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Regularization by noise in finite dimension

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Chapter 1

Introduction

1.1 Regularization by noise: definition, aim and motivation

We say that a regularization by noise phenomenon occurs for a possibly ill-posed ODE or PDE if this equation becomes well-posed under addition of noise. The main aim of this thesis is to study such a phenomenon in the context of finite-dimensional ODEs, with additive noise, and related linear transport-type PDEs.

Many ODEs and related linear PDEs with irregular coefficients, and many nonlinear PDEs, exhibit ill-posedness, for example explosion (non existence of suitable solution), non uniqueness, lack of regularity, lack of stability. For the ODE case, the most prominent example is given in dimension 1 by

$$\mathrm{d}X = 2\mathrm{sign}(X)|X|^{1/2}\mathrm{d}t, \quad X_0 = 0,$$

which has infinitely many solutions, namely all the solution that stays in 0 up to a time t_0 (possibly 0 or $+\infty$) and then leave 0 as $\pm(t-t_0)^2$. When we perturb the equation with a suitable (even small) noise, it can happen that the equation gains well-posedness, in a strong, pathwise sense: for almost every realization of the noise, the perturbed equation exhibits existence, uniqueness and, in some sense, stability. In the example above, we can add additive noise, i.e. a 1-dimensional Brownian motion W: the SDE

$$\mathrm{d}X = 2\mathrm{sign}(X)|X|^{1/2}\mathrm{d}t + \mathrm{d}W, \quad X_0 = 0$$

gains existence and pathwise uniqueness.

We aim here at describing this phenomenon in the case of finite-dimensional SDEs with additive noise, i.e.

$$\mathrm{d}X = b(t,X)\mathrm{d}t + \mathrm{d}W$$

on $\mathbb{R}^d$, where W is a d-dimensional Brownian motion, and of the associated linear transport-type SPDEs, like the stochastic transport equation

$$\partial_t v + b \cdot \nabla v + \nabla v \circ \dot{W} = 0,$$

where v is the random solution and $\circ$ denotes Stratonovich integration. Our main goal is to show regularization by noise for a wide class of vector field b, often where only integrability conditions on b are imposed. We will mainly look at restoring uniqueness, for ODEs with more than one solution (like the example above), and/or avoiding concentration of particles, for ODEs with non-injective flows, and/or restoring regularity of the flow, for ODEs with space-irregular flows. We will give an intuition on the phenomenon and proofs from different points of view and with different techniques, some based on the SDEs some on the SPDEs, some using the regularizing properties at fixed realization of the noise some exploiting the martingale structure.

One motivation to look at this problem comes from physics, where irregular vector fields are present in many situation, especially in fluid dynamics. The SDE above can be thought describing a motion of a particle subject to this irregular vector field and perturbed by noise. On the mathematical side, the main interest in this phenomenon comes from its counterintuitive nature (at a first glance, one may not expect that a time-irregular Brownian perturbation can correct a space-irregular vector field); the analysis of this phenomenon can therefore give more insight on the effect of noise on ill-posed system. This topic can also provide new techniques to solve SPDEs, and it can be considered as a first step before investigating the non-linear case (much more difficult in general). Finally, it may give some hints on the problem of zero noise: if we add $\epsilon \mathrm{d}W$ to the ODE and we let ϵ go to 0, which solutions of the ODE do we select? This might be a way to select the physically relevant solutions, although this is far from the scopes of this thesis.

1.2 Classes of examples

Before going into the types of examples, we underline one point of our analysis: we look at the SDE above mainly at a fixed realization $W(\omega)$ of the noise, but letting the initial datum x_0 vary; notice indeed that the SDE has a natural interpretation at ω fixed, without stochastic integration. In other words, at least at a formal level, we keep ω, the random input, fixed and we look at the flow $X^\omega(t, x_0)$, i.e. the map $X^\omega : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ such that, for every x_0, $X^\omega(\cdot, x_0)$ solves the SDE, for ω fixed, starting from x_0. We are less interested here at the law of the process $X(\cdot, x_0)$ for x_0 fixed.

The propotype questions we want to answer are the following one:

- for fixed ω, for given x_0, is the solution to the SDE above, at ω, starting from x_0, unique?
- for fixed ω, does the flow X^ω have spatial regularity, of Lipschitz type or Sobolev type?
- for fixed ω, does the flow X^ω avoid concentration (particles starting from distant points that come very close one to each others)?
- for fixed ω, if concentration is avoided but uniqueness may fail for single x_0, is there at least uniqueness among flows which do not concentrate particles?
- in possible non-uniqueness situation, is there a unique way to select the trajectories solution to the SDE (i.e. a unique, in some sense, measure supported on the set of solutions)?

Depending on the answer to these questions and on the type of ill-posedness of the ODE, we have different classes of examples (with some overlapping among classes). For each class, we give the corresponding regularization by noise result proved in the thesis, under some simplified assumptions (for example, we assume that the drift is time-independent) and referring to the true results after for precise statement and notation.

In the first class, the most restrictive one where the strongest regularization by noise occurs, the ODE shows non-uniqueness but noise restores uniqueness for ω and x_0 fixed and Lipschitz regularity of the flow. To get an idea of this, imagine a drift b which is regular everywhere but in 0. Under certain integrability conditions on the singularity in 0, the noise is much faster than any possible solutions to the deterministic ODE, and so moves the particle away from 0 before any non-uniqueness (or concentration or irregularity) phenomenon can appear.

The main result in this class is the following one (from Theorem 8.1, see also 9.1 for the associated PDE and 11.5 for a general Hölder continuous drift):

Theorem 1.1. *Assume that b is in $C^\alpha_{x,b}$ for some $\alpha > 1/2$. Then, for ω fixed, for any initial datum x, existence, uniqueness and (local) Lipschitz regularity in space hold for the solution to the SDE.*

In the second class, a similar phenomenon happens, for ω fixed, but at the level of flows rather than of fixed x_0: the noise avoids concentration of particles and restore uniqueness and Sobolev regularity among the flows that do not concentrate particles. In principle, the noise might not be so strong to restore existence and uniqueness from a single initial datum.

The main result in this class is the following one (from Theorems 10.1, 11.1 and 12.1):

Theorem 1.2. *Assume that b is in L^p_x for some $p > d \vee 2$. Then, for ω fixed, existence, uniqueness and $W^{1,m}_{x,loc}$ regularity in space holds among flows, solution to the SDE, with spatial density in $L^m_{x,loc}$ (precisely in $L^m_{t,x,loc}$), for m finite high enough.*

In the third class, the ODE shows concentration of particles, noise avoids it as before but the flow does not need to be weakly differentiable anymore.

The main result in this class is the following one (from Theorem 6.4):

Theorem 1.3. *Assume that b is in $W^{1,\tilde{p}}_x$ for some $\tilde{p} > 1$, $\mathrm{div} b$ is in L^p_x for some $p > d/2 \vee 1$ and b has compact support. Then, for ω fixed, existence and uniqueness hold among flows, solution to the SDE, with spatial density in $L^m_{x,loc}$ (precisely in $L^m_{t,x,loc}$), for m finite high enough.*

In the fourth class, the weakest and largest one, the noise at ω fixed may not produce uniqueness, but uniqueness in law (at x_0 fixed or with a diffuse initial datum X_0) provides a unique probability measure on the set of solution at ω, which is given by filtering the unique law with respect to W.

Regularization by noise, strictly speaking, deals with the first three classes of examples, which we are mainly interested in, but we will also investigate the fourth type of examples, on the continuity equation (which will be stated in the next section). We get indeed this result (from Theorem 7.12):

Theorem 1.4. *Assume that b is in $L^{\tilde{p}}_x$ for some $\tilde{p} > 2$, $\mathrm{div} b$ is in L^q_x for some $q > p/2 \vee 1$ and b has compact support. Then existence and uniqueness hold for the stochastic continuity equation in L^m among solutions in a suitable weighted $L^m_{t,x,\omega}$ space, for m finite high enough, and adapted to the Brownian (completed) filtration.*

1.3 The linear SPDEs

We have mentioned we look at the SDE for ω fixed and consider, formally, uniqueness and concentration properties of the flow solution to the SDE. Now we want to be more precise on these concepts; this brings to the formulation of two stochastic PDEs, the continuity equation and the transport equation.

An SDE with irregular coefficients may not even have a flow (since there could be more than one solution starting from any point). But we can still speak of uniqueness and concentration properties when we introduce a mass: we take μ_0, an initial finite measure on $\mathbb{R}^d$, and look at the evolution of μ_0 under the SDE (at ω fixed). In the case we have a flow X, this means looking at

$$\mu_t^\omega = (X_t^\omega)_\# \mu_0, \tag{1.1}$$

the image measure of μ_0 under X_t, but we can extend the definition also when X is replaced by any family $X(x)$ of solutions parameterized by the initial datum x (and to suitable superposition of such families). We stress the fact that μ_t is not the law of the SDE of the probability measure P, for a fixed initial point x, but is the "law" under X of the initial measure μ_0, for a fixed ω. The family of measures $(\mu_t)_t$ satisfies the (stochastic) continuity equation

$$\partial_t \mu + \mathrm{div}(b\mu) + \mathrm{div}(\mu \circ \dot{W}) = 0,$$

in the sense of distributions in space and in the integral formulation in time, where $\circ$ denotes Stratonovich integration. Using the equation, we can still formulate the problems of uniqueness and concentration. For example, the non-concentration property translates into the existence of a solution μ which stays, at any time t, absolutely continuous with respect to Lebesgue measure (starting from μ_0 absolutely continuous); one can specialize this concept asking that μ_t has a density in L^m_{loc} for some m. The uniqueness property starting from a single point x translates into uniqueness for the continuity equation starting from δ_x, while the uniqueness property among non-concentrating flows translates into uniqueness among solutions which are absolutely continuous with respect to Lebesgue measure (starting from a diffuse initial condition). In particular, the continuity equation can capture easily well-posedness in cases where this holds among non-concentrating flows but not among single solutions. Therefore it represents a well-suited framework where to study some of our questions.

We also consider the (stochastic, backward) transport equation

$$\partial_t v + b \cdot \nabla v + \nabla v \cdot \circ \dot{W} = 0,$$

again in the sense of distributions in space and in the integral formulation in time. This equation is well-suited for studying the regularity of the flow. Indeed we have the formal representation formula

$$v_{s,t}(x) = v_t(X_{s,t}(x)), \tag{1.2}$$

where $v_{s,t}$ is the solution, at time s of the transport equation, with final time t and final datum v_t, and $X_{s,t}$ is the flow with s, t as initial and final times. So any differentiability property of the flow (in space x) translates into a differentiability property for the transport equation (with regular final datum).

A general advantage of these SPDEs is the linearity, which helps in various situations, for example for a priori estimates. Maybe the most relevant context where such linearity is exploited is the duality method (which will be explained later): the continuity equation and the transport equation are formally dual one to each other. This provides one formal way to prove uniqueness, which can be made rigorous with not so much effort, under suitable regularity assumptions. More important, the duality technique has the advantage to be essentially deterministic, which helps for results at ω fixed.

We will give some proofs which use only flows (Lagrangian approach), some which use only PDEs (Eulerian approach) and can be of interest on their own, for linear SPDEs, without the link with the flows.

1.4 Different kinds of uniqueness

The SDE uniqueness can be of different kinds, we state and explain them. Notice that the SDE has an easy interpretation at $W(\omega)$ fixed, with no need (for the definition) of stochastic integration. To make it clearer, defining $\tilde{X} = X - W$, we get the random ODE

$$\mathrm{d}\tilde{X} = \tilde{b}(\tilde{X})\mathrm{d}t,$$

where $\tilde{b}$ is the random coefficient $\tilde{b}^{\omega}(t, x) = b(t, x + W_t(\omega))$.

We start with solutions with a fixed initial datum x in $\mathbb{R}^d$.

1. The strongest kind of uniqueness is the so called path-by-path uniqueness. It means roughly what follows: for a.e. realization $W(\omega)$ of the Brownian motion, the SDE driven by $W(\omega)$ has at most one solution. This concept makes sense because of the interpretation of the SDE at $W(\omega)$ fixed.

