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# Structural Reflection and the Ultimate L as the true, noumenal universe of mathematics


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## 1. Introduction

In this dissertation I am going to discuss structural reflection and the philosophy of set theory. We can find many different kinds of reflection principles in set theory, but I found structural reflection, as conceived by Joan Bagaria, an interesting and powerful method to characterize large cardinals in terms of reflection. In fact, structural reflection can produce a proper class of supercompact cardinals and a proper class of extendible cardinals (by using different conditions). Thus, structural reflection is important because it provides an intrinsic philosophical justification of large cardinals. In fact, the large cardinals, that we are able to interpret as principles of structural reflection are fundamental for $\Omega$-logic and second-order arithmetic. By adopting structural reflection, we reflect an internal structural property of the membership relation. We can try to clarify immediately what one may mean by reflecting an internal structural property of the membership relation by following Bagaria's thought. We could answer that it is a property of some structure of the form $(X, \in, (R_i)_{i \in I})$, where X is a set or a proper class and $(R_i)_{i \in I}$ is a family of relations on X, and where I is a set that may be empty. So, an internal structural property of $\in$ would be formally given by a formula $\phi(x)$, possibly with parameters, that defines a class of structures of the form $(X, \in, (R_i)_{i \in I})$. We might interpret this fact by saying that there exists an ordinal $\alpha$ that reflects $\phi$ and such that for every structure A in the class (that is, for every structure A that satisfies $\phi$) there exists a structure B also in the class which belongs to $V_\alpha$ and is like A. Since, in general, A may be much larger than any B in $V_\alpha$, the closest resemblance of B to A will be attained in the case that B can be elementarily embedded into A. Thus we can now formulate the principle of structural reflection as follows:

Definition 1. *(Bagaria) (Structural reflection, SR) For every definable (in the first order language of set theory, with parameters) class of structures C of the form* $(X, \in$

*, $(R_i)_{i \in I}$), there exists $\alpha$ such that $\alpha$ reflects C, i.e. $C^{V_\alpha} = C \cap V_\alpha$ and for every A in C there exists B in $C \cap V_\alpha$ and an elementary embedding from B into A.*

From a mathematical perspective, the main objective of this dissertation is the application of structural reflection to the canonical inner model for a measurable cardinal, namely $L[U]$. following Bagaria's thought and his results [Bagaria 13], I will prove that structural reflection for $\Pi_1$ classes of structures definable in V relativized to this canonical inner model is equivalent to the existence of $0^\dagger$. Then, I will prove that structural reflection for classes of structures (whatever complexity) definable within $L[U]$ is implied by the existence of $0^\dagger$. The mathematical result concerning classes of structures $\Pi_1$ definable in V will support my philosophical thesis that considers Woodin's Ultimate L as the true, noumenal universe of mathematics very close to V. In fact, I will prove that if we apply structural reflection with ($\Pi_1$ definable in V) classes of structures to a weak extender model for a supercompact cardinal, we do not get transcendence over this inner model. At the same time, I will conjecture that if we apply structural reflection to a canonical inner model for a strong cardinal, we obtain $0^\P$. Bagaria [Bagaria 13] proved that $0^\sharp$ existence is equivalent to structural reflection (with $\Pi_1$ definable classes of structures in V) relativized to Gödel's constructible universe L. Therefore, on one side, if we relativize structural reflection with $\Pi_1$ definable classes of structures in V to Gödel's constructible universe, inner model of iterated sharps, inner model of measurability, inner model of iterated daggers and inner model for a strong cardinal we obtain a sharp for these specific inner models. On the other side, if we relativize structural reflection for $\Pi_1$ classes of structures definable in V to a weak extender model for a supercompact cardinal, we do not get transcendence. So, we may assert that structural reflection supports the philosophical thesis claiming that Woodin's Ultimate L (not yet constructed) is very close to V and it can be considered as the true, noumenal (I will clarify immediately the philosophical meaning of this word) universe of mathematics where undecided mathematical statements, such as the Continuum

Hypothesis, are settled. There are many aspects that justify structural reflection itself. First of all, $\Sigma_1$ structural reflection can be proved from the axioms of ZFC. So, structural reflection is a feature of the universe of sets. Secondly, we have the concept of richness. When we relativize structural reflection to inner models, we are able to transcend the specific inner model and obtain a bigger, richer universe of sets. Thus, the philosophical concept of richness justifies structural reflection. On the one side, structural reflection provides intrinsic philosophical justifications for large cardinals. On the other side, we have philosophical reasons that support structural reflection and render structural reflection a powerful method to interpret large cardinals as principles of reflection. Therefore, from a philosophical perspective, structural reflection is able to justify intrinsically large cardinal notions such as infinitely-many Woodin cardinals and a proper class of Woodin cardinals, that are fundamental for second-order arithmetic and $\Omega$-logic. The precedent two cardinal notions reduce the phenomenon of incompleteness which arises within the universe of sets. Surely, infinitely-many Woodin cardinals and a proper class of Woodin are justified also extrinsically (fruitfullness of the results, i. e., what they are able to prove).

Structural reflection is fundamental from a philosophical perspective because it gives us an intrinsic philosophical justification of large cardinals. Intrinsic philosophical justifications are based on the conceptual analysis of the sets themselves. In fact, as we will see, reflection is an essential feature of the universe of sets. Bill Tait's reflection [Koellner 09] could not overcome the barrier represented by Gödel's constructible universe and Philip Welch's [Welch 10] global reflection implies embeddings of proper classes which can be seen as problematic mathematical objects. Therefore, structural reflection seems to be more natural and a more powerful method to characterize very large cardinal numbers. The best feature of this kind of reflection is that it seems to improve on Tait's and Welch's methods of reflection. On the one side, it produces a proper class of supercompact cardinals and so it transcends Gödel's constructible universe. On the other side structural reflection implies embeddings of sets without mentioning proper classes, which are problematic mathematical objects.

There are two main objectives that characterize this dissertation. First, I want to justify large cardinals by assuming structural reflection and I want to show that Woodin's Ultimate L is very close to V (the universal class). We will see that structural reflection produces a proper class of supercompact cardinals. Then I will prove two small lemmas suggesting that structural reflection supports the philosophical thesis that Woodin's Ultimate L (inner model for a supercompact cardinal) is very close to V if the Ultimate L conjecture is true.

Second, in this dissertation (as I was saying before) I will apply structural reflection to inner models. In fact, I will introduce the philosophical concept of richness. When we relativize structural reflection to inner models, we transcend these inner models by producing sharps. We have thus a richer universe. Richness can also be used as a justification for structural reflection itself. To understand richness we have to become aware that each inner model is a universe and that when we transcend it, we get a richer picture of the universe. I must stress also at this point that richness is a different concept from maximality. I will argue that structural reflection forces us to sustain weak metamathematical potentialism concerning the universe of sets. In fact, we can speak of weak metamathematical potentialism concerning $\Pi_1$ structural reflection relativized to inner models. This is because when we relativize $\Pi_1$ structural reflection to specific canonical inner models such as Gödel constructible model, inner models for iterated sharps, inner models of measurability, inner models of iterated daggers and an inner model for a strong cardinal, we obtain transcendence over these inner models. Whereas I will prove that when we reach the level of a supercompact cardinal and we relativize $\Pi_1$ structural reflection to a weak extender model N, because of the closure properties of this inner model we do not get transcendence over this inner model. Thus, the main objective of this dissertation is that structural reflection supports mathematically my philosophical belief that Woodin's Ultimate L (inner model for a supercompact cardinal) can be considered as the true, noumenal, universe of mathematics and is very close to V. Surely, V is still the usual universe of mathematics but if the Ultimate L conjecture were true, by Woodin's theorem (Transference theorem) [Woodin 10b] the Ultimate L would be very close to V and it could be considered as the true, noumenal universe of mathematics. This is an important aspect because within the Ultimate L, undecided mathematical statements such as CH would be settled.

I will apply a metaphysical Kantian distinction to set theory. Thus, to express my philosophical position, I apply a Kantian distinction between phenomenal reality and noumenal reality to set theory. For Kant, the phenomenal reality is the realm of appearance and it is not what it is really (the reality in itself). While the noumenal reality is what it is really. I will argue that in set theory the phenomenal reality is created by human mind and is represented by metamathematical models such that $L[U], K^{DJ}, V[G]$, etc. While the noumenal reality is the immutable, eternal, true world of sets itself independent from human mind and where sets are not interpreted. Thus, I will argue that we have to distinguish within set theory between the phenomenal metamathematical models (the phenomenal reality of set theory) and the true noumenal universe of mathematics. In fact, we have to distinguish between the mathematics of models concerning the phenomenal reality of set theory and the mathematics concerning the true noumenal universe of sets[1]. I will argue that this distinction disappears within the universe of mathematics if the Ultimate L conjecture is true. In fact I will say that if we have an inner model (strategic variation), namely $L^{\Omega}_{S}$, for a supercompact cardinal, this inner model, although a phenomenal reality created by human mind, would be close the true noumenal universe of sets. This inner model would be very close to V (the universal class) since it would be like L in the case that $0^{\sharp}$ does not

[1]This idea is originated from a profound and intense discussion that I had with Hugh Woodin at the Isaac Newton Mathematical Institute in Cambridge.

exist and for a suitable extender inner model $\mathbb{M}$ strong large cardinal axioms transfer down from V to $\mathbb{M}$. So, if the Ultimate conjecture L is true, a phenomenal reality would be very close to the noumenal universe of sets V. In this case, the inner model of a supercompact cardinal would be the true universe of mathematics. So we have to distinguish between the phenomenal set theory (mathematics of models) and noumenal set theory (mathematics concerning the Ultimate L structure).

The phenomenal mathematics of models instead is characterized by all metamathematical models, inner and outer models (forcing extensions). However, if the ultimate L conjecture is true, all consistent enlargements of L (canonical inner models where condensation can be seen as a noumenal property) can be seen as noumenal approximations to the true, noumenal universe of mathematics (the Ultimate L), while the phenomenal mathematics of models, where we combinatorially explore all possibilities for mathematics, is essentially characterized by outer models (forcing extensions). In this picture, within the phenomenal mathematics of models, we have the failure of the Continuum Hypothesis. Instead, if the Ultimate L conjecture is true, the Continuum Hypothesis holds within the Ultimate L. Therefore, we have to distinguish between phenomenal truths, characterizing the mathematics of models, and noumenal truths characterizing the true noumenal universe of mathematics if the Ultimate L conjecture is true.

In order to decide questions within the universe of sets, we should capture the notion of the noumenal, true, arbitrary set. We have two extreme methods to interpret the notion of the noumenal, arbitrary set that lie on the notion of power set. On the one side, we have strict definabilism represented by Gödel's constructible universe L, where we take all definable subsets at the successor stage. In this case, definabilism is strict because few large cardinal notions are consistent with L. On the other side, when we construct forcing extensions, we extend the notion of arbitrary set. In fact, by adopting forcing extensions, we add new sets. Thus, we should ask ourselves when we capture the notion of the noumenal set. We have a solution if the Ultimate L conjecture is true. In fact, in this case we would have a kind of extended definabilism. In fact, all known large cardinals would be consistent with the inner model of a supercompact cardinal. Then, since definabilism is kind of strong predicativism, the Ultimate L would be the true, noumenal universe of mathematics characterized by predicativism. If we want to develop a modal logic for the universe of sets and if the Ultimate L conjecture is true, truths concerning the Ultimate L would be necessary truths such as $2 + 2 = 4$ and so, if the Ultimate L conjecture is true, we would have that the Continuum Hypothesis is a necessary truth. In fact, the Continuum Hypothesis would be a necessary truth like $3 + 3 = 6$.

If the Ultimate L conjecture were not true, I would argue that we do not have access to the true, noumenal world of sets. In this case, we have to accept a strong form of pluralism. We would have only a plurality of phenomenal metamathematical models or phenomenal universes with their specific own truths. We would not have noumenal truths but only phenomenal truths. In this case, the solution to the continuum hypothesis is that we do not have a solution to the continuum hypothesis [Hamkins 10], but the countinuum hypothesis would be true in some phenomenal models or phenomenal universes and it would be false in other phenomenal universes. In this case, I will argue that we can make a philosophical choice and choose a specific phenomenal model. I will argue that the Bounded Proper Forcing Axiom does settle CH but this would be a phenomenal truth that holds in a phenomenal universe. So, If the Ultimate L conjecture were false, we would have only phenomenal set theory, a plurality of phenomenal models with their specific phenomenal truths. I would argue that a phenomenal model, where the Bounded Proper Forcing Axiom holds, is philosophically preferable. In fact, we need an $\Sigma_2$-reflecting cardinal, whose inner model is L, to prove the consistency of BPFA. So if the Ultimate L conjecture were false, among the plurality of all phenomenal metamathematical models we would select specific models supporting our choice with philosophical justifications . If the Ultimate L conjecture were false, we would have no access to the true, noumenal world of sets. Maybe, some mathematicians might be concerned that if the Ultimate L conjecture is true, the mathematical game of set theory is over. I would argue that this is not the case. In fact, the goal of mathematicians would be discovering the richness of the Ultimate L structure. If we relativize $\Pi_1$-structural reflection to a weak extender model, N, for a supercompact cardinal, we do not get transcendence over this inner model, but all embeddings of structures are within this inner model. In fact, we have the following theorem:

THEOREM 1 (Woodin 10). *Suppose that* $o^N_{Long} = \infty$. *Suppose that* $\gamma \in Ord$,

$$j : N \cap V_{\gamma+1} \longrightarrow N \cap V_{j(\gamma)+1}$$

*is an elementary embedding with critical point* $\kappa \geq \delta$. *Then* $j \in N$ .

So, now we can state the theorem that witnesses the closure properties of a weak extender model for a supercompact cardinal.

THEOREM 2. *Suppose* $o^N_{Long} = \infty$, *N is a weak extender model for* $\delta$ *supercompact, N is definable and C is a class of structures* $\Pi_1$ *definable (with parameters) in V. Then all embeddings of classes of structures relativized to N belong to N.*

I will prove this theorem in the following sections. Instead, If we relativize $\Pi_1$ structural reflection to inner models such as L, inner models of iterated sharps, inner model of measurability, inner models of iterated daggers and inner model for a strong cardinal, we get transcendence over these inner models. Whereas, if we relativize $\Pi_1$-structural reflection to inner model of a supercompact cardinal, we do not get transcendence over this inner model. Thus, we can speak (as I was saying before) of weak metamathematical potentialism concerning $\Pi_1$ structural reflection relativized to these inner models.

I argue that principles of structural reflection transfer down from V to a suitable extender model $\mathbb{M}$.

DEFINITION 2 (Bagaria 10). *A cardinal $\kappa$ is $C^{(n)}$-extendible if for every $\lambda$ greater than $\kappa$ there exists an elementary embedding*

$$j : V_\lambda \longrightarrow V_\mu$$

*some $\mu$, with crit(j)=$\kappa$, and $V_{j(\kappa)}$ is a $\Sigma_n$-elementary substructure of V.*

We can state Bagaria's theorem:

THEOREM 3 (Bagaria 10). *The following are equivalent:*

*(1) SR, i. e, $\boldsymbol{\Sigma_n}$-SR for all n.*

*(2) There exists a $C^{(n)}$-cardinal, for every n.*

*(3) Vopěnka's principle.*

Since Hugh Woodin [Woodin 10], by assuming that the Ultimate L exists, is able to transfer down from V to a suitable extender model very large cardinal notions, we should be able to transfer down from V to $\mathbb{M}$ a proper class of $C^{(n)}$-extendible cardinals (weaker large cardinals than what Woodin is able to transfer down). I argue that within a suitable extender model $\mathbb{M}$ there exists $C^{(n)}$-extendible cardinal for every n and so, since they are equivalent, also $\boldsymbol{\Sigma_n}$-SR and Vopěnka's principle hold in $\mathbb{M}$. In fact we have the following theorem that implies that stronger large cardinals numbers than $C^{(n)}$-extendible cardinals transfer down from V to $\mathbb{M}$.

THEOREM 4 (Woodin 10). *Suppose $2 < \kappa < \omega$, $\mathbb{M}$ is a suitable extender model, and*

$$j : V_\lambda \longrightarrow V_\lambda$$

*is an elementary embedding such that $\delta_{\mathbb{M}}$-supercompact $<$ crit(j) and such that $V_\lambda \prec_{\Sigma_\kappa} V$. Then, there exists a $\lambda' \leq \lambda$ and a nontrivial elementary embedding*

$$j^1 : \mathbb{M} \cap V_{\lambda'} \longrightarrow \mathbb{M} \cap V_{\lambda'}$$

*such that* $\mathbb{M} \cap V_{\lambda'} \prec_{\Sigma_\kappa} \mathbb{M}$ *and such that* $j^1 \in \mathbb{M}$.

Woodin is able to transfer down these very large cardinal numbers so we have to readapt his proof to transfer a proper class of $C^{(n)}$-extendible cardinals down from V to $\mathbb{M}$.

THEOREM 5. *Assume that for every n, there exists a* $C^{(n)}$*-extendible cardinal in V. Then in* $\mathbb{M}$ *(suitable extender model), for every n, there exists a* $C^{(n)}$*-extendible cardinal.*

Since $\boldsymbol{\Sigma_n}$-SR and Vopěnka's principle hold within a suitable extender model, structural reflection witnesses that the Ultimate L is very close to V if the ultimate L conjecture is true. In fact, these principles of structural reflection that hold in V, hold within a suitable extender model.

As i was saying before, by applying structural reflection to inner models we get transcendence over these inner models. If we apply structural reflection to L, we obtain $0^\sharp$ and then we can iterate this operation. In the following sections, I will introduce the metamathematical operation $Inn^{M,n}$ by which we can form a canonical inner model when we apply it to a sharp. Then, I will introduce the finite structural reflection hierarchy, a metamathematical hierarchy which at successor stage is constituted by the application of structural reflection to canonical inner models and by the application of the operation $Inn^{M,n}$ to the sharp produced in the first step by structural reflection. The finite structural reflection hierarchy belongs to the Dodd-Jensen core model $K^{DJ}$ and it is equivalent to an initial segment of the hierarchy of iterated mice. Then, following a result of Neeman [Neeman 06], I will introduce the following conjecture:

(SRC) For every natural number n, one can build a canonical inner model $K$ for n-Woodin cardinals, so that some form of structural reflection for this $K$ is equivalent to $\Pi^1_{n+1}$-determinacy.

The first part of this dissertation is devoted to see why the phenomenon of incompleteness is an essential feature of mathematics. In the first chapter, we will examine first-order, second-order and third-order arithmetic. We will consider many attempts, conceived by mathematicians, to avoid incompleteness. In fact, we will see that there are unprovable mathematical statements within first-order, second-order and third-order arithmetic and we will become aware that by introducing particular large cardinal axioms, we are able to reduce the phenomenon of incompleteness. Thus, we must justify intrinsically these large cardinals. I will argue that the justification of Determinacy axioms and Forcing axioms follows from the justification of large cardinals. I will argue also that if the Ultimate L conjecture is false we have a phenomenal solution to the Continuum Hypothesis. The second chapter is devoted to reflection. In this chapter we will examine all different kinds of reflection that occur in mathematics. In the third chapter, I will discuss structural reflection. This chapter is the most innovative from mathematical perspective. Here, I will apply structural reflection to inner models and I will get transcendence over these inner models. At the end of this chapter, I will discuss also the philosophy of mathematics which I sustain. The fourth chapter is characterized by philosophical ideas. In this chapter, I will attempt to apply philosophical ideas to mathematical results and mathematical truths to philosophical theories.

CHAPTER 1

# The Dream of Completeness

**0.1. Preliminaries to this chapter.** In this chapter I will discuss the phenomenon of incompleteness in arithmetic and set theory. I will discuss how the phenomenon of incompleteness, discovered by Gödel, appears in first-order arithmetic, second-order arithmetic and, finally, third-order arithmetic where the Continuum Hypothesis is formulated. This chapter is fundamental since we will be aware that some truths cannot be proved. Along the way towards third-order arithmetic, I will examine different axioms that were assumed by mathematicians to settle undecided questions. In the first section, I will introduce Gödel's incompleteness theorems. Gödel's sentences are unprovable truths of first-order arithmetic. A fundamental aspect will be explained in section four when I will discuss set theory. In fact, we will see that a problem formulated by Luzin, considered an unprovable truth at the beginning of the last century, was settled by introducing an axiom which asserts the existence of infinitely many Woodin cardinals. In second section I will explain Turing's completeness result about transfinite progressions. Turing, by going into the transfinite, attempted to settle first-order arithmetical sentences including Gödel's sentences. Unfortunately, Turing's attempt was doomed to fail because of a problem connected with ordinal notation, as we will see. In the third section I will discuss the phenomenon of incompleteness in set theory. Departing from second-order arithmetic we will introduce the continuum hypothesis, formulated in third-order arithmetic, which the axioms of ZFC theory do not settle. In this section, I will discuss an axiom, namely the Bounded Proper Forcing Axiom [Woodin 10b], which, according to my philosophical beliefs, can be seen as a phenomenal solution to the continuum hypothesis if the Ultimate L conjecture is false. At the end of this section, I will review Woodin's result about $\Omega$-logic. Woodin attempted to formulate a complete theory for third-order arithmetic which depends on the $\Omega$-conjecture.

## 1. Gödel's theorems

**1.1. Prerequisites to this section.** The language of arithmetic consists of first-order logic apparatus and the following symbols: 0-ary function symbol (costant) 0, unuary function symbol S (the successor function), two binary function symbols $+, \times$, two binary relation symbols $=, <$ and for each n, infinitely many n-ary predicate symbols $X_n$. Now we can introduce Levy's hierarchy. A formula $\phi$ is $\Sigma_0$ or $\Pi_0$ ($\Delta_0$) if and only if it does not contain unbounded quantificators. For $n \geq 1$, by recursion, we assert that $\phi$ is $\Sigma_n$ if and only if has the form $\exists \tilde{x} \psi(\tilde{x})$ where $\psi(\tilde{x})$ is $\Pi_{n-1}$. and that $\phi$ is $\Pi_n$ if and only if it has the following form $\forall \tilde{x} \psi(\tilde{x})$ where $\psi(\tilde{x})$ is $\Sigma_{n-1}$. Therefore, when we assert that a formula is $\Sigma_n$, we want to say, first of all, that it consists of a $\Delta_0$ formula which has n blocks of existential quantificators in front . Secondly, this formula starts with a block of existential quantificators. Thirdly, this formula is characterized by an alternation of blocks of universal quantificators and blocks of existential quantificators. A formula is $\Delta_1$ if it is equivalent to both a $\Sigma_1$ and a $\Pi_1$ formula. Usually, we will use also superscripts that point out to the order of formulas. For example a $\Pi_1^0$ formula starts with an unbounded block of universal quantificators and it is a first-order formula. Let $n > 0$ be a natural number and let us consider the nth order predicate calculus. There are variables of orders 1, 2, .....,n and the quantifiers are applied to variables of all orders. An nth order formula contains, in addition to first-order symbols and higher order quantifiers, predicates $X(z)$ where $X$ and $z$ are variables of order $\kappa + 1$ and $\kappa$ respectively (for any $\kappa < n$). Satisfaction for an nth order formula in a model $M = (A, P, ......, f, ......., c, ......, )$ is defined as follows: variables of first-order are interpreted as elements of the set A, variables of second-order as elements of P(A) (as subsets of A), etc; variables of order n are interpreted as elements of $P^{n-1}(A)$. The predicate $X(z)$ is interpreted as $z \in X$. A $\Pi_m^n$ formula is a formula of order $n+1$ of the form $\forall X \exists Y........\psi$ (m quantifiers) where X, Y, are $(n+1)th$ order variables and $\psi$ is such that all quantified variables are of order at most n. Similarly, a $\Sigma^n_m$ formula is the same but with $\exists$ and $\forall$ interchanged. See [Jech 06]

**1.2. Preliminaries to this section.** In section 1.3 we will consider two arithmetical statements that cannot be proven by PA. Sometimes mathematicians say that Gödel's sentences are not mathematically interesting. So, I want to consider Goodstein's theorem and an extension of the finite Ramsey theorem, two arithmetical statement which PA cannot prove. So, we can say that the phenomenon of incompleteness is an essential feature of first-order arithmetic. We will see in the following sections that the phenomenon of incompleteness also appears naturally in second-order arithmetic and in third-order arithmetic. To escape from incompleteness, we have to make very strong assumptions. In section 1.5 I will present some notions of computability. I will define the notions of primitive recursive functions and partial recursive functions. Then, I will explain Church's thesis and I will discuss it philosophically in connection with the consistency of ZFC and Intuitionism. Finally, I will introduce Turing's Universe and Turing's degrees of computability. Gödel's first incompleteness theorem establishes that there is a missmatch between truth and theoremhood within PA. This section aims at showing what is the distance between truth and theoremhood within PA in terms of Turing's degrees of computability. In this section, I will introduce also some notions related to intuitionism. In fact, I will argue that Church's thesis can be considered as potentially true but it cannot be seen as an atemporal truth. In section 1.6 I will discuss Gödel's incompleteness theorems. I will show how it is possible to construct a Gödel's sentence. In this section we will discuss how the phenomenon of incompleteness was discovered by Gödel in 1931. In the first section we have discussed statements unprovable within PA mathematically interesting (Goodstein's theorem and the extended finite Ramsay theorem), in this section we will examine the original construction of Gödel.

**1.3. Brief introduction to unprovable truths that are mathematically interesting.** I entitled this chapter the dream of completeness because at the beginning of the last century many mathematicians believed that all mathematical truths could be proved. The axiomatic systems, such as Peano arithmetic and Zermelo-Frankel axiomatic set theory, were considered to be complete. We could prove all truths by deducing them from the axioms. A theory is complete if for every formula, the theory can prove the formula itself or its negation. Unfortunately, in 1930, Kurt Gödel proved that no consistent axiomatic theory that is sufficiently strong is negation complete. There are truths that cannot be proved. The day after Gödel communicated his famous result to a philosophical meeting in Könisberg, in September 1930, David Hilbert could be found in another part of the same city delivering the opening address to the Society of German Scientists and Physicians, famously declaring:

> For the mathematician there is no Ignorabimus, and, in my opinion, not at all for natural science either......... The true reason why (no one) has succeeded in finding an unsolvable problem is, in my opininion, that there is no unsolvable problem. In contrast to the foolish Ignorabimus, our credo avers: We must know, We shall know.

For the first incompleteness theorem there is a sentence (Gödel sentence) that is true but unprovable within Peano axiomatic number system. Gödel sentence says that *I am unprovable* and it is true because it is unprovable. At the first look, it can seem a self-referential sentence which is similar to the liar paradox, but it is not the case. In fact, for Gödel's coding (as we will see later), Gödel sentence is an arithmetical sentence expressed in the language of arithmetic. Only at the moment that we decode the sentence we discover that this sentence says of itself to be unprovable. So Peano axiomatic system, which aims at pinning down the structure of natural numbers is incomplete. There are truths that cannot be proved.

Let us introduce the axioms of Peano's first-order axiomatic system (PA).

The language of PA is a first-order language whose non-logical vocabulary includes the constant 0 (zero), the one-place function S (the successor function) and the two-place functions + (addition) and × (multiplication). The axioms are the following:

1) $\forall x(0 \neq Sx)$

2) $\forall x \forall y(Sx = Sy \longrightarrow x = y)$

3) $\forall x(x + 0 = x)$

4) $\forall x \forall y(x + Sy = S(x + y))$

5) $\forall x(x \times 0 = 0)$

6) $\forall x \forall y(x \times Sy = (x \times y) + x)$

7) (Induction schema) $\phi(0) \wedge \forall x(\phi(x) \longrightarrow \phi(S(x)) \longrightarrow \forall x \phi(x)$, for every formula.

The most problematic axiom is the Induction schema, since by assuming this axiom, we are refering to numerical properties. Thus, ideally we should be able to quantify over numerical properties (sets). So we should adopt a second-order version of it. But in first-order axiomatic system, quantifiers range over the domain of numbers, so we are forced to adopt first-order language. The solution is represented by the fact that we use a schema. Thus, any first-order formula expressing a property which fits the template is an induction axiom.

An important subsystem of Peano axiomatic system is Robinson's arithmetic, (**Q**), which has the following axioms:

1) $\forall x(0 \neq Sx)$

2) $\forall x \forall y(Sx = Sy \longrightarrow x = y)$

3) $\forall x(x \neq 0 \longrightarrow \exists y(x = Sy))$

4) $\forall x(x + 0 = x)$

5) $\forall x \forall y(x + Sy = S(x + y))$

6) $\forall x(x \times 0 = 0)$

7) $\forall x \forall y(x \times Sy = (x \times y) + x)$

**Q** is a sound theory, its axioms are all true in the standard model of arithmetic and its logic is truth-preserving. But, **Q** is incomplete. There are very simple true quantified sentences that **Q** cannot prove. It cannot prove universal generalizations. Since **Q** lacks the induction schema, it cannot handle all quantified sentences. However, although Robinson's arithmetic is a weak theory, it is very interesting. In fact, **Q** is sufficiently strong. This weak subsystem of Peano's arithmetic is $\Sigma_1$-complete. It can prove all true $\Sigma_1$ sentences. Furthermore, all primitive recursive functions can be expressed by a $\Sigma_1$ formula in Q [1] sentences. Therefore, **Q** can represent all primitive recursive functions including the demonstrability predicate, fundamental in the construction of the undecidable Gödel sentence. Suppose a theory of arithmetic is formally axiomatized, consistent and can prove everything that **Q** can prove (a very weak requirement). Then this theory will be sufficiently strong and so will be incomplete since it will be possible within this theory to construct Gödel's undecidable sentence.

The first incompleteness theorem undermines *Principia Mathematica*'s logicism.[2] However in 1931, the logicist project was over. Instead, the dominant project was Hilbert's program which aimed at showing that infinitary mathematics was not contradictory. Hilbert was thinking that we should divide mathematics into a core of uncontentious real mathematics and a superstructure of ideal mathematics. Propositions of real mathematics are simply true or false. Four plus two is six and two plus one is three. We could say according to the simplicity of the statements [Smith 06] that $\Pi_1$-statements of arithmetic belong to Hilbert's uncontentious real mathematics. We will discover later that many $\Pi_1$-statement are unprovable, such as Gödel sentence, the consistency statement (Gödel second incompleteness theorem) and Goldbach's conjecture whereas other $\Pi_1$ statements are provable

[1]In the language of arithmetic $\Delta_0$ formulas are bounded formulas built up using identity, the less-then-or-equal relation, propositional connectives and bounded quantifiers. $\Sigma_1$ formulas are unbounded existential quantifications of $\Delta_0$ formulas and $\Pi_1$ are universal unbounded quantifications of $\Delta_0$ formulas.

[2]We mean by Logicism a theory which implies that all arithmetical truths can be derived from basic, self-evident, logical truths. This theory aims at constructing mathematics upon logic.

such as the Last theorem of Fermat. By contrast, ideal mathematics shouldn't be thought of as having representational content and its sentences aren't strictly-speaking true or false. In pursuing this idea, Hilbert took a very restricted view of real mathematics. Influenced by Kant, Hilbert thought that the most certain of arithmetic was grounded on intuition, which enabled us to understand finite sequences of numbers and results when we manipulated them. Hilbert's view is characterised by two components, namely strict finitism and a formalistic approach towards mathematics. For the German mathematician mathematics is represented by finite strings of symbols that we manipulate. Maybe we can identify what Hilbert was thinking by using the term *real core mathematics*, with the theory PRA, namely first-order arithmetic plus primitive recursive functions. In fact from one side PRA is a theory about arithmetic and from the other side it is strong enough to capture all primitive recursive functions. So according to Hilbert's view, we must distinguish real core mathematics from its ideal superstructure (such as set theory). Then you want to know which bits of ideal mathematics are safe to use, are real-sound, namely what ideal mathematics proves is true. For this one has to find which parts of ideal mathematics can be proved finitistically consistent. A corollary of the first Gödel incompleteness theorem was the second Gödel incompleteness theorem which states: no consistent sufficiently strong theory can prove its own consistency. Robinson's arithmetic (**Q**) and Peano arithmetic (PA) cannot have a proof of their own consistency. So no modest formal arithmetic can establish the consistency of a fancy ideal theory. So we cannot have consistency proofs for branches of ideal mathematics. Therefore, Hilbert's project of trying to establish the real soundness of ideal mathematics by giving consistency proofs using real and contentual mathematics was demolished by Gödel's second incompleteness theorem.

Returning to Gödel's first incompleteness theorem, we have that Gödel sentence is unprovable or undecidable. We can also say that it is incomputable. We use the term computable for functions, namely computable by a Turing machine or by recursion, when the informal instructions of an algorithm are made formal. Using the term computable truth means that we can give a proof of that truth (tree proof or linear sequence proof). At this point, we have to clarify the concept of truth in mathematics: why a mathematical sentence is true? We could answer that a mathematical sentence is true because it is proved within the axiomatic system such as PA, or outside the system, or because there is an independent mathematical reality which makes the sentence true. However, mathematical truth is a definite and precise mathematical property that we express by inductive definitions. Alfred Tarski introduced inductive definitions of truth which made the notion of truth a precise mathematical property. Gödel proved his two incompleteness theorems by looking outside the formal system[3] and when we come accross Gödel sentence, we discover that it is true because it is unprovable. So there is a strong link between truth and provability in mathematics, but thanks to Gödel's theorem we can say that there is a miss-match between truths and proofs. I entitled this section *the dream of completeness* yet around 1929 many mathematicians were believing that it would have been possible that Peano axiomatic system was negation-complete. In fact in 1929 Mojźesz Presburger proved that the theory P (PA Peano arithmetic minus multiplication) was negation-complete. In the same year, Thoralf Skolem proved that a theory with multiplication, but lacking addition, was negation complete. Therefore, many mathematicians were hoping that also Peano arithmetic was negation-complete. It is interesting to know that Presburger used in his proof a model-theoretic procedure (quantifier elimination) which also Alfred Tarski later adopted to show that the theory of real closed fields is negation-complete. Therefore in 1929 many mathematicians were thinking that also Peano arithmetic PA would be a negation-complete theory. In fact, even Gödel attempted to prove the completeness of Peano arithmetic. But if arithmetic with multiplication minus addition, and arithmetic with addition minus multiplication, are negation-complete theories we should ask ourselves why when we put together these two operations we have the phenomenon of incompleteness. The reason is

[3]In 1938 Hilbert and Bernays gave a formal proof of Gödel's theorems within the system.

that thanks to addition and multiplication we can construct a chain of primitive recursive functions and we can show at the end that the predicate of demonstrability **Bew** is primitive recursive. Since in Peano arithmetic all primitive recursive functions are representable, also the predicate of demonstrability is representable and so we can construct Gödel's sentence which says of itself to be unprovable. Sometimes mathematicians assert that Gödel sentences are not mathematically interesting.

**1.4. Unprovable mathematical statements that are mathematically interesting.** Paris and Kirby proved that an arithmetical statement (mathematically interesting) was undecidable by PA. Goodstein theorem is expressible in PA by a $\Pi_2$ [4] sentence. But we can ask ourselves if Goodstein theorem can be proven in PA. Maybe we need a long proof but at the end we can prove Goodstein theorem within PA. Unfortunately the answer is negative for the following theorem:

THEOREM 6 (Kirby-Paris 82). *If PA is consistent, then Goodstein theorem is undecidable in PA.*

Therefore the arithmetical proposition which expresses Goodstein theorem cannot be proved in PA. Goodstein theorem is an example of an arithmetical statement unprovable in PA and it is mathematically interesting . In 1977, Jeff Paris and Leo Harrington found another arithmetical statement that PA could not prove. This statement is an extension of the finite Ramsey theorem (Ext(FRT)). The extension of FRT (Ext(FRT)) is true in the standard model ($N \models Ext(FRT)$) but it cannot be proven within PA. So Ext(FRT) is another example of an arithmetical proposition undecidable in PA. To start, we have to prove Ext-FRT, but in this proof we need to prove the infinite Ramsey theorem and this proof cannot be accomplished within PA because this proof requires König Lemma which

[4] Alan Turing, in 1934, when he was working on transfinite progressions, he was really interested in obtaining a completeness result for $\Pi_2$ sentences, even if he was able to prove only the completeness of transfinite progressions for $\Pi_1$ sentences.

cannot be formalised within PA. If we introduce a theory S by adding individual constants to PA and we will have:

$$PA \vdash Con(S) \longrightarrow Con(PA)$$

then we will have:

$$PA \vdash Ext(FRT) \longrightarrow Con(S)$$

Therefore if Ext(FRT) were proved within PA, it would be possible to prove the consistency of PA, but this is impossible by Gödel's second incompleteness theorem. So as in the case of Goodstein's theorem, we can say that Gödel's second incompleteness theorem is fundamental for obtaining undecidable arithmetical sentences. In the case of Goodstein theorem, we have that if PA could prove it, then PA would be able to prove its own consistency by Gentzen's proof-theoretic reasoning. In the case of the finite Ramsey theorem, we have that if PA could prove it, then PA would be able to prove its own consistency because the extension of the finite Ramsey theorem implies the consistency of the theory S and the consistency of the theory S implies the consistency of PA. But in both cases, this is impossible by Gödel's second incompleteness theorem. Thus, to prove that these two interesting arithmetical sentences are undecidable within PA, it is fundamental to assume Gödel's second incompleteness theorem. Gödel's sentence (**G**) and the consistency statement (Con(PA)) are both $\Pi_1$ undecidable sentences. Between these sentences there is a strong connection. In fact, the impossibility of proving Con(PA) derives directly (it is a corollary) from the impossibility of proving Gödel sentence (**G**). If we take Gödel's sentence, Goodstein's theorem and the finite Ramsey theorem, they are all undecidable sentences but they are separated, there is not a direct connection between them. However, the impossibility of proving Goodstein's theorem and the finite Ramsey theorem within

PA is based on the impossibility of proving Con(PA). Therefore we can say that the phenomenon of incompleteness of PA stems from a combination of both Gödel's theorems. The impossibility of proving Gödel's sentence renders impossible to prove Con(PA) and the impossibility of proving Con(PA) makes impossible to prove Goodstein and the finite Ramsey theorem within PA. The fact, that these two arithmetical sentences (mathematically interesting) are undecidable, is based essentially on Gödel's second incompleteness theorem. Gödel's second incompleteness theorem is also important in the theory ZFC (Zermelo-Frankel axiomatic set theory). In fact, if $\kappa$ is a large cardinal, $V_\kappa$ would be a model of ZFC and so the existence of this large cardinal cannot be proved in ZFC because of Gödel's second incompleteness theorem. Large cardinals can exist in ZFC universe but their existence cannot be proved in ZFC because of Gödel's second incompleteness theorem. The fact that we cannot prove directly Con(ZFC), forces us to have relative consistency proof. Assuming only that a stronger theory is consistent (ZFC + Axiom(one)), we prove Con(ZFC). Gödel's second incompleteness theorem forces to go higher in the large cardinal hierarchy. By introducing a new large cardinal [5] axiom, a stronger theory, (ZFC + Axiom(one)) we can prove the consistency of a weaker theory, namely Con(ZFC). Then by introducing a large cardinal $\lambda > \kappa$ we can prove the consistency statement (Con(ZFC + Axiom($\kappa$)) and so on. Mathematicians would say that we introduce large cardinals to settle undecided questions (Gödel's program). In fact Gödel's second incompleteness theorem renders Con(ZFC), Con(ZFC+ Axiom($\kappa$)), Con(ZFC + Axiom($\lambda$)) all undecidable sentences respectively within ZFC, ZFC + Axiom($\kappa$) and ZFC + Axiom($\lambda$) and we are forced to introduce larger and larger cardinal numbers to settle all these undecidable sentences. Therefore to sum up the structure of Paris and Harrington's proof, we can state the following formal expressions:

$$PA \vdash Con(S) \longrightarrow Con(PA)$$

[5]Even if it is too early for large cardinals, I want to introduce this idea related to arithmetic

then we have:

$$PA \vdash Ext(FRT) \longrightarrow Con(S)$$

Therefore if Ext(FRT) were proved within PA, it would be possible to prove the consistency of PA, but this is impossible by Gödel's second incompleteness theorem.

At this point, let's consider *Isaacson's conjecture* that can be seen as a limit to the acceptability of Kirby-Paris and Paris-Harrington theorems. We have to notice that Paris-Kirby theorem involves a kind of reasoning that goes beyond what is required for understanding the basic arithmetic of finite numbers. In fact, in order to prove that Goodstein's theorem is independent from PA, we need to adopt transfinite induction up to $\epsilon_0$. We can say the same also about Paris-Harrington theorem. To prove that the extension of the finite Ramsey theorem is independent from PA, we need König's lemma, namely an infinite tree that only branches finitely at any point must have an infinite path through it. So we can state *Isaacson's conjecture*:

If we are to give a rationally compelling proof of any true sentence which is independent of PA, then we will need to appeal to ideas that go beyond those which are constitutive of basic arithmetic.

Also to understand the truth of undecidable Gödel's sentences for PA, it seems to require conceptual skills which go beyond our practise of elementary operations applied to finite natural numbers. The problem that we face when we evaluate *Isaacson's conjecture* is the same as when we try to understand what Hilbert was thinking for real mathematics by adopting finitistic methods. What do we mean for pure arithmetical knowledge? it is difficult to say what are the contents of pure arithmetical knowledge and what are the limits of pure arithmetical reasoning. Therefore, the truth of *Isaacson's conjecture* depends on our personal and subjective evaluation of what we consider as pure arithmetical knowledge.

When we were speaking about Goodstein's theorem we have quoted Gentzen's consistency proof of PA. Now, I want to highlight the theory which is able to handle Gentzen's proof-theoretical reasoning. Gentzen was able to prove the consistency of PA by appealing to a theory that was weaker then PA in some respects and stronger then PA in others. In fact, he could not use a stronger theory which contained PA since all doubts about the consistency of PA would become doubts about the stronger theory. Furthermore, he could not use a weaker theory since Gödel second incompleteness theorem shows that no weaker theory contained in PA can prove PA consistency. For his proof Gentzen adopted transfinite induction up to $\epsilon_0$. It is possible to show that we can handle Gentzen's proof by appealing to the theory $PRA_0$ (quantifier-free primitive recursive arithmetic) and by adding to this theory enough transfinite induction to deal with quantifier-free formulae. In fact, in this theory we have all primitive recursive functions and we can cope with transfinite induction for quantifier free-formulae. We can say that the theory $PRA_0 + TI(\epsilon_0)$ is enough to show the consistency of PA. This theory is neither contained in PA (since it can prove Con(PA) by Gentzen's proof theoretic reasoning, which PA cannot), nor it contains PA (since it cannot prove quantifier-involving instances of the Induction schema). It is important to notice that if we use the notation $\omega_\kappa$ as a tower of $\kappa$-times $\omega$, namely $\omega_\kappa = \omega^{\omega^{\omega^{\omega^{\omega^{\cdots^{\omega_\kappa}}}}}}$ (we have to recall that $\epsilon_0$ is a tower of $\omega$ - times $\omega$, namely $\epsilon_0 = \omega^{\omega_1^{\omega_2^{\omega_3^{\omega_4^{\omega_5^{\cdots^{\omega_\omega}}}}}}}$), we can prove that the following induction principle:

$$TI(\omega_\kappa) = \forall x((\forall y < xP(y) \longrightarrow P(x)) \longrightarrow \forall x < \omega_\kappa P(x))$$

can be derived in PA, for every $\kappa$. Whereas this other principle:

$$TI(\epsilon_0) = \forall x((\forall y < xP(y) \longrightarrow P(x)) \longrightarrow \forall x P(x))$$

cannot be proved in PA. This principle is another sentence that it is true but independent from PA.[6]

**1.5. Notions of computability, Turing's universe and Intuitionism.** At this point, before constructing Gödel's sentence, I want to speak a little about computability. This section aims at showing what is the distance between truth and theoremhood within PA in terms of Turing's degrees of computability. Computability is strongly connected to completeness. Actually, we should say that incompleteness is a subclass of incomputability. To compute a function, we need the notion of algorithm which is a set of finite informal instructions. If we want to compute a function, we have to follow all informal steps of an algorithm. However, we have always to cope with informal instructions. Alan Turing and Kurt Gödel were focusing at rendering the informal notion of algorithm formal. At this point, let's introduce Church's thesis.

DEFINITION 3. *(Church) $f$ is effectively computable if and only if it is partial recursive.*

Thanks to this thesis, the informal side of computation (algorithm) is combined with the formal side of computation (partial recursive functions). $f$ is effectively computable if there exists some description of an algorithm, in some language, which can be used to compute any value $f(x)$ for which $f(x)\downarrow$. Church's thesis is independent from the language for computing. We establish a strong equivalence between all models of computations and formulate Church's thesis for all these different models (Lambda calculus, Turing machine, and unlimited register machine). Functions, that can be computed, are the same independently of the model of computation that we adopt. Church's thesis states that if someone can give a description of an algorithm for computing $f$, then there is a description of $f$ as a partial recursive function or a Turing machine or in Lambda calculus or as an unlimited

[6] It is interesting to notice that in 1931, Jaques Herbrand was able to prove the consistency of a fragment of arithmetic. Kurt Gödel considered this result as the most important partial result for the Hilbert program; Herbrand result was based on his fundamental theorem which implies a quantifier elimination procedure, namely a reduction of predicate calculus to propositional calculus.

register machine. Church's thesis is true until now, because nobody has been able to find a counterexample to this thesis. However, it is possible to conceive a counterfactual situation or, possible world, where someone is capable of constructing an algorithm for computing $f(x)$ which does not have a formal description as a partial recursive function or as a Turing machine. By considering Church's thesis as true, we are introducing a temporal component in our world of mathematics. Church's thesis is true until now, but we cannot exclude that in the future someone will disprove it (finding a particular informal algorithm). Furthermore, we can say that Church's thesis is potentially true and has a temporal component (I will clarify these notions immediately after the introduction of some ideas related to intuitionism). When someone proves a theorem, according to classical mathematics, this theorem is atemporally true and actual true (I will explain this notion immediately). In classical mathematics, a truth does not have the dimension of time and is atemporal, because a proposition is true also before that a proof is constructed. Truths are outside the dimension of time and by constructing proofs, according to the classical vision of mathematics, we simply discover and capture them. In the case of Church's thesis, there is a temporal component, namely until now it is true. Church's thesis has a temporal component. Maybe, we should adopt a different conception of mathematics, such as intuitionism where the notion of time comes into the realm of mathematics. As Church's thesis, also the consistency of ZFC has a temporal component. Because of Gödel's second incompleteness theorem, we cannot prove directly the consistency of ZFC. Of course, we can trust the ZFC system, but we cannot exclude that in the future someone will discover a contradiction in it. Thus, ZFC is consistent until now. It has a temporal component. For the consistency of ZFC as for the truth of Church thesis, there is a temporal component which forces us to consider intutionism. To clarify this conception, I want to discuss some ideas related to intuitionism. Brouwer, the father of intuitionism, considered mathematics as activity of mental construction independent from the language. So, for Brouwer, Logic was not essential to mathematics. For Brouwer, a mathematical proposition is true when we can show a construction of it. At the beginning of his thought, Brouwer was rejecting hypothetical constructions and contradictions, but then he adopted the same view of Heyting, the other father of mathematical intuitionism. According to Heyting, $\neg A$ is true if the hypothesis that $A$ is true causes a contradiction. This is the *hypothetical interpretation* of negation which features the conception of Heyting. In 1923, Brouwer accepted hypothetical constructions and contradictions. In fact, he took position against mathematics without negation conceived by Griss. While for Brouwer mathematics was an activity without need of any languages, for Heyting language was essential for mathematics in order to communicate mathematical constructions. In fact, Heyting developed intuitionistic logic because he was thinking to render mathematics communicable in a formal language. According to Heyting, the fundamental activity of our mind is that of creating entities. This construction of abstract entities is the foundation of intuitionistic mathematics. Heyting rejects a platonistic-realistic philosophy of mathematics. In fact, in 1939, he wrote:

> An intuitionistic mathematician would not take position against a philosophy which holds that mind, during his creative activity, reproduces entities of a transcendent world, but he would consider this doctrine too speculative as foundation of pure mathematics. [Heyting 39]

Heyting rejects the idea that there is a transcendent world of mathematics independent from human mind, which renders mathematical propositions true or false, but for Heyting mathematics is a **creation** of human mind. Furthermore, he wants to change the classical vision of mathematics by saying that truth is not anymore the fundamental notion but intuitionistic mathematics is based on the notion of knowledge. For Heyting, a mathematical proposition is true when we know that proposition because it is evident or by showing a construction (proof) of it. So, intuitionistic mathematics there are not truths independent from our act of knowing them or are preexisting to our knowledge. There are not atemporal truths in mathematics but there are only temporal truths. We could say that according to intuitionism, a mathematical proposition starts to be true because it is evident or after that we show a proof (construction) of it. In 1958, Heyting formulated the positive principle which states that every mathematical theorem is the result of a successful construction. For Brouwer and Heyting truth becomes a temporal property of propositions. When we have an actual proof or construction of a proposition, we can consider that proposition as true. Martin-Löf [Martin-Lof 91], combining Heyting's view with the classical mathematics' point of view, distinguishes between actual truth and potential truth of a proposition (he reconsiders the Aristotelian distinction between act and potentiality). So, a proposition is actual true if we have a construction or a proof of it. However, the same proposition was potentially true also before a proof of it and it will be potentially true even if nobody will prove it. So, for Martin-Löf a potential truth is independent from human knowledge and it is atemporal. Instead, following Heyting, he sustains that actual truths are dependent from human knowledge and are temporal. Also Prawitz [Prawitz 77] wants to combine intutionism with the belief that there are eternal-atemporal truths. Prawits introduces a proof-theoretic platonism. He believes that there is an independent world of proofs. Therefore, for Prawitz, proofs are actual existent but only potentially knowable by human beings. So, there might be atemporal mathematical truths because there are actual proofs in Prawitz's independent world of proofs, but we do not know them. Thus, Prawitz, in order to save atemporality in mathematics by adopting intuitionism as a point of view, he assumes a realistic-platonic philosophy of mathematics which Heyting and Brouwer would reject.

Now, we can discuss Church's thesis and the consistency of ZFC. For Heyting and Brouwer, since we do not have a construction or a proof of these two mathematical propositions, Church's thesis and Con(ZFC) cannot be considered as truths neither temporal truths.

Heyting and Brouwer would have said that we do not know these mathematical propositions and so we do not know their truth values. If we adopt Martin-Löf conception, we can say that Church's thesis and Con(ZFC) are potential truths. Thus, they are atemporal truths only because they are potential. However, they are not actual truths since we do not have yet a construction or proof of them. If we adopt Prawitz's view, we can say that, maybe, there exist atemporal proofs or constructions of Church's thesis and Con(ZFC) in the realm of the platonic-proof theoretic world independent from human mind, but we do not know these constructions. Even if I have a semi-realistic conception of mathematics (as you will see in the following chapters), I believe that what makes a mathematical proposition an atemporal-actual truth is the effective construction or proof of it. So, Church's thesis and Con(ZFC) are only potentially true. Maybe, they are atemporal truths only potentially. Until now, they can be considered only temporal truths because even if they are very convincing, we cannot exclude that in the future we will be able to find a counterexample to Church thesis or a contradiction within ZFC. We can believe in them, but if we do not have a construction or a proof of them, we cannot consider them as actual-atemporal truths. As you will see in the following chapters, I believe that if the Ultimate L conjecture is false, then the Continuum Hypothesis is settled by the Bounded Proper Forcing Axiom even if this would be a phenomenal solution according to my philosophical beliefs. The fact that the continuum is $\aleph_2$ (if the Ultimate L conjecture is false) is an actual-atemporal truth and we have a proof of it. However, the mathematical community does not accept completely this result. I have to say that within set theory, actual-atemporal truths are dependent from the assumptions (phenomenal model) that a mathematician makes. So the fact that the continuum is $\aleph_2$ is an atemporal-actual proof relative to the assumption of the Bounded Proper Forcing Axiom. Thus, in set theory, we do not have an absolute conception of actual-atemporal truths, but we have a relativistic conception of truths, since actual-atemporal truth depends on actual proof relative to the assumptions (phenomenal

model) that a mathematician makes. Sometimes, assumptions might be rejected by some mathematicians and accepted by other mathematicians.
At this point we should look for examples of incomputable sets. However, before addressing this issue, we must introduce the following definitions:

DEFINITION 4. *$A \subseteq \mathbb{N}$ is computably enumerable (c.e.) if there is an effective process for enumerating all the members of $A$. $A$ is computably enumerable if there is a computable function $f$ such that $A = \{f(0), f(1), f(2), f(3), f(4), f(5), ....\} =$ range$(f)$.*

Now we should explain how the notion of being computably enumerable relates with the notion of being computable. In 1944, Emil Post answered to this question by proving the following theorem:

THEOREM 7. *(Emil Post) If $A \subseteq N$ is computable, then $A$ is also computably enumerable.*

PROOF. We say that A is computable, so that we can effectively decide if $x \in A$ for any given $x \in N$. Then we can effectively enumerate the members of A by asking, in turn, is $0 \in A$, is $1 \in A$, is $2 \in A$, is $3 \in A$,........., and each time we get yes to the question: is $x \in A$?. Enumerating $x$. □

At this point, we can introduce the following theorem:

THEOREM 8. *$A \subseteq \mathbb{N}$ is computable iff both $A$ and $A^*$ (the complement of $A$) are computably enumerable.*

We can restate the notion of computably enumerable in the following manner:

THEOREM 9. *If $W$ is an effectively enumerable set of natural numbers, then there is some effectively decidable numerical relation $R$ such that $n \in W$ if and only if $\exists xRxn$.*

We might also restate the notion of computably enumerable set by adopting the informal side of computation in the following way:

THEOREM 10. *W is an effectively enumerable set of numbers if and only if it is the numerical domain of some algorithm* $\Pi$.

Now we can introduce the first example of incomputable set. In fact, we state the following theorem:

THEOREM 11. *There is an effectively enumerable set of numbers K such that its complement* $K^*$ *is not effectively enumerable.*

PROOF. set $K =_{def} \{e|e \in W_e\}$. For any $e$, by definition $e \in K^*$ if and only if $e \notin W_e$. Thus, $K^*$ cannot be identical to any of the $W_e$. Therefore, $K^*$ is not one of the effectively enumerable sets (since the $W_e$ are all of them) . □

At this point, I want to present a sort of phenomenology of the latter theorem's proof. In this case, we have another example of diagonalization procedure. In fact, we have the following:

$$\begin{bmatrix} 0 \in W_0?_* \; 1 \in W_0? \; 2 \in W_0? \; 3 \in W_0? \\ 0 \in W_1? \; 1 \in W_1?_* \; 2 \in W_1? \; 3 \in W_1? \\ 0 \in W_2? \; 1 \in W_2? \; 2 \in W_2?_* \; 3 \in W_2? \\ 0 \in W_3? \; 1 \in W_3? \; 2 \in W_3? \; 3 \in W_3?_* \end{bmatrix}$$

In the proof, if $W_x$ are all enumerable sets, K is the diagonal set or diagonal line which is marked by the symbol $*$. $K^*$ (the complement of K) is the antidiagonal set or antidiagonal line and it does not belong to the list. In fact if $x \in K$, $x \notin K^*$ by definition. Diagonalization is a very important tool in mathematical logic. If we enumerate a list of numbers, functions, sets or properties we might always diagonalise out . Furthermore if the members of the list such as numbers, functions, sets or properties share a distinctive feature, when we diagonalise out and we form the antidiagonal set, we can establish that the antidiagonal set does not have any more that distinctive feature. So, the first step to diagonalise out is to enumerate a list of numbers, sets, functions. For example we cannot diagonalise out from $\mu$-recursive functions, because there is not an effective procedure to determine if the search of the $\mu$-operator terminates. So we cannot diagonalise out from partial recursive functions and we cannot contradict Church's Thesis.

