\, dV_{\bar g}\right|\le c\varepsilon^{n-1} \tag{3.59}$$

*and the hypersurface in $(\log\varepsilon+2,+\infty)\times\Sigma$ given by*

$$S(\varepsilon,\rho):=\{(\rho+\psi(\theta;\varepsilon,\rho),\theta)\,|\,\theta\in\Sigma\}$$

*has constant mean curvature equal to $(n-1)\tanh(n\rho/2)$ computed with respect to the metric $g_\varepsilon$ defined by (3.36) and the unit normal vector pointing toward the infinity.*

*Proof.* In this proof it is useful to introduce $\bar x_j:=\fint_\Sigma x_j\in\mathbb{R}$. The constant $C_1,C_2,\ldots$ will be independent of $\varepsilon$, $\rho$ and $j\in\mathbb{N}$ but may depend on $R$. We also understand that all of the integrals and the Hölder norms of this proof are computed with respect to $(\Sigma,\bar g)$. First notice that for $j=0$ the function $x_0=0$ satisfies the two inequalities (3.58). We are going to prove (3.58) by induction on $j$, but to do so we need the preliminary estimates (3.61) and (3.62) to establish a good choice for $c$ and $\varepsilon_R$. The first non-trivial function in the iterative scheme is $x_1$ and is given by

$$L(\varepsilon,\rho)[x_1]=E(0;\varepsilon,\rho).$$

Taking the mean integral, since $\Sigma$ is close, we get

$$-2n(n-1)\varepsilon^2\phi^{2-n}(\rho)\bar x_1=\fint E(0;\varepsilon,\rho).$$

By Lemma 3.17 it follows

$$|\bar x_1|\le C_1\varepsilon^{n-1}.$$

On the other hand, Lemma 3.18 implies

$$\|x_1-\bar x_1\|_{\mathscr{C}^{2,\alpha}(\Sigma)}\le C\,\|E(0;\varepsilon,\rho)\|_{\mathscr{C}^{0,\alpha}(\Sigma)}$$

and by Lemma 3.17 we get

$$\|x_1-\bar x_1\|_{\mathscr{C}^{2,\alpha}(\Sigma)}\le C_2\varepsilon^{n+1}.$$

We recall that both the constant which controls the error term and the constant which controls the quadratic term depend on $R$ and this is precisely why we can not prove this result for $\rho\in[\log\varepsilon+3,+\infty)$. By definition

$$L(\varepsilon,\rho)[x_{j+1}]=E(x_j;\varepsilon,\rho)+\varepsilon^2\phi^2(\rho)Q(x_j;\varepsilon,\rho),$$

it follows immediately that

$$L(\varepsilon,\rho)[x_{j+1}-x_1]=E(x_j;\varepsilon,\rho)-E(0;\varepsilon,\rho)+\varepsilon^2\phi^2(\rho)Q(x_j;\varepsilon,\rho). \tag{3.60}$$

Therefore if we assume that $\|x_j - \bar{x}_j\|_{\mathscr{C}^{2,\alpha}(\Sigma)} < \varepsilon^2$ and $\|x_j\|_{\mathscr{C}^{2,\alpha}(\Sigma)} < 1$, then from Lemma 3.17, Lemma 3.18 and Lemma 3.20 we get

$$\begin{aligned}
&\|(x_{j+1} - \bar{x}_{j+1}) - (x_1 - \bar{x}_1)\|_{\mathscr{C}^{2,\alpha}(\Sigma)} \\
&\leq C \left(\|E(x_j;\varepsilon,\rho)\|_{\mathscr{C}^{0,\alpha}(\Sigma)} + \|E(0;\varepsilon,\rho)\|_{\mathscr{C}^{0,\alpha}(\Sigma)} + \varepsilon^2\phi^2(\rho)\|Q(x_j;\varepsilon,\rho)\|_{\mathscr{C}^{0,\alpha}(\Sigma)}\right) \\
&\leq C_3 \left(\varepsilon^{n+1} + \varepsilon^2\|x_j\|^2_{\mathscr{C}^{2,\alpha}} + \|x_j\|_{\mathscr{C}^{2,\alpha}}\|x_j - \bar{x}_j\|_{\mathscr{C}^{2,\alpha}} + \varepsilon^2\|x_j - \bar{x}_j\|_{\mathscr{C}^{2,\alpha}}\right),
\end{aligned}$$

while taking the integral of (3.60) one deduces similarly that

$$\begin{aligned}
&|\bar{x}_{j+1} - \bar{x}_1| \\
&\leq C_4 \left(\varepsilon^{n-1} + \|x_j\|^2_{\mathscr{C}^{2,\alpha}} + \varepsilon^{-2}\|x_j\|_{\mathscr{C}^{2,\alpha}}\|x_j - \bar{x}_j\|_{\mathscr{C}^{2,\alpha}} + \|x_j - \bar{x}_j\|_{\mathscr{C}^{2,\alpha}}\right).
\end{aligned}$$

We have proved that it holds

$$\begin{cases} |\bar{x}_1| \leq C_5\varepsilon^{n-1} \\ \|x_1 - \bar{x}_1\|_{\mathscr{C}^{2,\alpha}} \leq C_6\varepsilon^{n+1} \end{cases} \tag{3.61}$$

and that if it holds $\|x_j - \bar{x}_j\|_{\mathscr{C}^{2,\alpha}(\Sigma)} < \varepsilon^2$ and $\|x_j\|_{\mathscr{C}^{2,\alpha}(\Sigma)} < 1$ for every $j \geq 1$, then

$$\begin{cases} |\bar{x}_{j+1}| \leq |\bar{x}_{j+1} - \bar{x}_1| + |\bar{x}_1| \\ \qquad \leq C_7 \left(\varepsilon^{n-1} + \|x_j\|^2_{\mathscr{C}^{2,\alpha}} + \varepsilon^{-2}\|x_j\|_{\mathscr{C}^{2,\alpha}}\|x_j - \bar{x}_j\|_{\mathscr{C}^{2,\alpha}} + \|x_j - \bar{x}_j\|_{\mathscr{C}^{2,\alpha}}\right), \\ \|x_{j+1} - \bar{x}_{j+1}\|_{\mathscr{C}^{2,\alpha}} \leq \|(x_{j+1} - \bar{x}_{j+1}) - (x_1 - \bar{x}_1)\|_{\mathscr{C}^{2,\alpha}} + \|x_1 - \bar{x}_1\|_{\mathscr{C}^{2,\alpha}} \\ \qquad \leq C_8 \left(\varepsilon^{n+1} + \varepsilon^2\|x_j\|^2_{\mathscr{C}^{2,\alpha}} + \|x_j\|_{\mathscr{C}^{2,\alpha}}\|x_j - \bar{x}_j\|_{\mathscr{C}^{2,\alpha}} + \varepsilon^2\|x_j - \bar{x}_j\|_{\mathscr{C}^{2,\alpha}}\right). \end{cases} \tag{3.62}$$

The constants $C_1, \ldots, C_8$ do not depend on $j$ since they do not depend on the function $x_j$. Now we can prove (3.58) by induction choosing $c = c(R) = C_5 + C_6 + 2C_7 + 2C_8$ and $\varepsilon_R > 0$ small enough so that

$$2c^2\varepsilon^{n-1} + c^2\varepsilon^{n+3} + 3c^2\varepsilon^{n+1} + c\varepsilon^2 < 1.$$

In fact this guarantees $|\bar{x}_1| < c\varepsilon^{n-1}$ and $\|x_1 - \bar{x}_1\|_{\mathscr{C}^{2,\alpha}} < c\varepsilon^{n+1}$, and if we assume that (3.58) holds for $j$ then it holds $\|x_j - \bar{x}_j\|_{\mathscr{C}^{2,\alpha}(\Sigma)} < \varepsilon^2$ and $\|x_j\|_{\mathscr{C}^{2,\alpha}(\Sigma)} < 1$, and then

$$\begin{aligned}
|\bar{x}_{j+1}| &\leq C_7 \left(\varepsilon^{n-1} + \|x_j\|^2_{\mathscr{C}^{2,\alpha}} + \varepsilon^{-2}\|x_j\|_{\mathscr{C}^{2,\alpha}}\|x_j - \bar{x}_j\|_{\mathscr{C}^{2,\alpha}} + \|x_j - \bar{x}_j\|_{\mathscr{C}^{2,\alpha}}\right) \\
&\leq \frac{c}{2}\left(\varepsilon^{n-1} + c^2(\varepsilon^{n-1} + \varepsilon^{n+1})^2 + \varepsilon^{-2}c(\varepsilon^{n-1} + \varepsilon^{n+1})c\varepsilon^{n+1} + c\varepsilon^{n+1}\right) \\
&= \frac{c}{2}\varepsilon^{n-1}\left(1 + 2c^2\varepsilon^{n-1} + c^2\varepsilon^{n+3} + 3c^2\varepsilon^{n+1} + c\varepsilon^2\right) < c\varepsilon^{n-1}.
\end{aligned}$$

Similarly we obtain

$$\|x_{j+1} - \bar{x}_{j+1}\|_{\mathscr{C}^{2,\alpha}(\Sigma)} \leq \frac{c}{2}\varepsilon^{n+1}\left(1 + 2c^2\varepsilon^{n-1} + c^2\varepsilon^{n+3} + 3c^2\varepsilon^{n+1} + c\varepsilon^2\right) < c\varepsilon^{n+1}.$$

This proves (3.58) by induction. Then it is sufficient to apply Ascoli-Arzelà and, up to subsequence, $x_j \to \psi(\cdot\,;\varepsilon,\rho)$ in $\mathscr{C}^2(\Sigma)$ as $j \to \infty$ and it holds (3.59). Moreover, passing to the limit from (3.57), the function $\psi$ verify

$$L(\varepsilon,\rho)[\psi] = \varepsilon^2\phi^2(\rho)Q(\psi;\varepsilon,\rho) + E(\psi;\varepsilon,\rho)$$

and this is equivalent to say that $S(\varepsilon,\rho)$ has constant mean curvature equal to $H(\rho) = (n-1)\tanh(n\rho/2)$ with respect to $g_\varepsilon$. □

### 3.6.8 Weak stability

Due to the convergence of the iterative scheme in the previous subsection, we built hypersurfaces $S(\varepsilon,\rho) = \{r = \rho + \psi(\cdot\,;\varepsilon,\rho)\}$ in $(M, h_\varepsilon)$ with constant mean curvature equal to $(n-1)\tanh(n\rho/2)$ satisfying Theorem 3.15-($i$),($ii$). In this subsection we prove that it also holds Theorem 3.15-($iii$) and we will make use of the arguments and notation of Subsection 3.6.2. First we notice that since $S(\varepsilon,\rho)$ has constant mean curvature with respect to $g_\varepsilon = \mathrm{e}^{2u_\varepsilon}h_\varepsilon$, then it holds

$$\Phi\left(\rho + \fint_\Sigma \psi(\cdot\,;\varepsilon,\rho), u_\varepsilon, \psi(\cdot\,;\varepsilon,\rho) - \fint_\Sigma \psi(\cdot\,;\varepsilon,\rho)\right) = 0.$$

By (3.41) and (3.39), up to reduce $\varepsilon_R$, we can suppose that the function $x(\sigma, u_\varepsilon) \in \mathscr{C}^{2,\alpha}(\Sigma)$ introduced in Subsection 3.6.2 is defined for

$$\sigma := \rho + \fint_\Sigma \psi(\cdot\,;\varepsilon,\rho),$$

for every $\rho \in [\log\varepsilon + 3, -\log\varepsilon + R]$. But we know that the $x = x(\sigma, u_\varepsilon)$ is (locally) the unique $\mathscr{C}^{2,\alpha}(\Sigma)$-function with zero average for which $\Phi(\sigma, u_\varepsilon, x) = 0$ by the implicit function theorem. Since for small $\varepsilon_R$ the function $\psi$ is close to zero, then this implies

$$S(\varepsilon,\rho) = \hat{S}\left(u_\varepsilon, \rho + \fint_\Sigma \psi(\cdot\,;\varepsilon,\rho)\right).$$

Namely, we showed that every single hypersurface $S(\varepsilon,\rho)$ coincides with a leaf of the foliation formed by the $\hat{S}(u_\varepsilon,\sigma)$'s. This implies the weak stability for the $S(\varepsilon,\rho)$'s. On the other hand two different hypersurfaces of the $S(\varepsilon,\rho)$'s can not correspond to the same leaf of the foliation formed by the $\hat{S}(u_\varepsilon,\sigma)$'s since they have different mean curvatures. Moreover $S(\varepsilon,\rho)$ depends continuously on the parameter $\rho$, so we deduce that the $S(\varepsilon,\rho)$'s hypersurfaces provide a foliation of a compact subset of $(M, h_\varepsilon)$, coinciding with a part of the foliation constructed in Subsection 3.6.2, and the point ($iii$) of Theorem 3.15 follows.

### 3.6.9 The volume of the leaves

In this subsection we prove Theorem 3.15-($iv$), which is the estimate for the area of the minimal leaf $S(\varepsilon, 0)$ in terms of $\varepsilon^{n-1}$. Notice that the volume form of $h_\varepsilon$ can be obtained multiplying the volume form of $g_\varepsilon$ by $\mathrm{e}^{nu_\varepsilon}$, therefore

$$|\mathrm{Area}_{g_\varepsilon}(S) - \mathrm{Area}_{h_\varepsilon}(S)| \leq \left(\max_S |\mathrm{e}^{nu_\varepsilon} - 1|\right) \mathrm{Area}_{h_\varepsilon}(S) \tag{3.63}$$

for every hypersurface $S$. Using (3.39), in particular we deduce that

$$(1 - C\varepsilon^n)\mathrm{Area}_{h_\varepsilon}(S(0,\varepsilon)) \leq \mathrm{Area}_{g_\varepsilon}(S(0,\varepsilon)) \leq (1 + C\varepsilon^n)\mathrm{Area}_{h_\varepsilon}(S(0,\varepsilon))$$

for some $C > 0$ independent of $\varepsilon$. Since $S(0,\varepsilon)$ has zero mean curvature, it is a local minimum for $\mathrm{Area}_{g_\varepsilon}$ in a neighbourhood of $\{r = 0\}$. On the other hand, by construction $g_\varepsilon = \mathrm{e}^{2u_\varepsilon} h_\varepsilon$ and the hypersurface $\{r = 0\}$ is a (global, as we will see in the next section) minimum for $\mathrm{Area}_{h_\varepsilon}$. This implies

$$\begin{aligned}&(1 - C\varepsilon^n)\mathrm{Area}_{h_\varepsilon}(\{r = 0\}) \leq (1 - C\varepsilon^n)\mathrm{Area}_{h_\varepsilon}(S(\varepsilon, 0)) \leq \mathrm{Area}_{g_\varepsilon}(S(\varepsilon, 0))\\ &\mathrm{Area}_{g_\varepsilon}(S(\varepsilon, 0)) \leq \mathrm{Area}_{g_\varepsilon}(\{r = 0\}) \leq (1 + C\varepsilon^n)\mathrm{Area}_{h_\varepsilon}(\{r = 0\}).\end{aligned}$$

We just shown that $\mathrm{Area}_{g_\varepsilon}(S(\varepsilon, 0))$ is comparable with $\mathrm{Area}_{h_\varepsilon}(\{r = 0\})$. A rapid computation shows that the last quantity is equal to $2^{\frac{2(n-1)}{n}} \mathrm{Area}_{\bar{g}}(\Sigma)\, \varepsilon^{n-1}$, as we wanted to prove.

In dimension $n = 3$ this estimate on the area of the minimal leaf can be improved with a sharper estimate, namely the Riemannian Penrose inequality of Section 4.3. Precisely, in this setting it will follow $\mathrm{Area}_{g_\varepsilon}(S(\varepsilon, 0)) < 2^{4/3}\mathrm{Area}_{\bar{g}}(\Sigma)\, \varepsilon^2$.

## 3.7 Extension of the foliation to the infinity

In this subsection we will prove the delicate point ($v$) of Theorem 3.15. In a similar situation, involving the construction of a CMC foliation near the infinity of asymptotically Anti-de Sitter manifolds, Ambrozio [5] adapted the mean curvature approach of Rigger [77] for the existence and the approach of Neves and Tian [73] for the uniqueness. Differently, we will adapt a variational approach.

Since we are going to study the infinity of $(M, g_\varepsilon)$, it will be useful to work with the coordinate $t = r + \log \varepsilon \in (2 \log \varepsilon + 2, +\infty)$. With this choice

$$g_\varepsilon = \mathrm{e}^{u_\varepsilon} h_\varepsilon = \mathrm{e}^{u_\varepsilon(t,\theta)} \left(dt^2 + \mathrm{e}^{2t}(1 + \varepsilon^n \mathrm{e}^{-nt})^{4/n}\bar{g}\right),$$

for every $(t, \theta) \in (2 \log \varepsilon + 2, +\infty) \times \Sigma$. We recall that by the inequality (3.40) we know that for every $(n - 1)/2 < \delta < n$ there exists $C > 0$ such that

$$\sup_{t \in (-3,+\infty)} \sup_{\Sigma} \mathrm{e}^{\delta t} \left(|u_\varepsilon| + |\nabla u_\varepsilon| + |\nabla^2 u_\varepsilon|\right) < C\varepsilon^n, \tag{3.64}$$

where the geometric quantities are computed with respect to $dt^2 + \mathrm{e}^{2t}\bar{g}$. However, as it will be clearer in next of this subsection, it would be sufficient that (3.64) holds for some $\delta > n-1$ in order to guarantee the existence of a foliation near the infinity of $g_\varepsilon$.

Suppose that it holds the following lemma:

**Lemma 3.22.** *Let $(\Sigma, \bar{g})$ be a compact Riemannian manifold of dimension $n-1$, $n \geq 2$. There exist $T > 0$ (large) with the following property. For every $\varepsilon \in (0,1)$ the manifold $(2\log\varepsilon + 2, +\infty) \times \Sigma$ equipped with the metric*

$$\mathrm{e}^{2u_\varepsilon(t,\theta)}\left(dt^2 + \mathrm{e}^{2t}(1+\varepsilon^n \mathrm{e}^{-nt})^{4/n}\bar{g}\right),$$

*where $u_\varepsilon$ is a smooth function satisfying (3.64) for some $\delta > n-1$, admits a unique weakly stable foliation by CMC hypersurfaces $\{S(t,\varepsilon)\}_{t\geq T}$ such that every leaf $S(\tau,\varepsilon)$ is the graph of the function $t = \tau + x$ for some $x \in \mathscr{C}^\infty(\Sigma)$ with zero average and $\|x\|_{\mathscr{C}^{2,\alpha}(\Sigma)} < 1$. The (constant) mean curvature of $S(t,\varepsilon)$, which is computed with respect to the normal vector pointing toward the infinity, strictly increase as $t \to +\infty$ and tends to $n-1$.*

Then it follows Theorem 3.15-($v$). Indeed it is sufficient to apply the lemma above to $(M, g_\varepsilon)$ and then Theorem 3.15-($i$),($ii$),($iii$) with $R > T$ and $\varepsilon < \min\{\varepsilon_R, 1\}$. This causes an overlapping of the foliations provided by these two mentioned results. By uniqueness, the foliation near the infinity provided by Lemma 3.22 must be an extension of the compact foliation provided by Theorem 3.15-($i$),($ii$),($iii$), and the claim would follow. Hence the aim of this section is to prove Lemma 3.22. The proof is so organized:

- First we fix a function $x \in \mathscr{C}^{2,\alpha}(\Sigma)$ with zero mean value and we compute the area $V_{\tau,\varepsilon}(x)$ of the hypersurface $\{t = \tau + x\}$ with respect to $g_\varepsilon = \mathrm{e}^{u_\varepsilon(t,\theta)}\left(dt^2 + \mathrm{e}^{2t}(1+\varepsilon^n\mathrm{e}^{-nt})^{4/n}\bar{g}\right)$, then we compare it with the area $W_{\tau,\varepsilon}(x)$ of the same hypersurface computed with respect to the model metric $h_\varepsilon = dt^2 + \mathrm{e}^{2t}(1+\varepsilon^n\mathrm{e}^{-nt})^{4/n}\bar{g}$;

- Then we check that $x = 0$ is a regular minimum for $W_{\tau,\varepsilon}(x)$ and that the convexity of $W_{\tau,\varepsilon}$ at $x = 0$ does not degenerate as $\tau$ grows;

- Then we observe that the operator $V_{\tau,\varepsilon} - W_{\tau,\varepsilon}$ converges to zero as $\tau \to +\infty$ up to the second order, uniformly for bounded and positive $\varepsilon$;

- We use the previous convergence to deduce that $V_{\tau,\varepsilon}$ has a unique minimum $x(\cdot\,;\tau,\varepsilon)$, so that $S(\tau,\varepsilon) = \{t = \tau + x(\cdot\,;\tau,\varepsilon)\}$ is the unique hypersurface with constant mean curvature of the form $\{t = \tau + x\}$. Finally we check that for large $\tau$ the just built family of the CMC hypersurfaces $\{S(\tau,\varepsilon)\}_\tau$ provides the unique CMC foliation required. In particular, we discuss why these hypersurfaces form a smooth foliation, why they are weakly stable and why the mean curvatures of the leaves increase toward the infinity and tend to $n-1$.