2. A still strong kind of uniqueness is the pathwise uniqueness: given a probability space with a filtration and a Brownian motion, any two processes, adapted to that filtration, solutions to the SDE, must coincide.

3. The weakest form of uniqueness is uniqueness in law: any two adapted processes solutions to the SDE must have the same law.

Notice the difference between path-by-path uniqueness and pathwise uniqueness: intuitively, in both cases one can take two families X^ω, Y^ω solutions to the SDE (at ω), but, for pathwise uniqueness, the two families must be adapted processes (with respect to suitable filtrations) to coincide, while, for path-by-path uniqueness, they coincide independently of any other assumption (they do not need even to be measurable in ω). In particular, path-by-path uniqueness implies pathwise uniqueness. Also pathwise uniqueness implies uniqueness in law, by the classical Yamada-Watanabe theorem.

Similar definitions can be given for uniqueness among flows which do not concentrate.

In the following, we are mainly interested in path-by-path uniqueness, among single solutions and among flows. Notice that path-by-path uniqueness among flows can provide information on pathwise uniqueness even among single solutions. Indeed path-by-path uniqueness among flows, together with existence, can provide existence of strong (i.e. adapted to Brownian filtration) solutions, at least starting from a.e. initial datum. If moreover we have uniqueness in law from a fixed initial datum (which can be proved in many cases via Girsanov theorem or via PDEs arguments), then pathwise uniqueness among solutions holds by Yamada-Watanabe theorem.

As for PDEs, path-by-path uniqueness, both among solutions and among flows, has a translation at the level of the continuity equation. Indeed, there exists a proper transformation of the continuity equation, which can be read at $W(\omega)$ fixed, and where a path-by-path uniqueness definition can be given. Similarly for the transport equation.

Intuitively, also pathwise uniqueness and uniqueness in law can be translated for continuity equation, but the link here is not completely clear and we do not investigate this for the moment. Let us just mention that the translation of uniqueness in law for continuity equation is the so called Wiener uniqueness, namely uniqueness among solutions which are adapted to the Brownian filtration.

1.5 Methods and tools

The main methods used are the following ones:

- **Uniqueness from regularity via duality**: This method is based on the formal relation
$$\frac{d}{ds}X_{s,t}(X_{0,s}) = 0,$$
where $X_{s,t}$ is the flow from time s to t and $X_{0,s}$ is any solution (starting at time 0) at time s. At the level of PDEs, recalling the representation

formulae (1.1) and (1.2), this relation becomes

$$\frac{d}{ds}\int_{\mathbb{R}^d} v_{s,t}\mathrm{d}\mu_s = 0,$$

where μ solves the continuity equation and v the transport equation. The first, resp. the second relation, if true, implies easily uniqueness for X, resp. for μ. To make the relations rigorous, we need some regularity assumptions either on the drift b or on v. In particular, as soon as we have a priori estimates on the space derivative of v (or equivalently X), we have uniqueness. This reasoning works at ω fixed; hence it implies not only path-by-path uniqueness, but, in principle, also an identification of the "good" set of Brownian trajectories.

- **Improved space regularity of $\tilde{b}$ and Young estimates, for the SDE and the continuity equation**: The modified random drift $\tilde{b}$ enjoys 1 degree more of space regularity, at the price of losing 1/2 time derivative (as it can be seen, for example, by heat equation estimates). Hence the random ODE driven by $\tilde{b}$ can be studied via Young integration arguments. This brings to Lipschitz (in space) a priori estimates which imply regularity. A similar reasoning, directly for uniqueness, can be done for the continuity equation, using Young integration arguments for transport-type PDEs. An adaptation of this method (for flows) can be done via Girsanov theorem.

- **Itô-Kunita transformation of the SDE**: The transformation of the SDE via diffeomorphism uses Itô formula, which involves a Laplacian and therefore makes one hope for elliptic regularization effects. Indeed, using PDEs arguments, one can suitably transform the SDE into a new one, with more regular coefficients.

- **A priori estimates for SPDEs, via parabolic PDEs**: Given a solution v to the transport equation, then $E[v]$ satisfies a parabolic PDE. The same is essentially true for $E[v^m]$ and $E[|\nabla v|^m]$, at a formal level and with a system of parabolic PDEs (rather than a single PDE). This gives Sobolev bounds on v via parabolic estimates.

- **Wiener chaos decomposition of SPDEs**: By the linearity of v, the projection of the SPDE on the Wiener chaos spaces bring to a system of SPDEs, which actually reduce to the correspondent parabolic PDE. This allows to deduce Wiener uniqueness for the SPDE from uniqueness for the parabolic PDE.

Some tools have already been mentioned. For completeness we give a list of (what we think as) the main tools:

- **duality among linear equations**;
- **Young integration theory**, for an analysis of the SDE based on the improved regularity of $\tilde{b}$;
- **the martingale structure**, both for Itô-Kunita transformation and a priori estimates for SPDEs (they both involve the Laplacian in Itô formula), and also to get the improved regularity of $\tilde{b}$ (although the martingale structure is not essential for this point);
- **Girsanov theorem**, for an adaptation of Young estimates and (as we will see) in the Itô-Kunita transformation method;
- **the renormalization property** (v^m satisfies a transport-type equation), used for a priori estimates for SPDEs;
- **the parabolic PDE estimates**, used whenever the martingale structure comes into play;
- **Wiener chaos decomposition**.

Among these methods and tools, let us remark the two big approaches:

- **The pathwise Young argument**. This approach decouples the problem into two: first we find (and identify when possible) a full-measure set where the modified drift $\tilde{b}$ has good regularity properties, then we use such regularity properties to get well-posedness. As observed by M. Gubinelli, this is an approach à la rough paths: we identify an element in the equation (in this case $\tilde{b}$), which governs the equation in a continuous way. This approach has the advantage of being applicable to a wide class of perturbations and, in some cases, of identifying the exceptional set of Brownian trajectories where things might go wrong; this may be relevant for non-linear arguments (if the exceptional set is independent of b for example). The continuity with respect to $\tilde{b}$ may help also in the zero noise problem (a suitable rescaling of $\tilde{b}$ may suggest which solutions are selected in the small noise limit).
- **The martingale argument**. This approach uses deeply the martingale structure: more precisely, it exploits Itô formula and the regularizing properties of the associated second order operator (the Laplacian in our case). This approach has the advantage of exploiting simpler tools

and (more important) of being applicable to larger classes of drifts b, which cannot be reached (at least not easily) by the Young integration approach.

Notice that the two arguments can be mixed (as actually we do here). For example, one can use a martingale argument to show the improved regularity for $\tilde{b}$ and then Young integration bounds for the a priori estimates on the flow. One can also identify another element rather than $\tilde{b}$ which governs the equation, like $Db(X)$ via Girsanov theorem.

We remark that we speak here (with some abuse) of martingale structure since we can apply Itô formula and get a deterministic integral (i.e. defined without stochastic integration) with second order terms and a stochastic integral with first order terms and with zero expectation: this can be done in the context of martingale theory. On the other hand, we also use that the deterministic integral is written in terms of a deterministic second order operator: this property is not related to martingales, but may be rather connected to Markov property (via Dirichlet forms).

1.6 A historical overview on the subject

Regularization by noise is a wide subject nowadays and it includes many works. We give a short overview of the topic, recalling some of the works, from our perspective and without any claim of completeness. We start with the papers that mainly contributed to this thesis.

Before going into regularization by noise, we recall a few relevant results for ODEs and associated linear PDEs with irregular coefficients. DiPerna and Lions [DL89] introduce the use of the transport equation to study the corresponding ODE and get existence and uniqueness for the transport equation, for drifts with Sobolev regularity and bounded divergence. Ambrosio [Amb04] extends the result to drifts with bounded variation and introduce the concept of Lagrangian flow (see also [AC14] for a revision). From the Lagrangian point of view, Crippa and De Lellis [CDL08] prove a similar result using flows. Finally, we recall the approach by Figalli [Fig08] on martingale Lagrangian flows for SDEs.

Regularization by noise, in a weak sense, is known from a long time: for example, Girsanov theorem allows to restore uniqueness in law for a SDE with bounded drift; Strook and Varadhan [SV06] have generalize this approach to bounded continuous coefficients with uniform elliptic diffusion. The analysis of the law of SDEs with non-smooth coefficients in one dimension can be performed in one dimension via scale function and speed measure, see for example [Bre92].

If we consider strong uniqueness, we can say the story starts with the works by Zwonkin [Zvo74] and Veretennikov [Ver80]; in the last one, strong uniqueness for SDEs is proved when the drift is bounded. The case of unbounded drift was treated by Portenko [Por82], although for weak solutions, and by Gyöngy and Martínez [GM01]. Later, Krylov and Röckner [KR05] show strong uniqueness for the SDE under only a certain integrability of the drift (the Krylov-Röckner condition). This result is revisited and extended, in the context of stochastic flows, by Fedrizzi and Flandoli [FF11], [FF13a], who prove existence and regularity of the flow, using the Itô-Tanaka trick and the regularity properties of the Kolmogorov equation. At the level of the SDE, this represents an optimal result (see the counterexample in Chapter 2 when the Krylov-Röckner condition does not hold) and is essentially what we call the martingale approach in the Lagrangian setting. A similar approach in the case of Hölder continuous drift, with a stronger result in terms of regularity, is given in [FGP10]. The case of general elliptic diffusion coefficients is dealt in [Bah99] and in [Zha05] among others.

In this line, but with a different approach, Chen and Li [CL14] give a regularization by noise result, which uses the heat kernel properties to get a priori estimates on the derivative; this approach could be suited for generalization e.g. to manifolds. Again in this line, Champagnat and Jabin [CJ13] decouple the problem into estimates on the Fokker-Planck equation and uniqueness given such estimates.

It is worth mentioning that, in one dimension, the problem can be studied through special tools: the first work in this direction is by Flandoli and Russo [FR02], who use a version of the Itô-Tanaka trick in one dimension. Local time can also be used, see for example [Att10] and [AP12]. See also [CE05] on a classification of uniqueness/non-uniqueness situations for one-dimensional SDEs.

In parallel with the development of the SDE theory, results for the associated linear PDEs started. The first intuition in this direction is by Flandoli, Gubinelli and Priola [FGP10], who prove pathwise well-posedness of the stochastic transport equation for Hölder continuous drifts, relying on the differentiability of the related stochastic flows and the method of characteristics (which is related to the duality method). Other works follow this direction and extend this result to prove regularity or uniqueness under the Krylov-Röckner condition: see for example [FF13b] (regularity of the transport equation), [NO15] and [MO15] (uniqueness and renormalization property for continuity/transport equation), [MNP15] and [BN14] (improved regularity for transport equation for irregular and regular drift), [FMN14] for the vector advection equation, [Rez14] for an application to Navier-Stokes equations.

Another direction instead uses directly PDE methods (without appealing to the flow) to get pathwise well-posedness. One of the first work in this direction was by Attanasio and Flandoli [AF11], who combine the renormalization theory (for deterministic theory, close to duality here) and properties of the Kolmogorov equation and prove uniqueness for the transport equation for drifts with bounded variation and with discontinuous initial condition. A martingale-based approach is proposed in [BFGM14], where regularity for the transport equation is proved via a priori estimates (following an idea in [BF13]), under the Krylov-Röckner condition and actually even slightly beyond (uniqueness is also proved via duality and the result is transferred to the flow). This is the martingale approach in the Eulerian setting.

Outside the tools that exploit the martingale structure, Proske and coauthors started a line of proof (for SDEs), which exploits Girsanov theorem and regularizing properties of the law of the Brownian motion. In [MPMBN$^+$13] this approach is used to get Malliavin differentiability of the solution, in case of bounded drift. The approach is extended in [Rez14] to the Krylov-Röckner condition (a geometric interpretation is also given). The method can be applied also to wider class of noises, at least when a Girsanov-type result holds, and indeed it has been applied in [BNP15].