We encounter another example of diagonalization when we discuss Richard paradox. In logic, Richard's paradox is a semantical antinomy in set theory and natural language first described by the french mathematician Jules Richard in 1905. The original statement of the paradox has a relation to Cantor's diagonal argument of the uncountability of real numbers. The paradox begins with the observation that ceratain expressions in English unambiguosly define real numbers, while other expressions in English do not. Thus, there is an infinite list of english phrases that unambiguosly define real numbers; at this point we can use Cantor's diagonal argument to see how Richard's paradox works; arrange this list by lenght and then order lexicographically, so that the ordering is canonical. This yields an infinite list of the corresponding real numbers: $r_1$, $r_2$......,etc. Since real numbers are dense (between two real numbers, there is always a third real number), we can consider real numbers in the interval $[0, 1]$. Then we can write real numbers in binary digits in the following way:

$$\begin{bmatrix} r_1 : \mathbf{0}110001010011... \\ r_2 : 0\mathbf{1}11110101011... \\ r_3 : 11\mathbf{0}1110010111... \\ r_4 : 101\mathbf{0}111100011.... \\ r_5 : 1110\mathbf{1}11010001.... \end{bmatrix}$$

Go down the diagonal, taking the n-th digit of the n-th real number $r_n$ (in our example produces 01001) and flip each digit, swapping 0s and 1s (in our example produces 10110). By construction, this flipped diagonal real number differs from $r_1$ in the first place, from $r_2$ in the second place and so on. So our diagonal construction defines a new real (a richardian real) which differs from all the other reals. Now define a real number (Richardian real) in the following way: **the n-th digit of the n-th real number $r_n$ is the opposite** (if it is 0, it is 1 and if it is 1, it is 0). This definition is an expression in English which unambiguosly defines a real number **r** (a richardian real number). Thus **r** must be one of the $r_n$ numbers. However, **r** was constructed so that it cannot equal any of the $r_n$. This is a paradoxical contradiction. If we take formalised languages, it is possible to say that a formula $\phi(x)$ defines a real number if there is exactly one real number $r$ such that $\phi(r)$ holds. Then it is not possible to define, in ZFC, the set of all formulas that define real numbers. For, if it were possible to define this set, it would be possible to diagonalize over it to produce a new definition of a real number, following the outline of Richard's paradox above.

One problem in logic is the nature of many irrational numbers. We do not know how they are. Alan Turing was very keen on computing real numbers but we do not know their nature. At this point, iI want to discuss this philosophical thought. When you have a matrix of real numbers, namely a list of real numbers, you can form the antidiagonal set (a Richardian real). Now we can think to add this antidiagonal set to the precedent matrix, then we have a new matrix. We can diagonalise out from this matrix and form a new antidiagonal set (the second Richardian real). By accomplishing this operation, we form the third, the fourth Richardian real and so on. This operation can be iterated through the infinite and it does not have any bound. So, maybe we can think that we might charcterise a large part of irrational numbers as Richardian reals. If this operation does not have a bound, we can always diagonilise out until the set of Richardian reals overlaps the set of irrational numbers.

At this point, we can return to the original issue of computability. Let's consider the notion of creative set.

Definition 5 (Cooper 07). *We can say that* $A \subseteq N$ *is creative if and only if 1)* $A$ *is c.e., and 2) there is a computable function* $f$ *such that for each* $e$, $W_e \subset A^* \rightarrow f(e) \in A^* - W_e$. *If* $A$ *satisfies 1) e 2), we call* $f$ *the creative function for* $A$.

Now we can state the following theorem:

Theorem 12 (Cooper 07). *Creative sets do exist. In particular* $K$ *is creative.*

Now we have seen an example of incomputable set within the realm of computability. Do we have examples of incomputable sets outside the theoretical framework of computability ?

Following the greek mathematician Diophantus, Hilbert stated his famous problem: Given any polynominal equation in one or more variables, with integer coefficients, find a solution consisting entirely of integers, namely solve any Diophantine equation. (Hilbert's tenth problem) Find a general way of telling effectively whether a given Diophantine equation has a solution or not. Now we can introduce the concept of Diophantine set:

Definition 6. *A set* $A \subseteq N$ *is Diophantine if*

$$A = \{x \in N | (\exists y_1, ........, y_n \in N\}[p_A(x, y_1, .......y_n) = 0]\}$$

*for some polynominal* $p_A(x, y_1, ......, y_n)$ *(with integer coefficients).*

Martin Davis in 1950 found the key to solve Hilbert's tenth problem. Davis, Matiasevich, Putnam and Robinson proved later that the answer was negative. The strategy of Martin Davis was focused on proving that every computably enumerable set is Diophantine. In fact if K (the creative set and so incomputable) is diophantine, we obtain a negative solution to Hilbert's tenth problem. At the end, in pursuing the objective of proving the diophantine nature of larger and larger classes of computably enumerable sets, Julia Robinson, Yury Matiasevich and Hilary Putnam were able to prove the following theorem:

THEOREM 13. *(Davis, Matiasevich, Putnam, Robinson)*

*1) Every computable enumerable set is Diophantine.*

*2) There is not any positive solution to Hilbert's tenth problem.*

At this point, we can start to compare the computability of different sets of numbers A and B. Now we can introduce the following definition:

DEFINITION 7. *(Emil Post) We say B is many-one reducible (or m-reducible) to A (written* $B \leq_m A$*) if and only if there is a computable function f such that for all* $x \in N$*:*

$$x \in B \leftrightarrow f(x) \in A.$$

Now we can introduce the following two theorems:

THEOREM 14 (Cooper 07). *The ordering* $\leq_m$ *is reflexive and transitive.*

THEOREM 15 (Cooper 07). *1)If* $B \leq_m A$ *and A is computable, then B is computable. 2) If* $B \leq_m A$ *and A is computably enumerable, then B is computably enumerable.*

At this point, we can collect different sets which cannot be distinguished from each other by adopting many-one reducibility:

DEFINITION 8. *We write* $A \equiv_m B$ *(A many-one equivalent to B) if* $A \leq_m B$ *and* $B \leq_m A$*.*

LEMMA 1. $\equiv_m$ *is an equivalence relation.*

The ordering $\leq_m$ induces a structure on the equivalence classes under $\equiv_m$. Thus, we can introduce the following definition:

DEFINITION 9. *(Turing) An equivalence class under* $\equiv_m$ *is called an* $m-degree$*(or many-one degree). We write* $a_m = deg_m(A) = \{X \subseteq N | A \equiv_m X\}$

*and $D_m$ = the of all m-degrees.*

*2) We write $b_m \leq_m a_m$ if and only if $B \leq_m A$ for some $A \in a_m, B \in b_m$*

At this point, we can induce a partial ordering on $D_m$ in the following manner:

DEFINITION 10. *Let $a_m, b_m, c_m \in D_m$. then $\leq$ satisfies:*

*1) ($\leq$ is reflexive) $a_m \leq a_m$.*

*2)($\leq$ is transitive) $a_m \leq b_m \wedge b_m \leq c_m \longrightarrow a_m \leq c_m$.*

*3) ($\leq$ is antisymmetric) $a_m \leq b_m \wedge b_m \leq a_m \longrightarrow a_m = b_m$.*

*the properties 1)- 3) make $\leq$ a partial ordering on $D_m$.*

$D_m$ does have a least element but it does not have the greatest element. In fact, we can state the following corollary:

LEMMA 2. *$D_m$ has a least element $0_m$ consisting of all computable sets (other than $\emptyset$ and N).*

We do not know yet the fatness of $D_m$ and the exact contents of $0^1_m$ where we can locate all unsolvable problems, the creative sets. All computable sets, as we have said before, are located in $0_m$, the least element of $D_m$ (Turing universe). $0^1_m$ is the greatest element of all computably enumerable sets. In fact, between $0_m$ and $0^1_m$ we can find all computable enumerable sets. The fundamental point to highlight for the following discussion is that incomputability is located very low in Turing Universe. In fact, already at the level of $0^1_m$ we encounter incomputable sets, the creative sets. Since $D_m$ (Turing universe) does not have a greatest element, we can have a sequence of degrees without any bound, namely $0_m, 0^1_m, 0^2_m, 0^3_m$........$0^{\omega}_m$.....$0^{\omega+\omega}_m$........$0^{\epsilon_0}_m$....., and we are able to find computable sets only at the level of $0_m$. Already at the level of $0^1_m$ we find the creative sets and the phenomenon of incomputability arises. Therefore, we can state that Turing universe is essentially characterised by the phenomenon of incomputability and computability covers only a tiny part of this universe which includes computable and incomputable problems. There are much more incomputable sets than computable sets. So, we should ask ourselves why we have few examples of incomputable problems and many examples of computable problems. One reason might be the fact that we are always looking for computable problems. When we state a problem or a question to solve, there is already in the question a way to compute or to solve the problem. For instance, Gödel (Gödel sentences), Cohen (continuum hypothesis), Turing (halting problem) and Church (undecidability of first-order logic) were looking at all these problems in order to compute them, but at the end, these problems turn out to be unsolvable. To prove incomputability is much more difficult than proving computability. We look at the problems, at least initially, with the eyes of computability. Now, we can introduce an important lemma:

LEMMA 3. *(John Myll) The set of all creative sets is exactly* $= 0^1_m$.

I want to conclude this part about computability with the following observation. PA (first-order arithmetic) is creative for Gödel incompleteness theorems and so it is contained in $0^1_m$. The theory True(PA), the theory of true first-order arithmetic, is not even axiomatisable, for Tarski's theorem about the undefinability of truth. However, can we locate the degree of True(PA) ? So we could understand better how much of arithmetic, our axiomatic theories do capture (the main purpose of this section). By adopting Barry Cooper's words:

> Well, it turns out that the theorems of PA hardly scrape the surface of true arithmetic. [Cooper 07]

In fact, the following theorem shows the degree of True(PA) and its distance from PA contained in $0^1_m$.

THEOREM 16 (Cooper 07). *The degree of True(PA) is* $0^\omega$.

Therefore, the distance between theoremhood of PA and truths which PA attempts to capture is huge. Furthermore, we might suppose that there are many other truths which our axiomatic theories are not capable of capturing.

**1.6. Gödel's sentences undecidable within PA.** At this point, we can come back to our original issue and we can construct Gödel sentence. Before of that, we must introduce Gödel numerical coding. By adopting this method, syntactic properties will become simple numerical properties. Then it will be easy to show that these numerical properties are primitive recursive. Thanks to Gödel coding, we can define a numerical property **Bew**(m,n) which holds just when m is the code number in our scheme of a PA-derivation of the sentence with number n. By adopting Gödel coding, we create a new language where all syntactic properties such as being an axiom, being a sentence and being a sentence derived by modus ponens, become numerical propreties. We have a numerical language which, unlike natural languages, is precise and does not have problem of denotation. The self referential Gödel sentence which says of itself to be unprovable, becomes a Gödel number and it is self referential only when we translate it back from Gödel numbering. So, it recalls the liar paradox only after a procedure of decoding, in fact, before it is simply a number (very large). Now we can get an idea of Gödel numbering by associating odd numbers to function symbols, costants, quantifiers, separating symbols and even numbers to variables in the following manner:

$$\neg = 1$$

$$\wedge = 3$$

$$\vee = 5$$

$$\longrightarrow = 7$$

$$\leftrightarrow = 9$$

$$\forall = 11$$

$$\exists = 13$$

$$= = 15$$

$$( = 17$$

$$) = 19$$

$$0 = 21$$

$$S = 23$$

$$+ = 25$$

$$\times = 27$$

$$x = 2$$

$$y = 4$$

$$z = 6$$

Now we can use the fundamental theorem of arithmetic (factorization in prime factors) to obtain Gödelian numbering. Let the expression T be the sequence of $\kappa + 1$ symbols and variables $s_0, s_1, s_2, s_3, ........, s_\kappa$. Then T's Gödel number is calculated by taking the basic code-number $c_i$ for each $s_i$ in turn, using $c_i$ as an exponent for the $i + 1$-th prime number $\pi_i$, and then multiplying the results to get:

$$\pi_0^{c_0} \times \pi_1^{c_1} \times \pi_2^{c_2} \times \pi_3^{c_3} ........ \times \pi_\kappa^{c_\kappa}.$$

Now we can give some examples of Gödel numbering taken from Peter Smith's book:

The single symbol "S"(the successor function) has the Gödel number: $2^{23}$.

The standard numeral $SS0$ has the Gödel number: $2^{23} \times 3^{23} \times 5^{21}$.

the sentence $\exists y(S0+y) = SS0$ has the Gödel number: $2^{13} \times 3^{4} \times 5^{17} \times 7^{23} \times 11^{21} \times 13^{25} \times 17^{4} \times 19^{19} \times 23^{15} \times 29^{23} \times 31^{23} \times 37^{21}$. [Smith 07]

If we adopt a Hilbert style axiomatic system of logic, proof-arrays are simply linear sequences of sentences. A good way of coding these is by what we call *super Gödel numbers* [Smith 07]. Given a sequence of sentences or other expressions

$$T_0, T_1, T_2, ......, T_n$$

we first code each $T_i$ by a regular Gödel number $g_i$ to produce a sequence of numbers

$$g_0, g_1, g_2, ......, g_n$$

Now we encode this sequence of regular Gödel numbers using a single *super Gödel number*, by multiplying powers of primes to get:

$$2^{g_0} \times 3^{g_1} \times 5^{g_2} \times 7^{g_3} \times ......... \times \pi_n^{g_n}$$

At this point, we can define the proof relation **Bew**(m, n): **Bew**(m,n) holds just if m is the super Gödel number of a sequence of sentences that is a PA-proof of the closed sentence with regular Gödel number n. Now we have to introduce the following important definitions and theorems:

Definition 11. *A one place numerical function f is expressed by $\phi(x, y)$ in a arithmetical language, just if, for any m, n:*
*if f(m)=n, then $\phi(\tilde{m}, \tilde{n})$ is true.*
*if $f(m) \neq n, then \neg\phi(\tilde{m}, \tilde{n}) is$ true.*

Definition 12. *A one-place function f is captured by $\phi(x, y)$ in the theory like PA just if, for any m, n:*

*if* $f(m) = n$, *then* $PA \vdash \phi(\tilde{m}, \tilde{n})$

*If* $f(m) \neq n$, *then* $PA \vdash \neg\phi(\tilde{m}, \tilde{n})$

Definition 13. *A theory like Q (Robinson arithmetic) or PA (Peano arithmetic) is sufficiently strong if for every primitive recursive function f, there is a corresponding* $\phi$ *in Q or PA that captures it*

Theorem 17. *Q and so also PA can capture all* $\Sigma_1$ *functions.*

Theorem 18. *Every primitive recursive function is* $\Sigma_1$.

Theorem 19. *Q (Robinson arithmetic) and PA (Peano arithmetic) are sufficiently strong (they can capture all primitive recursive functions).*

Gödel's construction aims at taking an open sentence **G**(y) which contains y free. This sentence has as Gödel number|**G**| and Gödel substitutes the Gödel number for **G** for the free variable in **G**. So Gödel forms the sentence**G**(|**G**|). This is another example of diagonalization. At this point we can introduce the following theorem:

Theorem 20. *There is a primitive recursive function diag(n) which, when applied to a number n which is the Gödel number of some sentence, produces the Gödel number of that sentence's diagonalization.*

Now we can deepen our analysis about the numerical relation **Bew**(m,n). We have already said that this relation holds when m is the super Gödel number of a PA proof of the sentence with Gödel number n. We can state a fundamental theorem which renders Q and PA able to capture this numerical relation:

Theorem 21. ***Bew****(m,n) is primitive recursive.*

From this relation we can obtain the following predicate: $Prov(n) = \exists v Bew(v, n)$ which holds when the sentence with Gödel number n is provable. However, we cannot define the provability property by some bounded quantification such as $(\exists v \leq B) Bew(v, n)$. If we could, then the provability property would be primitive recursive, but it is not. The predicate *Prov(n)* is not even $\mu - recursive$. In fact, we can state the following theorem:

THEOREM 22. *No open formula in the theory Q and in the theory PA can capture the corresponding numerical property* $Prov_T$.

So Q and PA are not recusively decidable because they cannot capture the property $Prov_T$ which is not recursive. From this fact, we obtain a theorem which gives a negative solution to the Entscheidungsproblem:

THEOREM 23. *(Church) The property of being a theorem of first-order logic is recursively undecidable.*

PROOF. Suppose first-order theoremhood is recursively decidable. In other words, suppose that the property of numbering a logical theorem is $\mu - recursive$. Let $Q^*$ be the conjunction of the seven non-logical axioms of Q (Robinson arithmetic), and let $\phi$ be any sentence of the first-order language. By our supposition, there is a $\mu - recursive$ function which decides whether $(Q^* \longrightarrow \phi)$ is a logical theorem. But $(Q^* \longrightarrow \phi)$ is a logical theorem just if $\phi$ is a Q-theorem. So our supposition implies that there is a $\mu - recursive$ function which decides what is a theorem of Q. But we have just seen that there cannot be such function, given Q's consistency. So the supposition must be false. [Smith 07] □

Therefore, on one side we have Gödel completeness theorem for first-order logic and on the other side we have Church's result about the undecidability of first-order logic. Thus, we should ask ourselves how we can combine these opposite results. If a first-order formula is valid for Gödel completeness theorem we are sure that we can find a proof of the formula itself. However, if the first-logic formula is invalid, for Church's result, we will enter an everlasting loop in searching for a proof which does not ever terminate. Now we can come back to Gödel theorem, we start with the following definition:

DEFINITION 14. *The relation $Bew^*(m,n)$ which holds just when $m$ is the super Gödel number for a PA proof of the diagonalization of the formula with Gödel number $n$ is also primitive recursive.*

Now we can construct Gödel sentence:

$$G(y) = \forall x \neg Bew^*(x,y)$$

Finally we diagonalise G itself to give :

$$G^1 = \exists y(y = |G| \land G(y))$$

. This is our Gödel sentence for PA and it is a $\Pi_1^0$ sentence. $G^1$ is equivalent to $G(|G|)$ or to $\forall x \neg Bew^*(x,|G|)$. It follows that $G^1$ is true if and only if it is unprovable in PA.

THEOREM 24. *(Gödel) If PA is consistent, $G^1$ is true if and only if it is unprovable in PA.*

PROOF. $G^1$ is true if and only if there is no number m such that $Bew^*(m,|G|)$. Therefore, $G^1$ is true if and only if there is no number m such that m is the code number for a PA proof of $G^1$ itself (for the diagonalization). But, if $G^1$ is provable, some number would be the code number of a proof of it. Hence $G^1$ is true if and only if it is unprovable in PA. □

Now to complete the proof we need the following definition from which we can obtain the concept of $\omega - consistency$:

DEFINITION 15. *An arithmetic theory $T$ is $\omega - inconsistent$ if, for some open sentences $\phi(x)$, $T$ can prove each $\phi(\tilde{m})$ and $T$ can also prove $\neg\forall x\phi(x)$.*

So now we can state the following theorem:

THEOREM 25. *(Gödel) [Smith 06] If PA is $\omega - consistent$, $PA \nvdash \neg G^1$.*

PROOF. Suppose that PA is $\omega-consistent$ but $\neg G^1$ is provable in PA. That's equivalent to assuming (1) $PA \vdash \exists x Bew^*(x, |G|)$. But if PA is $\omega - consistent$, it is consistent. So if $\neg G^1$ is provable, $G^1$ is not provable. Hence for any m, m cannot code for a proof for $G^1$. But $G^1$ is the formula you get by diagonalizing G. Therefore, by the definition of $Bew^*$, our assumptions imply that $Bew^*(m, |G|)$ is false, for each m. So we have (2) $PA \vdash \neg Bew^*(\tilde{m}, |G|)$ for each m. But (1) and (2) together make PA $\omega - inconsistent$ after all, contrary to hypothesis. Hence, if PA is $\omega - consistent$, $\neg G^1$ is unprovable [Smith 07] □

At this point, we should ask ourselves if we can avoid the condition of $\omega - consistency$ and adopt simply the condition of consistency. Before doing that, we have to introduce the important theorem called the fixed point theorem:

THEOREM 26. *(Kleene) If a theory like PA is consistent and $\phi(x)$ is any formula of its language with one free variable, then there is a sentence $\kappa$ of PA's language such that $PA \vdash \kappa \leftrightarrow \phi(|\kappa|)$.*[7]

It does not matter what condition we take, so long as it can be expressed in PA's language. There will be a sentence which PA shows is true if and only if satisfies that condition. Thanks to the fixed point theorem, we can formally prove Gödel first incompleteness theorem within PA.[8] In fact, we can form the following biconditional:

$$PA \vdash G \leftrightarrow \neg Prov(|G|).$$

To avoid the condition of $\omega - consistency$ we have to introduce Rosser provability predicate which informally says that *if i am provable there is already a proof of my negation.* To construct Rosser predicate we have to introduce the numerical relation $\tilde{Bew}(m, n)$ which holds when m is the super Gödel number of a PA proof of the negation of the sentence

[7] I remind to the reader that the symbol $|\kappa|$ means the Gödel number of $\kappa$.
[8] We have proved Gödel theorem informally by looking at the formal system PA from the outside.

with Gödel number n. This numerical relation is primitive recursive. Now we can introduce Rosser provability predicate:

DEFINITION 16. *(Rosser)*

$$RProve_{PA} = \exists v(Bew(v,x) \wedge (\forall w \leq v)\neg \tilde{Bew}(w,x)).$$

Thus, if it has a proof, there is not smaller proof of its negation. Now we can apply the fixed point theorem in the following manner:

$$PA \vdash R_{PA} \leftrightarrow \neg Rprove_{PA}(|R_{PA}|)$$

In other words, $R_{PA}$ is true just if, if it is provable, there is already a proof of its negation. Rosser sentence is another undecidable sentence and it avoids the condition of $\omega - consistency$. To show that Rosser sentence is independent from PA, it is enough to assume the condition of consistency.

We have proved the first incompleteness theorem that can be represented by the following sentence:

*if PA is consistent, then G is not provable in PA.*

We can formalize what we have written by adopting the following sentence:

$$Con \longrightarrow \neg Prov(|G|).$$

Thus, we can formalize half of the first incompleteness theorem inside PA in the following manner:

$$(A) \quad PA \vdash Con \longrightarrow \neg Prov(|G|).$$

We have reasoned about the first incompleteness theorem by looking informally at PA from the outside, but now by adopting the precedent sentence and constructing the Gödel sentence formally, we are working inside PA. As we have seen before we can construct

Gödel sentence by using the fixed point theorem in the following manner:

$$(B) \quad PA \vdash G \leftrightarrow \neg Prov(|G|).$$

Now suppose (for reductio) that :

$$(1) \quad PA \vdash Con.$$

Then given the formalised First theorem, **Modus Ponens** yields:

$$(2) \quad PA \vdash \neg Prov(|G|).$$

But (B) tells us that $\neg Prov(|G|)$ and G are provably equivalent in PA. Therefore:

$$PA \vdash G.$$

But, this contradicts the First theorem. So supposition (1) is false, unless PA is inconsistent. Thus, assuming the formalized First incompleteness theorem, we can state the Second incompleteness theorem:

THEOREM 27. *(Gödel-Von neumann) If PA is consistent,* $PA \nvdash Con_{PA}$.[9]

*Con* is another true but unprovable $\Pi^0_1$ sentence, independent from PA. The theorem tells us that even PA is not enough to deduce the consistency of PA, secondly that no weaker theory than PA can deduce the consistency of PA, thirdly that we cannot use PA to prove the consitency of a stronger theory such as ZFC (Hilbert's program fails) and finally that if we are going to produce a consistency proof PA, we should adopt a theory which is weaker in some respects and stronger in others than PA (Gentzen Proof by using the theory $PRA_0 + \epsilon_0$-transfinite induction, as we have seen before).

[9]A story tells us that this theorem was discovered firstly by Von neumann. The Hungurian mathematician was in train, after that he had listened to Gödel's conference about the first incompleteness theorem, and he was able to deduce the Second incompleteness theorem from the First. At the same time, also Gödel himself was able to prove his theorem

So we have seen that for the First incompleteness theorem the dream of having a complete theory of first-order arithmetic fails. Furthermore, we have seen that for the Second incompleteness theorem, Hilbert's program was demolished. However, while Hilbert's program is doomed to fail, we should ask ourselves whether the dream of having a complete theory of first-order arithmetic can be rescued. Thus, this aspect will be the topic of the next section. Alan Turing in his doctoral dissertation in 1939 under the supervision of Alonzo Church attempted to answer positively to this question by going through the transfinite.

## 2. Transfinite Progressions

**2.1. Preliminaries to this section.** In section 2.2 I will introduce Fregean definite descriptions and I will connect Fregean senses to the issue of completeness. I will show that proving propositions or conjectures in mathematics fixes new Fregean senses (definite descriptions) to objects in mathematics. In this section I will explain some issues connected with the philosophy of language and I will discuss the problem of denotation caused by improper definite descriptions. I will introduce the solution for improper definite descriptions conceived by Russell. In section 2.3 I will discuss Turing's attempt to obtain a complete theory by going through the transfinite. I will introduce Turing's completeness theorem for $\Pi^0_1$-statements. I will explain the problem with transfinite progressions. In fact I will highlight the problem connected with ordinal notation. Finally, I will introduce Feferman's completeness theorem for $\Pi^0_2$-statements. This result faces the same problems as Turing's result. In fact, we have the problem of finding a unique ordinal notation. In this part, I will classify some mathematical conjectures or propositions such as the twin prime conjecture and Riemann hypothesis in terms of hierarchy of formulas. I will conclude by asserting that Turing's dream of obtaining a complete theory for first-order arithmetical statement unproved within PA, by going through the transfinite, is doomed to fail because of the problems represented by ordinal notation.

**2.2. Gottlob Frege's definite descriptions and completeness.** Before speaking about transfinite progressions, I want to address an issue represented by the Goldbach conjecture since it can be expressed by a $\Pi_1^0$ sentence, as the consistency statement. Goldbach's conjecture is one of the oldest and renown unsolved problems in arithmetic. It states:

Definition 17. *(Goldbach) Every even integer greater than 2 can be expressed as the sum of two primes.*

It has been proven that the conjecture holds until $4 \times 10^{18}$ but it remains unproved. In fact it is another statement that we do not know whether it is true or false. The conjecture was originally formulated by the Christian mathematician Goldbach at the end of the eighteenth century. In June 1742, Goldbach wrote a letter to the mathematician Euler in which he conceived the following conjecture:

Definition 18. *Every integer which can be written as the sum of two primes, can also be written as the sum of as many primes as one wishes, until all terms are units.*

He then proposed a second conjecture in the margin of his letter:

Definition 19. *Every integer greater than 2 can be written as the sum of three primes.*

All these three definitions are equivalent but the Goldbach conjecture remains unproved.
The number two is denoted by the two following definite descriptions: (1) the number that is the smallest prime and (2) the number that is the cube root of eight. So, even if these two different definite descriptions are two different Fregean senses of number two, they denote the same referement. But if the Goldbach conjecture were true, the two definite descriptions namely (1) The number that is even and greater than two and (2) The number that is even and the sum of two primes, would be two different Fregean senses that denote the

same set of numbers. However, we do not know whether the Goldbach conjecture is true. Now I want to address and explain the following thesis: Proving the truth of conjectures or propositions fixes new Fregean senses to objects in mathematics.

If we could prove Goldbach conjecture, we would have two different definite descriptions (two different Fregean senses) which denote the same set.

The distinction between sense and meaning (referement) was very important for Frege who was able to explain it in his article Über Sinn und Bedeutung. In order to clarify this distinction, we have to depart from singular terms which are constituted by proper names and definite descriptions. Proper names are usual names such us Plato, Kant, Gödel and Cantor. Definite descriptions are singular terms characterised by the fact that they begin with determinative article such as *The teacher of Aristotle*, *The author of critique of pure reason*, *The discoverer of the incompleteness of arithmetic* and *The creator of the paradise of transfinite numbers.* The meaning or referement of a Proper name is the object to which the Proper name is referring. While the meaning or referement of a definite description is the object which the definite description is describing. The notion of Sense (Sinn) is elusive, but we can say that it is *the way in which the referement is given.* In the case of proper names, the notion of Fregean Sense is obscure, while in the case of definite descriptions is clear since they characterise, describe and show the object to which they are referring. However, Saul Kripke affirms that for the Fregean theory each Proper name is synonymous of a definite description. Even if Frege has never identified the Sense (Sinn) of a name with a definite description, he gave examples in which he was always identifying the Sense of a name with a definite description. Bertrand Russell said that each proper name such as Socrates, is an abbreviation of a definite description such as *The teacher of Plato.* So, we can assert that definite descriptions are the Fregean senses of singular terms since they characterise the object (meaning or referement) which they are denoting. It is not true to say that each Fregean sense determines the meaning or referement. In fact, if we say *The* *biggest odd number*, this definite description is not denoting any number. Furthermore, For the Fregean semantics, each Fregean sense is connected only to one meaning or referement, but different Fregean senses might have the same referement. Gottlob Frege was able to explain this aspect with the following example:

1) Hesperus (the night star) = Hesperus (the night star).

2) Hesperus (the night star) = Phosphorus (the morning star).

Hesperus and Phosphorus are two names of Venus. The sentence (2) has got more informative value. Even if Hesperus and Phosphorus have the same meaning or referement (Venus), they have two different Fregean senses. The Sense of Hesperus characterises Venus as *The star that you can see in the night in that part of the sky* and the Sense of Phosphorus characterises Venus as *The star that you can see in the morning in that part of the sky.* The fact that Hesperus and Phosphorus are two Fregean senses of the same referement, namely Venus, cannot be established apriori, but only after that we have obtained empirical evidence.

Now the Fregean sense of a sentence is the thought (Gedanke) expressed by the sentence itself. While the meaning or referement of sentence is its truth value. For Frege, sentences can be only true or false. Moreover, truth and falsity are two objects denoted by sentences. The Fregean thoughts, which are the senses (Sinnen) of sentences, belong to an atemporal and anti-psychological third reign.

The Fregean meaning or referement of predicates are concepts which Frege consider as functions to be completed. *being a prime number* or *being even* are predicates which denote two concepts, namely two functions, and when they are completed by two arguments (singular terms such as the number 2 or 3) give, as values of the functions (concepts), the value of truth or falsity, because they form sentences. Frege called improper definite descriptions those descriptions which lack of a meaning or referement and for the German philosopher sentences which contain improper definite descriptions are neither true neither false. In

1905 Bertrand Russell in his article On denoting tried to argue against this Fregean thesis. For the Nobel prize, also improper definite descriptions have a truth value. So we have to examine the following sentence:

**(1) The actual king of France is bald**.

The problem of this sentence is that France does not have any actual king, it is a republic. Russell wants to eliminate definite descriptions by preferring the logical form of a sentence to its grammatical form. For Russell, the logical form of (1) is the following:

$$\exists x(Actual\ King\ of\ France(x) \wedge \forall y(Actual\ King\ of\ France(y) \longrightarrow y = x) \wedge Bald(x)).$$

For Frege the sentence is neither true or false. Instead if we continue the analysis of Russell, we will discover that it has got a truth value. Russell asks himself what is the negation of (1). Russell says that we have two different interpretations of the negation symbol. We start with the first one:

$$\exists x(Actual\ King\ of\ France(x) \wedge \forall y(Actual\ King\ of\ France(y) \longrightarrow y = x) \wedge \neg Bald(x)).$$

According to Russell, this is a wrong interpretation of the logical form, since France does not have any King whether bald or not. So Russell introduces the right logical form of (1)

$$\neg\exists x(Actual\ King\ of\ France(x) \wedge \forall y(Actual\ King\ of\ France(y) \longrightarrow y = x) \wedge Bald(x)).$$

Hence, this sentence is true since the actual King of France does not exist whereas (1) is false. So, by changing the scope of the negation symbol, Russell was able to give a convincing answer to the problem of improper definite descriptions.

Now if the Goldbach conjecture were true, we could form the Goldbach set, namely all even numbers greater than two and sum of two primes. We could fix two different Fregean senses to the Goldbach set, namely *The set containing all even numbers greater than two* and *The set containing all even numbers sum of two primes.* Proving conjecture or propositions fix

Fregean senses to objects in mathematics such as the Goldbach set. Furthermore, we can relativise all Fregean senses of the Goldbach set to each element of the Goldbach set. For instance, the Fregean senses, namely *The number that is even and greater than two* and *The number that is even and the sum of 3 and 5*, will denote the number 8 which would belong to the Goldbach set. We can see also definite descriptions (Fregean senses) extensionally. They might express properties which define a set and we can see these definite descriptions extensionally. Since definite descriptions might define a set, we can see, in this case, a connection between Fregean senses and Gödelian definitions in the constructible universe, namely L, when Fregean semantics is relativised to the language of mathematics. If the continuum hypothesis is proved to be true, we could fix to the set of real numbers the following Fregean sense, namely *The set is large as the first aleph, namely* $\aleph_1$. Whereas if we prove that the cardinality of the continuum is a precise aleph different from the first aleph, we disprove the continuum hypothesis but we could fix to the set of real numbers the following Fregean sense *The set is large as the second or the third or the forth..... aleph.* if we are thinking about it, we could say that when we are taming the infinite, we are using finite fregean senses, namely finite string of symbols, which denote, as referement, the infinite. We are calculating and discovering properties about the infinite by using finite Fregean senses that denote the infinite. [10]

**2.3. Transfinite progressions.** At this point we can come back to our original issue, namely transfinite progressions. Gödel's second incompleteness theorem makes us able to see the phenomenon of **inexhaustibility** of mathematics. The sentence that formalises the consistency of PA, namely Con(PA), is independent from PA, even if it is true. However if a theory is sound (it does not prove false propositions), also a theory $PA^1$, obtained by adding to PA the sentence Con(PA) as new axiom, will be sound and will be strictly stronger than PA because it will prove Con(PA). However, also $PA^1$ will be incomplete

[10]Also in Religion, we could say that all different religions adopt all different Fregean senses ( different Holy books, different theories..etc) that denote the same referement, namely God.

and so, we cannot prove $Con(PA^1)$ within $PA^1$, but we can form a new and stronger theory $PA^2$ which proves more things than $PA^1$ such as $Con(PA^1)$ by adding to $PA^2$ the true sentence $Con(PA^1)$ and so on through the infinite. In his phd dissertation in 1937, Alan Turing formalised this intuition by introducing ordinal logic and a surprising idea to overcome the phenomenon of incompleteness by iterating through the transfinite the procedure of adding undecidable sentences to a theory, such as reflection sentences or consistency statements, hoping to obtain at certain point a complete theory. Thus, Alan Turing was dreaming a complete theory by traveling through the transfinite.

We will have a sequence of theories where, for example, $T_{i+1} = T_i + \ Con(T_i)$ and $T_{\omega+1} = T_\omega + \ Con(T_\omega)$. At limit passages $\gamma$ we accomplish the following operation: $\bigcup_{\beta<\gamma} T_\beta$. If $T_i$ is sound, then, since $Con(T_i)$ is true, also $T_{i+1}$ is sound. So we can associate ordinals to theories. First of all we need a $\Sigma_1$ formula which defines the axioms of theories that constitute the sequence. Then we need an ordinal notation.

We must think of a limit ordinal as a sequence of ordinals which tend to it, enumerated by a function $\phi_e(x)$, so that for the definition of axioms of a theory indexed by a limit ordinal it satisfies the following condition: $\rho_{lim(e)}(y)$ if and only if there exists an n such that $\rho_{\phi_e(n)}(y)$. By an ordinal logic we mean a sequence of theories $T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}$....... where each $\alpha$ is a name for an ordinal, namely a number of the Kleene's class **O**. However, a Limit ordinal can have different notations. For instance, $\omega$ is the limit of computable, strictly increasing, sequences of natural numbers. Furthermore, it may happen that even if $\alpha_i$ and $\alpha_j$ denote the same ordinal, the theories $T_{\alpha_i}$ and $T_{\alpha_j}$ prove different theorems. When theories with different ordinal notations prove the same theorems, we say that the ordinal logic is invariant. Turing proved the following dichotomy: an ordinal logic can be either invariant or complete for $\Pi^0_1$ statements, but not both at the same time.

An ordinal is said to be computable if it is isomorphic to a recursive well-order. It is well-known that there is at least a countable ordinal, but not computable and the least countable ordinal but not computable is denoted by $\omega_1^{CK}$.

In 1936 Church and Kleene introduced the notion of constructive ordinals. The class **O** of constructive ordinals overlaps perfectly the set of computable ordinals. The class **O** is a system of notation, namely a system of codes for countable ordinals. If we write $\tilde{\alpha_i}$ for the ordinal denoted by $\alpha_i$, then we can assert that $\omega_1^{CK}$ is the least ordinal that does not belong to the class **O**.

At this point, we can define inductively the class **O** and a partial order $<_o$ on it.

DEFINITION 20. *(1) 0 has notation 1.*

*(2) Suppose that you have already defined $<_o$ on ordinals smaller than $\alpha$ and that you have assigned a notation to them:*

*(a) If $\alpha = \beta + 1$ and $\beta$ has the notation b, assign the notation $2^b$ to $\alpha$ and add the pairs $(z, 2^b)$ to the relation $<_o$, for each $z \leq_o b$;*

*(b) If $\alpha$ is a limit ordinal, it can be interpreted as a sequence of ordinals that tend to it. Suppose that this sequence can be enumerated by a function $\phi_e$ such that for each n, $\phi_e(n) <_o \phi_e(n+1)$, where $\phi_e(n) = a_n$ and the increasing sequence $\tilde{a_1}, \tilde{a_2}, \tilde{a_3}, ......$ has got as limit ordinal $\alpha$. Then $3 \times 5^e$ is a notation for $\alpha$; add each pair $(z, 3 \times 5^e)$ such that $z \leq_o \phi_e(n)$ for some n, to the relation $<_o$.*

$<_o$ is not a linear order but it is a tree. In fact, each limit ordinal smaller than $\omega_1^{CK}$ can receive infinite different notations. At each point which corresponds to a limit ordinal, this order splits in infinite branches. Whereas natural numbers have a fixed notation, ordinals can receive different notations. Alan Turing was able to obtain a completeness result.

THEOREM 28. *(Turing) [Franzen 04] For each progression, for every $\Pi_1$ true statement, exists a notation $a \in$ **O** such that $\tilde{a} = \omega + 1$ and $\forall x \psi(x)$ can be proved in $T_a$.*

PROOF. we denote with $S(a)$ the successor code $2^a$ and with $lim(a)$ the code for limit ordinal $3 \times 5^e$. We define by recursion the following function:

$$(1)\phi_e(n) = n \; if \; for \; every \; \kappa \leq n, \psi(\tilde{n}) \; is \; true$$

$$(2)\phi_e(n) = S(lim(e)) \; if \; exists \; \kappa \leq n \; such \; that \; \psi(\tilde{n}) \; is \; false.$$

where $\psi(x)$ is decidable. For hypothesis, $\forall(x)\psi(x)$ is a true $\Pi_1$ statement, then for every n, we have $\phi_e(n) = n$ and so the sequence of values $\phi_e(0), \phi_e(1), \phi_e(2)$, is the sequence 0, 1, 2 and $lim(e)$ is an element of **O** which denotes $\omega$. Now we can reason within $T_{S(Lim(e))}$, checking in this theory that if $T_{Lim(e)}$ is consistent, then the statement $\forall(x)\psi(x)$ is true: we suppose that the statement $\forall(x)\psi(x)$ is false; then we have for some number $n$ that the satement $\psi(\tilde{n})$ is false. Then the theory $T_{Lim(e)}$ for some $n$ and for each $\kappa \geq n$ (from a certain point ) will determine that $\phi(\tilde{\kappa}) = S(lim(e))$; So from a certain point, $T_{S(lim(e))}$ and $T_{Lim(e)}$ will be the same, and so $T_{Lim(e)}$ will prove its own consistency; then for Gödel's second incompleteness theorem follows that $T_{Lim(e)}$ is inconsistent. But $T_{S(Lim(e)}$ proves the consistency of $T_{Lim(e)}$. Therefore, $T_{S(Lim(e))}$ proves $\forall(x)\psi(x)$. We have to notice that $S(Lim(e))$ denotes $\omega + 1$. □

Now, we can continue our discussion by looking at Solomon Feferman's completeness result. Before of that, we must introduce two fundamental aspects of Feferman's conception. Alan Turing did not consider his completeness result valuable. He was thinking that his result was useless from a mathematical perspective. In fact, Turing's approach shifts the question if a $\Pi_1$-sentence true to the question if a number a belongs to **O** and this last problem is a far more complex computable issue.

Solomon Feferman introduced the concept of autonomous progressions namely, collections of theories $T_\alpha$, where $\alpha$ can be proved to belong to **O** within a system $T_\beta$ already accepted. Instead of iterating the consistency statement, Solomon Feferman was adding to theories the unlimited uniform reflection principle. This principle states: *every sentence provable* *in T is true.* We can, while staying within the language of arithmetic, add the principle of uniform $\Sigma_n$-reflection for every n:

$$\forall(x)(Theorem(x) \wedge \ \Sigma_n sentence(x) \longrightarrow True\Sigma_n(x)).$$

In 1962 Feferman proved a completeness theorem for progressions based on unlimited uniform reflection:

THEOREM 29. *(Feferman) [Franzen 04] For any uniform reflection progression, there is a branch in* ***O*** *such that there is, for any true arithmetical sentence $\phi$, an a in B with $|a| < \omega^{\omega^{\omega+1}}$ for which $\phi$ is provable in $T_a$.*

Therefore, there is a reflection sequence of length $\omega^{\omega^{\omega+1}}$ based on PA where every true arithmetical sentence is provable. However, there is no hint in the proof of the theorem of any way in which arithmetical truths in general can be formally derived from axioms that we recognize as valid. Unfortunately, also in the case of Feferman's completeness theorem, many problems remain for the ordinal notation at the limit passage. Feferman completeness theorem strengthens Turing's result, because it refers to $\Pi^0_2$ sentences. At this point, we can state Feferman completeness theorem for $\Pi^0_2$ sentences:

THEOREM 30. *(Feferman) For any progressions based on the uniform reflection principle and every true $\Pi^0_2$ -sentence $\phi$, there is an a with $|a| = \omega^2 + \omega + 1$ such that $\phi$ is provable in $T_a$.*

At this point, i want to adopt Torkel Franzen's words to explain the importance that primitive recursive functions might have had in Feferman's completeness proof:

> Primitive recursive functions play a large role in the proof of the $\Pi^0_2$-completeness theorem, for reasons shown by the following argument. Suppose $\forall x \exists y \psi(x, y)$ is a true $\Pi^0_2$-sentence. Then for every n there is a smallest

> proof $f(n)$ in $T_0$ of $\exists\psi(\tilde{n}, y)$, by the $\Sigma_1$-completeness theorem. $f$ is computable, but may or may not be primitive recursive. Suppose $f$ is primitive recursive. Then the formaliztion $\phi$ *for every* $n$*,* $f(n)$ *is a proof in* $T_0$ *of* $\exists y\psi(\tilde{n}, y)$ is equivalent in $T_0$ to a $\Pi$-formula, and we can apply Turing's completeness theorem to conclude that $\phi$ is provable in some $T_a$, where $|a| = \omega + 1$. We can then use the uniform reflection principle for $T_0$ to prove $\forall x\exists y\psi(x, y)$ in $T_a$. Unfortunately such a proof cannot be carried out in general, because $f$, although computable, is not in general primitive recursive...........[Franzen 04]

What Alan Turing really hoped to obtain was a completeness theorem for $\Pi^0_2$-statements in $\forall\exists$ (for all, there exists)-form. He called these statements *number-theoretical* problems. These problems which can be expressed by a $\Pi^0_2$-statement include the *twin prime conjecture.* Now we can introduce the twin prime conjecture. First of all, we have to say that a twin prime is a prime number that has a prime gap of two. In other words, it differs from another prime number by two, for example the twin prime pair (41, 43). Sometimes the term twin prime is used for a pair of twin primes; an alternative name for this is prime twin or prime pair. At this point we can introduce the twin prime conjecture which does not have a solution:

Definition 21. *(Twin prime conjecture)There are infinitely many primes p such that p + 2 is also prime.*

Furthermore, Alan Turing pointed out that the question whether a given program for one of his machine computes a total function is in $\forall\exists$-form (it can be expressed by a $\Pi^0_2$-statement). In a note of his dissertation, Alan Turing pointed out that also Riemann Hypothesis can be expressed by a $\Pi^0_2$-statement. Years later, Georg Kreisel showed that Riemann Hypothesis can also be expressed by a $\Pi^0_1$-statement. However, Turing's class of number-theoretical problems does not include such statements as finiteness of the number of solutions of diophantine equation which can be expressed by a $\Sigma_2^0$-statement ($\exists\forall$-form, Hilbert's tenth problem that we have seen in the precedent section) or the Waring's problem [11] which can be expressed by a $\Pi_3^0$-statement ($\forall\exists\forall$-form). In dealing with $\Pi_2^0$-number theoretical problems, Alan Turing introduced a new kind a computation, namely a computation relative to an *oracle* (o-machines). We can conclude this section by saying that both Turing and Feferman were dreaming complete theories by going through the transfinite. The main problem of their solutions was based essentially on the impossibility of giving a unique notation to the ordinals at the limit passages. In fact, the problem of ordinal notation has a greater computational complexity than the problem of proving arithmetical truths.

## 3. Set theory

**3.1. Preliminaries to this section.** This section is devoted to set theory. We will see how the phenomenon of incompleteness characterizes second-order and third-order arithmetic. We will examine also the solution adopted by mathematicians to prove undecidable statements. We will focus our attention on Luzin's problem, which characterises second-order arithmetic, and the Continuum Hypothesis, that characterizes third-order arithmetic. While Luzin's problem has been solved positively, the Continuum Hypothesis remains undecidable for many mathematicians. However, from my philosophical perspective, I will argue that the Continuum Hypothesis has been settled. In section 3.2 (Prerequisites) I will introduce the basic concepts of set theory necessary for the following sections. I will explain the ZFC axioms. Then I will introduce the concepts of ordinals and cardinals. In section 3.3 I will show how to reduce all different systems of numbers to sets. This section is important because it highlights the fact that we define all mathematical objects in terms of sets. Set theory represents the foundation of mathematics. In fact ZFC Universe shaped

[11]In number theory, Waring's problem asks whether each natural number $\kappa$ has an associated positive integer s such that every natural number is the sum of at most s $\kappa^{th}$ powers of natural numbers. For example, every natural number is the sum of at most 4 squares, 9 cubes, or 19 fourth powers.

by Zermelo-Frankel axioms system can be seen as the universe of mathematics. Algebra and Analysis can be accomplished within ZFC Universe. Mathematics can be accomplished by adopting the language of set theory and the ZFC axioms. In section 3.4 I will introduce two kinds of large cardinal numbers. In this section, I will explain Gödel's constructible universe, the minimal inner model. It is interesting to say that the precedent two large cardinal notions are consistent with the axiom of constructibility, namely V=L. In section 3.5 I will introduce the concepts related to descriptive set theory. We will see in this section that Luzin's problem, a mathematical statement undecided by ZFC axioms, is settled by a large cardinal axiom. In this section, I will introduce the axiom of determinacy and the axioms of definable determinacy. This section is important because I will highlight how an undecided mathematical statement, formulated in second-order arithmetic, was settled by a large cardinal assumption. In section 3.6 I will explain the method of forcing. In this section I will show Paul Cohen's independence proof by which he was able to construct a meta-mathematical model within which assuming the consistency of ZFC, the axioms of ZFC are consistent with $\neg CH$. This section is important because we will see that the Continuum Hypothesis is an undecided mathematical statement from ZFC axioms and, so third-order arithmetic is doomed to be incomplete. The ZFC axioms do not settle the Continuum Hypothesis. In section 3.7 I will introduce Forcing Axioms. We will see that these axioms do settle the Continuum Hypothesis. In this section I will argue that the Bounded Proper Forcing Axiom may represent a phenomenal solution to the Continuum Hypothesis. In fact, the Bounded Proper Forcing Axiom does settle the Continuum Hypothesis. For this axiom, the cardinality of the Continuum is $\aleph_2$ as Kurt Gödel was thinking. At the end of this section, I will introduce a Kantian distinction between phenomenal and noumenal reality applied to set theory. According to this distinction, meta-mathematical models of set theory belong to the phenomenal reality of set theory. In section 4.8 I will explain Woodin's program. I will discuss the $\Omega$-logic and I will introduce the $\Omega$-conjecture.

This section is important because if we assume the existence of a proper class of Woodin cardinals and that the $\Omega$ Conjecture holds, we have an $\Omega$-complete picture of the structure $H(\omega_2)$ reducing the phenomenon of incompleteness for third-order arithmetic. Then I will introduce Woodin's Maximum and I will highlight the importance of this axiom. I will conclude this section by comparing Turing's Conjecture with the $\Omega$ Conjecture. Then, I will compare the $\Omega$ Conjecture with Church's thesis and the consistency of ZFC.

**3.2. Prerequisites: ZFC axioms, ordinal and cardinal numbers.** In set theory, not every property can define a set, by Russell's paradox. Thus, we have to make a distinction between sets and classes. So we can define a class:

DEFINITION 22. *A class is a collection of the form $\{x$: $x$ is a collection with property $P\}$.*

In order to construct new sets from old ones, we must introduce Zermelo's axioms. Ernest Zermelo in 1905 was motivated to formulate the axioms in order to reach an important mathematical result, namely the *Well-ordering theorem.* Paradoxes, such as Russell's paradox or Burali-Forti's paradox, were not the main concern for the German mathematician. In order to construct a well-order on the set of real numbers, Zermelo introduced the axiom of choice. At this point, we can introduce ZFC axioms:

(1) Axiom of extensionality:

$$\forall x \forall y (x = y \leftrightarrow \forall z (z \in x \leftrightarrow z \in y))$$

Two sets are equal if and only if they contain the same elements.

(2) The empty set axiom:

$$\exists x \forall y \; y \notin x$$

There is a set with no elements.

(3) The axiom of pairs:

$$\forall x \forall y \exists z \forall w (w \in z \leftrightarrow (w = x \vee w = y))$$

For any two sets, there is a set whose elements are exactly these sets.

(4) The axiom of separation:

$$\forall x \exists y \forall z (z \in y \leftrightarrow (z \in x \wedge \phi(z)))$$

where $\phi(z)$ is any condition expressed in the first order language of set theory with free variable $z$ ($\phi(z)$ may contain other free variables). For any set $x$ there is a set consisting of all $z$ in $x$ for which $\phi(z)$ holds. So, this axiom avoids Russell's paradox, since the property applies to an already given set, namely $x$. Since we are speaking about definability, we can say that the axiom of separation mirrors the successor stage which Gödel adopted in the creation of the constructible universe, namely L.

(5) The Power set axiom:

$$\forall x \exists y \forall z (z \in y \leftrightarrow z \subseteq x)$$

For any set $x$ there is a set consisting of all subsets of $x$, called the power set of x and denoted by $P(x)$.

(6) The Union axiom:

$$\forall x \exists y \forall z (z \in y \leftrightarrow \exists w (z \in w \wedge w \in x))$$

For any set $x$ there is a set which is the union of all the elements of $x$.

(7) The axiom of infinity:

$$\exists x (\emptyset \in x \wedge \forall y (y \in x \longrightarrow y \cup \{y\} \in x))$$

There is an inductive set.

(8) The axiom of foundation:

$$\forall x \exists y((x \neq \emptyset) \longrightarrow y \in x \wedge x \cap y = \emptyset)$$

Every set is well-founded, it contains an $\in$-minimal element.

(9) The axiom of replacement:

$$\forall x \exists y \forall y^1 (y^1 \in y \leftrightarrow \exists x^1 (x^1 \in x \wedge \phi(x^1, y^1)))$$

where $\phi(s,t)$ is a formula such that

$$\forall s \exists t (\phi(s,t) \wedge \forall t^1 (\phi(s,t^1) \longrightarrow t^1 = t))$$

If $\phi(s,t)$ ($\phi(s,t)$ may have other free variables) is a class function, then when its domain is restricted to a set x, the resulting images form a set y. The axiom of replacement (9) was not included by Zermelo in the original formulation of the axioms. It was added in order to prove the existence of sets like

$$\{\mathbb{N}, P(\mathbb{N}), P(P(\mathbb{N})), P(P(P(\mathbb{N}))), ........\}$$

By the axiom of replacement, we can define functions on $\mathbb{N}$ and also, we can define functions on the ordinals. Furthermore, we can prove that the axioms of large cardinals are generalizations of the axiom of replacement plus the axiom of infinity. Now, we can introduce the last axiom, namely the axiom of choice:

Suppose that $F$ is a family of non empty sets. Then, there is a function $h : F \longrightarrow \bigcup F$ such that for each $A \in F$, $h(A) \in A$. $h$ is said to be a choice function for $F$.

Zermelo formulated the axiom of choice in order to prove that every set can be well-ordered. The axiom of choice is equivalent to Zorn's lemma which we can define in the following way:

Definition 23. *(Zorn's lemma) Let $P$ be a non-empty set partially ordered by $R$ with the property that every chain $C$ in $P$ has an upper bound in $P$. Then $P$ contains at least one maximal element.*

At this point, we can introduce the concept of ordinals. A set is transitive iff it contains all elements of its elements.

Definition 24. *An ordinal number is a transitive set that is well-ordered by $\in$.*

Furthermore, if $\alpha$ and $\beta$ are ordinals, then $\alpha \in \beta$ if and only if $\alpha \subset \beta$. Therefore, $\alpha \in \beta$ if and only if $\alpha$ is a proper initial segment of $\beta$. From this , it is implied that $\alpha$ is exactly the set of all its $\in$-predecessors, which are themselves ordinals. Thus, for all ordinal numbers $\alpha$ and $\beta$, either $\alpha < \beta$ or $\beta < \alpha$ or $\alpha = \beta$.
The successor of an ordinal $\alpha$ is the ordinal $\alpha \cup \{\alpha\}$, usually denoted by $\alpha + 1$. A limit ordinal is an ordinal which is neither empty nor a successor. The natural numbers are finite ordinals. The set $\mathbb{N}$ is identified with the first infinite ordinal, which is a limit ordinal, and it is denoted by $\omega$. We can add that an ordinal is countable if it is either finite or bijectable with $\omega$. The set of all countable ordinals is not countable and is, therefore, the first uncountable ordinal denoted by $\omega_1$. The set of all ordinals bijectable with some $\alpha \leq \omega_1$ is an ordinal not bijectable with any $\alpha \leq \omega_1$ and it is denoted by $\omega_2$. We can continue in this way. A limit ordinal $\alpha$ is called regular if there is no function: $F : \beta \longrightarrow \alpha$ with $\beta < \alpha$ and $range(F)$ unbounded in $\alpha$. Otherwise, $\alpha$ is called singular. The cofinality of $\alpha$ is the least $\beta \leq \alpha$ for which there exists $F : \beta \longrightarrow \alpha$ with range unbounded in $\alpha$. Thus, $\alpha$ is regular if and only if $cof(\alpha) = \alpha$.
Now we can introduce cardinal numbers by the following definition:

Definition 25. *A cardinal number is an ordinal that is not bijectable with any smaller ordinal.*

Every infinite cardinal is a limit ordinal. Given an infinite cardinal $\kappa$, the set of all ordinals which are bijectable with some $\lambda \leq \kappa$ is a cardinal. It is the least cardinal greater than $\kappa$ and it is denoted by $\kappa^+$. The transfinite sequence of all infinite cardinals is denoted, according to Cantor, by the Hebrew letter $\aleph$ indexed by ordinals. Thus,

$$\aleph_0, \aleph_1, \aleph_2, ..........., \aleph_\omega, \aleph_{\omega+1}, ............, \aleph_\alpha, ..........$$

The Well-ordering Principle implies that every set has a cardinality.
In ZFC one can prove that the universe of all sets $V$ forms a comulative hierarchy. Every set belongs to some $V_\alpha$, for some ordinal where the $V_\alpha$ are defined as follows:

(1) $V_0 = \emptyset$
(2)$V_{\alpha+1} = P(V_\alpha)$, the power set of $V_\alpha$
(3)$V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$, if $\lambda$ is a limit ordinal.
(4)$V = \bigcup_{\alpha \in On} V_\alpha$ is the universe of all sets.

We can prove that all $V_\alpha$ are transitive sets.

**3.3. Reduction of all systems of numbers to the notion of set.** The first problem of Hilbert's list was the continuum problem, namely the cardinality of $\mathbb{R}$ (the set of real numbers). The main problem [Goldrei 96] with the real numbers was to explain irrational numbers. Dedekind and Cantor define irrational numbers in terms of rational numbers. If $r$ is irrational, each rational number lies either to the left or to the right of $r$. So $r$ cuts $\mathbb{Q}$ into two subsets L and R, where L consists of all the rationals to the left of $r$ and R consists of rationals to its right. L and R are both non-empty, disjoint, any rational in L is less than any rational in R and both L and R contain rationals arbitrarily close to $r$.