### 3.7.1 Zero-order estimates

We will assume w.l.o.g. that $\mathrm{Area}_{\bar{g}}(\Sigma) = 1$. It is well known that a generic hypersurface of type $\{t = \tau + x\}$, where $x \colon \Sigma \to \mathbb{R}$ is a regular function with zero average, has constant mean curvature with respect to $g_\varepsilon$ if and only if $x$ minimize the area functional relative to the metric $g_\varepsilon$. In the following, we will set

$$h_\varepsilon = dt^2 + \mathrm{e}^{2t}\xi^2(t)\bar{g}, \quad \xi(t) := (1 + \varepsilon^2 \mathrm{e}^{-nt})^{2/n}$$

so that $g_\varepsilon = \mathrm{e}^{2u_\varepsilon} h_\varepsilon$. Notice that $\xi$ depends on $\varepsilon$, however it will be clear in the next computation that what counts is its convergence to 1 as $t \to +\infty$ uniformly for bounded and positive $\varepsilon$. Let $S = \{t = \tau + x\}$ be an hypersurface in $(2\log\varepsilon + 2, +\infty) \times \Sigma$ and assume $x \in \mathscr{C}^{2,\alpha}(\Sigma)$ with $⨍_\Sigma x = 0$. Then the volume form induced by $h_\varepsilon$ on $S$ is

$$dV_S = \sqrt{\det(\partial_i x \partial_j x + \mathrm{e}^{2(\tau+x)}\xi^2(\tau + x)\bar{g}_{ij})}\, d\theta^1 \dots d\theta^{n-1},$$

where $(\theta^1, \dots, \theta^{n-1})$ are local coordinates for $\Sigma$ and we set for simplicity $\partial_i := \frac{\partial}{\partial \theta^i}$. By the matrix determinant lemma, we get

$$\begin{aligned} dV_S &= \sqrt{1 + \mathrm{e}^{-2(\tau+x)}\xi^{-2}(\tau + x)|\nabla x|^2}\, \mathrm{e}^{(n-1)(\tau+x)}\xi^{n-1}(\tau + x) dV_{\bar{g}} \\ &= \mathrm{e}^{(n-2)(\tau+x)}\xi^{n-2}(\tau + x)\sqrt{\mathrm{e}^{2(\tau+x)}\xi^2(\tau + x) + |\nabla x|^2_{\bar{g}}}\, dV_{\bar{g}}. \end{aligned}$$

Therefore the area $W_{\tau,\varepsilon}(x)$ of the hypersurface $\{t = \tau + x\}$ computed with respect to $h_\varepsilon$ is given by

$$W_{\tau,\varepsilon}(x) = \int_\Sigma \mathrm{e}^{(n-2)(\tau+x)}\xi^{n-2}(\tau + x)\sqrt{\mathrm{e}^{2(\tau+x)}\xi^2(\tau + x) + |\nabla x|^2_{\bar{g}}}\, dV_{\bar{g}}. \quad (3.65)$$

Similarly one can compute the area $V_{\tau,\varepsilon}(x)$ of the hypersurface $\{t = \tau + x\}$ with respect to $g_\varepsilon$ obtaining

$$V_{\tau,\varepsilon}(x) = \int_\Sigma \mathrm{e}^{nu_\varepsilon(\tau+x,\theta)}\mathrm{e}^{(n-2)(\tau+x)}\xi^{n-2}(\tau+x)\sqrt{\mathrm{e}^{2(\tau+x)}\xi^2(\tau + x) + |\nabla x|^2_{\bar{g}}}\, dV_{\bar{g}}. \quad (3.66)$$

One easily notice that $x = 0$ is a global minimum for $W_{\tau,\varepsilon}$ among the functions with zero average, in fact $\mathrm{e}^t\xi(t)$ is convex and using the Jensen inequality one has

$$\begin{aligned} W_{\tau,\varepsilon}(x) &\geq \int_\Sigma \mathrm{e}^{(n-1)(\tau+x)}\xi^{n-1}(\tau + x) dV_{\bar{g}} \\ &\geq \mathrm{e}^{(n-1)\int_\Sigma(\tau+x)dV_{\bar{g}}}\xi^{n-1}\left(\int_\Sigma (\tau + x) dV_{\bar{g}}\right) \\ &= \mathrm{e}^{(n-1)\tau}\xi^{n-1}(\tau) = W_{\tau,\varepsilon}(0) \end{aligned}$$

with equality if and only if $x = 0$. This fact confirms that the slices $\{t = \text{const.}\}$ have constant mean curvature with respect to $h_\varepsilon$.

We will assume the operators above to be defined on the functional space

$$A := \left\{ x \in \mathscr{C}^{2,\alpha}(\Sigma) \quad \text{s.t.} \quad \fint_\Sigma x = 0 \right\},$$

and sometimes we will restrict the domain to $A_1$, which is the closed ball of $A$ with radius equal to 1 in the $\mathscr{C}^{2,\alpha}(\Sigma)$-norm. As first step, we check that $V_{\tau,\varepsilon}$ well approximates $W_{\tau,\varepsilon}$ for large $\tau$, uniformly in $\varepsilon$. In the next subsections we will extend this result up to the second order.

**Lemma 3.23.** *In the setting of this section, one has*

$$\lim_{\tau\to+\infty} \sup_{\varepsilon\in(0,1)} \sup_{x\in A_1} |V_{\tau,\varepsilon}(x) - W_{\tau,\varepsilon}(x)| = 0.$$

*Proof.* Since $x$ is bounded in $\mathscr{C}^{2,\alpha}(\Sigma)$ and since $\xi(t)$ is also bounded for $t \to +\infty$, we can conclude that both $\mathrm{e}^{\tau+x}\xi(\tau+x)$ and $\sqrt{\mathrm{e}^{2(\tau+x)}\xi^2(\tau+x) + |\nabla x|^2_{\bar{g}}}$ are positive quantities bounded by $2\mathrm{e}^{\tau+x}$. Then subtracting (3.65) from (3.66) we infer that

$$|V_{\tau,\varepsilon}(x) - W_{\tau,\varepsilon}(x)| \leq 2^{n-1} \int_\Sigma \left| \mathrm{e}^{nu_\varepsilon(\tau+x,\theta)} - 1 \right| \mathrm{e}^{(n-1)(\tau+x)}\, dV_{\bar{g}}.$$

In view of the hypothesis (3.64), we have $|\mathrm{e}^{nu_\varepsilon(\tau+x,\cdot)} - 1| < 2\mathrm{e}^{-\delta(\tau+x)}$. It follows

$$|V_{\tau,\varepsilon}(x) - W_{\tau,\varepsilon}(x)| \leq 2^n \int_\Sigma \mathrm{e}^{(n-1-\delta)(\tau+x)}\, dV_{\bar{g}}.$$

We got the thesis for boundedness of $x$ and the hypothesis $n-1-\delta < 0$. $\square$

### 3.7.2 First-order estimates

In order to avoid long lines in the next formulas, we introduce the shorter notation

$$I = \mathrm{e}^{\tau+x}\xi(\tau+x), \quad J = \sqrt{\mathrm{e}^{2(\tau+x)}\xi^2(\tau+x) + |\nabla x|^2_{\bar{g}}} \quad \text{and} \quad K = [1+\xi^{-1}\xi'](\tau+x). \tag{3.67}$$

In this way (3.65) and (3.66) become

$$W_{\tau,\varepsilon}(x) = \int_\Sigma I^{n-2} J\, dV_{\bar{g}} \quad \text{and} \quad V_{\tau,\varepsilon}(x) = \int_\Sigma I^{n-2} J\, \mathrm{e}^{nu_\varepsilon(\tau+x,\theta)}\, dV_{\bar{g}}. \tag{3.68}$$

Using that $DI_x[v] = IKv$ and $DJ_x[v] = J^{-1}(I^2K + \bar{g}(\nabla x, \nabla v))$, for every $x \in A_1$ the Fréchet differential of $V_{\tau,\varepsilon}$ at $x$ is the linear operator $D(V_{\tau,\varepsilon})_x \colon A \to \mathbb{R}$

given by

$$\begin{aligned} D(V_{\tau,\varepsilon})_x[v] = n &\int_\Sigma I^{n-2} J \mathrm{e}^{n u_\varepsilon(\tau+x,\cdot)} \partial_r u_\varepsilon(\tau+x,\cdot) v \, dV_{\bar{g}} \\ &+ (n-2) \int_\Sigma I^{n-2} J K \mathrm{e}^{n u_\varepsilon(\tau+x,\cdot)} v \, dV_{\bar{g}} \\ &+ \int_\Sigma I^n J^{-1} K \mathrm{e}^{n u_\varepsilon(\tau+x,\cdot)} v \, dV_{\bar{g}} \\ &+ \int_\Sigma I^{n-2} J^{-1} \mathrm{e}^{n u_\varepsilon(\tau+x,\cdot)} \bar{g}(\nabla x, \nabla v) \, dV_{\bar{g}}. \end{aligned} \tag{3.69}$$

Similarly, the Fréchet differential of $W_{\tau,\varepsilon}$ at $x$ is given by

$$\begin{aligned} D(W_{\tau,\varepsilon})_x[v] = (n-2) &\int_\Sigma I^{n-2} J K v \, dV_{\bar{g}} \\ &+ \int_\Sigma I^n J^{-1} K v \, dV_{\bar{g}} \\ &+ \int_\Sigma I^{n-2} J^{-1} \bar{g}(\nabla x, \nabla v) \, dV_{\bar{g}} \end{aligned} \tag{3.70}$$

and can be obtained by formula for $D(V_{\tau,\varepsilon})$ replacing $u_\varepsilon$ by zero. Notice that $D(W_{\tau,\varepsilon})_0[v] = 0$ for every $v \in A$, and this follows from the observation that $x = 0$ is a global minimum for $W_{\tau,\varepsilon}$. The following lemma shows that $D(V_{\tau,\varepsilon})$ well approximates $D(W_{\tau,\varepsilon})$ for large $\tau$, uniformly in $\varepsilon$.

**Lemma 3.24.** *In the setting of this section, one has*

$$\lim_{\tau \to +\infty} \sup_{\varepsilon \in (0,1)} \sup_{x \in A_1} \sup_{\|v\|_{\mathscr{C}^{2,\alpha}(\Sigma)} = 1} |D(V_{\tau,\varepsilon})_x[v] - D(W_{\tau,\varepsilon})_x[v]| = 0.$$

*Proof.* As observed in the proof of Lemma 3.23, both $I$ and $J$ are positive quantities bounded by $2\mathrm{e}^{\tau+x}$. Subtracting (3.70) from (3.69) and using the boundedness of $x$ and $v$, we easily get

$$\begin{aligned} |D(V_{\tau,\varepsilon})_x[v] - D(W_{\tau,\varepsilon})_x[v]| \leq n 2^{n-1} &\int_\Sigma \mathrm{e}^{(n-1)(\tau+x)} \mathrm{e}^{n u_\varepsilon(\tau+x,\cdot)} |\partial_r u_\varepsilon(\tau+x,\cdot)| \, dV_{\bar{g}} \\ &+ (n-2) 2^{n-1} \int_\Sigma \mathrm{e}^{(n-1)(\tau+x)} K |\mathrm{e}^{n u_\varepsilon(\tau+x,\cdot)} - 1| \, dV_{\bar{g}} \\ &+ 2^n \int_\Sigma \mathrm{e}^{n(\tau+x)} J^{-1} K |\mathrm{e}^{n u_\varepsilon(\tau+x,\cdot)} - 1| \, dV_{\bar{g}} \\ &+ 2^{n-2} \int_\Sigma \mathrm{e}^{(n-2)(\tau+x)} J^{-1} |\mathrm{e}^{n u_\varepsilon(\tau+x,\cdot)} - 1| \, dV_{\bar{g}}. \end{aligned} \tag{3.71}$$

Since $\xi(t) \to 1$ and $\xi'(t) \to 0$ as $t \to +\infty$, we can observe that $0 < K < 2$ and $0 < J^{-1} < 2\mathrm{e}^{-(\tau+x)}$. Moreover by (3.64) we have deduce $|\mathrm{e}^{n u_\varepsilon(\tau+x,\cdot)} - 1| <$

$2\mathrm{e}^{-\delta(\tau+x)}$, $|\partial_r u_\varepsilon(\tau+x,\cdot)| < 2\mathrm{e}^{-\delta(\tau+x)}$ and $|\mathrm{e}^{nu_\varepsilon(\tau+x,\cdot)}| < 2$. The previous estimates imply

$$|D(V_{\tau,\varepsilon})_x[v] - D(W_{\tau,\varepsilon})_x[v]| \le c(n)\int_\Sigma \mathrm{e}^{(n-1-\delta)(\tau+x)}\, dV_{\bar g}. \tag{3.72}$$

for some constant $c(n) > 0$. We got the thesis for boundedness of $x$ and the hypothesis $n-1-\delta < 0$. □

### 3.7.3 Second-order estimates

Now we want to consider a further differentiation and study the bilinear and symmetric operators $D^2(W_{\tau,\varepsilon})_x, D^2(V_{\tau,\varepsilon})_x \colon A \times A \to \mathbb{R}$. We have

$$\begin{aligned}
&D^2(V_{\tau,\varepsilon})_x[v,w] = \\
&2n\int_\Sigma \left((n-2)I^{n-2}J + I^nJ^{-1}\right)K\mathrm{e}^{nu_\varepsilon(\tau+x,\cdot)}\partial_r u_\varepsilon(\tau+x,\cdot)vw\, dV_{\bar g} \\
&+n\int_\Sigma I^{n-2}J^{-1}\,\mathrm{e}^{nu_\varepsilon(\tau+x,\cdot)}\partial_r u_\varepsilon(\tau+x,\cdot)\left(\bar g(\nabla x,\nabla w)v + \bar g(\nabla x,\nabla v)w\right)\, dV_{\bar g} \\
&+n\int_\Sigma I^{n-2}J\,\mathrm{e}^{nu_\varepsilon(\tau+x,\cdot)}\left[n(\partial_r u_\varepsilon(\tau+x,\cdot))^2 + \partial_r^2 u_\varepsilon(\tau+x,\cdot)\right]vw\, dV_{\bar g} \\
&+\int_\Sigma I^{n-2}\left((n-2)^2JK^2 + 2(n-1)I^2J^{-1}K^2 + (n-2)JDK + I^2J^{-1}DK\right)\mathrm{e}^{nu_\varepsilon(\tau+x,\cdot)}vw\, dV_{\bar g} \\
&+(n-2)\int_\Sigma I^{n-2}J^{-1}K\mathrm{e}^{nu_\varepsilon(\tau+x,\cdot)}(\bar g(\nabla x,\nabla w)v + \bar g(\nabla x,\nabla v)w)\, dV_{\bar g} \\
&-\int_\Sigma I^{n-2}J^{-3}\,\mathrm{e}^{nu_\varepsilon(\tau+x,\cdot)}(I^2Kv + \bar g(\nabla x,\nabla v))(I^2Kw + \bar g(\nabla x,\nabla w))\, dV_{\bar g} \\
&+\int_\Sigma I^{n-2}J^{-1}\,\mathrm{e}^{nu_\varepsilon(\tau+x,\cdot)}\bar g(\nabla w,\nabla v)\, dV_{\bar g}
\end{aligned} \tag{3.73}$$

and, replacing $u_\varepsilon$ by zero, we also get

$$\begin{aligned}
&D^2(W_{\tau,\varepsilon})_x[v,w] = \\
&\int_\Sigma I^{n-2}\left((n-2)^2JK^2 + 2(n-1)I^2J^{-1}K^2 + (n-2)JDK + I^2J^{-1}DK\right)vw\, dV_{\bar g} \\
&+(n-2)\int_\Sigma I^{n-2}J^{-1}K(\bar g(\nabla x,\nabla w)v + \bar g(\nabla x,\nabla v)w)\, dV_{\bar g} \\
&-\int_\Sigma I^{n-2}J^{-3}(I^2Kv + \bar g(\nabla x,\nabla v))(I^2Kw + \bar g(\nabla x,\nabla w))\, dV_{\bar g} \\
&+\int_\Sigma I^{n-2}J^{-1}\bar g(\nabla w,\nabla v)\, dV_{\bar g}.
\end{aligned} \tag{3.74}$$

In the above equation we introduced the short notation $DK := [-\xi^{-2}(\xi')^2 + \xi^{-1}\xi''](\tau + x)$ so that $DK_x[w] = DKw$. It is important to point out that in particular for every $\tau \in \mathbb{R}$ large enough and for every $\varepsilon \in (0, 1)$ we have

$$\begin{aligned} D^2(W_{\tau,\varepsilon})_0[v, w] &= (n-1)\mathrm{e}^{(n-1)\tau}\xi^{n-1}(\tau) \left[(n-1)K^2 + DK\right]_{x=0} \int_\Sigma vw \, dV_{\bar{g}} \\ &\quad + \mathrm{e}^{(n-3)\tau}\xi^{n-3}(\tau) \int_\Sigma \bar{g}(\nabla w, \nabla v) \, dV_{\bar{g}} \geq \int_\Sigma vw \, dV_{\bar{g}}, \end{aligned} \tag{3.75}$$

namely the convexity of $W_{\tau,\varepsilon}$ at its minimum $x = 0$ does not degenerate to zero as $\tau \to +\infty$.

Similarly to the two previous subsections, we show that $D^2(V_{\tau,\varepsilon})$ well approximates $D^2(W_{\tau,\varepsilon})$ for large $\tau$, uniformly in $\varepsilon$.

**Lemma 3.25.** *In the setting of this section, one has*

$$\lim_{\tau \to +\infty} \sup_{\varepsilon \in (0,1)} \sup_{x \in A_1} \sup_{\|v\|_{\mathscr{C}^{2,\alpha}(\Sigma)}=1} \sup_{\|w\|_{\mathscr{C}^{2,\alpha}(\Sigma)}=1} |D^2(V_{\tau,\varepsilon})_x[v, w] - D^2(W_{\tau,\varepsilon})_x[v, w]| = 0.$$

*Proof.* This computation is very similar to the proves of Lemmas 3.23 and 3.24, but has longer expressions. For this reason we omit the computation, but we recall how this result can be obtained. First one subtract (3.74) from (3.73). Then one has to recall that:

- The quantity $I = \mathrm{e}^{\tau+x}\xi(\tau+x)$ satisfies $0 < I < 2\mathrm{e}^{\tau+x}$ for boundedness of $\|x\|_{\mathscr{C}^{2,\alpha}(\Sigma)}$ and of $\xi(t) = (1 + \varepsilon^2\mathrm{e}^{-nt})^{2/n}$ as $t \to +\infty$;
- The quantity $J = \sqrt{\mathrm{e}^{2(\tau+x)}\xi^2(\tau + x) + |\nabla x|^2_{\bar{g}}}$ satisfies $0 < J < 2\mathrm{e}^{\tau+x}$ for the same arguments above, and $0 < J^{-1} < 2\mathrm{e}^{-(\tau+x)}$ for boundedness of $\xi^{-1}(t)$ as $t \to +\infty$;
- The quantities $K = [1+\xi^{-1}\xi'](\tau+x)$ and $DK = [-\xi^{-2}\xi'+\xi^{-1}\xi''](\tau+x)$ satisfy $0 < K, DK < 2$ for boundedness of $\|x\|_{\mathscr{C}^{2,\alpha}(\Sigma)}$ and the fact that $\xi(t) \to 1$, $\xi'(t) \to 0$ and $\xi''(t) \to 0$ as $t \to +\infty$;
- In view of the hypothesis (3.64), we have

  $$|\mathrm{e}^{nu_\varepsilon(\tau+x,\cdot)} - 1|, |\partial_r u_\varepsilon(\tau + x, \cdot)|, |\partial_r^2 u_\varepsilon(\tau + x, \cdot)| < 2\mathrm{e}^{-\delta(\tau+x)},$$

  and $|\mathrm{e}^{nu_\varepsilon(\tau+x,\cdot)}| < 2$.

With these observations, one get

$$|D^2(V_{\tau,\varepsilon})_x[v, w] - D^2(W_{\tau,\varepsilon})_x[v, w]| \leq c(n) \int_\Sigma \mathrm{e}^{(n-1-\delta)(\tau+x)} \, dV_{\bar{g}}. \tag{3.76}$$

exactly as in the zero-order and first-order estimates. We got the thesis for boundedness of $x$ and the hypothesis $n - 1 - \delta < 0$. □

### 3.7.4 Construction of the CMC foliation

From Lemma 3.23, Lemma 3.24 and Lemma 3.25, we obtain that

$$\lim_{\tau\to+\infty}\sup_{\varepsilon\in(0,1)}\|V_{\tau,\varepsilon}-W_{\tau,\varepsilon}\|_{\mathscr{C}^2(A_1,\mathbb{R})}=0, \tag{3.77}$$

which is the uniform convergence of $V_{\tau,\varepsilon}-W_{\tau,\varepsilon}$ to zero up to the second order as $\tau\to+\infty$, uniformly in $\varepsilon$. On the other hand we observed before that $W_{\tau,\varepsilon}$ admits a unique regular minimum $x=0$ and that the convexity of $W_{\tau,\varepsilon}$ at this minimum does not degenerate as $\tau\to+\infty$. As a consequence there exists $T>0$ such that for $\tau>T$ and $\varepsilon\in(0,1)$ the functional $V_{\tau,\varepsilon}$ admits a unique minimum $x(\cdot\,;\tau,\varepsilon)$ in $A_1$ which converges to 0 as $\tau$ grows, namely

$$\|x(\cdot\,;\tau,\varepsilon)\|_{\mathscr{C}^{2,\alpha}(\Sigma)}\to 0 \quad \text{as } \tau\to+\infty \tag{3.78}$$

for every $\varepsilon\in(0,1)$. Notice that $x(\cdot\,;\tau,\varepsilon)$ is actually smooth on $\Sigma$ since resolves the second-order elliptic PDE $D(V_{\tau,\varepsilon})_x=0$. This also implies that $x(\cdot\,;\tau,\varepsilon)$ is regular in $\tau$ and $\varepsilon$ and as a consequence $\|\partial_\tau x(\cdot\,;\tau,\varepsilon)\|_{\mathscr{C}^{2,\alpha}(\Sigma)}\to 0$ as $\tau\to+\infty$.

The hypersurface $S(\tau,\varepsilon):=\{t=\tau+x(\cdot\,;\tau,\varepsilon)\}$ has been constructed with the property of having constant mean curvature. We want to check that they provide a weakly stable CMC foliation of the infinity, with strictly increasing mean curvatures tending to $n-1$ as the leaves approach the infinity. This is done using the considerations of the previous subsection: essentially it will be enough to check these properties for the model manifold $(M,h_\varepsilon)$ and consider $\tau>T$ for $T$ large enough. Indeed, if $T$ is large enough then: From the observations of the previous subsection, if $T$ is large enough then:

- the family $\{S(\tau,\varepsilon)\}_{\tau>T}$ is a foliation, in fact it holds $\partial_\tau(\tau+x(\cdot\,;\tau,\varepsilon))>0$;
- every leaf $S(\tau,\varepsilon)$ is weakly stable. This can be proved with the same arguments that we used for the foliation of the compact subset of the end. In fact the Jacobi operator of $S(\tau,\varepsilon)$ approximates the Jacobi operator of $\{t=\tau\}$ in $h_\varepsilon$ for large $\tau$, and this last example is weakly stable (cfr. Subsection 3.6.1);
- the mean curvature of $S(\tau,\varepsilon)$ computed with respect to $g_\varepsilon$ is strictly increasing and tends to $n-1$ as $\tau\to+\infty$. In fact it is sufficient to notice that the same statement holds for the slices $\{t=\tau\}$ with respect to $h_\varepsilon$, as a consequence of Lemma 5.4.