The path-by-path problems were less investigated up to recent years, one possible reason being the lack of a proper calculus without probability. The first result in this direction is by Davie [Dav07], who proves path-by-path uniqueness for the SDE for bounded drifts. This result, up to now, is still almost optimal and the proof is not easy to reproduce. A breakthrough came with the paper by Catellier and Gubinelli [CG12]: in the context of Hölder continuous drifts, they use Young integration techniques and better regularity of the modified drift to get well-posedness for a.e. Brownian path (with the exceptional set being, in some cases, independent of the drift). Their approach is deterministic in nature and in fact it is applied to a wide class of noises, like fractional Brownian motion. This is the pathwise Young approach in the Lagrangian case (which we consider, in this thesis, limited to the Brownian case).

This approach opens the way to path-by-path analysis also for PDEs. In [Cat15] and [Nil15] existence and (in the first paper) uniqueness is proved for the stochastic (better: rough) transport equation at a fixed realization of the noise, even for a more general noise; the proofs exploit the method of characteristics. A path-by-path method based directly on the PDE is in [GM], where the recently developed approach on unbounded Young/rough drivers, by Bailleul and Gubinelli [BG15], is used in this context. This is the pathwise Young approach in the Eulerian case.

The duality method is another approach to the path-by-path uniqueness problem. The advantage of such a method is that one can get regularity by other means (for example martingale tools) and still get path-by-path uniqueness. In this form the method is used in the already cited [BFGM14], in [Cat15] and also in [Sha14], where a simple formulation involving only flows is given.

We close this part of the overview with Wiener uniqueness. A result in this direction is given in [Mau11] for the stochastic transport equation under the Krylov-Röckner condition (restricted to time-independent vector fields). The method exploited, the Wiener chaos decomposition, is adapted from the paper [LJR02] by Le Jan and Raimond, who use it in the context of non-smooth diffusion coefficients. Another approach, based on Wiener exponentials, is proposed by Fedrizzi, Neves and Olivera [FNO14] and extended in [MO]. Here this result is presented but using the Wiener chaos approach.

So far, the papers who are directly linked to this thesis. Of course, many other approaches, in this or other contexts, have been developed and they definitely constitute other lines of research, with new viewpoints and new applications. They did not find space here only because they are not in the directions explored by this thesis. We mention some of them, starting from the SDE case:

- Regularization by noise for infinite-dimensional SDEs: The works [DPF10], [DPFPR13] and [DPFRV14] (among others) explore this topic extending, for continuous or bounded vector fields, the approach in martingale Eulerian approach.

- Regularization by noise for degenerate noise: This case, where the previous techniques do not extend easily, is explored for example in [CDR12], for a class of Hölder continuous drifts and a noise of weak Hörmander type, in [LTS15], where the case of degeneracy in all dimensions but one is treated, and in [BBC07], for an example of one dimensional degenerate noise with reflection.

- Regularization by noise for other kinds of noise: Apart for the already cited [CG12] and [BNP15], we mention [HP13] (among others) for a PDE-based approach to regularization with Lévy noise, and [Pri15] for path-by-path uniqueness, via duality, and other results again for Lévy noise.

Here are some topics which are no more regularization by noise for ODEs or linear PDEs, but are related at least in the spirit (one adds a noise to the system and sees what happens):

- Regularization by noise for nonlinear PDEs, with singular interacting kernel and transport-type noise: Concerning transport-type noise, this topic is much more difficult than the linear case and the behaviour of the system is different. For example, in many equations coming from physics, the drift is driven by an interaction kernel of the form $|x-y|^{\alpha}$, for some power α usually smaller than 1. Thinking from a Lagrangian point of view, the main difficulty in such cases is that, when a noise of the form $+\mathrm{d}W$ is added to the system, W independent of x, then the interaction $X-Y$ between two particles is poorly affected by this noise (the additive noise on X and Y is cancelled in the difference) and this prevents the kernel, and so the drift, to be regularized (oppositely to the linear case where the drift is regularized). The situation seems not to change for noises as $\sigma(x)W$, at least for regular σ, maybe space-irregular noises may help. Anyway, there are a few works on this topic, like [FGP11], where noise regularizes because it avoids very rare (measure zero) initial configurations bringing to explosion (rather than because it is stronger than the deterministic drifts). In [DFV14] the case of the Vlasov-Poisson system is treated and a regularization by noise phenomenon for special initial conditions is proved.

- Regularization by noise for nonlinear PDEs, with general noise: In other works, regularization by noise is shown with a particular form of the noise, in order to respect the structure of the equation and to produce regularization at the same time; we recall some of papers on this topic. A first class of works is on Navier-Stokes equations with additive noise: we mention [DPD03] on the associated Kolmogorov equation, [FR08] and [Rom08] on existence of Markov selections and links between uniqueness and invariant measures. Another relevant context is that of dispersive equations: we mention [dBD05] on blow-up for stochastic Schrödinger equation, [BSDDM05] on numerical results on nonlinear Schrödinger equation with multiplicative white noise (which suggest a regularization effect), [DT11] with improved Strichartz estimates for nonlinear Schrödinger equation with dispersive noise, [CG15] and [CG14] on nonlinear Schrödinger and Korteweg–de Vries equations with dispersive noise, by Young-type techniques. A third class of works is on parabolic second-order PDEs in one spatial dimension (which include reaction-diffusion equations and Burgers equation): we mention [AG01], [BGP94] and [Gyö98]. Outside these contexts, we mention [BF13] for parabolic systems, [GS14] for scalar conservation laws, [DLN01] and [SY04] for mean curvature flows.

- Regularization by noise for fluid-dynamical models: Many papers deal with regularization by noise for some models from fluid dynamics (not necessarily coming from SPDEs), mainly from turbulence. We mention [Bia13] and [BBF14] among others; see also the book [Fla11] for an overview.

- Other "unexpected" effects of noise on SDEs: Here the field is huge if we take into account any effect but, just to stay at the level of SDEs with additive noise, we mention [HM14a]-[HM14b] on SDEs with drifts which grow more than linearly, [FGS14] on synchronization by noise and [ACW83] on stabilization by noise of linear ODEs.

- Regularization by noise in law, for distributional drifts: Here one uses the Wiener measure to make sense of SDEs where the drift is just a distribution. We mention, among other works, [BC03] for drifts in the Kato class, [FIR14] for distributional drifts with Young techniques, [DD14] (in one dimension) and [CC15] (in general dimension) again for distributional (but more irregular) drifts with rough paths and para-controlled distributions techniques.

- Regularization by noise for the density of the laws: Existence of regular densities for the laws of some S(P)DEs driven by irregular vector fields can be shown. We mention (among other papers) [DR14] on densities for Navier-Stokes equations and [BC14] on densities of SDEs with Hölder continuous coefficients. This type of results and the techniques used may be applied to regularization by noise (at least in the martingale approach).

- Zero-noise selection: For the ODE case, we take an ODE with more than one solution, we add $+\epsilon W$ to the ODE, which becomes well-posed, and then we let ϵ go to 0 to see which deterministic solutions are selected. The problem is non-trivial and has been solved, in many cases, only in one dimension, by Bafico and Baldi [BB82], see also (among various papers) [AF11], [DF14], [Tre13] and [KM13] and [PP15] for a multidimensional example. In the PDE case, the idea is similar (and of course difficult), an example for a special case of Vlasov-Poisson is in [DFV14]. Zero noise limits also appear in other contexts, see [MP14] and [DLN01].

Finally, it is dutiful to recall the first book on regularization by noise: [Fla11]. Here one can find an overview and insights into several topics (intuitions, model from fluid dynamics, infinite-dimensional case and others, beside the finite-dimensional case), as well as many references.

1.7 Organization of the thesis

The thesis is organized as follows:

- In Chapter 2, we classify the types of irregular drifts, depending on the corresponding regularization by noise results. We also give an intuition of the phenomenon and an example of very irregular drift, where regularization by noise does not occur.

- In Chapter 3, we give definitions, basic properties and technical issues and some stability results on the SDE and the associated linear SPDEs. This Chapter does not contain regularization by noise results but is the technical basis for the rest of the thesis.

- In Chapter 4, we show the renormalization/duality method. As a result of this Chapter and the previous ones, we give some "black-box" results which provide (path-by-path) well-posedness given suitable regularity estimates on the flow or on the drift.

- In Chapter 5, we give the a priori estimates and a uniqueness result on parabolic PDEs which are needed in the following chapters.

- In Chapter 6, we give a first regularization by noise result, for existence of solutions to the stochastic continuity equation. We obtain, as a consequence, a well-posedness result in the class of non-concentrating solutions.

- In Chapter 7, we prove Wiener uniqueness for the stochastic continuity equation, starting from uniqueness for the Fokker-Planck equation.

- In Chapter 8, we show regularization by noise in the case of α-Hölder continuous drifts with $\alpha > 1/2$. We use Young integration techniques to show a priori Lipschitz estimates on the flow solution to the SDE.

- In Chapter 9, in the context of the previous Chapter, we show path-by-path uniqueness for the stochastic continuity equation, using a Young integration approach but purely based on PDE techniques.

- In Chapter 10, we extend the approach in the Chapter 8, via Girsanov theorem and a SPDE argument (which will be fully developed in Chapter 12).

- In Chapter 11, we show regularization by noise for a class of unbounded vector fields. We use a suitable transformation of the SDE, involving the associated parabolic PDE, to show a priori Sobolev estimates on the flow solution to the SDE. The analogous Lipschitz estimates for Hölder continuous drifts are also shown.

- In Chapter 12, we prove a similar result, but with a purely PDE method, based on Sobolev energy estimates and the renormalization property for the stochastic transport equation. We also apply this method to study another linear SPDE, namely the stochastic vector advection equation (a linearization of the three-dimensional stochastic Euler equation with multiplicative noise).

- In the Appendix, we give some technical facts on (joint) measurability, we recall the main properties of the spaces of functions that we use and finally we recall the main facts on Young integration.

1.8 Notation

Here we recall some frequently used notation.

- $(\Omega, \mathcal{A}, (\mathcal{F}_{s,t})_{0\le s\le t\le T}, P)$ is a probability space with a two-side filtration (i.e. $(\mathcal{F}_{s,t}) \subseteq \mathcal{F}_{s',t'}$ for $s' \le s \le t \le t'$); we can take $\mathcal{A} = \mathcal{F}_{0,T}$ without loss of generality. We assume that $\mathcal{A}$ is countably generated up to P-null sets, in particular all the σ-algebrae $\mathcal{F}_{s,t}$ are countably generated up to P-null sets (see the Appendix, Sections A.1 and A.2 for more details). We also assume the standard assumption, i.e. $\mathcal{F}_{t,t}$ contains all the P-null sets and $(\mathcal{F}_{s,t})_{s\le t}$ is right continuous with respect to t and left continuous with respect to s. We sometimes fix s and consider only the forward filtration.

- The variables s, t, r denote times in $[0, T]$; s usually is the initial time for forward equations, t is the final time for backward equations (but in Chapter 9, 0 is the initial time and T the final time). The variables $x, y, \dots$ denote space points on $\mathbb{R}^d$; $x \cdot y$ denotes the canonical scalar product on $\mathbb{R}^d$; B_R and $\bar{B}_R$ denote the centered resp. open ad close balls in $\mathbb{R}^d$ of radius R. The variable ω denotes the probabilistic datum in Ω.

- The process W is a d-dimensional Brownian motion with respect to the filtration $(\mathcal{F}_{s,t})_{s,t}$ (i.e. $W_t - W_s$ is independent of $\mathcal{F}_{0,s}$ and of $\mathcal{F}_{t,T}$). We denote by $\mathcal{F}^W$ the Brownian completed filtration, namely $\mathcal{F}^W_{s,t}$ is generated by $W_{t'} - W_{s'}$ for $s \le s' \le t' \le t$ and by the P-null sets. We use the notation $\circ \mathrm{d}W$ for the Stratonovich integration; sometimes we write $\circ\dot{W}$, as a short form for the integral notation. We use the same symbol for integration in the forward direction and in the backward direction.