Dedekind defined a real number to be a partition of $\mathbb{Q}$ in two non-empty subsets, L and R, with the property that every element of L is less than every element of R. This partition is called a **Dedekind cut** and $\mathbb{R}$ is defined to be the set of all such partitions. We shall use in the following definition only the left side, namely L:

DEFINITION 26. *A Dedekind left cut (or Dedekind left set) is a subset $r$ of $\mathbb{Q}$ with the following properties:*
*(1) $r$ is proper, non-empty subset of $\mathbb{Q}$, so that $\emptyset \neq r \neq \mathbb{Q}$.*
*(2) $r$ is closed to the left, if $q \in r$ and $p <_{\mathbb{Q}} q$, then $p \in r$.*
*(3) $r$ has no maximum element, for any $p \in r$ there is some $q \in r$ with $q <_{\mathbb{Q}} p$.*
*A real number is a Dedekind left set and $\mathbb{R}$ is the set of all such real numbers.*

Surely, real numbers include rational numbers. But a rational number is not a Dedekind left set, namely a Dedekind left set is a set of rational numbers. So we have to specify which real numbers are going to correspond to the rationals.

DEFINITION 27. *Let $q \in \mathbb{Q}$. Then the real number corresponding to $q$ is*

$$q = \{p \in \mathbb{Q} : p <_{\mathbb{Q}} q\}.$$

As an alternative construction, Cantor used Cauchy sequences of rationals. Cantor's idea was based on the idea that any irrational number could be regarded as the limit of a Cauchy sequence of rationals. Cantor defined a real number as the set of all sequences of rationals whose terms get arbitrarily close to the terms of this sequence (real numbers).

DEFINITION 28. *The sequence $[q_n]$ of rationals is a Cauchy sequence if for each $\epsilon >_{\mathbb{Q}} 0$ where $(\epsilon \in \mathbb{Q})$ there is an $N \in \mathbb{N}$ such that*

$$|q_i - q_j| <_{\mathbb{Q}} \epsilon, \; for \; all \; i, j \geq_{\mathbb{N}} .N$$

Now we have to capture the idea of two such sequences getting arbitrarily close to each other, and so that they are in some sense equivalent.

DEFINITION 29. *Let* $[a_n]$ *and* $[b_n]$ *be Cauchy sequences of rationals. We shall say that they are equivalent and write* $[a_n] =_{Ca} [b_n]$ *if for each* $\epsilon >_{\mathbb{Q}} 0$ *there is an* $N \in \mathbb{N}$ *such that*

$$|a_n - b_n| <_{\mathbb{Q}} \epsilon, \; for \; all \; n \geq N$$

.

Therefore we can introduce the following definition:

DEFINITION 30. *A real number in Cantor's definition is any equivalence class under the relation* $=_{Ca}$*, namely any set of the form* $\{[b_n] : [b_n] =_{Ca} [a_n]$*, where* $[a_n]$ *is a Cauchy sequence*$\}$*. We write such a class as* $[[a_n]]$*. We use* $\mathbb{R}_c$ *to stand for the set of all Cantor real numbers. Given a rational number* $q$*, the corresponding Cantor real number,* $q_C$ *is defined by*

$$q_C = [[q_n]]$$

*where* $[[q_n]]$ *is an equivalence class of sequences of rational numbers.*

Walking along this way of reduction, we might explain rational numbers in terms of integers [Goldrei 96]. A rational number could be described by an ordered pair $(a, b)$ with $b$ positive corresponding to the fraction $\frac{a}{b}$. This would create the problem of representing a rational by several distinct pairs of integers in the set $\mathbb{Z} \times \mathbb{Z}^+$ as $\frac{1}{2} = \frac{2}{4} = \frac{7}{14} = \frac{13}{26}$. However in $\mathbb{Z}$, $\frac{a}{b} = \frac{c}{d}$ can be interpreted as

$$a \times d = b \times c$$

Thus, we can introduce the following definition:

DEFINITION 31. *For any* $a, b, c, d \in \mathbb{Z}$ *with* $b, d > 0$*, we shall write* $(a, b) =_{\mathbb{Z}} (c, d)$ *when* $a \times d = b \times c$*.*

At this point we can define rational numbers in terms of integers:

Definition 32. *Let* $[[a,b]]$ *be the equivalence class of the ordered pair* $(a,b)$ *of integers under the equivalence relation* $=_{\mathbb{Z}}$*, i.e., the set*

$$\{(c,d) \in \mathbb{Z} \times \mathbb{Z}^{+} : (a,b) =_{\mathbb{Z}} (c,d)\}.$$

*A rational number is such an equivalence class and* $\mathbb{Q}$ *is the set of all these equivalence classes.*

Now we can try to define integers in terms of natural numbers $\mathbb{N}$. The problem is how to represent the negative integers without using subtraction, which is not a closed operation on $\mathbb{N}$. We can represent the integer $n$ by a pair $(a,b)$ of natural numbers such that in $\mathbb{Z}$, $a-b=n$. So for instance $-3$ could be represented by $(1,4)$ or $(7,10)$. However, we have the problem of representing the integer by a single object. We can accomplish that by observing that $a-b=c-d$ can be written in equivalent way as $a+d=b+c$. So we can introduce the following definition:

Definition 33. *For any* $a,b,c,d \in \mathbb{N}$ *we shall write* $(a,b) =_{\mathbb{N}} (c,d)$ *if* $a+d=b+c$*.*

Thus, we can define integers in terms of natural numbers:

Definition 34. *Let* $[[(a,b)]]$ *be the equivalence class of the ordered pair* $(a,b)$ *of natural numbers under the equivalence relation* $=_{\mathbb{N}}$*, i.e, the set*

$$\{(c,d) \in \mathbb{N} \times \mathbb{N} : (a,b) =_{\mathbb{N}} (c,d)\}.$$

*An integer is such an equivalence class and* $\mathbb{Z}$ *is the set of all these equivalence classes.*

So from a philosophical point of view, since we can define $\mathbb{R}$ in terms of $\mathbb{Q}$, $\mathbb{Q}$ in terms of $\mathbb{Z}$ and $\mathbb{Z}$ in terms of $\mathbb{N}$, we can affirm that natural numbers are ontologically the foundation of all other numbers. We can construct all other numbers departing form natural numbers. Furthermore, concerning the foundation of mathematical knowledge, since to the set $\mathbb{N} - \{0, 1\}$ we can apply the fundamental theorem of arithmetic, namely the factorization in prime factors, we can affirm that prime numbers are the **atoms** of arithmetic. Each positive integer can be represented as a product of prime factors. We can continue this process of reduction and we can try to define natural numbers in terms of sets. If we can express natural numbers in terms of sets, we have a single foundation for mathematics. The basic property of sets would be the membership relation, namely $\in$. Now we can introduce the following definition:

DEFINITION 35. *Given a set $x$,the successor of $x$, written $x^+$, is the set $x^+ = x \cup \{x\}$.*

At this point, we can follow von Neumann idea and represent natural numbers by sets in the following manner:

$0 = \emptyset$.

$1 = \emptyset^+ = \emptyset \cup \{\emptyset\}$.

2$= \emptyset^{++} = (\emptyset^+)^+ = \{\emptyset\} \cup \{\{\emptyset\}\} = \{\emptyset, \{\emptyset\}\}$.

3$= \emptyset^{+++} = (\emptyset^{++})^+ = \{\emptyset, \{\emptyset, \}\} \cup \{\{\emptyset, \{\emptyset\}\}\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$. Now we can introduce the following definition:

DEFINITION 36. *The set $y$ is inductive if $\emptyset \in y$ and $x^+ \in y$ whenever $x \in y$.*

We are going to define $\mathbb{N}$ as the intersection of all inductive sets, so that it will be the smallest inductive set.

DEFINITION 37. *The set of natural numbers $\mathbb{N}$ is the intersection of all inductive subsets of any inductive set $y$,*

$$\mathbb{N} = \bigcap \{z : \ z \ is \ an \ inductive \ subset \ of \ y\}$$

$$= \{x : x \in z \ for \ all \ inductive \ z \subseteq y\}.$$

*A natural number is a member of* $\mathbb{N}$.

Thus we can introduce the following theorem :

THEOREM 31. *The set* $\mathbb{N}$ *is inductive.*

**3.4. The first large cardinal numbers and the Constructible universe L.** At this point we can introduce inaccessible cardinals which are the smallest large cardinals in the large-cardinal hierarchy:

DEFINITION 38. *A cardinal* $\kappa$ *is (strongly) inaccessible if it is uncountable, regular, and a strong limit, namely for every cardinal* $\lambda < \kappa, 2^{\lambda} < \kappa$.

One can prove in ZFC that $\kappa$ is inaccessible if and only if it is regular and $V_\kappa$ is a model of ZFC. Therefore, for Gödel's second incompleteness theorem, it is impossible to prove in ZFC the existence of inaccessible cardinals. An inaccessible cardinal is a model of ZFC because it represents a closure point for ZFC axioms and, more precisely, for the axiom of replacement. We might highlight that all large cardinals numbers are generalizations of the axiom of replacement plus the axiom of infinity. If $\kappa$ is inaccessible, then the set $C$ of all strong limit cardinals smaller than $\kappa$ is a closed unbounded subset of $\kappa$. So if $\kappa$ is the least inaccessible cardinal, then all cardinals in C must be singular, for otherwise there would be an inaccessible cardinal below $\kappa$. At this point, we can introduce the notion of Mahlo cardinal:

DEFINITION 39. *An inaccessible cardinal is called Mahlo if the set of inaccessible cardinals smaller than* $\kappa$ *is stationary. Thus,* $\kappa$ *is Mahlo if and only if it is inaccessible and every closed unbounded subset of* $\kappa$ *contains an inaccessible cardinal. Therefore, the first Mahlo cardinal, if it exists, is much greater than the first inaccessible cardinal.*

At this point, I want to introduce Gödel constructible universe. In 1938, Kurt Gödel published an article where he introduced the model L which is based on the idea of constructible set. Ernest Zermelo did not characterize arbitrary sets, instead Gödel introduced the concept of constructible set which implies the use of first order logical formulas. Gödel's sets are constructible thanks to first order formulas which define a set itself. By shaping the constructible universe, Gödel gave a relative consistency proof of the generalized continuum hypothesis. Within model L, ZFC axioms are consistent with the generalized continuum hypothesis. Surely we have a relative consistency proof. Infact, we have firstly to assume that ZF is consistent and then we have to prove that a stronger theory has a model. F. Drake [Drake 74] argues that ZFC axioms, since do not characterize the power set operation as Gödel does with the definable power set, cannot solve the continuum hypothesis. There is a strong connection between the concept of constructible set and the axiom of separation. The class L of all constructible sets is a transitive model of ZFC and it is the smallest transitive model which contains all ordinals. A set $X$ is definable in a model $(M, \in)$ if there is a formula $\phi \in$ FORM (the set of all well-formed first order logical formulas) and some $a_1, .......a_n \in M$ such that:

$$X = \{x \in M : (M, \in) \models \phi(x, a_1, .....a_n\}$$

and

$$Def(M) = \{X \subset M : X \ is \ definable \ in \ (M, \in)\}.$$

Now we can define by transfinite induction the class L:

(1)$L_0 = \emptyset$.

(2) $L_{\alpha+1} = \ Def(L_\alpha)$.

(3) $L_\alpha = \bigcup_{\beta<\alpha} L_\beta \ if \ \alpha \ lim.$

(4) $L = \bigcup_{\alpha \in ORD} L_\alpha$.

The definable class L is the class of all constructible sets. The statement $V = L$, namely *every set is constructible*, is the axiom of constructibility. Gödel, in 1938, believed that this axiom was true, but, then, he changed his mind by accepting also arbitrary sets. First of all, by the forcing method (Cohen's model), the negation of the axiom of constructibility, namely $\neg V = L$, is consistent with the axioms ZF and so, the axiom $V = L$ is an undecidable statement for the ZF axioms. ZF axioms do not decide whether all sets are constructible. Secondly, the axiom which asserts the existence of a measurable cardinal implies the negation of the axiom of constructibility. According to Drake [Drake 74], we should speak of hypothesis of constructibility. Now we can introduce the following theorem:

THEOREM 32 (Kunen 06). *For every $\alpha$, $\alpha \subset L_\alpha (L_\alpha \cap\ Ord\ = \alpha)$.*

The following Lemma is fundamental:

LEMMA 4. *$Ord\ \subset L$.*

The class L of all constructible sets, since it contains the proper class of all ordinals, is itself a proper class. L is the minimal transitive model of ZF. At this point, we can introduce the structural properties of L:

THEOREM 33. *(Gödel) The following properties characterize constructible sets:*

*(A)* $Def(L_\alpha) \subseteq P(L_\alpha)$.

*(B)* $(\alpha \leq \beta) \longrightarrow (L_\alpha \subseteq L_\beta$.

*(C)* $(x \in y \in L_\alpha) \longrightarrow (x \in L_\alpha)$.

*(D)* $\forall \alpha, L_\alpha \subseteq V_\alpha$ .

*(E)* $(\alpha \leq \omega) \longrightarrow (L_\alpha = V_\alpha)$.

*(F)* $(\alpha \geq \omega) \longrightarrow (|L_\alpha| = |\alpha|)$.

*(G)* $(\alpha < \beta) \longrightarrow (\alpha, L_\alpha \in L_\beta)$.

At this point, we can compare the class of all constructible sets with Von neumann's hierarchy which we can define again:

Definition 40. *We can define Von neumann's hierarchy again in the following way:*
$V_0 = \emptyset$.
$V_{\alpha+1} = P(V_\alpha)$.
$V_\alpha = \bigcup_{\beta<\alpha} V_\beta$ *if* $\alpha$ *lim.*

By assuming that L satisfies the axiom of foundation, we can say that L, as Von neumann's hierarchy, is a comulative hierarchy for structural properties (B) and (C). For the structural property (E), we can say that in the finite and at $\omega$, L and V are the same. L and V start to be different at the level $\omega + 1$ . In fact,, whereas $|V_{\omega+1}| = P(V_\omega) > \aleph_0$, for the structural property (F) $|L_{\omega+1}| = |Def(L_\omega)| = |\omega + 1| = \aleph_0$. Within V, the power set operation increases cardinality at each successor stage for Cantor's theorem. Within L, instead, the definable power set does not increase cardinality, but cardinality increases at the level of uncountable ordinals. In few words, $|L_\omega| = |L_{\omega+2}| = |L_{\omega+\omega}| = |L_{\omega^\omega}| = \aleph_0$, whereas $|L_{\omega_1}| = |\omega_1| > \aleph_0$. If we represent graphically L and V by adopting an enlarging cone, we can say that the cone of L is thinner and higher than the cone of V. The cone of V enlarges at each successor stage, while the cone of L enlarges only at the level $L_{\omega_1}$. The cone of L does not enlarge for the definable power set operation but it enlarges for the intrinsic property of uncountable ordinals. Thus, for this fact, we might assert that $|P^L(\omega) \leq |L_{\omega_1}|$. V and L are very different. While $V_{\omega+1}$ contains all arbitrary subsets of $V_\omega$, $L_{\omega+1}$ contains only some definable subsets of $L_\omega$. For instance, at the level $L_{\omega+3}$, $L_{\omega+7}$ and $L_{\omega+\omega}$ there might be some definable subsets of $L_\omega$.
Thus, L grows gradually (this justifies the height of L) and for the structural property (F) is not enough to prove the consistency of the continuum hypothesis, since there might be some definable subsets of $L_\omega$ at the level $L_{\omega_1}$, $L_{\omega_2}$, $L_{\omega_7}$, etc. Thus to prove that $P^L \subset L_{\omega_1}$, it is necessary to set an upper bound to the gradual growth of definable subsets of $L_\omega$ within L. This can be accomplished by combining the Mostowsky's transitive collapse theorem and the downward Löwenheim-Skolem's theorem.

If we take the generalized continuum hypothesis, Gödel proved that if X is a constructible subset of $\omega_\alpha$ then we have a $\gamma < \omega_{\alpha+1}$ such that $X \in L_\gamma$. Therefore, $P^L(\omega_\alpha) \subset L_{\omega_{\alpha+1}}$ and since we have that $|L_{\omega_{\alpha+1}}| = \aleph_{\alpha+1}$, we have that $|P^L(\omega_\alpha)| \leq \aleph_{\alpha+1}$. Gödel connected constructible sets with countable first-order language and so, the cardinality does not increase for the operation of the definable power set. Within L, cardinality increases for the intrinsic characteristic of uncountable ordinals.
At this point, we can examine the notion of absoluteness.

Definition 41. *Take a formula $\phi$ with $x_1, \ldots.., x_n$ free.*
*(1) If $M \subset N$, $\phi$ is absolute for M and N if*

$$\forall x_1.....x_n \in M(\phi^M(x_1......x_n) \leftrightarrow \phi^N(x_1.......x_n)).$$

*(2) $\phi$ is absolute for M if and only if $\phi$ is absolute for M and V:*

$$\forall x_1......x_n \in M(\phi^M(x_1.......x_n) \leftrightarrow \phi(x_1, ......x_n)).$$

In (1) if $\phi$ is absolute, then $\phi$ is true in M and in N. In (2) the notion of absoluteness refers to the whole universe of sets, namely V. So, if $\phi$ is true in M, then $\phi$ is true in every metamathematical model or set-theoretic interpretation. The notion of being an ordinal number or the operation of union are absolute. So, ordinal numbers are the same for every metamathematical model of set theory. On the contrary, the notion of being a cardinal number and the power set operation are not absolute. If we take two metamathematical models X and Y, an uncountable cardinal within X might be countable within Y. The independence of the continuum hypothesis is based on the fact that the power set operation is not absolute. Now we can introduce the following lemma:

Lemma 5. *If M is a transitive model and $\phi$ is $\Delta_0$ formula, then for all $x_1, \ldots.., x_n$, $\phi^M \leftrightarrow \phi(x_1, \ldots...., x_n)$. The formula $\phi$ is absolute for the transitive model M.*

At this point, the following lemma is fundamental:

LEMMA 6. *The following formulas are formalized by $\Delta_0$ formulas and so, are absolute for transitive models:*

*(1)* $x = \{u, z\}$.

*(2)* $x =< u, z >$.

*(3)* $x = \emptyset$.

*(4)* $x \subset y$.

*(5)* $x$ *is transitive.*

*(6)* $x$ *is an ordinal.*

*(7)* $x$ *is a limit ordinal.*

*(8)* $x$ *is a natural number.*

*(9)* $x = \omega$.

*(10)* $Z = X \times Y$.

*(11)* $Z = X - Y$.

*(12)* $Z = X \cap Y$.

*(13)* $Z = \bigcup Y$.

*(14)* $Z = Dom(X)$.

*(15)* $Z = Ran(X)$.

*(16)* $X$ *is a relation.*

*(17)* $f$ *is a function.*

*(18)* $y = f(x)$.

Now we can introduce the concepts of upward absoluteness and downward absoluteness. If M is a transitive model, we can assert that $\phi(\tilde{x})$ is downward absolute if and only if

$$(\forall \tilde{x} \in M)(\phi(\tilde{x}) \longrightarrow \phi^M(\tilde{x})).$$

On the other side, we can assert that $\phi(\tilde{x})$ is upward absolute if and only if

$$(\forall \tilde{x} \in M)(\phi^M(\tilde{x}) \longrightarrow \phi(\tilde{x})).$$

Now we can reintroduce Levy's hierarchy. A formula $\phi$ is $\Sigma_0$ or $\Pi_0$ ($\Delta_0$) if and only if it does not contain unbounded quantificators. For $n \geq 1$, by recursion, we assert that $\phi$ is $\Sigma_n$ if and only if has the form $\exists \tilde{x} \psi(\tilde{x})$ where $\psi(\tilde{x})$ is $\Pi_{n-1}$. and that $\phi$ is $\Pi_n$ if and only if it has the following form $\forall \tilde{x} \psi(\tilde{x})$ where $\psi(\tilde{x})$ is $\Sigma_{n-1}$. Therefore, when we assert that a formula is $\Sigma_n$, we want to say, first of all, that it consists of a $\Delta_0$ formula which has n blocks of existential quantificators in front . Secondly, this formula starts with a block of existential quantificators. Thirdly, This formula is characterized by an alternation of blocks of universal quantificators and blocks of existential quantificators. A formula is $\Delta_1$ if it is both $\Sigma_1$ and $\Pi_1$. Now we can introduce the following lemma:

Lemma 7. *(1) In a transitive model M, if $\phi$ is $\Sigma_1$, then $\phi$ is upward absolute.*
*(2) In a transitive model M, if $\phi$ is $\Pi_1$, then $\phi$ is downward absolute.*
*(3) In a transitive model M, if $\phi$ is $\Delta_1$, then $\phi$ is absolute.*

Now we can examine the power set operation. We can formalize this operation in the following way:

$$y = P(x) \leftrightarrow \forall x(z \in y \longrightarrow z \subseteq x) \land \forall z(z \subseteq x \longrightarrow z \in y).$$

Since the second conjunct contains an unbounded universal quantificator, this formula is $\Pi_1$. So, the power set operation is downward absolute. This operation cannot be also $\Sigma_1$. In fact, in this case, it would be $\Delta_1$ and so it would be absolute and this is impossible. Now we can examine the notion of being a cardinal number. First of all, we can formalize the fact that a set is bigger than another set . In a transitive model M, $|X| \leq |Y| \leftrightarrow \exists f \phi(f, X, Y)$ where $\phi$ is $\Delta_0$. This formula implies that there is an injective function from X into Y. At this point we can formalize the notion of being a cardinal number:

$$\alpha \textit{ is a cardinal } \leftrightarrow \neg\exists f(\exists\beta\in\alpha)\phi(\alpha,\beta,f)$$

where $\phi$ is $\Delta_0$. This formula implies that if $\alpha$ is a cardinal, then there cannot be a bijective function from $\alpha$ to its proper part, namely $\beta$. This formula is $\Pi_1$ and it is downward absolute. In fact, if $\alpha \in M$, and if $\alpha$ is a cardinal, then $M \models \alpha$ *is a cardinal*. The fact that the notion of being a cardinal is downward absolute mirrors some difficulties that we encounter when we use forcing methods. In fact to show that a regular cardinal in $M[G]$ (Cohen extended model) is a regular cardinal also in M (countable transitive model) is not problematic. On the contrary to prove that a regular cardinal in M (countable transitive model) is also a regular cardinal in $M[G]$ (Cohen extended model), is problematic because that cardinal can be collapsed. In fact, it is necessary to add that the poset $\mathbb{P}$ satisfies the countable chain condition (c.c.c.) to avoid the collapse of cardinals as we will see in the following section. Now, we can introduce the concept of extensional relation:

Definition 42. *R is extensional on A if and only if* $\forall x, y \in A(\forall x \in A(\forall z \in A(zRx \leftrightarrow zRy) \longrightarrow z = y)$.

So, if $\in$ is interpreted as R, we are asserting that the axiom of extensionality is true in A. If M is a transitive model, then $\in$ is extensional on M. An inner model of ZF is a transitive class, which contains all ordinals and satisfies ZF axioms. L is an inner model of ZF axioms and it is the smallest inner model.

Theorem 34. *(Gödel) L is a model of ZF.*

L satisfies also the axiom of choice and the generalized continuum hypothesis. These proofs are based on the fact that L is a model of the axiom of constructibility, namely V=L, and this axiom implies the axiom of choice (AC) and the generalized continuum hypothesis (GCH). However, it is clear that V=L implies the axiom of choice since it is easy to define a well-order of L. It might seem a banal fact that L is a model of V=L. However, in order to satisfy V=L within L, we have to prove that the property *every set is constructible* is absolute for L, namely for every $x \in L$, we have that $(x = \textit{ constructible set})^L$. At this point, we can show that the property *x is a constructible set* is absolute for inner models of ZF.

Lemma 8. *The function $\alpha \longrightarrow L_\alpha$ is $\Delta_1$.*

This lemma establishes that the function, which has as domain ordinals and as range constructible sets, is absolute for transitive models.

Lemma 9. *The property x is constructible is absolute for inner models of ZF.*

Now we can introduce the following fundamental theorem:

Theorem 35. *(Gödel) L satisfies the axiom of constructibility, namely V=L, and L is the smallest inner model of ZF.*

Now, we can see that L satisfies AC.

Theorem 36. *(Gödel) AC holds in L*

We end this section with the following fundamental theorem:

Theorem 37. *(Gödel) $L \models$ GCH.*

We have to say that the last theorem characterizes all inner models. In fact, GCH holds in all inner models including the Ultimate L if the ultimate L conjecture is true, since all these inner models satisfy the condensation principle.

**3.5. Descriptive set theory, the axioms of determinacy and Luzin's problem formulated in second-order arithmetic.** Surely, the Continuum Hypothesis (CH) (first problem in Hilbert's list) is the most famous unsolvable problem. We can state the Continuum Hypothesis in the following manner:

DEFINITION 43. *(The Continuum Hypothesis) Suppose that $X \subseteq \mathbb{R}$ is an uncountable set. Then there exists a bijection $j : X \longrightarrow \mathbb{R}$.*

The Continuum Hypothesis is independent from the axioms ZFC. If $\sigma$ is an independent arithmetical statement from the axioms ZFC and ZFC is consistent, ZFC does not prove $\sigma$ and ZFC does not prove $\neg\sigma$. In 1938, Kurt Gödel by creating the constructible universe, namely L, was able to prove the consistency of the Continuum Hypothesis:

THEOREM 38. *(Gödel) Assume ZFC is consistent. Then so is $ZFC + CH$.*

In 1963, Paul Cohen by introducing the method of forcing, was able to prove the consistency of the negation of the continuum hypothesis:

THEOREM 39. *(Cohen) Assume ZFC is consistent. Then so is $ZFC + \neg CH$ .*

The Continuum Hypothesis is formulated in third-order arithmetic and it can be expressed by a $\Sigma^2_1$-statement. Large-cardinal assumptions do not settle the continuum hypothesis. We will see that Woodin's program based on $\Omega$-logic does settle the continuum hypothesis but there are many issues to be considered in order to accept Woodin's result. Therefore, third-order arithmetic seems to stand beyond human ability to prove theorems. So we should ask ourselves if we can have a complete theory for third-order arithmetic or we can only dream this bind of completeness. Maybe, we can depart from second-order arithmetic and see if in the realm of second-order arithmetic there are undecidable arithmetical statements. So, we can ask ourselves if we can have a complete theory for second-order arithmetic. In order to examine second-order arithmetic, we have to explain descriptive set theory. First of all, we have to introduce some topological notions. A continuous function with a continuous inverse function is called an Homeomorphism in topology. Homeomorphisms are isomorphisms in the category of topological spaces. They are the mappings that preserve all the topological properties of a given space. In topology, a metric space is a set for which distances of the set are defined. The real line is a metric space with the metric $d(a, b) = |a - b|$. A metric space is separable if it has a countable dense set. It is complete if every Cauchy-sequence converges. Now we can introduce a fundamental topological space, namely Polish space.

Definition 44. *(Polish space) A Polish space is a topological space that is homeomorphic to a separable, complete, metric space.*

$\mathbb{R}$, Baire space $(N)$, Cantor space and the unit interval $[0, 1]$ are examples of Polish space. In descriptive set theory, it is fundamental the notion of Baire space (N). Before giving the definition of this important topological space, it is necessary to introduce other notions fundamental for descriptive set theory. Firstly, we can introduce the concept of algebra of sets and, then, we can define Borel sets.

Definition 45. *(algebra of sets) An algebra of sets is a collection $C$ of subsets of a given set $S$ such that:*
*(1) $S \in C$,*
*(2) if $X \in C$ and $Y \in C$ then $X \cup Y \in C$,*
*(3) if $X \in C$ then $S - X \in C$.*

A $\sigma$-algebra is additionally closed under countable unions (and intersections):

$$If\ X_n \in C\ for\ all\ n,\ then\ \bigcup_n^{\infty} X_n \in C$$

For any collection $A$ of subsets of $S$ there is a smallest algebra ($\sigma$-algebra) $C$ such that $A \subset C$. At this point, we can define Borel sets:

Definition 46. *A set of reals $B$ is Borel if it belongs to the smallest $\sigma$-algebra $C$ of sets of reals that contains all open sets.*

Now we can introduce Baire space (N):

DEFINITION 47. *(Baire space) the Baire space is the space $N = \omega^\omega$ of all infinite sequences of natural numbers, $(a_n : n \in \mathbb{N})$, with the following topology:*

$$O(s) = \{f \in N : s \subset f\} = \{(c_\kappa : \kappa \in \mathbb{N}) : (\forall \kappa < n) c_\kappa = a_\kappa\}.$$

The sets $O(s)$ form a basis for the topology of N. Each $O(s)$ is closed.

Now we can start by highlighting some regularity properties which definable subset of reals should have. The idea of measure of a subset of $\mathbb{R}^n$ clarified intuitions about the length of an interval of $\mathbb{R}$ or a curve in $\mathbb{R}^2$ and the volume of a solid in $\mathbb{R}^3$. In Lebesgue theory, a measure is a function $\mu$ from some set $Y$ of subsets of $\mathbb{R}^n$ to $\mathbb{R} \cup \{\infty\}$ with the following properties:

(1) $\mu(X) \geq 0$, for any subset X in Y;

(2) If X and Y are congruent subsets of $\mathbb{R}^n$, then $\mu(X) = \mu(Y)$;

(3) $\mu$ is countably additive: if $X_0, X_1, X_2 .......X_n........$ are countably many pairwise disjoint subsets of $\mathbb{R}^n$ then

$$\mu(\bigcup_{n \in N} X_n) = \Sigma_{n=0}^{\infty} \mu(X_n)$$

We should ask ourselves if there is a measure on all subsets of $\mathbb{R}^n$. Using the axiom of choice the answer is no. In fact, we can state the following theorem that is seen as a paradox:

THEOREM 40. *(Banach-Tarski) Let $S$ be the unit ball in $\mathbb{R}^3$, namely the set of all points within a sphere of radius 1. Then $S$ can be partitioned into finitely many subsets which can be moved, using translations and rotations, to produce two unit balls.*

The proof uses AC and is non-constructive. Most pieces of the ball are non-measurable sets. Banach-Tarski theorem belongs to second-order arithmetic. Now, we must say that Lebesgue measurable sets form a $\sigma$-algebra and contain all open intervals. Thus, all Borel sets are Lebesgue measurable.

Now we can continue to list other properties that feature definable subsets of reals, such as the following:

DEFINITION 48. *(Perfect set property) A set of reals is perfect iff it is nonempty, closed and contains no isolated points. A set of reals is said to have the perfect set property if it is either countable or contains a perfect subset*

The perfect set property is linked with the Continuum Hypothesis. In fact, sets of reals with the perfect set property satisfy the continuum hypothesis. In 1883, Cantor and Bendixson proved that all closed sets have the perfect set property.
The third regularity property of sets of reals, we will consider, is the property of Baire. A set of reals is nowhere dense iff its closure contains no open sets. Equivalently, a nowhere dense set is a set that is not dense in any nonempty open sets. A set of reals is meager iff it is the countable union of nowhere dense sets.

DEFINITION 49. *(Property of Baire) A set of reals $A$ has the property of Baire iff it is almost open in the sense that there is an open set $O$ such that the region where $O$ and $A$ do not overlap (symmetric difference: $O \triangle A$) is meager.*

We shall consider next another structural property of definable sets of reals, namely uniformization.

DEFINITION 50. *(Uniformization) Let $A$ and $B$ be subsets of the plane $(\omega^\omega)^2$ (Baire space N). $A$ uniformizes $B$ iff $A \subseteq B$ and for all $x \in \omega^\omega$, there exists $y$ such that $(x, y) \in B$ iff there is a unique $y$ such that $(x, y) \in A$.*

To sum up, A produces a choice function for the set of fibers of B. If we adopt AC, every set B has a uniformizing set A.
At this point, we can introduce definable subsets of reals. Borel sets of reals are obtained by starting with the closed subsets of $\omega^\omega$ (or $(\omega^\omega)^\kappa$ for some $\kappa < \omega$) and closing under the

operation of countable union and complements. This can be accomplished level by level in the following manner: Let $\kappa < \omega$. Let $\Sigma_1^0$ consist of the open subsets of $(\omega^\omega)^\kappa$ and let $\Pi_1^0$ be the set of closed subsets of $(\omega^\omega)^\kappa$. For each ordinal $\alpha$ such that $0 < \alpha < \omega_1$, recursively define $\Sigma_\alpha^0$ to be the set of sets that are countable unions of sets belonging to some $\Pi_\beta^0$, for $\beta < \alpha$, and define $\Pi_\alpha^0$ to be the set of sets that are countable intersections of sets appearing in some $\Sigma_\beta^0$, for $\beta < \alpha$. All these sets form the Borel hierarchy. By using Cantor's diagonalization procedure, Lebesgue, in 1905, proved that the Borel hierarchy constitutes a proper hierarchy.

The projective sets of reals are obtained by beginning with closed subsets of $(\omega^\omega)^\kappa$ and iterating the operations of complementation and projection. For $A \subseteq (\omega^\omega)^\kappa$, the complement of A is simply $(\omega^\omega)^\kappa - A$. For $A \subseteq (\omega^\omega)^{\kappa+1}$, the projection of A is:

$$p[A] = \{(x_1, ......., x_\kappa) \in (\omega^\omega)^\kappa | \exists y (x_1, ......., x_\kappa, y) \in A\}.$$

The projective hierarchy is defined in the following way: Let $\Sigma_0^1 = \Sigma_1^0$ and let $\Pi_0^1 = \Pi_1^0$. For each n such that $0 < n < \omega$, recursively define $\Pi_n^1$ to be the set of the complements of sets in $\Sigma_n^1$, and define $\Sigma_{n+1}^1$ to be the set of the projections of sets in $\Pi_n^1$. The projective sets form an hierarchy. A set of reals is $\Delta_n^1$ iff it is both in $\Sigma_n^1$ and in $\Pi_n^1$. In 1917, Suslin proved that the Borel sets are precisely the $\Delta_1^1$ sets. Thus, the projective hierarchy extends the Borel hierarchy.

At this point, we can introduce a different hierarchy based on the concept of definability and iterated into the transfinite: For a set $X$, let $Def(X)$ consist of the subsets of $X$ that are definable over $X$ using parameters from $X$. This is the definable power-set operation. We can form the hierarchy $L(\mathbb{R})$ by starting with $\mathbb{R}$ and iterating the definable power set operation along the ordinals. Thus,

(1) $L_0(\mathbb{R}) = V_{\omega+1}$.

(2) $L_{\alpha+1}(\mathbb{R}) = Def(L_\alpha(\mathbb{R}))$.

(3) $L_\lambda(\mathbb{R}) = \bigcup_{\alpha<\lambda} L_\alpha(\mathbb{R})$ for limit ordinals $\lambda$.

(4) $L(\mathbb{R}) = \bigcup_{\alpha \in On} L_\alpha(\mathbb{R})$.

For more details see [Koellner 11]. At this point, we can introduce a fundamental theorem in descriprive set theory discovered in 1917:

THEOREM 41. *(Luzin, Suslin) All $\Sigma^1_1$ sets have the perfect set property, the property of Baire, and are Lebesgue measurable.*

Thus, in particular, Borel sets have all the claimed regularity properties of definable subsets of reals. So, we can ask ourselves whether all projective sets have these regularity properties. For example: are all projective sets Lebesgue measurable (PM)? This problem, which belongs to second-order arithmetic, cannot be settled by ZFC. In 1925, Luzin was already thinking about the negative answer to PM when he declared:

> One does not know and one will never know of the projective sets whether or not they are each Lebesgue measurable [Luzin 1925].

Now we can introduce two results which establish that PM is an undecidable statement for the axioms of ZFC. The first result was established by Kurt Gödel by introducing the constructible universe, known as L:

THEOREM 42. *(Gödel) Assume ZFC + V=L. Then there are $\Sigma^1_2$ sets that do not have the property of Baire, are not Lebesgue measurable, and there are $\Pi^1_1$ sets that do not have the perfect set property.*

The second result was obtained by Robert Solovay in 1965:

THEOREM 43. *(Solovay) Assume ZFC and there is a strongly inaccessible cardinal. Then there is a forcing extension in which all projective sets have the perfect set property, the property of Baire and are Lesbegue measurable.*

Peter Koellner explains the undecidability regarding projective sets in the following way:

> If ZFC is consistent, then ZFC cannot determine whether all $\Sigma^1_2$ sets have the property of Baire and are Lesbegue measurable. If ZFC + there is a strongly inaccessible cardinal is consistent, then ZFC cannot determine whether all $\Pi^1_1$ sets have the perfect set property [Koellner 11].

Now we shall introduce the concepts of infinite games, winning strategy and determinacy, namely having a winning strategy. Let $X$ be a non-empty set. For $A \subseteq X^\omega$, $G_X(A)$ points out to the following infinite two-person game with perfect information: There are two players, player 1 and player 2. Player 1 initially selects an $x(0) \in X$; then player 2 selects an $x(1) \in X$; then player 1 selects an $x(2) \in X$; then player 2 selects an $x(3) \in X$; and so forth. Each selection is a move of the game, and each player before making each of his moves is informed about all the precedent moves (perfect information). The resulting $x \in X^\omega$ is a play of the game, an initial segment of a $x$ a partial play, and player 1 wins if $x \in A$, and otherwise player 2 wins. A, which is the payoff for the game $G_X(A)$. $G_X(A)$, is determined if a player has a winning strategy. At this point, we can define a winning strategy:

Definition 51. *A **strategy** for player (1) is a function $\sigma : \bigcup_{n\in\omega} X^{2n} \longrightarrow X$ that tells him what move to make given the previous moves, so that a (partial) play according to $\sigma$ is a (partial) play of the form:*
***Player (1)*** $\sigma(\emptyset)$, ***Player (2)*** $y(0)$, ***Player (1)*** $\sigma((\sigma(\emptyset), y(0)))$, ***Player (2)*** $y(1)$.....,
***Player (1)*** $\sigma((\sigma(\emptyset), y(0), \sigma((\sigma(\emptyset), y(0))), y(1)))$....
*Player (2)'s moves are enumerated by $y \in X^\omega$, and this play is denoted by*

$$\sigma * y$$

*$\sigma$ is a winning strategy for Player (1) iff*

$$\{\sigma * y | y \in X^\omega\} \subseteq A$$

*i.e no matter what moves Player (2) makes, plays according to $\sigma$ always result in a member of A. [Kanamori 09]*

Now, we can introduce the axiom of determinacy:

DEFINITION 52. *(Mycielsky, Steinhaus) (AD) every set of reals is determined.*

Peter Koellner lists the following examples of determined sets:

> If A is the set of all reals, then clearly player 1 has a winning strategy; if A is empty, clearly player 2 has a winning strategy; if A is countable, then player 2 has a winning strategy by diagonalising. This might lead one to expect that all sets of reals are determined. However, it is straightforward to use the axiom of choice (AC) to construct a non-determined set (by listing all winning strategies and diagonalising across them). For this reason AD was never really considered as a serious candidate for a new axiom. [Koellner 11]

However, the axioms of definable determinacy are consistent with the axiom of choice. Now we can introduce $\Delta^1_1$-determinacy (Borel determinacy):

DEFINITION 53. *$\Delta^1_1$-determinacy is the statement that all Borel sets are determined.*

In 1974, Donald A. Martin proved the following theorem:

THEOREM 44. *(Martin) $\Delta^1_1$-determinacy is provable in ZFC.*

We can consider the axioms of definable determinacy which fall outside the scope of ZFC.

Definition 54. *(PD) All projective sets are determined.*

Furthermore, we can relativise the full axiom of determinacy to $L(\mathbb{R})$ in the following way:

Definition 55. *($AD^{L(\mathbb{R})}$) All sets of reals in $L(\mathbb{R})$ are determined.*

It is interesting to notice that Solovay and Takeuti pointed out that there is a natural subuniverse, namely $L(\mathbb{R})$, in which AD could hold, consistently with assuming AC in the full universe.
At this point we can state something very important: If one assumes PD then all projective sets have the regularity properties and, furthermore, if one assumes $AD^{L(\mathbb{R})}$ then all of the sets of reals in $L(\mathbb{R})$ have the regularity properties.
At this point we can introduce the central large cardinal hypothesis for the completeness of second-order arithmetic:

Definition 56. *(Woodin) A strongly inaccessible cardinal $\delta$ is a Woodin cardinal if for each function $f : \delta \longrightarrow \delta$ there exists an elementary embedding $j : V \longrightarrow M$ with critical point $\gamma < \delta$ such that $f[\gamma] \subset \gamma$ and $V_{j(f)(\gamma)} \subset M$.*

If $\delta$ is the least Woodin cardinal then $\delta$ itself is not a very large cardinal in the usual sense. For example it is not weakly compact. We can assert that sets witnessing that a Woodin cardinal $\delta$ is Woodin exist in $V_\delta$, which shows in particular that if $\delta$ is a measurable Woodin cardinal then there are other Woodin cardinal below $\delta$. Moreover, we can add that supecompact cardinals are Woodin cardinals. So, if we have to justify philosophically Woodin cardinals, we can adopt Bagaria's structural reflection. In fact, Bagaria's structural reflection produces, as we will see in the next section, a proper class of supercompact cardinals. Therefore, in this case, Woodin's $\Omega$-logic (that we will examine in the following sections) would be justified intrinsically depending on the concept of set since reflection is an essential property shared by sets and the universe of sets itself. Surely, Woodin cardinals are justified also extrinsically, since, thanks to this large cardinal hypothesis, we can have a complete second-order arithmetic and PM, which ZFC cannot decide, is settled. So, Woodin cardinals have fruitful consequences. Now we can introduce two fundamental theorems:

Theorem 45. *(Shelah-Woodin) Assume there exist infinitely-many Woodin cardinals. Then every projective set is Lebesgue measurable.*

Thus, infinetely many Woodin cardinals settle PM, an undecidable statement of second-order arithmetic. By introducing the following theorem, we can see the strong link between large-cardinal hypotheses and the axioms of definable determinacy:

Theorem 46. *(Martin-Steel) Assume there exist infinitely-many Woodin cardinals. Then every projective set is determined.*

So infinitely-many Woodin cardinals imply the axiom of projective determinacy. If we assume infinitely-many Woodin cardinals, projective sets have the regularity properties of definable subsets of reals. The following theorem establishes a link between inner model theory and projective determinacy:

Theorem 47. *(Woodin) The following are equivalent:*
*(1) PD (schematic).*
*(2) For every* $n < \omega$*, there is a fine-structural, countably iterable inner model M such that* $M \models$ *There are n Woodin cardinals.*

Infinitely-many Woodin cardinals are sufficient to prove projective determinacy and inner models of Woodin cardinals are necessary to prove projective determinacy. I believe that the philosophical justification of the axiom of projective determinacy stems from a philosophical justification of large cardinal axioms such as the existence of infinitely-many Woodin cardinals. We can assert that the dream of having a complete second-order arithmetic was accomplished thanks to the large cardinal axiom that asserts the existence of infinitely-many Woodin cardinals. Luzin had negative feelings towards the solution of PM, but then, Woodin and Shelah gave a positive solution to PM. Now, I want to conclude this section devoted to second-order arithmetic with the following beautiful words expressed by Hugh Woodin:

> The fact that from infinitely-many Woodin cardinals one can prove that projective sets are Lesbegue measurable is a strong evidence that from the same assumption one should be able to prove Projective Determinacy. In 1985, using techniques developed in the inner model program, Martin-Steel succeeded in doing this. Surprisingly, the combinatorial properties of Woodin cardinals responsible for their discovery, for example, those aspects yielding the measurability of all projective sets, play no role in this determinacy proof (theorem 47) [Woodin 01] .

**3.6. The method of forcing and Paul Cohen's independence proof.** Before speaking about Woodin's program on Ω-logic and how he attempted to extend the completeness of second-order arithmetic to third-order arithmetic, I want to talk about the method of forcing based on boolean valued models since Ω-logic is essentially featured by boolean models of the universe.

The method of forcing was conceived by Paul Cohen (in 1963) in his proof of independence of the Continuum hypothesis and of the axiom of choice. The basic idea of forcing is to extend a transitive model M of set theory (the ground model) by adding a new set G (a generic set) in order to have a larger transitive model of set theory $M[G]$ called a generic extension. The generic set is approximated by forcing conditions in the ground model, and a particular choice of forcing conditions determines what is true in the generic extension.

Cohen's idea was to begin with a countable transitive model M of ZFC (with a particular set of forcing conditions in M). He established that a generic set $G$ exists and $M[G]$ is a model of ZFC and CH fails in $M[G]$.

It is also possible to take as the ground model the universe V itself and consider a generic imaginary extension of the universe, namely $V[G]$.

Let M (I will focus for the forcing construction on the countable transitive model M) be a transitive model of ZFC, called the ground model. In M, we can consider a partially ordered set $(\mathbb{P}, <)$, or poset, and the elements of $\mathbb{P}$ are called forcing conditions. We say that $p$ is stronger than $q$ if $p \leq q$. If $p$ and $q$ are conditions and there exists an $r$ such that both $r \leq p$ and $r \leq q$, then $p$ and $q$ are said to be compatible; otherwise they are incompatible. We say that a set $A \subset \mathbb{P}$ is an antichain if its elements are pairwise incompatible. We say that a set $S \subseteq \mathbb{P}$ is dense in $\mathbb{P}$ if for every $p \in \mathbb{P}$ there is a $q \in D$ such that $q \leq p$.

Definition 57. *A set $F \subset \mathbb{P}$ is a filter on $\mathbb{P}$ if (1) $F$ is non empty, (2) if $p \leq q$ and $p \in F$, then $q \in F$, (3) if $q, p, \in F$, then there exists $r \in F$ such that $r \leq p$ and $r \leq q$.*

Now we can introduce the second fundamental definition:

Definition 58. *A set of conditions $G \subset \mathbb{P}$ is generic over $M$ if (1) $G$ is a filter on $\mathbb{P}$, (2) If $D$ is dense in $\mathbb{P}$ and $D \in M$, then $G \cap D \neq \emptyset$.*

We can see how forcing works with the following example:

> Let $\mathbb{P}$ be the following notion of forcing: The elements of $\mathbb{P}$ are finite 0-1 sequences $(p(0), ......., p(n-1))$ and a condition $p$ is stronger than $q$ $(p < q)$ if $p$ extends $q$. Clearly, $p$ and $q$ are compatible if either $p \subset q$ or $q \subset p$. Let M be the ground model and let $G \subset \mathbb{P}$ be generic over M. Let $f = \bigcup G$. Since $G$ is a filter, $f$ is a function. For every $n \in \omega$, the set $D_n = \{p \in \mathbb{P} : n \in dom(p)\}$ is dense in $\mathbb{P}$. Thus, it meets G, and so $dom(f) = \omega$.The 0-1 function $f$ is the characteristic function of a set $A \subset \omega$. We claim that

> the function $f$ (or the set A) is not in the ground model. For every $0-1$ function $g$ in M, let $D_g = \{p \in \mathbb{P} : p \not\subset g\}$. The set $D_g$ is dense, hence it meets G, and it follows that $f \neq g$. [Jech 06]

This example highlights in which way we can adjoin a new set of natural numbers to the ground model. A set $A \subset \omega$, which we obtain in this way, is called a *Cohen generic real.* We have to add that a generic set over a transitive model need not exist in general. However, if the grounded model is countable, then generic sets do exist. At this point we can see how forcing works, by taking an example from Kunen's book [Kunen 06]. Before quoting Kunen, we have to say that while the notion of ordinal is absolute between transitive models, the notion of cardinal is not absolute.

> A simple application of this kind of partial order is that the notion of cardinal need not be absolute for $M, M[G]$. Thus, let $\kappa$ be an uncountable cardinal of M; i.e, $\kappa \in M$ and $(\kappa\ \textit{is an uncountable cardinal})^M$. Let $\mathbb{P} = \{p : |p| < \omega \wedge p \text{ is a function} \wedge dom(p) \subset \omega \wedge ran(p) \subset \kappa\}$, and let G be $\mathbb{P}$-generic over M. Then $\bigcup G \in M[G]$ by absoluteness of $\bigcup$, and G is a function from $\omega$ onto $\kappa$, so in $M[G]$, $\kappa$ is a countable cardinal. We say that $\mathbb{P}$ collapses $\kappa$. [Kunen 06]

At this point, I want to quote again Kunen's words that explain in a beautiful manner the concept of forcing:

> People living in M cannot construct a $G$ which is $\mathbb{P}$-generic over M. They may believe on faith that there exists a being to whom their universe, M, is countable. Such a being will have a generic $G$ and an $f_G = \bigcup G$. The people in M do not know what $G$ and $f_G$ are but they have names for them, $\Gamma$ and $\Phi$. [Kunen 06]

Now we can introduce a fundamental theorem about generic models:

Theorem 48. *(Cohen) Let M be a transitive model of ZFC and let* $(\mathbb{P}, <)$ *be a notion of forcing in M. If* $G \subset \mathbb{P}$ *is generic over* $\mathbb{P}$*, then there is a transitive model* $M[G]$ *such that:*

*(1)* $M[G]$ *is a model of ZFC.*

*(2)* $M \subset M[G]$ *and* $G \in M[G]$*.*

*(3)* $On^{M[G]} = On^M$*.*

*(4) If* $N$ *is a transitive model of ZF such that* $M \subset N$ *and* $G \in N$*, then* $M[G] \subset N$*.*

Each element of $M[G]$ has a name in M which describes how it was constructed. In fact, $M[G]$ can be described in the ground model M. We can define a forcing language and introduce a forcing relation $\vdash_f$ which are defined in the ground model. People, who live in M, will be able to comprehend a name,$\tau$, for an object in $M[G]$, but they will not in general be able to decide the object, $\tau_G$, that $\tau$ names, since it will be necessary knowledge of $G$.

Definition 59. $\tau$ *is a* $\mathbb{P}$*-name iff* $\tau$ *is a relation and*

$$\forall(\sigma, p) \in \tau(\sigma \textit{ is a } \mathbb{P} - \textit{name} \wedge p \in \mathbb{P}).$$

*[Kunen 06]*

One can define the characteristic function of $\mathbb{P}$-names, namely $H(\mathbb{P}, \tau)$.

Definition 60. $H(\mathbb{P}, \tau) = 1$ *iff* $\tau$ *is a relation* $\wedge\ \forall(\sigma, p) \in \tau(H(\mathbb{P}, \sigma) = 1 \wedge p \in \mathbb{P})$
$H(\mathbb{P}, \tau) = 0$ *otherwise. [Kunen 06]*

The concept $\tau$ is a $\mathbb{P}$-name is absolute for transitive models of $ZF-$ power set.

Definition 61. $V^{\mathbb{P}}$ *is the class of* $\mathbb{P}$*-names. If* $M$ *is a transitive model of* $ZFC$ *and* $\mathbb{P} \in M$*.* $M^{\mathbb{P}} = V^{\mathbb{P}} \cap M$*. Or by absoluteness,*

$$M^{\mathbb{P}} = \{\tau \in M : (\tau \textit{ is a } \mathbb{P} - \textit{name})^M\}.$$

*[Kunen 06]*

If we force over M, we use only $\mathbb{P}$-names in $M^{\mathbb{P}}$, which we may think that are defined within M .

Definition 62. *$val(\tau, G) = \{val(\sigma, G) : \exists p \in G((\sigma, p) \in \tau\}$.*
*We write also $\tau_G$ for $val(\tau, G)$.*

Definition 63. *If M is a transitive model of $ZFC$, $\mathbb{P} \in M$, and $G \subset \mathbb{P}$, then*

$$M[G] = \{\tau_G : \tau \in M^{\mathbb{P}}\}.$$

*[Kunen 06]*

The key point is given by the forcing theorem:

Theorem 49. *(forcing theorem) Let $(\mathbb{P}, <)$ be a poset in the ground model M. If $\phi$ is a sentence of the forcing language, then for every $G \subset \mathbb{P}$ generic over M,*
*$M[G] \models \phi$ if and only if $(\exists p \in G)p \vdash_f \phi$ .*

We can establish also properties of forcing as in the following examples:
If $p$ forces $\phi$ and $q \leq p$ then $q \vdash_f \phi$.
No $p$ forces both $\phi$ and $\neg\phi$.
$p \vdash_f \neg\phi$ if and only if no $q \leq p$ forces $\phi$, etc.
At this point, we may introduce Boolean-valued models. We begin with the following definition:

Definition 64. *A poset or a partially ordered set $(\mathbb{P}, <)$ is separative if for all $p, q \in \mathbb{P}$, if $p \not\leq q$ then there exists an $r \leq p$ that is incompatible with $q$.*

The forcing notions that we have already seen are separative. If B is a Boolean algebra, then $(B, <)$ is a separative partial order $\mathbb{P}$. The forcing relation can be defined for $\mathbb{P}$ and, we can link a Boolean algebra to $\mathbb{P}$. Let B be a complete Boolean algebra. A Boolean valued model $V^B$ consists of the universe V and functions of two variables with values in B:

$||x = y||$ and $||x \in y||$ (the Boolean values of $=$ and $\in$), which satisfy the following:

$$||x = x|| = 1$$

$$||x = y|| = ||x = y||$$

$$||x = y|| \times ||y = z|| \leq ||x = z||$$

$$||x \in y|| \times ||u = x|| \times ||w = y|| \leq ||u \in w||.$$

For every formula $\phi(x_1, .......x_n)$ we define the Boolean value of $\phi$:

$$||\phi(a_1, ....a_n)|| \quad where(a_1, ......a_n) \in A \ (a \ set)$$

as follows: For atomic formulas we have the precedent definitions. If the formulas are built with connectives, define the Boolean values as follows:

$$||\neg\phi(a_1, .......a_n)|| = -||\phi(a_1, ......., a_n)||$$

$$||(\phi \wedge \psi)(a_1, ........, a_n)|| = ||\phi(a_1, ....., a_n)|| \times ||\psi(a_1, .......a_n)||$$

$$||(\phi \vee \psi)|| = ||\phi(a_1, .......a_n)|| + ||\psi(a_1, ......a_n)||$$

$$||(\phi \longrightarrow \psi)|| = ||(\neg\phi \vee \psi)(a_1, .....a_n)||$$

$$||(\phi \leftrightarrow \psi)|| = ||((\phi \longrightarrow \psi) \wedge (\psi \longrightarrow \phi))(a_1, ......, a_n)||$$

If $\phi$ is of the form such as $\exists x\psi$ or $\forall x\psi$, then:

$$||\exists x\psi(x, a_1, ......., a_n)|| = \Sigma_{a\in A}||\psi(a, a_1, ........, a_n)||$$

$$||\forall x\psi(x, a_1, ......, a_n)|| = \Pi_{a\in A}||\psi(a, a_1, ......., a_n)||$$

If B is the trivial algebra$\{0, 1\}$, then a Boolean-valued model is just a two valued model. The Boolean value of $\phi$ is just a generalization of the satisfaction predicate, namely $\models$. We assert that $\phi(a_1, .....a_n)$ is valid in $V^B$ if $||\phi(a_1, .....a_n)|| = 1$ [Jech 06]. All axioms of first-order logic are valid in Boolean valued models. Boolean-valued models can be used in consistency proofs. Let $V^B$ be a Boolean-valued model (if it exists) such that all the axioms of ZFC are valid in $V^B$. Let $\phi$ be a set-theoretical statement and assume $||\phi \neq 0||$. Then we can conclude that $\phi$ is consistent relative to ZFC, and, so, it cannot be disproved. Each Boolean-valued model can be transformed into a two valued model. At this point we can introduce the Boolean-valued model $V^{\mathbb{B}}$. Let B be a complete Boolean algebra. We consider Boolean-valued sets, i.e., functions that assign Boolean-values to its elements. We define Boolean-valued sets by recursion on the ordinals in the following way:

(1) $V_0^{\mathbb{B}} = \emptyset$,

(2) $V_{\alpha+1}^{\mathbb{B}} =$ the set of all functions $x$ with $dom(x) \subset V_\alpha^{\mathbb{B}}$ and values in B,

(3) $V_\alpha^{\mathbb{B}} = \bigcup_{\beta<\alpha} V_\beta^{\mathbb{B}}$ if $\alpha$ is a limit ordinal,

(4) $V^{\mathbb{B}} = \bigcup_{\alpha \in On} V_\alpha^{\mathbb{B}}$.