This concludes the proof of Lemma 3.22.

We conclude this subsection with an observation about the precise value of the mean curvature of every leaf $S(\tau,\varepsilon)$, in the spirit of the previous section. We recall that we just know that the mean curvature of $S(\tau,\varepsilon)$ is constant and that this value increase to $n-1$ as $\tau \to +\infty$. Therefore once fixed the leaf $S(\tau,\varepsilon)$ there exists a unique $\rho = \rho(\tau,\varepsilon) \in \mathbb{R}$ such that the mean curvature of $S(\tau,\varepsilon)$ is $(n-1)\tanh(n\rho/2)$. If we set $\psi := \tau + x(\cdot;\tau,\varepsilon) - \rho - \log\varepsilon$, then $S(\tau,\varepsilon)$ can be written as $\{r = \rho + \psi\}$, with $t = r + \log\varepsilon$, and the last is written with the notation used for building the local CMC foliation in the previous section. In fact with the argument above we are able to find for every $\rho \in \mathbb{R}$ (large enough) the correspondent CMC leaf with constant mean curvature equal to $(n-1)\tanh(n\rho/2)$, which can be written in the form $\{r = \rho + \psi(\cdot;\varepsilon,\rho)\}$. Studying the correspondence $\rho = \rho(\tau,\varepsilon)$, one can observe that (3.78) is equivalent to

$$\|\psi(\cdot\,;\varepsilon,\rho)\|_{\mathscr{C}^{2,\alpha}(\Sigma)} \to 0 \quad \text{as } \rho \to +\infty. \tag{3.79}$$

# Chapter 4

# Physical applications

## 4.1 Einstein's constraint equations

In the 1916 Einstein published the well-known formula

$$\mathrm{Ric}_\gamma - \frac{1}{2}\mathrm{R}_\gamma \gamma + \Lambda\gamma = 8\pi G T \tag{4.1}$$

to model our 4-dimensional pseudo-Riemannian spacetime $(\mathcal{M}, \gamma)$ of signature $(-, +, +, +)$. In the above *Einstein's field equation* (4.1), $G$ denotes the Newton's constant of gravitation, $\Lambda$ denotes the cosmological constant and $T$ is the energy-momentum tensor, usually depending on $\gamma$. In general, it is very difficult to find solutions of (4.1), which is a set of ten non-linear equations on the metric coefficients and their derivatives (actually only six of them are linearly independent, since we have four degrees of freedom to make coordinate transformations). A fruitful method to build examples of solutions is the *Cauchy formulation* (or *method by initial data*), whose origins descend from different works of Choquet-Bruhat and Geroch in the 60's. The idea of this method is to build the space-time beginning from a space-like embedded hypersurface with prescribed metric and extrinsic curvature. Before introducing this method, we recall some important examples of Einstein's spacetimes.

### 4.1.1 Famous solutions of the Einstein's field equation

Here it is a crude list of remarkable solutions of (4.1) in the *vacuum* case $T = 0$. The first three examples are usually used to describe the infinity of our universe. The main difference between this three models is connected with the sign of the cosmological constant $\Lambda$.

- The *Minkowski spacetime* is the simplest example of Einstein's spacetime ($\Lambda = 0$). It is defined as $\mathbb{R}^4$ equipped with the flat Lorentzian metric $\gamma = -dt^2 + dx^2 + dy^2 + dz^2$. This metric can be also presented in a double-warped form (using spherical coordinates, which are smooth on $\mathbb{R}^4$ but the origin) as

$$\gamma = -dt^2 + dr^2 + r^2 g_{S^2}.$$

This is an example of *static* universe, namely all the slices $\{t = \text{const.}\}$ are metrically equal.

- The *de Sitter spacetime* can be regarded as an example of Einstein's spacetime in the vacuum case ($\Lambda = 3$) since it satisfies $\mathrm{Ric}_\gamma = 3\gamma$. It is topologically $\mathbb{R} \times S^3$ and the metric is given by

  $$\gamma = -dt^2 + \cosh^2(t)[dr^2 + \sin^2(r) g_{S^2}]$$

  with singularities descending from the choice of the coordinates. It is possible to consider rescaling of this metric to have solutions of the vacuum Einstein's field equation with arbitrary positive cosmological constant.

- Similarly as the previous point, the *anti-de Sitter spacetime* can be regarded as an example of Einstein's spacetime in the vacuum case ($\Lambda = -3$) since it satisfies $\mathrm{Ric}_\gamma = -3\gamma$. It is topologically $S^1 \times \mathbb{R}^3$ and the metric is given by

  $$\gamma = -dt^2 + \cos^2(t)[dr^2 + \sinh^2(r) g_{S^2}]$$

  with singularities descending from the choice of the coordinates. In particular, with a change of coordinate the singularities at $t = \pm\frac{\pi}{2}$ turn out to be apparent, and the anti-de Sitter spacetime can be written in the static form

  $$\gamma = -\cosh^2(r) dt'^2 + dr^2 + \sinh^2(r) g_{S^2}.$$

  It is possible to consider rescaling of this metric to have solutions of the vacuum Einstein's field equation with arbitrary negative cosmological constant.

While the Minkowski, de Sitter and anti-de Sitter spacetimes can be used to describe the distribution of matter for the large scale universe (and in fact a good property for general Einstein spacetimes is the one of being asymptotically close to these three models), there are other important spacetimes which are usually used to describe the local geometry of the universe. The following examples share the asymptotic behaviour of the three cases considered above, but are drastically different in a compact zone.

- The *Schwarzschild* is the unique solution of the Einstein's field equation (with $\Lambda = 0$) describing a spherically symmetric empty ($T = 0$)

spacetime around a spherical symmetric massive body. It is static and given by

$$\gamma = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 g_{S^2}.$$

Here the constant $m > 0$ represents the gravitational mass. This metric is defined for $r > 2m$ and present an apparent singularity at $r = 2m$ which depends on the choice of the coordinates. Notice that as $r \to +\infty$ this metric approximate the Minkowski spacetime.

- The *Schwarzschild-de Sitter* spacetime (resp. *Schwarzschild-anti de Sitter*) generalizes the Schwarzschild metric for Einstein's spacetime with positive (resp. negative) cosmological constant $\Lambda$. They are defined as

$$\gamma = -\left(1 - \frac{\Lambda}{3}r^2 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{\Lambda}{3}r^2 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 g_{S^2}$$

for $1 - \frac{\Lambda}{3}r^2 - \frac{2m}{r} > 0$. Despite the Schwarzschild-de Sitter and anti-de Sitter spacetimes can be described with the same formula, the sign of $\Lambda$ drastically changes the metrical structure of the space.

- Further generalizations of the metrics above are given by

$$\gamma = -\left(k - \frac{\Lambda}{3}r^2 - \frac{2m}{r}\right) dt^2 + \left(k - \frac{\Lambda}{3}r^2 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 g_B$$

where $(B, g_B)$ is a 2-dimensional Riemannian manifold with constant sectional curvature equal to $k$, replacing the sphere of the previous point.

Finally, we want to focus the attention on a very special class of solutions. We observe that the spacetime $\mathbb{R} \times \mathbb{R} \times T^2$ equipped with

$$\gamma_\varepsilon = -\frac{\varepsilon^2 \sinh^2(3r/2)}{\cosh^2(3r/2)} dt^2 + dr^2 + \varepsilon^2 \left(2\cosh(3r/2)\right)^{4/3} g_{T^2},$$

where $\varepsilon$ is a positive real number and $g_{T^2}$ is a flat metric defined on a torus $T^2$, provides a solution of (4.1) with $\Lambda = -3$. To see that, it is sufficient to set $m := 2\varepsilon^3$ and $r' := \varepsilon(2\cosh(3r/2))^{2/3}$, then we have

$$\gamma_\varepsilon = -\left(r'^2 - \frac{2m}{r'}\right) dt^2 + \left(r'^2 - \frac{2m}{r'}\right)^{-1} dr'^2 + r'^2 g_{T^2},$$

which falls inside the last example of spacetimes introduced above. Similarly, we can consider the Einstein's solution

$$\gamma_0 = -r^2 dt^2 + r^{-2} dr^2 + r^2 g_{T^2}$$

and set $r = \mathrm{e}^{-s}$, so that

$$\gamma_0 = -\mathrm{e}^{-2s} dt^2 + ds^2 + \mathrm{e}^{-2s} g_{T^2}.$$

The time-slices $\{t = \text{const.}\}$ of $\gamma_\varepsilon$ and $\gamma_0$ are respectively the funnel-like ends and the cusps of dimension $n = 3$ considered in this text.

### 4.1.2 The Einstein's constraint equations

In this subsection we present a famous method to build solutions $(\mathcal{M}, \gamma)$ of the Einstein's field equation (4.1) of arbitrary dimension $n + 1 \geq 3$. Consider any space-like hypersurface $(M, g)$ embedded in a Lorentzian manifold $(\mathcal{M}, \gamma)$ of signature $(1, n)$ and let $g$ and II denote the induced norm and the second fundamental form induced on $M$ respectively. Fix $p \in M$ and let $(\nu, \partial_1, \ldots, \partial_n)$ be an orthonormal frame for $T_p\mathcal{M}$ such that $\partial_i$ is tangent to $M$ for $i = 1, \ldots, n$. By the Gauss, Codazzi and Mainardi equations one can compute that

$$\mathrm{Ric}_\gamma(\nu, \nu) - \frac{1}{2}\mathrm{R}_\gamma \gamma(\nu, \nu) = \mathrm{Ric}_\gamma(\nu, \nu) + \frac{1}{2}\mathrm{R}_\gamma = \frac{1}{2}\left(\mathrm{R}_g + (\mathrm{tr}_g \mathrm{II})^2 - |\mathrm{II}|_g^2\right)$$

and that

$$\mathrm{Ric}_\gamma(\nu, \partial_i) - \frac{1}{2}\mathrm{R}_\gamma \gamma(\nu, \partial_i) = \mathrm{Ric}_\gamma(\nu, \partial_i) = g^{kj}\nabla_j \mathrm{II}_{ik} - g^{jk}\nabla_i \mathrm{II}_{jk}.$$

In particular if $(\mathcal{M}, \gamma)$ is an Einstein spacetime in the sense that it solves (4.1), then the last formulas imply that

$$\frac{1}{2}\left(\mathrm{R}_g + (\mathrm{tr}_g \mathrm{II})^2 - |\mathrm{II}|_g^2\right) - \Lambda = 8\pi G T(\nu, \nu)$$

and

$$g^{kj}\nabla_j \mathrm{II}_{ik} - g^{jk}\nabla_i \mathrm{II}_{jk} = 8\pi G T(\nu, \partial_i).$$

If we introduce the *mass density* $\mu := 8\pi G T(\nu, nu)$ and the *current density* $J_i := 8\pi G T(\nu, \partial_i)$, then we just find the *Einstein's constraint equations*

$$\begin{cases} \mathrm{R}_g + (\mathrm{tr}_g \mathrm{II})^2 - |\mathrm{II}|_g^2 - 2\Lambda = 2\mu \\ \mathrm{div}_g \mathrm{II} - d(\mathrm{tr}_g \mathrm{II}) = J. \end{cases} \tag{4.2}$$

The matter-free case $T = 0$ gives raise to the *(vacuum) Einstein's constraint equations*

$$\begin{cases} \mathrm{R}_g + (\mathrm{tr}_g \mathrm{II})^2 - |\mathrm{II}|_g^2 = 2\Lambda \\ \mathrm{div}_g \mathrm{II} = d(\mathrm{tr}_g \mathrm{II}). \end{cases} \tag{4.3}$$

The power of these equations is that if on one hand any space-like hypersurface of an Einstein's empty spacetime $(\mathcal{M}, \gamma)$ satisfies (4.3), then on the other hand any solution $(M, g, \mathrm{II})$ of (4.3), given by a Riemannian manifold

$(M, g)$ of dimension $n \geq 2$ and a symmetric $(0,2)$-tensor II, can be seen as a space-like hypersurface of an Einstein empty spacetime $(\mathcal{M}, \gamma)$ and, with respect to this immersion, $g$ is precisely $\gamma|_M$ and II coincides with the second fundamental form of $M$ in $\mathcal{M}$. Moreover there exists a unique maximal spacetime associated to a solution $(M, g, \mathrm{II})$ of (4.3). This was showed by Choquet-Bruhat, which studied the well-posedness of (4.1) with Cauchy data $(M, g, \mathrm{II})$ satisfying (4.3).

Observe that the time-slices of the Minkowski, De Sitter and Anti-de Sitter spacetimes, which can be regarded as Cauchy data for the Einstein's equations, are respectively the euclidean space (with a totally geodesic embedding), the sphere (with an umbilical embedding) and the hyperbolic space (with an umbilical embedding), all of dimension 3.

### 4.1.3 The conformal method

Now that we know the relevance of (4.3), whose solution can be used as initial data for building Einstein spacetimes, we can talk about the most famous method to product solutions $(M, g, \mathrm{II})$ of the Einstein's constraint equations, called *conformal method* (Lichnerowicz 1944, then generalized by Choquet-Bruhat, York and collaborators). For simplicity we will talk about the vacuum case $T = 0$ in the physically relevant dimension $n + 1 = 4$. The idea of this method is to consider a 3-dimensional manifold $M$ and to look for a metric $g$ and a symmetric $(0,2)$-tensor II written in a specific form. Precisely, one suppose to have fixed:

- a Riemannian metric $g_0$ on $M$;
- a regular function $\tau\colon M \to \mathbb{R}$;
- a symmetric $(0,2)$-tensor $\sigma$ in $(M, g_0)$ with $\mathrm{div}_{g_0}\sigma = 0$ and $\mathrm{tr}_{g_0}\sigma = 0$.

Then look for solutions of the Einstein's constraint equations in the form

$$g = \phi^4 g_0 \quad \text{and} \quad \mathrm{II} = \phi^{-2}(\sigma + \mathcal{L}W) + \frac{\tau}{3}\phi^4 g_0 \tag{4.4}$$

for some smooth positive function $\phi\colon M \to \mathbb{R}$ and some vector field $W$ defined on $M$. Here $\mathcal{L}$ denotes the conformal Killing operator, so that in local coordinates

$$(\mathcal{L}W)_{ij} = (g_0)_{jk}\nabla_i W^k + (g_0)_{ik}\nabla_j W^k - \frac{2}{3}(g_0)_{ij}\nabla_k W^k,$$

where the $\nabla$'s are computed with respect to $g_0$. The advantage of (4.4) is that, in view of the formulas for the conformal transformations of a metric (cfr. Section 5.4), the constraint equations (4.3) becomes

$$\begin{cases} \mathrm{div}(\mathcal{L}W) = \frac{2}{3}\phi^6 d\tau \\ \Delta\phi = \frac{1}{8}\mathrm{R}_{g_0}\phi - \frac{1}{8}\phi^{-7}|\sigma + \mathcal{L}W|^2 + \frac{1}{12}\tau^2\phi^5, \end{cases} \tag{4.5}$$

seen as equations in $\phi$ and $W$. In (4.5) the metrical objects are computed with respect to the metric $g_0$. In summary, the purpose of the conformal method is to find solutions of the constraint equations in the form (4.4) once fixed the *conformal data* $M$, $g_0$, $\tau$ and $\sigma$. This is possible if and only if the conformal data permits to find solutions $(\phi, W)$ of (4.5). The existence of a solution for (4.5) is false in general and it strongly depend on the choice of the conformal data. However, it can be proven that there exist solutions of (4.5) with conformal data $(M, g_0, \tau, \sigma)$ if and only if there exist solutions of (4.5) with conformal data $(M, \theta^4 g_0, \tau, \theta^{-2}\sigma)$. Therefore we can restrict our attention to specific metrics in the conformal class of $g_0$, for instance in view of the Yamabe problem it is not restrictive to suppose to have fixed $g_0$ with constant scalar curvature. So, *which conditions on the conformal data allow to find solutions of* (4.5)? This problem has been overstudied in the last years, but some points are still open. The point is that several hypothesis can be imagined on the conformal data, for instance

- one can assume some conditions on $M$. It is possible to consider $M$ as *compact manifold*, eventually *with boundary*, or one can suppose that it presents some ends which can be *asymptotically euclidean* or *asymptotically hyperbolic* or *asymptotically conical* or *asymptotically cylindrical* and so on. Every choice open the street for different research and results;
- as mentioned above, one can make hypothesis on $g_0$. A common choice is to take a metric $g_0$ with constant scalar curvature, however the sign of the curvature may influence the existence of a solution of (4.5);
- one can also make assumptions on $\sigma$. Usually the literature distinguish that case $\sigma \equiv 0$ from the general case;
- one can also make assumption on $\tau$. For instance one can suppose that $\tau$ is constant. This is called the *CMC conformal problem* since by (4.4) it holds $\mathrm{tr}_g \mathrm{II} = \tau$. This is the most understood background for the conformal method. There are some results also in the *Near-CMC conformal problem*, where $\tau$ is not more assumed to be constant but there is a suitable control of $|d\tau|$. For arbitrary $\tau$, the conformal problem is in general an open problem, with few exceptions. This last case is usually called *Far-CMC conformal problem.*

Moreover, it is also important to specify the space of functions where we are looking for solutions. Also, there is an analogous problem in the non-vacuum cases. This leads to further classifications depending on the energy-momentum $T$, which can assume distinct aspects according to the physical problem that one wants to study. For an overlooking on what is known and what is not, we suggest [48].

## 4.2 Apparent horizons

### 4.2.1 Definition

Let $(M, g, \mathrm{II})$ be a solution of the Einstein constraint equations and assume $\partial M \neq \emptyset$. Then $\partial M$ is called *apparent horizon* iff

$$-\mathrm{tr}_g \mathrm{II} + \mathrm{II}(\nu, \nu) + (n-1)H = 0,$$

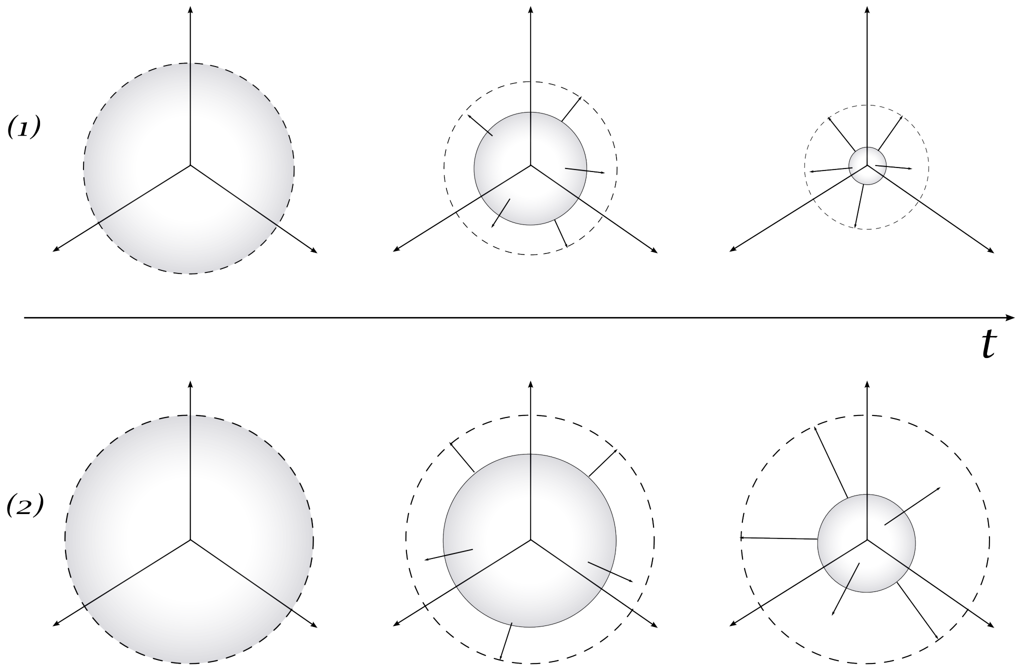
where $\nu$ denotes the out-warding unit vector field perpendicular to $\partial M$ and $H$ is the mean curvature of $\partial M \subset M$ computed with respect to the metric $g$ and the normal vector field $\nu$. In the next we briefly discuss the origin and the meaning of this definition. As basic example (supported by Figure 4.2.1), imagine that the ECE's solution $(M, g, \mathrm{II})$ is the initial data for a spacetime $\mathbb{R} \times M$ which is not static, but which is collapsing in future. Let $S$ be an hypersurface of $M$ at a fixed time (for instance at $t = 0$). For simplicity we can suppose that $S$ is a sphere in the physically relevant case $n + 1 = 4$. Now imagine that at $t = 0$ the sphere $S$ sends a light signal toward infinity, in direction perpendicular to $S$. After some time these light rays got a little away from $S$ but, since the spacetime is collapsing, it is possible that they are still inside the region originally (meaning at $t = 0$) included in $S$. If this happens for all the times, the sphere $S$ is called *future marginally trapped*. In the very special case in which all of the light rays (for all the times) lay exactly on the surface originally defined by $S$, then $S$ is called an apparent horizon. A similar situation can be described in a more general setting and it turns out that $\theta_+ := -\mathrm{tr}_g \mathrm{II} + \mathrm{II}(\nu, \nu) + (n-1)H$ is the expansion of the vector field generated by the light rays orthogonal to $S$. Therefore any hypersurface $S$ of $M$, such as $\partial M$, is marginally trapped (in the sense heuristically described before) if $\theta_+ \leq 0$. The outermost of the marginally trapped hypersurfaces satisfies $\theta_+ = 0$ and this motivates the definition of apparent horizon as described at the beginning of this section.