- We say that a real-valued function f is (ν-)measurable with respect to a measure space $(E, \mathcal{E}, \nu)$ if, for every open set G, $f^{-1}(G)$ is in the completion of $\mathcal{E}$ with respect to ν. We say that a real-valued process is progressively measurable if it is $(\mathcal{L}^1|_{[0,T]} \otimes P$-)measurable with respect to the progressive σ-algebra $\mathcal{P}$, which is generated by all the sets of the form $A \times B$, where A is in $\mathcal{B}([0,t])$ and B is in $\mathcal{F}_t$ for some t.

- The space $\mathcal{M}_x$ is the space of finite signed measures on $\mathbb{R}^d$; $\mathcal{M}_{x,+}$ is the subset of non-negative finite measures on $\mathbb{R}^d$. The image measure of a measure ν through a map f is denoted by $f_{\#}\nu$ or by $\nu \circ f^{-1}$.

- The spaces L^p_x, L^p_t, L^p_ω, $L^p_{t,x}$, ... denote $L^p(\mathbb{R}^d)$, $L^p([0,T])$ (or $L^p([s,T])$ or $L^p([0,t])$ when appropriate), $L^p(\Omega)$, $L^p([0,T] \times \mathbb{R}^d)$, ... When a domain D of $\mathbb{R}^d$ is considered, $L^p_{x,D}$ denotes $L^p(D)$. The space $L^p_{x,loc}$ is the space of functions which are in L^p_{x,B_R} for every $R > 0$. However we consider it a symbol rather than a space: when we say that a certain property is satisfies in $L^p_{x,loc}$ (for example, a functional is continuous on $L^p_{x,loc}$), we mean that, for every $R > 0$, this property is valid in L^p_{x,B_R}. We also introduce weighted spaces: given a strictly positive C^∞_x weight χ, $L^p_{x,\chi,D}$ is the space of measurable functions f with

$$\|f\|_{L^p_{x,D}} = \left(\int_D |f(x)|^p \chi(x) \mathrm{d}x \right)^{1/p} < +\infty.$$

- Given a Banach space $V = U^*$, the dual space of a separable Banach space U, the space $L^p_x(V)$ denotes $L^p(\mathbb{R}^d; V)$, the space of weakly-* measurable functions on $\mathbb{R}^d$ with values in V, with p-integrable V norm. It is a Banach space with the norm

$$\|f\|_{L^p_x(V)} = \| \|f(x)\|_V \|_{L^p_x}. \tag{1.3}$$

Similarly for $L^p_t(V)$, $L^p_\omega(V)$, $L^p_{t,x}(V)$, ...; the notation $L^p_{x,D}(V)$, $L^p_{x,loc}(V)$ is used as before. More details on such spaces are in the Appendix, Sections A.1 and A.2. The space V is usually $\mathcal{M}_x$, $L^p_{x,D}$ (for $p > 1$), $W^{1,p}_{x,D}$ (for $1 < p < +\infty$). Similarly, given a separable Banach space U, the space $L^p(U)$ denotes the space of weakly measurable functions on $\mathbb{R}^d$ with values in U, with p-integrable U norm; see again the Appendix for more details. The space U is usually C_{x,B_R}, $C_{x,0}$ (see below for these notations). The norm (1.3) is also used, when possible, outside the cases V separable space or dual of a separable space.

- Given a Banach space V, the spaces $C_t(V)$, $C_x(V)$ denote $C([0,T];V)$, $C(\mathbb{R}^d;V)$, the spaces of V-values continuous functions. Similarly, for $0<\alpha<1$, the spaces $C_t^\alpha(V)$, $C_x^\alpha(V)$ denote $C^\alpha([0,T];V)$, $C^\alpha(\mathbb{R}^d;V)$, the spaces of *locally* α-Hölder continuous functions. Given a domain D of $\mathbb{R}^d$, $C_{x,D}(V)$, $C^\alpha_{x,D}(V)$ denote resp. the space of V-valued continuous bounded functions on D and the space of V-valued bounded, globally α-Hölder continuous functions on D. They are Banach spaces with the norms

$$\|f\|_{C_{x,D}(V)} = \sup_{x\in D}\|f(x)\|_V,$$
$$\|f\|_{C^\alpha_{x,D}(V)} = \sup_{x\in D}\|f(x)\|_V + \sup_{x,y,\in D,x\neq y}\frac{\|f(x)-f(y)\|_V}{|x-y|^\alpha}.$$

When $D=\mathbb{R}^d$, we use the notation $C_{x,b}(V)$, $C^\alpha_{x,b}(V)$. For k in $\mathbb{N}$, the space $C^{k+\alpha}_{x,D}(V)=C^{k,\alpha}_{x,D}(V)$ denotes the space of k-times differentiable functions: we say that f, V-valued function, is differentiable in x if $f(y)=f(x)+Df(x)(y-x)+o(|y-x|)$ for some $Df(x)$ linear map from $\mathbb{R}^d$ to V; we say that f is twice differentiable if Df is differentiable and so on. For $V=\mathbb{R}$ or $\mathbb{R}^n$, we omit V in the notation. $C^\infty_{x,c}$ is the space of $\mathbb{R}$-valued C^∞ functions in x with compact support. The space $C_{x,0}$ is the closed subspace of $C_{x,b}(\mathbb{R})$ of continuous functions vanishing at infinity. The space $C^k_{x,lin}$ denotes the space of $\mathbb{R}$-valued C^k functions on $\mathbb{R}^d$, with bounded derivatives of order $1,\dots k$ (*lin* stands for "at most linear growth"). It is a Banach space with the norm

$$\|f\|_{C^k_{x,lin}} = \sup_{x\in\mathbb{R}^d}\frac{|f(x)|}{1+|x|} + \sum_{j=1}^k \sup_{x\in\mathbb{R}^d}|D^j f(x)|.$$

- The space $W^{k,p}_x$ is the Sobolev space on $\mathbb{R}^d$ of order k and exponent p. For $R>0$, $W^{k,p}_{x,B_R}$ is the Sobolev space on B_R of order k and exponent p, i.e. the space of functions whose distributional derivatives (up to order k) in the domain B_R are in L^p (see the Appendix for more details). $W^{1,\infty}_{x,B_R}$ can be identified with the space of Lipschitz function on B_R.

- For $R>0$, $0<\alpha<1$, $1\le p<+\infty$, $W^{\alpha,p}_{x,B_R}$ is the fractional Sobolev space on B_R of order α and exponent p: it is the space of functions in

L^p_{x,B_R} such that

$$\|f\|_{W^{\alpha,p}_{x,B_R}} = \|f\|_{L^p_{x,B_R}} + \left(\int_{B_R}\int_{B_R}\frac{|f(x)-f(y)|^p}{|x-y|^{d+\alpha p}}\mathrm{d}x\mathrm{d}y\right)^{1/p} < +\infty.$$

The definition is extended to $p=+\infty$ to the space of functions in L^∞_{x,B_R} such that

$$\|f\|_{W^{\alpha,\infty}_{x,B_R}} = \|f\|_{L^\infty_{x,B_R}} + \operatorname{ess}\sup_{x,y\in B_R, x\neq y}\frac{|f(x)-f(y)|}{|x-y|^\alpha} < +\infty;$$

it can be identified with the space of α-Hölder continuous functions on B_R. The space $W^{k+\alpha,p}_{x,B_R}$ is the space of functions in $W^{k,p}_{x,B_R}$ whose k-th order derivative is in $W^{\alpha,p}_{x,B_R}$; it has the norm $\|f\|_{W^{k+\alpha,p}_{x,B_R}} = \|f\|_{W^{k,p}_{x,B_R}} + \|D^k f\|_{W^{\alpha,p}_{x,B_R}}$. All the definition can be extended to the whole space $\mathbb{R}^d$, replacing B_R with $\mathbb{R}^d$.

- Given a Banach space V, for $R>0$, $0<\alpha\leq 1$, $1\leq p\leq +\infty$, the $W^{\alpha,p}_{x,B_R}(V)$ norm can be defined as before, replacing $|f(x)-f(y)|$ with $\|f(x)-f(y)\|_V$.

- The space of Borel bounded functions on $\mathbb{R}^d$ and on a domain D is denoted resp. by BB_x and by $BB_{x,D}$.

- The product $\langle\cdot,\cdot\rangle$ denotes the L^2 duality product, usually with respect to the x variable. When two variables x,y are involved in the product, we use the notation $\langle\cdot,\cdot\rangle_x$, $\langle\cdot,\cdot\rangle_y$, $\langle\cdot,\cdot\rangle_{x,y}$.

- The arguments $t,x,\omega,\dots$ of the functions are sometimes indicated as subscripts (or superscripts in case of ω), sometimes omitted.

- The letters b, X, μ, v are mainly used resp. for the drift, the solution to the SDE, the solution to the (stochastic) continuity equation, the solution to the (stochastic) transport equation. We sometimes use the notation μ^{b,s,μ_s}, $v=v^{b,t,v_t}$, ... to keep track of the dependence of μ on the drift b, the initial datum s and the initial measure μ_s, and similarly for v. The letter X is actually used both for a solution to the SDE and the flow solution, namely $X=X(s,t,x,\omega)=X^{s,\omega}_t(x)$.

- We use the notation $\tilde{b}$, $\tilde{X}$, $\tilde{\mu}$, $\tilde{v}$ for the transformed drift and the transformed solutions to the SDE, the SCE and the STE at the level of the random ODE and the corresponding linear random PDEs; in Chapter 9, we still keep the "tilde" notation but the link between b and $\tilde{b}$, X and $\tilde{X}$, ... is not used (see the first Section of that Chapter for details).

- The function ρ is a non-negative even $C^\infty_{x,c}$ function on $\mathbb{R}^d$, normalized to have $\|\rho\|_{L^1_x} = 1$. For $\epsilon > 0$, ρ_ϵ is the function defined by $\rho_\epsilon(x) = \epsilon^{-d}\rho(\epsilon^{-1}x)$; similarly $(\nabla\rho)_\epsilon(x) = \epsilon^{-d}\nabla\rho(\epsilon^{-1}x)$. For a function f and a measure ν, we use the notation $f^\epsilon = f * \rho_\epsilon$, $\nu^\epsilon = \nu * \epsilon$.

- A weight (strictly positive C^∞_x function on $\mathbb{R}^d$) is usually denoted by χ, conditions on the weight are given case by case. However the following weights are recurrent: χ_R, which is ≤ 1 globally, $= 1$ on B_R, $= 0$ on B_{2R} and with $|\nabla\chi| \leq 2/R$; $\chi_\eta(x) = (1+|x|^2)^{\eta/2}$; $\chi_{R,\eta}$, which is $= 1$ on B_R and $= (1+|x|^2)^{\eta/2}$ on B^c_{R+1}.

- Integrability exponents are usually denoted by p, q, r, m, $\tilde{m}$, ... The superscript $'$ denotes the conjugate exponent:
$$\frac{1}{p} + \frac{1}{p'} = 1.$$

- The expression "there exists a locally bounded function $C(\|f\|_V)$ ..." is a convenient (thought not usual) way to say that we have a priori bounds, uniform in terms of $\|f\|_V$.

- The constants are usually denoted by C. For simplicity of notation, when we are not interested in the value or the precise dependence of the constant, we use the same letter for different constants even in the same proof.

- We say that a normed space V, contained in another normed space U, has a set A as a mildly U-dense subset if A is contained in V and, for every f in V, there exists a sequence $(f^n)_n$ in A which converges to f in U and is bounded in V.

- $\mathrm{Lin}(V,U)$ denotes the linear continuous functionals from V to U Banach spaces.

- The notation $\mathrm{div} f$ denotes the distributional divergence of f.

- We say that f belongs to $C^{\beta-1}_t$, for $0 < \beta < 1$, if $\int_0^t f_r \mathrm{d}r$ belongs to C^β. We say that f belongs to $C^{\alpha+}_t$ if f belongs to $C^{\alpha+\epsilon}_t$ for some $\epsilon > 0$.

Chapter 2

Examples

In this chapter we give examples of regularization by noise. We show that, in the deterministic case, ill-posedness or lack of regularity occur, while they do not appear (at least partially) in the stochastic case. We classify these examples depending on the kind of uniqueness, non concentration and regularity the noise brings. Together whit the examples, we define the classes of drifts we are interested in and that will appear in the results. Finally we show a counterexample where regularization by noise does not occur.