Each $x \in V^{\mathbb{B}}$ is assigned the rank in $V^{\mathbb{B}}$,

$$\rho(x) \ = \ \textit{the least } \alpha \textit{ such that } x \ \in V_{\alpha+1}^{\mathbb{B}}$$

Now we can introduce the following theorem:

THEOREM 50 (Jech 06). *Every axiom of ZFC is valid in* $V^{\mathbb{B}}$*.*

Now, coming back to our original forcing construction (the countable transitive model M), we can think of this in the following way: Suppose that there is a countable transitive model of ZFC. Using the poset $\mathbb{P} \in M$, there exists a $\mathbb{P}$-generic filter over M, and $M[G]$ is a transitive model that satisfies $\neg CH$. Thus, $\neg CH$ is consistent relative to ZFC and it cannot be disproved by these axioms. We could have supposed that the axioms of ZFC held in the universe V itself and we could have constructed a filter G over V, forming the model

$V[G]$, that can be considered as an imaginary-virtual forcing extension of the universe. At this point, we can see how the proof of Paul Cohen for the independence of CH works:

THEOREM 51. *(Cohen) There is a generic extension $V[G]$ that satisfies $2^{\aleph_0} > \aleph_1$.*

PROOF. We feature the poset that produces a generic extension with the desired property. Let $\mathbb{P}$ be the set of all functions $p$ such that: (1) $dom(p)$ is a finite subset of $\omega_2 \times \omega$, (2) $ran(p) \subset \{0,1\}$, and let $p$ be stronger than $q$ if and only if $q \subset p$. If $G$ is a generic filter of conditions, we let $f = \bigcup G$. We assert that:

$$(1)\ f\ is\ a\ function$$

$$(2)\ \ dom(f) = \omega_2 \times \omega$$

We can say that $\omega_2$ means $\omega_2$ in the ground model. (1) holds because $G$ is a filter. For (2), the sets $D_{\alpha,n} = \{p \in \mathbb{P} : (\alpha, n) \in dom(p)\}$ are dense in $\mathbb{P}$, hence $G$ meets each of them, and so $(\alpha, n) \in dom(f)$ for all $(\alpha, n) \in \omega_2 \times \omega$.
Now, for each $\alpha < \omega_2$, let $f_\alpha(n) = \omega \longrightarrow \{0,1\}$ be the function defined as follows: $f_\alpha(n) = f(\alpha, n)$. If $\alpha \neq \beta$, then $f_\alpha \neq f_\beta$; this is because the set $D = \{p \in \mathbb{P} : p(\alpha, n) \neq p(\beta, n)\ for\ some\ n\}$ is dense in $\mathbb{P}$ and hence $G \cap D \neq \emptyset$. Thus, in $V[G]$ we have a one-to-one mapping $\alpha \longrightarrow f_\alpha$ of $\omega_2$ into $\{0,1\}^\omega$. [Jech 06] □

Each $f_\alpha$ is the characteristic function of a set $a_\alpha \subset \omega$. We call these sets *Cohen generic reals*. Hence $\mathbb{P}$ adds $\aleph_2$ *Cohen generic reals* to the ground model. However, we need to introduce a theorem that establishes that $\mathbb{P}$ preserves cardinals, namely the cardinal $\kappa_2^V$ is the cardinal $\aleph_2$ in $V[G]$. We start with the following definition:

DEFINITION 65. *A Poset $\mathbb{P}$ satisfies the countable chain condition (c.c.c.) if every antichain in $\mathbb{P}$ is at most countable.*

THEOREM 52. *(Cohen) If $\mathbb{P}$ satisfies the countable chain condition, then $V$ and $V[G]$ have the same cardinals and cofinalities.*

We can shape the poset $\mathbb{P}$ as we want. We can use the combinatorial properties of $\mathbb{P}$ to force the cardinality of the continuum to be in $V[G]$ any alephs of uncountable cofinality. In fact in the theorem of Cohen if we put

$$dom(f) = \omega_{\omega_{\omega_1}} \times \omega \ .$$

we have that

$$2^{\aleph_0} = \aleph_{\omega_{\omega_1}}.$$

in $V[G]$.

**3.7. Forcing Axioms, BPFA assumed as a phenomenal solution to the continuum hypothesis and a Kantian metaphysical distinction.** Before speaking about Woodin's program, we must introduce forcing axioms since these axioms do settle the continuum hypothesis. Another fundamental aspect that forces me to introduce these axioms is represented by the fact that Bounded Proper Forcing Axiom (BPFA) may represent a phenomenal solution to the Continuum Hypothesis .

Forcing axioms were conceived in order to saturate the universe of all sets by considering the forcing method. We have to introduce a different hierarchy represented by $H(\kappa)$ sets.

DEFINITION 66. *A set $X$ is transitive if each element of $X$ is also a subset of $X$. The transitive closure of a set $X$ is the set* $\cap\{Y|Y$ *is transitive and* $X \subseteq Y\}$

DEFINITION 67. *Suppose $\kappa$ is an infinite cardinal. $H(\kappa)$ denotes the set of all sets $X$ whose transitive closure has cardinality less than $\kappa$.*

For strongly inaccessible cardinal, this hierarchy and Von Neuman hierarchy are the same. For example $V_\omega = H_\omega$. Now suppose that $\mathbb{D}$ is a collection of dense subsets of a partial order $\mathbb{P}$. If we can show, in V, the existence of a generic filter $G \subseteq \mathbb{P}$ which meets every element of $D$, then we do not need to go to a generic extension $V[G]$ to have a generic filter for $\mathbb{D}$. Now we can introduce the axiom of forcing:

Definition 68. *$FA(\Gamma, \kappa)$ holds if for every partial order $\mathbb{P}$ with the property $\Gamma$ and every collection $\mathbb{D} = \{D_\alpha \subseteq \mathbb{P} : \alpha \leq \kappa\}$ of dense subsets of $\mathbb{P}$, there exists a filter $G \subseteq \mathbb{P}$ that meets every $D_\alpha, \alpha \leq \kappa$.*

The axiom of forcing says that by forcing method it is possible to construct a generic extension in which G exists. The most studied classes of forcing ($\Gamma$) are the following: Countable chain condition (c.c.c.), proper, and stationary set preserving (SSP).

Definition 69. *we say that $\mathbb{P}$ has the c.c.c. if, for every maximal antichain $A$ (i.e, $A \subseteq \mathbb{P}$ is such that $\forall p, q \in A, \neg\exists r(r \leq p \wedge r \leq q)$ and is maximal for this property) we have that $|A| \leq \aleph_0$.*

Definition 70. *A partial order $\mathbb{P}$ is proper if for every uncountable regular cardinal $\kappa > 2^{|\mathbb{P}|}$, and for every $M \prec H(\kappa)$, with $\mathbb{P} \in M$, every condition $p \in \mathbb{P} \cap M$ has an extension $q \leq p$ which is $(M, \mathbb{P})$-generic. Where $q$ is called $(M, \mathbb{P})$-generic if for every $D \subseteq \mathbb{P}$ dense and in $M$ and for every $r \leq q$ exists a condition $d \in D \cap M$ compatible with $r$, i.e, $D \cap M$ is predense below $q$.*

Definition 71. *A partial order $\mathbb{P}$ is SSP, if every $S \subseteq \omega_1$ stationary, remains stationary in every generic extension. By $\mathbb{P}$-Stationary set means that $S \cap C \neq \emptyset$ for every closed unbounded set $C \subseteq \omega_1$.*

All these notions of forcing (i.e, c.c.c, Proper, SSP) preserve $\omega_1$ in the generic extension. By defining $\mathbb{P}_\Gamma = \{\mathbb{P}\text{: forcing with the property } \Gamma\}$ we have the following theorem:

Theorem 53. *We have the following chain of inclusions:*

$$\mathbb{P}_{c.c.c.} \subseteq \mathbb{P}_{proper} \subseteq \mathbb{P}_{SSP}.$$

The first (historically) introduced forcing axiom was Martin Axiom (MA):

DEFINITION 72. *$MA(\kappa)$: if $\mathbb{P}$ is a partial ordering or poset with c.c.c. and $\mathbb{D}$ is a family of $\leq \kappa$-many dense subsets of $\mathbb{P}$, then there is a filter $G \subseteq \mathbb{P}$ such that $G \cap D \neq \emptyset$, for every $D \in \mathbb{D}$. MA is the statement: $\forall \kappa < 2^{\aleph_0} MA(\kappa)$.*

Now we can introduce the following theorem:

THEOREM 54. *$MA(2^{\aleph_0})$ is contradictory.*

PROOF. Let

$$\mathbb{P} = \{p : \omega \longrightarrow 2 \wedge |p| < \omega\}$$

with ordering relation $p \leq q$ if $q \subseteq p$. $\mathbb{P}$ has c.c.c.
Now we set

$$H = \{h_\alpha : \alpha \in 2^{\aleph_0}\}$$

a list of all functions from $\omega$ into $\{0, 1\}$. If we define for every $n \in \omega$:

$$D_n = \{p : n \in dom(p)\}$$

and for every $\alpha \in 2^{\aleph_0}$

$$E_\alpha = \{p : \exists\, n \in dom(p) \text{ such that } p(n) \neq h_\alpha(n)\}$$

then these sets are all dense sets in $\mathbb{P}$. Therefore, since

$$|\{D_n : n \in \omega\} \cup \{E_\alpha : \alpha \in 2^{\aleph_0}\}| = |2^{\aleph_0}|$$

there exists by $MA(2^{\aleph_0})$, a filter $G \subseteq \mathbb{P}$ which meets all these dense sets. Then $\bigcup G$ would be a total function from $\omega$ into $\{0, 1\}$ which does not belong to H. A contradiction. □

Thus, $MA(\aleph_1)$ disproves CH, the Continuum Hypothesis.
Baumgartner formulated the Proper Forcing Axiom (PFA). This axiom can be considered as generalization of MA applied to the class of proper posets with the necessary restriction

that the family $\mathbb{D}$ of dense open subsets of the poset $\mathbb{P}$ be of cardinality at most $\aleph_1$. Without this restriction this axiom (PFA) would be inconsistent with ZFC. Baumgartner also proved that PFA is consistent with ZFC, assuming the consistency of ZFC with the existence of a supercompact cardinal. At the beginning Baumgartner conceived Axiom A, a property of partial orderings or posets weaker than the c.c.c. condition. Properness is even weaker than the axiom A property. Now we can introduce the Proper Forcing Axiom:

Definition 73. *Proper Forcing Axiom, PFA, is the statement* $FA(\mathbb{P}_{proper}, \aleph_1)$ *.*

A strenghtening of the Proper Forcing Axiom is Martin's Maximum (MM).

Definition 74. *Martin Maximum, MM, is the statement* $FA(\mathbb{P}_{SSP}, \aleph_1)$ *.*

Lemma 10. *We have the following chain of implications:*

$$MM \longrightarrow PFA \longrightarrow MA(\aleph_1)$$

Martin's Maximum is the strongest forcing axiom. However, recently, Matteo Viale [Viale 15] introduces $MM^{++}$ and $MM^{+++}$ which strenghten Martin's Maximum. See also Asperó for a different kind of strenghtening [Aspero 12]. As for the consistency strenght of Martin Maximum (MM) and the Proper Forcing Axiom (PFA), we have the following theorem:

Theorem 55. *(Foreman, Magidor, Shelah) Assuming the existence of a supercompact cardinal, there is a generic extension which satifies MM, hence also PFA .*

Thus, Martin's Maximum and the Proper Forcing Axiom are consistent relative to the consistency of the axiom which asserts the existence of a supercompact cardinal. Martin's Maximum does settle the cardinality of the continuum. In fact, we have the following two theorems:

THEOREM 56. *(Foreman, Magidor, Shelah) For every regular cardinal $\kappa \geq \aleph_2$, MM implies that $\kappa^{\aleph_1} = \kappa$.*

THEOREM 57. *(Foreman, Magidor, Shelah) MM implies that $2^{\aleph_0} = \aleph_2$.*

PROOF. For theorem 57 we have that $2^{\aleph_0} \leq 2^{\aleph_1} \leq \aleph_2^{\aleph_1} = \aleph_2$, but MM implies $MA(\aleph_1)$, then $\aleph_1 < 2^{\aleph_0}$, so $2^{\aleph_0} = \aleph_2$ □

Also, the Proper Forcing Axiom (PFA) does settle the cardinality of the continuum.

THEOREM 58. *(Todorcevic, Velickovic) PFA implies $2^{\aleph_0} = \aleph_2$.*

We cannot strenghten the axioms MM and PFA by increasing the cardinality of $\mathbb{D}$, namely the family of dense sets which meet with the generic filter $G$. In fact, we have the following:

THEOREM 59. *$FA(\mathbb{P}_{proper}, \aleph_2)$ and $FA(\mathbb{P}_{SSP}, \aleph_2)$ are inconsistent.*

PROOF. Since $FA(\mathbb{P}_{SSP}, \aleph_2)$ implies $FA(\mathbb{P}_{proper}, \aleph_2)$ , we show this fact for $FA(\mathbb{P}_{proper}, \aleph_2)$. $FA(\mathbb{P}_{proper}, \aleph_2)$ implies $FA(\mathbb{P}_{proper}, \aleph_1) =$ PFA, which implies $2^{\aleph_0} = \aleph_2$. Thus $FA(\mathbb{P}_{proper}, \aleph_2)$ $= FA(\mathbb{P}_{proper}, 2^{\aleph_0})$ which implies $FA(\mathbb{P}_{c.c.c.}, 2^{\aleph_0}) = MA(2^{\aleph_0})$. However we have seen that $MA(2^{\aleph_0})$ is contradictory. □

At this point we shall introduce bounded forcing axioms, $FA(\Gamma, \kappa, \lambda)$, where $\Gamma$ is a property of partial orders and $\kappa$, $\lambda$ are cardinals:

DEFINITION 75. *(Bounded Forcing Axioms) $FA(\Gamma, \kappa, \lambda)$: for every partial order or poset $\mathbb{P}$ with the property $\Gamma$ and for every collection $\mathbb{I}$ of $\kappa$-many maximal antichains of $\mathbb{P}$ such that $|I| \leq \lambda$, for every $I \in \mathbb{I}$, there exists a filter $G$ which meets every $I \in \mathbb{I}$.*

$MA(\aleph_1)$ can be seen as the first bounded forcing axiom. We are interested in bounded forcing axioms such as $FA(\Gamma, \omega_1, \omega_1)$ where $\Gamma$ is proper or stationary preserving. So, we have BPFA (Bounded Proper Forcing Axiom) and BMM (Bounded Martin's Maximum). We have that $BMM \longrightarrow BPFA \longrightarrow MA(\aleph_1)$. Bounded Forcing Axioms can be formulated as principles of generic absoluteness. For the absoluteness theorem of Levy-Shoenfiled, a $\Sigma_1$-formula, which has parameters in $H(\omega_2)$, is absolute, namely is true in all transitive models only if it is true in one such model containing the parameters. We have $H(\omega_2, \in) \prec_1 (V, \in)$. So we have the following equivalence between forcing axioms and principles of generic absoluteness:

THEOREM 60. *(Bagaria) Let* $\Gamma$ *be a class of partial orderings and let* $\kappa$ *be an infinite cardinal with uncountable cofinality, then the following two statements are equivalent:*

$$(1) FA(\Gamma, \kappa, \kappa)$$

$$(2)(H(\kappa^+), \in)) \prec_1 (V^{\mathbb{P}}, \in) \; for \; every \; \mathbb{P} \in \Gamma.$$

This theorem is valid also for MA, since MA can be seen as a bounded forcing axiom. MA says that every $\Sigma_1$-formula with parameters in $H(\kappa)$, where $\kappa < 2^{\aleph_0}$, forced with $\mathbb{P}_{c.c.c.}$ is valid in V. Bounded forcing axioms do settle the continuum hypothesis. In fact, we have the following two theorems:

THEOREM 61. *(Todorcevic) BMM implies* $2^{\aleph_0} = \aleph_2$.

THEOREM 62. *(Moore) BPFA implies* $2^{\aleph_0} = \aleph_2$.

Woodin proved the consistency of BMM relative to the consistency of $\omega + 1$-many Woodin cardinals. Goldstern and Shelah prove that BPFA is consistent relative to the consistency of the axiom asserting the existence of a $\Sigma_2$-reflecting cardinal (I will define this notion immediately). If $\kappa$ is a strongly inaccessible cardinal, we have $V_\kappa \prec_{\Sigma_1} V$. $V_\kappa$ reflects all $\Sigma_1$-sentences with parameters. Now we can consider $\Sigma_2$-sentences. Suppose that $\kappa$ is a strongly inaccessible cardinal such that $V_\kappa \prec_{\Sigma_2} V$, i.e, it reflects all $\Sigma_2$ sentences with parameters. We can conclude that $\kappa$ is an inaccessible cardinal, a limit of inaccessible cardinals and much more. We can consider for every $n$ the existence of a regular cardinal such that $V_\kappa \prec_{\Sigma_n} V$. Such a cardinal is called an $n$-reflecting cardinal. For $n < m$, if $\kappa$ is an $m$-reflecting cardinal then it is also an $n$-reflecting cardinal. However, by Tarski's theorem about the undefinability of truth, there cannot be a definable $\kappa$ such that $V_\kappa$ reflects all sentences. A Mahlo cardinal $\kappa$ is inaccessible and in $V_\kappa$ there is a stationary class of $\Sigma_\omega$-reflecting cardinals, namely $\Sigma_n$-reflecting for every $n$. BPFA is consistent relative to the consistency of the axiom asserting the existence of $\Sigma_2$-reflecting cardinal. This is a very weak large cardinal axiom between the axiom asserting the existence of a strongly inaccessible cardinal and the axiom asserting the existence of a Mahlo cardinal. Since Martin Maximum (MM) and the Proper Forcing Axiom are consistent relative to the existence of a supecompact cardinal and we do not have yet an inner model for supercompact cardinal, even if these axioms do settle the continuum hypothesis, we cannot consider their answer to the first problem in Hilbert's list as decisive in the case that the Ultimate L conjecture were false. Instead, since BPFA is consistent relative to the existence of a $\Sigma_2$-reflecting cardinal and we have an inner model of this large cardinal notion, namely L, which forces us to trust this cardinal notion, we can state that BPFA may represent a phenomenal solution to the continuum hypothesis if the ultimate L conjecture were false and the cardinality of $\mathbb{R}$ is $\aleph_2$ as Gödel was thinking, although this would be a phenomenal truth according to my philosophical beliefs. Now to express my philosophical position, I have to apply a Kantian distinction between phenomenal reality and noumenal reality to set theory. Kantian noumenon is a posited object or reality that is known (if at all) without the use of physical senses. The term noumenon is used in relation with phenomenon which refers to an object apprehended by physical senses. The noumenal world may exist but it is completely unknowable to humans. The noumenal reality is the reality in itself or thing-in-itself . As expressed in Kant's Critique of Pure Reason [Kant 781], Human understanding is structured by innate categories of understanding that mind uses in order to make sense of raw unstructured experience (the phenomenal interpretation of reality). For Kant, when we employ a concept to categorize noumena (the things-in-themselves) we are categorizing phenomena (the observational manifestations of noumena). For Kant, we can categorize phenomena, but we can never directly know noumena. Even if noumena are unknowable, they are still needed as a limiting concept. The existence of the noumenal world limits reason to what he perceives to be its proper bounds, making many metaphysical questions unaswerable by reason. For Kant, the phenomenal reality based on physical senses' apprehension structured, then, by categories of understanding is the realm of appearance and it is not what it is really (the reality in itself). While the noumenal reality is what it is really. According to my philosophy, in set theory the phenomenal reality is constructed by human mind and is represented by metamathematical models such that $L[U], HOD, V[G]$, etc, in which we interpret arbitrary sets and we have different set-theoretic concepts. While the noumenal reality is the immutable, eternal world of sets itself independent from human mind. We have truths relative to the models (the phenomenal reality). Within canonical inner models with the notion of definable subsets or within outer models with the notion of generic filter, we interpret sets and we obtain truths specific or relative to the models. We construct metamathematical models (the phenomenal reality). Contrary to what Kant was thinking about the sensible world, I believe that we can know the noumenal reality of sets (the world of sets in it self) if the Ultimate L conjecture were true. In this case, the inner model for a supercompact cardinal, although a phenomenal model, would be very close to the universe of sets V and its structural content would be equivalent to the universe of sets V. Thus, we can consider the Ultimate L as the true, noumenal universe of sets where CH is settled. If the Ultimate L conjecture were false, we would have a plurality of phenomenal metamathematical models and among these models, within some of them, we would prove specific, phenomenal truths. If the Ultimate L conjecture were false, all metamathematical phenomenal models would be characterized by possible truths. However, some of them would represent a phenomenal solution of phenomenal truths and so I would argue that a specific phenomenal metamathematical model would prove a specific, phenomenal truths (BPFA for CH). So, in this case (the ultimate L conjecture is false), I would agree with Hamkins but I would argue that some mathematical statements, such as CH, have a phenomenal truth value within a phenomenal model. We would have phenomenal pluralism. In this case the noumenal, set theoretic reality would be inaccessible to us. If the ultimate L conjecture were false, the set theoretic noumenon would be inaccessible. However, as I will argue, if the Ultimate L conjecture were true, the true, noumenal universe of mathematics is represented by the ultimate L, a phenomenal reality constructed by human mind that would coincide with the noumenal universe V. Furthermore, I do not think that the notion of arbitrary set and the notion of full power set of arbitrary sets are precise mathematical concepts. In fact, they are subjected to the phenomenon of vagueness. Instead, I prefer the notion of definable set and definable power set. We should prefer a universe of mathematics totally constructible where all sets are definable making the notion of set precise and avoiding impredicative mathematical objects (predicativism). So if the Ultimate L conjecture were true, the true, noumenal universe of mathematics would be the Ultimate L (a phenomenal reality created by human mind), namely the inner model of a supercompact cardinal (phenomenal reality) which contains all large cardinals, where the generalised continuum hypothesis is true and where all mathematical notions are precise.

**3.8. Woodin's program applied to third-order arithmetic, Woodin's Maximum and a comparison between Turing's completeness and Woodin's completeness.** At this point, we can explain briefly Woodin's program. Woodin wants to add axioms to ZFC in order to have a solution to problems formulated in $H(\omega_2)$, namely third-order arithmetic. We have seen that infinitely-many Woodin cardinals and the axiom of Projective Determinacy do settle many problems in $H(\omega_1)$, namely second-order arithmetic. So, Woodin wants to find an axiom (Woodin's Maximum) which decides the whole theory of $H(\omega_2)$. We can quote Woodin's words in order to understand his program:

> The answer to the continuum problem lies in understanding $H(\omega_2)$, where $\omega_2$ is the smallest cardinal greater than $\omega_1$. This suggest an incremental approach. One attempts to understand in turn the structures $H(\omega), H(\omega_1)$, and then $H(\omega_2)$. A little, more precisely, one seeks to find the relevant axioms for these structures. Since the Continuum Hypothesis concerns the structure of $H(\omega_2)$, any reasonably complete collection of axioms for $H(\omega_2)$ will resolve the Continuum Hypothesis. [Woodin 01]

Woodin is dreaming a complete theory for third-order arithmetic. This dream, as we will see, is not so far from reality. The first step towards a complete theory is the theorem of Shoenfield which we have already seen:

Theorem 63. *(Shoenfield) If $\phi$ is $\Sigma^1_2$-formula then every transitive model of ZFC satisfies $\phi$ or every transitive model of ZFC satisfies $\neg\phi$.*

Woodin wants to obtain a similar result for the structure $H(\omega_2)$. Now we can introduce the semantic relation which features $\Omega$-logic.

Definition 76. *Suppose that $T$ is a countable theory in the language of set theory and $\phi$ a sentence. then*

$$T \models_\Omega \phi$$

*if for all complete boolean algebras $\mathbb{B}$ and for all ordinals $\alpha$,*

$$if\ V^{\mathbb{B}}_\alpha \models T\ then\ V^{\mathbb{B}}_\alpha \models \phi$$

This semantic notion is strong since large cardinal axioms imply an important absolute result:

THEOREM 64. *(Woodin) Assume ZFC and that there is a proper class of Woodin cardinals. Suppose that $T$ is a countable theory and $\phi$ a sentence. then for all complete Boolean algebras $\mathbb{B}$,*

$$T \models_\Omega \phi \ \ iff \ \ V^{\mathbb{B}} \models T \models_\Omega \phi$$

To explain this result, which implies that by assuming the existence of a proper class of Woodin cardinals we cannot alter the truth of $\phi$ by going to a forcing extension, we quote Koellner words:

> It follows immediately from the above that $\Omega$-satisfiability is also generically invariant. To underscore just how remarkable this is we note the following consequence: Suppose that there is a proper class of Woodin cardinals and let $\phi$ be a $\Sigma_2$-sentence. The statement that $\phi$ holds in a generic extension is generically absolute. For example, suppose that $\phi$ is the $\Sigma_2$-statement asserting that there is a Huge cardinal. Let $V^{\mathbb{B}}$ be a generic extension where the huge cardinal is collapsed. It follows from the above that it is possible to further force to *resurrect* the huge cardinal. [Koellner 09]

At this point in order to introduce a quasi-syntactic proof theoretic relation in $\Omega$-logic, we need to define the notion of universally Baire set:

DEFINITION 77. *Suppose $A \subseteq \omega^\omega$ and $\delta$ is a cardinal. The set $A$ is $\delta$-universally Baire if for all posets or partial orders $\mathbb{P}$ of cardinality $\delta$ there exist trees $S$ and $T$ in $\omega \times \kappa$ for some $\kappa$ such that*

*(1) $A = p[T]$.*

*(2) If $G \subseteq \mathbb{P}$ is $V$-generic then in $V[G]$,*

$$p[T] = \omega^\omega / p[S].$$

*The set $A$ is universally Baire if it is $\delta$-universally Baire for all $\delta$.*

Universally Baire sets have an absolute interpretation in generic extensions $V[G]$. With the following condition, universally Baire sets are totally preserved in generic extensions only if the model M is robust enough.

Definition 78. *Suppose that $A \subseteq \omega^\omega$ is universally Baire and that M is a countable transitive model of ZFC. Then M is a strongly A-closed if for all set generic extensions $M[G]$ of M,*

$$A \cap M[G] \in M[G]$$

The notion of proof in $\Omega$-logic is not really syntactic, but model-theoretic (as we have seen). There is no proof calculus and no proof rules. The crucial notion for this quasi-syntactic proof theoretic relation is that of an A-closed model, where A is a universally Baire set. Asking for the model to be A-closed means that the model is robust enough with respect to A. It interprets absolutely A in all its generic extensions. A is preserved in all generic extensions, $M[G]$, of M.

Definition 79. *Suppose there is a proper class of Woodin cardinals, T is a countable theory in the language of set theory and $\phi$ is a sentence, then $T \vdash_\Omega \phi$ iff there exists a set $A \subseteq \omega^\omega$ such that*

*(1) A is universally Baire,*

*(2) For all countable transitive models M, if M is strongly A-closed and $T \in M$, then $M \models T \models_\Omega \phi$*

Like the semantic notion we have a quasi-syntactic notion linked to large cardinals:

Definition 80. *(Woodin) Assume there is a proper class of Woodin cardinals. Suppose T is a countable theory in the language of set theory, $\phi$ is a sentence, and $\mathbb{B}$ is a complete Boolean algebra. Then*

$$T \vdash_\Omega \phi \quad iff \quad V^{\mathbb{B}} \vdash_\Omega \phi.$$

While the soundness theorem is known to hold (Woodin) for $\Omega$-logic, it is an open-problem if the completeness theorem holds. So we can state the $\Omega$-conjecture:

DEFINITION 81. *($\Omega$-conjecture). Assume ZFC and that there is a proper class of Woodin cardinals. Then for each sentence $\phi$,*

$$\emptyset \models_\Omega \phi \quad iff \quad \emptyset \vdash_\Omega \phi.$$

We can say that a theory T is $\Omega$-complete if it decides all questions, since for a collection of sentences to which $\phi$ belongs, we have that $T \models_\Omega \phi$ or $T \models_\Omega \neg\phi$. Now we have to state a fundamental aspect of $\Omega$-logic.

DEFINITION 82. *A theory $T$ is $\Omega$-complete for a collection of sentences $\Gamma$ if for each $\phi \in \Gamma$, $T \models_\Omega \phi$ or $T \models_\Omega \neg\phi$.*

We state the result on generic absoluteness of $L(\mathbb{R})$

THEOREM 65. *(Woodin) Assume ZFC and that there is a proper class of Woodin cardinals. Then ZFC is $\Omega$-complete for the collection of sentences of the form $L(\mathbb{R}) \models \phi$.*

We have the completeness at the level of $L(\mathbb{R})$. Unfortunately, that the acutal large cardinals axioms are not $\Omega$-complete at the level of third-order arithmetic where the Continuum Hypothesis is formulated.

THEOREM 66. *Assume $A$ is a standard large cardinal axiom. Then $ZFC + A$ is not $\Omega$-complete for $\Sigma^2_1$ statements.*

However, by assuming CH (the Continuum Hypothesis), one can attain such $\Omega$-complete picture for $\Sigma^2_1$ statements.

THEOREM 67 (Woodin 10b). *Assume ZFC and that there is a proper class of measurable Woodin cardinals. Then $ZFC + CH$ is $\Omega$-complete for $\Sigma^2_1$ statements.*

Furthermore, up to $\Omega$-equivalence, $CH$ is the unique $\Sigma_1^2$ statement that is $\Omega$-complete for $\Sigma_1^2$ statements.

LEMMA 11. *Suppose $A$ is a $\Sigma_1^2$ sentence, $ZFC + A$ is $\Omega$-satisfiable, and $ZFC + A$ is $\Omega$-complete for $\Sigma_1^2$. Then*

$$(1)\ ZFC + CH \models_\Omega A \text{ and}$$

$$(2)\ ZFC + A \models_\Omega CH.$$

If one changes perspective from $\Sigma_1^2$ to $H(\omega_2)$ there is a companion result for $\neg CH$, assuming the Strong $\Omega$-conjecture.

THEOREM 68 (Woodin 10b). *Assume that there is a proper class of Woodin cardinals and that the Strong $\Omega$-conjecture holds. (1) There is an axiom $A$ such that*

$$(1) ZFC + A\ is\ \Omega - satisfiable$$

$$(2) ZFC + A\ is\ \Omega - complete\ for\ the\ structure\ H(\omega_2).$$

*Any such axiom $A$ has the feature that*

$$ZFC + A \models_\Omega H(\omega_2) \models \neg CH.$$

Thus, assuming that there is a proper class of Woodin cardinals and that the Strong $\Omega$ Conjecture holds, we have an $\Omega$-complete picture of $H(\omega_2)$ and within this picture CH fails. For the precedent two theorems, we have an apparent bifurcation at the level of CH. In fact, if our point of view is $H(\omega_2)$, every $\Omega$-complete theory states that CH fails. If our point of view is $V_{\omega+1}$ (second-order arithmetic), by assuming CH we have $\Omega$-completeness for $\Sigma_1^2$ statements. However, there is a limitative result established by Hugh Woodin. In fact, if there is a proper class of Woodin cardinals and the Strong $\Omega$ Conjecture holds then one cannot have an $\Omega$-complete picture of third-order arithmetic.

THEOREM 69. *(Woodin) Assume that there is a proper class of Woodin cardinals and that the Strong $\Omega$ Conjecture holds. Then there is no recursive theory $A$ such that $ZFC + A$ is $\Omega$-complete for $\Sigma^2_3$ statements.*

It is an open question if there is a recursively enumerable theory that is $\Omega$-complete for $\Sigma^2_2$ statements. It is established that CH will not be sufficient:

THEOREM 70. *(Jensen, Shelah) $ZFC + CH$ is not $\Omega$-complete for $\Sigma^2_2$ statements.*

At the level of third-order arithmetic, we might not have a unique $\Omega$-complete picture, but if there is one such $\Omega$-complete picture then there must be another, incompatible $\Omega$-complete picture. At this point, I want to conclude this section devoted to $\Omega$-logic by quoting Koellner [Koellner 09] about the status of $\Omega$-conjecture:

> There is evidence that the $\Omega$-conjecture holds. There are two key points. First, many of the meta-mathematical consequences of the $\Omega$-conjecture follow from the non-trivial $\Omega$-satisfiability of the $\Omega$-conjecture. This later statement is a $\Sigma_2$ statement and there are no known examples of $\Sigma_2$-statements that are provably absolute and not settled by large cardinals. So it is reasonable to expect this statement to be settled by large cardinal axioms. Moreover, it seems unlikely that the $\Omega$ Conjecture be false while its non-trivial $\Omega$-satisfiability be true. Second, recent results have shown that if inner model can reach one supercompact cardinal then it can reach all the traditional large cardinal axioms and, moreover, the $\Omega$ Conjecture holds in all these models. This provides evidence that no traditional large cardinal can refute the $\Omega$-satisfiability of the $\Omega$-conjecture and (by the first point) this is evidence that the $\Omega$ conjecture is true. Thus there is evidence that the above form of bifurcation will not occur. In fact, there is evidence that

> the Strong $\Omega$ Conjecture holds and thus there is evidence that bifurcation cannot even occur at the level of third-order arithmetic [Koellner 09]

Now I want to introduce briefly Woodin's Maximum. In order to highlight the importance of this axiom, i am going to quote Bagaria's words:

> Woodin has isolated an axiom we may call Woodin's Maximum (WM), that brings together the power of large cardinals and the Bounded Forcing Axioms. WM has the astonishing property that decides in $\Omega$-logic the whole theory of $H(\omega_2)$. WM asserts the following: (1) There exists a proper class of Woodin cardinals, and (2) A strong form of BMM holds in every inner model M of ZFC that contains $H(\omega_2)$ and thinks that there is a proper class of Woodin cardinals. [Bagaria 04]

Now we can introduce briefly Woodin's Maximum (WM). Recall that the dual of the closed unbounded filter is the ideal of non-stationary sets, the non-stationary ideal $I_{NS}$. $I_{NS}$ is $\kappa$-complete and it is closed under diagonal unions. At this point, we need to introduce briefly the forcing notion $\mathbb{P}_{max}$. I focus on $\mathbb{P}_{max}$ because if $NS_{\omega_1}$ is saturated then every member of $H(\omega_2)$ is in the iteration of a countable model of a fragment of ZFC. Since these countable models are elements of $L(\mathbb{R})$, their iterations induce a partial order in $L(\mathbb{P})$. This partial order, $\mathbb{P}_{max}$, produces an extension of $L(\mathbb{R})$ where $H(\omega_2)$ is the direct limit of the structures $H(\omega_2)$ of models satisfying every forceable theory. The structure $H(\omega_2)$ in the $\mathbb{P}_{max}$ extension of $L(\mathbb{R})$ by assuming $AD^{L(\mathbb{R})}$ satisfies every $\Pi_2$ sentence. $\mathbb{P}_{max}$ is based on iterated generic elementary embeddings. Suppose that I is a normal, uniform, proper ideal on $\omega_1$. Thus, I is a proper subset of $P(\omega_1)$ containing all the countable subsets, and such that whenever A is an I positive set (i.e, in $P(\omega_1/I)$) and $f : A \longrightarrow \omega_1$ is a regressive function, f is constant on a I positive set. Then forcing with the Boolean algebra $P(\omega_1/I$ produces a V-normal ultrafilter on $\omega_1^V$. So, we generate the ultrapower construction $Ult(V, U)$. The corresponding elementary embedding $j : V \longrightarrow Ult(V, U)$ has critical point

$\omega_1^V$, and since I is normal for each $A \in P(\omega_1)^V$, $A \in U$ if and only if $\omega_1^V \in j(A)$. Now take $ZFC^*$ to be ZFC - Power set - Replacement + $P(P(\omega_1))$ exists.

DEFINITION 83. *Let M be a model of $ZFC^*$ and let I be an ideal on $\omega_1^M$ which is normal in M. Let $\gamma$ be an ordinal less than or equal to $\omega_1$. An iteration of (M, I) of length $\gamma$ consists of models $M_\alpha (\alpha \leq \gamma)$, sets $G_\alpha (\alpha < \gamma)$ and a commuting family of elementary embeddings $j_{\alpha\beta} : M_\alpha \longrightarrow M_\beta (\alpha \leq \beta \leq \gamma)$ such that:*
*$M_0 = M$.*
*Each $G_\alpha$ is an $M_\alpha$- generic filter for $(P(\omega_1/j^{M_\alpha}_{0\alpha(I)}))$.*
*Each $j_{\alpha\alpha}$ is the identity map.*
*Each $j_{\alpha(\alpha+1)}$ is the utrapower embedding induced by $G_\alpha$.*
*For each limit ordinal $\beta \leq \gamma$, $M_\beta$ is the direct limit of the system $\{M_\alpha, j_{\alpha\delta} : \alpha \leq \delta < \beta\}$, $j_{\alpha\beta}$ is the induced embedding.*

We can prove that there is a unique iteration and each model in the iteration is well-founded. The forcing construction $\mathbb{P}_{max}$ was invented by Hugh Woodin in 1990. An important result of this construction is the $\Pi_2$-maximality of the $\mathbb{P}_{max}$ extension which is stated in the following theorem:

THEOREM 71 (Woodin 10b). *Suppose that there exists a proper class of Woodin cardinals, $A \subseteq \mathbb{R}$, $A \in L(\mathbb{R})$, $\phi$ is $\Pi_2$ in the extended language containing two additional unary predicates, and in some set forcing extension*

$$(H(\omega_2), \in, I_{NS_{\omega_1}}, A^*) \models \phi$$

*(where $A^*$ is the reinterpretation of A in this extension). Then*

$$L(\mathbb{R})^{\mathbb{P}_{max}} \models (H(\omega_2), \in, I_{NS_{\omega_1}}, A) \models \phi.$$

Forcing with $\mathbb{P}_{max}$ does not add reals, so there is no need to reinterpret A in the last line of the theorem. The theorem says that any such $\Pi_2$-statement that we can force in any extension must hold in the $\mathbb{P}_{max}$ extension of $L(\mathbb{R})$, so $H(\omega_2)$ of $L(\mathbb{R})^{\mathbb{P}_{max}}$ is maximal, or complete, in a certain sense, among other things. We define the theory $T_0$:

Definition 84. *$T_0$ is $ZFC - Replacement - Powerset$ plus $P(P(\omega_1))$ exists plus the scheme that definable trees of height $\omega_1$ have maximal branches.*

We define $\mathbb{P}_{max}$:

Definition 85. *The partial order $\mathbb{P}_{max}$ is the set of pairs ((M,I, a)) such that*
*(1) M is a countable transitive model of $T_0 + MA_{\aleph_1}$.*
*(2) (M,I) is an iterable pair (all iterations, ultrapower iterations, are well-founded).*
*(3) $a \in P(\omega_1)^M$ and $\exists x \in P(\omega)^M$ such that $\omega_1^{L[x,a]} = \omega_1^M$.*

The order on $\mathbb{P}_{max}$ is as follows: $((M, I), a) < ((N, J)b)$ if $N \in H(\omega_1)^M$ and there exists an iteration $j : (N, J) \longrightarrow (N^*, J^*)$ such that:
$j(b) = a$.
$j, N^* \in M$.
$I \cap N^* = J^*$.
We assert that a pair (M, I) is a $\mathbb{P}_{max}$- precondition if there exists an a such that $(M, I), a)$ is in $\mathbb{P}_{max}$. At this point we state the following fundamental definition:

Definition 86. *Let A be a set of reals. If M is a transitive model of $ZFC^*$ and I is an ideal on $\omega_1^M$ which is normal and precipitous (all ultrapowers are well-founded) in M, then the pair (M, I) is A-iterable if*
*(M,I) is iterable,*
*$A \cap M \in M$,*
*$j(A \cap M) = A \cap M^*$ whenever $j : (M, I) \longrightarrow (M^*, I^*)$ is an iteration of (M,I).*

We reach the full effect of $\mathbb{P}_{max}$ over a given model such as $L(\mathbb{R})$ because it has been proved that for each $A \subseteq \mathbb{R}$ in the model there exists a $\mathbb{P}_{max}$ precondition (M,I) such that:
(M, I) is A-iterable.
$(H(\omega_1)^M, A \cap M) \prec (H(\omega_1), A)$.
With the existence of A-iterable conditions (for all sets A in $L(\mathbb{R})$) we can see that $\mathbb{P}_{max}$ is an extension of $L(\mathbb{R})$. With $\delta^1_2$ we point out to the supremum of the lenghts of the $\Delta^1_2$-definable prewellorderings of the reals. Now we can state a fundamental theorem:

Theorem 72. *(Woodin) [Woodin 10b] (ZF). Assume that for every $A \subseteq \mathbb{R}$ there exists a $\mathbb{P}_{max}$ condition $((M, I)a)$ such that $(M, I)$ is a A-iterable and*

$$(H(\omega_1)^M, A \cap M) \prec (H(\omega_1), A)$$

*Suppose that $G \subseteq \mathbb{P}_{max}$ is a V-generic filter. Then in V[G] the following hold:*
*$P(\omega_1) = P(\omega_1)_G$.*
*$NS_{\omega_1} = I_G$.*
*$\delta^1_2 = \omega_2$.*
*$NS_{\omega_1}$ is saturated.*

So, in $\mathbb{P}_{max}$, we have the failure of CH. It has been proved that the $\mathbb{P}_{max}$ extension of $L(\mathbb{R})$ satisfies the axiom of choice (AC). In particular, it satisfies an equivalent form of the axiom of choice called $\psi_{AC}$. Now we can state a fundamental theorem:

Theorem 73. *(Woodin) [Woodin 10b] Suppose that $\delta$ is a limit of Woodin cardinals, and $\kappa > \delta$ is measurable. Let $A$ be a set of reals in $L(\mathbb{R})$. Suppose that $\phi$ is a $\Pi_2$ sentence in the expanded language with two additional unary predicates, and that $P$ is a partial order in $V_\delta$ forcing that $\phi$ holds in the structure $(H(\omega_2), \in, A(G))$. Then $\phi$ holds in the structure $(H(\omega_2), \in A)$ in the $\mathbb{P}_{max}$ extension of $L(\mathbb{R})$.*

Now we can define Woodin Maximum:

DEFINITION 87. *(WM) The axiom of determinacy (AD) holds in $L(\mathbb{R})$ and $L(P(\omega_1))$ is a $\mathbb{P}_{max}$-extension of $L(\mathbb{R})$, namely there is some $G$ which is $\mathbb{P}$-generic over $L(\mathbb{R})$ and*

$$L(P(\omega_1)) = L(\mathbb{R})[G].$$

If we assume Woodin Maximum, we can prove that a second axiom called $Woodin\ Maximum^{**}$ holds in $L(P(\omega_1))$. Recently, David Asperó and Ralf Shindler [Aspero 12] have isolated an axiom which implies Woodin's Maximum. This axiom is called $A{-}Bounded\ Martin's\ Maximum^{++}$ where A points out to a universally Baire set.

DEFINITION 88. *Given a universally Baire set $A \subset \mathbb{R}$, the axiom ($A - BMM^{++}$) says that for every stationary set preserving poset $\mathbb{P}$ and every $\mathbb{P}$-generic filter $G$ over $V$, we have that:*

$$(H^V_{\omega_2}, \in, (I_{NS_{\omega_1}})^V, A) \prec_{\Sigma_1} (H^{V[G]}_{\omega_2}, \in, (I_{NS_{\omega_1}})^{V[G]}, A^*)$$

*where $A^*$ is the $V[G]$ version of $A$.*

At this point we can connect $\mathbb{P}_{max}$, $\Omega$ logic and Woodin Maximum. A sentence $\phi$ is $\Omega_{ZFC}$ consistent if $ZFC \not\vdash_\Omega \neg\phi$. So, we can state:

THEOREM 74 (Woodin 10b). *Suppose that there is a proper class of Woodin cardinals and that there is an inaccessible cardinal which is a limit of Woodin cardinals. Then the theory*

$$ZFC + \ Woodin\ Maximum$$

*is $\Omega_{ZFC}$ consistent.*

Then:

THEOREM 75 (Woodin 10b). *If there is a proper class of Woodin cardinals, then for every set of reals $A$ in $L(\mathbb{R})$, every $\Omega_{ZFC}$ consistent $\Pi_2$ sentence for $(H(\omega_2), NS_{\omega_1}, A, \in)$ holds in the $\mathbb{P}_{max}$ extension of $L(\mathbb{R})$.*

We conclude with the following decidability result concerning $\Omega$ logic.

THEOREM 76. *(Woodin) [Woodin 10b] Suppose that there is a proper class of Woodin cardinals. Then for every sentence $\phi$, either*

$$ZFC + \textit{ Woodin Maximum } \vdash_{\Omega} L(P(\omega_1)) \models \phi$$

*or*

$$ZFC + \textit{ Woodin Maximum } \vdash_{\Omega} L(P(\omega_1)) \not\models \phi.$$

If we could prove the $\Omega$-conjecture, we would have a complete theory respect to $\models_{\Omega}$. In fact, thanks to Woodin's Maximum, $\models_{\Omega}$ would be a natural notion of logical consequence to adopt in order to decide every problem in $H(\omega_2)$. We shall now compare the result of completeness of Turing for transfinite progressions, that we have seen in section 2.3, and Woodin's result for $\Omega$-logic. Firstly, both Turing's and Woodin's approaches share a weak similarity. In fact, both approaches imply a maximality principle. In transfinite progressions (that we have seen in section 2.3), we take all theories until $\omega + 1$ and in $\Omega$-logic we take all forcing extensions. To compare these two approaches by abstracting from their particular formulation and by accomplishing a sort of phenomenology, we have to evaluate their success in deciding undecidable mathematical statements. Surely, in the case of Turing's completeness theorem, we attempt to prove $\Pi^0_1$ statements or, in the case of Feferman $\Pi^0_2$ statements while in $\Omega$-logic we attempt to have a complete theory of the structure $H(\omega_2)$ and decide statements such as the Continuum Hypothesis which has the complexity of $\Sigma^2_1$ statement. The success of $\Omega$-logic is based on the fact that the $\Omega$-conjecture holds. Thus, in order to compare Turing's approach and Woodin's approach, we must introduce and formulate Turing's Conjecture. This Conjecture may be formulated in the following way:

Definition 89. *(Turing's Conjecture) There exists a unique ordinal notation in order to index theories univocally.*

As we have seen in section 2.3, this is the main problem for transfinite progressions. Unlike proved theorems that are atemporal truths, Conjectures are unproved mathematical statements which do not possess the criteria of atemporality. In mathematics a proved, atemporal theorem cannot be dismissed, while a Conjecture may be disproved. We might assert that we believe that a specific Conjecture is true and it is probable that it is true, but we cannot assert that is an atemporal truth (Recall that we have examined the notion of atemporal truth in section 1.3 relating this notion to Intutionism). So, now we can compare Turing's Conjecture and the $\Omega$ Conjecture by asking ourselves which Conjecture is more probable to be true and which Conjecture can be believed to be true with more certainty. Church's thesis and the consistency of ZFC are other two conjectures very probable to be true. In fact, it is almost impossible to think of an informal algorithm which cannot be formalized as a partial recursive function and thanks to relative consistency proofs, it is very improbable that a contradiction will be discovered within ZFC. So, we can believe in Church's thesis and in the consistency of ZFC with the possible, highest degree of certainty. On the contrary, Turing's Conjecture, on which is based Turing's completeness theorem, is less probable to be true. We can believe in Turing's Conjecture with a lower degree of certainty. In fact establishing that we have a unique ordinal notation is a mathematical problem that has a greater computational complexity than the problem of establishing if a truth is a theorem (theoremhood). So, now we can ask ourselves what is the status of the $\Omega$ conjecture. Firstly, the $\Omega$-satisfiability of the $\Omega$-conjecture is a $\Sigma_2$ statement and there are no known examples of $\Sigma_2$-statements that are provably absolute and not settled by large cardinals. So it is reasonable to expect this statement to be settled by large cardinal axioms. Furthermore, it seems unlikely that the $\Omega$ Conjecture be false while its non-trivial $\Omega$-satisfiability be true. Secondly, if an inner model of a supercompact cardinal (the Ultimate

L) will be constructed, then this model can reach all the traditional large cardinal axioms and, moreover, the $\Omega$ Conjecture holds in all these models. So, there is a strong evidence that the $\Omega$-conjecture is true and it reasonable that the $\Omega$-conjecture will be proved to be true, becoming a theorem and so, an atemporal truth. Thus, there is a strong evidence in favor of the $\Omega$-Conjecture. We might add that if there is a proper class of Woodin cardinals and that for every $A \subseteq \mathbb{R}$, if A is OD then A is universally Baire then $HOD \models \Omega\ conjecture$. So we may assert that the satisfaction of the $\Omega$ *conjecture* rests on the satisfaction of other conjectures such as the HOD conjecture and the Strong $(\omega_1 + 1)$ Iteration Hypothesis or the Strong Unique Branch Hypothesis. We can conclude this section by saying that the $\Omega$ conjecture is more probable to be true than Turing's Conjecture. We can believe in the $\Omega$-Conjecture with an higher degree of certainty than Turing's Conjecture degree of certainty. Now we may compare the $\Omega$-Conjecture with Church's thesis and the consistency of ZFC. In fact, we can ask ourselves if it is possible for all these Conjectures becoming proved, atemporal truths, or simply mathematical theorems. We can say that Church's thesis is impossible to become a theorem. In fact, we should be able to collect all possible informal algorithms and then formalized them as partial recursive functions. It is impossible to collect all possible algorithms. Also it is impossible that we will have a direct proof of the consistency of ZFC, but we can have only relative consistency proofs. In this case, we have a theorem, namely Gödel's second incompleteness theorem, that makes impossible to have a direct proof of the consistency of ZFC. So, while even if it is almost impossible, it might be possible to collect all algorithms and prove Church's thesis, to prove directly the consistency of ZFC is impossible because of another atemporal truth, namely Gödel's second incompleteness theorem. On the contrary, it is very probable that the $\Omega$ Conjecture will become an atemporal, proved truth as all other theorems of mathematics. In fact, it is very probable that a large cardinal axiom will settle the $\Omega$ conjecture or that the Ultimate L will be constructed implying the truth of the $\Omega$ Conjecture.

Therefore, we have seen that at the beginning of the last century, the fact of having complete theories for first, second, third-order arithmetic was a dream. Even if Gödel sentences doom theories to be incomplete, we can say that all problems in second-order arithmetic are settled and the continuum hypothesis may have a noumenal solution if the ultimate L conjecture is true and a phenomenal solution, thanks to BPFA, if the ultimate L conjecture is false. So, the dream of proving undecidable truths, is not anymore only a dream, but it has become an important result of mathematics. I want to conclude this long section with the words of Hugh Woodin who explains that we are only at the beginning for the study of the infinite:

> What about the general continuum problem; what about $H(\omega_3), H(\omega_4), H(\omega_{\omega+2010})$, etc? The view that progress towards resolving the Continuum Hypothesis must come with progress on resolving all instances of the generailised Continuum Hypothesis seems too strong. The understanding of $H(\omega)$ did not come in concert with an understanding of $H(\omega_1)$, and the understanding of $H(\omega_1)$ failed to resolve even the basic mysteries of $H(\omega_2)$. The universe of sets is a large place. **We have just barely begun to understand it**. [Woodin 012]

CHAPTER 2

# Reflection

**0.1. Preliminaries to this chapter.** In this chapter, we will discuss the Reflection Principle and higher-order linguistic principles. I will conclude this chapter by examining Welch's Global Reflection principle, a kind of reflection, which implies embeddings and the use of proper classes. This chapter is important because I will highlight that the phenomenon of reflection characterises essentially the universe of sets. In fact, Reflection principles can be used as intrinsic philosophical justification for new axioms in set theory. On the contrary, extrinsic philosophical justifications are based on the success of accepting new axioms. Intrinsic justification are characterised by a conceptual analysis of the notion of set. If we imagine a counter-mathematical possible world where we have two axioms of set theory and we have to choose one of them, we should prefer intrinsic justification because this kind of justification involves the concept of set itself. The iterative conception of set is a kind of intrinsic justification. In fact, the operation *set of* (the power set operation) iterated characterizes essentially the universe of sets and characterizes the first large cardinals in the hierarchy of large cardinals (inaccessible and Mahlo) by taking fixed points of aleph function. Also the Reflection Principle is an essential feature of the universe of sets and it is not simply an epiphenomenon. In fact, in section 1 I will discuss Levy-Montague theorem about reflection principle. The axioms ZF prove that the Reflection Principle holds in the universe of sets. Furthermore, if we assume the Reflection Principle as an axiom together with the remaining axioms of ZFC, we can derive the axiom of infinity and the axiom of replacement. We have to notice that the large-cardinal axioms can be seen as generalizations of the axiom of infinity plus the axiom of replacement. Thus, the Reflection Principle characterises essentially ZFC universe and beyond. So, if we can interpret large-cardinal axioms as principles of reflection, they will be intrinsically justified. ZFC axioms determine a universe and it is like a partition of the Human thought. In fact, only sets, which we can construct from the axioms, belong to this universe that i have called ZFC universe. It is a partition because no all sets belong to this universe. For example, the Russell's class and ill-founded sets are excluded, respectively by the separation axiom and by well-founded axiom, from the ontology of ZFC universe. Axioms determine a partition of Human thought and sets, that are outside this partition, do not belong to the universe shaped by the axioms (in our case ZFC axioms). If we eliminate the well-founded axiom and we add the anti-foundation axiom, we have a new partition of Human thought and we have a new universe shaped by this new axiom, where ill-founded sets are admitted. As we have seen in the precedent chapter, $\Delta^1_1$-determinacy is proved by the axioms of ZFC (Martin). So, Borel determinacy is an essential feature of ZFC universe. From the axioms of ZFC, evident truths about sets, it is possible to derive Borel determinacy. Thus, Borel determinacy is a truth that characterises sets within ZFC universe. Surely, the Reflection Principle characterises ZFC universe in a stronger sense since by assuming this principle together with the other remaining ZFC axioms is possible to derive the axiom of infinity and the axiom of replacement. We should say that the reflection principle is an essential feature of ZFC universe and it can assume the status of an evident truth regarding sets. However, even if Borel determinacy characterises ZFC universe in a weaker sense (in comparison with the Reflection Principle), it is still a derivable truth from evident truths (axioms) regarding sets, and so it is a truth concerning sets. As we will see in the next chapter, also $\Sigma_1$-structural reflection is provable from ZFC axioms. So, as in the case of Borel determinacy, we can say that $\Sigma_1$-structural reflection is another truth that is an essential feature of ZFC universe.