*Example* 4.1. We want to characterize the apparent horizons for solutions of the constraint equations in the form $(M, g, \lambda g)$. So we replace II by $\lambda g$ in $\theta_+ := -\mathrm{tr}_g \mathrm{II} + \mathrm{II}(\nu, \nu) + (n-1)H$, then the condition $\theta_+ = 0$ for an being apparent horizon becomes $H = \lambda$. In this special case an apparent horizon is equivalent to a boundary with mean curvature equal to $\lambda$, with respect to the out-warding unit vector field.

### 4.2.2 Application to our problem

Assume that you are looking for a solution $(M, g, \mathrm{II})$ of the (vacuum) Einstein's constraint equations

$$\begin{cases} \mathrm{R}_g + (\mathrm{tr}_g \mathrm{II})^2 - |\mathrm{II}|_g^2 = 2\Lambda \\ \mathrm{div}_g \mathrm{II} = d(\mathrm{tr}_g \mathrm{II}) \end{cases}$$


(1)
(2)
t

such that the embedding of $M$ in its associated spacetime is totally umbilical. Namely, we look for a solution of the form $(M, g, \lambda g)$ for some $\lambda \in \mathbb{R}$. Then the constraint equations reduce to the single scalar equation

$$\mathrm{R}_g + n^2\lambda^2 - n\lambda^2 = 2\Lambda$$

since the last constraint equation is trivially satisfied. We recall that $n$ is the dimension of $M$. We just proved that the totally umbilical solutions $(M, g, \lambda g)$ of the Einstein's constraint equations have constant scalar curvature, so we set $\mathrm{R}_g = kn(n-1)$ for some $k \in \mathbb{R}$, and that the unique condition to be satisfied is

$$\lambda^2 = -k + \frac{2\Lambda}{n(n-1)}, \tag{4.6}$$

relating the constant $\lambda$ defining the second fundamental form, the cosmological constant $\Lambda$ and the constant $k$ defining the scalar curvature of $g$. If $M$ has a boundary, we observed that it is an apparent horizon if and only if it has mean curvature $H = \lambda$. In this text we worked with manifolds with constant negative scalar curvature $\mathrm{R} = -n(n-1)$, which is $k = -1$. From the observation above we infer the following result.

**Theorem 4.1.** *Let $(M, g)$ be Riemannian manifold consisting of a compact core and finitely many cusps, with dimension $n \geq 2$, and let $g_\varepsilon$ be the metrics provided by Theorem 3.1. If the cosmological constant $\Lambda$ verifies*

$$-\frac{n(n-1)}{2} \leq \Lambda < \frac{n^2(n-1)(n-2)}{2} \tag{4.7}$$

*and $\lambda \in [0, n-1)$ is defined by*

$$\lambda := \sqrt{1 + \frac{2\Lambda}{n(n-1)}}, \tag{4.8}$$

*then the Riemannian manifold with boundary $\hat{M}_\varepsilon$ obtained truncating each end of $(M, g_\varepsilon)$ along the unique hypersurfaces with mean curvature equal to $\lambda \in [0, n-1)$ (given by Theorem 3.1) provides a solution $(\hat{M}_\varepsilon, g_\varepsilon, \lambda g_\varepsilon)$ of the ECE with apparent horizons. These solutions with boundary converge in the Gromov-Hausdorff sense to $(M, g, \lambda g)$ as $\varepsilon \to 0$, which is a solution of the ECE (without boundary).*

## 4.3 The Riemannian Penrose inequality

### 4.3.1 Introduction

Although the ends of the asymptotically hyperbolic manifolds we considered (in dimension $n = 3$) are diffeomorphic to $(r_0, +\infty) \times \mathbb{T}^2$, the Riemannian

Penrose inequality is usually stated for manifolds whose ends are diffeomorphic to $(r_0, +\infty) \times S^2$. Let us begin with this case. Assume that $(M, h)$ is a 3-dimensional asymptotically hyperbolic Riemannian manifold whose ends are diffeomorphic to semicylinders with a spherical base. If $\mathrm{R}_h \geq -6$ then it is possible to define a mass $m > 0$ for $(M, h)$ in virtue of the positive mass theorem in the asymptotically hyperbolic case. If $M$ has a non-empty boundary which is a minimal hypersurface (assume outermost and connected for simplicity), then it is conjectured (*Penrose inequality conjecture*) that

$$\left(\frac{\mathrm{Area}_h(\partial M)}{16\pi}\right)^{1/2} + 4\left(\frac{\mathrm{Area}_h(\partial M)}{16\pi}\right)^{3/2} \leq m.$$

Until now there are not counterexamples, and in many examples the Penrose inequality also holds with rigidity (namely it is an equality in those models and holds strictly in their perturbations). The validity of this conjecture will have strong implications about the cosmic censorship hypothesis.

When one consider ends diffeomorphic to semicylinders with a compact surface $\Sigma$ as base, instead then $S^2$, then the Penrose inequality conjecture becomes

$$(1-\gamma)\left(\frac{\mathrm{Area}_h(\partial M)}{16\pi}\right)^{1/2} + 4\left(\frac{\mathrm{Area}_h(\partial M)}{16\pi}\right)^{3/2} \leq m,$$

where $\gamma$ is the genus of $\Sigma$ (see [24]). Notice that the constant $16\pi$ assumes less significance in the generic case (when $\Sigma$ is not $S^2$) and on the other hand the mass is usually defined up to rescale. A common approach to prove the validity of the Penrose inequality involves the existence of a CMC foliation $\{\Sigma_t\}_t$ of the ends (for simplicity, a single end). This should not surprising since the mass can be computed with this approach, as done by [46]. We adopted this approach, following [24], [5] and others, in the next subsection.

### 4.3.2 Application to our problem

**Theorem 4.2.** *Let $(M, g)$ be Riemannian manifold consisting of a compact core and finitely many cusps, with dimension $n = 3$, and let $g_\varepsilon$ be the metrics provided by Theorem 3.1. This implies that every end of $(M, g)$ is isometric to $((s_0, +\infty) \times \mathbb{T}^2, ds^2 + \mathrm{e}^{-2s} g_{\mathbb{T}^2})$ for some $s_0 \in \mathbb{R}$, where $(\mathbb{T}^2, g_{\mathbb{T}^2})$ is a flat torus. Let $\{\Sigma_t\}_{t>0}$ be the CMC foliation near the infinity of one end of $(M, g_\varepsilon)$ (given by Theorem 3.1) and set*

$$\sigma(t) := \frac{\sqrt{\mathrm{Area}_{g_\varepsilon}(\Sigma_t)}}{4\left(\mathrm{Area}_{g_{\mathbb{T}^2}}(\mathbb{T}^2)\right)^{3/2}} \int_{\Sigma_t} \left(R_t - \frac{1}{2}H_t^2 + 2\right) d\Sigma_t$$

*where $R_t$ and $d\Sigma_t$ denotes respectively the scalar curvature and the volume form induced by $g_\varepsilon$ on $\Sigma_t$, and $H_t$ is the mean curvature of the embedding of*

*$\Sigma_t$ in $(M, g_\varepsilon)$ computed with respect to the unit normal vector pointing toward the infinity of the end. Then it holds the Riemannian Penrose inequality $\sigma(t) < 2\varepsilon^3$ and $\sigma(t) \to 2\varepsilon^3$ as $t \to +\infty$.*

It follows the proof of this theorem.

*REMARK* 4.1 (Comparison with the AdS-case). In [5] Ambrozio faced a problem similar to Theorem 3.1-(4). Precisely he proved that there is a weakly stable CMC foliation for metrics that are suitable perturbation of the 3-dimensional Schwarzschild anti-de Sitter space of mass $\mu > 0$. This last metric can be written as

$$g = dt^2 + \left(\sinh(t)^2 + \frac{2\mu}{3\sinh(t)} + O\left(\mathrm{e}^{-5t}\right)\right) g_{S^2},$$

where $g_{S^2}$ is the round metric of the sphere and $t > 0$. The analogy with our case is that the metric $h_\varepsilon$, with the coordinate change $t = r + \log\varepsilon$, can be written as

$$h_\varepsilon = dt^2 + \left(\mathrm{e}^{2t} + \frac{4\varepsilon^n}{n}\mathrm{e}^{-(n-2)t} + O\left(\mathrm{e}^{-2(n-1)t}\right)\right)\bar{g}.$$

The expression of $h_\varepsilon$ with $n = 3$ is really comparable with the Schwarzschild anti-de Sitter space of mass $\mu = 2\varepsilon^3$, the main difference is that the compact cross-sections in our case turn out to be tori and not spheres (they are compact surfaces carrying a flat metric). In [5] Ambrozio also proved that a Penrose inequality held for his considered class of manifolds. Due to the analogies with his work, we are able to prove Theorem 4.2 in the following.

In dimension $n = 3$, which is the most interesting case in GR, each end of $(M, h_\varepsilon)$ is isometric to a semicylinder $(\log\varepsilon + 2, +\infty) \times \mathbb{T}^2$ and

$$h_\varepsilon = dr^2 + \varepsilon^2\left(2\cosh(3r/2)\right)^{4/3} g_{\mathbb{T}^2},$$

where the base $(\mathbb{T}^2, g_{\mathbb{T}^2})$ of the cylinder is a flat torus. We recall that this end presents a CMC foliation given by $\{r = \rho\}$ as $\rho$ varies in $(\log\varepsilon + 2, +\infty)$ and the leaf $S(\rho) := \{r = \rho\}$ has constant mean curvature equal to $H_\rho := 2\tanh(3\rho/2)$ computed with respect to the normal vector pointing toward the infinity. As a consequence of this fact and of the Gauss-Bonnet formula, the quantities

$$\begin{aligned}\sigma_{h_\varepsilon}(\rho) &:= \frac{\sqrt{\mathrm{Area}_{h_\varepsilon}(S(\rho))}}{4\left(\mathrm{Area}_{g_{\mathbb{T}^2}}(\mathbb{T}^2)\right)^{3/2}} \int_{S(\rho)} \left(R_{S(\rho)} - \frac{1}{2}H_\rho^2 + 2\right) dS(\rho) \\ &= \frac{1}{2}\left(\frac{\mathrm{Area}_{h_\varepsilon}(S(\rho))}{\mathrm{Area}_{g_{\mathbb{T}^2}}(\mathbb{T}^2)}\right)^{3/2} \frac{1}{\cosh^2(3\rho/2)},\end{aligned} \tag{4.9}$$

where $R_{S(\rho)}$ and $dS(\rho)$ denotes respectively the scalar curvature and the volume form induced by $h_\varepsilon$ on $S(\rho)$, are constantly equal to $2\varepsilon^3$. Here one has to use that $\mathrm{Area}_{h_\varepsilon}(S(\rho)) = \varepsilon^2 \left(2\cosh(3\rho/2)\right)^{4/3} \mathrm{Area}_{g_{\mathbb{T}^2}}(\mathbb{T}^2)$. This value $2\varepsilon^3$ is called *Hawking mass* of $(M, h_\varepsilon)$. Similarly we can consider the metric $g_\varepsilon$ of Theorem (3.1), which has been built as $g_\varepsilon = \mathrm{e}^{2u_\varepsilon} h_\varepsilon$ imposing the Yamabe equation $R_{g_\varepsilon} = -6$ on $M$. In Theorem 3.15 we proved the existence of a weakly stable CMC foliation $\{S(\varepsilon, \rho)\}$ for the ends of $(M, g_\varepsilon)$, defined for $\rho \geq \log \varepsilon + 3$, such that each leaf $S(\varepsilon, \rho) = \{r = \rho + \psi(\cdot\,; \varepsilon, \rho)\}$ has constant mean curvature equal to $H_\rho = 2\tanh(3\rho/2)$ computed with respect to the normal vector pointing toward the infinity. We introduce as before the quantity

$$\begin{aligned} \sigma_{g_\varepsilon}(\rho) &:= \frac{\sqrt{\mathrm{Area}_{g_\varepsilon}(S(\varepsilon,\rho))}}{4\left(\mathrm{Area}_{g_{\mathbb{T}^2}}(\mathbb{T}^2)\right)^{3/2}} \int_{S(\varepsilon,\rho)} \left( R_{S(\varepsilon,\rho)} - \frac{1}{2} H_\rho^2 + 2 \right) dS(\varepsilon,\rho) \\ &= \frac{1}{2} \left( \frac{\mathrm{Area}_{g_\varepsilon}(S(\varepsilon,\rho))}{\mathrm{Area}_{g_{\mathbb{T}^2}}(\mathbb{T}^2)} \right)^{3/2} \frac{1}{\cosh^2(3\rho/2)} \end{aligned} \tag{4.10}$$

where $R_{S(\varepsilon,\rho)}$ and $dS(\varepsilon, \rho)$ denotes respectively the scalar curvature and the volume form induced by $g_\varepsilon$ on $S(\varepsilon, \rho)$. In this case $\sigma_{g_\varepsilon}(\rho)$ is not constant in $\rho$ and the Hawking mass of $(M, g_\varepsilon)$ is defined as

$$m_{\mathrm{Haw}}(M, g_\varepsilon) := \lim_{\rho \to +\infty} \sigma_{g_\varepsilon}(\rho).$$

In our setting above the *Riemannian Penrose inequality* asserts that

$$\sigma_{g_\varepsilon}(\rho) < 2\varepsilon^3 \tag{4.11}$$

for every $\rho \geq 0$. In particular when $\rho = 0$ it implies $\mathrm{Area}_{g_\varepsilon}(S(\varepsilon, 0)) < \mathrm{Area}_{h_\varepsilon}(S(0))$, namely that the area of the minimal leaf in the end relative to $g_\varepsilon$ is smaller than the area of the minimal leaf of the end relative to $h_\varepsilon$, improving Theorem 3.15-($iv$). The proof of (4.11) will easily follow from this two results:

- It holds $m_{\mathrm{Haw}}(M, g_\varepsilon) = 2\varepsilon^3$;
- It holds $\sigma'_{g_\varepsilon}(\rho) > 0$ for $\rho > 0$.

We recall that the function $u_\varepsilon$ defining $g_\varepsilon$ satisfies (3.39)-(3.40), however - in order to prove the two points above - it will be sufficient to observe that from (3.40) and (3.41) one has

$$\max_{S(\varepsilon,\rho)} |\mathrm{e}^{3u_\varepsilon} - 1| \leq C\varepsilon^2 \mathrm{e}^{-\delta\rho} \tag{4.12}$$

for large $\rho$. Differently from the construction of the CMC foliation on $(M, g_\varepsilon)$, in this section we will make use of $R_{g_\varepsilon} = -6$.

### 4.3.3 Computing the Hawking mass

In this subsection we prove that $m_{\mathrm{Haw}}(M, g_\varepsilon) = 2\varepsilon^3$. By definition of Hawking mass, this is equivalent to prove that

$$\lim_{\rho\to+\infty} \sigma_{g_\varepsilon}(\rho)^{2/3} - \sigma_{h_\varepsilon}(\rho)^{2/3} = 0.$$

Since $\cosh(3\rho/2)^{4/3}$ is proportional to $\mathrm{Area}_{h_\varepsilon}(S(\rho))$, then the problem is equivalent to prove

$$\lim_{\rho\to+\infty} \frac{\mathrm{Area}_{g_\varepsilon}(S(\varepsilon,\rho))}{\mathrm{Area}_{h_\varepsilon}(S(\rho))} = 1. \tag{4.13}$$

By (3.63) and (4.12) for large $\rho$ it holds

$$(1-C\varepsilon^2 \mathrm{e}^{-\delta\rho})\mathrm{Area}_{h_\varepsilon}(S(\varepsilon,\rho)) \le \mathrm{Area}_{g_\varepsilon}(S(\varepsilon,\rho)) \le (1+C\varepsilon^2\mathrm{e}^{-\delta\rho})\mathrm{Area}_{h_\varepsilon}(S(\varepsilon,\rho)),$$

whereas from (3.65) one can easily notice that

$$\mathrm{Area}_{h_\varepsilon}(S(\rho-\|\psi(\cdot\,;\varepsilon,\rho)\|_{\mathscr{C}^0(\mathbb{T}^2)})) \le \mathrm{Area}_{h_\varepsilon}(S(\varepsilon,\rho)) \le \mathrm{Area}_{h_\varepsilon}(S(\rho+\|\psi(\cdot\,;\varepsilon,\rho)\|_{\mathscr{C}^0(\mathbb{T}^2)}))$$

for large $\rho$. Since $1 \pm C\varepsilon^2\mathrm{e}^{-\delta\rho}$ tends to 1 as $\rho \to +\infty$ and since it holds (3.79), then we get the thesis from the bounds above.

### 4.3.4 Monotonicity of $\sigma_{g_\varepsilon}$

In this subsection we prove that $\sigma'_{g_\varepsilon}(\rho) > 0$ for $\rho > 0$. It will be important to use the facts that $R_{g_\varepsilon} = -6$ and that $\{S(\varepsilon,\rho)\}_\rho$ is weakly stable. Consider the function $F\colon (\log\varepsilon + 3, +\infty) \times \Sigma \to \mathbb{R}\times\Sigma$ defined by $F(\rho,\theta) = (\rho + \psi(\theta;\varepsilon,\rho),\theta)$. This function parametrizes the CMC foliation in the sense that $F(\{\rho\}\times\Sigma) = S(\varepsilon,\rho)$. Let $\varphi_\rho$ be the function defined on $S(\varepsilon,\rho)$ by $\partial_\rho F = \varphi_\rho \nu_\rho$, where $\nu_\rho$ is the unit vector perpendicular to $S(\varepsilon,\rho)$ and directed to $r = +\infty$. By construction $\varphi > 0$. We claim that

$$\sigma'_{g_\varepsilon}(\rho) = \frac{\sqrt{\mathrm{Area}_{g_\varepsilon}(S(\varepsilon,\rho))}}{4\left(\mathrm{Area}_{g_{\mathbb{T}^2}}(\mathbb{T}^2)\right)^{3/2}} \int_{S(\varepsilon,\rho)} \varphi_\rho \left(-\Delta_{S(\varepsilon,\rho)} H_\rho + Q_\rho H_\rho\right) dS(\varepsilon,\rho), \tag{4.14}$$

where $\Delta_{S(\varepsilon,\rho)}$ denotes the Laplacian induced by $g_\varepsilon$ on $S(\varepsilon,\rho)$ and

$$Q_\rho = \frac{1}{2}R_{S(\varepsilon,\rho)} - \frac{1}{2}|\mathrm{II}|^2 + \frac{1}{4}H_\rho^2.$$

Here $|\mathrm{II}|^2$ denotes the square norm of the second fundamental form of the embedding $S(\varepsilon,\rho) \subset \mathbb{R}\times\Sigma$ with respect to the metric $g_\varepsilon$. To prove (4.14) it is sufficient to notice that by (4.10) the derivative of $4\left(\mathrm{Area}_{g_{\mathbb{T}^2}}(\mathbb{T}^2)\right)^{3/2}\sigma_{g_\varepsilon}(\rho)$ with respect to $\rho$ is

$$\sqrt{\mathrm{Area}_{g_\varepsilon}(S(\varepsilon,\rho))} \int_{S(\varepsilon,\rho)} \left(\frac{3}{2}\left(2 - \frac{1}{2}H_\rho^2\right)\frac{d}{d\rho}dS(\varepsilon,\rho) - H_\rho \frac{d}{d\rho}H_\rho dS(\varepsilon,\rho)\right).$$

In [45] Huisken and Polden proved that

$$\frac{d}{d\rho}dS(\varepsilon,\rho) = -\varphi_\rho H_\rho dS(\varepsilon,\rho) \quad \text{and} \quad \frac{d}{d\rho}H_\rho = L_{S(\varepsilon,\rho)}(\varphi_\rho), \tag{4.15}$$

where $L_{S(\varepsilon,\rho)} = \Delta_{S(\varepsilon,\rho)} + |\mathrm{II}|^2 + \mathrm{Ric}_{g_\varepsilon}(\nu_\rho,\nu_\rho)$ is the Jacobi operator of $S(\varepsilon,\rho)$. Thus the derivative of $4\left(\mathrm{Area}_{g_{\mathbb{T}^2}}(\mathbb{T}^2)\right)^{3/2}\sigma_{g_\varepsilon}(\rho)$ with respect to $\rho$ is equal to

$$\sqrt{\mathrm{Area}_{g_\varepsilon}(S(\varepsilon,\rho))}\int_{S(\varepsilon,\rho)}\left(-\frac{3}{2}\left(2-\frac{1}{2}H_\rho^2\right)\varphi_\rho H_\rho - H_\rho L_{S(\varepsilon,\rho)}(\varphi_\rho)\right)dS(\varepsilon,\rho).$$

Integrating by parts and using the Gauss equation

$$-6 = R_{g_\varepsilon} = R_{S(\varepsilon,\rho)} + 2\mathrm{Ric}_{g_\varepsilon}(\nu_\rho,\nu_\rho) + |\mathrm{II}|^2 - H_\rho^2,$$

we get (4.14). Since $H_\rho$ is constant on $S(\varepsilon,\rho)$, then from (4.14) we get

$$\begin{aligned}
4\left(\mathrm{Area}_{g_{\mathbb{T}^2}}(\mathbb{T}^2)\right)^{3/2}\sigma'_{g_\varepsilon}(\rho) &= \sqrt{\mathrm{Area}_{g_\varepsilon}(S(\varepsilon,\rho))}H_\rho\int_{S(\varepsilon,\rho)}Q_\rho\varphi_\rho\,dS(\varepsilon,\rho)\\
&= \sqrt{\mathrm{Area}_{g_\varepsilon}(S(\varepsilon,\rho))}H_\rho\int_{S(\varepsilon,\rho)}(\Delta_{S(\varepsilon,\rho)}+Q_\rho)(\varphi_\rho-\bar\varphi_\rho)\,dS(\varepsilon,\rho)\\
&+ \sqrt{\mathrm{Area}_{g_\varepsilon}(S(\varepsilon,\rho))}H_\rho\bar\varphi_\rho\int_{S(\varepsilon,\rho)}Q_\rho\,dS(\varepsilon,\rho),
\end{aligned}$$

where we set $\bar\varphi_\rho := \int_{S(\varepsilon,\rho)}\varphi_\rho\,dS(\varepsilon,\rho)$. Now notice that $H_\rho > 0$ for $\rho > 0$ and by Gauss-Bonnet and Cauchy-Schwartz $\int_{S(\varepsilon,\rho)}Q_\rho dS(\varepsilon,\rho) \geq 0$. Moreover $\Delta_{S(\varepsilon,\rho)} + Q_\rho$ and $L_{S(\varepsilon,\rho)}$ differ by a constant on $S(\varepsilon,\rho)$. It follows that

$$4\left(\mathrm{Area}_{g_{\mathbb{T}^2}}(\mathbb{T}^2)\right)^{3/2}\sigma'_{g_\varepsilon}(\rho) \geq \sqrt{\mathrm{Area}_{g_\varepsilon}(S(\varepsilon,\rho))}H_\rho\int_{S(\varepsilon,\rho)}L_{S(\varepsilon,\rho)}(\varphi_\rho-\bar\varphi_\rho)\,dS(\varepsilon,\rho).$$

Since $H_\rho$ is constant on $S(\varepsilon,\rho)$, then the same holds for $L_{S(\varepsilon,\rho)}(\varphi_\rho) = \partial_\rho H_\rho$. So by weak stability of $S(\varepsilon,\rho)$ we have

$$\begin{aligned}
0 &\leq -\int_{S(\varepsilon,\rho)}(\varphi_\rho-\bar\varphi_\rho)L_{S(\varepsilon,\rho)}(\varphi_\rho-\bar\varphi_\rho)dS(\varepsilon,\rho)\\
&= \int_{S(\varepsilon,\rho)}(\varphi_\rho-\bar\varphi_\rho)L_{S(\varepsilon,\rho)}(\bar\varphi_\rho)dS(\varepsilon,\rho)\\
&= \bar\varphi_\rho\int_{S(\varepsilon,\rho)}L_{S(\varepsilon,\rho)}(\varphi_\rho-\bar\varphi_\rho)dS(\varepsilon,\rho),
\end{aligned}$$

thus $\sigma'_{g_\varepsilon}(\rho) \geq 0$ for $\rho > 0$. It remains to show that in our setting we can not have $\sigma'_{g_\varepsilon}(\rho) = 0$. In fact, this would imply

$$\int_{S(\varepsilon,\rho)}(\varphi_\rho-\bar\varphi_\rho)L_{S(\varepsilon,\rho)}(\varphi_\rho-\bar\varphi_\rho)dS(\varepsilon,\rho) = \int_{S(\varepsilon,\rho)}Q_\rho dS(\varepsilon,\rho) = 0,$$

namely we deduce that $2|\mathrm{II}|^2 = H_\rho^2$ and that $L_{S(\varepsilon,\rho)}\bar\varphi_\rho$ is constant on $S(\varepsilon,\rho)$. This is equivalent to assert that $S(\varepsilon,\rho)$ is totally umbilical and (by the Gauss equation) has constant scalar curvature equal to zero. Therefore $g_\varepsilon$ should be isometric to $h_\varepsilon$, which is not possible since $u_\varepsilon \neq 0$.