In all the examples, we will assume, even if not explicitly said, at most linear growth of b outside a ball, namely:

Condition 2.1. *There exists $R_0 > 0$ such that $b/(1+|x|)$ is in $L^\infty_t(BB_{x,B^c_{R_0}})$, where BB denotes the set of Borel bounded functions.*

2.1 First class

In this class, we have examples of regularization by noise in the strongest sense: noise restores path-by-path uniqueness and Lipschitz regularity of the flow. The class of drifts where this regularization holds is the following one:

Condition 2.2. *The drift b is bounded Hölder continuous, in the sense that b is $C^\beta_t(C^\alpha_{x,b})$ for some $\beta > 0$, $\alpha > 0$.*

As a prototype example, we consider, on $\mathbb{R}^d$, for $0 < \alpha < 1$,

$$b(x) = g(x/|x|)|x|^\alpha 1_{0<|x|\leq 1} + g(x/|x|)1_{|x|>1}, \tag{2.1}$$

where $g : \mathbb{S}^{d-1} \to \mathbb{R}^d$ is a C^∞ function. In the deterministic case, depending on g, concentration, non uniqueness, irregularity of the flow can appear even in the same example.

For example, for $g(\hat{x}) = \hat{x}$, the deterministic ODE shows non-uniqueness from 0 and non-continuity of the flow. Indeed:

- if $x_0 \neq 0$, then there exists a unique solution Y to the ODE starting from x_0, namely $Y(t) = (|x_0|^{1-\alpha} + (1-\alpha)t)^{1/(1-\alpha)}\hat{x}_0 1_{t \le t_1} + e^{t-t_1}\hat{x}_0 1_{t>t_1}$, where t_1 is the first time that $|Y| = 1$ ($t_1 = 0$ if $|x_0| > 1$);

- if $x_0 = 0$, then there is an infinite number of solution to the ODE starting from 0, namely any $Y(t) = 1_{t>t_a}((1-\alpha)(t-t_0))^{1/(1-\alpha)}\hat{x}_a 1_{t\le t_1} + e^{t-t_1}\hat{x}_a 1_{t>t_1}$ for some t_0 in $[0,\infty]$ (for $t_a = \infty$, we find the null solution) and some x_a in $\mathbb{S}^{d-1}$ (and with t_1 as before).

We also have non-uniqueness for the transport equation with discontinuous (at 0) initial datum.

For $g(\hat{x}) = -\hat{x}$ instead, for every initial x_0, there exists a unique solution Y to the ODE, but this solution reaches 0 in finite time and then stays in 0. Thus concentration appears in 0 for the flow and the continuity equation.

For $d = 2$, $g(\hat{x}) = \hat{x}_1\hat{x}$, in the deterministic case the flow can exhibit discontinuity, non-uniqueness and concentration of the mass in 0: every particle which starts from $x_1 < 0$ reaches 0 in finite time, where it can stay up to any time t_0 in $[0, +\infty]$ and then leave 0.

Finally, consider $g(\hat{x}) = R\hat{x}$, where R is a unitary $d \times d$ matrix. For this choice, already in the deterministic case we have existence and uniqueness, from every initial datum, the solution being given by $Y(t) = \exp[|x_0|^\alpha tR]x_0$. However, the flow is not Lipschitz at 0. Notice that b, in this case, is a vector field in $W^{1,1}_{x,loc}$ with zero divergence, for which results for the deterministic case ([DL89], [Amb04]) already imply existence and uniqueness among non-concentrating flows.

For these examples and all the examples belonging to 2.2, our results 8.1 and 11.5 apply and restore: path-by-path uniqueness from a single initial datum, Lipschitz regularity for the flow (and one may also prove regularity for the transport equation); furthermore, 9.1 gives a PDE-based proof of path-by-path uniqueness for the SCE and 12.1 gives non-concentration for the flow and the continuity equation. [Actually Theorem 11.5 only gives a priori Lipschitz estimates for the flow, but one can use these estimates to prove path-by-path uniqueness and Lipschitz regularity.]

2.2 Second class

In this class, noise restores existence of non-concentrating flow, Sobolev regularity of the flow, uniqueness among flows which do not concentrate. We do not have Lipschitz regularity or uniqueness among single trajectories (in the sense that we do not know whether they hold or not).

We have two conditions for this class, the second slightly larger than the first one:

Condition 2.3. *The drift b belongs to the Krylov-Röckner (KR) class, namely $L^q_t(L^p_x)$ for some p,q satisfying*

$$2 < p, q < +\infty, \quad \frac{d}{p} + \frac{2}{q} < 1. \tag{2.2}$$

Condition 2.4. *The drift b belongs to the Ladyzhenskaya-Prodi-Serrin (LPS) class, plus a regular but at most linear term, namely $b = b^{(1)} + b^{(2)}$, where $b^{(2)}$ belongs to $L^\infty_t(C^1_{x,lin})$ and $b^{(1)}$ satisfies the following assumptions:*

- *$b^{(1)}$ is in $L^q_t(L^p_x)$ for some p,q satisfying*

$$2 \le p, q \le +\infty, \quad \frac{d}{p} + \frac{2}{q} \le 1; \tag{2.3}$$

- *1) $p > d$ or 2) $p = d \ge 3$ and $\|b^{(1)}\|_{L^\infty_t(L^d_x)}$ is small enough or 3) $p = d \ge 3$ and $b^{(1)}$ is in $C_t(L^d_x)$.*

We say that p, q satisfy Condition 2.3, resp. 2.4 if they satisfy (2.2), resp. (2.3).

Remark 2.5. *Notice that, under Condition 2.4 and $p = d \ge 3$ and $b^{(1)}$ in $C_t(L^d_x)$, we can assume, without loss of generality, $\|b^{(1)}\|_{L^\infty_t(L^d_x)}$ small enough: indeed, if $b^{(1)}$ is in $C_t(L^d_x)$, then, for every $\epsilon > 0$, there exists a vector field $\bar{b}$ in $C_t(C^\infty_{x,c})$ with $\|b^{(1)} - \bar{b}\|_{C_t(L^d_x)} < \epsilon$; therefore, we can take a new decomposition of b, replacing $b^{(1)}$ with $b^{(1)} - \bar{b}$ and $b^{(2)}$ with $b^{(2)} + \bar{b}$, to get $\|b^{(1)}\|_{L^\infty_t(L^d_x)}$ small enough.*

The first prototype example is given again by

$$b(x) = g(x/|x|)|x|^\alpha 1_{0<|x|\le 1} + g(x/|x|)1_{|x|>1}$$

but for $-1 < \alpha < 0$, where again $g : \mathbb{S}^{d-1} \to \mathbb{R}^d$ is a C^∞ function. These examples show very similar pathologies to the case $0 < \alpha < 1$.

A second example is given, in dimension $d = 2$, by

$$\begin{aligned} b_1(x) &= -\frac{1}{4} x_1 sgn(x_2)|x_2|^{-1/4}, \\ b_2(x) &= -5|x_2|^{4/5}. \end{aligned}$$

The deterministic ODE exhibits non-uniqueness and concentration in 0 (and existence of non-continuous generalized flows solutions). Indeed, the following paths are solutions of the ODE starting from x with $x_2 > 0$, for any real C:

$$\begin{aligned} X_1^C(t,x) &= x_1 \exp[x_2^{-1/20}] \exp[-(x_2^{1/5} - t)^{-1/4}] 1_{t \le x_2^{1/5}} + \\ &\quad + C \exp[-(t - x_2^{1/5})^{-1/4}] 1_{t > x_2^{1/5}}, \\ X_2^C(t,x) &= (x_2^{1/5} - t)^5. \end{aligned}$$

The counterpart at the level of the continuity equation is non-uniqueness and a concentration phenomenon.

A third example is given in [Aiz78] by Aizenman, where bounded divergence-free drifts on $\mathbb{R}^3$ are produced, which exhibits non-existence of measure-preserving flows or non-uniqueness of such flows.

For these examples and all the examples belonging to 2.3 or to 2.4, our results 10.1, 11.1 and 12.1 apply and restore: existence and path-by-path uniqueness for non-concentrating flows (and non-concentrating solutions to the continuity equation), Sobolev regularity for the flow and the transport equation.

We should say, for the examples above and in general in the class 2.4, we do not know whether these results are optimal or a stronger regularization by noise result holds, like Lipschitz regularity or path-by-path uniqueness among single trajectories (apart for the example in [Aiz78], since a result of Davie [Dav07] ensures path-by-path uniqueness among single paths for bounded drifts).

2.3 Third class

In this class, noise restores still existence and uniqueness of non-concentrating flows, but regularity could not hold. More precisely, noise restores existence for these flows, while uniqueness holds already for the deterministic equation (the point is that there might be no non-concentrating solutions in the deterministic case).

The condition for this class is the following one:

Condition 2.6. *The drift b belongs to $L_t^{m'}(W_{x,loc}^{1,m'})$ for some $m' > 1$ and has compact support. Its divergence satisfies $|\mathrm{div} b|^2$ in the Krylov-Röckner class.*

The compact support assumption (here and also in the fourth class) is due to technical reasons and may be replaced by suitable conditions at infinity.

For all the examples belonging to 2.6, our result 6.4 applies and restores: non-concentration for the flow and the continuity equation, while path-by-path uniqueness among non-concentrating flows is ensured already in the deterministic case.

Once more, the examples of the form (2.1), for $0 < \alpha < 1$ belong to this class. However for these examples, noise restores also path-by-path uniqueness among single trajectories and Lipschitz regularity for the flows, so that these example are not peculiar of this class. On the other hand, if one wants a simple proof only of existence and uniqueness among non-concentrating solutions, this method provides such a proof.

2.4 Fourth class

In this class, there might not be path-by-path uniqueness, but noise gives a unique selection by filtering with respect to Brownian motion.

Condition 2.7. *The drift b belongs to $L^{\bar{m}}_t(L^{\bar{m}}_{x,loc})$ for some $\bar{m} > 2$ and has compact support. Its divergence satisfies $|\mathrm{div} b|^2$ in the Krylov-Röckner class.*

For all the examples belonging to 2.7, our result 7.12 applies and restores: uniqueness among the non-concentrating solutions to continuity equations which are adapted to the Brownian filtration.

2.5 An intuition

Let us come back to the prototype examples of the first and second classes, namely

$$b(x) = g(x/|x|)|x|^\alpha 1_{0<|x|\leq 1} + g(x/|x|)1_{|x|>1},$$

for $-1 < \alpha < 1$. Here it is possible to have an intuitive idea, “by hands”, of what happens. If one consider the ODE without noise starting from 0, any solution Y grows near 0 no faster than $t^{1/(1-\alpha)}$; on the contrary, the Brownian motion W near 0 grows as $t^{1/2}$ (this is false, but only for a logarithmic correction, which does not affect the intuition). Heuristically, we could say that the “speed” of Y near 0 which is caused by the drift is like $t^{\alpha/(1-\alpha)}$, while the one caused by W is like $t^{-1/2}$. So what we expect to happen is that the Brownian motion moves the particle immediately away from 0, faster than the action of the drift, and this prevents the formation of non-uniqueness or singularities. At least in the one-dimensional case, this can be seen also through speed measure and scale function, see [Bre92], see also [CE05].

This explains also intuitively why $\alpha > -1$. Indeed, this bound is optimal, as we see from the following Section.

2.6 A counterexample

Here we take the drift

$$b(x) := -\beta|x|^{-2}x1_{x\neq 0},$$

with $\beta > 1/2$. Notice that this drift is at the borderline, but outside, of the LPS class. For this particular SDE, we have

Proposition 2.8. *For some $T > 0$ and $M > 0$, if X_0 is a random variable, independent of W and uniformly distributed on B_M, then there does not exist a weak solution, starting from X_0.*

Proof. **Step 1: the SDE does not have a weak solution for $X_0 = 0$ (for any $T > 0$).** The method is taken from [CE05].