In order to justify intrinsically Projective Determinacy, we must firstly justify the axiom that asserts the existence of infinitely-many Woodin cardinals. In fact, this axiom implies Projective Determinacy. So, we must interpret Woodin cardinals as principles of reflection. At this point, I want to stress the following aspect. If we assume the existence of the universe of the totality of mathematical abstract concepts, ZFC axioms partition this universe and create a sub-universe. ZFC axioms are evident truths that imply the existence of simple sets. The ontology of ZFC universe is determined by sets whose existence is implied by the axioms or sets whose existence is derivable from these axioms. Surely, we can extend ZFC ontology by introducing large-cardinal axioms. Large cardinals may exist within ZFC universe but their existence cannot be proved within ZFC. Contradictions limit ZFC ontology. For instance, the Fregean full axiom of comprehension cannot be accepted as a evident truth regarding sets because of Russell's Paradox. In fact, we must introduce the axiom of separation in order to avoid Russell's paradox. Russell's class does not belong to ZFC ontology. It is the same also for the class of all ordinals or the class of all cardinals. In fact, we must introduce in ZFC, the distinction between sets and proper classes. Proper classes do not belong to the ZFC ontology. By introducing large-cardinal axioms we extend the ZFC ontology since the existence of these large cardinals cannot be proved within ZFC. So, we create new universes ($ZFC\ +\ Large\ cardinal\ axiom$) with a different ontology. However, contradictions determine also if these new and different universes may exist. Ackermann [Ackermann 56] declared that the notion of set (Menge) is not a well-defined notion and also the distinction between sets and classes is not well-defined. On the contrary, I believe that the notion of set is well defined. In fact, I believe that sets are mathematical objects that belong to ZFC ontology that determines which set exist. Existent sets are those whose existence is implied by ZFC axioms or it is derivable from ZFC axioms. The same reasoning is valid for different universes when we assume the existence of a large cardinal. Only contradictions limit ZFC ontology or the existence of different universes such as $ZFC +\ Reinhardt\ cardinal\ exists$ universe. Classes are mathematical objects that cause contradictions and so they do not belong to $ZFC$ ontology. The act of creating new and different universes from $ZFC$ universe by introducing large-cardinal axioms is essentially determined by Godel's second incompleteness theorem. In fact, by this theorem, we cannot prove the existence of large cardinals within ZFC because they would be metamathematical models of ZFC axioms and they would be direct proof of the consistency of ZFC axioms. In section 2 I will introduce the concept of indescribability. In this section, we will see how higher-order reflection can be used to interpret the first large cardinal axioms. We will see that weakly compact cardinals are $\Pi^1_1$ indescribable. In section 3 I will discuss Koellner's limitative result concerning linguistic reflection principles. In this section, I will introduce briefly combinatorial set theory in order to understand $\eta_\omega$-Erdös cardinal barrier. In fact, we will see that by Koellner's theorem, linguistic reflection principle cannot overcome this barrier. I must highlight that the axiom asserting the existence of $\eta_\omega$-Erdös cardinal is consistent with $V = L$. Therefore, to interpret the axiom asserting the existence of measurable Woodin cardinals as a principle of reflection, we must conceive a different kind of principles of reflection. In section 4 I will introduce Welch's Global Reflection principle. Welch's approach is based on an embedding of a substructure into a superstructure. This aspect is shared also by Bagaria's structural reflection as we will see in the next chapter. Welch by assuming Global Reflection Principle is able to produce a proper class of measurable Woodin cardinals, fundamental for $\Omega$-logic. Thus, Welch is able to justify intrinsically measurable Woodin cardinals. However, we must say that Welch's principle implies the use of proper classes. In this section, I will express my philosophical doubts about the use of proper classes within mathematical discourse.

## 1. The Reflection Principle

Levy and Montague proved the Reflection Principle for ZF. This Principle is similar to the Löwenheim-Skolem theorem. While the latter theorem proves that every model has an elementary submodel, the Reflection Principle asserts that for any finite number of formulas, a set $V_\kappa$ is like an elementary submodel of the universe $V$ with respect to the given formulas. The Principle is proved without the axiom of choice.

THEOREM 77. *(Levy-Montague) [Jech 06] (The Reflection Principle) Let* $\phi(x_1, ........x_n)$ *be a formula. For each* $M_0$ *there exists a set* $M_0 \subset M$ *such that*

$$\phi^M(x_1, ....x_n) \leftrightarrow \phi(x_1, ....x_n)$$

*for every* $x_0, ....., x_n \in M$*. (We say that M reflects* $\phi$*). Furthermore, M is transitive and reflects* $\phi$*. Moreover there is a limit ordinal* $\alpha$ *such that* $M_0 \subset V_\alpha$ *and* $V_\alpha$ *reflects* $\phi$*.*

The proof works for any finite number of formulas and not just one. As a consequence of the Reflection Principle and of Gödel's second incompleteness theorem, it follows that ZF is not finitely axiomatizable. Any finite number of theorems of ZF has a model by the Reflection Principle, while the existence of a model of ZF is not provable by Gödel's theorem. Also no consistent extension of ZF is finitely axiomatizable. The axiom of infinity and the axiom of replacement are provable from the Reflection Principle and the other remaining axioms. Therefore, the Reflection Principle is not an epiphenomenon of set theory, but it is an essential property of the universe of set theory (ZF). It features directly the concept of set itself. So, Reflection can be used as an intrinsic justification based on a conceptual analysis of large cardinals. If Woodin cardinals can be interpreted as principles of reflection, then these large cardinal numbers are intrinsically justified. At the same time Woodin cardinals are also extrinsically justified since as we have seen before, by assuming these large cardinals, we can obtain important results in set theory. I believe that linguistic reflection (indescribability, Tait-Koellner) and the iterative conception of set (the iteration of the operation of set of and fixed points of aleph function) are fundamental to justify philosophically and intrinsically large cardinals. However, for linguistic reflection we have two barriers, namely $\Pi^1_n$-indescribability and $\eta_\omega$-Erdös cardinal that we cannot overcome.

In fact, with linguistic reflection we are forced to stay within Gödel's constructible universe because linguistic reflection is able to produce only large cardinals consistent with L. We can adopt a Hilbert's distinction between safe mathematical reasoning and less safe mathematical reasoning. Hilbert asserted that finite arithmetic was a safe mathematical reasoning while ideal mathematics (set theory) was unsafe. In my opinion, all notions which are within Gödel's constructible universe are safe because they can be intrinsically justified while all notions which are beyond Gödel's constructible universe are less safe. In fact, linguistic reflection and the iterative conception of set are my preferred methods to justify directly large cardinals. However, we can use different kinds of reflection (Welch's Global reflection and Bagaria's structural reflection) and we can overcome $\eta_\omega$- Erdös cardinal barrier and we can interpret Woodin cardinals as principles of reflection. However, I think that these kinds of reflection are less powerful methods from a philosophical point of view than direct linguistic reflection or the iterative conception of set in order to justify intrinsically large cardinals. We shall now examine linguistic reflection. For $\kappa$ a regular cardinal the following are equivalent:

$$(1) V_\kappa \models ZFC$$

$$(2) V_\kappa \prec_{\Sigma_1} V.$$

$V_\kappa$ reflects all $\Sigma_1$ sentences with parameters, which means that for every $a_1, .......a_\kappa \in V_\kappa$ and every $\Sigma_1$-formula $\phi(x_1, ......., x_\kappa)$,

$$V_\kappa \models \phi(a_1, ......a_n) \;\; iff \;\; \phi(a_1, .....a_\kappa)$$

A regular cardinal satisfying (1) or (2) is inaccessible. As we have seen before, by considering $\Sigma_2$-sentences, we obtain the notion of reflecting cardinal. More generally, for every $n$

one may consider the existence of a regular cardinal $\kappa$ such that

$$V_\kappa \prec_n V$$

Such cardinal is called $n$-reflecting cardinal. A strengthening of the notion of inaccessible cardinal, is the notion of Mahlo cardinal. $\kappa$ is a Mahlo cardinal if it is regular and the set of inaccessible cardinals below $\kappa$ is stationary, namely every closed unbounded subset of $\kappa$ contains an inaccessible cardinal. A Mahlo cardinal $\kappa$ is inaccessible and in $V_\kappa$ there is a stationary class of $\Sigma_\omega$-reflecting cardinals, namely $\Sigma_n$-reflecting for every $n$. $\kappa$ is Mahlo iff $\kappa$ is regular, $V_\kappa \models ZFC$ and the set of regular cardinals $\lambda < \kappa$ such that $V_\lambda \models ZFC$ is stationary. Therefore, if we accept inaccessible and reflecting cardinals, we have to accept also Mahlo cardinals because they are the next natural step in the process of extending the linguistic reflection properties of the universe of all sets.

## 2. Indescribability

Before speaking about indescribability, I have to introduce a little combinatorial set theory to understand the notions of weakly compact cardinals and Erdös cardinals. This part about combinatorial set theory is also important in order to understand $\eta_\omega$-Erdös cardinal barrier fundamental for Koellner's limitative result about linguistic reflection.
A partition of a set S is a pairwise disjoint family $P = (X_i : i \in I)$ such that $\bigcup_{i \in I} X_i = S$. With the partition $P$ we can associate a function $F : S \longrightarrow I$ such that $F(x) = F(y)$ if and only if x and y are in the same $X \in P$. $[A]^n := \{X \subset A : |X| = n\}$ is the set of all subsets of A that have exactly $n$ elements. If $\{X_i : i \in I\}$ is a partition of $[A]^n$, then a set $H \subset A$ is *homogenous* for the partition if for some $i$, $[H]^n$ is included in $X_i$, namely all n-element subsets of $H$ are in the same piece of partition. We can start with the following theorem:

THEOREM 78. *(Ramsey)* $\aleph_0 \longrightarrow (\aleph_0)^n_\kappa \quad (n, \kappa \in \omega)$.

So, for the infinite Ramsey theorem, a partition of an infinite countable set gives as a result an infinite *homogenous* set. Now we have to introduce two important lemmas:

LEMMA 12. *For all* $\kappa$, $2^\kappa \not\longrightarrow (\omega)^2_\kappa$

LEMMA 13. *For every* $\kappa$, $2^\kappa \not\longrightarrow (\kappa^+)^2_2$

Therefore we have that $\aleph_1 \not\longrightarrow (\aleph_1)^2_2$. So, the natural generalization of Ramsey theorem is false. Now we can define weakly compact cardinals:

DEFINITION 90. *A cardinal* $\kappa$ *is weakly compact if it is uncountable and satisfies the partition property* $\kappa \longrightarrow (\kappa)^2_2$.

We have the following Lemma:

LEMMA 14. *Every weakly compact cardinal is inaccessible.*

We shall now introduce the concept of a tree, which is fundamental, in order to characterise weakly compact cardinals.

DEFINITION 91. *A tree is a partially ordered set* $(T, <)$ *with the property that for each* $x \in T$*, the set* $\{y : y < x\}$ *of all predecessors of* $x$ *is well-ordered by* $<$*. The* $\alpha$*-level of* $T$ *consists of all* $x \in T$ *such that* $\{y : y < x\}$ *has order-type* $\alpha$*. The height of* $T$ *is the least* $\alpha$ *such that the* $\alpha$*-level of* $T$ *is empty. The* $\alpha$*-level is the height of the well-founded relation* $<$*. A branch in* $T$ *is a maximal linearly ordered subset of* $T$*.*

Now we can define two kinds of trees:

DEFINITION 92. *A tree is a Suslin tree if the height of* $T$ *is* $\omega_1$*, every branch in* $T$ *is at most countable and every antichain in* $T$ *is at most countable.*

DEFINITION 93. *An Aronszajn tree is a tree of height* $\omega_1$ *all of whose levels are at most countable and which has has no uncountable branches.*

The following is a fundamental property shared by weakly compact cardinals:

Definition 94. *(The tree property) A regular uncountable cardinal $\kappa$ has the tree property if every tree of height $\kappa$ whose levels have cardinality $< \kappa$ has a branch of cardinality $\kappa$.*

We have the following theorem:

Theorem 79 (Jech 06). *If $\kappa$ is weakly compact, then $\kappa$ has the tree property and if $\kappa$ is inaccessible and has the tree property, then $\kappa$ is weakly compact.*

Now we can introduce Ramsey cardinals:

Definition 95. *A cardinal $\kappa$ is a Ramsey cardinals if $\kappa \longrightarrow (\kappa)_2^{<\omega}$*

Clearly, every Ramsey cardinal is weakly compact. At this point, we can introduce Erdös cardinals:

Definition 96. *For every limit ordinal $\alpha$, the Erdös cardinal $\eta_\alpha$ is the least $\kappa$ such that $\kappa \longrightarrow (\alpha)_2^{<\omega}$.*

Notice that $\kappa$ is a Ramsey cardinal if and only if $\kappa = \eta_\kappa$. Now we can introduce two fundamental theorems concerning Erdös cardinals:

Theorem 80. *If $\eta_\omega$ exists then there exists a weakly compact cardinal below $\eta_\omega$.*

The next theorem shows that $\eta_\omega$ (Erdös cardinal) is consistent with $V = L$.

Theorem 81 (Jech 06). *If $\kappa \longrightarrow (\omega)^{<\omega}$ then $L \models \eta_\omega$ (it exists in L.)*

In fact we have the following theorem:

Theorem 82. *If there is a cardinal $\kappa$ such that $\kappa \longrightarrow (\omega_1)_2^{<\omega}$ then $0^\sharp$ exists. Therefore $\eta_{\omega_1}$ is consistent with $V \neq L$.*

As we will see later, the $\eta_\omega$-Erdös cardinal constitutes a barrier for linguistic reflection principles. In fact, linguistic reflection principles are either below this barrier or are inconsistent by Koellner's [Koellner 091] dichotomy theorem.

If we increase the order of the variables of the sentences reflected, we obtain the notion of indescribable cardinals. The notion of indescribability implies a kind of higher-order reflection. We can start with the following definition:

DEFINITION 97. *A cardinal $\kappa$ is $\Pi^n_m$-indescribable if whenever $R \subset V_\kappa$ and $\sigma$ is a $\Pi^n_m$ sentence such that $(V_\kappa, \in, R) \models \sigma$, then for some $\alpha < \kappa$, $(V_\alpha, \in, R \cap V_\alpha) \models \sigma$.*

The following theorem asserts that to be indescribable, a cardinal must be inaccessible:

THEOREM 83. *If $\kappa$ is not inaccessible, then it is describable by a first-order sentence, i.e., $\Pi^0_m$-describable for some $m$.*

The following is a fundamental theorem which links the notion of indescribability with the notion of weak compactness:

THEOREM 84. *(Hanf, Scott) [Jech 06] A cardinal $\kappa$ is $\Pi^1_1$-indescribable if and only if it is weakly compact.*

We have also the following lemma:

LEMMA 15. *Every weakly compact cardinal $\kappa$ is Mahlo , and the set of Mahlo cardinals below $\kappa$ is stationary.*

Unlike measurable cardinals, weakly compact cardinals and indescribable cardinals are consistent with $V = L$. However, indescribability is consistent also with measurability. In fact, we have with the following theorem:

THEOREM 85. *Every measurable cardinal is $\Pi^2_1$-indescribable.*

Surely, if $V = L$, a $\Pi^2_1$-indescribable cardinal would not be measurable in L.

## 3. Koellner's limitative results about Tait's reflection principles

Peter Koellner [Koellner 091] at the beginning of his article asserts that Reflection Principles aim to articulate the informal idea that the height of the universe V is *absolutely infinite* and hence cannot be characterized form below. Moreover, these principles assert that any statement true in V is true in some smaller $V_\alpha$. Towards V, the universe of all sets, we can have two perspectives. The first is the actualist perspective which sustains that totality of all sets is a completed totality. This perspective has not any problem to justify Reflection Principles since we can articulate the idea that the totality of all sets cannot characterized from below. However, the actualist perspective has problems to justify higher-order reflection. In fact, according to the actualist perspective, there are no sets beyond the totality of all sets, so for this perspective it is impossible to accept full higher-order quantification over the universe of sets. Thus, assuming higher-order reflection, we are considering full higher-order quantification over the universe of sets and so we are taking as parameters of formulas in linguistic reflection also sub-classes of the universe V itself. On the contrary, the potentialist perspective sustains that the totality of all sets does not constitute a completed totality. Thus, the potentialist view has problems to justify Reflection Principles but it does not have any problem to justify higher-order reflection and so, full higher-order quantification over the universe of sets since the universe of sets is not a completed totality. The actualist and the potentialist perspective face opposite problems of philosophical justification. However, concerning reflection principles, we face immediately Reinhardt limitative result about third-order and higher-order parameters. Third-order parameter imply that they are sets and they have sets as elements. Now we can see this limitative result in the following way: To see this let

$$A^{(3)} = \{(\sigma|\sigma < \alpha)^{(2)}|\alpha \in \Omega\}^{(3)}$$

and let $\phi(A^{(3)})$ be the statement that each element of $A^{(3)}$ is bounded. This is true in V but for each $\alpha \in \Omega$ the reflected version of the statement, $\phi^{\alpha}(A^{(3),\alpha})$, is false since $(\sigma|\sigma < \alpha)^{(2)} \in A^{(3),\alpha}$ is unbounded. This result of Reinhardt suggests that we must forgo statements where the order of the parameters is $\geq 2$. Thus Tait introduces the following definition:

Definition 98. *(Tait) A formula in the language of finite orders is positive iff it is build up by means of the operations* $\vee, \wedge, \forall, \exists$, *and from atoms of the form* $x = y, x \neq y, x \in y, x \notin y, x \in Y^{(2)}, x \notin Y^{(2)}$ *and* $X^m = Y^m$ *and* $X^m \in Y^{m+1}$, *where* $m \geq 2$.

Referring to this restricted language, Tait introduces $\Gamma_n^{(2)}$ class of formulas:

Definition 99. *(Tait) For* $0 < n < \omega$, $\Gamma_n^{(2)}$ *is the class of formulas of the form*

$$\forall X_1^{(2)} \exists Y_1^{(\kappa_1)} .........\forall X_n^{(2)} \exists Y_n^{\kappa_n} \phi(X_1^{(2)}, Y_1^{(\kappa_1)}, ......., X_n^{(2)}, Y_n^{(\kappa_n)}, A^{(l_1)}, ......, A^{(l_{n^1})})$$

*where* $\phi$ *does not have quantifiers of second- or higher-order and* $\kappa_1, ....., \kappa_n, l_1, ....., l_{n^1}$ *are natural numbers.*

Then he introduces his $\Gamma_n^{(2)}$ reflection principle:

Definition 100. *For* $0 < n < \omega$, $\Gamma_n^{(2)}$*-reflection is the schema asserting that for each sentence* $\phi \in \Gamma_n^{(2)}$, *if* $V \models \phi$ *then there is a* $\delta \in \Omega$ *such that* $V_\delta \models \phi^\delta$.

Now we have to introduce the notion of n-ineffable cardinal:

Definition 101. *(Baumgartner) For* $0 < n < \omega$, $\kappa$ *is n-ineffable iff for any* $(K_{a_1,......,a_n}|\alpha_1 < ....... < \alpha_n < \kappa)$ *with* $K_{a_1,......,a_n} \subseteq \alpha_1$ *for* $\alpha_1 < ...... < \alpha_n < \kappa$, *there is an* $X \subseteq \kappa$ *and an* $S$ *stationary in* $\kappa$ *such that for* $\beta_1 < ........ < \beta_n$, *all in* $S$, $X \cap \beta_1 = K_{\beta_1,....\beta_n}$.

Theorem 86. *(Tait) Suppose* $n < \omega$ *and* $V_\kappa \models \Gamma_n^{(2)} - reflection$. *Then* $\kappa$ *is n-ineffable.*

THEOREM 87. *(Tait) Suppose $\kappa$ is a measurable cardinal. Then, for each $n < \omega, V_\kappa \models \Gamma_n^{(2)} - reflection$.*

Therefore we can ask ourselves how strong is $\Gamma_n^{(2)} - reflection$ and if we can allow universal quantifiers of order greater than 2. Peter Koellner answers to these questions with the following two theorems:

THEOREM 88 (Koellner 091). *Assume $\eta_\omega$ exists (Erdös cardinal). Then there is a $\delta < \eta_\omega$ such that $V_\delta$ satisfies $\Gamma_n^{(2)} - reflection$ for all $n < \omega$.*

The existence of the cardinal $\eta_\omega$ produces an $\omega$-sequence of indiscernibles for $(V_\kappa, \in, R)$ for any finitary relation R on $V_\kappa$. Each such indiscernible $\sigma$ will determine that $(V_\sigma, \in) \prec (V_\kappa, \in)$ and will have unbounded reflection properties since it is indiscernible all the way up to $V_\kappa$. Secondly, from such cardinal one can construct a countable transitive model M of ZFC (namely the transitivisation of the Skolem Hull of such an indiscernible set) and a non-trivial elementary embedding $j$ with $j : M \longrightarrow M$. Now any internal reflection principle provable in $j : V \longrightarrow V$ would then be provable from such $j : M \longrightarrow M$ and, thus, will not break the $\eta_\omega$-barrier. At this point, we can introduce the second theorem of Peter Koellner:

THEOREM 89 (Koellner 091). *$\Gamma_1^{(3)} - reflection$ is inconsistent.*

So, according to Koellner, linguistic reflection principles can be divided into two classes: (1) weak: $\Gamma_n^{(2)} - reflection$, for $n < \omega$, (2) inconsistent: $\Gamma_n^{(m)} - reflection$, for $m > 2$ and $n \geq 1$. We conclude this section with Koellner words:

> Since $\Gamma_1^{(3)}$ comes directly after $\bigcup_{n<\omega} \Gamma_n^{(2)}$, this classification is exhaustive and we have a dichotomy theorem: Reflection principles are either weak or inconsistent. [Koellner 091]

## 4. Welch's Global Reflection

Welch's reflection [Welch 12] is very strong and it is able to produce a proper class of measurable Woodin cardinals, fundamental for $\Omega$-logic. So by examining this kind of reflection, we are able to overcome the barrier represented by Gödel's constructible universe. Welch [Welch 12] uses proper classes (Cantor inconsistent multiplicities) such as ON or V or Card which cause paradoxes in set theory and cannot be considered as sets. Welch collects all these classes in $C$, he considers the global universe $(V, \in, C)$ and he wants to reflect this global universe down to some initial segments, namely $V_\alpha$ together with the collection of all its parts (the classes over $V_\alpha$) which we may identify as $V_{\alpha+1}$. The elements of $V_{\alpha+1}$ play the role of classes for $V_\alpha$. Now we can introduce Welch's Global Reflection:

Definition 102. *(Global Reflection) (Welch) There is a $\kappa \in ON$ and there is $j \neq id$ elementary, $crit(j) = \kappa$,*

$$j : (V_\kappa, \in, V_{\kappa+1}) \longrightarrow (V, \in, C)$$

$Crit(j) = \kappa$ ensures that $j(\beta) = \beta$ for any $\beta < \kappa$ but $\kappa$, as a member of $V_{\kappa+1}$, is sent to $On$, as a member of $C : j(\kappa) = On$. The elementarity of the embedding ensures that the embedding preserves the whole structure $< \kappa$. This principle of reflection can be applied to any $\alpha$. Now suppose that Global reflection principle holds as witnessed by a $j$ with critical point $\kappa$. Define a $U$ on $P(\kappa)$ by

$$X \in U \leftrightarrow \kappa \in j(X)$$

The strong inaccessibility of $\kappa$ yields the $\delta$-additivity of $U$ for any $\delta < \kappa$ (all $\delta$ take measure 0). $U$ is a non-principal ultrafilter. Thus $U$ witnesses that $\kappa$ is a measurable cardinal. But then, as Welch proves assuming Global Reflection,

$\forall\alpha < \kappa(V, \in) \models \exists\kappa > \alpha(\kappa \textit{ a measurable cardinal}) \longrightarrow$

$(V_\kappa, \in) \models \forall\alpha\exists\lambda > \alpha(\lambda \textit{ a measurable cardinal}) \longrightarrow$

$(V, \in) \models$ *There is proper class of measurable cardinals*). Welch is able to prove the following Lemma:

LEMMA 16. *(Global Reflection) (Welch)* $(V, \in) \models \forall\alpha\exists\lambda > \alpha(\lambda \; a \; measurable \; Woodin \; cardinal)$.

PROOF. Let $f \in \kappa^\kappa \subseteq V_{\kappa+1}$, be arbitrary and consider $\tilde{f} = j(f)$. Then $\tilde{f} = On \longrightarrow On$; $Range(\tilde{f}) \subseteq \kappa$. Take $\lambda > \kappa$ a sufficiently large inaccessible, so that $\tilde{f} < \lambda$, and consider the $\lambda$-strong extender derived from $j$:

$$\{a \in [\lambda]^{<\omega} : E_a = \{z \in P([\kappa]^{|a|}) : a \in j(z)\}\}$$

This has the following properties:
$\Upsilon = (E_a : a \in [\lambda]^{<\omega})$ is a $(\kappa, \lambda)$-extender with $j(f)(\kappa) = j_\Upsilon(f)(\kappa) < \lambda$, and such that $Ult((V, \in), \Upsilon)$ is well-founded and if $\kappa : V \longrightarrow N \equiv Ult((V, \in), \Upsilon)$, is the unique transitivisation collapse map, then $V_\lambda = V_\lambda^N$.
This may be formalised as a first-order property and we abbreviate it as $\Phi(\kappa, \lambda, j(f), \Upsilon)$. about the displyed objects. Then:
$(V, \in, C) \models \exists\alpha[\exists\lambda\exists\Upsilon(Range(j(\alpha)) \subseteq \alpha \wedge \Phi(\alpha, \lambda, j(f), \Upsilon)]$.
We can shorten this as
$(V, \in C) \models \exists\alpha\phi(j(f), \alpha)$
and this is a first-order statement about $j(f)$. By Global Reflection Principle:
$(V_\kappa, \in, V_{\kappa+1}) \models \exists\alpha\phi(f, \alpha)$.
Thus, $\alpha$ witnesses that $\kappa$ is a Woodin cardinal in the case of $f$. Let vary $f$ over $\kappa^\kappa$ and we can see that $\kappa$ is Woodin. Thus, we have

$$(V, \in) \models \kappa \; is \; Woodin \; cardinal \; and \; measurable$$

and such measurable Woodin cardinals are unbounded in both $\kappa$ and $On$. □

However I believe that we should limit the use of proper classes. In fact, a direct use of proper classes or sub-classes of proper classes is not a precise mathematical operation. Taking proper classes or subclasses does not seem a legitimate mathematical operation. From one side, proper classes cause paradoxes. From the other, arbitrary subclasses of proper classes are not precise mathematical objects. I think that if we want to take subclasses of proper classes, we should take only definable subclasses of proper classes. In chapter 5, I will explain the operation of taking definable subclasses of proper classes and my attempt to extend the Universe avoiding Cantor's paradox and Burali-Forti's paradox. I prefer to adopt, as we will see in the following chapter, an indirect use of proper classes. In fact, I believe that we can use proper classes as indexes of iterated structural reflection relativized to inner models. Surely, also this use can be seen as problematic since proper classes as indexes can still cause paradoxes. However, in chapter 5, I will try to legitimate proper classes by extending the universe and attempting to avoid paradoxes.

CHAPTER 3

# Structural reflection

**0.1. Preliminaries to this chapter.** In section 1 I will introduce the notion of structural reflection. I am going to state a theorem of Joan Bagaria which asserts that structural reflection produces a proper class of supercompact cardinals and a proper class of extendible cardinals. This fact is fundamental, since we can interpret a proper class of supercompact cardinals as principles of reflection and so they can be intrinsically justified. This aspect is important for second-order arithmetic and $\Omega$-logic (as we have seen). We can say that structural reflection is an essential feature of the ZFC universe, since $\Sigma_1$ structural reflection is provable from the axioms of ZFC. So, structural reflection is a characteristic of the universe of sets. In this section I will introduce the philosophical concept of richness which constitutes a justification for structural reflection. When we relativize a class of structures, that is $\Pi_1$ definable, to an inner model, we transcend this inner model and so we have a richer universe. In section 2 I will introduce some well-known canonical inner models and the concept of relative constructibility, as conceived by Levy. After that, I will introduce the inner model of measurability and the technique of iterated ultrapowers. In section 3 I will relativize structural reflection to L and I will show that we can transcend this inner model by producing $0^\sharp$. In this section I will explain how to iterate structural reflection and apply structural reflection to the inner model containing $0^\sharp$, namely $L[0^\sharp]$. Obviously, this operation can be iterated again and the process does not have a bound. Moreover, I will introduce the finite transcendental structural reflection hierarchy, which forms a metamathematical sequence of inner models. In section 4 I will introduce briefly the theory of $0^\dagger$ and I will relativize structural reflection to the model $L[U]$. Also in this case, when we relativize structural reflection to $L[U]$, we transcend this inner model, thus producing $0^{\dagger}$. In section 5 I will describe a canonical inner model for a strong cardinal and I will relativize structural reflection to this model, thus producing the sharp for this inner model, namely $0^{\P}$. Also in this case, we transcend the inner model containing a strong cardinal. In section 6 I will discuss Woodin's HOD conjecture. Then, I will introduce the Wholeness axioms [Corazza 00]. In section 7, I will introduce Woodin's Ultimate L model. I will conclude this chapter by discussing the philosophy of mathematics that I sustain (section 8).

## 1. Structural reflection and the philosophical concept of richness

According to Gödel, the fundamental guiding principle in setting up new axioms of set theory is the unknowability of the absolute, and so any new axiom should be based on such principle [Wang 96]. Gödel's program consisted, therefore, in formulating stronger and stronger systems of set theory by adding to the base theory new principles. So the question is how should one understand and formulate the idea of reflection embodied in Ackermann's principle [Bagaria 13]. Some light is provided by the following quote of Gödel where he asserts that the indefinability of V should be the source of all axioms of infinity.

> Generally, I believe that, in the last analysis, every axiom of infinity should be derivable from the (extremely plausible) principle that V is indefinable, where definability is to be taken in a more and more generalised and idealised sense. [Wang 96]

One possible interpretation of Gödel's principle of the indefinability of V is an unrestricted version of the Levy-Montague reflection theorem. Namely, every formula, with parameters, in any formal language with the membership relation, that holds in V, must also hold in some $V_\alpha$. This has been indeed the usual way to interpret Gödel's view of reflection as a justification for the axioms of large cardinals [Bagaria 13]. In a recent article, Peter

Koellner, as we have already seen, actually identifies reflection principles with generalised forms of the Levy-Montague reflection theorem:

> Reflection principles aim at articulating the informal idea that the height of the universe is absolutely infinite and hence cannot be characterised from below. These principles assert that any statement true in V is true in some smaller $V_\alpha$. [Koellner 091]

Koellner explicitly interprets Gödel's view of reflection as a source of large cardinals in this way:

> Since the most natural way to assert that V is undefinable is via reflection principles and since to assert this in a more and more generalised and idealised sense is to move to languages of higher order with higher order parameters, Gödel is espousing the view that higher-order reflection principles imply all large cardinals axioms. [Koellner 091]

The main problem of the program of finding an intrinsic justification of large cardinal axioms via principles of reflection lies, we believe, on a too restrictive interpretation of the notion of reflection according to which the reflection properties of V are exhausted by generalised forms of Levy-Montague reflection theorem to higher order logics. Thus, we should think about a different way to conceive reflection principles as Bagaria [Bagaria 10] [Bagaria 13] and Welch [Welch 12] did (we have already seen the case of Welch's global reflection principle). Furthermore, we might interpret in a different way another claim of Gödel:

> The universe of sets cannot be uniquely characterised (i.e., distinguished from all its initial segments) by any internal structural property of the membership relation in it which is expressible in any logic of finite or transfinite type, including infinitary logics of any cardinal number. [Wang 96]

Bagaria interprets this Gödel's quotation in the following sense:

> This does not immediately suggest that the uncharacterizability of V should be interpreted in the sense of Levy-Montague kind of reflection. Rather, what the quote seems to suggest is some sort of reflection, not (only) of formulas, but of structural properties of the membership relation. Thus, what one would like to reflect is not the theory of V, but rather the structural content of V [Bagaria 13]

At this point, we can try to clarify what one may mean by reflecting an internal structural property of the membership relation by following Bagaria's thought. We could answer that it is a property of some structure of the form $(X, \in, (R_i)_{i \in I})$, where X is a set or a proper class and $(R_i)_{i \in I}$ is a family of relations on X, and where I is a set that may be empty. So, an internal structural property of $\in$ would be formally given by a formula $\phi(x)$, possibly with parameters, that defines a class of structures of the form $(X, \in, (R_i)_{i \in I})$. We might interpret this fact by saying that there exists an ordinal $\alpha$ that reflects $\phi$ and such that for every structure A in the class (that is, for every structure A that satisfies $\phi$) there exists a structure B also in the class which belongs to $V_\alpha$ and is like A. Since, in general, A may be much larger than any B in $V_\alpha$, the closest resemblance of B to A will be attained in the case that B can be elementarily embedded into A . Thus we can now formulate the principle of structural reflection as follows:

Definition 103. *(Bagaria) (Structural reflection) (SR) For every definable (in the first order language of set theory, with parameters) class of structures C of the form* $(X, \in, (R_i)_{i \in I})$*, there exists* $\alpha$ *such that* $\alpha$ *reflects C, i.e.* $C^{V_\alpha} = C \cap V_\alpha$ *and for every A in C there exists B in* $C \cap V_\alpha$ *and an elementary embedding from B into A.*

We must notice that if C is a set, then the principle becomes trivial. Thus, we should assume that the SR principle is for proper classes of structures. Bagaria [Bagaria 13] formulates the SR principle in the first-order language of set theory as an axiom schema, to wit an axiom, for each natural number n.

DEFINITION 104 (Bagaria 10). $\Sigma_n$ *structural reflection* ($\Sigma_n - SR$) *: for every* $\Sigma_n$ *definable, with parameters, class* $C$ *of structures of the form* $(X, \in, ....)$, *there exists and ordinal* $\alpha$ *that reflects* $C$.

$\Pi_n$-SR is defined analogously. The first observation is that $\Sigma_1$-SR is provable in ZFC.

THEOREM 90 (Bagaria 10). $\Sigma_1$-*SR holds. In fact every uncountable cardinal* $\kappa$ *with* $V_\kappa = H_\kappa$ *and such that* $V_\kappa$ *contains the parameters of some* $\Sigma_1$ *definition of a given class* $C$ *of structures reflects* $C$.

But $\Pi_1$-SR is already very strong. We have the following

THEOREM 91 (Bagaria 10). *the following are equivalent :*

*1)* $\mathbf{\Pi_1}$-*SR*

*2)* $\mathbf{\Sigma_2}$-*SR*

*3) There exists a* ***proper class*** *of supercompact cardinals.*

For the next level of complexity we have the following:

THEOREM 92 (Bagaria 10). *the following are equivalent:*

*1)* $\mathbf{\Pi_2}$-*SR*

*2)* $\mathbf{\Sigma_3}$-*SR*

*3) There exists a* ***proper class*** *of extendible cardinals.*

Now it is the moment to discuss an important issue concerning structural reflection. If we apply structural reflection to classes of structures relativised to inner models like $L, L[0^\sharp], L[U]$, $L[0^\dagger]$,etc., we are able to obtain transcendence over inner models. At this point I want to introduce a new concept, namely richness. Transcendence over inner models points out to richness. If we have $\Pi_1$-classes of structures definable in V, we may relativise them to inner models such as L, inner models of iterated sharps, inner model of measurability, etc. By doing this we produce the specific sharp and we transcend these inner models. This process can be ascribed to the philosophical concept of richness. Richness can be seen as a kind of justification for structural reflection principles, since when we transcend inner models, we obtain a richer and bigger universe. In fact, the philosophical concept of richness may be seen as a justification of structural reflection principles since, as Penelope Maddy [Maddy 97] argues, we should always prefer axioms or principles that give us a richer picture of the universe of sets. Structural reflection seems to imply that the universe of sets is essentially uniform. In fact, Bagaria's structural reflection seems to imply that structures, which are located lower in the hierarchy of the universe, resemble structures which are higher in the universe. Since structural reflection applies to classes of structures within inner models, instead of writing SR(C) we can write directly SR(M) by meaning that we are applying structural reflection to classes of structures within the inner model M. Later we will prove a theorem (general case) discovered by Joan Bagaria, namely:

THEOREM 93 (Bagaria 13). *The following are equivalent for any set of ordinals X:*
*(1) SR(C), where C is the* $\Pi_1$ *definable class of structures of the form* $(L_\alpha[X], \in, \beta)$*, where* $\alpha > \beta$ *and are cardinals in (V)*
*(2) SR(C) for any definable (in V), with parameter X, class of structures C,* $C \subseteq L[X]$ *i.e.,* $SR(L[X])$.
*(3)* $X^\sharp$ *exists.*

Following similar arguments, we may obtain the following:
(1) SR(L) if and only if $0^\sharp$ exists
(2) $SR(L[0^\sharp])$ if and only if $0^{\sharp\sharp}$ exists
(3) SR(L[U]) if and only if $0^\dagger$ exists
(4) $SR(L[U][0^\dagger])$ if and only if $0^{\dagger\dagger}$ exists.

I sustain weak metamathematical potentialism concerning the universe of sets. We may introduce the following hypothesis: If we relativize $\Pi_1$ structural reflection to any inner model that contains no supercompact cardinal we get transcendence over this inner model. In contrast, by the closure properties of a weak extender model for a supercompact cardinal, if we relativize $\Pi_1$ structural reflection to this inner model, we do not get transcendence over it. This is a plausible conjecture that I will discuss in the section devoted to Woodin's Ultimate L. We will argue that by using a theorem of Woodin [Woodin 10b], when we apply $\Pi_1$ structural reflection to a weak extender model for a supercompact cardinal we do not get transcendence over this model. This is a general hypothesis, since we do not know if in the future we will discover new cardinals strictly less than a supercompact cardinal whose inner models will have closure properties similar to a weak extender model for a supercompact cardinal. We can speak of *weak metamathematical potentialism* concerning $\Pi_1$ structural reflection relativized to inner models. I said *metamathematical potentialism* because when we relativize $\Pi_1$ structural reflection to inner models containing cardinals strictly less than a supercompact cardinal, we obtain transcendence over these inner models and we do not have a resting point. I said *weak* because when we reach the level of a supercompact cardinal and we have a weak extender model N, for the closure properties of this inner model we do not get transcendence over this inner model. In order to understand these results regarding structural reflection applied to inner models which produces sharps, in the Appendix I present the theory of Silver indiscernibles. In the next sections, I will speak about inner models and then I will focus my attention on the theory of $0^\dagger$.

## 2. Beginning inner model theory

The minimal inner model is L, and $L \subseteq M$ for any inner model M since $L^M = L$ [1]. Andras Hajnal and Azriel Levy in their doctoral dissertations developed basic generalizations of L which are the basis for the construction of inner models beyond L. For a given

[1] see [Kanamori 09]

set A the constructible closure $L(A)$, i.e., the smallest inner model M such that $A \subseteq M$. $L(\mathbb{R})$ is an example of this kind of construction. More precisely: given A define

1) $L_0(A) = tc(A)$

2) $L_{\alpha+1}(A) = Def(L_\alpha(A))$

3)$L_\gamma(A) = \bigcup_{\alpha<\gamma} L_\alpha(A)$ for limit $\gamma > 0$

4) $L(A) = \bigcup_{\alpha \in Ord} L_\alpha(A)$.

Although $L(A)$ is indeed an inner model, unless $tc(A)$ has a well-ordering in $L(A)$, $L(A)$ does not satisfy the axiom of choice. $|L_\alpha(A)| = |tc(A)|.\ |\alpha|$, for $\alpha \geq \omega$, a result established by induction on $\alpha$.

Levy introduced also for a given set A, the inner model $L[A]$ of sets constructible relative to A, i.e. the smallest inner model M such that for every $x \in M$, $A \cap x \in M$. For the inner model program, Levy's construction is fundamental as we will see. The idea is to define a relativised hierarchy where assertion about membership in A can be made of sets defined so far, such as within Gödel's constructible universe. This construction implies a strict form of predicativism. Let:

$Def^A(x) = \{y \subseteq x | y\ is\ definable\ over\ (x, \in, A \cap x)\}$

making $A \cap x$ available as a unary relation for definitions. In analogy with L, one defines the following hierarchy:

1)$L_0[A] = \emptyset$

2) $L_{\alpha+1}[A] = Def^A(L_\alpha[A])$

3) $L_\gamma[A] = \bigcup_{\alpha<\gamma} L_\alpha[A]$ for limit $\gamma > 0$.

4) $L[A] = \bigcup_{\alpha \in Ord} L_\alpha[A]$.

Unlike the case of $L(A)$, in the case of $L[A]$ what remains of A is only $A \cap L[A]$. However, $L[A]$ is more constructive since knowledge of A is incorporated through the hierarchy of definitions, and like L, $L[A]$ satifies the axiom of choice. Here we have $|L_\alpha[A]| = |\alpha|$ for $\alpha \geq \omega$, a result established by induction on $\alpha$.

One fundamental aspect of the theory of large cardinals was the investigation of the smallest inner models in which they maintain their essential features. The first milestone of the inner model program was the construction of inner models of measurability.

Gödel's fundamental work on L is the beginning of the inner model program. For large cardinals like inaccessible, reflecting, Mahlo, $\eta_\omega$-Erdös and weakly compact cardinals, the corresponding inner model is L itself. Scott's result that measurable cardinals contradict $V = L$ forced mathematicians to think about inner models for measurability, first considered by Solovay. Let U be a $\kappa$-complete, nonprincipal ultrafilter over $\kappa > \omega$. Since the measurability of $\kappa$ implies the introduction of the set U (the ultrafilter), Solovay took in consideration $L[U]$, Levy's inner model of sets constructible relative to $U$. $U^* = U \cap L[U] \in L[U]$ and so $L[U^*] = L[U]$, and the following hold:

Theorem 94. *(Solovay) [Jech 06]* $L[U] \models U^*$ *is a $\kappa$-complete ultrafilter over $\kappa$.*

Theorem 95. *(Solovay) [Jech 06] If U is normal, then* $L[U] \models U^*$*is normal.*

Thus, $\kappa$ is measurable in $L[U]$, and like L with respect to ZF, it is consistent with $\kappa$ being measurable that $V = L[U]$, so that $U$ could have been $U^*$ all along. Focusing on these inner models of measurability, $(L[U], \in, U)$ is a $\kappa$-model iff $(L[U], \in, U) \models$ "$U$ is a normal ultrafilter over $\kappa$". Thus, $U \in L[U]$ is incorporated from the beginning as a unary relation. Then $U \in L[U]$ implies $L[U]^{L[U]} = L[U]$ and hence that $L[U] \models V = L[U]$.

Theorem 96. *(Solovay) [Kanamori 09] Suppose that* $(L[U], \in, U)$ *is a $\kappa$-model. Then the following hold in* $L[U]$*:*

*1)* $\forall \gamma \geq \kappa (2^\gamma = \gamma^+)$

*2) $\kappa$ is the only measurable cardinal.*

Silver obtained the first substantial result on $\kappa$-models, namely:

Theorem 97. *(Silver) [Kanamori 09] Suppose that* $(L[U], \in, U)$ *is a $\kappa$-model. then* $L[U] \models GCH$.

At this point, in order to understand Kunen's result, we have to introduce the concept of iterated ultrapowers[2]. Let $\kappa$ be a measurable cardinal and let $U$ be a $\kappa$-complete non-principal ultrafilter on $\kappa$. Using $U$, we construct an ultrapower of V, modulo U; and since the ultrapower is well-founded, we identify the ultrapower with its transitive collapse, a transitive model $M \cong Ult_U(V)$. Let us denote this transitive model by $Ult^1_U(V)$, or just $Ult^1$. Let $j^0 = j_U$ be the canonical embedding of V in $Ult^1$, and let $\kappa^1 = j^0(\kappa)$ and $U^1 = j^0(U)$.

In the model $Ult^1$, the ordinal $\kappa^1$ is a measurable ordinal and $U^1$ is a $\kappa^1$-complete ultrafilter on $\kappa^1$. Thus, working inside $Ult^1$, we can construct an ultrapower mod $U^1$: $Ult_{U^1}(Ult^1)$. Let us denote this ultrapower $Ult^2$, and let $j^1$ be the canonical embedding of $Ult^1$ in $Ult^2$ given by this ultrapower. Let $\kappa^2 = j^1(\kappa^1)$ and $U^2 = j^1(U^1)$.

We can continue this procedure and obtain transitive models: $Ult^1, Ult^2, ..........., Ult^{(n)}$. $(n < \omega)$.

Thus we get a sequence of models $Ult^{(n)}$, $n < \omega$ (where $Ult^{(0)} = V$). For any $n < m$, we have an elementary embedding $i_{n,m} : Ult^{(n)} \longrightarrow Ult^{(m)}$ which is the composition of the embeddings $j^{(n)}, j^{(n+1)}....., j^{(m-1)}$:

$i_{n,m}(x) = j^{(m-1)} j^{(m-2)} ..... j^{(n)}(x) \quad (x \in Ult^{(n)})$.

These embeddings form a commutative system; that is :

$i_{m,\kappa} \times i_{n,m} = i_{n,\kappa} \quad (m < n < \kappa)$.

We also let $\kappa^{(n)} = i_{0,n}(\kappa)$, and $U^{(n)} = i_{0,n}(U)$. Note that $\kappa^{(0)} < \kappa^{(1)} < \kappa^{(2)} < ......... < \kappa^{(n)}..$, and $Ult^{(0)} \supset Ult^{(1)} \supset Ult^{(2)} \supset ........ \supset Ult^{(n)}, .....$

Theorem 98. *(Kunen) [Jech 06] 1) If $V = L[U]$ and $U$ is a normal measure on $\kappa$, then $\kappa$ is the only measurable cardinal and $U$ is the only normal measure on $\kappa$.*

*2) For every ordinal $\kappa$, there is at most one $U \subset P(\kappa)$ such that $U \in L[U]$ and $L[U] \models$ "$U$ is a normal measure on $\kappa$".*

[2] See [Jech 06]

*3) If $\kappa_1 < \kappa_2$ are ordinals and if $U_1, U_2$ are such that $L[U_i] \models$ "$U_i$ is a normal measure on $\kappa_i$" (i= 1, 2), then $L[U_2]$ is an iterated ultrapower of $L[U_1]$; i.e, there is $\alpha$ such that $L[U_2] = Ult^{(\alpha)}_{U_1}(L[U_1])$, and $U_2 = i_{0,\alpha}(U_1)$.*

There are three kinds of inner models that occur in inner model theory: coarse inner models, fine-structural inner models, and core models. Historically, for a given large cardinal hypothesis, a coarse inner model was first discovered, and this served as a precursor to the more involved fine-structural inner model, which in turn served to the even more involved core model.

Since we have studied before coarse inner model with one measurable cardinal, the next natural step is to construct a coarse inner model with more than one measurable cardinal. Models of the form $L[U]$ are unsuited for this purpose since (as we have seen before, by Kunen's theorem) they can contain at most one measurable cardinal. Instead, the right thing to do is replace $U$ with a sequence $W$ of normal measures $U$, each of which witnesses the measurability of a different measurable cardinal in V. The next step is to ensure that one can capture measurable cardinals of high-order and this requires allowing $W$ to contain many measures concentrating on a single cardinal. Mitchell developed this theory in 1974 and constructed the model $L[W]$ with many measures on a single cardinal.

## 3. Relativising structural reflection to inner models

Structural reflection produces a proper class of supercompact cardinals. Now we can ask ourselves if structural reflection can produce other large cardinal notions. Since $\mathbf{\Pi_1}-SR$ implies already the existence of a proper class of supercompact cardinals, we must look for particular $\Pi_1$-definable classes (with parameters) of structures relativised to inner models. So, we might consider the principle of structural reflection restricted to particular definable classes of structures. Recall:

DEFINITION 105 (Bagaria 13). *Structural reflection for C (SR(C)): There exists an ordinal $\alpha$ that reflects C, where C is $\Pi_1$ class of structures definable (with parameters) in V. For every A in C relativized to the canonical inner model M, there exists B in $C \cap M^{\alpha}$ and an elementary embedding j from B into A.*

At this point we can apply structural reflection to L and $0^{\sharp}$. Let C be the class of structures of the form $(L_\beta, \in, \gamma)$, where $\gamma$ and $\beta$ are cardinals (in V) and $\gamma < \beta$. Clearly, C is $\Pi_1$ definable (without parameters).

THEOREM 99 (Bagaria 13). *1) SR(C)*[3] *if and only if* $0^{\sharp}$ *exists.*
*(2)* $0^{\sharp}$ *implies SR(D), for all classes D of structures of the same type that are definable in L.*

PROOF. (1): Suppose first that $\alpha$ reflects C. Pick cardinals $\gamma$ and $\beta$, with $\gamma$ a cardinal in V, such that $\alpha < \gamma < \beta$. Then there are cardinals $\gamma'$ and $\beta'$, with $\gamma'$ a cardinal in V and $\gamma' < \beta' < \alpha$, and an elementary embedding :

$$j : (L_{\beta'}, \in, \gamma') \longrightarrow (L_\beta, \in, \gamma)$$

Since $j(\gamma') = \gamma$, $j$ is not the identity. Let $\kappa$ be the critical point of $j$. Thus, $\kappa \leq \gamma' < \beta$. Hence by Kunen's theorem (see [Kanamori 09], 21.1) $0^{\sharp}$ exists.
Now suppose that $0^{\sharp}$ exists. Let $\alpha$ be a limit cardinal in V. We claim that $\alpha$ reflects C. For suppose $(L_\beta, \in, \gamma) \in C$ with $\alpha \leq \beta$. Let $\gamma'$ and $\beta'$ be cardinals in V such that $\gamma' < \beta' < \alpha$ and $\gamma' \leq \gamma$. Let I denote the class of Silver indiscernibles. Let $j : I \cap [\gamma', \beta'] \longrightarrow I \cap [\gamma, \beta]$ be order preserving such that $j(\gamma') = \gamma$ and $J(\beta') = \beta$ . Then $J$ generates an elementary embedding:

$$j : (L_{\beta'}, \in, \gamma') \longrightarrow (L_\beta, \in, \gamma)$$

[3] For what we said before, we can write SR(L) implying that we are speaking of classes of structures relativised to L.

as required.

(2) Fix a D and a formula $\phi(x)$, possibly with ordinals $\alpha_0 < ...... < \alpha_m$ as parameters, that defines it in L. Let $\kappa$ be a limit of Silver indiscernibles greater than $\alpha_m$ and such that $\kappa$ is correct for D, that is, if $A \in V_\kappa$, then $\phi(A)$ holds if and only if $V_\kappa \models \phi(A)$. We claim that $\kappa$ reflects D. For suppose $B \in D$. Without loss of generality, $B \notin L_\kappa$. Since $0^\sharp$ holds there exists an increasing sequence of Silver indiscernibles $i_0, ......, i_n, i_{n+1}$, with $\kappa \leq i_n$ and a formula $\psi(y, z_0......z_n)$, without parameters, such that

$$B = \{y : L_{i_{n+1}} \models \psi(y, i_0......, i_n)\}$$

Choose indiscernibles $j_0 < ....... < j_n < j_{n+1} < \kappa$ with $\alpha_m < j_0$ and let

$$A = \{y : L_{j_{n+1}} \models \psi(y, j_0, ....j_n)\}$$

Thus $A \in L_\kappa$ . We have that $L \models \phi(B)$. That is

$$L \models \forall x(\forall y(y \in x \leftrightarrow L_{i_{n+1}} \models \psi(y, i_0, ...i_n)) \longrightarrow \phi(x))$$

By indiscernibility,

$L \models \forall x(\forall y(y \in x \leftrightarrow L_{j_{n+1}} \models \psi(y, j_0, ...j_n)) \longrightarrow \phi(x))$

which implies $L \models \phi(A)$, i.e $A \in D$.

Let $j : L \longrightarrow L$ be an elementary embedding that sends $i_\kappa$ to $j_\kappa$, all $\kappa \leq n + 1$. Then by indiscernibility, the map $j|A : A \longrightarrow B$ is an elementary embedding. □

The following theorem gives a similar result, relativised to any set of ordinals.

THEOREM 100 (Bagaria 13). *$SR(L[X])$ iff $X^\sharp$ exists (for any set $X$ of ordinals).*

Now we are in a position to apply structural reflection to $L[0^\sharp]$, namely the inner model containing $0^\sharp$. Let C be the class of structures of the form $(L[0^\sharp]_\beta, \in, \gamma)$, where $\gamma$ *and* $\beta$ are cardinals (in V) and $\gamma < \beta$.

LEMMA 17. *SR($L[0^\sharp]$)) if and only if $0^{\sharp\sharp}$ exists.*

These arguments suggest that when we apply structural reflection to class of structures relativised to inner models, we obtain always a transcendence over inner models. As examples, we have the following:

SR($L[0^{\sharp\sharp}]$) if and only if $0^{\sharp\sharp\sharp}$ exists.

SR($L[0^{\sharp\sharp\sharp}]$) if and only if $0^{\sharp\sharp\sharp\sharp}$ exists, etc.

It is possible to consider structural reflection as a **transcendental successor** function with respect to inner models. When we apply structural reflection to an inner model, we obtain a sharp that points out to the fact that we transcend the inner model and that the inner model itself is not rigid (there is an embedding of the inner model into itself). For example:

(1) SR ($L$) if and only if $0^{\sharp 1} = 0^\sharp$ exists

(2) SR($L[0^{\sharp 1}]$) if and only if $0^{\sharp 2}$ exists.

The simpler hierarchy of sharps mirrors this hierarchy in the following sense:

(1) $0^{\sharp 1} = 0^\sharp$

(2) $0^{\sharp(\alpha+1)} = (0^{\sharp\alpha})^\sharp$

(3) If $\alpha$ is a limit ordinal then $0^{\sharp\alpha}$ represents $(0^{\sharp\gamma} : \gamma < \alpha)$

Thinking about the partial hierarchy of structural reflection, we should ask ourselves what all these sharps are. The answer is simple: these sharps are sets of ordinals. So when we apply structural reflection to an inner model, we are adding a set of ordinals to the metamathematical sequence of these inner models. In fact by transcending an inner model, we are producing a sharp (a set of ordinals) that belongs to the sequence. Then we form an inner model containing this sharp and by applying structural reflection to this inner model, we transcend this inner model again and we produce another sharp. We have two operations, namely transcending inner models by applying structural reflection to them and then forming a new inner model by using the sharp produced by structural reflection in the precedent operation. But by applying structural reflection to inner models, we are adding sets of ordinals to this sequence of inner models.

Now we are in a position to conceive the total transcendental hierarchy. We have to introduce the following two operations: $Inn^{M,\alpha}$ which applied to a specific sharp, produces an inner model containing this sharp and SR(structural reflection), which applied to an inner model, produces a sharp and so a transcendence over this inner model (the successor stage). Structural reflection can be seen as an analogous operation to the power set operation in Von Neumann's cumulative hierarchy. However, with structural reflection we are in the realm of metamathematics.

The transcendental hierarchy is shaped in the following way:

(1) $Inn^{M,0}(0^{\sharp,0}) = L$

(2) SR$(0^{\sharp,0}) = SR(L)$ if and only if $0^{\sharp 1}$ exists

(3) $Inn^{M,1}(0^{\sharp,1}) = L[0^{\sharp 1}]$

(4) SR $(Inn^{M,1}(0^{\sharp,1})) = SR(L[0^{\sharp 1}])$ if and only if $0^{\sharp 2}$ exists.

(5) $Inn^{M,2}(0^{\sharp,2}) = L[0^{\sharp,2}]$.

The precedent hierarchy highlights that in this sequence of inner models we have two fundamental operations, namely the application of structural reflection to inner models and the operation of forming a new inner model containing the sharp produced by structural reflection. These two operations mirror Cantor's distinction.

## 4. The Core Model and Structural Reflection

The first core model was Dodd and Jensen's construction, namely $K^{DJ}$. This model (the core model up to a measurable cardinal) is an inner model that contains much of the large cardinals below a measurable cardinal. It is characterized by the following features: (1) $K^{DJ}$ has a definable well-ordering, satisfies GCH and some combinatorial principles such as $\square$. (2) There is a non-trivial elementary embedding $j : K^{DJ} \longrightarrow K^{DJ}$ if and only if $L[U]$ exists, (3) If $L[U]$ does not exist then the Covering theorem (we will see this later) holds for $K^{DJ}$.

If $L[U]$ exists then $K^{DJ}$ has a simple definition:

$$K^{DJ} = \bigcap_{\alpha \in Ord} Ult^{\alpha}_{U}(L[U]).$$

Let us define the notion of a mouse (mice are the building blocks of $K^{DJ}$):

DEFINITION 106. *A mouse is a transitive model $M = L^U_\alpha$ such that:*

*(1) $U$ is a normal $\kappa$-complete iterable $M$-ultrafilter on some $\kappa < \alpha$.*

*(2) All iterated ultrapowers of $L^U_\alpha$ by $U$ are well-founded.*

*(3) $M = H^M_1(\gamma \cup \rho)$ for some $\gamma < \kappa$ and some finite $\rho \subset \alpha$.*

Now we can state the following theorem:

THEOREM 101 (Dodd Jensen 81). *We have the following:*

*(1) $K^{DJ}$ is an inner model of ZFC and has a $\Sigma_2$ well-ordering.*

*(2) $K^{DJ}$ satisfies GCH.*

*(3) $\mathbb{R}^{DJ}$ has a $\Sigma^1_3$ well-ordering.*

*(4) $K^K = K$, and $K^{V[G]} = K$ for every generic extension.*

*(5) In $K^{DJ}$, $L[U]$ does not exist.*

*(6) If $0^\sharp$ does not exist then $K = L$. If $0^\sharp$ exists then $0^\sharp \in K^{DJ}$. More generally, for every $x \in K^{DJ}$, if $x^\sharp$ exists then $x^\sharp \in K^{DJ}$.*

LEMMA 18 (Dodd Jensen 81). *A mouse exists if and only if $0^\sharp$ exists.*

LEMMA 19 (Dodd Jensen 81). *If mice exist then $K^{DJ} = \bigcup\{M :$ is a mouse$\}$.*

The Covering theorem implies that if $L[U]$ does not exist (the sharp for $K^{DJ}$), then $K^{DJ}$ is very close to V.