## 4.4 Generalizations

In this section we give some ideas on possible generalizations of Theorem 3.1, without proof.

### 4.4.1 Preserving some cusps

We think that it is possible to give a slightly stronger version of Theorem 3.1. Precisely, once considered $(M, g)$ as in the hypothesis of the theorem, we would like to build metrics $(g_\varepsilon)$ which replace with funnel-like ends only finitely-many (chosen) cusps. So the difference is that we want to preserve some cusps, instead of replacing all of them by funnel-like ends. The existence of such a metric should follow from Theorem 3.1 itself. It seems to be sufficient to initially replace all the cusps as in the original statement, and then letting to 0 the parameters relative to the cusp we want to preserve.

### 4.4.2 Gluing cusps

With the same argument of this thesis, we think that it is possible to glue two different manifolds whose ends are cusps through cusps, as it is known to be possible for hyperbolic surfaces. This should be done with arguments similar to the ones of this text, using a “piece of funnel ”to connect the cusps. More generally, this result should be generalized to the gluing of several couples of cusps.

# Chapter 5

# Appendix

This appendix collects some of the classic results in Riemannian Geometry, focusing mainly in those arguments which concern the manifold's curvatures, and other results - such as Lemma 5.4 - which are known but not so common in literature and that have been used in the thesis. We also use this appendix to fix the notation, which is however pretty standard.

## 5.1 Remarks about Riemannian Geometry

### Curvatures

In dimension $n = 2$ there is essentially a unique definition of intrinsic curvature for a metric defined on a surface: the Gaussian curvature (cfr. Theorema Egregium, 1827). In higher dimension this concept of curvature, which is a scalar function defined on the surface, can be replaced by a well-known curvature tensor, introduced in the following.

Let $(M, g)$ be a smooth Riemannian manifold of real dimension $n \geq 2$. We recall that this means that $M$ is equipped with an atlas of smooth charts and $g$ is a symmetric $(0, 2)$-tensor, which can be written in local coordinates $(x^1, x^2, \dots, x^n)$ as

$$g = g_{ij}\, dx^i \otimes dx^j,$$

with the property that $(g_{ij}(p))$ is a positive definite matrix at each point $p \in M$. We use to denote by $\nabla_g$, or simply by $\nabla$, the *Levi-Civita connection* of $g$ extended on tensors, which is the unique connection on $M$ such that:

*Torsion-free*: for every vector fields $X$,$Y$ on $M$ one has $\nabla_X Y - \nabla_Y X = [X, Y]$, with $[X, Y]$ denoting the Lie brackets of $X$ and $Y$;

*Preserving $g$*: for every vector fields $X$, $Y$, $Z$ on $M$ one has $\nabla_X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$.

In local coordinates $(x^1, x^2, \dots, x^n)$ the Levi-Civita connection is determined by its *Christoffel symbols*

$$\Gamma_{ij}^k = \Gamma_{ji}^k = \frac{1}{2} g^{kl} \left( \partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \right),$$

meaning that $\nabla_{\partial_i} \partial_j = \Gamma_{ij}^k \partial_k$, where we set $\partial_i = \frac{\partial}{\partial x^i}$ for $i = 1, \dots, n$. More generally, if $K$ is a $(r, s)$-tensor, then $\nabla K$ is the $(r, s+1)$-tensor defined in local coordinates by

$$\begin{aligned}\nabla_k K_{i_1,\dots,i_s}^{j_1,\dots,j_r} = \partial_k K_{i_1,\dots,i_s}^{j_1,\dots,j_r} &+ \sum_{\rho=1}^{r} \Gamma_{kl}^{j_\rho} K_{i_1,\dots,i_s}^{j_1,\dots,j_{\rho-1},l,j_{\rho+1},\dots,j_r} \\ &- \sum_{\sigma=1}^{s} \Gamma_{ki_\sigma}^{l} K_{i_1,\dots,i_{\sigma-1},l,i_{\sigma+1},\dots,i_s}^{j_1,\dots,j_r}.\end{aligned}$$

For instance $\nabla_k g_{ij} = \partial_k g_{ij} - \Gamma_{ik}^l g_{lj} - \Gamma_{kj}^l g_{il}$ and therefore the fact that the Levi-Civita connection preserves $g$ can be equivalent stated as $\nabla g = 0$. The ***curvature tensor*** (or *Riemann tensor*) $R$ of $(M, g)$ is the $(1, 3)$-tensor defined by

$$R(X, Y)Z := [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z,$$

for every vector field $X, Y$ and $Z$ defined on $M$. Due to the presence of a metric $g$ on $M$, the curvature tensor can also be introduced as the $(0, 4)$-tensor

$$R(X, Y, Z, W) = g\left(R(X, Y)Z, W\right),$$

for every vector field $X, Y, Z$ and $W$ defined on $M$. The main algebraic properties of the curvature tensor are listed below.

Let $X, Y, Z, V, W \in C^\infty(TM)$ be vector fields, then:

(1) $R(X, Y, Z, W) = -R(Y, X, Z, W) = -R(X, Y, W, Z)$;

(2) $R(X, Y, Z, W) = R(Z, W, X, Y)$;

(3) $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$;

(4) $\nabla R(X, Y, Z, V, W) + \nabla R(Y, Z, X, V, W) + \nabla R(Z, X, Y, V, W) = 0$.

Properties (3) and (4) are usually called *first and second Bianchi identity* respectively.

In local coordinates $(x^1, x^2, \dots, x^n)$ the expression of the curvature tensor is

$$R(\partial_i, \partial_j)\partial_k = R_{kij}^l \partial_l = \left( \partial_i \Gamma_{jk}^l - \partial_j \Gamma_{ik}^l + \Gamma_{jk}^m \Gamma_{im}^l - \Gamma_{ik}^m \Gamma_{jm}^l \right) \partial_l,$$

and $R(\partial_i, \partial_j, \partial_k, \partial_l) = g_{ml} R_{kij}^m$. The curvature tensor in some sense measures the deviation of $g$ from the flat euclidean metric, therefore the origin of its name, but why should this definition be the analogue of the Gaussian curvature in high dimension? The answer descends from a second (equivalent!) way to introduce the notion of curvature for $(M, g)$, which clearly extends the Gaussian curvature in every dimension. Let $p \in M$ and consider a 2-plane $\pi \subset TM$. The ***sectional curvature*** of $\pi$ is defined to be the Gaussian curvature of the surface $\exp_p(\pi)$ in $p$ and is given by

$$\sec_p(\pi) := \frac{R(w, v, v, w)}{g(v, v)g(w, w) - g(v, w)^2},$$

for any two vectors $v, w \in T_pM$ generating $\pi$ (the definition does not depend on the choice of $v$ and $w$). The sectional curvature contains the same information of $R$, in fact it is possible to compute the sectional curvature from the curvature tensor and viceversa.

There are other curvature tensors which can be derived from $R$. Tracing the curvature tensor, we get a symmetric $(0, 2)-$tensor called *Ricci tensor*. In local coordinates it is given by

$$\mathrm{Ric}_{kj} = R^l_{klj} = \partial_l \Gamma^l_{jk} - \partial_j \Gamma^l_{lk} + \Gamma^m_{jk}\Gamma^l_{lm} - \Gamma^m_{lk}\Gamma^l_{jm}.$$

Taking the trace again we get the *scalar curvature*, which is the function defined on $M$ by $\mathrm{R}_g := g^{kj}\mathrm{Ric}_{kj}$. If $n > 2$ then the scalar curvature obviously contains less information about $(M, g)$ than the Ricci curvature, which in turns contains less information than the sectional curvature. If $n = 2$ this is not true since it holds $\mathrm{R}_g(p) = 2\sec_p(T_pM)$ for every $p \in M$, namely the scalar curvature is twice the Gaussian curvature. All the concepts of curvature above are *metrical invariants*, namely they are preserved under isometries. Given two equidimensional Riemannian manifolds $(M, g)$ and $(M', g')$, a *local isometry* is a smooth map $F\colon M \to M'$ such that $F^*g' = g$. An *isometry* is a local isometry which is also a diffeomorphism.

## Geodesics

A Riemannian metric $g$ on $M$ induces a structure of metric space. In fact given a smooth curve $\gamma\colon [a, b] \subset \mathbb{R} \to M$ one can define

$$\mathrm{Length}_g(\gamma) := \int_a^b \sqrt{g\left(\dot\gamma(t), \dot\gamma(t)\right)}dt$$

and for $p, q \in M$. Then one defines $d(p, q)$ as the infimum of $\mathrm{Length}_g(\gamma)$ among the smooth curves $\gamma$ connecting $p$ to $q$. It turns out that $d$ is a distance on $M$ and the topology of $M$ as differential manifold coincides with the topology induced by $d$. In general this infimum is not a minimum, meaning that for two general points $p, q \in M$ may not exist a curve $\gamma$ connecting $p$ and $q$ with length $d(p, q)$. However such a curve exists and is unique if $p$ and $q$ are sufficiently close and, in any case, when a smooth curve

$\gamma$ connects $p$ an $q$ and has length equal to $d(p,q)$ then, up to parametrize $\gamma$ by arc-length, it holds $\nabla\gamma' = 0$. This is a consequence of the first variation formula. In local coordinates $\nabla\gamma' = 0$ is equivalent to the second-order system of ODE on $\gamma$ given by

$$\ddot{\gamma}^k(t) + \Gamma^k_{ij}(\gamma(t))\dot{\gamma}^i(t)\dot{\gamma}^j(t) = 0, \quad \forall k = 1, \ldots, n.$$

A curve $\gamma$ is called *geodesic* if it holds $\nabla\gamma' = 0$ at each point $\gamma(t)$. Equivalently, a geodesic is a curve $\gamma$ such that (locally) the distance between two points belonging to its support is realized by the curve $\gamma$ itself. A geodesic $\gamma$ is necessarily parametrized by arc-length, indeed $\frac{d}{dt}g(\gamma',\gamma') = 2g(\gamma',\nabla\gamma') = 0$ since the Levi-Civita connection preserves $g$. By the fundamental theorem of local existence and uniqueness for ODE applied to $\nabla\gamma' = 0$, for every $p \in M$ it is well-defined a map

$$\exp_p\colon U_p \to M, \quad \exp_p(v) = \gamma_v(1) \quad \forall v \in U_p$$

where $U_p$ is a starred neighbourhood of $0 \in T_pM$ and $\gamma_v$ is the unique geodesic satisfying $\gamma_v(0) = p$ and $\gamma_v'(0) = v$. More generally, one can notice that $\exp_p(tv) = \gamma_v(t)$, so that $\exp_p(tv)$ is in particular a geodesic for every $v \in U_p$. It can be proved that the *exponential map* $\exp_p$ is smooth and a local diffeomorphism in $0 \in T_pM$ since it holds $d(\exp_p)_0 = \mathrm{id}_{T_pM}$. The *injectivity radius* of $p$ is defined as

$$\mathrm{injrad}_p(M,g) := \sup\{\rho > 0 \mid \exp_p|_{B_\rho(0)} \text{is a diffeomorphism onto its image}\},$$

where $B_\rho(0)$ denotes the ball of $T_pM$ of radius $\rho$ (with respect to $g_p$) centred at $0 \in T_pM$. The injectivity radius is always positive for each $p$, however it may happen that there is not a lower bound for $\mathrm{injrad}_p(M,g)$ as $p$ varies in $M$. Notice that if $d(p,q) < \mathrm{injrad}_p(M,g)$, then there is a unique minimizing geodesic from $p$ to $q$ and is of the type $\exp_p(tv)$ for some $v \in T_pM$. A further fundamental property of the exponential map is described in the Gauss Lemma. It asserts that $\exp_p\colon U_p \to M$ induces a *radial isometry*, namely

$$g(\nabla\exp_p(\partial_r), \nabla\exp_p(v)) = g_p(\partial_r, v), \quad \forall v \in U_p$$

where the function $r$ is defined as the distance from $p$. In particular the image via the exponential map of a ball of $T_pM$ centred at $0 \in T_pM$ of radius $R < \mathrm{injrad}_p(M,g)$ is precisely the ball centred at $p \in M$ of radius $R$. In general a geodesic curve can not be defined for all times $t \in \mathbb{R}$, however the famous Hopf-Rinow theorem answers to this problem. It states that for every Riemannian manifold $(M,g)$ the following properties are equivalent:

1. $(M,d)$ is complete as metric space;

2. all the geodesics are defined for every time;

3. for every $p \in M$ the exponential map $\exp_p$ is defined in the whole $T_pM$;

4. there exists $p \in M$ such that all the geodesics passing through $p$ are defined for every time;

5. every closed and bounded subset of $M$ is compact.

Moreover, each of the equivalent points above implies that there is a minimizing geodesic connecting any couple of points of $M$. In particular, if $M$ is compact then all the points above hold.

### Special metrics

One of the most important applications of the Riemannian Geometry should be to find the "best "metric $g$ on a given $n$-dimensional manifold $M$ encoding the most topological aspects of $M$. Such a result will be a great advantage for classifying class of manifolds. For example it is well-known that the unique topological invariant for an oriented compact surface $S$ is its Euler characteristic $\chi(S) = 2 - 2\gamma$, where $\gamma$ is the genus of $S$. On the other hand for every metric $g$ on $S$ it holds the Gauss-Bonnet formula

$$\int_S \mathrm{R}_g dS = 4\pi\chi(S).$$

This means that if we know that some oriented compact surface admits a metric with constant curvature, then we can deduce its Euler characteristic and therefore its topological structure. In higher dimension, even in the compact case, the problem of finding the best metric on a given manifold is still open and very rich from the point of view of the research. Special classes of metrics are given in the following definition.

**Definition 5.1.** A Riemannian metric $g$ on $M$ is said to have ***constant curvature*** $k \in \mathbb{R}$ if $\sec_p(\pi) = k$ for any $p \in M$ and $\pi \subset T_pM$ 2-plane. It is said to be ***E*instein** if $\mathrm{Ric} = k(n-1)g$. It is said to have **constant scalar curvature** if $\mathrm{R}_g = kn(n-1)$.

Any Riemannian manifold with constant curvature is Einstein and the converse in general is false in dimension $n > 3$ (ex. [75], pag.38). Any Einstein manifold has constant scalar curvature but the converse is in general false in dimension $n > 2$ (ex. [75], chp. 3). The constant $k \in \mathbb{R}$ appearing in the definition above is preserved in all the previous implications.

As a first approach, given a Riemannian manifold $M$, one can try to find a metric with constant curvature. However, such a metric may not exist. In fact, the only manifolds admitting such a metric are quotients of the euclidean space, the hyperbolic space or the sphere by an isometry action (Uniformization Theorem). On the other hand if one relax this requirement and looks for metrics with constant scalar curvature, then they are too much to encode topological properties (cfr. [21]). Einstein metrics seems to be a good compromise. Moreover the relevance of studying Einstein's metrics emerges also from a physical interest. Indeed the spacetime we live is conjectured to be an Einstein manifold of signature $(-,+,+,+)$.

## 5.2 Teichmüller Theory

In dimension $n \geq 3$ there exists at most one complete hyperbolic metric on a given manifold (Mostow rigidity, 1968), determined by its fundamental group. In dimension $n = 2$ this is not true and a given surface can admit several non-isometric hyperbolic structures. The Teichmüller theory studies these possibilities (we suggest [59] [83] for a good introduction). The typical setting for this theory is the case of oriented surfaces of finite-type. Probably, its most famous result looks like that: the space of complete hyperbolic metrics with finite volume and geodesic boundary, up to isometries isotopic to the identity (called *Teichmüller space*), on an oriented compact surface of genus $\gamma$ deprived by $p$ points and by $b$ small open disks (namely on a *surface of finite-type*) is diffeomorphic to $\mathbb{R}^{6\gamma+2p+3b-6}$. A similar result was also adapted to non-orientable surfaces and to complete metrics with infinite volume too. In the next we give some comments about the result above emphasizing those tools and objects which are used in the rest of the thesis.

The first step of the Teichmüller theory consists on building very special hyperbolic surfaces with boundary, called *pair of pants*. This is done gluing particular hexagons in the hyperbolic space. In dimension $n = 2$ the hyperbolic space is isometric to the *Poincaré disk* $(D, g_D)$, where $D = \{(x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 < 1\}$ and

$$g_D = \frac{4}{(1-\rho^2)^2} g_{\mathrm{eucl}}, \quad \rho := \sqrt{x^2 + y^2}.$$

It is well-known that geodesics on Poincaré disks consist on arcs of circumference perpendicular to $\partial D$, including the degenerate case of straight lines through the origin. By the Uniformization Theorem, any complete hyperbolic surface can be seen as the quotient of $(D, g_D)$ with respect to a suitable group of isometries. Therefore one can build examples of complete hyperbolic surfaces using special polygons of $(D, g_D)$. A *right angled hexagon* is defined as a six-sided regular polygon on $D$ whose sides are arcs of geodesics and such that two adjacent sides are perpendicular. It can be shown that for any $a, b, c > 0$ there exist a unique right angled hexagon up to isometries with three not-adjacent sides of length $a$, $b$ and $c$. For $a, b, c > 0$ the ***pair of pants*** with boundary lengths $2a$, $2b$ and $2c$ is defined to be the Riemannian manifold with boundary obtained by taking two copies of a right angled hexagon with with three not-adjacent sides of length $a$, $b$ and $c$ and identifying the respective remaining sides. A pair of pants is actually a hyperbolic

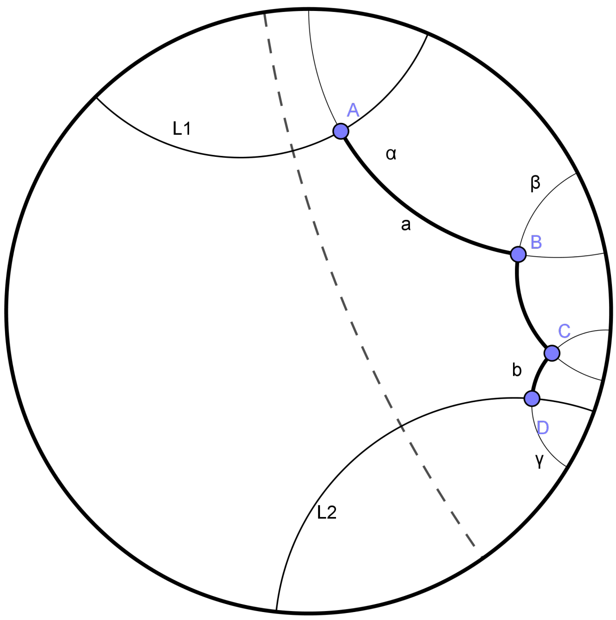



Figure 5.1: Here is how to build the right angled hexagon with three not-adjacent sides of given length $a$, $b$ and $c$ inside the Poincaré disk. (1) Consider any geodesic $\alpha$ and any two points $A$ and $B$ on it with hyperbolic distance equal to $a$. (2) Let $L1$ be the unique geodesic perpendicular to $\alpha$ and passing through $A$, and let $\beta$ be the unique geodesic perpendicular to $\beta$ and passing through $B$. (3) Now let $C \neq B$ be a point of $\beta$. The choice of this point in $\beta$ is a degree of freedom that we initially assume on this construction. Let $\gamma$ be the geodesic perpendicular to $\beta$ and passing through $C$. (4) Let $D$ be the point of $\gamma$ such that the hyperbolic distance between $C$ and $D$ is equal to $b$ and such that $A$ and $D$ belong to the same half-space determined by $\gamma$. (5) Define $L2$ the geodesic perpendicular to $\gamma$ and passing through $C$. (6) If the hyperbolic distance of $C$ from $B$ is large enough, then $L1$ and $L2$ are hyper-parallel. Therefore there exists a unique geodesic $l$ which is perpendicular to both $L1$ and $L2$. We define the intersection points $E = L2 \cap l$ and $F = L1 \cap l$. (7) The hyperbolic distance between $L1$ and $L2$ corresponds to the hyperbolic distance between $E$ and $F$ and depends on our initial choice of $C$ (the degree of freedom). It is always possible to find a unique point $C$ such that this distance is equal to $c \geq 0$. (8) The points $A$, $B$, $C$, $D$, $E$ and $F$ form the required hexagon. This figure was built with GeoGebra.