Assume, by contradiction, that (X, W) is a weak solution on $[0, T]$, i.e. there is a filtered probability space $(\Omega, \mathcal{A}, \mathcal{F}_t, P)$, an $\mathcal{F}_t$-Brownian motion W in $\mathbb{R}^d$, an $\mathcal{F}_t$-adapted continuous process $(X_t)_{t\geq 0}$ in $\mathbb{R}^d$, such that $\int_0^T |b(X_t)|\mathrm{d}t < +\infty$ and, a.s.,

$$X_t = \int_0^t b(X_r)\mathrm{d}r + W_t.$$

Hence X is a continuous semimartingale, with quadratic covariation $\langle X^i, X^j\rangle_t = \delta_{ij}t$ between its components. By Itô formula, we have

$$\mathrm{d}[|X_t|^2] = -2\beta 1_{X_t\neq 0}\mathrm{d}t + 2X_t\cdot \mathrm{d}W_t + \mathrm{d}t = (1_{X_t=0} - (2\beta-1)1_{X_t\neq 0})\mathrm{d}t + 2X_t\cdot \mathrm{d}W_t.$$

We now claim that

$$\int_0^T 1_{X_r=0}\mathrm{d}r = 0 \tag{2.4}$$

holds with probability one. This implies

$$|X_t|^2 = -(2\beta-1)\int_0^t 1_{X_r\neq 0}\mathrm{d}r + \int_0^t 2X_r\cdot \mathrm{d}W_r.$$

Therefore $|X_t|^2$ is a positive local supermartingale, null at $t = 0$. This implies $|X_t|^2 \equiv 0$, hence $X_t \equiv 0$. But this contradicts the fact that $\langle X^i, X^j\rangle_t = \delta_{ij}t$.

It remains to prove the claim (2.4). Consider the random set $\{t \in [0, T] | X_t = 0\}$. It is a subset of $A_1 = \{t \in [0, T] | X^1_t = 0\}$, so it is sufficient to prove that the Lebesgue measure of A_1 is zero, P-a.s. and this is equivalent to $P\left(\int_0^T 1_{X^i_r=0}\mathrm{d}r = 0\right) = 1$. Since X is a continuous semimartingale, with quadratic covariation $\langle X^i, X^j\rangle_t = \delta_{ij}t$, also X^1 is a continuous semimartingale, with quadratic covariation $\langle X^1, X^1\rangle_t = t$. Hence, by the occupation times formula (see [RY99], Chapter VI Corollary 1.6)

$$\int_0^T 1_{X_r^1=0}\mathrm{d}r = \int_{\mathbb{R}} 1_{a=0} L_T^a(X^1)\mathrm{d}a$$

where $L_T^a(X^1)$ is the local time at a on $[0,T]$ of the process X^1. Hence, a.s., $\int_0^T 1_{X_r^1=0}\mathrm{d}r = 0$.

Step 2: the SDE does not have a weak solution starting from X_0 uniformly distributed on B_M (for some $T > 0$ and $M > 0$). Again, we suppose by contradiction that there exists such a solution X (associated with some filtration $\mathcal{G}$), on a probability space $(\Omega, \mathcal{A}, P)$. Let τ be the first time when X hits 0 (it is a stopping time with respect to $\mathcal{G}$), with $\tau = \infty$ where X does not hit 0. We now claim that

$$P(\tau < \infty) > 0. \tag{2.5}$$

Assuming this, we can construct a new process Y, which is a weak solution to the SDE, starting from $Y_0 = 0$. This is in contradiction with Step 1. The process Y is built as follows. Take $\tilde{\Omega} = \{\tau < \infty\}$, $\tilde{\mathcal{A}} = \{A \cap \tilde{\Omega} | A \in \mathcal{A}\}$, $Q = P(\tilde{\Omega})^{-1} P|_{\tilde{\mathcal{A}}}$, on $\tilde{\Omega}$ define $Y_t = X_{t+\tau}$ and $\tilde{W}_t = W_{t+\tau} - W_\tau$, $\mathcal{H}_t = \sigma\{\tilde{W}_s, Y_s | s \le t\}$ (this is a σ-algebra on $\tilde{\Omega}$). Then we observe the following facts:

- $\tilde{W}$ is a natural Brownian motion on the space $(\tilde{\Omega}, \mathcal{A}, Q)$, i. e., for every positive integer n, for every $0 < t_1 < \dots t_n$ and for every $f_1, \dots f_n$ in $C_b(\mathbb{R}^d)$, there holds

$$\begin{aligned} &E\left[1_{\tilde{\Omega}} \prod_{j=1}^n f_j(W(t_j+\tau) - W(t_{j-1}+\tau))\right] \\ &= P(\tilde{\Omega}) \prod_{j=1}^n \int_{\mathbb{R}^d} f_j d\mathcal{N}(0, (t_j - t_{j-1})I), \end{aligned} \tag{2.6}$$

  where $\mathcal{N}(m, A)$ is the Gaussian law of mean m and covariance matrix A. This can be verified, for a general $\mathcal{G}$-stopping time, with a standard argument: first one proves (2.6) when τ is a stopping time with discrete range in $[0, \infty]$, then, for the general case, one uses an approximation of τ with stopping times τ_k with discrete range such that $\tau_k \downarrow \tau$ (as $k \to \infty$) and $\{\tau = \infty\} = \{\tau_k = \infty\}$ for every k.

- $\tilde{W}$ is a Brownian motion with respect to the filtration $\mathcal{H}$, i.e., for every $0 = t_0 < t_1 < \ldots t_n \leq s < t$ and for every $f, g_1, \ldots g_n$ in $C_b(\mathbb{R}^d)$, there holds

$$E\Big[1_{\tilde{\Omega}} f(W(t+\tau) - W(s+\tau)) \prod_{j=0}^{n} g_j(X(t_j+\tau))\Big]$$
$$= \int_{\mathbb{R}^d} f d\mathcal{N}(0,(t-s)I) E\Big[\prod_{j=0}^{n} g_j(X(t_j+\tau))\Big].$$

  Again this can be shown by approximation (with stopping times with discrete range).

- Y is a weak solution to the SDE, starting from $Y_0 = 0$. This follows immediately from

$$X_{s'} = X_s + \int_s^{s'} b(X_r)\mathrm{d}r + W_{s'} - W_s,$$

  setting $s' = t + \tau$ and $s = \tau$.

It remains to prove the claim (2.5). We suppose by contradiction that $\tau = \infty$ a.s.; this implies that, for every t, $P(X_t \neq 0) = 1$. Then, computing $E[|X|^2]$ by the Itô formula, we get

$$\frac{\mathrm{d}}{\mathrm{d}t} E[|X_t|^2] = -2\beta P\{X_t \neq 0\} + 1 = -2\beta + 1 < 0,$$

hence there exists a time $t_0 > 0$ such that $E[|X_{t_0}|^2] < 0$, which is a contradiction. This completes the proof. $\square$

Chapter 3

Some general facts on SDEs and associated SPDEs

In this chapter we want to discuss in detail some general facts on the SDEs and the associated linear stochastic PDEs (SDPEs).

Before starting, we recall some assumptions and notation we use in this Chapter and in the following ones. We always assume that b is a fixed function (not an equivalence class) and is in $L^1_t(L^1_{x,loc})$ (with a little abuse of notation, we use the symbol L^p also for the function b, although it is not an equivalence class). We denote by $(\Omega, (\mathcal{F}_{s,t})_{0\le s\le t}, P)$ a given two-sided filtered probability space, satisfying the standard assumptions ($F_{s,s}$ contains the P-null sets for every s, $(\mathcal{F}_{s,t})_t$ is right continuous for fixed s, $(\mathcal{F}_{s,t})_s$ is left continuous for fixed t). The process W is a d-dimensional Brownian motion with respect to $(\mathcal{F}_{s,t})_{s,t}$ (i.e. W is a d-dimensional Brownian motion with respect to its natural completed filtration and, for any $s' \le s \le t \le t'$, $\mathcal{F}_{s',s}$, $W_t - W_s$ and $\mathcal{F}_{t,t'}$ are independent). For technical reasons, we assume the following Condition (see the Appendix, Sections A.1 and A.2 for more details):

Condition 3.1. *The measure space $(\Omega, \mathcal{F}_{0,T})$ is countably generated, up to P-null sets, i.e. there exists a countable subset $\mathcal{C}$ of $\mathcal{F}_{0,T}$ such that the completion (with respect to P) of the σ-algebra generated by $\mathcal{C}$ is $\mathcal{F}_{0,T}$.*

As a consequence of this Condition, all the σ-algebrae $\mathcal{F}_{s,t}$ are countably generated up to P-null set.

This condition is satisfied, for example, if $(\Omega, \mathcal{F}_{s,t}, P)$ is the canonical space $C([0,T])$, endowed with the filtration generated by π_r (the evaluation map at time r) and by the P-null sets, with the Wiener measure P. For this reason, this assumption is not restrictive for existence and for path-by-path uniqueness (since we can work with the canonical space).

3.1 Definitions of solutions

The main equations we are interested in are the (deterministic or stochastic) ordinary differential equation and its associated (deterministic or stochastic) continuity and transport equation.

We start with the deterministic case: we consider:

- the ordinary differential equation (ODE) on $\mathbb{R}^d$: $X = (X_t)_t$, t in $[s, T]$, solves
$$\mathrm{d}X = b(X)\mathrm{d}t,$$
with given initial datum $X_s = x$;

- the continuity equation (CE) on $\mathbb{R}^d$: $\mu = (\mu_t)_t$, t in $[s, T]$, is a family of signed measures on $\mathbb{R}^d$ solving
$$\mathrm{d}\mu + \mathrm{div}(b\mu)\mathrm{d}t = 0,$$
with given initial datum μ_s;

- the (backward) transport equation (TE) on $\mathbb{R}^d$: $v = v(s, x)$, s in $[0, t]$, x in $\mathbb{R}^d$, is a scalar field solving
$$\mathrm{d}v + b \cdot \nabla v \mathrm{d}s = 0,$$
with given final datum v_t.

Their stochastic counterparts are (with σ real constant):

- the stochastic (ordinary) differential equation (SDE) on $\mathbb{R}^d$: $X = (X_t^\omega)_{t,\omega}$, t in $[s, T]$, ω in Ω, is an adapted process solution to
$$\mathrm{d}X = b(X)\mathrm{d}t + \sigma \mathrm{d}W,$$
with given initial datum $X_s = x$;

- the stochastic continuity equation (SCE) on $\mathbb{R}^d$: $\mu = (\mu_t^\omega)_{t,\omega}$, t in $[s, T]$, ω in Ω, is an adapted family of signed measures on $\mathbb{R}^d$ solving
$$\mathrm{d}\mu + \mathrm{div}(b\mu)\mathrm{d}t + \sigma \mathrm{div}(\mu) \circ \mathrm{d}W = 0,$$
with given initial datum μ_s.

- the stochastic (backward) transport equation (STE) on $\mathbb{R}^d$: $v = v(s, x, \omega) = v_s^\omega(x)$, s in $[0, t]$, x in $\mathbb{R}^d$, ω in Ω, is an adapted scalar field solving
$$\mathrm{d}v + b \cdot \nabla v \mathrm{d}t + \sigma \nabla v \circ \mathrm{d}W = 0.$$
with given final datum v_t.

We recall the formal links between these three equations, which hold when coefficients and solutions are regular (precise definitions will be given after). First, let $X = X(s,t,x,\omega)$ be the stochastic flow solving the SDE (which, at a formal level, is a family of processes $(X(s,\cdot,x,\cdot))_{s,x}$ indexed by the initial time and datum); we call $X^\omega_{s,t}$, omitting sometimes ω, the map from $\mathbb{R}^d$ into itself given by the flow, with initial time s, evaluated at time t and with ω fixed. We have formally:

- the family μ, defined by
$$\mu^\omega_t := (X^\omega_{s,t})_\# \mu_s, \tag{3.1}$$
is the solution to the SCE (the symbol $(X^\omega_{s,t})_\# \mu_s$ denotes the image measure of μ_s under $(X^\omega_{s,t})$);
- the family v, defined by
$$v^\omega_s := v_t(X^\omega_{s,t}), \tag{3.2}$$
is the solution to the backward STE.