THEOREM 102 (Dodd Jensen 81). *The following are equivalent:*

*(1) $L[U]$ exists.*

*(2) There exists a non-trivial elementary embedding $j : K^{DJ} \longrightarrow K^{DJ}$.*

THEOREM 103. *(The Covering theorem) [Dodd Jensen 81] If $L[U]$ does not exist, then for every uncountable set $X$ of ordinals there exists a set $Y, with X \subset Y$ in $K^{DJ}$ such that $|Y| = |X|$.*

A mouse can be iterated. Furthermore, [Schimmerling 01] the theory of embeddings $L[0^\sharp] \longrightarrow L[0^\sharp](0^{\sharp\sharp}$ exists) and the iterated mice $M_0^{\sharp\sharp}$ run parallel. We may define the iterated mice $M_0^{\sharp,\dots,\sharp}$. If $0^\sharp$ exists then a mouse $M_0^\sharp$ exists for Dodd Jensen theorem. Moreover, if $0^{\sharp\sharp}$ exists then the iterated mouse $M_0^{\sharp\sharp}$ exists and so on. At this point, let's reintroduce the structural reflection. Let's consider the finite structural reflection hierarchy, namely $SR^{<\omega}$. At this point, we can simplify and reformulate the finite structural reflection hierarchy in the following way, for $n < \omega$:

$(A) SR^0 = Inn^{M,0} = L.$

(B)

$$SR^{n+1} = \begin{cases} SR(L[0^{\sharp n}]) = 0^{\sharp n+1} \\ Inn^{M,n+1}[0^{\sharp n+1}] = L[0^{\sharp n+1}] \end{cases}$$

In the finite structural reflection hierarchy, the first step is the construction of Gödel's constructible universe, namely L. The successor stage within this hierarchy is constituted by two steps. The first step is the application of structural reflection to a specific inner model producing a sharp and the second step it is the formation of the inner model by adopting the operation $Inn^{M,n}$ containing that sharp. So we have two steps at successor stage. In fact, the successor stage is constituted by two passages, namely applying structural reflection to inner model and, then, forming the inner model that contains the sharp obtained by the precedent step. We should ask ourselves what is the relationship between the finite structural reflection hierarchy, namely $SR^{<\omega}$ and the core model. Now we state the following theorem:

THEOREM 104. *The finite structural reflection hierarchy, namely $SR^{<\omega}$, is properly contained in $K^{DJ}$ and the finite structural reflection hierarchy is equivalent to the hierarchy of iterated mice within $K^{DJ}$*

PROOF. Assume that $\Pi_1$ structural reflection relativized to L holds. Then we produce $0^\sharp$. Then iterate this operation and form the finite structural reflection hierarchy, namely $SR^{<\omega}$. Relativize the finite structural reflection hierarchy to $K^{DJ}$. Since by Dodd and Jensen theorem, for every $x \in K^{DJ}$, if $x^\sharp$ exists then $x^\sharp \in K^{DJ}$, the finite structural reflection hierarchy is properly contained in $K^{DJ}$. The successor step in the structural reflection produces a sharp that is equivalent by Dodd and Jensen theorem to the formation of a mouse $M_0^\sharp$. If we iterate the structural reflection operation, we produce $0^{\sharp\sharp}$, which is equivalent to the iterated mouse $M_0^{\sharp\sharp}$. Thus, the finite structural reflection hierarchy, namely $SR^{<\omega}$, is equivalent to the hierarchy of iterated mice within $K^{DJ}$. □

The interesting aspect of this theorem is that the structural reflection hierarchy, which is external to $K^{DJ}$ since we pick always cardinals in V, is equivalent to the hierarchy of mice which is internal to $K^{DJ}$. Thus, by assuming the finite structural reflection hierarchy, namely $SR^{<\omega}$, although each embedding implies that we pick cardinals in V, we are working inside $K^{DJ}$. At this point, we can state a conjecture that establishes an equivalency between structural reflection and determinacy. This equivalency is based on some results obtained by Italy Neeman [Neeman 06]. First of all, we must clarify some notations. By $G_\omega(A)$ we denote the length $\omega$ game with payoff A and by $W(B)$ we mean the set $\{x \in \mathbb{R}|$ player 1 has a winning strategy in $G_\omega(B_x)\}$. For determinacy, see chapter 1 of this dissertation section 3.

THEOREM 105 (Neeman 06). *Let $B_i (i < \omega)$ be a recursive enumeration of the $W^{(n)}(< \omega^2 - \Pi^1_1)$ sets. Then the sharp for n Woodin cardinals and $\{ \ i \mid$ player 1 has a winning strategy in $G_\omega(B_i)\}$ are each recursive in the other.*

Now we can state the Structural Reflection Conjecture:

(SRC) For every natural number n, one can build a canonical inner model $K$ for n-Woodin cardinals, so that some form of structural reflection for this $K$ is equivalent to $\Pi^1_{n+1}$-determinacy.

We may say that this conjecture (if true) could represent a case for philosophical realism. In fact, if we prove this conjecture, we will establish equivalencies between embeddings of structures and infinite games. Thus, departing from different points within the mathematical universe, we describe the same mathematical objects. This conjecture seems to suggest that mathematical objects are independent from human mind since by adopting very different theories, we describe the same objects. Even if we adopt different descriptions, we have always the same mathematical objects.

## 5. The theory of $0^\dagger$

Having developed a detailed analysis of the transcendence over L, we consider a canonical formulation of transcendence over inner models of measurability. If $(L[U], \in, U)$ is a $\kappa$-model for some ordinal $\kappa$, then there exists under sufficient assumptions a set $U^\sharp \subseteq \kappa$ analogous to $0^\sharp$ that generates a closed unbounded class of indiscernibles for $(L[U], \in, U, \gamma)_{\gamma \leq \kappa}$ . However, because the $\kappa$-models for various $\kappa$ are merely iterates of each other, one might expect a unifying transcendence principle. In fact, soon after the isolation of $0^\sharp$, Solovay formulated such a principle: the existence of the set of integers $0^\dagger$ (zero-dagger) [Kanamori 09]. The theory of $0^\dagger$ is specular in many aspects to the theory of $0^\sharp$. The idea behind $0^\dagger$ is to develop a canonical theory for structures of form $(L[U], \in, U) \models$ "$U$ is a normal

ultrafilter over $\kappa$", with **two** sets of indiscernibles, one below $\kappa$ and one above, that together generate the structure. For M a structure and X and Y subsets of the domain M so that $X \cup Y$ is linearly ordered by a relation $<$, $(X, Y, <)$ is a double set of indiscernibles for M iff for every formula $\phi(\sigma_1, .....\sigma_{n+s})$ in the language of M; $x_1 < ......... < \ x_n$ and $\gamma_1 < ......... < \ \gamma_n$ all in X; and $y_1 < ....... < \ y_s$ and $\beta_1 < ....... < \ \beta_s$ all in Y,
$M \models \phi(x_1, ......x_n, y_1......y_s)$ *if and only if* $M \models \phi(\gamma_1........\gamma_n, \beta_1.......\beta_s)$. Let the language $L^*$ be the language of set theory, together with a predicate for $U$, augmented by constants $(c_\epsilon|\epsilon \in \omega) \cup (d_\epsilon|\epsilon \in \omega)$.
A remarkable well-founded model is the theory in $L^*$ of some structure $(L_\gamma[U], \in, U, x_\epsilon, y_\epsilon)_{\epsilon\in\omega}$ where $\gamma$ is a limit ordinal greater than $\omega$; for some ordinal $\kappa$, $(L_\gamma[U], \in, U) \models U$ is a normal ultrafilter over $\kappa$; and $(x_\epsilon|\epsilon \in \omega), (y_\epsilon|\epsilon \in \omega)$ is a double set of ordinal indiscernibles for $(L_\gamma[U], \in, U)$ such that for every $\epsilon \in \omega$,

$$x_\epsilon < x_{\epsilon+1} < \kappa < y_\epsilon < y_{\epsilon+1}$$

The canonical Skolem terms $t_\phi$ for $\phi$ a formula of $L^*$ and corresponding Skolem Hulls are equal to those formed for the theory of $0^\sharp$ (that we have examined in the precedent sections). So $0^\dagger$ exists if there is a remarkable well-founded model for inner models of measurability. Now we can state the following fundamental theorem proved by Solovay.

THEOREM 106. *(Solovay) [Kanamori 09] (1)* $0^\dagger$ *exists iff there is a $\kappa$-model for some ordinal $\kappa$ that has un uncountable set of indiscernibles whose minimum element is greater than $\kappa$. Hence, if there is a $\kappa$-model for some $\kappa$ and a Ramsey cardinal greater than $\kappa$, then* $0^\dagger$ *exists.*
*(2)* $0^\dagger$ *exists iff for every uncountable cardinal $\lambda$, there is a $\lambda$-model and a double class (X,Y) of indiscernibles for it such that : $X \subseteq \lambda$ is closed and unbounded, $Y \subseteq On - (\lambda + 1)$ is a closed unbounded class, $X \cup \{\lambda\} \cup Y$ contains every uncountable cardinal and the Skolem hull of $X \cup Y$ in the $\lambda$ model is again in the model.*

Since the theory of $0^\dagger$ is similar to the theory of $0^\sharp$, Solovay obtained the following theorem:

THEOREM 107. *(Solovay)[Kanamori 09]* $0^\dagger$ *is absolute for transitive models of ZF.*

THEOREM 108. *(Solovay) [Kanamori 09] The following are equivalent :*
*(1)* $0^\dagger$ *exists.*
*(2) There is a* $\kappa$*-model for some* $\kappa$ *and an elementary embedding of that model into itself with critical point greater than* $\kappa$*.*

Therefore if $0^\dagger$ exists, $V = L[U]$ fails. Now we are in a position to apply structural reflection to the model $L[U]$.

THEOREM 109. *Let C be the class of structures of the form* $(L[U]_\beta, \in, \gamma)$ *where* $\beta$ *and* $\gamma$ *are cardinals (in V) and* $\gamma < \beta$*. Being* $U$ *a predicate, C is* $\Pi_1$ *definable, with parameter* $U$*. Then SR(L[U]) if and only if* $0^\dagger$ *exists.*

PROOF. Suppose first that $\alpha$ reflects C where $\beta, \gamma$ are cardinals in $V$ and $\gamma < \beta$. Pick cardinals $\gamma$ and $\beta$, with $\gamma$ a cardinal in V, such that $\alpha < \gamma < \beta$. Then there are cardinals $\gamma'$ and $\beta'$, with $\gamma'$ a cardinal in V and $\gamma' < \beta' < \alpha$, and an elementary embedding

$$J : (L[U]_{\beta'}, \in, \gamma') \longrightarrow (L[U]_\beta, \in, \gamma)$$

Since $J(\gamma') = \gamma$, $J$ is not the identity. Let $\kappa$ be the critical point of $J$. Thus, $\kappa \leq \gamma' < \beta'$. Hence by an application of Kunen's theorem to $L[U]$ (see [Kanamori 09] 21.1) $0^\dagger$ exists.
Now suppose that $0^\dagger$ exists. $\kappa$ is a measurable cardinal and $L[U]$ is the inner model for $\kappa$. I is a closed unbounded set of indiscernibles below $\kappa$ and J is a closed unbounded class of indiscernibles above $\kappa$ such that $I \cup J$ contains all uncountable cardinals except $\kappa$. Every set $X \in L[U]$ is definable in $L[U]$ from $I \cup J$ and the elements of $I \cup J$ are indiscernibles

for $L[U]$. The truth value of

$$L[U] \models \phi(\alpha_1, ........., \alpha_n, \beta_1, .......\beta_m)$$

is independent of the choice of $\alpha_1 < ....... < \alpha_n \in I$ and $\beta_1 < ...... < \beta_m \in J$. Every set in $L[U]$ is definable from $I \cup J$. If $a \in L[U]$, there exists an increasing sequence $(\gamma_1, .......\gamma_n)$ of Silver indiscernibles and a formula $\phi$ such that

$$L[U] \models \text{ } a \textit{ is the unique } x \textit{ such that } \phi(x, \gamma_1, ....., \gamma_n).$$

Let $\alpha$ be a limit ordinal in V. We claim that $\alpha$ reflects C ($\Pi_1$ definable class of structures in V). Fix a set of indiscernibles S below $\kappa$ such that S is a proper subset of I, namely $S \subset I$ and $|S| < |\alpha|$. Let $\gamma'$ and $\beta'$ cardinals in V and let $\{\gamma', \beta'\} \cap S \neq \emptyset$. Suppose $(L[U]_\beta, \in, \gamma) \in C$ with $\alpha \leq \beta$. Now let $\gamma'$ and $\beta'$ be cardinals in V such that $\gamma' < \beta' < \alpha$ and $\gamma' \leq \gamma$ . Let $\beta$ and $\gamma$ cardinals in V such that $\{\beta, \gamma\} \cap \{I \cup J\} \neq \emptyset$. Let $J : S \cap (\gamma', \beta') \longrightarrow \{I \cup J\} \cap (\gamma, \beta)$ be order preserving and such that $J(\gamma') = \gamma$ and $J(\beta') = \beta$. Then $J$ generates an elementary embedding

$$J : (L[U]_{\beta'}, \in, \gamma') \longrightarrow (L[U]_\beta, \in, \gamma)$$

as required. □

There is a difference between external structural reflection where we take classes of structures definable in V which, then, are relativized to a canonical inner model and inner structural reflection where we take classes definable within a specific inner model. The following theorem points out to inner model-theoretic structural reflection.

Theorem 110. $0^\dagger$ *exists implies* $SR(D)$*, for all classes of structures of the same type that are definable in* $L[U]$*, being* $U$ *a predicate,* $D$ *is definable, with parameter* $U$*.*

Proof. Suppose that $0^\dagger$ exists. $\kappa$ is a measurable cardinal and $L[U]$ is the inner model for $\kappa$. I is a closed unbounded set of indiscernibles below $\kappa$ and J is a closed unbounded class of indiscernibles above $\kappa$ such that $I \cup J$ contains all uncountable cardinals except $\kappa$. Every set $X \in L[U]$ is definable in $L[U]$ from $I \cup J$ and the elements of $I \cup J$ are indiscernibles for $L[U]$. The truth value of

$$L[U] \models \phi(\alpha_1, ........., \alpha_n, \beta_1, .......\beta_m)$$

is independent of the choice of $\alpha_1 < ....... < \alpha_n \in I$ and $\beta_1 < ...... < \beta_m \in J$. Every set in $L[U]$ is definable from $I \cup J$. If $a \in L[U]$, there exists an increasing sequence $(\gamma_1, .......\gamma_n)$ of Silver indiscernibles and a formula $\phi$ such that

$$L[U] \models \textit{ a is the unique x such that } \phi(x, \gamma_1, ....., \gamma_n).$$

Fix $D$ and a formula $\phi(x)$, possibly with ordinals $\alpha_0, ........, \alpha_m$ as parameters, that defines it in $L[U]$. Let $\sigma$ be a limit of Silver indiscernibles greater than $\alpha_m$ and such that $\sigma$ is correct for $D$, namely, if $A \in V_\sigma$, then $\phi(A)$ holds if and only if $V_\sigma \models \phi(A)$. We claim that $\sigma$ reflects $D$. For suppose $B \in D$. Without loss of generality, $B \notin L[U]_\sigma$. Since $0^\dagger$ holds, there exist two increasing sequences of indiscernibles: $i_0, ........, i_n, i_{n+1} \in I$ and $b_0, ......, b_n, b_{n+1} \in J$ with $\sigma \leq b_n$. Fix a set of indiscernibles S below $\kappa$ such that S is a proper subset of I, namely $S \subset I$ and $|S| < |\sigma|$. Let be that in S there exists an increasing sequence of indiscernibles $j_0, ......., j_n, j_{n+1}$ with $\alpha_m < j_0$. Now there exists a formula $\psi(y, z_0, ....., z_n)$ without parameters, such that

$$B = \{y : L[U]_{b_{n+1}} \models \psi(y, i_0, ......, i_n, ....., b_0, ........, b_n)\}$$

Now pick indiscernibles $j_0, ......j_n, j_{n+1} \in S$ so that $j_0 < ....... < j_n < j_{n+1} < \sigma$ with $\alpha_m < j_0$ and let

$$A = \{y : L_{j_{n+1}} \models \psi(y, j_0, .....j_n)\}$$

Thus $A \in L[U]_\sigma$. We have that $L[U] \models \phi(B)$. That is,

$$L[U] \models \forall x(\forall y(y \in x \leftrightarrow L[U]_{b_{n+1}} \models \psi(y, i_0, ...., i_n, ......, b_0, ..., b_n)) \longrightarrow \phi(x)).$$

By indiscernebility,

$$L[U] \models \forall x(\forall y(y \in x \leftrightarrow L[U]_{j_{n+1}} \models \psi(y, j_0, ....., j_n)) \longrightarrow \phi(x)).$$

which implies $L[U] \models \phi(A)$, i. e., $A \in D$.
let $J : L[U] \longrightarrow L[U]$ be an elementary embedding that sends $b_\sigma$ or $i_\sigma$ to $j_\sigma$, all $\sigma \leq n+1$. Then by indiscernibility, the map $J|A : A \longrightarrow B$ is an elementary embedding. □

The theory of structural reflection applied to $L[U]$ is similar to the theory of structural reflection applied to L. Therefore when we form the model $L[0^\dagger]$ and we apply structural reflection to it, we obtain the following result: $SR(L[0^\dagger])$ if and only if $0^{\dagger\dagger}$ exists. By introducing the operation $Inn^M$ and by interpreting structural reflection as the transcendental successor function (as we have seen before ), we can form the finite structural reflection hierarchy, namely $SR^{<\omega}_{0^\dagger}$. The first step of this hierarchy is $L[U]$. We can formulate the finite structural reflection hierarchy for daggers in the following way, for $n < \omega$:
$(A) SR^0 = Inn^{M,0} = L[U]$.
(B)

$$SR^{n+1} = \begin{cases} SR(L[0^{\dagger n}]) = 0^{\dagger n+1} \\ Inn^{M,n+1}[0^{\dagger n+1}] = L[0^{\dagger n+1}] \end{cases}$$

So we examine the Finite structural reflection hierarchy for daggers, namely $SR^{<\omega}_{0^\dagger}$. We should ask ourselves which is the core model that contains this finite structural reflection hierarchy. Let $K[U]$ be Mitchell's core model for sequences of measures, then we conjecture that the following holds:
(SRHCD) The Finite Structural Reflection Hierarchy for daggers, namely $SR^{<\omega}_{0^\dagger}$, is properly

contained by $K[U]$.

Now we will examine the canonical model $L[E^*]$, the canonical inner model for a strong cardinal. If we apply structural reflection to this inner model, we will produce $0^{\P}$, known in set theory as zero-pistol, which is equivalent to the existence of a non-trivial elementary embedding $j : L[E^*] \longrightarrow L[E^*]$.

## 6. The theory of $L[E^*]$, extender models

The next step above the hierarchy of measurable cardinals is the hierarchy leading to a strong cardinal [4].

DEFINITION 107. *A cardinal $\kappa$ is $\lambda$ -strong if there is an elementary embedding $j : V \longrightarrow M$ such that $\kappa = crit(j)$, $\lambda < j(\kappa)$, and $P^{\lambda}(\kappa) \subseteq M$. A cardinal $\kappa$ is strong if it is $\lambda$-strong for every ordinal $\lambda$.*

A cardinal is 1-strong if and only if it is measurable. An extender is a generalised ultrafilter designed to represent the strong embeddings needed for strong cardinals.

A $(\kappa, \lambda)$ extender corresponds to an elementary embedding $\pi : M \longrightarrow N$ where M and N are transitive models of $ZF^-$, $\kappa = crit(\pi)$, and $\lambda \leq \pi(\kappa)$.

The model M need not be a model of ZF; indeed we can typically assume that $\kappa$ is the largest cardinal in M since $P^M(\kappa)$ is the only part of M which will be used for the ultrapower construction. Extenders are so called because the embedding $\pi$ can be extended to an embedding on a full Model $M'$ of set theory, provided that the subsets of $\kappa$ in $M'$ are contained in those of M.

Suppose that $\pi : M \longrightarrow N$ is an extender and $M'$ is a transitive model of set theory such that $P^{M'}(\kappa) \subseteq P^M(\kappa)$.

If $a, a' \in [\lambda]^{<\omega}$, and $f$and $f'$ are functions in $M'$ with domains $[\kappa]^{|a|}$ and $[\kappa]^{|a'|}$ respectively,

[4]See [Mitchell 11]

then we say that $(f,a) =_\pi (f',a')$ if and only if $(a,a') \in \pi(\{(v,v') \in [\kappa]^{|a|} \times [\kappa]^{|a'|} : f(v) = f'(v)\})$. We write $[f,a]_\pi$ for the equivalence class $\{(f',a') : (f,a) =_\pi (f',a')\}$.
Finally we write $Ult(M',\pi)$ for the model with universe

$$\{[f,a]_\pi : f \in {M'}^{\kappa} \cap M' \ and \ a \ \in \lambda^{<\omega}\}$$

and with the membership relation $\in_\pi$ defined by

$$[f,a]_\pi \in_\pi [f',a']_\pi \ if \ (a,a') \in \pi(\{(v,v') : f(v) \in f'(v')\}$$

The ultrapower embedding $i^\pi : M^1 \longrightarrow Ult(M',\pi)$ is defined by $i^\pi(x) = [x,\emptyset]_\pi$. Here x is regarded as a constant, that is, a 0-ary function [Mitchell 11].
We will only be interested in extenders such that $Ult(M',\pi)$ is well-founded and hence isomorphic to a transitive model, and we will identify $Ult(M',\pi)$ with the transitive model to which is isomorphic. The ordinal $\lambda$ is called the length of the $(\kappa,\lambda)$-extender $\pi$, and it it written $len(\pi)$.

Theorem 111. *Suppose that $\phi(v_0,......,v_{n-1})$ is a formula of set theory, and $a_i \in [\lambda]^{<\omega}$ for $i < n$ and $f_i : [\kappa]^{|a_i|} \longrightarrow \lambda$. Then*

$$Ult(M',\pi) \models \phi([f_0,a_0]_\pi.........[f_{n-1},a_{n-1}]_\pi)$$

*if and only if*

$$(a_0,........a_{n-1}) \in \pi(\{(v_0,......v_{n-1}) : M' \models \phi(f_0(v_0),............f_{n-1}(v_{n-1}))\})$$

This statement suggests the alternate definition of an extender as a sequence E of ultrafilters. The ultrafilter sequence representing a $(\kappa,\lambda)$-extender $\pi$ is the sequence $E^\pi = (E_a : a \in [\lambda]^{<\omega})$ of ultrafilters defined by

$$E_a = \{x \subseteq \kappa^a : a \in \pi(\{ran(v) : v \in x\})\}.$$

Now we can introduce the notion of countable completeness for extenders which is more complicated than that for ultrafilters.

An $(\kappa, \lambda)$-extender E is countably-complete if for each sequence $(a_i : i \in \omega)$ of sets $a_i \in [\lambda]^{<\omega}$ and each sequence $(X_i : \ i < \omega)$ of sets $X_i \in E_a$, there is a function $v : \bigcup_i a_i \longrightarrow \kappa$ such that $v|a_i \in X_i$ for each $i < \omega$.

THEOREM 112. *If $E^*$ is a collection of countably complete extenders then any iterated ultrapower using extenders in $E^*$ is well-founded.*

This completes the preliminary exposition of extenders. The following definition points out to the property of coherence satisfied by a sequence of extenders.

A coherent sequence of nonoverlapping extenders is a function $E^*$ with domain of the form $\{(\kappa, \beta) : \beta < o^{E^*}(\kappa)\}$ (where $o(\kappa)$ is the order of $\kappa$) such that

1) if $o^{E^*}(\kappa) > 0$ then $o^{E^*}(\lambda) < \kappa$ for every $\lambda < \kappa$

and if $\beta < o^{E^*}(\kappa)$ then

2) $E^*(\kappa, \beta)$ is a $(\kappa, \kappa + 1 + \beta)$ extender E

3) $i^{E^*(\kappa,\beta)}(E^*|(\kappa + 1)) = E^*|(\kappa, \beta)$.

The term nonoverlapping refers to clause 1. We will see that nonoverlapping sequences are adequate to construct models with a strong cardinal. Cardinals very much larger than a strong cardinal require extender sequences with overlapping extenders. If $E^*$ is a coherent nonoverlapping sequence of extenders in V and M is a inner model such that the restriction of $E^*$ to M is a member of M, then $E^*$ is coherent in M. Now we need to start with a weaker version of coherence in order to obtain long extender sequences which are coherent in $L[E^*]$.

A sequence $E^*$ of extenders is weakly coherent if each extender $E = (\kappa, \beta)$ is a $(\kappa, \kappa+1+\beta)$ extender such that $o^{i^E E^*}(\kappa) = \beta$.

At this point I will not discuss the part about the comparison of iterations.

THEOREM 113 (Mitchell 11). *Suppose that* $E^*$ *is a weakly coherent extender sequence and that* $E$ *is a countably complete* $(\kappa, \kappa+1, \beta)$*-extender in* $L[E^*]$ *such that* $o^{i^E E^*}(\kappa) = \beta$. *Then* $E = E^*(\kappa, \beta)$

We continue with three important theorems.

THEOREM 114 (Mitchell 11). *If* $E^*$ *is a weakly coherent extender sequence of countably complete extenders, then* $E^*$ *is coherent in* $L[E^*]$.

THEOREM 115 (Mitchell 11). *If* $\kappa$ *is a strong cardinal, then there is a weakly coherent sequence* $E^*$ *of countably complete extenders such that there is a strong cardinal* $\kappa^1 \leq \kappa$ *in* $L[E^*]$.

THEOREM 116 (Mitchell 11). *If* $E^*$ *is a coherent sequence of countably complete extenders in* $L[E^*]$ *then* $L[E^*] \models GCH$.

Now we can introduce $0^{\P}$ the sharp for the inner model $L[E^*]$. This sharp (if it exists) implies a transcendence over the inner model containing a strong cardinal. In fact, if $0^{\P}$ exists, we have the following non-trivial elementary embedding:

$$L[E^*] \longrightarrow L[E^*].$$

Since $0^{\P}$ has similar properties to $0^{\dagger}$, I make the following conjecture (SRS). When we relativize $\Pi_1$ definable classes (with parameters) of structures to $L[E^*]$ we may obtain the following:

(SRS conjecture) $SR(L[E^*])$ if and only if $0^{\P}$ exists.

Since the theory of $0^{\P}$ is similar to the theory of $0^{\dagger}$ I conjecture that we can form the model $L[0^{\P}]$ and apply structural reflection to this inner model in order to obtain $0^{\P\P}$. Like for $0^{\dagger}$ we can continue. Like for the theory of $0^{\sharp}$ and $0^{\dagger}$, I conjecture that by starting with $0^{\P}$ we can construct the Finite Structural Reflection Hierarchy for zero pistols. In fact, We can form a finite transcendental hierarchy of inner models with these sharps by adopting the operation $Inn^M$ and structural reflection at successor stage.

## 7. The HOD conjecture and the Wholeness axioms

Jensen's covering lemma says that if $0^\sharp$ does not exist and A is an uncountable set of ordinals, then there exists $B \in L$ such that $A \subseteq B$ and $|A| = |B|$. The conclusion implies that if $\gamma$ is a singular cardinal, then it is a singular cardinal in L. Moreover, if $\beta$ is a singular cardinal, then $(\beta^+)^L = \beta^+$. Jensen covering lemma implies that L is close to V. In contrast, if $0^\sharp$ exists and $\beta$ is an uncountable cardinal, then $\beta$ is inaccessible in L. In this case, L is very far from V. Thus, the covering lemma implies the following theorem that does not mention $0^\sharp$:

THEOREM 117. *(Jensen) Exactly one of the following holds:*
*(1) L is correct about singular cardinals and computes their successors correctly.*
*(2) Every uncountable cardinal is inaccessible in L.*

Canonical inner models other than L have been defined and proved to satisfy similar covering properties and corresponding dichotomies. Canonical inner models are contained in HOD.

DEFINITION 108. *A set X is ordinal-definable if there is a formula $\phi$ such that*

$$X = \{u : \phi(u, \alpha_1, ......, \alpha_n)\}$$

*for some ordinal numbers $\alpha_1, ......, \alpha_n$.*

OD is the class of ordinal definable sets. HOD is the class of hereditarily ordinal-definable sets.

DEFINITION 109. *HOD denotes the class of hereditarily ordinal-definable sets:*

$$HOD = \{x : \ Transitive\ Closure(\{x\}) \subset OD\}$$

*The class HOD is transitive and contains all ordinals.*

THEOREM 118 (Jech 06). *HOD is a transitive model of ZFC.*

The following theorem highlights the fact that either HOD is close to V or HOD is far form V.

THEOREM 119 (Woodin 12). *Assume that $\delta$ is an extendible cardinal. Then exactly one of the following hold:*
*(1) For every singular cardinal $\gamma > \delta$, $\gamma$ is singular in HOD and $(\gamma^+)^{HOD} = \gamma^+$.*
*(2) Every regular cardinal greater than $\delta$ is measurable in $HOD$.*

The above theorem states the HOD dichotomy without mentioning an anologue of $0^\sharp$ for HOD. In fact, since no analogue of $0^\sharp$ is mentioned for HOD and we cannot transcend HOD as in the case of L, we may conjecture that (2) of HOD dichotomy fails. Before introducing the HOD conjecture, we have to define the notion of $\omega$-strongly measurable cardinals:

DEFINITION 110. *(Woodin) Let $\lambda$ be an uncountable regular cardinal. Then $\lambda$ is $\omega$-strongly measurable in HOD iff there is a $\kappa < \lambda$ such that:*
*(1) $(2^\kappa)^{HOD} < \lambda$ and*
*(2) There is no partition $(S_\alpha | \alpha < \kappa)$ of $cof(\omega) \cap \lambda$ into stationary sets such that $(S_\alpha | \alpha < \kappa) \in HOD$.*

We state the HOD conjecture [Woodin 12]:

DEFINITION 111. *(Woodin) (HOD conjecture) There is a proper class of regular cardinals that are not $\omega$-strongly measurable in HOD.*

Building a canonical inner model with a supercompact cardinal is a major problem for set theory (as we will see in the next section). For a canonical inner model of a supercompact cardinal we have to use weak extender models [Woodin 12]:

DEFINITION 112. *(Woodin) A transitive class N model of ZFC is called a weak extender model for $\delta$ supercompact iff for every $\gamma > \delta$ there exists a normal fine measure $U$ on $P_\delta(\gamma)$ such that:*

*(1) $N \cap P_\delta(\gamma) \in U$ and*

*(2) $U \cap N \in N$.*

We conclude this brief section with the following theorem:

THEOREM 120 (Woodin 12). *Let $\delta$ be an extendible cardinal. The following are equivalent:*

*(1) The HOD conjecture.*

*(2) HOD is a weak extender model for $\delta$ supercompact.*

*(3) Every singular cardinal $\gamma > \delta$, is singular in HOD and $\gamma^+ = (\gamma^+)^{HOD}$.*

If $\delta$ is an extendible cardinal, then no non-trivial elementary embedding maps a weak extender model for $\delta$ supercompact to itself. For this we recall Kunen's theorem:

THEOREM 121. *(Kunen) [Jech 06] Let $\kappa$ be an ordinal. Then there is no non-trivial elementary embedding*

$$j : V_{\kappa+2} \longrightarrow V_{\kappa+2}$$

We state now Woodin's theorem:

THEOREM 122 (Woodin 12). *If N is a weak extender model for $\delta$ supercompact, then there is no elementary embedding $j : N \longrightarrow N$ with $\delta \leq crit(j)$ and $j \neq id$.*

We will now introduce the Wholeness axioms proposed by Paul Corazza [Corazza 00] and [Hamkins 99]. They are weakenings of Kunen's theorem in order to avoid inconsistency and to have a non trivial embedding of V into itself. The Wholeness axioms are formalized in the language $\{\in, j\}$, augmenting the usual language of set theory $\{\in\}$ with an additional unary function symbol $j$ to represent the embedding. ZFC is expressed in the smaller language $\{\in\}$. Corazza's first Wholeness axiom, namely $WA_0$, asserts that $j$ is a non-trivial amenable elementary embedding from the universe $V$ to itself. Elementarity is expressed by the scheme $\phi(x) \longrightarrow \phi(j(x))$, where $\phi$ runs through the formulas of the usual set theory. We can state non-triviality by the following formula $\exists x j(x) \neq x$. Furthermore, amenability is simply the assertion that $j$ restricted to a set B is a set for every set B. Corazza [Corazza 00] formulates also the version of the Wholeness axiom, namely $WA_\infty$, which asserts in addition that the full separation axiom holds in the language $\{\in, j\}$. This axiom is the endpoint of a hierarchy of axioms, namely $WA_0, WA_1, WA_2, .........., WA_\infty$ which represent the Wholeness axioms. Now, we can define the Wholeness axiom $WA_n$:

DEFINITION 113. *The Wholeness Axiom $WA_n$ consists of the following formulas:*
*(1) Elementarity: All instances of $\phi(x) \leftrightarrow \phi(j(x))$ for $\phi$ in the language $\{\in\}$.*
*(2) Separation: all instances of the Separation axiom for $\Sigma_n$ formulae in the full language $\{\in, j\}$.*
*(3) Non-triviality: The axiom $\exists x j(x) \neq x$.*

Kunen's theorem does not apply because the Wholeness axioms schemes do not have instances of the axiom of replacement in the full language with $j$. In fact, Kunen uses the Replacement Axiom in the full language to know the the critical sequence $\{\kappa_n | n \in \omega\}$, defined by $\kappa_0 = \kappa = cp(j)$ and $\kappa_{n+1} = j(\kappa_n)$, is a set. Now we can state a fundamental theorem:

THEOREM 123 (Corazza 00). *If there is an $I_1$ embedding $j : V_{\lambda+1} \longrightarrow V_{\lambda+1}$, then $WA_\infty$ is consistent with $HOD = V$. Hence also $WA_0$ is consistent with $HOD = V$.*

So, as a corollary, the Wholeness axioms, whose upper bound on consistency strength is $I_3$ hypothesis (there is a nontrivial elementary embedding of $V_\lambda$ into itself) are consistent with the Ultimate L conjecture, namely a consistent Ultimate enlargement of Gödel's constructible universe.

## 8. The Ultimate L model

This section aims at explaining briefly the ideas underlying Woodin's construction of an inner model for a supercompact cardinal (not yet constructed). This program is based on Woodin's results from [Woodin 10]. One of the main motivation for the search of an ultimate consistent enlargement of L is the validation of the Ω-conjecture. Woodin [Woodin 10], in order to find an inner model of a supercompact cardinal, adopts the concept of long extenders. Suppose that

$$j : V \longrightarrow M$$

is an elementary embedding with critical point $\kappa$. Suppose that $\eta$ is an ordinal, $\eta > \kappa$, and let $\eta^*$ be the least ordinal such that $\eta \leq j(\eta^*)$. From $j$ one can define the extender of length $\eta$. If the ordinal $\eta^*$ is greater than the critical point $\kappa$, then E is a long extender. Recall that the formal definition of the extender E specifies a family of ultrafilters. For each finite set $s \subseteq \eta$ let

$$E_s = \{A \subseteq [\eta^*]^{|s|} | s \in j(A)\}.$$

Thus $E_s$ is an ultrafilter. The set

$$E = \{(s, A) | s \in [\eta]^{<\omega} \textit{ and } A \in E_s\}$$

is the extender of length $\eta$ derived by $j$, it is also the $(\kappa, \eta)$-extender derived from $j$.

Definition 114. *Suppose that E is an extender.*

*(1) CRT(E) is the critical point of the elementary embedding*

$$j_E : V \longrightarrow M_E$$

*given by E.*

*(2) LTH (E) is the length of the extender E.*

*(3) For each* $\alpha < LTH(E)$ *let* $SPT(E;\alpha)$ *be the least ordinal* $\beta$ *such that* $j_E(\beta) > \alpha$ *and let* $SPT(E) = sup\{SPT(E;\alpha)|\alpha < LTH(E)\}$.

*(4)* $\rho(E) = sup\{\eta|V_\eta \subseteq M_E\}$.

Woodin [Woodin 10] explains that $CRT(E)$ is the completeness of the ultrafilters associated to extender, E. $LTH(E)$ is the domain of E. $SPT(E)$ is the space of an extender. Then Woodin [Woodin 10] defines a premouse as follows:

Definition 115. *a premouse is a pair* $(M, \delta)$ *such that:*

*(1)* $M \models ZF + \Sigma_2 - replacement$.

*(2) Suppose that* $F : M_\delta \longrightarrow M \cap Ord$ *is definable from parameters in M, then F is bounded in M.*

*(3)* $\delta$ *is strongly inaccessible in M.*

We give next the definition of iteration tree:

Definition 116. *Suppose that* $(M, \delta)$ *is a premouse. An iteration tree, T, on* $(M, \delta)$ *of length* $\eta$ *is a tree order* $<_T$ *on* $\eta$ *with minimum element 0 and which is a suborder of the standard order, together with a sequence*

$$(M_\alpha, E_\beta, j_{\gamma,\alpha} : \alpha < \eta, \beta + 1 < \eta, \gamma <_T \alpha)$$

*such that the following hold.*

*(1)* $M_0 = M$.

*(2)* $j_{\gamma,\alpha} : M_\gamma \longrightarrow M_\alpha$ *for all* $\gamma <_T \alpha < \eta$.

*(3) Suppose that* $\alpha + 1 < \eta$. *Then* $\alpha + 1$ *has an immediate predecessor,* $\alpha^*$, *in the tree order* $<_T$ *and:*

*(a)* $E_\alpha \in j_{0,\alpha}(M \cap V_\delta)$ *and* $M_\alpha \models E_\alpha$ *is an extender model which is not* $\omega - huge$.

*(b) If* $\alpha^* < \alpha$ *then* $SPT(E_\alpha) + 1 \leq min\{\rho(E_\beta) | \alpha^* \leq \beta < \alpha\}$*.*

*(c)* $M_{\alpha+1} = Ult(M_{\alpha^*}, E_\alpha)$ *and*

$$j_{\alpha^*,\alpha+1} : M_{\alpha^*} \longrightarrow M_{\alpha+1}$$

*is the associated embedding.*

*(4) If* $0 < \beta < \eta$ *is a limit ordinal then the set of* $\alpha$ *such that* $\alpha < \beta$ *is cofinal in* $\beta$ *and* $M_\beta$ *is the limit of the* $M_\alpha$ *where* $\alpha <_T \beta$ *relative to the embeddings;* $j_{\alpha,\beta}$*.*

Hugh Woodin [Woodin 10] wants to generalize the notion of iteration tree for the case of long extenders. We need a suitable generalization since the most natural generalization leads to the failure of iterability. Then we come to the definition of $(+\Theta)$-iteration tree where $\Theta \in Ord$:

DEFINITION 117 (Woodin 10). *Suppose that* $(M, \delta)$ *is a premouse and that* $T$ *is an iteration tree on* $(M, \delta)$ *with associated sequence,*

$$(M_\alpha, E_\beta, j_{\gamma,\alpha} : \alpha < \eta, \beta + 1 < \eta, \gamma <_T \alpha).$$

*Suppose that* $\Theta \in Ord$*. Then the iteration tree,* $T$*, is a* $(+\Theta)$*-iteration tree if for all* $\alpha + 1 < \eta$*,*

$$sup\{SPT(E_\beta) | \alpha + 1 \leq \beta \text{ and } \beta^* \leq \alpha\} + \Theta \leq \rho(E_\alpha)$$

*where* $\beta + 1 < \eta$*,* $\beta^*$ *is the* $T$ *predecessor of* $\beta + 1$*.*

Woodin [Woodin 10] is able to adapt the proof of the following theorem to iteration trees of length $\alpha$. Firstly, he introduces the following definition.

DEFINITION 118. *Suppose that* $(M, \delta)$ *is a premouse,*

$$\pi : M \longrightarrow V_\Theta$$

*is an elementary embedding, $T$ is an iteration tree on $(M, \delta)$, and $b$ is a maximal branch of $T$. Let $M_b$ be the direct limit given by $b$ and let*

$$j_b : M \longrightarrow M_b$$

*be the associated embedding. The branch $b$ is $\pi$-realizable if there exists an elementary embedding,*

$$\pi_b : M_b \longrightarrow V_\Theta$$

*such that $\pi = \pi_b . j_b$.*

Now we can introduce a fundamental theorem that Woodin [Woodin 10] is able to prove for iteration trees of length $\alpha$ where $\alpha \in Ord$:

THEOREM 124 (Woodin 10). *Suppose that $(M, \delta)$ is a countable premouse,*

$$\pi : M \longrightarrow V_\Theta$$

*is an elementary embedding,*

$$T = (M_\alpha, E_\beta, j_{\gamma,\alpha} : \alpha < \eta, \beta + 1 < \eta, \gamma <_T \alpha)$$

*is a countable $(+2)$-iteration tree on $(M, \delta)$ and that $T$ has no proper maximal $\pi$-realizable branch. Then $\eta = \gamma + 1$ and for all extenders $E \in M_\gamma \cap V_{j_{0,\gamma}(\delta)}$, for all $\gamma^* \leq \gamma$, if $\gamma^* < \gamma$ and*

$$SPT(E) + 2 \leq min\{\rho(E_\alpha) | \gamma^* \leq \alpha < \gamma\},$$

*then $Ult(M_{\gamma^*}, E)$ is well-founded and moreover the corresponding maximal branch of the induced iteration tree of length $\gamma + 2$ is $\pi$-realizable.*

It is possible to prove the precedent theorem for iterated trees of finite length. Steel [Woodin 10] proved the theorem for iteration trees of length $\omega$. Woodin [Woodin 10] is able to prove the theorem for iteration trees of length $\alpha$, where $\alpha$ is any ordinal. Martin and Steel [Woodin 10] proposed two hypotheses concerning iteration trees on V.

(UBH) The Unique Branch Hypothesis:

Suppose that $T$ is an iteration tree on a premouse $(V_\Theta, \delta)$. Then T does not have two distinct cofinal well-founded branches.

(CBH) The Cofinal Branch Hypothesis:

Suppose that T is an iteration tree on a premouse $(V_\Theta, \delta)$, then:

(1) If T has a limit length then T has a cofinal branch;

(2) If T has a successor length, $\eta + 1$, then T can be freely extended to an iteration tree of length $\eta + 2$.

Unfortunately if there is a supercompact cardinal then these hypotheses are false. So, Woodin [Woodin 10] formulates other three hypotheses. Firstly, we introduce the following definition:

Definition 119. *An iteration tree, T, is strongly closed if:*

*(1) T is a (+ 1)-iteration tree; and*

*(2) each extender, E, occurring in T is LTH(E)-strong in the model from which it is selected and LTH(E) is strongly inaccessible in that model.*

The first hypothesis is the following:

Definition 120. *(Strong $(\omega_1 + 1)$-Iteration Hypothesis) [Woodin 10] Suppose that $(M, \delta)$ is a countable premouse and that*

$$\pi : M \longrightarrow V_\Theta$$

*is an elementary embedding. Then $(M, \delta)$ has an iteration strategy of order $\omega_1 + 1$ for strongly closed iteration trees on $(M, \delta)$.*

Definition 121. *(Strong Iteration Hypothesis) [Woodin 10] Suppose that* $(M,\delta)$ *is a premouse,* $\kappa < \delta$*, and that*

$$\pi : M \longrightarrow V_\Theta$$

*is an elementary embedding such that there is a strong cardinal below* $\pi(\kappa)$*. Suppose that there is a proper class of Woodin cardinals. Then* $(M,\delta)$ *has an iteration strategy of order* $\omega_1$ *which is universally Baire in the codes, for strongly closed iteration trees with all critical points above* $\kappa$*.*

Definition 122. *(Strong Unique Branch Hypothesis) [Woodin 10] Suppose that* $(V_\Theta,\delta)$ *is a premouse such that* $T$ *is a countably strongly closed iteration tree on* $(V_\Theta,\delta)$ *of limit length. Then* $T$ *has at most one cofinal well-founded branch.*

By assuming this strong hypothesis, we have the following:

Theorem 125 (Woodin 10). *Suppose that* $(V_\Theta,\delta)$ *is a premouse and that Strong Unique Branch Hypothesis holds.*
*(1) Suppose that* $T$ *is a countable strongly closed iteration tree on* $(V_\Theta,\delta)$ *of limit length. Then* $T$ *has a cofinal well-founded branch.*
*(2) Suppose that*

$$T = (M_\alpha, E_\beta, j_{\gamma,\alpha} : \alpha < \eta+1, \beta+1 < \eta+1, \gamma <_T \alpha)$$

*is a countably strongly closed iteration tree on* $(V_\Theta,\delta)$*. Suppose that* $\eta^* < \eta$ *and that*

$$SPT(E_\eta) + 1 \leq min\{j_{E_\beta}(CRT(E_\beta)) | \eta^* \leq \beta < \eta\}.$$

*then* $Ult(M_{\eta^*}, E_\eta)$ *is well-founded.*

We have other two theorems regarding the Strong Unique Branch Hypothesis:

THEOREM 126 (Woodin 10). *Suppose that the Strong Unique Branch Hypothesis holds and that $\delta_0$ is a supercompact cardinal. Then the Strong Unique Branch Hypothesis holds at all strong cardinals $\delta \geq \delta_0$.*

THEOREM 127 (Woodin 10). *Suppose that $\delta_0$ is supercompact and that the Strong Unique Branch Hypothesis holds. Suppose that $(V_\Theta, \kappa)$ is a premouse with $\delta_0 < \kappa$. Then for each ordinal $\gamma$ there is an iteration strategy for $(V_\Theta, \kappa)$ of order $\gamma$ restricting to iteration trees with all critical points above $\delta_0$.*

Now we present a fundamental theorem that was Woodin's original motivation for the search of the Ultimate L model.

THEOREM 128 (Woodin 10). *Suppose that there is a proper class of Woodin cardinals, there is a strong cardinal, and that the Strong Iteration Hypothesis holds. Then the $\Omega$-conjecture holds.*

Now we will examine the closure properties of a weak extender model N (that we have defined in the precedent section) for a supercompact cardinal. We start with the following definition:

DEFINITION 123. *Suppose that $\Phi$ is a class.*
*(1) $o^{\Phi}_{mLONG}(\delta) = \infty$ if for all $\gamma > \delta$ there exists an extender $E \in \Phi$ such that*
*(a) $SPT(E) < \delta$ and $\rho(E) > \gamma$,*
*(b) $j_E((CRT(E)) = \delta$.*
*(2) $o^{\Phi}_{sLONG} = \infty$ if for all $\gamma > \delta$ there exists an extender $E \in \Phi$ such that*
*(a) $CRT(E) = \delta$,*
*(b) $SPT(E) > \gamma$.*

We have that $o^{V}_{mLONG} = \infty$ if and only if $o^{V}_{sLONG} = \infty$. At this point we state two theorems that witness the closure properties of a weak extender model N for a supercompact cardinal:

THEOREM 129 (Woodin 10). *Suppose that* $o^N_{LONG}(\delta) = \infty$. *Suppose that* $\gamma > \delta$ *and* $\gamma$ *is a cardinal of* $N$. *Suppose that*

$$j : (H(\gamma^+))^N \longrightarrow (H(j(\gamma)^+))^N$$

*is an elementary embedding with critical point* $\kappa \geq \delta$. *Then* $j \in N$.

THEOREM 130 (Woodin 10). *Suppose that* $o^N_{LONG}(\delta) = \infty$. *Suppose that* $\gamma \in Ord$,

$$j : N \cap V_{\gamma+1} \longrightarrow N \cap V_{j(\gamma)+1}$$

*is an elementary embedding with critical point* $\kappa \geq \delta$. *Then* $j \in N$.

Now we can define the suitable extender model:

DEFINITION 124. *Suppose that* $\mathbb{M}$ *is a transitive class such that for some* $\delta$, $o^{\mathbb{M}}_{LONG} = \infty$.

*(1)* $\delta_{\mathbb{M}}$ *denotes the least* $\kappa \leq \delta$ *such that* $o^{\mathbb{M}}_{LONG}(\kappa) = \infty$.

*(2)* $\mathbb{M}$ *is a suitable extender model if the following hold:*

*(a) There exists a cofinal set* $I_{\mathbb{M}} \subset \delta_{\mathbb{M}}$ *and a sequence* $(E_\alpha : \alpha \in I_{\mathbb{M}})$ *in* $V_{\delta_{\mathbb{M}}}$ *witnessing that* $\delta_{\mathbb{M}}$ *is a Woodin cardinal (in V) such that*

$$(E_\alpha \cap \mathbb{M} : \alpha \in I_{\mathbb{M}}) \in \mathbb{M}$$

*and such that for all* $\alpha \in I_{\mathbb{M}}$,
$j_{E_\alpha}((E_\beta : \beta \in CRT(E_\alpha) \cap I_{\mathbb{M}}))|LTH(E_\alpha) = (E_\beta : \beta \in LTH(E_\alpha) \cap I_{\mathbb{M}},\ \rho(E_\alpha) = LTH(E_\alpha) = \alpha$, *and such that* $\alpha = CRT(E_\beta)$ *for some* $\beta \in I_{\mathbb{M}}$.

*(b) (Weak* $\Sigma_2$*-definability) There exists* $X \in V_{\delta_{\mathbb{M}+1}}$ *and a formula* $\phi(x_0, x_1)$ *such that for all* $\beta < \eta_1 < \eta_2 < \eta_3$, *if* $X \in V_\beta$ *and if*

$$(\mathbb{M})^{V_{\eta_1}} \cap V_\beta = (\mathbb{M})^{V_{\eta_3}} \cap V_\beta$$

*then*

$$(\mathbb{M})^{V_{\eta_1}} \cap V_\beta = (\mathbb{M})^{V_{\eta_2}} \cap V_\beta = (\mathbb{M})^{V_{\eta_3}} \cap V_\beta,$$

*where for all* $\gamma > \delta_{\mathbb{M}}$,

$$(\mathbb{M})^{V_\gamma} = \{a \in V_\gamma | V_\gamma \models \phi(a, X)\}$$

The following theorems point out to the closure properties of a suitable extender model and show that it is possible to transfer down from V to $\mathbb{M}$ very large cardinal notions:

THEOREM 131 (Woodin 10). *Suppose that* $\mathbb{M}$ *is a suitable extender model and*

$$j : V_\lambda \longrightarrow V_\lambda$$

*is an elementary embedding such that* $\delta_{\mathbb{M}} < crit(j)$ *and such that* $V_\lambda \prec_{\Sigma_2} V$. *Then* $j(\mathbb{M} \cap V_\lambda) = \mathbb{M} \cap V_\lambda$ *and for all* $\gamma < \lambda$,

$$j(\mathbb{M} \cap V_\gamma) \in \mathbb{M}.$$

THEOREM 132 (Woodin 10). *Suppose* $\mathbb{M}$ *is a suitable extender model and*

$$j : V_\lambda \longrightarrow V_\lambda$$

*is an elementary embedding such that* $\delta_{\mathbb{M}} < CRT(j)$ *and such that* $V_\lambda \prec_{\Sigma_2} V$. *then there exists* $\lambda' \leq \lambda$. *and a nontrivial elementary embedding*

$$j' : \mathbb{M} \cap V_{\lambda'} \longrightarrow \mathbb{M} \cap V_{\lambda'}$$

*such that* $j' \in \mathbb{M}$.

THEOREM 133 (Woodin 10). *Suppose that* $2 < n < \omega$, $\mathbb{M}$ *is a suitable extender model, and*

$$j : V_\lambda \longrightarrow V_\lambda$$

*is an elementary embedding such that* $\delta_{\mathbb{M}} < CRIT(j)$ *and such that* $V_\lambda \prec_{\Sigma_n} V$. *Then there exists* $\lambda' \leq \lambda$ *and a non trivial embedding*

$$j' : \mathbb{M} \cap V_{\lambda'} \longrightarrow \mathbb{M} \cap V_{\lambda'}$$

*such that* $\mathbb{M} \cap V_{\lambda'} \prec_{\Sigma_n} \mathbb{M}$ *and such* $j' \in \mathbb{M}$.

Now I apply structural reflection to the Ultimate L model. f we relativize structural reflection to a weak extender model, N, for a supercompact cardinal, we do not get transcendence over this inner model, but all embeddings of structures are within this inner model. Firstly, we restate the following theorem:

THEOREM 134 (Woodin 10). *Suppose that* $o^N_{Long} = \infty$. *Suppose that* $\gamma \in Ord$,

$$j : N \cap V_{\gamma+1} \longrightarrow N \cap V_{j(\gamma)+1}$$

*is an elementary embedding with critical point* $\kappa \geq \delta$. *Then* $j \in N$ .

We may state the theorem that witnesses the closure properties of a weak extender model for a supercompact cardinal.

THEOREM 135. *Suppose* $o^N_{Long} = \infty$, *N is a weak extender model for* $\delta$ *supercompact, N is definable and C is a class of structures* $\Pi_1$ *definable (with parameters) in V. Then all embeddings of classes of structures relativized to N belong to N.*

PROOF. Let N be a weak extender model for $\delta$ supercompact. Let C be the class of structures of the form $(N_\beta, \in, \gamma)$, where $\gamma$ and $\beta$ are cardinals (in V) and $\gamma < \beta$. Suppose that $\alpha$ reflects C. Pick cardinals $\gamma$ and $\beta$, with $\gamma$ a cardinal in V, such that $\alpha < \gamma < \beta$. Then there are cardinals $\gamma'$ *and* $\beta'$, with $\gamma'$ a cardinal in V and $\gamma' < \beta' < \alpha$, and an elementary embedding :

$$j : (N_{\beta'}, \in, \gamma') \longrightarrow (N_\beta, \in, \gamma)$$

Since $j(\gamma') = \gamma$, $j$ is not the identity. Let $\kappa$ be the critical point of $j$. If $\kappa < \delta$ then $j \in N$ and if $\kappa \geq \delta$ for Woodin's theorem [Woodin 10] $j \in N$. Thus, all embeddings of classes of structures within N belong to N. □

I argue that principles of structural reflection transfer down from V to a suitable extender model $\mathbb{M}$.

DEFINITION 125 (Bagaria 10). *A $C^{(n)}$-extendible cardinal $\kappa$ is $C^{(n)}$-extendible if for every $\lambda$ greater than $\kappa$ there exists an elementary embedding*

$$j : V_\lambda \longrightarrow V_\mu$$

*some $\mu$, crit(j)=$\kappa$, and $V_{j(\kappa)}$ is a $\Sigma_n$-elementary substructure of V.*

We can state Bagaria's theorem:

THEOREM 136 (Bagaria 10). *The following are equivalent:*

*(1) SR, i. e, $\boldsymbol{\Sigma_n}$-SR for all n.*

*(2) There exists a $C^{(n)}$-cardinal, for every n.*

*(3) Vopěnka's principle.*

Since Hugh Woodin [Woodin 10], by assuming that the Ultimate L exists, is able to transfer down form V to a suitable extender model very large cardinal notions, we should be able to transfer down from V to $\mathbb{M}$ a proper class of $C^{(n)}$-extendible cardinals (weaker large cardinals than what Woodin is able to transfer down). We restate Woodin's theorem that implies that stronger large cardinals numbers than $C^{(n)}$-extendible cardinals transfer down from V to $\mathbb{M}$.

THEOREM 137 (Woodin 10). *Suppose $2 < n < \omega$, $\mathbb{M}$ is a suitable extender model, and*

$$j : V_\lambda \longrightarrow V_\lambda$$

*is an elementary embedding such that* $\delta_{\mathbb{M}}$*-supercompact* $<$ *crit(j) and such that* $V_\lambda \prec_{\Sigma_n} V$*. Then, there exists a* $\lambda' \leq \lambda$ *and a nontrivial elementary embedding*

$$j' : \mathbb{M} \cap V_{\lambda'} \longrightarrow \mathbb{M} \cap V_{\lambda'}$$

*such that* $\mathbb{M} \cap V_{\lambda'} \prec_{\Sigma_n} \mathbb{M}$ *and such that* $j' \in \mathbb{M}$*.*

Woodin is able to transfer down this very large cardinal numbers so we have to readapt his proof to transfer a proper class of $C^{(n)}$-extendible cardinals down from V to $\mathbb{M}$.