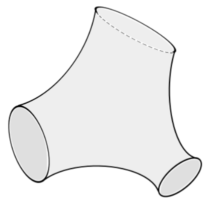


Figure 5.2: A pair of pants is an example of hyperbolic surface with three boundary components and Euler characteristic $-1$. Its volume is equal to $2\pi$.

surface with smooth boundary since the angles of the hexagon measure $\pi/2$. Its three boundary components are geodesics isometric to circumferences of lengths $2a$, $2b$ and $2c$. The name "pair of pants" descends from its shape: it looks like a sphere deprived of three disks. Since the right angled hexagons are essentially unique once fixed the lengths of three non-adjacent sides, then it turns out that (up to isometries) there exists a unique pair of pants with fixed boundary lengths $2a, 2b, 2c > 0$. Actually this result still holds for the degenerate case $a, b, c \geq 0$. In fact one can consider hexagons with a side of degenerate length zero and then build pair of pants where one, two or three boundary components have degenerated to a *cusp*. A cusp in a degenerate pair of pants is always isometric to

$$ds^2 + \mathrm{e}^{-2s} d\theta^2, \quad (s, \theta) \in (s_0, +\infty) \times S^1$$

for some $s_0 > 0$ large enough. Now observe that gluing equivalent (non-degenerate) sides of pair of pants it is possible to build a lot of examples of smooth hyperbolic surfaces, eventually with geodesic boundaries or cusps.

The key point of Teichmüller theory is that this process can be inverted, namely all the complete hyperbolic structures with finite volume can be obtained gluing pairs of pants. We conclude this section with a sketch of the proof for the main result of Teichmüller theory. For simplicity we consider

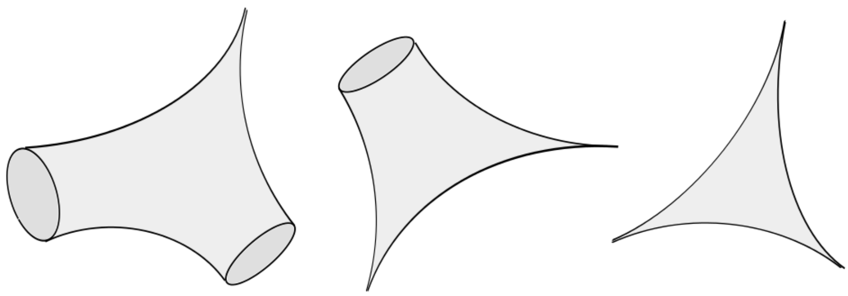


Figure 5.3: Degenerate pair of pants with one, two or three cusps. The last one is the unique complete hyperbolic metric with finite volume on the 3-punctured sphere.

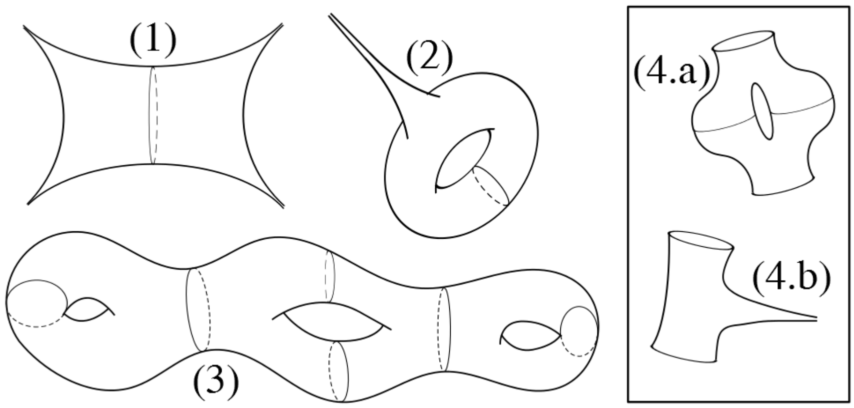



Figure 5.4: (1) It is represented a decomposition of the sphere with 4 holes in two pairs of pants with two cusps each. (2) The torus admits no hyperbolic metrics unless you remove at least one point. Examples of complete hyperbolic metrics of finite volume on the punctured torus are obtained identifying the boundaries of a single pair of pants with one cusp and two sides of the same length. (3) For genus grater than 1 there are examples of compact hyperbolic metrics obtained gluing non-degenerate pairs of pants. (4) Given any surface $S$ of genus $\gamma$ with $n$ punctures obtained gluing pairs of pants, it is possible to obtain a hyperbolic surface with genus $\gamma+1$ and $n$ punctures or a surface with genus $\gamma$ and $n+1$ punctures just entering between two glued pants of $S$ the surface $(4.a)$ or $(4.b)$ respectively, with the proper boundary lengths.

the case without boundary.

Let $S_{\gamma,p}$ be the $p$-punctured surface of genus $\gamma$. By Gauss-Bonnet Theorem if there exist a hyperbolic metric with finite volume on $S_{\gamma,p}$ then it must hold

$$\chi(S_{\gamma,p}) := 2 - 2\gamma - p < 0.$$

So we suppose $2 - 2\gamma - p < 0$. Under this assumption it is possible to show that every complete hyperbolic metric with finite volume on $S_{\gamma,p}$ can be obtained by gluing of pair of pants. Precisely, one need $-\chi(S_{\gamma,p})$ pair of pants and $p$ of them are degenerate with a single cusp. On the other hand, if you want to build hyperbolic surfaces gluing $-\chi - p$ non-degenerate pairs of pants and $p$ pairs of pants with a cusp, you have a total amount of $3(-\chi-p)+2p$ boundaries and for each couple of them you have two degrees of freedom for identifying them: the length of the boundary and a torsion parameter for the identification, descending from the possible isometry of $S^1$ to itself. This gives you $3(-\chi-p)+2p = 6\gamma+2p-6$ degree of freedom. With this argument it is possible to show that the Teichmüller space of $S_{\gamma,p}$ is isomorphic to $\mathbb{R}^{6\gamma+2p-6}$. We emphasize that we are considering metrics with finite volume. If one drops this assumption, then it is clearly possible to build examples of complete hyperbolic metrics also in some case $\chi(S_{\gamma,p}) \geq 0$. As an example consider the Poincaré disk ($\gamma = 0$, $p = 1$) or the cylinder

($\gamma = 0$, $p = 2$) equipped with

$$dt^2 + \mathrm{e}^{2t} d\theta^2, \quad (t, \theta) \in \mathbb{R} \times S^1$$

or with

$$dt^2 + 4\varepsilon^2 \cosh(r)^2 d\theta^2, \quad (t, \theta) \in \mathbb{R} \times S^1,$$

for fixed $\varepsilon > 0$. The last example restricted to $[0, +\infty) \times S^1$ is often called *funnel* and presents a geodesic boundary at $\{t = 0\}$ of length $4\pi\varepsilon$. Actually, if one consider a complete hyperbolic surface with negative Euler characteristic and infinite volume, then all of the ends with infinite volume of that surface are funnels.

## 5.3 Warped products

Let $(\Sigma, \bar{g})$ be a Riemannian manifold of dimension $n - 1$. This section studies the geometric properties of the *warped product* $\mathbb{R} \times \Sigma$ equipped with the metric

$$g = dr^2 + \psi(r)^{4/n} \bar{g},$$

where $\psi \colon \mathbb{R} \to \mathbb{R}$ is a smooth positive function. This class of metrics is largely present in this thesis, since cusps and funnels are examples of warped products. We are going to prove the following result:

**Proposition 5.1.** *Consider the warped product $g = dr^2 + \psi(r)^{4/n} \bar{g}$ defined on $\mathbb{R} \times \Sigma$, where $(\Sigma, \bar{g})$ be a Riemannian manifold of dimension $n - 1$. For $j = 1, 2, 3, 4$ let $\bar{X}_j$ be a vector field on $\Sigma$ and let $X_j = x_j \partial_r + \bar{X}_j$ be a vector field on $\mathbb{R} \times \Sigma$. Then the following formulas hold:*

- *Formula for the curvature tensor:*

$$\begin{aligned} \psi^{-4/n} R(X_1, X_2, X_3, X_4) &= \bar{R}(\bar{X}_1, \bar{X}_2, \bar{X}_3, \bar{X}_4) \\ &\quad + \frac{2}{n} \left( \frac{\ddot{\psi}}{\psi} - \frac{n-2}{n} \frac{\dot{\psi}^2}{\psi^2} \right) S_1 + \frac{4}{n^2} \psi(r)^{4/n - 2} \dot{\psi}(r)^2 S_2, \end{aligned} \tag{5.1}$$

  *with $S_1 = x_1 x_3 \bar{g}(\bar{X}_2, \bar{X}_4) - x_1 x_4 \bar{g}(\bar{X}_2, \bar{X}_3) + x_2 x_4 \bar{g}(\bar{X}_1, \bar{X}_3) - x_2 x_3 \bar{g}(\bar{X}_1, \bar{X}_4)$ and $S_2 = \bar{g}(\bar{X}_1, \bar{X}_3) \bar{g}(\bar{X}_2, \bar{X}_4) - \bar{g}(\bar{X}_2, \bar{X}_3) \bar{g}(\bar{X}_1, \bar{X}_4)$;*

- *Formula for the Ricci tensor*

$$\begin{aligned} \mathrm{Ric}(X_1, X_2) &= \overline{\mathrm{Ric}}(\bar{X}_1, \bar{X}_2) + \frac{2(n-1)}{n} \left( \frac{n-2}{n} \frac{\dot{\psi}^2}{\psi^2} - \frac{\ddot{\psi}}{\psi} \right) x_1 x_2 \\ &\quad - \frac{2}{n} \psi^{4/n} \left( \frac{n-2}{n} \frac{\dot{\psi}^2}{\psi^2} + \frac{\ddot{\psi}}{\psi} \right) \bar{g}(\bar{X}_1, \bar{X}_2); \end{aligned} \tag{5.2}$$

- *Formula for the scalar curvature*

$$\mathrm{R}_g = \psi(r)^{-4/n}\mathrm{R}_{\bar{g}} - \frac{4(n-1)}{n}\frac{\ddot{\psi}(r)}{\psi(r)} \tag{5.3}$$

Let us prove the proposition above. We consider local coordinates $(x^i) = (r, \theta^\alpha) =$ on $\mathbb{R} \times \Sigma$, where $(\theta^\alpha)$ are local coordinates on $\Sigma$. Latin indexes run from 1 to $n$, greek indexes from 1 to $n-1$. Since $g_{rr} = 1$, $g_{r\alpha} = 0$ and $g_{\alpha\beta} = \psi(r)^{4/n}\bar{g}_{\alpha\beta}$, then the Christoffel symbols of $g$ are given by:

$$\begin{aligned}
&\Gamma^r_{rr} = \Gamma^r_{r\beta} = \Gamma^r_{\beta r} = \Gamma^\alpha_{rr} = 0\\
&\Gamma^r_{\alpha\beta} = -\frac{1}{2}\partial_r g_{\alpha\beta} = -\frac{2}{n}\psi(r)^{4/n-1}\dot{\psi}(r)\bar{g}_{\alpha\beta}\\
&\Gamma^\alpha_{r\beta} = \Gamma^\alpha_{\beta r} = \frac{1}{2}g^{\alpha\gamma}\partial_r g_{\beta\gamma} = \frac{2}{n}\psi(r)^{-1}\dot{\psi}(r)\delta^\alpha_\beta\\
&\Gamma^\alpha_{\beta\gamma} = \bar{\Gamma}^\alpha_{\beta\gamma}.
\end{aligned}$$

The overlined objects refer to $\bar{g}$. We can then compute the curvature tensor $R_{ijkl} = g_{ml}R^m_{kij}$, which is given by:

$$\begin{aligned}
&R_{rrrr} = 0\\
&R_{rrr\alpha} = -R_{rr\alpha r} = R_{r\alpha rr} = -R_{\alpha rrr} = 0\\
&R_{rr\alpha\beta} = R_{\alpha\beta rr} = 0\\
&R_{r\alpha r\beta} = -R_{r\alpha\beta r} = R_{\alpha r\beta r} = -R_{\alpha rr\beta} = \frac{2}{n}\psi(r)^{4/n}\left(\frac{\ddot{\psi}(r)}{\psi(r)} - \frac{n-2}{n}\frac{\dot{\psi}(r)^2}{\psi(r)^2}\right)\bar{g}_{\alpha\beta}\\
&R_{r\alpha\beta\gamma} = -R_{\alpha r\beta\gamma} = R_{\beta\gamma r\alpha} = -R_{\beta\gamma\alpha r} = 0\\
&R_{\alpha\beta\gamma\delta} = \psi(r)^{4/n}\bar{R}_{\alpha\beta\gamma\delta} + \frac{4}{n^2}\psi(r)^{8/n-2}\dot{\psi}(r)^2(\bar{g}_{\alpha\gamma}\bar{g}_{\delta\beta} - \bar{g}_{\beta\gamma}\bar{g}_{\alpha\delta}).
\end{aligned}$$

From these formulas we can compute the sectional curvature (only for this computation we assume $n > 2$). Assume that $p \in \mathbb{R} \times \Sigma$ has coordinates $(r, \theta)$ and let $\pi$ be a 2-plane of $T_p(\mathbb{R} \times M)$. Let $v = x\partial_r + v^\alpha\partial_\alpha$ and $w = y\partial_r + w^\alpha\partial_\alpha$ be a $g$-orthonormal frame for $\pi$, namely $x^2 + \psi(r)^{4/n}\bar{g}_{\alpha\beta}v^\alpha v^\beta = 1$, $y^2 + \psi(r)^{4/n}\bar{g}_{\alpha\beta}w^\alpha w^\beta = 1$ and $xy + \psi(r)^{4/n}\bar{g}_{\alpha\beta}v^\alpha w^\beta = 0$, and assume that $\mathrm{pr}_\Sigma(\pi) := \langle v^\alpha\partial_\alpha, w^\alpha\partial_\alpha\rangle$ is a 2-piano in $T_\theta\Sigma$. It follows that

$$\overline{\sec}_\theta(\mathrm{pr}_\Sigma(\pi)) = \frac{w^\alpha v^\beta v^\gamma w^\delta \bar{R}_{\alpha\beta\gamma\delta}}{\psi(r)^{-8/n}[(1-x^2)(1-y^2) - (-xy)^2]}$$

is the sectional curvature of $\mathrm{pr}_\Sigma(\pi)$ with respect to $(\Sigma, \bar{g})$. On the other hand $\sec_p(\pi) = R(w, v, v, w)$ and by the formulas for the components of $R$

we get

$$\begin{aligned}\sec_p(\pi) = &-\frac{2}{n}(y^2+x^2)\left(\frac{\ddot\psi(r)}{\psi(r)} - \frac{n-2}{n}\frac{\dot\psi(r)^2}{\psi(r)^2}\right)\\ &+\frac{4}{n^2}\frac{\dot\psi(r)^2}{\psi(r)^2}(-1+y^2+x^2)\\ &+\psi(r)^{4/n}w^\alpha v^\beta v^\gamma w^\delta \bar R_{\alpha\beta\gamma\delta}.\end{aligned}$$

Equivalently, we proved

$$\begin{aligned}\sec_p(\pi) = &\frac{2}{n}(y^2+x^2)\left(\frac{\dot\psi(r)^2}{\psi(r)^2} - \frac{\ddot\psi(r)}{\psi(r)} - \frac{n}{2}\frac{\overline{\sec}_\theta(\mathrm{pr}_\Sigma(\pi))}{\psi(r)^{4/n}}\right)\\ &-\frac{4}{n^2}\frac{\dot\psi(r)^2}{\psi(r)^2} + \frac{\overline{\sec}_\theta(\mathrm{pr}_\Sigma(\pi))}{\psi(r)^{4/n}}.\end{aligned}$$

The formula for the Ricci tensor descends from the expression of the components of $R$, indeed $\mathrm{Ric}_{jk} = g^{lm}R_{ljkm}$, thus

$$\begin{aligned}\mathrm{Ric}_{rr} &= \frac{2(n-1)}{n}\left(\frac{n-2}{n}\frac{\dot\psi(r)^2}{\psi(r)^2} - \frac{\ddot\psi(r)}{\psi(r)}\right)\\ \mathrm{Ric}_{r\alpha} &= \mathrm{Ric}_{\alpha r} = 0\\ \mathrm{Ric}_{\alpha\gamma} &= \overline{\mathrm{Ric}}_{\alpha\gamma} - \frac{2}{n}\psi(r)^{4/n}\left(\frac{n-2}{n}\frac{\dot\psi(r)^2}{\psi(r)^2} + \frac{\ddot\psi(r)}{\psi(r)}\right)\bar g_{\alpha\gamma}.\end{aligned}$$

Tracing these formulas we get the expression for the scalar curvature

$$\mathrm{R}_g = \psi(r)^{-4/n}\mathrm{R}_{\bar g} - \frac{4(n-1)}{n}\frac{\ddot\psi(r)}{\psi(r)}$$

and the proposition is proved.

**Warped products with constant sectional curvature**

We want to study all the warped products $dr^2 + \psi(r)^{4/n}\bar g$ with sectional curvature constant to $k \in \mathbb{R}$. In virtue of the previous results, this is equivalent to require that for every 2-piano $\bar\pi$ in $T\Sigma$ and for every $x, y \in \mathbb{R}$ it holds

$$\frac{4}{n^2}\frac{\dot\psi^2}{\psi^2} - \frac{\overline{\sec}_\theta(\bar\pi)}{\psi^{4/n}} + k = \frac{2}{n}(y^2+x^2)\left(\frac{\dot\psi^2}{\psi^2} - \frac{\ddot\psi}{\psi} - \frac{n}{2}\frac{\overline{\sec}_\theta(\bar\pi)}{\psi^{4/n}}\right).$$

Necessarily the sectional curvature of $\bar g$ must be constant to some $\lambda \in \mathbb{R}$ and $\psi$ must satisfy

$$\begin{cases}\frac{4}{n^2}\frac{\dot\psi^2}{\psi^2} - \frac{\lambda}{\psi^{4/n}} + k = 0\\ \frac{\dot\psi^2}{\psi^2} - \frac{\ddot\psi}{\psi} - \frac{n}{2}\frac{\lambda}{\psi^{4/n}} = 0.\end{cases}$$

Actually the second equation follows deriving the first one, thus the warped products with constant sectional curvature equal to $k$ all arise from a base $(\Sigma, \bar{g})$ with constant sectional curvature equal to $\lambda$ and a warped function $\psi$ which is a positive solution of

$$\frac{4}{n^2}\frac{\dot{\psi}^2}{\psi^2} - \lambda\psi^{-4/n} + k = 0.$$

If one set $y = \psi^{2/n}$, then the equation above becomes $\dot{y}^2 + ky^2 = \lambda$. Remarkable solutions of the equation above (which is invariant by translation) are given by $\psi(r) = (\lambda/k)^{n/4}\cos(\sqrt{k}r)^{n/2}$ if $k, \lambda > 0$, by $\psi(r) = (-\lambda/k)^{n/4}\sinh(\sqrt{-k}r)^{n/2}$ if $k < 0$ and $\lambda > 0$, by $\psi(r) = (\lambda/k)^{n/4}\cosh(\sqrt{-k}r)^{n/2}$ if $k, \lambda < 0$, and there are not solutions if $k > 0$ and $\lambda < 0$. If we assume $\lambda = 0$ then necessarily $k \leq 0$ and $\psi(r) = \mathrm{e}^{\sqrt{-k}nr/2}$ is a solution. If we assume $k = 0$ then necessarily $\lambda \geq 0$ and $\psi(r) = (1 + \lambda r)^{n/2}$ is a solution.

### Einstein warped products

We want to study all the warped products $dr^2 + \psi(r)^{4/n}\bar{g}$ which are Einstein, namely such that $\mathrm{Ric} = k(n-1)g$ for some $k \in \mathbb{R}$. In virtue of the previous results, this is equivalent to require that

$$k(n-1) = \frac{2(n-1)}{n}\left(\frac{n-2}{n}\frac{\dot{\psi}^2}{\psi^2} - \frac{\ddot{\psi}}{\psi}\right)$$

and

$$k(n-1)\psi^{4/n}\bar{g}_{\alpha\beta} = \overline{\mathrm{Ric}}_{\alpha\gamma} - \frac{2}{n}\psi^{4/n}\left(\frac{n-2}{n}\frac{\dot{\psi}^2}{\psi^2} + \frac{\ddot{\psi}}{\psi}\right)\bar{g}_{\alpha\gamma}.$$

From the second one we see that necessarily $\bar{g}$ is Einstein, namely $\overline{\mathrm{Ric}} = \lambda(n-2)g$ for some $\lambda \in \mathbb{R}$, and $\psi$ must satisfy

$$\begin{cases} \frac{n}{2}k = \frac{n-2}{n}\frac{\dot{\psi}^2}{\psi^2} - \frac{\ddot{\psi}}{\psi} \\ k(n-1)\psi^{4/n} = \lambda(n-2) - \frac{2}{n}\psi^{4/n}\left(\frac{n-2}{n}\frac{\dot{\psi}^2}{\psi^2} + \frac{\ddot{\psi}}{\psi}\right), \end{cases}$$

which is equivalent to the single equation

$$\frac{4}{n^2}\frac{\dot{\psi}^2}{\psi^2} - \lambda\psi^{-4/n} + k = 0.$$

We have already encountered this equation in the previous paragraph, therefore the Einstein warped products can be obtained from as we did for the warped products with constant sectional curvature, but relaxing the condition on the base $(\Sigma, \bar{g})$.