Outside the regular case, we are interested in solutions to the SCE with no regularity assumptions (distributional solutions) and we want to prove uniqueness among this solutions: indeed this uniqueness gives uniqueness among certain flows solving the SDE.

As for the STE, we are mainly (but not exclusively) interested in solutions to the STE with Sobolev space regularity (differentiable solutions): indeed this regularity allows to prove existence of flows with Sobolev space regularity.

In the following, we only give the definitions for the stochastic continuity and transport equation. However, the definitions of solution for the deterministic continuity and transport equation can be recovered from the ones in the stochastic case, by dropping all the terms with the stochastic integral and the terms with a second order derivative (i.e. $\sigma = 0$) and removing ω (and the progressively measurability) from the assumptions.

We will refer to the deterministic case as the case $\sigma = 0$ (dropping ω), while we will take, in the stochastic case, $\sigma = 1$.

Moreover we adopt the convention that, where dealing with the distributional/differentiable solutions to (S)CE/(S)TE, the assumptions given in the corresponding definitions are always satisfied.

3.1.1 Continuity equation

Definition 3.2. *Fix $s \geq 0$ and let μ_s be in $\mathcal{M}_x$. A distributional solution to the stochastic continuity equation (SCE) is a family of measures $(\mu^\omega_t)_{t\geq s,\omega}$, in*

$L^\infty_{t,\omega}(\mathcal{M}_x)$, bounded-weakly progressively measurable (with respect to $(\mathcal{F}_{s,t})_t$), with $|b||\mu|$ in $L^1_t(\mathcal{M}_{x,loc})$ for a.e. ω, such that, for every φ in $C^\infty_{x,c}$, it holds for a.e. (t,ω),

$$\begin{aligned}&\langle \mu_t, \varphi\rangle \\ &= \langle \mu_s, \varphi\rangle + \int_s^t \langle \mu, b\cdot\nabla\varphi\rangle \mathrm{d}r + \sigma\int_s^t \langle \mu, \nabla\varphi\rangle\cdot \mathrm{d}W + \frac{1}{2}\sigma^2\int_s^t\langle \mu, \Delta\varphi\rangle \mathrm{d}r.\end{aligned} \tag{3.3}$$

A classical solution is a distributional solution which is in $C_s(C^2_{x,loc})$ for a.e. ω.

We recall that the space $L^\infty_{t,\omega}(\mathcal{M}_x)$ is the space of weakly-* measurable functions from $[0,T]\times\Omega$ with values in $\mathcal{M}_x$. The bounded-weak progressive measurability means that: for every f bounded Borel function on $\mathbb{R}^d$, $(t,\omega)\mapsto \langle \mu^\omega_t, f\rangle$ is progressively measurable. The condition $|b||\mu|$ in $L^1_t(\mathcal{M}_{x,loc})$ for a.e. ω ensures that the term $\int_s^t \langle \mu, b\cdot\nabla\varphi\rangle \mathrm{d}r$ is well-defined. For more details on these technical conditions, see the Appendix, Sections A.1 and A.2.

For simplicity, here and in the following, we assume the initial datum for the SCE (and for the forward STE) and the final datum for the backward STE to be deterministic.

Remark 3.3. *The initial time can be reduced to 0 by the following transformation: if μ is a solution to the SCE with initial time s, then $\overrightarrow{\mu}$, defined as $\overrightarrow{\mu}(r,\omega) = \mu(r+s,\omega)$ is a solution to the the SCE starting at 0*

$$\mathrm{d}\overrightarrow{\mu} + \mathrm{div}(\overrightarrow{b}\,\overrightarrow{\mu})\mathrm{d}t + \sigma\mathrm{div}(\overrightarrow{\mu})\circ \mathrm{d}\overrightarrow{W} = 0,$$

with respect to the filtration $(\overrightarrow{\mathcal{F}}_r)_r$, where $\overrightarrow{b}(r,x) = b(r+s,x)$, $\overrightarrow{W}_r = W_{r+s} - W_s$ and $\overrightarrow{\mathcal{F}}_r = \mathcal{F}_{s,s+r}$.

3.1.2 Backward transport equation, distributional solution

Similarly to the CE case, we define the distributional solution for the backward STE. Since the form of the equation is different (and its meaning as well), we need an additional hypothesis on the divergence of b. The stochastic integral has to be understood in the backward direction.

Definition 3.4. *Fix $t\geq 0$, let v_t be in L^∞_x and let b in $L^1_{s,x,loc}$ with* div$b$ *in $L^1_{s,x,loc}$. A distributional solution to the stochastic backward transport equation (STE) is a map v in $L^\infty_{s,\omega}(L^\infty_x)$ (with $s\leq t$), weakly-* progressively* *measurable (with respect to $(\mathcal{F}_{s,t})_s$), such that, for every φ in $C^\infty_{x,c}$, it holds for a.e. (s,ω),*

$$\begin{aligned}&\langle v_s, \varphi\rangle \\ &= \langle v_t, \varphi\rangle - \int_s^t \langle v, b\cdot\nabla\varphi\rangle \mathrm{d}r - \int_s^t \langle v, \varphi \mathrm{div} b\rangle \mathrm{d}r + \\ &- \sigma\int_s^t \langle v, \nabla\varphi\rangle \mathrm{d}W + \frac{1}{2}\sigma^2\int_s^t \langle v, \Delta\varphi\rangle \mathrm{d}r.\end{aligned} \tag{3.4}$$

We recall that the space $L^\infty_{s,\omega}(L^\infty_x)$ is the space of weakly-* measurable, essentially bounded functions from $[0,t]\times\Omega$ with values in L^∞_x. It can be identified with the space $L^\infty_{s,x,\omega}$ (see the Appendix, Sections A.1 and A.2).

3.1.3 Backward transport equation, differentiable solution

We will deal with solutions of the (S)TE with Sobolev differentiable. In this case, the formulation above changes and it is possible to avoid the integrability assumption on the divergence of b (bringing the derivative on v). Again the stochastic integral has to be understood in the backward direction.

Definition 3.5. *Fix $t\geq 0$, m in $]1,+\infty]$, let v_t be in $C^1_{x,b}$ and let b in $L^1_{s,x,loc}$ with* $\mathrm{div}b$ *in $L^1_{s,x,loc}$. A $W^{1,m}_x$ differentiable solution to the stochastic transport equation (STE) is a map v in $L^\infty_{s,\omega}(L^\infty_x)$ (with $s\leq t$), weakly-* progressively measurable (with respect to $(\mathcal{F}_{s,t})_s$), in $L^m_{s,\omega}(W^{1,m}_{x,loc})$, such that, for every φ in $C^\infty_{x,c}$, it holds for a.e. (s,ω),*

$$\begin{aligned}&\langle v_s, \varphi\rangle \\ &= \langle v_t, \varphi\rangle + \int_s^t \langle b\cdot\nabla v, \varphi\rangle \mathrm{d}r + \sigma\int_s^t \langle \nabla v, \varphi\rangle \mathrm{d}W - \frac{1}{2}\sigma^2\int_s^t \langle \nabla v, \nabla\varphi\rangle \mathrm{d}r.\end{aligned} \tag{3.5}$$

A classical solution is a differentiable solution which is in $C_s(C^2_{x,loc})$ for a.e. ω.

For m finite, for $R>0$, the space $L^m_{s,\omega}(W^{1,m}_{x,B_R})$ is the space of weakly (or equivalently strongly) measurable functions from $[0,t]\times\Omega$ with values in $W^{1,m}_{x,B_R}$ and with finite $L^m_{s,\omega}(W^{1,m}_{x,B_R})$ norm. It can be identified with a subspace of $L^m_{s,x,\omega,[0,T]\times B_R\times\Omega}$. For $m=+\infty$, the space $L^\infty_{s,\omega}(W^{1,\infty}_{x,B_R})$ is the subspace of $L^2_{s,\omega}(W^{1,2}_{x,B_R})$ with finite $L^\infty_{s,\omega}(W^{1,\infty}_{x,B_R})$ norm. See the Appendix, Sections A.1 and A.2), for more details.

3.1.4 Forward transport equation

In the above definitions, we always considered the backward case, as the TE will be used in the backward form. However, it is more convenient to deal with the forward formulation (one usually is more familiar with calculations in the forward case). Instead of giving the definition of solution in the forward case, which is completely analogous to the backward one, we give the link between the two cases.

Remark 3.6. *If v is a (distributional or differentiable) solution to the backward STE, with final time t, then $\overrightarrow{v}$, defined as $\overrightarrow{v}(r,x,\omega) = v(t-r,x,\omega)$ is a (distributional or differentiable) solution to the forward STE*

$$\mathrm{d}\overrightarrow{v} - \overrightarrow{b} \cdot \nabla \overrightarrow{v} \mathrm{d}t - \sigma \nabla \overrightarrow{v} \circ \mathrm{d}\overrightarrow{W} = 0,$$

with respect to the filtration $(\overrightarrow{\mathcal{F}}_r)_r$, where $\overrightarrow{b}(r,x) = b(t-r,x)$, $\overrightarrow{W}_r = W_t - W_{t-r}$ and $\overrightarrow{\mathcal{F}}_r = \mathcal{F}_{t-r,t}$.

3.1.5 Time continuity

Before going on, we give a couple of results which allow to work with some nicer formulations (from a technical point of view) of the definitions above.

In all the previous definitions, the equality were always up to a set of measure zero. In the deterministic case, one can work with a suitable version of the solution, so that the PDE is satisfied for every t.

Lemma 3.7. *Let $(\mu_t)_t$ be a distributional solution to the CE. Then there exists a modification $(\bar{\mu}_t)_t$ (i.e. $\mu_t = \bar{\mu}_t$ for a.e. t), weakly-* continuous in time t, which satisfies* (3.3) *for every t and for every φ.*

Proof. We take the initial time $s = 0$ for simplicity. Fix D a countable set in $C_c^\infty(\mathbb{R}^d)$, which is dense in C_c^2 (in the sense that it is dense in $C^2(B_R)$ for every R), and let F be a full-measure set in $[0,T]$ such that, for every t in F, $\|\mu_t\|_{\mathcal{M}_x} \le \|\mu\|_{L_t^\infty(\mathcal{M}_x)}$ and μ_t satisfies (3.3) for every φ in D. For any t in $[0,T]$, let $(t_n)_n$ be a sequence in F converging to t (it exists because F is dense). Since $(\mu_{t_n})_n$ is a bounded sequence of signed measures, by Banach-Alaoglu theorem there exists $\bar{\mu}_t$ with $\|\bar{\mu}_t\|_{M_x} \le \|\mu\|_{L_t^\infty(M_x)}$ and a subsequence $(\mu_{t_{n_k}})_k$ converging weakly-* to $\bar{\mu}_t$. So we have, for every φ in D,

$$\begin{aligned} \langle \bar{\mu}_t, \varphi \rangle &= \lim_k \langle \mu_{t_{n_k}}, \varphi \rangle \\ &= \langle \mu_0, \varphi \rangle + \lim_k \int_0^{t_{n_k}} \langle \mu, b \cdot \nabla \varphi \rangle \mathrm{d}r = \langle \mu_0, \varphi \rangle + \int_0^t \langle \mu, b \cdot \nabla \varphi \rangle \mathrm{d}r. \end{aligned} \tag{3.6}$$

Since D is dense in $C_{x,c}$, the formula above determines $\bar{u}_t$ completely. This in particular does not depend on the choice of $(t_n)_n$ or (t_{n_k}) (not even on the version of μ chosen), since the RHS of (3.6) does not. Besides, for t in F, (3.6) implies that $\langle \bar{\mu}_t, \varphi \rangle = \langle \mu_t, \varphi \rangle$ for every φ in D and so $\bar{\mu}_t = \mu_t$, hence $\bar{\mu}$ is a version of μ. Finally, a simple density argument (using density of D and the uniform bound $\|\bar{\mu}_t\|_{\mathcal{M}_x} \leq \|\mu\|_{L^\infty_t(M_x)}$ for every t) allows to extend (3.6) to every φ in C^∞_c and to show continuity of $t \mapsto \langle \bar{\mu}, \varphi \rangle$ for every φ in $C_{x,c}$, i.e. weak-* continuity in time. The proof is complete. □

In a similar way, using the $L^\infty_{t,x,\omega}$ bound and in addition the lower semi-continuity properties of the norms of $L^\infty_{t,x}$ and $L^m_t(W^{1,m}_{x,B_R})$, one can prove:

Lemma 3.8. *Let $(v_s)_s$ be a distributional, resp. differentiable solution to the backward TE. Then there exists a modification $(\bar{v}_s)_s$ (i.e. $v_s = \bar{v}_s$ as functions in L^∞_x for a.e. s), weakly-* continuous in time, which satisfies* (3.4)*, resp.* (3.5) *for every s and for every φ. The analogous result holds for the forward TE.*

From now on we will work with these continuous modifications when possible, without using the notation with "bar". Notice however that, in the case of the transport equation, $\bar{v}$ is weakly-* continuous in t as an L^∞_x-valued map (in particular, it takes values in a space of equivalent classes), and we do not know whether there exists a representative $\bar{\bar{v}}$ (i.e. $\bar{\bar{v}}_t$ is in the class $\bar{v}_t$ for every t), which is jointly measurable in (s, x). A similar observation holds for the continuity equation, when the solution is in $L^m_{t,x}$ for some m.