THEOREM 138. *Assume that for every n, there exists a* $C^{(n)}$*-extendible cardinal in V (equivalent to: for every n, there exists a proper class of* $C^{(n)}$*-extendible cardinals). Then in* $\mathbb{M}$*, for every n there exists a* $C^{(n)}$*-extendible cardinal.*

PROOF. Suppose $\kappa$ is $C^{(n)}$-extendible. So for every $\lambda$ greater than $\kappa$ there exists an elementary embedding

$$j : V_\lambda \longrightarrow V_\mu$$

some $\mu$ and $V_{j(\kappa)} \prec_{\Sigma_n} V$. Assume $\delta_{\mathbb{M}}$ is a supercompact cardinal. Then we argue that there exists a $\rho \leq j(\kappa)$ and a non trivial embedding

$$j' : \mathbb{M} \cap V_\lambda \longrightarrow \mathbb{M} \cap V_\mu$$

some $\mu$ such that $\mathbb{M} \cap V_\rho \prec_{\Sigma_n} \mathbb{M}$ and $j' \in \mathbb{M}$. Fix $X \in V_{\delta_{\mathbb{M}}+1}$ and a formula $\phi(x_0, x_1)$ such that $\mathbb{M}$ is weakly $\Sigma_2$ definable in V from X. We have that $V_{j(\kappa)} \prec_{\Sigma_n} V$ and that $V_{j(\kappa)} \models ZFC$. So assuming $n \geq 2$,

$$\mathbb{M} \cap V_{j(\kappa)} = \{a \in V_{j(\kappa)} | V_{j(\kappa)} \models \phi[a, X]\}.$$

Let I be the set of all $\rho < j(\kappa)$ such that

$$V_\rho \prec V_{j(\kappa)}$$

and such that $X \in V_\rho$. Then I is cofinal in $j(\kappa)$ and $j(I) = I$. Note that for each $\rho \in I$,

$$\mathbb{M} \cap V_\rho \prec \mathbb{M} \cap V_{j(\kappa)}$$

and so for each $\rho \in I$,

$$\mathbb{M} \cap V_\rho \prec_{\Sigma_n} \mathbb{M}.$$

The theorem follows by absoluteness. But then
$\forall \alpha < \rho(\mathbb{M}, \in) \models \exists \rho > \alpha(\rho$ is a $C^{(n)}$-extendible cardinal),
then $(\mathbb{M} \cap \rho, \in) \models \forall \alpha \exists \lambda > \alpha(\lambda$ is a $C^{(n)}$-extendible cardinal),
then $(\mathbb{M}, \in) \models$ there is a proper class of $C^{(n)}$-extendible cardinal. $\square$

Since principles of structural reflection hold within a suitable extender model, structural reflection witnesses that the Ultimate L (if the ultimate L conjecture is true) is very close to V. Principles of structural reflection that hold in V hold also within the Ultimate L if the Ultimate L conjecture is true. Thus, the Ultimate L can be considered as the true, noumenal universe of mathematics as I will explain in the following section.

## 9. The philosophy of mathematics that I sustain

I argue that we have to distinguish within set theory between the phenomenal meta-mathematical models and the true noumenal universe of mathematics. Further, we have to distinguish between the mathematics of models concerning the phenomenal reality of set theory and the mathematics concerning the true noumenal universe of sets. To understand this we have to apply a metaphysical Kantian distinction to set theory. Thus, to express my philosophical position I have to apply a Kantian distinction to set theory between phenomenal reality and noumenal reality. Kantian noumenon is a posited object or reality that is known (if at all) without the use of physical senses. The term *noumenon* is used in relation with the term *phenomenon* which refers to an object apprehended by physical senses. The noumenal world may exist but it is completely unknowable to humans. The noumenal reality is the reality in itself or thing-in-itself. As expressed in Kant's Critique of Pure Reason [Kant 781], Human understanding is structured by innate categories of understanding that the mind uses in order to make sense of raw unstructured experience (the phenomenal interpretation of reality). For Kant, we can categorize phenomena, but we can never directly know noumena. Even if noumena are unknowable, they are still needed as a limiting concept. The existence of the noumenal world limits reason to what he perceives to be its proper bounds, making many metaphysical questions unanswerable by reason. For Kant, the phenomenal reality based on physical senses' apprehension structured, then, by categories of understanding is the realm of appearance and it is not what it is really (the reality in itself). While the noumenal reality is what it is really. I argue that in set theory the phenomenal reality is created by human mind and is represented by metamathematical models such as $L[U], K^{DJ}, V[G]$, etc. Thus, metamathematical models are created or constructed by Human mind according to my beliefs. Also the Ultimate L, if the Ultimate conjecture is true, belongs to the phenomenal reality of set theory and it is created or constructed by Human mind. While the noumenal reality is the immutable, eternal, true world of sets itself independent from human mind and where sets are not interpreted. I claim that this distinction disappears within the universe of mathematics if the Ultimate L conjecture is true. In fact I hold that if we have an inner model (strategic variation), namely $L_S^\Omega$, for a supercompact cardinal, this inner model, although a phenomenal reality, would coincide with the true noumenal universe of sets V. This inner model would be very close to V since it would be like L in the case that $0^\sharp$ does not exist and for a suitable extender $\mathbb{M}$ strong large cardinal axioms transfer down from V to $\mathbb{M}$. So, if the Ultimate conjecture is true, a phenomenal reality would coincide with the noumenal true universe of sets V. In this case, the inner model of a supercompact cardinal would be the true universe of mathematics. However, at the same time, we can still build models for set theory and accomplish the mathematics of model. Within the mathematics of models, we explore all possibilities for mathematics while if the ultimate conjecture is true, truths concerning the Ultimate L, would be necessary truths characterizing the true noumenal universe of mathematics. The mathematics of models is characterized by all metamathematical models, inner and outer models (forcing extensions). However, if the ultimate L conjecture is true, all consistent enlargements of L (inner models) can be seen as approximations to the true, noumenal universe of mathematics (the Ultimate L), while the mathematics of models, where we combinatorially explore all possibilities for mathematics, is essentially characterized by outer models (forcing extensions). Within the mathematics of models, I have focused my attention essentially on $\mathbb{P}_{max}, \Omega - logic$ and Woodin maximum that we have seen in the first chapter (section: set theory). We have to say that $\mathbb{Q}_{max}$ and stationary tower forcing $\mathbb{P}_{<\delta}, \mathbb{Q}_{<\delta}$ produce the same extension as $\mathbb{P}_{max}$ [Woodin 10b]. As we have seen in the first chapter (section: set theory), I have focused on $\mathbb{P}_{max}$ because if $NS_{\omega_1}$ is saturated then every member of $H(\omega_2)$ is in the iteration of a countable model of a fragment of ZFC [Woodin 10b]. Since these countable models are elements of $L(\mathbb{R})$, their iterations induce a partial order in $L(\mathbb{R})$. This partial order, $\mathbb{P}_{max}$, produces an extension of $L(\mathbb{R})$ where $H(\omega_2)$ is the direct limit of the structures $H(\omega_2)$ of models satisfying every forceable theory (as we have seen in the first chapter: section set theory). The structure $H(\omega_2)$ in the $\mathbb{P}_{max}$ extension of $L(\mathbb{R})$ by assuming $AD^{L(\mathbb{R})}$ satisfies every $\Pi_2$ sentence [Woodin 10b]. Taking point classes such that $\Gamma \subseteq P(\mathbb{R})$ we have considered $\mathbb{P}_{max}$ extensions of larger inner models, such as $L(\Gamma, \mathbb{R})$, than $L(\mathbb{R})$. We have considered also $\Omega$-logic since within this logic we take all partial orders. There is a strong connection (as we have seen) between $\mathbb{P}_{max}$, $\Omega$-logic and

Woodin Maximum. In fact, if there is a proper class of Woodin cardinals, then for every set of reals A in $L(\mathbb{R})$, every $\Omega_{ZFC}$-consistent $\Pi_2$ sentence for $(H(\omega_2), NS_{\omega_1}, A, \in)$ holds in the $\mathbb{P}_{max}$ extension of $L(\mathbb{R})$ [Woodin 10b] (as we have seen). Furthermore, suppose that there is a proper class of Woodin cardinals and there is an inaccessible cardinal which is a limit of Woodin cardinals, then the theory $ZFC + Woodin\ Maximum$ is $\Omega_{ZFC}$ consistent [Woodin 10b]. The phenomenal mathematics of model, where we explore combinatorially all possibilities for mathematics and so it is based on forcing constructions, can be characterized by $\mathbb{P}_{max}$ and $\Omega$-logic. In this picture, within the phenomenal mathematics of models, we have the failure of the Continuum Hypothesis. Instead, if the Ultimate L conjecture is true, the Continuum Hypothesis holds within the Ultimate L. Therefore, we have to distinguish between phenomenal truths, characterizing the mathematics of models, and noumenal truths characterizing the true noumenal universe of mathematics if the Ultimate L conjecture is true. However, the phenomenal reality and the true noumenal universe are connected according to set theory. The true noumenal universe of sets (the Ultimate L) influences the phenomenal reality of metamathematical of models. In fact, the truth of the Continuum Hypothesis produces some results within the combinatorial phenomenal mathematics of models. In fact, there are limits to any possible generalization of the $\mathbb{P}_{max}$ variations to the context of CH. Thus, if the Continuum Hypothesis holds then the theory $H(\omega_2)$ cannot be finitely axiomatized over ZFC in $\Omega$-logic [Woodin 10b]. Secondly, let $\phi(x)$ be a $\Sigma^2_1$ formula and let r be a real number. Suppose that $\kappa$ is a measurable Woodin cardinal. Then if $\mathbb{P}$ and $\mathbb{Q}$ are partial orders in $V_\kappa$ such that $\mathbb{P}$ forces $\phi(r)$ and $\mathbb{Q}$ forces the Continuum Hypothesis to hold, then $\mathbb{Q}$ forces $\phi(r)$. In particular, if $\kappa$ is a measurable cardinal and CH holds, then any $\Sigma^2_1$ statement true in some small (cardinality less than $\kappa$) generic extension of V is already true [Woodin 10b]. So, if the Ultimate conjecture is true, a true noumenal truth such as the Continuum Hypothesis, produces results within the phenomenal mathematics of models. Thirdly, if the Ultimate L conjecture is true, then the $\Omega$-conjecture would be true. So, in this case, we would have a determinate set of truths. The full pluralism within the mathematics of models and the freedom of creating different models with different truths would be limited. If the $\Omega$ conjecture is true, we would have a definable set of truths with determinate values and so we would limit the possibility having different models with different truths values for the phenomenal mathematics of models. We would not have different metamathematical models with different truths but we would have a set of definable, absolute set of truths shared by all metamathematical models. If the $\Omega$ Conjecture is true and there is a proper class of Woodin cardinals then the set $V_\Omega$ is definable in the structure $H(\delta^+)$ where $\delta$ is the least Woodin cardinal [Woodin 10b]. So the independence of a sentence is not a proof that the sentence has no answer as Hamkins [Hamkins 10] is arguing by assuming his multiverse philosophy (as we have seen) in the case of CH. In fact, Hugh Woodin [Woodin 09] argues that if the $\Omega$ conjecture is true all mathematical statements of complexity like CH have determinate truth values. The connection between the Ultimate L conjecture and the $\Omega$ Conjecture is established by the following theorem:

Theorem 139 (Woodin 10). *Suppose that there is a proper class of Woodin cardinals, there is a strong cardinal, and that the Strong Iteration Hypothesis holds. Then the $\Omega$ conjecture holds.*

In particular, by assuming the Strong $(\omega_1+1)$ Iteration hypothesis and that there is an extendible cardinal then there is a fine-structural suitable extender model $\mathbb{M} \subset HOD$. As corollaries, we obtain that the HOD conjecture must hold in V and the $\Omega$ conjecture holds within the suitable extender model. We have examined the status of the $\Omega$-Conjecture. If we could prove the $\Omega$-conjecture, we would have a complete theory respect to $\models_\Omega$. In fact, thanks to Woodin's Maximum, $\models_\Omega$ would be a natural notion of logical consequence to adopt in order to decide every problem in $H(\omega_2)$. We have compared the result of completeness of Turing for transfinite progressions (we have seen this in the first chapter:

section transfinite progressions) and Woodin's result for $\Omega$-logic (we have seen this in the first chapter: section set theory). Firstly, both Turing's and Woodin's approaches share a weak similarity. In fact, both approaches imply a maximality principle. In transfinite Turing's progressions (as we have seen), we take all theories until $\omega + 1$ and in $\Omega$-logic we take all forcing extensions. To compare these two approaches by abstracting from their particular formulation and by accomplishing a sort of phenomenology, we have to evaluate their success in deciding undecidable mathematical statements. Surely, in the case of Turing's completeness theorem (as we have seen), we attempt to prove $\Pi^0_1$ statements or, in the case of Feferman $\Pi^0_2$ statements while in $\Omega$-logic we attempt to have a complete theory of the structure $H(\omega_2)$ and decide statements such as the Continuum Hypothesis which has the complexity of $\Sigma^2_1$ statement. The success of $\Omega$-logic is based on the fact that the $\Omega$-conjecture holds. Thus, in order to compare Turing's approach and Woodin's approach, we must introduce and formulate Turing's Conjecture (as we have seen in the first chapter: section transfinite progressions). This Conjecture may be formulated in the following way:

Definition 126. *(Turing's Conjecture) There exists a unique ordinal notation in order to index theories univocally.*

As we will see, this is the main problem for transfinite progressions. Unlike proved theorems that are atemporal truths, Conjectures are unproved mathematical statements which do not possess the criteria of atemporality. In mathematics a proved, atemporal theorem cannot be dismissed, while a Conjecture may be disproved. We might assert that we believe that a specific Conjecture is true and it is probable that it is true, but we cannot assert that is an atemporal truth (we relate this notion to Intutionism). So, we can compare Turing's Conjecture and the $\Omega$ Conjecture by asking ourselves which Conjecture is more probable to be true and which Conjecture can be believed to be true with more certainty. Church's thesis and the consistency of ZFC are other two conjectures very probable to be true. In fact, it is almost impossible to think of an informal algorithm which cannot be formalized as a partial recursive function and thanks to relative consistency proofs, it is very improbable that a contradiction will be discovered within ZFC. So, we can believe in Church's thesis and in the consistency of ZFC with the possible, highest degree of certainty. On the contrary, Turing's Conjecture, on which is based Turing's completeness theorem, is less probable to be true. We can believe in Turing's Conjecture with a lower degree of certainty. In fact establishing that we have a unique ordinal notation is a mathematical problem that has a greater computational complexity than the problem of establishing if a truth is a theorem (theoremhood). So, now we can ask ourselves what is the status of the $\Omega$ conjecture. Firstly, the $\Omega$-satisfiability of the $\Omega$-conjecture is a $\Sigma_2$ statement and there are no known examples of $\Sigma_2$-statements that are provably absolute and not settled by large cardinals. So it is reasonable to expect this statement to be settled by large cardinal axioms. Furthermore, it seems unlikely that the $\Omega$ Conjecture be false while its non-trivial $\Omega$-satisfiability be true. Secondly, if an inner model of a supercompact cardinal (the Ultimate L) will be constructed, then this model can reach all the traditional large cardinal axioms and, moreover, the $\Omega$ Conjecture holds in all these models. So, there is a strong evidence that the $\Omega$-conjecture is true and it reasonable that the $\Omega$-conjecture will be proved to be true, becoming a theorem and so, an atemporal truth. Thus, there is a strong evidence in favor of the $\Omega$-Conjecture. We might add that if there is a proper class of Woodin cardinals and that for every $A \subseteq \mathbb{R}$, if A is OD then A is universally Baire then $HOD \models \Omega$ *conjecture*. So we may assert that the satisfaction of the $\Omega$ *conjecture* rests on the satisfaction of other conjectures such as the HOD conjecture and the Strong $(\omega_1 + 1)$ Iteration Hypothesis or the Strong Unique Branch Hypothesis. We can say that the $\Omega$ conjecture is more probable to be true than Turing's Conjecture. We can believe in the $\Omega$-Conjecture with an higher degree of certainty than Turing's Conjecture degree of certainty. We may compare the $\Omega$-Conjecture with Church's thesis (we have seen this in the first chapter: Gödel's theorems) and the consistency of ZFC. In fact, we can ask ourselves if it is possible for all these Conjectures becoming proved, atemporal truths, or simply mathematical theorems. We can say that Church's thesis is impossible to become a theorem. In fact, we should be able to collect all possible informal algorithms and then formalized them as partial recursive functions. It is impossible to collect all possible algorithms. Also it is impossible that we will have a direct proof of the consistency of ZFC, but we can have only relative consistency proofs. In this case, we have a theorem, namely Gödel's second incompleteness theorem, that makes impossible to have a direct proof of the consistency of ZFC. So, while even if it is almost impossible, it might be possible to collect all algorithms and prove Church's thesis, to prove directly the consistency of ZFC is impossible because of another atemporal truth, namely Gödel's second incompleteness theorem. On the contrary, it is very probable that the $\Omega$ Conjecture will become an atemporal, proved truth as all other theorems of mathematics. In fact, it is very probable that a large cardinal axiom will settle the $\Omega$ conjecture or that the Ultimate L will be constructed implying the truth of the $\Omega$ Conjecture.

In order to decide questions within the universe of sets, we should capture the notion of the noumenal, true, arbitrary set. We have two extreme methods to interpret the notion of the noumenal, arbitrary set that lie on the notion of power set. On one side, we have strict definabilism represented by Gödel's constructible universe, namely L, where we take all definable subsets at the successor stage. In this case, definabilism is strict because few large cardinal notions are consistent with L. On the other side, when we construct forcing extensions, we extend the notion of arbitrary set. In fact, by adopting forcing extensions, we add new sets. Thus, we should ask ourselves when we capture the notion of the noumenal set. We have a solution if the Ultimate L conjecture is true. In fact, in this case we would have a form of extended definabilism. In fact, all known large cardinals would be consistent with the inner model of a supercompact cardinal. Then, since definabilism is kind of strong predicativism and so mathematical notions such as the power set operation are more precise, the Ultimate L would be the true, noumenal universe of mathematics where all notions are more precise. We must say that if the Ultimate L conjecture is true, we do not have the dichotomy between phenomenal reality and noumenal reality within the universe of mathematics, because the phenomenal reality represented by the Ultimate L would coincide with the true, noumenal universe of sets V. If we want to develop a modal logic for the universe of sets and if the Ultimate L conjecture is true, truths concerning the Ultimate L would be necessary truths such as $2 + 2 = 4$. Truths concerning the mathematics of models would be counter-mathematical possible truths. In fact, if the Ultimate L conjecture is true, the failure of CH would be a truth for the phenomenal mathematics of models, but it would be a truth within a counter-mathematical possible world where mathematics is different, since CH would be a necessary truth within the Ultimate L.

If the Ultimate L conjecture were not true, I would argue that we have no access to the true, noumenal world of sets V. In this case, I argue that we cannot accede the world of sets . Specific phenomenal metamathematical models become a solution for specific phenomenal truths. If the Ultimate L conjecture were not true, I would argue that we do not have access to the true, noumenal world of sets. In this case, we have to accept a strong form of pluralism. We would have only a plurality of phenomenal metamathematical models or phenomenal universes with their specific own truths. We would not have noumenal truths but only phenomenal truths. In this case, the solution to the continuum hypothesis is that we do not have a solution to the continuum hypothesis [Hamkins 10], but the countinuum hypothesis would be true in some phenomenal models or phenomenal universes and it would be false in other phenomenal universes. In this case, I will argue that we can make a philosophical choice and choose a specific phenomenal model. I will argue that the Bounded Proper Forcing Axiom does settle CH but this would be a phenomenal truth that holds in a phenomenal universe. So, If the Ultimate L conjecture were false, we would have only phenomenal set theory, a plurality of phenomenal models with their specific phenomenal truths. I would argue that a phenomenal model, where the Bounded Proper Forcing Axiom holds, is philosophically preferable. In fact, we need an $\Sigma_2$-reflecting cardinal, whose inner model is L, to prove the consistency of BPFA. So if the Ultimate L conjecture were false, among the plurality of all phenomenal metamathematical models we would select specific models supporting our choice with philosophical justifications . If the Ultimate L conjecture were false, we would have no access to the true, noumenal world of sets V. So in this case (the ultimate L conjecture is false), I would agree with Hamkins but I would argue that some mathematical statements, such as CH, have a phenomenal truth value within a phenomenal model. We would have phenomenal pluralism. In this case the noumenal, set theoretic reality would be inaccessible to us. If the ultimate L conjecture were false, the set theoretic noumenon would be inaccessible according to my philosophical beliefs. From an ontological perspective, my pluralism is different from Hamkins' pluralism, because my pluralism is phenomenal and the models or universes are not real, but mere interpretations of noumenal set theoretic universe V, inaccessible to us if the Ultimate L conjecture were false.

Maybe, some mathematicians might be concerned that if the Ultimate L conjecture is true, the mathematical game of set theory is over. I would argue that this is not the case. In fact, the goal of mathematicians would be discovering the richness of the Ultimate L structure which the true, noumenal world of sets.

CHAPTER 4

# Philosophical Aspects

**0.1. Preliminaries to this chapter.** In section 1, I am going to discuss some issues in philosophy of set theory. I will introduce Cantor's absolute infinite. I will compare Cantor's conception of the infinite with the conception of the Absolute principle of two neoplatonic philosophers, namely Plotinus and Damascius. After that, I will compare reflection principles with the apophatic method conceived by these neoplatonic philosophers. At the end of this section, I will explain how to extend the universe of sets. This part is important because I want to extend the universe in order to legitimate the use of proper classes as indexes of iterated structural reflection applied to inner models. In section 2, I will stress philosophical aspects. I will reintroduce the philosophical distinction between the phenomenal and the noumenal reality within set theory. I will criticize again Hamkin's multiverse philosophy. Then, I will highlight the problematic nature of real numbers. I will conclude this section by criticizing the formalistic philosophy in mathematics and I restate that the Ultimate L is the right universe of mathematics. In section 3 I will compare two different axioms of set theory. Then I will introduce the distinction between the phenomenal and the noumenal power set. I will show how it may be possible to define the noumenal power set. At the end of this section, I will introduce the concept of maximization as a principle that can justify large cardinals. I will present a case, namely weakly compact cardinals, that it may represent a reason to believe in realism. At the end of this section, I will connect maximization with extrinsic justifications and the Cantor's conception of mathematical freedom. In section 4 I will compare a Melissus' quote with Cantor's theorem. In section 5, I am going to discuss Duns Scotus' idea about the infinite.

I will argue that we can see the hierarchy of large cardinal as a way to perfection. After introducing Scotus's conception of human infinite intellect, I will argue that since Human Mind can accede the abstract world of set theory, it cannot be reduced to the brain. Human Mind cannot be reduced and it is supervenient on the brain. In section 6 I will use a set-theoretical argument to support Anselm's ontological proof of the existence of God. I will argue by criticizing Kant that existence can be seen as a predicate of perfection for abstract spiritual objects. In section 7 I will discuss paradoxes and I will introduce the Curry-Liar paradox that I conceived.

## 1. Philosophy of the Infinite: comparison between Cantor's absolute infinite and the Absolute principle of Damascius and Plotinus.

Georg Cantor emphasizes the unknowability of the transfinite sequence of all ordinal numbers, which he thinks of as an appropriate symbol of the absolute: The Absolute can only be acknowledged, but never known, not even approximately known [Cantor 76]. The principle of the unknowability of the Absolute seems to have only a metaphysical meaning for Cantor. Cantor distinguishes the proper infinite (transfinite) from the improper infinite (potential infinite) and from the Absolute infinite. Cantor asserts that the notion of improper infinite, or potential infinite, was historically accepted [Cantor 76]. For the German mathematician, the potential infinite is not a kind of infinite but he considers it as a variable finite number. Cantor distinguishes the potential infinite from the actual infinite with the following words:

> While the potential infinite points out to an indeterminate magnitude, always finite, variable having values that become small or larger than any arbitrary and finite upper bound, the actual infinite is a fixed magnitude, constant, larger than any finite magnitude of the same kind. [Cantor 76]

The set of natural numbers is not only an example of the actual infinite, but also of the proper infinite or transfinite. Surely, the set of all natural numbers is not the Absolute infinite. This means that, although infinite, the set of all natural numbers is limited on the upper part by other sets which have larger magnitude (power or cardinality), such as the set of real numbers. Differently from the Absolute infinite, both the finite and the proper infinite or transfinite share the fact of being limited in their magnitude. The proper infinite or transfinite can be tamed mathematically, whereas the Absolute infinite is beyond the limits of human reason, it cannot be understood mathematically. Even if for many philosophers the actual infinite could not be tamed, for Cantor it was a fundamental part of mathematics while the Absolute infinite, although a kind of actual infinite, was beyond the limits of human reason. To Aristotle, who formulated the principle of number annihilation, namely for every $\alpha$, $\alpha+\infty = \infty$, Cantor responds by observing that $\omega+\alpha \neq \omega$. According to Cantor, the rejection of the proper infinite was based on the fact that it had to be subjected to the same laws of the finite. Many philosophers asserted that the number could be precise only in the realm of the finite, while the actual infinite belonged to the realm of God. Cantor, instead, conceived the idea that between the finite and the Absolute an unlimited hierarchy of concepts, the transfinite numbers, exists by whom, though, it is not possible to understand the Absolute infinite :

> Omnia seu finita seu infinita definita sunt et excepto Deo ab intellectu determinari possunt.[ quoted by Lolli 02]

The Absolute can be indicated, but we cannot have knowledge of It, not even approximately. This aspect suggests that the sequence of all transfinite numbers can represent the Absolute, anticipating the awareness that this sequence is not a set.

Cantor's distinction between the Absolute and the proper infinite (transfinite) mirrors the distinction of two neoplatonic philosophers, namely Iamblicus (245-325 a.c) and Damascius (458-538 a.c), between the two transcendent principles, generators of reality. From three passages of Damascius' De principiis we can understand the hierarchy of the supreme principles according to Iamblicus : The principle totally ineffable ($\pi\alpha\nu\tau\alpha\pi\alpha\sigma\iota\nu$ $\alpha\pi o\rho\rho\eta\tau o\varsigma$)

precedes the One true and proper ($\tau o\ \alpha\pi\lambda\omega\varsigma\ \epsilon\nu$) [Damascius 02]. Iamblicus' metaphysics retaken by Damascius overcame Plotinus (205-270 a.c) philosophy, the first neoplatonic philosopher. Plotinus set as foundation of reality a unique principle, the One, Who was ineffable, unutterable, unknowable and indescribable. However in Plotinus' metaphysics this One, even if ineffable, in order to create reality, had to be connected to the World and His attributes characterized the entire reality. So, in Plotinus' thought it can be found the following theoretical difficulty: From one side, the principle, the One, was unutterable and ineffable; from the other side, since He was connected to reality, was describable. In order to overcome this theoretical difficulty, Damascius (retaking Iamblicus' idea) introduced a second principle preceding to Plotinus' One, absolutely transcendent, ineffable and unutterable. The first principle absolutely unutterable generated the second principle, Plotinus' One, who in Damascius thought was utterable, coordinated to reality and describable. As Damascius overcame Plotinus and his followers by rendering the second principle, Plotinus' One, utterable and describable, Cantor was able to tame the proper infinite (the transfinite) mathematically inquirable. It is possible to say, instead, that between the Absolute infinite of Cantor and Damascius' absolutely transcendent principle there is a perfect conceptual identity. For both thinkers, God is absolutely unutterable, indescribable and beyond the limits of human reason. It is not possible to deny his existence (or prove his existence) because we cannot say anything. For Damascius, the Skeptics can doubt about the existence of the second principle, but not about the absolutely trascendent One, beyond human reason limits. Concerning Cantor, the Skeptics can reject the proper infinite, the transfinite, (the finite human mind cannot tame the actual infinite), but they cannot deny the existence of the Absolute infinite because He is beyond the limits of human reason limits (beyond the domain of human reason). In the history of ideas, Descartes made a mistake because with his ontological proof he put God under the scope of human reason. So in this way for human beings it is possible also to deny His existence or prove that God does not exist whereas in Cantor's or Damascius' thought this is impossible. Like Cantor and Damascius, a theologician Karl Barth [Barth 10] said that human beings cannot understand and know anything about God's nature because it is possible that God (beyond the human logic) put Adolf Hitler in heaven and Saint Francis in hell.

The total unutterability, ineffability of Damascius' principle and Cantor's Absolute infinite gave rise to two respectively different methodologies in order to speak about what is not possible to speak about. According to Hao Wang [Wang 96], Kurt Gödel asserts that:

> All principles to constitute the axioms of set theory should be reducible to a form of Ackermann's principle: the Absolute infinite is unknowable. The strength of this principle increases when we obtain systems of set theory increasingly stronger. The other principles are only heuristic. Thus, the central principle is the reflection principle, which will be understood better when our experience will increase.[Wang 96]

Peter Koellner [Koellner 091] uses the following words to describe reflection principles :

> The reflection principles aim at articulating the informal idea that the height of the universe is absolutely infinite, and so it cannot be characterised from below. These principles assert that every sentence true in V, is true in some smaller $V_\alpha$.[Koellner 091]

Reflection Principles derive, as we have seen, from the reflection theorem of Levy and Montague [1] For many authors reflection principles represent intrinsic justification of large cardinals axioms (large cardinal axioms). It is possible through reflection to speak about the Absolute infinite. Initial segments of the universe reflect properties which cannot characterize directly the Absolute infinite because He is unknowable. Reflection is an indirect method to speak about what cannot be characterized. The idea of reflecting properties

[1]Every formula of the first-order language of set theory true in V reflects to some $V_\alpha$. That is, for every formula $\phi(x_1......x_n)$ and every $a_1......a_n \in V$ there is an $\alpha$ such that: $V \models \phi(a_1.....a_n) if\ and\ only\ if\ V_\alpha \models \phi(a_1......a_n)$.

of the universe is based on Cantor metaphysical conception that the Absolute infinite is unknowable.

The neoplatonic philosophers, when they spoke about the absolutely transcendent Principle, which could not be characterized, used the apophatic method, or *via negationis*. From the impossibility of nominating correctly the Principle, or characterizing it with some positive attributes, takes origin the attempt of giving, anyway, a description enumerating all names and attributes which cannot refer to it. The apophatic method plays a fundamental role in Plotinus' Enneads where The One is described as being without limit, without figure and parts, neither in some place neither in any place, neither moved neither still, not in the time, lacking of qualities and lacking of being, neither One, etc [Plotinus 09]. The One (the principle absolutely transcendent) is beyond all positive determinations. It is very interesting the following assertion of Damascius [Damascius 02]: About the Supreme Principle we cannot say anything. This sentence seems to give rise to a semantic antinomy similar to the Liar paradox : The sentence talks about other sentences, forcing to avoid those sentences that have as object the Supreme Principle; However the sentence itself mentions the Supreme Principle, and, although negative, it describes The One. Both the apophatic method and reflection principles are methodologies to talk respectively about the Supreme Principle and the Absolute infinite, totally unknowable otherwise. The first methodology goes higher and higher. In fact, by negating every attribute, we put the Supreme Principle beyond every determination. Reflection goes lower and lower. Initial segments of the universe reflect properties which cannot characterize directly the Absolute infinite.

Joan Bagaria [Bagaria 13] uses the following words to describe reflection principles:

> This principle of the unknowability of the Absolute, which in Cantor's work seems to have only a metaphysical (non-mathematical) meaning, resurfaces again in the 1950's in the work of Ackermann and Levy (as we have seen

> before), taking the mathematical form of the principle of reflection. Thus, in Ackermann's set theory, in fact a theory of classes, which is formulated in the first-order language of set theory with an additional constant symbol for the universe of all sets V, the idea of reflection is expressed in the form of an axiom schema of comprehension: Ackermann's reflection : Let $\phi(x, x_1, z_1.......z_n)$ be a formula which does not contain the constant symbol V. Then for every $a_1, .......a_n \in V$, $\forall x(\phi(x, a_1......a_n) \longrightarrow x \in V) \longrightarrow \exists y(y \in V \wedge \forall x(x \in y \leftrightarrow \phi(x, a_1, ......a_n)))$.
> A consequence of Ackermann's reflection is that no formula can define V, or OR, and is therefore in agreement with Cantor's principle of the unknowability of the Absolute. However, Ackermann's set theory (with foundation) was shown by Levy (1959) and Reinhardt (1970) to be essentially equivalent to ZF, in the sense that both theories prove the same theorems about sets. Thus Ackermann's set theory did not provide any real advantage with respect to the simpler and intuitively clearer ZFC axioms and so it was eventually forgotten. [Bagaria 13]

But, from a philosophical perspective, Ackermann's set theory is very interesting. In fact, within this theory it is possible to prove the existence of classes like $P(V), PP(V), PPP(V)$. In a few words, in Ackermann's theory we can prove the existence of subclasses of proper classes. [2] Reinhardt, following Ackermann, introduced the theory of $\Omega$-classes where he admits classes like $On + 1$, $On + \omega$, $On + On$, et. At this point it immediately arises the following question : is it possible to inquire and to extend the universe? and moreover, is it possible to legitimate proper classes and to use them as indexes in iterated structural reflection? I believe that the possibility of extending the universe is connected with the possibility of giving a positive solution to Burali-Forti's antinomy and Cantor's antinomy. I

[2] It is also philosophically interesting that for Ackermann what a set (menge) is, is not a well-defined notion.

think, in fact, that these two antinomies are generated by the fact of putting a block to the natural first Cantorian generating principle of set theory, namely taking successor stage. If we apply Cantor's theorem to the universal class, namely V, we produce a paradox because at the same time V contains all other sets and there is a class, the power set (the sub-class) of V, that for Cantor's theorem is bigger and it is not contained in V. But if we could extend the universe of sets and V were only an initial segment of a larger universe, then even if we apply Cantor's theorem to the universal class, we would not have a paradox any more. Taking limit stage (the second generating principle of set theory according to Cantor) is not so immediate and natural as taking successor stage, so if we consider the class of all ordinals On or the universal class V and we put a block to the natural and essential operation of set theory, namely the generation of the successor number, we produce paradoxes that threaten the pillars on which set theory is built. Bertrand Russell believed that even if Cantor's antinomy, Burali-Forti's antinomy and Russell's paradox seem to be different, they have the same mathematical form. I strongly disagree with him. Cantor's and Burali's antinomies derive (according to my conception) from the fact of blocking the possibility of taking successor stages whereas Russell's paradox is based on a linguistic self-referential sentence. On one hand, we can consider Cantor's and Burali's antinomies as structural contradictions, on the other hand Russell's paradox seems to have a linguistic or semantic nature. So in order to legitimate the use of proper classes, such as On $+ \omega$, as indexes of iterated structural reflection and give a positive solution to Cantor's and Burali-Forti's paradox, we can see if it is possible to extend the universe. Cantor's antinomy is generated by the power set operation applied to the universal class. It is not very clear what taking all arbitrary subclasses of the universal class actually means. The power set of the universal class is not an operation which can be considered legitimate. When we consider proper classes, it is meaningless to imply operations which force us to take all subclasses without control. In fact, also the power set of $\omega$ is a vague operation.

*All subsets of* $\omega$ is a sentence which does not have a precise meaning. On one side, I can consider only definable subsets of $\omega$, as in L, thus making the continuum hypothesis true , on the other side i can construct models (forcing extensions) where all ZFC axioms are true and the cardinality of the power set of $\omega$ is $\aleph_\omega$, namely a very large number in ZFC. So if the power set of $\omega$ is a vague operation, the power set of the universal class is not a legitimate operation. Thus it is possible to inquire in which way we can extend Cantor's universe. Even if $P(V_{On})$ (the power set of Universal class) does not have a precise meaning and it is not a legitimate operation, the operation $P^{def}(V_{On})$ seems to be a meaningful operation. In a few words, taking all definable subclasses of proper classes is a legitimate operation. We are adding L (Gödel's constructible universe) upon the vertex of Cantor's universe. Surely, suppose that we have a theory for this extended universe. Suppose that we have axioms for this extended universe. Since it is an extension of Cantor's universe, we can call it the first constructible Gödelian universe. Considering all arbitrary subclasses of a proper class is meaningless, but referring to only definable subclasses of a proper class is a valid operation. Both the universal class (V) and the class of all ordinals On are definable by $\Delta_0$ formulas, so Cantor's universe (initial rank) is a member of the successor, namely the first constructible class of the first Gödelian universe. The first two initial ranks of the first constructible Gödelian universe are the following : $L_0^G = V_{On}$ and $L_1^G = P^{def}(V_{On})$. Since we take only definable subclasses of the universal class, in the first Gödelian universe we do not increase the number of sets comparing with the elements of the Universal class, V (Cantor's Universe).[3] Cantor's antinomy and Burali Forti's antinomies specific for Cantor's universe vanish completely in the first Gödelian universe. Cantor's paradox is a direct consequence of Cantor's theorem. Cantor's paradox states that there is no greatest cardinal number. To understand this paradox, we have to follow this reasoning: V is the universal class, $P(V) \subseteq V$ and so, $P(V) \leq V$; But this aspect contradicts the fact

[3]Cantor's universe is enlarging because at the successor stage we adopt the real power set operation for arbitrary sets, whereas in the first Gödelian universe at the successor stage we take only definable subclasses of a proper class and we do not add new elements which have a cardinality larger than the universe V itself

that, by Cantor's theorem, if V were a set, we would have $V < P(V)$. To give a solution to Cantor's paradox, we are forced to deny that Universal class exists as a set and we call V **a proper class**, a mathematical object which does not belong to the universe (or ontology) of ZFC. In the first Gödelian universe, we have the successor of Universe class and we are stating that the Universal class is the initial segment of this first Gödelian universe. The assumption $P(V) \leq V$ is not anymore true in the first Gödelian universe. In fact, we have $L_1^G \geq L_0^G$ where $L_1^G$ is the successor stage and $L_0^G$ corresponds to Cantor's universe or simply to the universal class. The concept of the greatest element becomes a relative concept: the universal class V is the greatest element relatively to Cantor's universe but it is not anymore the unique greatest element in the first Gödelian universe. So in the first Gödelian universe the universal class is not anymore a complicated mathematical object, but it belongs to the universe (or ontology) of this universe...... For Burali-Forti's paradox we can have the same reasoning. In fact, this paradox originates from the fact that if the class of all ordinals were an ordinal, it would be isomorphic to a proper initial segment of itself. But, if the class of all ordinals is simply an initial segment of a larger universe (Gödel's first universe), then the problematic issue regarding this paradox would disappear. However, we will have Cantor's antinomy and Burali Forti's paradox specific for the first Gödelian universe. Then to solve the antinomies specific for the first Gödelian universe we have to transcend it (extend it) by creating the second Gödelian universe and so on. I said before that the concept of the greatest element is a relative concept. Therefore, we would have the greatest element in the first Gödelian universe, namely $V_{On}^{G1}$ (the set of all sets and classes of Cantor's universe and the first Gödelian universe). So if we do not transcend (extend) the first Gödelian universe by creating the second Gödelian universe, Cantor's paradox applies again to the first Gödelian universe. So we can ask in which way we can transcend the first Gödelian universe. The answer is simple. We can take all definable subclasses of the Universal class of the first Gödelian universe, namely $V_{On}^{G1}$.

Surely, after creating the second Gödelian universe, we will face the same problem since Cantor's paradox applies also to this universe. So we have to transcend it (extend it) by generating the third Gödelian unverse. We must take all definable subclasses of the universal class of the second Gödelian universe namely, $V_{On}^{G2}$. Then, we continue in this way. We cannot stop creating new universes in order to escape from Cantor's paradox. The reasoning for Burali-Forti's paradox is the same. In a few words in order to solve the Cantor's and Burali-Forti's antinomies we must be potentialist concerning all these universes (all extensions of Cantor's universe). It is always possible to add a new universe.

## 2. Stressing philosophical aspects: the Kantian distinction again within set theory and the problematic nature of real numbers

Realism has been, maybe, the first philosophy of mathematics. In fact, we can find in Proclus' writings (4th century a.c) the following assertion : the idealizations of geometry are innate forms precedent and independent from any experience. Also Descartes uses the same words to describe geometry:

> When I imagine a triangle, although in any place of the world there is not a similar geometric figure outside my thought, and there has never been, however a certain nature, or form, or essence determined by this figure, which is immutable and eternal, neither I created, neither depends from my spirit, does not stop existing. [Descartes 641]

Following contemporary thought we can distinguish between simple realism which sustains that set theory, or mathematics, is the study of an objective universe, namely the universe of sets, and plentiful platonism which affirms the existence of different universes corresponding to different no-contradictory theories formulated in first-order logic. Philosophers, who support simple realism, believe that propositions such as the continuum hypothesis, which are undecidable in the accepted current theory, Zermelo-Frankel theory with AC or its extensions (large cardinal axioms) have a truth value, not yet known, in the universe of which ZFC axioms are a description, obviously incomplete. On the contrary, plentiful platonism sustains that these propositions do not have a truth value; there are universes in which they are true and universes in which they are false; there are different universes. Following plentiful platonism, Joel David Hamkins [Hamkins 10] develops the multiverse conception which asserts the existence of many universes of mathematics. First of all, according to Hamkins, whole mathematics can be reduced to set theory and since in set theory we have different models like L, $V[G]$(outer models), $L[U]$ (inner models), $L[W]$, $L[E]$, Ult(V), etc.. we can consider these models as different universes of mathematics with different true propositions. Even if Hamkins' theory is very interesting, I disagree with Hamkins for an ontological perspective. In fact, I believe that all these models belong to metamathematics and so they are simply phenomenal interpretations of the unique, immutable, noumenal universe of sets V. .

At this point, I am asking myself why we have many problems to discover the real nature of third-order arithmetic (to obtain absoluteness results in third-order arithmetic). I believe that the nature of irrational numbers is problematic. Real numbers are constituted by rational numbers and irrational numbers. We know that rational numbers are countable, so irrational numbers are uncountable.

In his poem on nature, Parmenides describes two views of reality (we shall focus only on the first view). In the way of truth, he explains how reality (called as what-is) is one, change is impossible, and existence is timeless, uniform, necessary, and unchanging. Natural numbers, integers (Z) and rational numbers are parmenidean because they are timeless, uniform, necessary and, above all, unchanging (I am not mentioning parmenidean monism). If we draw a straight line, all these numbers correspond to a precise point. On the contrary, irrational numbers might be better described by Heraclitus' maxim: No man ever steps in the same river twice (there is a ever-present change in the universe). Irrational numbers seem to move continuously and they do not seem to be timeless (the magnitude of an irrational number seems to be an everlasting process). Since after a decimal there is always another decimal, to calculate the magnitude of an irrational number, we have to imply a constant change. Furthermore, it is impossible to put an irrational number on a straight line. If we imagine an infinite Turing Machine that positions numbers on a straight line (a counterfactual situation), after an infinite amount of time, the Turing machine will be able to position all rational numbers, whereas for irrational numbers it will not produce any answer, but it will go on forever. There is also another issue involved in the conception of irrational numbers. They involve the concept of the infinite. The square root of two is a finite number, but the decimals continue forever. So I am asking myself how it is possible that a finite concept can involve infinity.

William O. Quine characterizes irrational numbers in the following way:

> Then it is discovered that the rules of our algebra can be much simplified by conceptually augmenting our ontology with some mythical entities, to be called irrational numbers [Quine 51]

So for Quine, irrational numbers even if mythical entities, since they are useful, they belong to the ontology of mathematics. It seems to me that Quine, like Gödel for strong axioms of infinity, is justifying extrinsically the existence of irrational numbers. They are mythical, they are not numbers like natural numbers, but they are useful so they can belong to the world of mathematics. The theoretical difficulty of extrinsic justification, which for Gödel corresponds to the fruitfulness of the consequences of adopting a peculiar axiom and for Quine corresponds to the usefulness of simplifying mathematical results, is the following: if mathematics is a creation of human mind, we can use extrinsic justification to enlarge our ontology or accept an axiom without problems, but if mathematics is a description (as I think) of an external, objective, independent, immutable and eternal world, extrinsic justification must be connected with intrinsic justification which is based essentially on a conceptual analysis. If we have to make a choice, first of all we should look at intrinsic justification and secondly at extrinsic justification.

Irrational numbers seem to be a creation of human spirit, they do not seem to belong to the world of sets and, so, maybe, the theoretical difficulty of finding a solution to the continuum problem is generated by the problematic nature of these numbers. However, it is possible to argue that we can relate the continuum problem solely to pure set theory and so, we can speak about subsets, countable and uncountable sets without mentioning any number systems. Unfortunately, there is a strong connection between irrational numbers (real numbers) and subsets of a countable set. In fact we can write an irrational number in binary expansion and we will have a string of 0s and 1s, e.g. 0100011111000000..... without an end. We can see this string as values of a characteristic function and so each string can correspond to an infinite subset of a countable set. Therefore, maybe, in order to find a solution to the continuum problem accepted by the whole mathematical community, we should clarify the nature of irrational numbers. Even if I believe in a unique, eternal, acasual world of sets and I hope that set theorists will obtain absoluteness results for third-order arithmetic, we can take, as Universe for the mathematical game, a Universe which belongs to the phenomenal metamathematics. Surely I consider metamathematics as a creation of human mind and exclusively a phenomenal interpretation of the real world of sets. But, if a model contains all large cardinal notions, it can be viewed as a satisfactory universe where we can accomplish mathematics. If a model contains all large cardinal notions, it is rich enough and we can hope that only few notions of the real, immutable, acausual world of sets are left aside. Since the power set operation for arbitrary sets is vague (I will speak about the noumenal power set later on), i prefer the definable power set operation, more precise for mathematics. Therefore Woodin's Ultimate L (if the Ultimate L conjecture is true), which is an inner model containing all large cardinal numbers and as all other inner models is characterised by definability (the power set operation is precise), can be considered as perfect Universe for mathematics. In this very large inner model, the continuum hypothesis is true and so also Analysis should mirror this result. So to make a joke, even if I am realist, I become intuitionist because I am forced to consider the Ultimate L, which is simply a creation of human mind, as the real universe of mathematics. If the Ultimate L conjecture is false, I believe that the continuum hypothesis is settled by the Bounded Proper Forcing Axiom, but this would be a phenomenal truth which holds within a phenomenal models that we choose philosophically among a plurality of different phenomenal models or universes.

At this point, I would like to combine structuralism with my semi-realistic conception. Structuralism affirms that mathematics is the study of structures. However, a structuralist does not say what structures really are. Nevertheless, structuralism must explain in which way structures are studied and inquiried mathematically. For instance, structuralism can use an informal semantics where the fundamental notions are primitive and not defined: structures are characterised by properties, concerning relations and functions, which are true in the structures; but neither the concept of truth nor the concept of property are analysed. Structures are considered as primitive logical concepts and the study of structures is accomplished in the informal semantics. Unfortunately, this approach does not take in consideration the development of mathematical logic in the last century and the use of informal semantics seems to say that mathematics studies what it studies. Another solution can be the axiomatic approach. We can avoid whole semantics and we can say that structures are characterised by axioms: when a structure is given, we postulate, by using symbolic writings, conditions which must be satisfied by operations and relations. Properties of structures are consequences which are deduced from the axioms. So, structuralism might resemble formalism or deductivism. The formalist position is assumed by Bourbaki [Lolli 02].

Gabriele Lolli in his fascinating book, at some point introduces platonic structuralism according to which structures are sets. I agree with this position and even if structuralism does not define structures, i think that platonic structuralism might overcome this problem in the following way : structures are families of sets bound together by the membership relation. A structure which is constituted by a family (very small o very large) of sets and characterised by the relations (membership) between the sets of the family, is a small universe. Each structure is a different small universe where we accomplish different calculations. In my semi-realistic conception, we have the immutable, independent, eternal, acasual world of sets and then we have metamathematical models and structures which are creation of human mind. There is a fundamental difference between metamathematical models and structures. Metamathematical models have a semantic pretension. In fact, a model aims at rendering all axioms true and some mathematical propositions true. Structures do not have this pretension. In my conception, we create structures by choosing specific sets and establishing conditions for the membership relation. Certainly we can know the nature of sets and the characteristics of relations in the structure, but this is the phenomenal reality [4] (like for metamathematical models). Even if in the case that the Ultimate L conjecture is true, I consider the Ultimate L the noumenal universe for mathematics, I still believe that the ultimate L is a phenomenal reality.

At this point, i would clarify why the formalistic conception of mathematics is not a good philosophy of mathematics. Formalism does not inquiry the essence or the meaning of numbers, but only how numbers are used. Mathematics, in the formalistic conception, is like playing with signs which are empty (meaningless). They get their meaning from certain rules within specific mathematical games. Mathematics is a game with its specific rules. According to J.Thomae [Lolli 02] arithmetic is like chess with different typology of rules, but still a game. David Hilbert considered arithmetical equations as strings of signs which were meaningless outside the formal system and meaningful only within the system. The formal system must be consistent. Contradictions should always be avoided. The formal

[4]Retaking a Kantian distinction, the phenomenal reality is the apparent world whereas the noumenal reality is the objective, independent, true universe of sets

system is characterised by a mechanical, blind deduction within the system, using the rules of the system. The other aspect is the construction of the system itself: the choice of the language, axioms and rules which delimit the mathematical game of all internal possible symbolic activities. I think that formalism is fascinating but we should distinguish between the instant of intuition and the moment of metacognition. To see that an axiom is true or to find a solution to a mathematical problem are mathematical actions which occur at the level of mathematical intuition. Maybe the proof itself is mechanical, but the starting point, namely the truth of an axiom, and the ending point, the solution to the problem, are not mechanical at all, but they belong to the realm of intuition. Nevertheless intuition is not a rational process, but it characterises our irrationality and, as Kant was thinking, it belongs to sensibility. Intuition is independent from games and rules. When we grasp the truth of an axiom, independently from the rules of the games, we are naturally referring to something (relations and mathematical entities) which can be independent objects or creations of human mind external to the formal system. Intuition makes us denoting something that is external, meaningful, independent from the formal system (the mathematical game). To understand the truth of an axiom, we cannot only see the string of symbols itself but we have to accede a reality which can be the immutable world of sets or a human spirit creation. We can say that what we have done mathematically is only a mechanical proof within the game and we have understood the meaning of the symbols only from the rules of the game, only at the level of metacognition. Solely when we reflect about what we have done, a moment after intuition, we can adopt a formalistic point of view. Formalism should explain the instant of intuition. For example to understand that the axiom which assert the existence of a strongly inaccessible cardinal constitutes a model of ZFC, we have to use intuition (at least I believe). First of all, we have to grasp the fact that a strongly inaccessible cardinal cannot be reached from below. To understand this, we are referring to an external, meaningful object. Secondly we have to grasp what the axiom of replacement is saying (the possibility of generating all functions within ZFC). At the end, we have to combine the truth of these two axioms and grasp the fact that no function can reach a strongly inaccessible cardinal (there is no cofinal function) and so a strongly inaccessible cardinal is a closure point for the axioms of ZFC. All these passages require intuition where we are denoting something independent. Only at the level of metacognition we can say that putting together two strings of symbols (the formulas for the two axioms) we are able to deduce mechanically within the ZFC game that a strong inaccessible cardinal is a model of ZFC.

At this point I want to come back to a problematic issue for realism. A simple realist is a realist not only for ontological entities but also for truth values. A plentiful realist is a realist for ontological entities but not for truth values since each structure can have different mathematical true propositions. So beyond ontological realism, there is also a truth value realism. But there is a very problematic theoretic difficulty for this kind of realism, namely Tarski's theorem about the indefinability of truth. For this theorem, the concept of mathematical truth is vague, if the language is informal, or we have to consider a huge hierarchy of metalanguages always more and more complicated. The main problem is the following: to define the truth of mathematical propositions of the universe of set (or mathematics), we need a theory in which this universe is the object, so a theory which can prove that the whole existent mathematics is not contradictory. This is impossible. But we can still speak of local truths relative to specific metamathematical models. However Tarski's theorem represents a serious problem for simple realism.

The Ultimate L can represent a solution (philosophical) for Tarski's theorem (when this theorem is used as a case against simple realism). In fact the nature of this metamathematical model is parodoxical : the Ultimate L contains at the same time local truths, since it is still a metamathematical model, and universal truths, since it has got all large cardinal notions and it can be seen as a close representation of the real universe of all sets.

So for Tarski's theorem (when it is used as a case against simple realism) we should adopt the Ultimate L as a solution.

## 3. The plausibility of a new axiom, namely $ZFC + 0^\sharp$ *exists*

Now I want to compare the axiom $0^\sharp$ *exists* with the axiom of constructibility. For Kunen's impossibility theorem, we cannot have an embedding of the universe V into itself ($j : V \longrightarrow V$ is inconsistent), but if $0^\sharp$ exists, we can have an embedding of L into itself. Therefore the existence of $0^\sharp$ contradicts the axiom of constructibility, namely $V = L$. So the choice between $ZFC + 0^\sharp$ *exists* and $ZFC + V = L$ is a fundamental philosophical question which for its importance it deserves to be treated immediately. Many set theorists reject the axiom of constructibility because they judge it too restrictive. In 1938 Kurt Gödel wrote the following :

> The axiom of constructibility added as a new axioms seems to give a natural completion of the axioms of set theory, in so far as it determines the vague notion of an arbitrary infinite set in a definite way. [Wang 96]

From 1947 on, Kurt Gödel changed his view about the axiom of constructibility and he rejected it. However, instead of following the majority of set theorists that reject this axiom, we should ask ourselves why Gödel during this initial period was accepting it. The main reason is that the notion of arbitrary set is vague. For instance, if we take the power set operation, we are forced to face a problem of vagueness. We do not know what is the meaning of taking all subsets of $\omega$. The word *All* is vague. By forcing method, the sentence *all subsets* of $\omega$ can be interpreted metamathematically and the number of subsets of $\omega$ can be equal with a large cardinal notion while in L by taking all definable subsets of $\omega$ the continuum hypothesis holds. In L, the vague notion of arbitrary set is made precise. In set theory, from one side we have a great variety of beautiful models, from the other side we cannot settle the continuum hypothesis. Towards the solution of CH, I believe that we have two options, namely the top down road or the bottom up road. The top down is the usual method which consists in searching for new axioms. I call this way top down because we go higher and higher in the universe to settle something like CH which is located lower in the hierarchy of the actual infinite. On the contrary, the bottom up method consists in deepening our analysis of the power set operation and the notion of arbitrary set that is connected to it. If we have to focus on the power set operation, we have three options. First of all, there is definabilism which corresponds to the construction of L (the constructible universe). From one side L avoids strictly impredicative definitions, but from the other side the construction of L is based on the original impredicative use of the class of all ordinals. Therefore even if L can be seen as an extreme form of predicativism, the constructible universe is characterised by an impredicative use of the class of all ordinals. Secondly we have arbitrariness that forces us to take all arbitrary subsets of a given set. While defininabilism is connected with L and other inner models like $L[U]$, the conception of arbitrariness is linked to outer models (forcing method). As it usually happens in the history of ideas that a third way between two options is preferable, so it seems that the third way represented by combinatorialism can give a solution to the power set operation. Now we have to clarify what combinatorialism is. Paul Bernays uses the following words in order characterise combinatorialism:

> Modern analysis, etc, abstracts from the possibility of giving definitions of sets, sequences and functions. These notions are used in a quasi-combinatorial sense by which I mean: in the sense of an analogy of the infinite to the finite. Consider, for example, the different functions which assign to each member of the finite series 1,2,..... n a number of the same series. There are nn functions of this sort, and each of them is obtained by n independent determinations. Passing to the infinite case, we imagine functions engendered by an infinity of independent determinations which assign to each integer an integer, and we reason about the totality of these functions. In the same

> way, one views a set of integers as the result of infinitely many independent acts deciding for each number whether it should be included or excluded. We add to this the idea of the totality of these sets. Sequences of real numbers and sets of real numbers are envisaged in an analogous manner. In [Maddy 97]

So according to combinatorialism, there is one function from reals to reals for every way of making $2^{\aleph_0}$ independent assignments of a real to a real. Ignasi Jané [Jane 05] applies the combinatorial method to the power set operation. Now I will try to describe his view. First of all, he deals with the Gödel *set of (power set) operation* which assigns to any given domain D a new domain D*, the power domain of D, which consists of the sets of objects in D. When D is a finite domain of n elements, not only we can tell that there are exactly $2^n$ distinct selections of D-objects, but we also know how to describe them explicitly. So in the finite case, the power domain D* can be described in full as the totality of D-sets. But no such procedure works for the infinite case. We can introduce the conception of combinatorial D-set, that is, of a plurality selected by arbitrarily and independently deciding for every object in the domain whether to select it or not. The problem is that we do not know what a combinatorial set is. Moreover the combinatorial approach to D-sets is meaningful and will single out a domain only under the assumption that such domain exists. For all these problems, Ignasi Jané asserts that we do not describe D* as the totality of all D-sets, but we postulate the existence of a domain called D* and we define a D-set to be an object of D*. We require that D* is maximally extensional over D, namely D* cannot be extended without loss of extensionality. Ignasi Jané explains his view with the following words:

> Strictly speaking, then, we do not know what all D-sets are and we do not know what D* is. Nevertheless, we can reason about D*, we can define some D-sets and we can argue for the existence of D-sets with certain properties.