**Warped products with constant scalar curvature**

We want to study all the warped products $dr^2 + \psi(r)^{4/n}\bar{g}$ with scalar curvature constant to $kn(n-1) \in \mathbb{R}$. In virtue of the previous results, this is equivalent to require that

$$kn(n-1) = \psi(r)^{-4/n}\mathrm{R}_{\bar{g}} - \frac{4(n-1)}{n}\frac{\ddot{\psi}(r)}{\psi(r)}.$$

Then necessarily $\mathrm{R}_{\bar{g}} = (n-1)(n-2)\lambda$ for some $\lambda \in \mathbb{R}$ and $\psi$ must satisfy

$$\frac{4}{n(n-2)}\frac{\ddot{\psi}(r)}{\psi(r)} - \lambda\psi(r)^{-4/n} + \frac{kn}{n-2} = 0.$$

Notice that we can assume $n > 2$, otherwise we reduce to the case of warped products with constant sectional curvature, which was already considered. If we set $y = \psi^{2/n}$, then the equation above becomes $\dot{y}^2 + \frac{2}{n-2}y\ddot{y} + \frac{kn}{n-2}y^2 = \lambda$. The function $\psi$ considered for Einstein warped products are also solutions of the equation above since they verify $\ddot{y} = -ky$. However this time we have a larger class of warped functions, which do not verify $\ddot{y} = -ky$. For instance if we assume $\lambda = 0$, then $\psi(r) = \cosh(nr/2)$ is a solution of the equation above for $k = -1$, $\psi(r) = \cos(nr/2)$ is a solution of the equation above for $k = 1$ and $\psi(r) = r$ is a solution of the equation above for $k = 0$. All of these examples do not correspond to warped functions for Einstein warped products.

## 5.4 Conformal geometry

Let $(M, g)$ be a Riemannian manifold of dimension $n \geq 2$. A *conformal transformation* of $g$ is the metric $\tilde{g} := \mathrm{e}^{2u}g$ obtained rescaling $g$ by a smooth function $u \colon M \to \mathbb{R}$. The study of conformal transformation has been really fruitful in many areas, probably the *Yamabe problem* is the most famous result concerning the conformal geometry (see Section 5.7). In this section we recall how to relate the curvature properties of $\tilde{g}$ (denoted with a tilde) with respect to $g$ and $u$. The formulas involve the Laplace-Beltrami operator $\Delta$, given in local coordinates by

$$\Delta u = g^{ij}\nabla_{ij}u.$$

**Proposition 5.2.** *Let $(M, g)$ be a Riemannian manifold of dimension $n \geq 2$, let $u \colon M \to \mathbb{R}$ be a smooth function and set $\tilde{g} := \mathrm{e}^{2u}g$. Then the following formulas hold:*

- *Formula for the $(0,4)$-curvature tensor:*

$$\tilde{R} = \mathrm{e}^{2u}\left(R - g \owedge (\nabla^2 u - du \otimes du + \frac{1}{2}|\nabla u|^2 g)\right), \tag{5.4}$$

*where $\owedge$ is the Kulkarni-Nomizu product defined in local coordinates for symmetric $(0,2)$-tensors by*

$$(g \owedge h)_{ijkl} = g_{ik}h_{jl} + g_{jl}h_{ik} - g_{il}h_{jk} - g_{jk}h_{il};$$

- *Formula for the Ricci tensor*

$$\tilde{\mathrm{Ric}} = \mathrm{Ric} - \left(\Delta u + (n-2)|\nabla u|^2\right) g + (n-2)(du \otimes du - \nabla^2 u) \quad (5.5)$$

- *Formula for the scalar curvature*

$$R_{\tilde{g}} = \mathrm{e}^{-2u}(R_g - 2(n-1)\Delta u - (n-1)(n-2)|\nabla u|^2). \quad (5.6)$$

*In dimension $n > 2$ the above formula simplifies to*

$$R_{\tilde{g}} = \mathrm{e}^{-2u}\left(R_g - \frac{4(n-1)}{n-2}\mathrm{e}^{-\frac{n-2}{2}u}\Delta \mathrm{e}^{\frac{n-2}{2}u}\right).$$

The proof of this proposition is a direct computation. Since $\tilde{g}_{ij} = \mathrm{e}^{2u}g_{ij}$, then $\tilde{g}^{ij} = \mathrm{e}^{-2u}g^{ij}$ and

$$\tilde{\Gamma}^k_{ij} = \Gamma^k_{ij} + \delta^k_i \partial_j u + \delta^k_j \partial_i u - g_{ij}\nabla^k u.$$

Therefore the curvature tensor is locally given by

$$\begin{aligned}
\tilde{R}_{ijkl} &= \tilde{g}_{ml}\left(\partial_i \tilde{\Gamma}^m_{jk} - \partial_j \tilde{\Gamma}^m_{ik} + \tilde{\Gamma}^p_{jk}\tilde{\Gamma}^m_{ip} - \tilde{\Gamma}^p_{ik}\tilde{\Gamma}^m_{jp}\right) \\
&= \mathrm{e}^{2u}R_{ijkl} + \mathrm{e}^{2u}g_{ml}\partial_i(\delta^m_k \partial_j u + \delta^m_j \partial_k u - g_{kj}\nabla^m u) \\
&- \mathrm{e}^{2u}g_{ml}\partial_j(\delta^m_k \partial_i u + \delta^m_i \partial_k u - g_{ki}\nabla^m u) \\
&+ \mathrm{e}^{2u}g_{ml}\Gamma^p_{kj}(\delta^m_i \partial_p u + \delta^m_p \partial_i u - g_{ip}\nabla^m u) \\
&+ \mathrm{e}^{2u}g_{ml}(\delta^p_k \partial_j u + \delta^p_j \partial_k u - g_{kj}\nabla^p u)\Gamma^m_{ip} \\
&+ \mathrm{e}^{2u}g_{ml}(\delta^p_k \partial_j u + \delta^p_j \partial_k u - g_{kj}\nabla^p u)(\delta^m_i \partial_p u + \delta^m_p \partial_i u - g_{ip}\nabla^m u) \\
&- \mathrm{e}^{2u}g_{ml}\Gamma^p_{ki}(\delta^m_j \partial_p u + \delta^m_p \partial_j u - g_{jp}\nabla^m u) \\
&+ \mathrm{e}^{2u}g_{ml}(\delta^p_k \partial_i u + \delta^p_i \partial_k u - g_{ki}\nabla^p u)\Gamma^m_{jp} \\
&+ \mathrm{e}^{2u}g_{ml}(\delta^p_k \partial_i u + \delta^p_i \partial_k u - g_{ki}\nabla^p u)(\delta^m_j \partial_p u + \delta^m_p \partial_j u - g_{jp}\nabla^m u).
\end{aligned}$$

After some simplifications, the above formula becomes

$$\begin{aligned}
\tilde{R}_{ijkl} &= \mathrm{e}^{2u}R_{ijkl} + \mathrm{e}^{2u}(g_{jl}\partial_{ik}u - g_{kj}\partial_{il}u + g_{kj}\Gamma^p_{il}\partial_p u) \\
&- \mathrm{e}^{2u}(g_{il}\partial_{jk}u - g_{ki}\partial_{jl}u + g_{ki}\Gamma^p_{jl}\partial_p u) \\
&+ \mathrm{e}^{2u}g_{il}\Gamma^p_{kj}\partial_p u + \mathrm{e}^{2u}(g_{il}\partial_j u \partial_k u - g_{ik}\partial_j u \partial_l u) \\
&- \mathrm{e}^{2u}g_{jl}\Gamma^p_{ki}\partial_p u - \mathrm{e}^{2u}(g_{jl}\partial_i u \partial_k u - g_{jk}\partial_i u \partial_l u) \\
&- \mathrm{e}^{2u}|\nabla u|^2 g_{kj}g_{il} + \mathrm{e}^{2u}|\nabla u|^2 g_{ki}g_{jl}.
\end{aligned}$$

Since $\nabla_{ij}u = \partial_{ij}u - \Gamma^p_{ij}\partial_p u$ and since by definition

$$\begin{aligned}
&(g \owedge (\nabla^2 u - du \otimes du + \frac{1}{2}|\nabla u|^2 g))_{ijkl} \\
&= g_{ik}(\nabla^2_{jl}u - \partial_j u \partial_l u + \frac{1}{2}|\nabla u|^2 g_{jl}) \\
&+ g_{jl}(\nabla^2_{ik}u - \partial_i u \partial_k u + \frac{1}{2}|\nabla u|^2 g_{ik}) \\
&- g_{il}(\nabla^2_{jk}u - \partial_j u \partial_k u + \frac{1}{2}|\nabla u|^2 g_{jk}) \\
&- g_{jk}(\nabla^2_{il}u - \partial_i u \partial_l u + \frac{1}{2}|\nabla u|^2 g_{il}),
\end{aligned} \tag{5.7}$$

we got the formula for the curvature tensor. The Ricci tensor $\tilde{\mathrm{Ric}}_{jk} = \tilde{g}^{il}\tilde{R}_{ijkl}$ is then given by

$$\begin{aligned}
\tilde{\mathrm{Ric}}_{jk} = g^{il}R_{ijkl} &+ g^{il}(g_{jl}\nabla_{ik}u - g_{kj}\nabla_{il}u) \\
&- g^{il}(g_{il}\nabla_{jk}u - g_{ki}\nabla_{jl}u) \\
&+ g^{il}(g_{il}\partial_j u \partial_k u - g_{ik}\partial_j u \partial_l u) \\
&- g^{il}(g_{jl}\partial_i u \partial_k u - g_{jk}\partial_i u \partial_l u) \\
&- |\nabla u|^2 g^{il} g_{kj} g_{il} + g^{il}|\nabla u|^2 g_{ki} g_{jl},
\end{aligned}$$

that is

$$\tilde{\mathrm{Ric}}_{jk} = \mathrm{Ric}_{jk} - (n-2)\nabla_{jk}u + (n-2)\partial_j u \partial_k u + \left(-\Delta u - (n-2)|\nabla u|^2\right) g_{jk}.$$

Tracing again, we get the formula for the scalar curvature

$$R_{\tilde{g}} = \mathrm{e}^{-2u}(R_g - (2n-2)\Delta u - (n-1)(n-2)|\nabla u|^2).$$

Finally we set $\psi := \mathrm{e}^{\frac{n-2}{2}u}$, and the above formula becomes

$$R_{\tilde{g}} = \psi^{-\frac{2}{n-2}}(R_g - \frac{4(n-1)}{n-2}\psi^{-1}\Delta\psi).$$

## 5.5 Hypersurfaces

The aim of this section is to compute the second fundamental form (extrinsic curvature) and the mean curvature of hypersurfaces that are graphs in special classes of Riemannian manifolds. Before doing so, we recall some definitions. Given a regular function $f\colon M \to \mathbb{R}$, the *gradient* of $f$ is defined as the vector field $\nabla f$ on $M$ satisfying $g(\nabla f, X) = df(X)$ for every vector field $X$ on $M$. In local coordinates $\nabla f = g^{ij}\partial_i f \partial_j$. It is called *Hessian* of $f$ the symmetric $(0,2)$-tensor defined by

$$\mathrm{Hess} f(X,Y) = g(\nabla_X \nabla f, Y)$$

for every vector field $X$, $Y$ on $M$. In local coordinates $\mathrm{Hess}f(\partial_i, \partial_j) = \partial_i\partial_j f - \Gamma_{ij}^k \partial_k f$. We already introduced the *Laplace-Beltrami operator* $\Delta f$ of $f$, defined as the trace of the Hessian of $f$. In local coordinates

$$\Delta f = \mathrm{tr}\,\mathrm{Hess}f = g^{ij}(\partial_i\partial_j f - \Gamma_{ij}^k \partial_k f).$$

Finally we recall that for a vector field $X$, the *divergence* of $X$ is defined to be the function $\mathrm{div}X = \mathrm{tr}_g \nabla X$, which is given in local coordinates by $\mathrm{div}X = \partial_i X^i + \Gamma_{ij}^j X^i$. In virtue of the previous definition, one can notice that $\Delta f = \mathrm{div}\nabla f$.

Now let $(M, g)$ be a Riemannian manifold of dimension $n$ and consider an hypersurface $S \subset M$. The ***second fundamental form*** of $S \subset M$ is defined by

$$\mathrm{II}(X, Y) = g(\nabla_\nu X, Y) = -g(\nabla_X Y, \nu)$$

for every vector fields $X, Y$ tangent to $S$, where $\nu$ denotes a unit vector field normal to $S$ (notice that the sign of the second fundamental form depends on the choice of a direction for $\nu$). The ***mean curvature*** of $S \subset M$ is defined to be the trace of the second fundamental form. The role of the second fundamental form is to relate the curvature of the hypersurface to those one of the ambient space.

**Proposition 5.3.** *Let $S$ be an hypersurface embedded in a Riemannian manifold $(M, g)$. Denote by $g_0$, $R_0$ and* II *respectively the metric induced by $(M, g)$ on $S$, the curvature tensor of $(S, g_0)$ and the second fundamental form of the embedding $S \subset M$. Then the following facts hold:*

- Gauss equation*: For any vector fields $X, Y, Z$ and $W$ tangent to $S$ one has*

  $$R(X, Y, Z, W) = R_0(X, Y, Z, W) - \mathrm{II}(Y, Z)\mathrm{II}(X, W) + \mathrm{II}(X, Z)\mathrm{II}(Y, W)$$

- Codazzi-Mainardi equation*: For any vector fields $X, Y$ and $Z$ tangent to $S$ one has*

  $$R(X, Y, Z, \nu) = -(\nabla_X \mathrm{II})(Y, Z) + (\nabla_Y \mathrm{II})(X, Z),$$

  *where $\nu$ denotes the unit vector field normal to $S$ for which the second fundamental form is computed.*

### Graphs on cylinders

As first example, consider the cylinder $(\mathbb{R} \times \Sigma, g)$, where $g = dr^2 + \bar{g}$ and $(\Sigma, \bar{g})$ is a Riemannian manifold of dimension $n - 1$. Every smooth function $\psi\colon \Sigma \to \mathbb{R}$ defines the graph hypersurface $S := \{r = \psi\} = \{(\psi(\theta), \theta) \,|\, \theta \in$

$\Sigma\}$. We can consider the unit vector $\nu$ perpendicular to $S$ and directed to $r = +\infty$, which is

$$\nu = \frac{\nabla(r-\psi)}{|\nabla(r-\psi)|} = \frac{\partial_r - \nabla_{\bar g}\psi}{\sqrt{1+|\nabla_{\bar g}\psi|^2_{\bar g}}}.$$

If $(\theta^\alpha) = (\theta^1, \dots, \theta^{n-1})$ denote local coordinates for $\Sigma$, then the metric $g$ on $S$ becomes $g|_S = (\partial_\alpha\psi\partial_\beta\psi + \bar g_{\alpha\beta})d\theta^\alpha \otimes d\theta^\beta$ and

$$v_\alpha := \partial_\alpha\psi\partial_r + \partial_\alpha$$

is a local frame for $TS$. Thus the second fundamental form is given by

$$\begin{aligned}\mathrm{II}_{\alpha\beta} &= -g\left(\nabla_{v_\alpha} v_\beta, \nu\right)\\ &= -\partial_\alpha\psi\partial_\beta\psi g\left(\nabla_{\partial_r}\partial_r, \nu\right) - \partial_\alpha\psi g\left(\nabla_{\partial_r}\partial_\beta, \nu\right) - \partial_{\alpha\beta}\psi g\left(\partial_r, \nu\right)\\ &\quad - \partial_\beta\psi g\left(\nabla_{\partial_\alpha}\partial_r, \nu\right) - g\left(\nabla_{\partial_\alpha}\partial_\beta, \nu\right).\end{aligned}$$

A direct computation shows that the unique non-vanishing Christoffel symbols of $g$ are the ones which do not involve the variable $r$, namely $\Gamma^\gamma_{\alpha\beta}$, which coincide with the Christoffel symbols $\bar\Gamma^\gamma_{\alpha\beta}$ of $\bar g$. Thus

$$\mathrm{II}_{\alpha\beta} = -\frac{\partial_{\alpha\beta}\psi - g\left(\nabla_{\partial_\alpha}\partial_\beta, \nabla_{\bar g}\psi\right)}{\sqrt{1+|\nabla_{\bar g}\psi|^2_{\bar g}}} = -\frac{\overline\nabla_{\alpha\beta}\psi}{\sqrt{1+|\nabla_{\bar g}\psi|^2_{\bar g}}}$$

is the formula for the second fundamental form. To compute the mean curvature $H$ of $S \subset M$, we have to notice that the inverse of $(g|_S)_{\alpha\beta} = \bar g_{\alpha\beta} + \partial_\alpha\psi\partial_\beta\psi$ is

$$(g|_S)^{\alpha\beta} = \bar g^{\alpha\beta} - \frac{\overline\nabla^\alpha\psi\overline\nabla^\beta\psi}{1+|\nabla_{\bar g}\psi|^2_{\bar g}}$$

and therefore

$$\begin{aligned}H &= -\left(\bar g^{\alpha\beta} - \frac{\overline\nabla^\alpha\psi\overline\nabla^\beta\psi}{1+|\nabla_{\bar g}\psi|^2_{\bar g}}\right)\frac{\overline\nabla_{\alpha\beta}\psi}{\sqrt{1+|\nabla_{\bar g}\psi|^2_{\bar g}}}\\ &= -\frac{\Delta_{\bar g}\psi}{\sqrt{1+|\nabla_{\bar g}\psi|^2_{\bar g}}} + \frac{\mathrm{Hess}_{\bar g}\psi(\nabla_{\bar g}\psi, \nabla_{\bar g}\psi)}{\left(1+|\nabla_{\bar g}\psi|^2_{\bar g}\right)^{3/2}}\\ &= -\mathrm{div}_{\bar g}\left(\frac{\nabla_{\bar g}\psi}{\sqrt{1+|\nabla_{\bar g}\psi|^2_{\bar g}}}\right).\end{aligned} \tag{5.8}$$

## Graphs on conformal manifolds

As second example, consider a conformal metric $\tilde{g} = \mathrm{e}^{2u} g$ where $(M, g)$ is a Riemannian manifold of dimension $n$. Let $S \subset M$ be an hypersurface, we want to relate the second fundamental form $\tilde{\mathrm{II}}$ and the mean curvature $\tilde{H}$ of $S$ in $(M, \tilde{g})$ with respect to the same objects II and $H$ relative to $(M, g)$. We notice that $\tilde{\nu} := \mathrm{e}^{-u}\nu$ is the $\tilde{g}$-unit vector perpendicular to $S$ if $\nu$ denotes the $g$-unit vector perpendicular to $S$. Then

$$\tilde{\mathrm{II}}(X, Y) = -\tilde{g}(\tilde{\nabla}_X Y, \tilde{\nu}) = -\mathrm{e}^u g(\tilde{\nabla}_X Y, \nu).$$

Using the formula for the transformation of the Christoffel symbols under a conformal change, one observe that

$$\tilde{\nabla}_X Y = \nabla_X Y + Y(u)X + X(u)Y - g(X, Y)\nabla u,$$

therefore

$$\tilde{\mathrm{II}}(X, Y) = \mathrm{e}^u \mathrm{II}(X, Y) + \mathrm{e}^u g(X, Y) g(\nabla u, \nu).$$

Taking the trace by $(\tilde{g}|_S)^{-1}$, we get the formula for the mean curvature

$$\tilde{H} = \mathrm{e}^{-u}(H + (n-1) g(\nabla u, \nu)).$$

## Graphs on warped products

Due to the results of the last two subsections, we are going to show the following result.