> Thus, no matter what plurality of D-objects we would ever acknowledge, there should be a D-set corresponding to it. In particular, in D* there is a set corresponding to each plurality of D-objects which we know how to specify in some given context, as there are D-sets corresponding to those pluralities specifiable in terms of other members of D*. D* is thus conceived as being closed under various operations, some of them inspired by the suggestion of combinatorial sets. In a sense, we can think of D* as the ideal completion of open-ended range of specifiable pluralities of D-objects. [Jane 05]

This approach is surely fascinating and it takes some aspects of definabilism even if in a combinatorial way (ideal completion of open-ended range of *definable* pluralities of D-objects). In my opinion a combinatorial set should be based on choice functions. Actually a combinatorial set should be a choice set abstracting from the possibility of giving definitions of set and using (if necessary) impredicative conditions. When a condition (definition) is given, which can be also impredicative, we abstract from that condition and we form the choice set by selecting all elements which satisfy that condition, making independent determinations. Therefore the problem of finding a combinatorial set is related to the plausibility of the axiom of choice. Kurt Gödel believed that this axiom was true. Georg Cantor was using a different principle that we can call the iterable choice principle. This principle was based on the idea that at time 1 we select one element, at time 2 we select another element, at time 3 a different element and so on throughout the infinite. Cantor's principle is more realistic [5] since it is characterised by a temporal component, while the axiom of choice is atemporal and changeless since in one shot we have a choice set. I said changeless because it seems to me that the choice set belongs to an immutable, atemporal, eternal and acausal (this last aspect brings up a philosophical problem about the perception of mathematical entities) world of mathematical forms.

[5]By using the word realistic, I mean that this principle belongs to the physical world and not to the changeless, eternal platonic world of mathematics.

So we have seen that the axiom of constructibility, even if restrictive, makes precise the notion of arbitrary set and it renders the power set operation fixed and definite. So maybe this is the main reason why Gödel was believing in this axiom. In fact, we have seen that many problems arise in the case of the power set operation. So it seems to me that (adopting a Kantian distinction) we have to distinguish between the phenomenal power set and the noumenal power set. The phenomenal power set is what we interpret metamathematically and renders the metamathematics of set theory so rich. The noumenal power set is the real operation that inhabits the platonic world of mathematics and it is beyond (for now) our understanding. It is possible to sum up my realistic conception, which makes the distinction between the phenomenal and the noumenal reality in set theory, with the following maxim: God gave us sets (noumenal reality), we (humans) metamathematically (phenomenally) interpret them. Before moving to the fundamental philosophical issue of this chapter, a striking question arises in my mind. I am always asking myself how it is possible that a finite human mind can grasp the concept (largeness) of, for example, a supercompact cardinal. In the Grundlagen, Cantor affirms that a finite human mind can understand the transfinite because the transfinite is subjected to some immutable laws that humans can conceive and assume. I believe, as I will explain later, that the fact that we can capture the concept of the infinite can be assumed to support my dualist thesis in philosophy of mind. At this point, coming back to our original question, namely the choice between the axiom of constructibility and the axiom asserting the existence of $0^\sharp$, we have to introduce some principle that would enable us to decide between these two axioms. The first principle that I want to discuss is maximization. Penelope Maddy [Maddy 97] describes maximization as follows:

> The idea is to motivate the case against restrictive theories by appeal to MAXIMIZE, so the central claim will be that restrictive theories somehow restrict isomorphism types [.....] There are things like $0^\sharp$ that are not in

> L. And not only is $0^\sharp$ not in L, its existence implies the existence of an isomorphism type that is not realized by anything in L. [......] So it seems that ZFC + V=L is restrictive because it rules out the extra isomorphism types available from ZFC + $0^\sharp$ exists. [Maddy 97]

So if we adopt the principle of maximization, we should choose the theory ZFC + $0^\sharp$ exists because it implies the existence of more sets (more isomorphism types). Surely, maximization is an important principle and many set theorists adopt it because they want to have a richer universe. So this principle justifies axioms that render the universe of set theory richer and richer. However we must always avoid inconsistency. I believe that maximization is related to three ideas, namely extrinsic justification, a realistic conception of mathematics, and the Cantorian conception of freedom. Gödel asserts that extrinsic justification is based on the fruitfulness of the results. In a few words, an axiom should be evaluated on the basis of the results that we can obtain from it. But as Maximization pushes set theorists further and further, so extrinsic justification forces set theorists to go further and further (infinitely many Woodin cardinals, proper class of Woodin measurable cardinals). So, maximization is related to extrinsic justification since from one side the constructible universe is too restrictive and from the other side, we have to transcend Gödel's universe to obtain fundamental results for second-order arithmetic and third-order arithmetic. It seems to me that maximization is related also to a realistic conception of mathematics. It seems that when we maximize by accepting intuitively a particular axiom, we are discovering a new reality as scientists discover new planets and new atomic particles. Certainly, some set theorists would respond that there is no objective reality of sets but only a reality which is created by human mind and we can call it, the intra-subjective mathematical reality. However, there is a case in set theory that supports my conviction in realism. When we study large cardinals and we deepen our analysis of weakly compact cardinals, we become aware of a striking aspect which forces us to believe that maybe weakly compact cardinals exist independently from our mind. Firstly, a weakly compact cardinal is uncountable and satisfies the partition property $\kappa \longrightarrow (\kappa)^2_2$. Secondly a weakly compact cardinal satisfies the tree property. Furthermore a cardinal $\kappa$ is $\Pi^1_1$ indescribable if and only if it is weakly compact. Moreover there is the issue of infinitary languages. A collection of $L_{\lambda\phi}$ sentences is satisfiable iff it has a model under the expected interpretation of infinitary conjuction, disjunction and quantification; and is $\kappa$-satisfiable iff every sub-collection of cardinality less than $\kappa$ is satisfiable. For a cardinal $\kappa > \omega$, $\kappa$ is weakly compact iff any collection of $L_{\kappa\kappa}$ sentences using at most $\kappa$ non-logical symbols, if $\kappa$ is satisfiable. So we have seen that the notion of weakly compact cardinal is derivable from totally different parts of set theory and we get the same notion. We depart from partition calculus, reflection or infinitary languages and we grasp the same notion. It is this interdefinability (or multidefinability) that forces me to believe that maybe this large cardinal notion exists independently of our mind.

At the end, maximality is connected with the Cantorian conception of freedom. In the Grundlagen, Cantor affirms that the main feature of mathematics is its freedom. A mathematician should be free to introduce new mathematical concept, unless contradictory. If we see the large cardinal hierarchy, we notice that every cardinal notion is the natural evolution (in many cases) of concepts that are located lower in the hierarchy. The more the model M[6] is similar to the universe V, the larger cardinal notion we obtain. Since no large cardinal notion causes contradiction until now (except for a Reinhardt cardinal in the presence of the axiom of choice), if we stop at the level of a measurable cardinal,

[6]The model M is the transitive collapse of some Ultrapower of the universe. After taking the ultrapower of the Universe V (well-founded), we generate a triangle of embeddings: an embedding of the universe V into the Ultrapower, an embedding of the ultrapower into its transitive collapse M and at the end, an embedding of the Universe directly into the model M itself. So when we generate the embedding of the universe V into the transitive model M, we discover the first measurable cardinal (the critical point of the embedding). Then departing from the first measurable cardinal and putting conditions on the image of the critical point, namely $j(\kappa)$, in M (enlarging M), we obtain larger and larger cardinal notions

then our freedom of introducing new concepts would be limited. The set theorist Menachem Magidor asserted that the intrinsic justification of the axioms is based mainly on the analysis of the concepts involved. If we deepen our analysis of the large cardinal notions, well, mathematical freedom can be seen as the intrinsic justification of new axioms. Mathematician must be free to introduce new mathematical concepts unless contradictory. If we look at the large cardinals hierarchy, a Reinhardt cardinal generates a contradiction in ZFC (Kunen's inconsistency result). Mathematical freedom fits perfectly with the large cardinals hierarchy. Measurable, strong, superstrong, supercompact, extendible cardinals represent a consistent enlargement of M, so if we are free, we cannot limits and we must be free to introduce them as axioms. Thus mathematical freedom is connected to maximality.

## 4. Melissus of Samo and Georg Cantor

At this point, I want to focus my attention on a thought that comes from ancient greek philosophy for two reasons. I would like to apply an idea that comes from an ancient greek philosopher to modern set theory.

The philosopher I want to speak about is Melissus of Samo. This philosopher was born around the sixth century B.C. Melissus was the last philosopher of the Eleatic school and his critical discussion about the Parmenidean principle (what it is,$\tau o\ \epsilon o \nu$) and his assertion about the infiniteness of this principle opened the way to the development of ancient greek philosophy.

The unity of $\tau o\ \epsilon o \nu$ was declared clearly from Parmenides when he defined it as One and continous. Parmenides also asserted that since the $\tau o\ \epsilon o \nu$ (the principle of reality) is one, nothing which could stay close to it could be born and this principle could not be divided because it is the same in all its parts. But since Parmenides attributed the finiteness to $\tau o\ \epsilon o \nu$, this aspect produced the following theoretical difficulty: If it is one and finite, it must admit something beyond itself. For Melissus the $\tau o\ \epsilon o \nu$ is not born, it was always present, it will be present forever, it does not have beginning, it does not have an end and it is infinite. Moreover, for Melissus the $\tau o\ \epsilon o\nu$ is One, otherwise it would confine to something else and infinite otherwise it would confine to the void. Let us consider the following Melissus' assertion: $\epsilon\iota\ \gamma\alpha\rho\ \alpha\pi\epsilon\iota\rho o\nu\ \epsilon\iota\eta,\ \epsilon\nu\ \epsilon\iota\eta\ \alpha\nu$ : $\epsilon\iota\ \gamma\alpha\rho\ \delta\upsilon o\ \epsilon\iota\eta,\ o\upsilon\kappa\ \alpha\nu\ \delta\upsilon\nu\alpha\iota\tau o\ \alpha\pi\epsilon\iota\rho\alpha\ \epsilon\iota\nu\alpha\iota, \alpha\lambda\lambda'\ \epsilon\chi o\iota\ \alpha\nu\ \pi\epsilon\iota\rho\alpha\tau\alpha\ \pi\rho o\sigma\ \alpha\lambda\lambda\eta\lambda\alpha$ (If it is infinite, it must be one: if they were two, they could not be infinite, but each of them would be the boundary of the other).

The first thing to notice is the word $\alpha\pi\epsilon\iota\rho o\nu$ which means infinite and derives from $\alpha$ (without) and $\pi\epsilon\iota\rho\alpha\varsigma,\ \alpha\tau o\varsigma,\ \tau o$ (end, boundary, limit). The $\alpha\pi\epsilon\iota\rho o\nu$ (indefinite, infinite, limitless) was the first principle ($\alpha\rho\chi\eta$) of reality for the presocratic philosopher Anaximander (611 B.C). For this thinker, The $\alpha\pi\epsilon\iota\rho o\nu$ was unlimited in its source, it could create without experiencing decay, so that genesis would never stop. The $\alpha\pi\epsilon\iota\rho o\nu$ was an abstract principle and it was no longer a point in time, but a source that could perpetually give birth to whatever will be.

Aristotle writes (Metaphysics, 3-4) that the Presocratics were searching for the element that constitutes all things. While each Presocratic philosopher gave a different answer as to the identity of this element (water for Thales and air for Anaximenes). Anaximander understood the beginning or first principle to be an endless, unlimited primordial mass ($\alpha\pi\epsilon\iota\rho o\nu$), subject to neither old age nor decay, that perpetually yielded fresh materials from which everything we perceive is derived.

Now coming back to Melissus' assertion, we can compare Melissus' thought with Georg Cantor's thought. In set theory, we do not have only one infinite, but a hierarchy of infinite cardinal numbers (alephs, the transfinite) where at each successor stage we obtain a bigger infinite. By Cantor's theorem, the set of natural numbers $\mathbb{N}$ is smaller than the set of real numbers $\mathbb{R}$. This aspect contradicts Melissus' maxim since for the greek philosopher we cannot have two things which are infinite otherwise they would be the boundary of each other (they would be finite). We could answer that natural numbers and real numbers belong to two different ontological planes and you are simply comparing them. The problem is that, by assuming AC, both the set of natural numbers and the set of real numbers can be well-ordered and so we represent them as aleph numbers. The set of natural numbers is the first aleph, namely $\aleph_0$, whereas by the forcing methods we can assign to the set of real numbers different alephs, namely $\aleph_1, \aleph_3, \aleph_\omega$......*etc.* If we assume Melissus' maxim, we can say the following: being two infinite sets, according to Melissus' maxim, the set of all subsets of the set of natural numbers seems to limit superiorly the set of natural numbers and the set of all subsets of all subsets of a countable set seems to limit superiorly the set of all subsets of a countable set , etc. According to Melissus maxim, if something is infinite, it must be unique and we cannot have infinite sets bigger than other infinite sets. Melissus would have said to modern set theorists that they look at the infinite with the eyes of finiteness. In fact, if all sets were countable, there would be one infinite and there would be no theoretical difficulty for Melissus. However, Melissus would disagree with modern set theory when we say that we have an infinite set bigger than another one. However, we have Cantor's theorem, an atemporal truth, that cannot be questioned if we introduce the actual infinite. Thus, we may say that the knowledge of the mathematical infinite for ancient philosophers was not correct. The set of real numbers does not limit superiorly the set of natural numbers, but thanks to Cantor's theorem, when we compare these two sets regarding cardinality, we become aware that the set of real numbers is bigger than the set of natural numbers because, thanks to diagonalization procedure, some real numbers do not correspond to natural numbers. Since the main feature of mathematics is iteration, the possibility of iterating specific operations, we can iterate Cantor's theorem and create the hierarchy of all alephs. This hierarchy, even if it contains infinite sets bigger than other sets, is fully justified because we have a theorem (Cantor's theorem) an atemporal, actual truth that it cannot be questioned. If we accept the concept of actual infinite, Cantor's theorem is not a conjecture that it can be potentially, temporally true and we may argue that it is false. Thus, we must conclude that Melissus' quote is not correct. The infinite is not unique, but we have a plethora of infinite sets.

## 5. John Duns Scotus, the infinite and philosophy of mind

John Duns Scotus (1270-1308), called the subtle doctor, was a Franciscan. It is commonly supposed that the scholastic philosophers (following Aristotle) believed in the idea that the infinite was potential, not actual. Here is a passage by Scotus suggesting that they did not.

> O Lord God, are not the things that can be known infinite in number and are they not all known actually by an intellect which knows all things? Therefore, that intellect is infinite which, at one and the same moment, has actual knowledge of all these things. Our God, yours is such an intellect. The nature that is identical with it then is also infinite. I show the antecedent and consequence of this enthymeme. The antecedent: Things potentially infinite in number (things, which if taken one at a time are endless) become actually infinite if they exist simultaneously. Now what can be known is of such a nature so far as a created intellectual is concerned, as is sufficiently clear. Now all that the created intellect knows successively, your intellect knows actually at one and the same time. Then, the actual infinite is known. I prove the major of this syllogism, although it seems evident enough. Consider these potentially infinite things as a whole. If they exist all at once, they are either actually infinite or actually finite. If finite, then if we take one after the other, eventually we shall actually know them all. But if we cannot actually know them all in this way, they will be actually infinite if known simultaneously. The consequence of this enthymeme I prove as follows. Whenever a greater number requires or implies

> greater perfection than does a smaller one, numerical infinity implies infinite perfection. For example, greater motive power is required to carry ten things than to carry five. Therefore, an infinite motive power is needed to carry an infinity of such things. Now in the point at issue, since the ability to know two things distinctly implies a greater perfection of intellect than the ability to know only one, what we proposed to prove follows. This last I prove to be so because the intellect must apply itself and concentrate if it is to understand the intelligible distinctly. If then it can apply itself to more than one, it is not limited to any one of them and if it can apply itself to an infinity of such it is completely unlimited. [Duns Scotus 82]

For Scotus the divine intellect is infinite. However for him the actual infinite exists also in nature. From the impossibility of counting all things of nature which are potentially in number, he concludes that all these things are actually infinite if taken simultaneously. This thought may be true. In fact since we cannot enumerate all things of the universe, maybe they are infinite if taken simultaneously. For Scotus this actual infinite in nature can be understood by only an infinite intellect. From this he derives that this infinite intellect must be infinite perfect and unlimited. Let us look closer at two aspects in Scotus' quotation. Firstly, Scotus asserts that a greater number implies a greater perfection than a smaller number does. This aspect may be considered in relation to the large cardinal hierarchy. Some large cardinal notions seem to perfection the features of large cardinals which are located lower in the hierarchy. For example, in the case of measurable cardinals, comparing V with M where $\kappa$ is the critical point of some elementary embedding $j : V \longrightarrow M$, we have that $V_\kappa$ is contained in M, but, instead, few elements of $V_{j(\kappa)}$ are present in M. Maybe the image of $\kappa$, namely $j(\kappa)$, is very high but M is very thin. Thus, we can perfect this aspect of measurability and establish that $V_{j(\kappa)} \subseteq M$. In this way, we obtain stronger large cardinal notions such as a **superstrong cardinal**. Therefore Scotus' idea of perfection can be found also in the large cardinal hierarchy where perfection implies the concepts of closure and completeness.

The other idea from Scotus's quote that deserves attention is the following : if an intellect applies itself to an infinity, it is completely unlimited. The Human intellect is able to accomplish mathematical calculation about the infinite and it can perceive all different kinds of infinity distinctly. The human intellect respects the laws of the infinite, as Cantor was asserting in the Grundlagen, and so human beings are able to tame the transfinite. For Cantor, the human mind is finite but it can understand the actual infinite. I start to think that since the human intellect can conceive and use distinctly all large cardinal numbers without causing contradictions (unless Kunen's theorem), maybe the human intellect is distinct from the brain. Moreover, the human intellect can construct inner and outer models for almost all large cardinal numbers, and so it can make precise calculations about the infinite. Maybe, the physical state of the brain cannot capture the idea of the infinite but the intellect (mind) that supervenes on it can know intuitively the infinite. Maybe our physical support is fundamental for our Mind which supervenes on it, but then Mind is irriducible to the brain because our intellect can accede the world of abstract objects that physical states cannot accede. In fact, from my study about the infinite, I start to support **Supervenience** in philosophy of mind. I believe that Mind must be separated from its physical support, namely the brain. On the contrary, reductive physicalism sustains the identity between Mind and Brain and implies the reduction of mental states to physical states. I believe, as I said before, in Supervenience that is a non-reductive physicalism. In fact, I believe that Mind supervenes on the Brain, but then Mind cannot be reduced to the brain. There is an asymmetric dependency between Mind and brain. Brain is fundamental for Mind, but after Mind supervenes on the Brain, it cannot be reduced to physical states. While there cannot be only mental possible worlds, there can be only physical worlds. I do not believe that physical states can understand or conceive abstract mathematical objects, while Mind can accomplish this action. So the fact that, according to Cantor, Mind can tame the infinite, renders Mind irridubible to physical states. I use the concept of the infinite to show that Mind is irriducible to brain. Descartes [Descartes 641] used the idea of the infinite to prove the existence of God. The study of the infinite forces me to support this kind of non-reductive physicalism, namely Supervenience. Putnam sustains that mental states cannot be reduced to physical states because a single mental state can have multiple physical states that realize it. By assuming the irriducibility of Mind, Putnam advocates functionalism, a theory in philosophy of mind which holds that mental types and properties are functional types located in a higher level of abstraction than physical states. Mental properties are second-order functional properties whereas physical states are first-order properties. Furthermore, Davidson holds that Mind is irriducible to the brain because mental states are anomalous. Mind is anomalous because it has its features completely distinctive from physical states. The anomality of Mind makes impossible to find Laws which can connect mental states to physical states. I believe in Putnam's and Davidson's arguments. I add to these arguments that Mind is irriducible to brain because Mind can accede the world of sets or the abstract world of mathematics. A possible world characterized by only physicalism (we have only cerebral states and we do not mental states which supervene on them) cannot accede the world of sets. After the Supervenience occurs, we have two completely different domains, namely the domain of Mind and the domain of Brain, and the domain of Mind cannot be reduced to Brain, at least according to my view. I believe that each mental supervenient state (second-order functional state) has multiple subvenient physical states which realize it, but this mental state cannot be reduced to a physical state. I believe also that supervenient *Qualia* (qualitative mental states, phenomenal properties) have multiple subvenient physical realizators but at the end *Qualia* cannot be reduced to physical states. The laws ruling mental states are completely different from the laws ruling physical states and so, we do not have bridge laws which connect mental domain with physical domain. A reduction is impossible. Surely, we have to explain the problem of mental causation. We should ask ourselves if mental states can cause physical states. I believe that the answer to this question is negative. I assume that the world is physically closed. We can have only a physical state or multiple physical states which cause other physical states. Causation must be only physical. We have to distinguish between phenomenal causation and noumenal causation. phenomenal causation occurs when we interpret supervenient mental states as causes of other mental states or of subvenient physical states. The noumenal causation occurs when we describe as subvenient physical states cause other physical states. Adopting the words of Block, mental states are causally epiphenomenal. Mind is causally irrelevant. I believe that Mind, even if it is irriducible, is causally an epiphenomenon. Jaegwon Kim [Kim 00] asserts that it is possible reduce mental states to physical states by revising the reduction model (bridge laws) introduced by Nagel. I believe that this is impossible since mental states are realized by multiple physical states that might have a complex network of logical implications that we cannot know. Now to conclude this section, I want to introduce the following thought. The principle aim of Artificial Intelligence was that of creating thinking machines or generating artificial minds. Historically, departing from universal Turing machines, computer scientists have been developing Software more and more complicated in order to create thinking machines. They based their research on the concept of Software. I believe that this approach is wrong. In fact, since I believe that Mind supervenes on the Brain and the physical support is necessary, i think that Artificial Intelligence must base his research on the Hardware. Firstly, It is essential to create artificial neural networks. I support in Artificial Intelligence the theory of connectionism. In 1962 Rosenblatt developed *Perceptron* the first neural artificial network capable of calculating many mathematical functions. Another example of connectionism is NETTALK constructed by Sejnowsky. This artificial neural network is capable of reading every english word. At the end, I believe that artificial Mind might supervene on complicated artificial neural networks and, maybe, it might be able to accede the abstract world of mathematics.

## 6. Reinhardt Cardinals and Anselm's argument

let us begin with some considerations about Anselm's ontological argument for the existence of God. The first ontological argument was proposed by Anselm of Canterbury in 1078 in his Proslogion. Anselm defined God as **....that than which nothing greater can be conceived.....** He suggested that even *the fool* can understand this concept, and this understanding itself means that the being must exist in the mind. The concept must exist either only in our mind, or in both our mind and in reality. If such a being exists only in our mind, then a greater being, which exists in the mind and in reality, can be conceived. Therefore, if we can conceive of a being than which nothing greater can be conceived, it must exist in reality. Thus, **a being than which nothing greater could be conceived**, which Anselm defined as God, must exist in reality. At this point we can see how the argument works. When Anselm pronounces the expression **that than which nothing greater can be conceived**, everyone can understand the meaning. This notion is in the intellect, but it cannot be only in the mind. In fact, that which exists only in the mind is less than that which exists both in the mind and in reality; Thus if *that than which nothing greater can be conceived* exists only in the mind, we can think about something greater, namely *that than which nothing greater can be conceived, which exists in the mind and in reality.* We have a contradiction. In fact, we would affirm *that than which nothing greater can be conceived is that than which something greater can be conceived.* Therefore God exists. Even if Anselm describes God with an expression, his conception is similar to that of Plotinus, Damascius and Iamblicus which we face in the precedent section. For Anselm, God is beyond human reason and he actually adopts the apophatic method or *via negationis.* In fact, Anselm's expression is still negative. It is not a positive sentence about God. The philosopher belongs to what we have called negative theology. Anselm explains negative theology in Proslogion in the following way: If God is not greater of everything that can be thought, then God is not that than which nothing greater can be conceived; but God is that than which nothing greater can be conceived, therefore God is greater than everything that can be thought. Here, by using his negative expression, Anselm is asserting that God is beyond every human thought. For Anselm, God is incomprehensible like for neoplatonic philosophers (precedent section). However, if we say that Anselm belongs only to the negative theology and he uses only the apophatic method or *via negationis*, we are making a mistake. Thanks to his negative expression, Anselm is able to describe God's nature with positive attributes. In fact, we should say that Anselm adopts the apophatic method only initially. God is still beyond human thoughts, but thanks to his initial negative expression, Anselm is capable of deriving the essential positive attributes that describe God's essence. For example Anselm affirms that God is the supreme good. But if God were not the supreme good, he would not be that than which nothing greater can be conceived. Then Anselm asserts that God is omnipotent. But if God were not omnipotent, he would be that than which something greater can be conceived. Anselm starts with something negative, but then he is able to derive logically from that expression all positive attributes which characterise God. I believe that Anselm's proof is logically convincing and now I will try to respond to other philosophers who criticized this proof. The first philosopher-theologician, who argued against Anselm's proof, was Gaunilo. The argument of this philosopher has got two issues. First of all we have the example of the most perfect island that can be thought. According to Gaunilo, you can think about the most perfect island, but this does not mean that the island exists also in reality. Gaunilo makes a mistake. In fact, he identifies *that than which nothing greater can be conceived* with that which is greater than everything. The idea that the island is greater than all other islands for richness of goods, does not have anything to do with the expression *that than which nothing greater can be conceived.* In fact I can always think about a greater island, because the idea of the most perfect island is finite and I can always add something to this idea. The second Gaunilo's critique is more persuasive. He says that we cannot understand in clear way Anselm's expression which defines God as *that than which nothing greater can be conceived*. For Gaunilo, the nature of God is totally incomprehensible and so also Anselm's expression is meaningless. We have an extreme case of negative theology. For Gaunilo, you cannot say anything about God, neither a negative expression. You cannot reject this objection. In this case you can agree or disagree with Gaunilo. If you say that human beings are not capable of understanding Anselm's negative expression because God's nature is totally and absolutely incomprehensible, then you agree with Gaunilo. I personally disagree with Gaunilo. I believe that we are able to comprehend the exact meaning of Anselm's negative expression. Then there is Kant's objection contained in his Critique of Pure Reason. The German philosopher proposed that the statement God exists must be analytic or syntetic - the predicate must be inside or outside of the subject, respectively. If the proposition is analytic, as the ontological argument takes it to be, then the statement would be true only because of the meaning given to the words. Kant claimed that this is merely a tautology and cannot say anything about reality. However, if the statement is synthetic, the ontological argument does not work, as the existence of God is not contained within the definition of God (and, as such, evidence for God would need to be found). Kant writes that *being* is obviously not a real predicate and cannot be part of the concept of something. He proposed that existence is not a predicate, or quality. This is because existence does not add to the essence of a being, but merely indicates its occurrence in reality. He stated that by taking the subject of God with all its predicates and then asserting that God exists, I add no new predicate to the conception of God. He argued that the ontological argument works only if existence is a predicate; if this is not so, then it is conceivable for a completely perfect being to not exist, thus defeating the ontological argument. I disagree with Kant. Before arguing against Kant, I must reflect on the concept of contradiction and introduce the notion of extendible and Reinhardt cardinals. In my opinion, contradictions limit the ontology of mathematics. For example, the universal class (for Cantor's antinomy) and the class of all ordinals (for Burali-Forti antinomy) are proper classes, are not sets and so, they do not exist in ZFC universe. Contradictions force us to exclude these mathematical objects from the ontology of the mathematical universe. The class of all sets which do not belong to themselves (for Russell paradox) is avoided in ZFC by limiting the abstraction principle conceived by Frege and by introducing the limited axiom of separation (only if a set is already given, then a property can define a subset of it). Therefore also in this case contradictions force us to exclude from the ZFC universe the Russellian class. A Reinhardt cardinal implies the existence of an elementary embedding of V into itself, but because of Kunen's inconsistency result we are forced to exclude Reinhardt cardinals from the ontology of ZFC. Also in this case, contradictions limit the ontology of the mathematical universe. Now it is the moment to introduce the notion of extendible cardinal in the following way:

Definition 127. *$\kappa$ is $\eta$-extendible iff there is a $\sigma$ and a $j : V_{\kappa+\eta} \prec V_\sigma$ with $crit(j) = \kappa$ and $\eta < j(\kappa)$. $\kappa$ is extendible iff $\kappa$ is $\eta$-extendible for every $\eta > 0$.*

An extendible cardinal is a very large cardinal notion, since the whole theory of $V_{\kappa+\eta}$ is preserved in the embedding. A Reinhardt cardinal would be larger since whole V is preserved in the embedding. But (as we saw above) a Reinhardt cardinal does not exist in the universe of ZFC. Therefore, in this case, existence is fundamental to establish an hierarchy. Whereas Reinhardt cardinals are excluded from the hierarchy of ZFC because of inconsistency, extendible cardinals belong to the hierarchy. Therefore we can assert that existence is a predicate for abstract and immaterial objects. I think that we have to distinguish between two ontological planes, namely the platonic, abstract and immaterial ontological plane for mathematical objects and God, and the factual plane for things in reality. Existence is a predicate in the immaterial and platonic ontological plane and forces us to consider extendible cardinal belonging to the hierarchy and so a preferable and better notion than a Reinhardt cardinal (in ZFC), whereas it is not a predicate in the factual plane for things of the physical reality. Therefore, existence can be seen as a predicate for large cardinal numbers and God.

## 7. Paradoxes and the Curry-Liar paradox

In the precedent sections I spoke about Cantor, Burali-Forti and Russell antinomies, now I want to discuss other paradoxes. The main reason for doing this is that I believe that through paradoxes we can characterize Cantor's absolute infinite. The first paradox that i want to examine is **Berry's paradox** which is connected with the problem of giving precise definitions in mathematics. The Berry's paradox is a self-referential paradox arising from an expression such as **the smallest positive integer not definable in fewer than twelve words** (note that this defining phrase has fewer than twelve words). Berry was a junior librarian at Oxford (like Boole was a librarian at the university of Cork) and he discussed this paradox with Russell. Berry's self referential sentence arises from the more limited paradox which arises from the expression *the first undefinable ordinal*. We can consider the following expression: **the smallest positive integer not definable in under eleven words**. If there are positive integers that satisfy a given property, then there is a smallest positive integer that satisfies that property; This is the integer to which the above expression refers. The above expression is only ten words long, so this integer is defined by an expression that is under eleven words long. This is a paradox: there must be an integer defined by this expression, but since is self-contradictory (any integer it defines is definable in under eleven words), there cannot be any integer defined by it. Berry paradox points out that definitions can be vague. Definitions must be precise. This antinomy is very similar to the Liar and Russell paradox. In fact, it is a self-referential sentence. A language that speaks about itself is very dangerous. If we remain at the same level of language, the Liar and Berry paradox do not have a solution because they generate a vicious circle, but if we form a stratification of languages (meta-languages), these antinomies can have a solution. They cannot have an ending point or a roof. In fact, we would have Russell and Berry paradox specific for the object language, then for the meta-language 1, then for the meta-language 2, then for the meta-language 3, etc. We would have solution to Berry and the Liar paradox, when higher levels of language (higher meta-languages) reflect on lower levels of language (lower meta-languages), but then we would have these antinomies specific for the higher levels of language. Therefore, the hierarchy of meta-languages must be potentially existent.

In logic, Richard's paradox is a semantical antinomy in set theory and natural language first described by the french mathematician Jules Richard in 1905. The original statement of the paradox has a relation to Cantor's diagonal argument of the uncountability of real numbers. The paradox begins with the observation that certain expressions in English unambigously define real numbers, while other expressions in English do not. Thus, there is an infinite list of english phrases that unambigously define real numbers; arrange this list by length and then order lexicographically, so that the ordering is canonical. This yields an infinite list of the corresponding real numbers: $r_1$, $r_2$......,etc. Since real numbers are dense (between two real numbers, there is always a third real number), we can consider real numbers in the interval $[0, 1]$. Then we can write real numbers in binary digits in the following way:

$$\begin{bmatrix} r_1 : \mathbf{0}110001010011... \\ r_2 : 0\mathbf{1}11110101011... \\ r_3 : 11\mathbf{0}1110010111... \\ r_4 : 101\mathbf{0}111100011.... \\ r_5 : 1110\mathbf{1}11010001.... \end{bmatrix}$$

Go down the diagonal, taking the n-th digit of the n-th real number $r_n$ (in our example produces 01001) and flip each digit, swapping 0s and 1s (in our example produces 10110). By construction, this flipped diagonal real number differs from $r_1$ in the first place, from $r_2$

in the second place and so on. So our diagonal construction defines a new real (a richardian real) which differs from all the other reals. Now define a real number (richardian real) in the following way: **the n-th digit of the n-th real number $r_n$ is the opposite** (if it is 0, it is 1 and if it is 1, it is 0). This definition is an expression in English which unambiguosly defines a real number **r** (a richardian real number). Thus **r** must be one of the $r_n$ numbers. However, **r** was constructed so that it cannot equal any of the $r_n$. This is a paradoxical contradiction. If we take formalised languages, it is possible to say that a formula $\phi(x)$ defines a real number if there is exactly one real number $r$ such that $\phi(r)$ holds. Then it is not possible to define, in ZFC, the set of all formulas that define real numbers. For, if it were possible to define this set, it would be possible to diagonalize over it to produce a new definition of a real number, following the outline of Richard's paradox above.

One problem in logic is the nature of many irrational numbers. We do not know how they are. Alan Turing was very keen on computing real numbers but we do not know their nature. At this point, i want to discuss this philosophical thought. When you have a matrix of real numbers, namely a list of real numbers, you can form the antidiagonal set (a richardian real). Now we can think to add this antidiagonal set to the precedent matrix, then we have a new matrix. We can diagonilise out from this matrix and form a new antidiagonal set (the second richardian real). By accomplishing this operation, we form the third, the fourth richardian real and so on. This operation can be iterated through the infinite and it does not have any bound. So, maybe we can think that we might charcterise a large part of irrational numbers as richardian reals. If this operation does not have a bound, we can always diagonilise out until the set of richardian reals overlaps the set of irrational numbers. So, maybe it is wrong, but irrational numbers could be seen as richardian reals.

At this point, I want to discuss Curry paradox. This antinomy occurs in naive set theory and naive logics, and allows the derivation of an arbitrary sentence from a self-referring sentence and some apparently innocuous logical deduction rule. For example, if we say **if this sentence is true, Catalunya is an independent European state**. Even if I hope that Catalunya will be independent, for the moment the consequent of this conditional is clearly false. The sentence **if this sentence is true, Catalunya is an independent European state** is itself true. The quoted sentence is of the form *if A then B* where A refers to the sentence itself and B refers to **Catalunya is an independent European state**. The usual method for proving a conditional sentence is to show that by assuming that hypothesis (A) is true, then the conclusion (B) can be proven from that assumption. Therefore, for the purpose of the proof, assume A. Because A refers to the overall sentence, this means that assuming A is the same as assuming *if A, then B*. Therefore, in assuming A, we have assumed both A and *if A, then B*. From these, we can obtain B by **modus ponens**. Therefore *Catalunya is an independent European state*, but we know that is false, which is a paradox. We can reason also in the following way. Suppose that the sentence A is false. Then, for the law of material implication, the only possibility admitted is that the antecedent of this conditional ( A is true) is true, whereas the consequent (then Catalunya is an independent European state) is false. But sustaining that the antecedent is true is the same as defining true A, contradicting in this way what we have said. Therefore, we must conclude that the sentence A is true, and this forces us to say that is true also the proposition **Catalunya is an independent European state**. Curry paradox is very important since it is the only paradox that is negation-free (this aspect is important for paraconsistent logics). People who think that, in order to avoid paradoxes, we should use only positive defining properties, should be aware of the existence of this paradox. This paradox is really problematic in mathematical logic. The paradoxical sentence is an apriori truth and so it can be true in every system of logic since it does not need to be supported by any other postulate. From this paradoxical sentence, it is possible to derive as true the sentence B (Catalunya is an independent Euoropean state) and its negation (Catalunya is not an independent European state). It is very similar to the law of Pseudo Scotus, namely from the absurd we can derive any propositions. In fact, we can apply this paradoxical sentence to each proposition and then prove it. By proving anything, you render the formal system inconsistent. Arthur Prior from the paradox of Curry derives the existence of god in the following way:

> C= If C is true, then God exists. To avoid the paradox, the consequent (God exists) must be true.

. Now I want to highlight a sentence that I conceived, namely **if this sentence is true, the consequent of this conditional is false**. I combine Curry paradox with the Liar paradox. The amusing thing about this paradox is that when we have Curry paradox because we interpret as false the consequent of this conditional, for the dynamic of the liar paradox (if the consequent is false, because it is saying that it is false then it is true) we escape from the Curry paradox. We have a second level of abstraction in the Curry-Liar paradox. When we say that the consequent is false and so at the first level of abstraction we have Curry paradox, then (for the dynamic of the liar paradox) at the second level of abstraction we do not have anymore Curry paradox. However if we judge the consequent as true and so at the first level of abstraction we do not have the Curry paradox, then (for the dynamic of the liar paradox) at the second level of abstraction we do have Curry paradox. If a contradiction implies another contradiction, you remain in the realm of absurdity, but with the Curry-Liar paradox (nested contradictions) we can escape from absurdity thanks to the liar paradox at the second level of abstraction.

CHAPTER 5

# Appendix

## 1. Silver indiscernibles

[1] 1970 Jack Silver used indiscernibles as a concept in set theory. When he was a student in Berkeley, he was able to isolate the concept of $0^\sharp$, a great divide in the landscape of large cardinals. The concept of $0^\sharp$ is originated by the analysis of L (the constructible universe) based on the construction of Silver indiscernibles. Now we can state the following important theorem:

THEOREM 140. *(Silver) If there is a Ramsey cardinal then: (1) if $\kappa$ and $\lambda$ are uncontable cardinals and $\kappa < \lambda$ then $(L_\kappa)$ is an elementary substructure of $L_\lambda$, (2) There is a unique closed unbounded class of ordinals I containing all uncoutable cardinals such that for every uncountable cardinal $\kappa$ : $|I \cup \kappa| = \kappa$, $I \cup \kappa$ is a set of indiscernibles for $L_\kappa$ and every $a \in L_\kappa$ is definable in $L_\kappa$ from $I \cup \kappa$.*

The elements of the class I are called Silver indiscernibles. Before going further, we need to focus on the reason why we have introduced the concept of Ramsey cardinal. This large cardinal notion comes from the partition calculus, as we have already seen. Let $\kappa$ be an infinite cardinal, let $\alpha$ be an infinite limit ordinal $\alpha \leq \kappa$, and let $m$ be a cardinal $2 \leq m < \kappa$. The symbol $\kappa \longrightarrow (\alpha)^{<\omega}_m$ denotes the property that for every partition F of the set $[\kappa]^{<\omega}$ (the finite subsets of $\kappa$) into $m$ pieces, there exists a set $H \subset \kappa$ of order-type $\alpha$ such that for each $n \in \omega$, F is constant on $[H]^n$. A cardinal $\kappa$ is a Ramsey cardinal if $\kappa \longrightarrow (\kappa)^{<\omega}$. Since we cannot have $\omega \longrightarrow (\omega)^{<\omega}$ because it is false, we have an homogeneous set $\kappa$ which

[1]See [Jech 06]

is uncountable and we can use it as a set of Silver indiscernibles.

By the Reflection Principle if $\phi$ is a formula, then there exists an uncountable cardinal $\kappa$ such that $L \models \phi(a_1......a_n)$ if and only if $L_\kappa \models \phi(a_1.......a_n)$, for every $a_1, ......a_n \in L_\kappa$. By the precedent theorem the right hand side holds if and only if $L_\lambda \models \phi(a_1.....a_n)$ for all cardinals $\lambda \geq \kappa$. Therefore if Silver indiscernibles are used to generate L, we have a great reflection phenomenon. Thus, by the precedent theorem, we have $(L_\kappa, \in) \prec (L, \in)$ for every uncountable cardinal $\kappa$. As a consequence of the Theorem, Silver indiscernibles are indiscernibles for L: if $\phi(v_1.....v_n)$ is a formula then $L \models \phi[a_1.....a_n]$ if and only if $L \models \phi[b_1.....b_n]$ whenever $a_1 < ........ < a_n$ and $b_1 < ........ < b_n$ are increasing sequences in I.

Every constructible set is definable from I. For what we have said before, every formula $\phi(v_1.......v_n)$ is either true or false in L for any increasing sequence of Silver indiscernibles. Moreover, the truth value concides with the truth value of $L_{\aleph_\omega} \models \phi[\aleph_1......\aleph_n]$ since $L_{\aleph_\omega} \prec L$ and $\aleph_1.......\aleph_n$ are Silver indiscernibles. At the beginning of this chapter, we have introduced the concept of $0^\sharp$, now we can define it in the following way: $0^\sharp = \{\phi : L_{\aleph_\omega} \models \phi[\aleph_1......\aleph_n]\}$. If Silver's theorem holds, then $0^\sharp$ exists. The set $0^\sharp$ is, strictly speaking, a set of formulas. But as formulas can be coded by natural numbers, we can regard $0^\sharp$ as a subset of $\omega$. Devlin-Paris showed how to get $0^\sharp$ from a combinatorial consequence of $\kappa \longrightarrow (\omega_1)_2^{<\omega}$. This large cardinal property has less consistency strength than a Ramsey cardinal. Now we can introduce the following two theorems:

THEOREM 141. *Assuming* $0^\sharp$*, every set in V definable in L without parameters is countable.*

PROOF. If $x \in L$ is definable in L by a formula $\phi$, then the same formula defines $x$ in $L_{\aleph_1}$, thus $x \in L_{\aleph_1}$. □

In particular, every ordinal number definable in L is countable.

THEOREM 142. *Assuming $0^\sharp$, every uncountable cardinal is inaccessible in L.*

Now, recall that a cardinal $\kappa$ is Mahlo if it is regular and the set of all inaccessible cardinal below $\kappa$ is stationary, namely this set intersects all closed unbounded subset of $\kappa$. We can introduce the following theorem

THEOREM 143. *Every uncountable cardinal is a Mahlo cardinal in L*

The proof of Silver's theorem [2] is based on a theorem of Ehrenfeucht and Mostowski in model theory, stating that every infinite model is elementarily equivalent to a model that has a set of indiscernibles of prescribed order-type. We shall use the canonical well-ordering of L to endow the models $(L_\lambda, \in)$ with definable Skolem functions. For each formula $\phi(\alpha, \beta_1........\beta_n)$, let $h_\phi$ be the n-ary function defined as follows: $h_\phi(\beta_1.......\beta_n) = \{the\ <-least\ \alpha\ such\ that\ \phi(\alpha, \beta_1.....\beta_n, ), \emptyset\ otherwise\}$.
We call $h_\phi$, $\phi \in FORM$, the canonical Skolem function. For each limit ordinal $\lambda$, $h_\phi^{L_\lambda}$ is an n-ary function on $L_\lambda$, the $L_\lambda$ interpretation of $h_\phi$, and it is definable in $(L_\lambda, \in)$. For each limit ordinal $\lambda$, the functions $h_\phi^{L_\lambda}$, $\phi \in FORM$, are Skolem functions for $(L_\lambda, \in)$ and so a set $M \subset L_\lambda$ is an elementary submodel of $(L_\lambda, \in)$ if and only if M is closed under the $h_\phi^{L_\lambda}$. If $X \subset L_\lambda$, then the closure of $X$ (the Skolem hull) under the $h_\phi^{L_\lambda}$ is the smallest elementary submodel $M \prec L_\lambda$ such that $X \subset M$, and is the collection of all elements of $L_\lambda$ definable in $L_\lambda$ from $X$ and $ordinals < \lambda$. The construction of the Skolem hull is very common in logic. The downward Löwenheim-Skolem theorem and Gödel's completeness theorem are based essentially on this construction. The Skolem hull implies the phenomenon of reflection. By Levy's reflection principle, $\Sigma_1$-formulas are reflected by an initial segment of the universe. From a philosophical perspective, this is very interesting, since all logical operations that we can accomplish in structures of higher cardinality, can be done in structures of lower cardinality. The Skolem hull, which implies a kind of structural reflection, makes us able

[2] I took many issues from [Jech 06]

to simplify our logical calculation.

Let $\lambda$ be a limit ordinal and let M=(A,E) be a model elementarily equivalent to $(L_\lambda, \in)$ . The set $On^M$ of all ordinal numbers of the model M is linearly ordered by E; let's use $x < y$ rather than x E y for $x, y \in On^M$. A set $I \subset On^M$ is a set of indiscernibles for M if for every formula $\phi$, $M \models \phi(x_1, ........, x_n)$ if and only if $M \models \phi(y_1, ......, y_n)$, whenever $x_1 <, ......, < x_n$ and $y_1 <, ......, < y_n$ are elements of I. Let $h^M_\phi$ denote the M-interpretation of the canonical Skolem functions. Given a set $X \subset A$, let us denote by $H^M(X)$ the closure of X under all $h^M_\phi$, $\phi \in Form$. The set $H^M(X)$ is the Skolem hull of X and is an elementary submodel of M.

If I is a set of indiscernibles for M, let $\Sigma(M, I)$ be the set of all formulas $\phi(v_1.....v_n)$ true in M for increasing sequences of elements of I:

$\phi(v_1......v_n) \in \Sigma(M, I) \leftrightarrow M \models \phi(x_1, .....x_n)$ for some $x_1......x_n \in I$ such that $x_1 < ...... < x_n$.

A set of formulas $\Sigma$ is called E.M set (Ehrenfeucht-Mostowski) if there exists a model M elementarily equivalent to some $L_\lambda$, $\lambda$ a limit ordinal, and an infinite set I of indescirnibles for M such that $\Sigma = \Sigma(M, I)$.

Lemma 20. *If $\Sigma$ is an E.M set and $\alpha$ is an infinite ordinal number, then there exists a model M and a set of indiscernibles I for M such that: 1) $\Sigma = \Sigma(M, I)$, 2) the order-type of I is $\alpha$, 3) $M = H^M(I)$*

Now I would like to pay attention to the third clause of the lemma which is asserting that the model M is equal to its Skolem hull. In this case the Skolem functions instantiate formulas by picking elements of I, namely the indiscernibles of the model. For each E.M set $\Sigma$ and each ordinal $\alpha$, let us call the $(\Sigma, \alpha) - model$ the unique pair (M,I) given by the precedent lemma. At the end we will show that the existence of a Ramsey cardinal implies the existence of an E.M. set $\Sigma$ having a certain syntactical property (remarkability) and such that every $(\Sigma, \alpha)$-model is well-founded. Let's start with well-foundeness first.

Lemma 21. *the following are equivalent, for an E.M set $\Sigma$: 1) for every ordinal $\alpha$, the $(\Sigma, \alpha)$ model is well-founded, 2) for some ordinal $\alpha \geq \omega_1$, the $(\Sigma, \alpha)$ model is well-founded, 3) for every ordinal $\alpha < \omega_1$, the $(\Sigma, \alpha)$-model is well-founded.*

We shall now define remarkability. We consider only $(\Sigma, \alpha)$-models where $\alpha$ is an infinite limit ordinal. Let us say that a $(\Sigma, \alpha) - model$ (M,I) is unbounded if the set I is unbounded in the ordinals of M, that is, if for every $x \in ORD^M$ there is $y \in I$ such that $x < y$.

Lemma 22. *The following are equivalent, for any E.M. set $\Sigma$: 1) for all $\alpha, (\Sigma, \alpha)$ is unbounded, 2) For some $\alpha$, $(\Sigma, \alpha)$ is unbounded, 3) For every Skolem term $t(v_1......v_n)$ the set $\Sigma$ contains the following formula : if $t(v_1.....v_n)$ is an ordinal, then $t(v_1......v_n) < v_{n+1}$.*

Thus we say that an E.M. set $\Sigma$ is unbounded if it contains the precedent formula for all Skolem terms t. Let $\alpha$ be a limit ordinal, $\alpha > \omega_1$, and let (M,I) be the $(\Sigma, \alpha)$-model. For each $\sigma < \alpha$, let $i_\sigma$ denote the $\sigma th$ element of I. We say that (M,I) is remarkable if it is unbounded and if every ordinal x of M less than $i_\omega$ is in $H^M(i_n : n \in \omega)$.

Lemma 23. *the following are equivalent for any unbounded E.M set $\Sigma$: 1) For all $\alpha > \omega$ the $(\Sigma, \alpha)$- model is remarkable, 2) for some $\alpha > \omega$ the $(\Sigma, \alpha)$ model is remarkable, 3) For every Skolem term $t(x_1..........x_m, y_1........y_n)$ the set $\Sigma$ contains the formula: if $t(x_1........x_m, y_1..........y_n)$ is an ordinal smaller than $y_1$, then $t(x_1.......x_m, y_1........y_n) = t(x_1.......x_m, z_1......z_n)$.*

Remarkability implies a conception of completeness since the indiscernibles are unbounded and for every ordinal in an initial segment there is a correspondent Skolem term. Coming back to the issue of the section, we have to say an E.M. set $\Sigma$ is remarkable if it is unbounded and contains the formula of the precent lemma (clause (3)) for all Skolem terms t. An important consequence of remarkability is the following: Let (M,I) be a remarkable

$(\Sigma, \alpha)$ model and let $\gamma < \alpha$ be a limit ordinal. Let J $= (i_\sigma : \sigma < \gamma)$ and let $B = H^M(J)$. Then $(B, J)$ is the $(\Sigma, \gamma)$ model and the ordinals of B form an initial segment of the ordinals of M.

We call an E.M set $\Sigma$ well-founded if every $(\Sigma, \alpha)$ model is well founded:

THEOREM 144. *(Silver) If there exists a Ramsey cardinal, then there exists a well-founded remarkable E.M. set.*

If there exists a Ramsey cardinal, then Theorem 84 holds. For every limit ordinal $\alpha$, the $(\Sigma, \alpha)$ model is a well-founded model elementarily equivalent to some $L_\gamma$, and so is isomorphic to some $L_\beta$.

LEMMA 24. *If $\kappa$ is un uncountable cardinal, then the universe of the $(\Sigma, \kappa)$ model is $L_\kappa$*

For each uncountable cardinal $\kappa$, let $I_\kappa$ be the unique subset of $\kappa$ such that $(L_\kappa, I_\kappa)$ is the $(\Sigma, \kappa)$ model . $I_\kappa$ is closed and unbounded in $\kappa$.

LEMMA 25. *if $\kappa < \lambda$ are uncountable cardinals, then $I_\lambda \cap \kappa = I_\kappa$ and $H^{L_\lambda}(I_\kappa) = L_\kappa$*

Using this lemma we can prove both (1) and (2) of Theorem 84, except for the uniqueness of Silver indiscernibles. We let $I = \bigcup\{I_\kappa : \kappa \textit{ is an uncountable cardinal}\}$. For each uncountable cardinal $\kappa$, $I \cap \kappa = I_\kappa$ is a closed unbounded set of order type $\kappa$, and is a set of indiscernibles for $L_\kappa$; moreover, every $\alpha \in L_\kappa$ is definable in $L_\kappa$ from $I_\kappa$ and it follows that $\kappa \in I_\lambda$; hence I contains all uncountable cardinals. Also, since $L_\kappa = H^{L_\lambda}(I_\kappa)$, we have $L_\kappa \prec L_\lambda$. The next two lemmas prove the uniqueness of Silver indiscernibles and of the corresponding E.M. set.

LEMMA 26. *(Silver) There is at most one well-founded remarkable E.M. set.*

We, therefore, define $0^\sharp$ in the following way: $0^\sharp$ is the unique well-founded remarkable E.M. set if it exists. The uniqueness of Silver indiscernibles now follows from:

Lemma 27. *For every regular uncountable cardinal $\kappa$ there is at most one closed unbounded set of indiscernibles $X$ for $L_\kappa$ such that $L_\kappa = H^{L_\kappa}(X)$.*

Thus we have pointed out that (1) and (2) of Theorem 84 hold under the assumption that $0^\sharp$ exists. On the other hand, if (2) of Theorem 84 holds, then $0^\sharp$ exists because, $(L_{\omega_1}, I \cap \omega_1)$ is a remarkable well-founded model with $\aleph_1$ indiscernibles. So if there is a Ramsey cardinal, then $0^\sharp$ exists. That will follow from the following lemma:

Lemma 28. *Let $\kappa$ be an uncountable cardinal. If there exists a limit ordinal $\lambda$ such that $(L_\lambda, \in)$ has a set of indiscernibles of order-type $\kappa$, then there exists a limit ordinal $\gamma$ and a set $I \subset \gamma$ of order-type $\kappa$ such that $(L_\gamma, I)$ is remarkable.*

It follows that if $\kappa$ is Ramsey, then $(L_\kappa, \in)$ has a set of indiscernibles of order-type $\kappa$. Then, there exists a remarkable model $(L_\lambda, I)$ where I has order-type $\kappa$. $\Sigma(L_\gamma, I)$ is well-founded and remarkable and hence $0^\sharp$ exists.
The set $0^\sharp$ is, strictly speaking, a set of formulas. But as formulas can be coded by natural numbers, we can regard $0^\sharp$ as a subset of $\omega$.

Lemma 29. *(Silver) The property ($\Sigma$ is a well-founded remarkable E.M. set) is absolute for every inner model of ZF. Hence $M \models 0^\sharp$ exists if and only if $0^\sharp \in M$ in which case $(0^\sharp)^M = 0^\sharp$.*

Since a well-founded ultrapower of the universe induces an elementary embedding $j_u : V \longrightarrow Ult$, and conversely, if $j : V \longrightarrow M$ is a nontrivial elementary embedding, then it is possible to define a normal measure on the least ordinal moved by j. Let j be a nontrivial elementary embedding of the universe, and let M be a transitive model of ZFC, containing all the ordinals. Let $N = j(M) = \bigcup_{\alpha \in Ord} j(M \cap V_\alpha)$. Then N is a transitive model of ZF and $j : M \longrightarrow N$ is elementary: $M \models \phi(\alpha_1.....\alpha_n)$ if and only if $N \models \phi(j(\alpha_1), .......j(\alpha_n))$. In particular, if $M = L$, then $j(V) \models (N\ is\ the\ constructible\ universe)$, and so $N = L$, and

$j|L$ is an elementary embedding of L in L. Thus if there exists an elementary embedding of L (into L), then $V \neq L$. If $0^\sharp$ exists, then there are nontrivial elementary embeddings of L. In fact, let j be any order-preserving from the class I of all Silver indiscernibles into itself. Then j can be extended to an elementary embedding of L; we simply let $j(t^L[\gamma_1.....\gamma_n]) = t^L[j(\gamma_1), ....., j(\gamma_n)]$ for every Skolem term t and any Silver indiscernibles $\gamma_1 < ......... < \gamma_n$. Also the converse is true, if there is a nontrivial elementary embedding of L, then $0^\sharp$ exists:

THEOREM 145. *(Kunen) The following are equivalent : 1)* $0^\sharp$ *exists, 2) There is a nontrivial elementary embedding* $j : L \longrightarrow L$.

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