**Lemma 5.4.** *Let $(\Sigma, \bar{g})$ be a Riemannian manifold of dimension $n-1$ and consider $M := \mathbb{R} \times \Sigma$ equipped with the metric*

$$g := \mathrm{e}^{2u}\left(dr^2 + \phi^2(r)\bar{g}\right),$$

*where $u\colon \Sigma \to \mathbb{R}$ and $\phi\colon \mathbb{R} \to \mathbb{R}$ are smooth functions, $\phi > 0$. Then the second fundamental form* II *of the graph of a smooth function $\psi\colon \Sigma \to \mathbb{R}$, with respect to $g$ and the unit normal pointing to $r = +\infty$, can be computed as*

$$\begin{aligned}\mathrm{e}^{-u}\sqrt{1 + \phi^{-2}|\nabla\psi|^2}\,\mathrm{II} = &-\mathrm{Hess}\,\psi + \phi^{-1}\dot{\phi}\, d\psi \otimes d\psi \\ &+ (\phi^2 \partial_r u + \phi\dot{\phi})(\phi^{-2} d\psi \otimes d\psi + \bar{g}) \\ &- \bar{g}(\nabla u, \nabla\psi)(\phi^{-2} d\psi \otimes d\psi + \bar{g}),\end{aligned} \tag{5.9}$$

*while the mean curvature $H$ can be computed as*

$$\begin{aligned}e^u H = &-\phi^{-1}\mathrm{div}\left(\frac{\nabla\psi}{\sqrt{\phi^2 + |\nabla\psi|^2}}\right) \\ &+ (n-1)\frac{\phi^{-1}\dot{\phi} + \partial_r u - \phi^{-2}\langle\nabla u, \nabla\psi\rangle}{\sqrt{1 + \phi^{-2}|\nabla\psi|^2}}.\end{aligned} \tag{5.10}$$

*In the above formulas all the geometric quantities are computed with respect to $\bar{g}$ and, with abuse of notation, we tacitly omitted the composition with $\psi$, so that $u$, $\partial_r u$, $\langle \nabla u, \nabla \psi\rangle$, $\phi$ and its derivative $\dot\phi$ are the functions defined for $\theta \in \Sigma$ by $u(\psi(\theta),\theta)$, $\partial_r u(\psi(\theta),\theta)$, $\bar{g}^{ij}\partial_i u(\psi(\theta),\theta)\partial_j\psi(\theta)$, $\phi(\psi(\theta))$ and $\dot\phi(\psi(\theta))$ respectively. Equivalently, (5.10) can be rewritten as*

$$\begin{aligned}\sqrt{1+\phi^{-2}|\nabla\psi|^2}e^u H = &-\phi^{-2}\Delta\psi + \frac{\phi^{-4}\mathrm{Hess}\psi(\nabla\psi,\nabla\psi)}{1+\phi^{-2}|\nabla\psi|^2}\\ &+\phi^{-1}\dot\phi\frac{\phi^{-2}|\nabla\psi|^2}{1+\phi^{-2}|\nabla\psi|^2}\\ &+(n-1)\phi^{-1}\dot\phi\\ &+(n-1)\left(\partial_r u - \phi^{-2}\langle\nabla u,\nabla\psi\rangle\right).\end{aligned} \tag{5.11}$$

Let us show the previous result. We have to compute the second fundamental form and the mean curvature of the hypersurface $S = \{r = \psi\}$ in $\mathbb{R}\times\Sigma$ equipped with the metric $\tilde{g} = \mathrm{e}^{2u}(dr^2 + \phi^2(r)\bar{g})$, with respect to the unit vector pointing to $r = +\infty$. If we introduce $v := u + \log\phi$ and consider the new variable $\rho$ in $\mathbb{R}$ defined by $\rho'(r) = 1/\phi(r)$ and $\rho(0) = 0$, then $\tilde{g} = \mathrm{e}^{2v}g$ with $g = d\rho^2 + \bar{g}$. With this notation the hypersurface $S$ becomes the graph $\{\rho = f\}$, where $f\colon \Sigma \to \mathbb{R}$ is defined by

$$f(\theta) = \int_0^{\psi(\theta)} \frac{dr}{\phi(r)}.$$

Due to the results of the previous paragraphs, we know that

$$\tilde{\mathrm{II}} = \mathrm{e}^v\mathrm{II} + \mathrm{e}^v g(\nabla v,\nu)g,$$

where

$$\mathrm{II} = -\frac{\mathrm{Hess}_{\bar{g}}f}{\sqrt{1+|\nabla_{\bar{g}}f|^2_{\bar{g}}}} \quad\text{and}\quad \nu = \frac{\partial_\rho - \nabla_{\bar{g}}f}{\sqrt{1+|\nabla_{\bar{g}}f|^2_{\bar{g}}}}.$$

Since $\partial_\alpha f = \partial_\alpha\psi/(\phi\circ\psi)$, then $\nabla_{\bar{g}}f = (\psi\circ\psi)^{-1}\nabla_{\bar{g}}\psi$ and $\mathrm{Hess}_{\bar{g}}f = (\phi\circ\psi)^{-1}\mathrm{Hess}_{\bar{g}}\psi - (\phi\circ\psi)^{-2}(\dot\phi\circ\psi)d\psi\otimes d\psi$. Then

$$\begin{aligned}\tilde{\mathrm{II}} = &-\mathrm{e}^v\frac{(\phi\circ\psi)^{-1}\mathrm{Hess}_{\bar{g}}\psi - (\phi\circ\psi)^{-2}(\dot\phi\circ\psi)d\psi\otimes d\psi}{\sqrt{1+(\phi\circ\psi)^{-2}|\nabla_{\bar{g}}\psi|^2_{\bar{g}}}}\\ &+\mathrm{e}^v g(\nabla v,\nu)((\phi\circ\psi)^{-2}d\psi\otimes d\psi + \bar{g}).\end{aligned}$$

Now we use that $v = u + \log\phi$, so that

$$g(\nabla v,\nu) = \frac{(\phi\circ\psi)\partial_r u(\psi,\cdot) + \dot\phi\circ\psi - (\phi\circ\psi)^{-1}\bar{g}^{\alpha\beta}\partial_\alpha u(\psi,\cdot)\partial_\beta\psi}{\sqrt{1+(\phi\circ\psi)^{-2}|\nabla_{\bar{g}}\psi|^2_{\bar{g}}}}.$$

This gives

$$\begin{aligned}
\mathrm{e}^{-u(\psi,\cdot)}\tilde{\mathrm{II}} = &-\frac{\mathrm{Hess}_{\bar g}\psi - (\dot\phi\circ\psi)(\phi\circ\psi)^{-1}d\psi\otimes d\psi}{\sqrt{1+(\phi\circ\psi)^{-2}|\nabla_{\bar g}\psi|^2_{\bar g}}} \\
&+ (\phi\circ\psi)\frac{(\phi\circ\psi)\partial_r u(\psi,\cdot) + \dot\phi\circ\psi}{\sqrt{1+(\phi\circ\psi)^{-2}|\nabla_{\bar g}\psi|^2_{\bar g}}}((\phi\circ\psi)^{-2}d\psi\otimes d\psi + \bar g) \\
&- \frac{\bar g(\nabla_{\bar g}u(\psi,\cdot),\nabla_{\bar g}\psi)}{\sqrt{1+(\phi\circ\psi)^{-2}|\nabla_{\bar g}\psi|^2_{\bar g}}}((\phi\circ\psi)^{-2}d\psi\otimes d\psi + \bar g).
\end{aligned}$$

It remains to check the formula for the mean curvature. It is sufficient to trace the formula above via the inverse of $(g|_S)_{\alpha\beta} = \mathrm{e}^{2u(\psi,\cdot)}(\partial_\alpha\psi\partial_\beta\psi + (\phi\circ\psi)^2\bar g_{\alpha\beta})$, which is

$$(g|_S)^{\alpha\beta} = \mathrm{e}^{-2u(\psi,\cdot)}\left((\phi\circ\psi)^{-2}\bar g^{\alpha\beta} - (\phi\circ\psi)^{-4}\frac{\overline\nabla^\alpha\psi\overline\nabla^\beta\psi}{1+(\phi\circ\psi)^{-2}|\nabla_{\bar g}\psi|^2_{\bar g}}\right).$$

In this way we obtain

$$\begin{aligned}
\sqrt{1+(\phi\circ\psi)^{-2}|\nabla_{\bar g}\psi|^2_{\bar g}}\,\mathrm{e}^{u(\psi,\cdot)}\tilde H = &-(\phi\circ\psi)^{-2}\Delta_{\bar g}\psi \\
&+ \frac{(\phi\circ\psi)^{-4}\mathrm{Hess}_{\bar g}\psi(\nabla_{\bar g}\psi,\nabla_{\bar g}\psi)}{1+(\phi\circ\psi)^{-2}|\nabla_{\bar g}\psi|^2_{\bar g}} \\
&+ (\phi\circ\psi)^{-3}(\dot\phi\circ\psi)\frac{|\nabla_{\bar g}\psi|^2_{\bar g}}{1+(\phi\circ\psi)^{-2}|\nabla_{\bar g}\psi|^2_{\bar g}} \\
&+ (n-1)(\phi\circ\psi)^{-1}(\dot\phi\circ\psi) \\
&+ (n-1)\left(\partial_r u(\psi,\cdot) - (\phi\circ\psi)^{-2}\bar g(\nabla_{\bar g}u(\psi,\cdot),\nabla_{\bar g}\psi)\right),
\end{aligned} \tag{5.12}$$

and the claim follows.

## 5.6 Elliptic operators

We recall the basic definitions and properties of elliptic operators on Riemannian manifolds. This section also explains the notation used in the thesis, which is standard in literature.

### Functional spaces

Let $(M,g)$ be a Riemannian manifold of dimension $n$. The volume form $dV_g$ induced by $g$ gives raise to a Borel measure on $M$. This allows one to define measurable functions from $M$ to $\mathbb{R}$ and weak derivatives. For $k\in\mathbb{N}$ and $p\in[1,+\infty]$, the ***Sobolev space*** $\mathrm{L}^{p,k}(M,g)$ is defined to be the set of those measurable functions $f\colon M \to \mathbb{R}$ (modulo equality almost everywhere) satisfying

$$\|f\|_{\mathrm{L}^{p,k}(M)} := \max_{j\leq k} \left(\int_M |\nabla^{(j)} f|_g^p dV_g\right)^{\frac{1}{p}} < +\infty$$

if $p < \infty$, and

$$\|f\|_{\mathrm{L}^{\infty,k}(M)} := \max_{j\leq k} \sup_M |\nabla^{(j)} f|_g^p < +\infty$$

for $p = +\infty$. For $k \in \mathbb{N}$ and $\alpha \in (0,1)$, the ***Hölder space*** $\mathscr{C}^{k,\alpha}(M,g)$ is defined to be the set of those measurable functions $f\colon M \to \mathbb{R}$ (again modulo equality almost everywhere) satisfying

$$\|f\|_{\mathscr{C}^{k,\alpha}(M,g)} = \max_{j\leq k} \sup_M |\nabla^{(j)} f| + \sup_{p\neq q\in M} \frac{|\nabla^{(k)} f(p) - \nabla^{(k)} f(q)|}{d(p,q)^\alpha} < \infty,$$

where the term $\sup_{p\neq q\in M}$ is considered among those $q \neq p$ belonging to a normal convex neighbourhood of $p \in M$ so that the parallel transport is well-defined and the expression $\nabla^{(k)} f(p) - \nabla^{(k)} f(q)$ makes sense. Both the Sobolev and Hölder space are examples of Banach spaces on $M$. Their definition strongly depends on the metric $g$, however if $M$ is compact then $g$ only influences the definitions of the norms but not Sobolev or Hölder spaces themselves, and it holds the following result.

**Proposition 5.5.** *If $(M,g)$ is a compact Riemannian manifold of dimension $n$, then the following are true.*

1. *For $p \in [1,+\infty)$ the space of smooth function $\mathscr{C}^\infty(M)$ is dense in $\mathrm{L}^{p,k}(M,g)$;*
2. *For $k, k' \in \mathbb{N}$ and $p, p' \in [1,+\infty)$ satisfying*
$$k \geq k' \quad \text{and} \quad k - n/p \geq k' - n/p', \tag{5.13}$$
*the space $\mathrm{L}^{p,k}(M,g)$ embeds continuously in $\mathrm{L}^{p',k'}(M,g)$, and it is a compact embedding if (5.13) hold strictly;*
3. *For $m, k \in \mathbb{N}$, $\alpha \in (0,1)$ and $p \in [1,+\infty)$ satisfying*
$$m - n/p \geq k + \alpha, \tag{5.14}$$
*the space $\mathrm{L}^{p,m}(M,g)$ embeds continuously in $\mathscr{C}^{k,\alpha}(M,g)$, and it is a compact embedding if (5.14) hold strictly;*
4. *For $k, k' \in \mathbb{N}$ and $\alpha, \alpha' \in (0,1)$ satisfying*
$$k + \alpha \geq k' + \alpha', \tag{5.15}$$
*the space $\mathscr{C}^{k,\alpha}(M,g)$ embeds continuously in $\mathscr{C}^{k',\alpha'}(M,g)$, and it is a compact embedding if (5.15) hold strictly.*

### Elliptic linear operators

Consider a compact Riemannian manifold $(M, g)$ of dimension $n$. A *second order operator* is an operator that takes a function $u$ on $M$ which is at least two-times differentiable (possibly in a weak sense) and maps it to a function $P(u)$ defined on $M$ depending only (and at least continuously) on $u$, $\nabla u$ and $\nabla^2 u$. The operator is called *linear* if $P(\lambda u + \mu v) = \lambda P(u) + \mu P(v)$ for $u, v$ functions and $\lambda, \mu \in \mathbb{R}$. In general a second-order operator $P$ is not linear, but it is possible to linearise it around a function $u$ letting

$$Lv := \lim_{t \to 0} \frac{P(u + tv) - P(u)}{t}$$

for a function $v$. Then $P$ is linear if and only if coincides with its linearization. A *second-order linear operator $L$ with $\mathscr{C}^{k,\alpha}(M)$-coefficients* can be written by definition as

$$Lu = \sum_{i,j=1}^{n} a^{ij} \nabla_{ij} u + \sum_{i=1}^{n} b^i \nabla_i u + cu,$$

where $a^{ij} = a^{ji}$, $b^i$ and $c$ belong to $\mathscr{C}^{k,\alpha}(M)$. It can be seen as an operator $L\colon \mathscr{C}^{k+2,\alpha}(M) \to \mathscr{C}^{k,\alpha}(M)$. The *principal symbol* of $L$ at $p \in M$ is the homogeneous polynomial of degree 2 defined as $\sigma(p, \xi) := \sum_{ij} a^{ij}(p) \xi_i \xi_j$. The operator $L$ is called *elliptic* if $\sigma(p, \xi) \neq 0$ for every $p \in M$ and every non-zero $\xi \in \mathbb{R}^n$. More generally a (non-linear) second order operator $P$ is called *elliptic at $u$* if its linearization at $u$ is elliptic. The most famous example of second order linear elliptic operator with smooth coefficients on $(M, g)$ is probably the Laplace-Beltrami operator $\Delta + \kappa$, $\kappa \in \mathbb{R}$, whose principal symbol is $\sigma(p, \xi) = |\xi|^2_{g(p)}$. A second well-known property of $\Delta + \kappa$ is to be self-adjoint. Let $L$ be a second-order linear operator with smooth coefficients, it turns out that there exists an unique linear operator $L^*$, called *adjoint* of $L$, such that

$$\int_M vLudV_g = \int_M uL^* vdV_g$$

for every $u, v \in \mathrm{L}^{2,2}(M, g)$. The Laplace-Beltrami operator is self-adjoint in the sense that $(\Delta + \kappa)^* = \Delta + \kappa$. Indeed we recall that since $M$ is close without boundary the it holds

$$\int_M v(\Delta u + \kappa u) dV_g = -\int_M g(\nabla v, \nabla u) dV_g + \kappa \int_M vudV_g = \int_M u(\Delta v + \kappa v) dV_g.$$

### Regularity and existence results

We list some very useful and classic properties of a linear elliptic operators (for simplicity, we focus on operators of the second order, but analogous results hold in general).

**Theorem 5.6** (Regularity)**.** *Let $(M, g)$ be a compact Riemannian manifold and $L$ a second-order elliptic linear operator on $M$ with smooth coefficients. Fix $p > 1$, $k \in \mathbb{N}$ and $\alpha \in (0, 1)$. Suppose that $Lu = v$ holds weakly for two integrable functions $u$ and $v$ on $M$. If $v \in \mathrm{L}^{p,k}(M)$ then $u \in \mathrm{L}^{p,k+2}(M)$ and*

$$\|u\|_{\mathrm{L}^{p,k+2}(M)} \leq C \left( \|v\|_{\mathrm{L}^{p,k}(M)} + \|u\|_{\mathrm{L}^{1,0}(M)} \right)$$

*for some $C > 0$ independent of $u$, $v$. If $v \in \mathscr{C}^{k,\alpha}(M)$ then $u \in \mathscr{C}^{k+2,\alpha}(M)$ and*

$$\|u\|_{\mathscr{C}^{k+2,\alpha}(M)} \leq C \left( \|v\|_{\mathscr{C}^{k,\alpha}(M)} + \|u\|_{\mathscr{C}^{0}(M)} \right)$$

*for some $C > 0$ independent of $u$, $v$.*

In particular, under the hypothesis of the theorem above one deduces that if $v$ is smooth then $u$ is smooth.

**Theorem 5.7** (Existence)**.** *Let $(M, g)$ be a compact Riemannian manifold and $L$ a second-order elliptic linear operator on $M$ with smooth coefficients. Fix $p > 1$, $k \in \mathbb{N}$ and $\alpha \in (0, 1)$. The operator $L$ can be seen as operator $\mathscr{C}^{k+2,\alpha}(M) \to \mathscr{C}^{k,\alpha}(M)$, $\mathrm{L}^{p,k+2}(M) \to \mathrm{L}^{p,k}(M)$ or $\mathscr{C}^{\infty}(M) \to \mathscr{C}^{\infty}(M)$. In all these cases* Ker$L$ *is a finite-dimensional subspace of the domain of $L$. Moreover for every $v \in \mathrm{L}^{p,k}(M)$ there exists $u \in \mathrm{L}^{p,k+2}(M)$ such that $Lu = v$ if and only if $\int_M vw dV_g = 0$ for every $w \in \mathrm{L}^{p,k}(M)$ such that $L^* w = 0$, and $u$ is unique if one requires $\int_M uw dV_g = 0$ for every $w \in \mathrm{L}^{p,k+2}(M)$ such that $Lw = 0$. Similarly, for every $v \in \mathscr{C}^{k,\alpha}(M)$ there exists $u \in \mathscr{C}^{k+2,\alpha}(M)$ such that $Lu = v$ if and only if $\int_M vw dV_g = 0$ for every $w \in \mathscr{C}^{k,\alpha}(M)$ such that $L^* w = 0$, and $u$ is unique if one requires $\int_M uw dV_g = 0$ for every $w \in \mathscr{C}^{k+2,\alpha}(M)$ such that $Lw = 0$.*

For instance, the previous result implies that $\Delta + \kappa$ is an isomorphism from $\mathscr{C}^{k+2,\alpha}(M)$ to $\mathscr{C}^{k,\alpha}(M)$ whenever $-\kappa$ is not an eigenvalue of $\Delta$. Similarly, it follows that the image of $\Delta$, whose kernel consists on constant functions, contains those functions with null mean.

The last result we recall is usually known as *Interior Schauder Estimates.*

**Theorem 5.8.** *Let $(M, g)$ be a compact Riemannian manifold and $L$ a second-order elliptic linear operator on $M$ with smooth coefficients. Let $\Omega \subset \Omega'$ be two relatively compact open subsets of $M$ such that $d_g(\Omega, \partial\Omega') > 0$. Fix $k \in \mathbb{N}$ and $\alpha \in (0, 1)$, then there exists a constant $C > 0$ such that if $u \in \mathscr{C}^{k+2}(\Omega', g)$ satisfies $Lu \in \mathscr{C}^{k,\alpha}(\Omega', g)$, then $u \in \mathscr{C}^{k+2,\alpha}(\Omega, g)$ and*

$$\|u\|_{\mathscr{C}^{k+2,\alpha}(\Omega,g)} \leq C \left( \|Lu\|_{\mathscr{C}^{k,\alpha}(\Omega',g)} + \|u\|_{\mathscr{C}^{0}(\Omega',g)} \right).$$

An analogue result holds in the Sobolev setting. The Interior Schauder Estimates above are stated under the hypothesis of compactness of $M$, however since they only involve $\Omega$ and $\Omega'$ one can expect the same result for a non-compact $M$. If $M$ is not compact, we have to assume that the symbol of $L$ satisfies $\sigma(p,\xi) > \lambda|\xi|^2_{g(p)}$ for some $\lambda > 0$ independent of $p$ (uniform ellipticity). Then the Interior Schauder Estimates holds as well but the constant $C$ will depend as previously on $(M,g)$, $d_g(\Omega, \partial\Omega')$, $k$, $\alpha$ and on upper bounds for the $\mathscr{C}^{k+2,\alpha}$-norms of the coefficients of $L$, but also on $\lambda$ and on the diameter of $\Omega'$ .

## 5.7 The Yamabe problem

Let $(M,g)$ be a compact Riemannian manifold of dimension $n \geq 2$. If $n = 2$ the uniformization theorem asserts that $g$ is conformal to a metric with constant scalar curvature (this is actually true also in the non-compact case). The Yamabe problem asks if the same holds in higher dimension. Precisely

**Yamabe problem, 1960** Let $(M,g)$ be a compact Riemannian manifold of dimension $n \geq 3$. Is there a metric $\tilde{g}$ in the conformal class $[g]$ of $g$ such that the scalar curvature $\mathrm{R}_{\tilde{g}}$ is constant?

Yamabe himself claimed to have found a solution of his problem, however Neil Trudinger found a mistake in Yamabe's proof. Some time later Trudinger and Aubin showed that the approach of Yamabe could work but with some hypothesis on $M$, precisely if the *Yamabe invariant* (the precise definition below) is smaller of a certain constant $\lambda(S^n)$. Moreover, Aubin showed that such conditions were verified by Riemannian manifolds of dimension $n \geq 6$ which are not locally conformally flat. The remaining cases was proved some years later, in 1984, by Richard Schoen, who gave a positive answer to the Yamabe problem.

Given two metrics $g$ and $\tilde{g} = \psi^{\frac{4}{n-2}} g$ in the same conformal class, their scalar curvatures are related by

$$R_{\tilde{g}} = \psi^{-\frac{2}{n-2}} \left( R_g - \frac{4(n-1)}{n-2} \psi^{-1} \Delta\psi \right)),$$

as we noticed in Section 5.4. Therefore the Yamabe problem is equivalent to find a positive smooth solution $\psi$ to the *Yamabe equation*

$$\frac{4(n-1)}{n-2} \Delta\psi - R_g \psi + \varepsilon \psi^{\frac{n}{n-2}} = 0,$$

for some $\epsilon \in \mathbb{R}$. By a variational point of view, the Yamabe problem is also equivalent to show the existence of stationary points of the operator $Q\colon [g] \to \mathbb{R}$ defined by

$$Q(\tilde{g}) := \frac{\int_M \mathrm{R}_{\tilde{g}}\, dV_{\tilde{g}}}{\mathrm{Vol}_{\tilde{g}}(M)^{\frac{n-2}{n}}}.$$

In fact it can be shown that a metric in [g] is a stationary point for $Q$ if and only if it has constant scalar curvature. It can be also shown that $Q$ is bounded below, so it makes sense to introduce the *Yamabe invariant*

$$\lambda(M) := \inf_{[g]} Q,$$

which only depends on $M$ and $[g]$. A *Yamabe metric* is a metric $\tilde{g} \in [g]$ such that $Q(\tilde{g}) = \lambda(M)$, namely a minimum for $Q$. The idea for showing the Yamabe problem was actually to prove the existence of a Yamabe metric on $[g]$. For what concerns the uniqueness, it can be shown that if $\lambda(M) \leq 0$, then there is a unique solution for the Yamabe problem up to homothety. In particular if $\lambda(M) < 0$ there exists a unique metric in $[g]$ with constant scalar curvature equal to $-1$, and if $\lambda(M) = 0$ there exists a unique metric in $[g]$ with constant scalar curvature equal to 0 and volume equal to 1. Differently, if $\lambda(M) > 0$ there may exists different metrics in $[g]$ with constant scalar curvature, which are not related by homothety. This depends on the possible existence of stationary points for $Q$ which are not Yamabe metrics.

The Yamabe problem in the non-compact setting is still open and false in general. This strongly depends on the behaviour of the geometry of the ends of the manifold.

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