Remark 3.9. *For the stochastic PDEs, the dependence on ω makes it difficult to deduce the existence of modifications of the solutions, which satisfy the equations for every t and φ (outside a P-null set in Ω independent of t and φ). However, for the continuity equation, one can easily deduce that, for every φ in C^∞_c, there exists a process $\mu(\varphi)$, modification of $\langle \mu, \varphi \rangle$ (i.e. $\mu(\varphi) = \langle \mu, \varphi \rangle$ for a.e. (t, ω)), with continuous trajectories and verifying* (3.3) *for every t, outside a P-null set in Ω independent of t (but possibly dependent on φ). A similar conclusion holds also for the stochastic transport equation, where we call $v(\varphi)$ the time-continuous modification of $\langle v, \varphi \rangle$.*

We conclude this section extending, in the deterministic case, the distributional formulation of the linear PDEs to the case of time-dependent test functions.

Lemma 3.10. *Let μ be a solution to the CE (we assume that it is the weakly-* continuous version). Then, for every φ in $C_t(C^1_{x,c})$, such that, for every x,*

$\varphi(\cdot, x)$ is in $W_t^{1,1}$ and $\partial_t\varphi$ is in $L_t^1(C_{x,c})$, it holds

$$\langle \mu_t, \varphi_t \rangle = \langle \mu_s, \varphi_s \rangle + \int_s^t \langle \mu, b \cdot \nabla\varphi \rangle \mathrm{d}r + \int_s^t \langle \mu, \partial_t\varphi \rangle \mathrm{d}r. \tag{3.7}$$

The analogous result holds also for the TE, in both distributional and differentiable formulations, again for every φ as above.

Here and in the following, ρ is a $C_{x,c}^\infty$ nonnegative even function, normalized to have $\|\rho\|_{L_x^1} = 1$ and, for any $\epsilon > 0$, $\rho_\epsilon(x) = \epsilon^{-d}\rho(\epsilon^{-1}x)$ is a standard mollifier. For any function f in $L_{x,loc}^1$ and for any locally finite signed measure ν, $f^\epsilon = f * \rho_\epsilon$ and $\nu^\epsilon = \nu * \rho_\epsilon$. Measurability and continuity property of f^ϵ and ν^ϵ, when f and ν depends on t and ω, are in the Appendix, Section A.2.

Proof. We prove the result for the CE, again with $s = 0$ for simplicity. Using $\rho_\epsilon(x - \cdot)$ as test function for φ, we find the following equation for μ^ϵ, valid for every (t, x):

$$\mu_t^\epsilon(x) = \mu_0^\epsilon(x) - \int_0^t (b_r\mu_r) * \cdot\nabla\rho_\epsilon(x)\mathrm{d}r.$$

Multiplying this equation by $\varphi(t, x)$ and using the chain rule, we get

$$\mu_t^\epsilon(x)\varphi_t(x) = \mu_0^\epsilon(x)\varphi_0(x) - \int_0^t (b_r\mu_r) * \cdot\nabla\rho_\epsilon(x)\varphi_r(x)\mathrm{d}r + \int_0^t \mu_r^\epsilon(x)\partial_t\varphi_r(x)\mathrm{d}r$$

Now we integrate this equality in x, exchange the integrals in t and x and then bring the convolution on φ, leads to

$$\langle \mu_t^\epsilon, \varphi_t \rangle = \langle \mu_0^\epsilon, \varphi_0 \rangle + \int_0^t \langle \mu, b \cdot \nabla\varphi^\epsilon \rangle \mathrm{d}r + \int_0^t \langle \mu, (\partial_t\varphi)^\epsilon \rangle \mathrm{d}r$$

Now we let ϵ go to 0. Using the conditions μ in $L_t^\infty(\mathcal{M}_x)$, $|b||\mu|$ in $L_t^1(\mathcal{M}_{x,loc})$, the uniform (in (t, x)) convergence of φ^ϵ and its space derivative and the $L_t^1(C_x)$ convergence of $\partial_t\varphi^\epsilon$, we get 3.7. □

3.2 Random equations and rigorous links with stochastic equations

In this section we introduce the random (ordinary and partial) differential equations corresponding to the stochastic equations SDE, SCE and STE.

These are deterministic equations, but with coefficients parameterized by ω in Ω, and they are obtained from the stochastic equation via a suitable transformation. As we will see, their usefulness is mainly in two facts (beside also other technical aspects):

- the random coefficients (the random drift $\tilde{b}$ below), obtained transforming the usual coefficients (the drift b) enjoy some unexpected regularity property, that gives an intuition and, in some cases, a proof of regularization by noise;
- in this setting we can study uniqueness among solutions for these equations, at frozen ω, that is path-by-path uniqueness.

The random coefficient that appears in every equation is

$$\tilde{b}(t,x,\omega) = b(t,x+W_t(\omega)).$$

The starting point is the following: if X solves the SDE, then $\tilde{X} = X - W$ solves the random ODE (rDE)

$$\mathrm{d}\tilde{X} = \tilde{b}(\tilde{X})\mathrm{d}t.$$

One can go also in the reverse direction. Precisely, the following lemma, of immediate proof, holds:

Lemma 3.11. *A process X satisfies the SDE if and only if $\tilde{X} = X - W$ is progressively measurable (with respect to $(\mathcal{F}_t)_t$) and satisfies the rDE for a.e. ω.*

Given the rDE, we consider the associated linear PDEs, namely the random continuity equation (rCE)

$$\mathrm{d}\tilde{\mu} + \mathrm{div}(\tilde{b}\tilde{\mu})\mathrm{d}t = 0;$$

and the random transport equation (rTE)

$$\mathrm{d}\tilde{v} + \tilde{b}\cdot\nabla\tilde{v}\mathrm{d}t = 0.$$

Given the stochastic PDEs and their random counterparts, one can ask for a direct link among them. From the link between ODEs and linear PDEs (3.1) and (3.2) (in both deterministic and stochastic cases) and the formula $\tilde{X} = X - W$, we can guess such a link: we have formally

$$\tilde{\mu}_t = (\tilde{X})_t)_{\#}\mu_0 = (X_t - W_t)_{\#}\mu_0 = (\cdot - W_t)_{\#}\mu_t$$

and similarly

$$\tilde{v}_t = v_0(\tilde{X}_t^{-1}) = v_0(\tilde{X}_t^{-1} - W_t) = v_t(\cdot - W_t).$$

In the following lemmata we prove these links rigorously in the general case, without appealing to the underlying flow X (which may not exist in general), but with a regularization procedure.

Lemma 3.12. *Let μ be a distributional solution to the SCE. Then $\tilde{\mu}$, defined by*

$$\tilde{\mu}_t = (\cdot - W_t)_{\#}\mu_t,$$

is a distributional solution to the rCE.

Notice that, for a.e. ω, $\tilde{\mu}$ makes sense as $\mathcal{M}_x$-valued map $t \mapsto \tilde{\mu}_t$, bounded-weakly progressively measurable and in $L_t^\infty(\mathcal{M}_x)$; moreover, $|\tilde{b}||\tilde{\mu}|$ is in $L_t^1(\mathcal{M}_{x,loc})$ for a.e. ω.

Proof. We start with the case of initial time 0. Fix φ in $C_c^\infty(\mathbb{R}^d)$. We claim that it holds, for a.e. (t, x, ω),

$$\begin{aligned}
&\mu_t^\epsilon(x)\varphi(x - W_t) &(3.8)\\
&= \mu_0^\epsilon(x)\varphi(x) - \int_0^t (\mu_r b_r) * \nabla\rho_\epsilon(x)\varphi(x - W_r)\mathrm{d}r + \\
&- \int_0^t \mu_r * \nabla\rho_\epsilon(x)\varphi(x - W_r)\cdot \mathrm{d}W_r + \frac{1}{2}\int_0^t \mu_r * \Delta\rho_\epsilon(x)\varphi(x - W_r)\mathrm{d}r + \\
&- \int_0^t \mu_r^\epsilon(x)\nabla\varphi(x - W_r)\cdot \mathrm{d}W_r + \frac{1}{2}\int_0^t \mu_r^\epsilon(x)\Delta\varphi(x - W_r)\mathrm{d}r + \\
&+ \int_0^t \mu_r * \nabla\rho_\epsilon(x)\cdot\nabla\varphi(x - W_r)\mathrm{d}r,
\end{aligned}$$

where all the addends but the stochastic integral are measurable in (t, x, ω) and the stochastic integrals have versions that are measurable in (t, x, ω) (these are the versions considered in the equality above). So, for a.e. (t, ω), integrating (3.8) in x, we get

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mu^\epsilon(t,x)\varphi(x-W_t)\mathrm{d}x \\
&= \int_{\mathbb{R}^d} \mu_0^\epsilon(x)\varphi(x)\mathrm{d}x - \int_0^t \int_{\mathbb{R}^d} (\mu_r b_r) * \nabla\rho_\epsilon(x)\varphi(x-W_r)\mathrm{d}x\mathrm{d}r + \\
&\ - \int_0^t \int_{\mathbb{R}^d} \mu_r * \nabla\rho_\epsilon(x)\varphi(x-W_r)\mathrm{d}x \cdot \mathrm{d}W_r + \\
&\ + \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} \mu_r * \Delta\rho_\epsilon(x)\varphi(x-W_r)\mathrm{d}x\mathrm{d}r + \\
&\ - \int_0^t \int_{\mathbb{R}^d} \mu_r^\epsilon(x)\nabla\varphi(x-W_r)\mathrm{d}x \cdot \mathrm{d}W_r + \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} \mu_r^\epsilon(x)\Delta\varphi(x-W_r)\mathrm{d}x\mathrm{d}r + \\
&\ + \int_0^t \int_{\mathbb{R}^d} \mu_r * \nabla\rho_\epsilon(x) \cdot \nabla\varphi(x-W_r)\mathrm{d}x\mathrm{d}r,
\end{aligned}$$

where we used Fubini theorem and stochastic Fubini theorem (precisely Theorem 2.2 in [Ver12]) to exchange the integrals. Using again Fubini theorem to bring the convolution on $\varphi(\cdot - W_t)$, we get (for a.e. (t,ω))

$$\begin{aligned}
&\langle \mu_t, \varphi^\epsilon(\cdot - W_t)\rangle \\
&= \langle \mu_0, \varphi^\epsilon\rangle - \int_0^t \langle \mu_r, b_r \cdot \nabla\varphi^\epsilon(\cdot - W_r)\rangle \mathrm{d}r + \\
&\ + \int_0^t \langle \mu_r, \nabla\varphi^\epsilon(\cdot - W_r)\rangle \cdot \mathrm{d}W_r + \frac{1}{2}\int_0^t \langle \mu_r, \Delta\varphi^\epsilon(\cdot - W_r)\rangle \mathrm{d}r + \\
&\ - \int_0^t \langle \mu_r, \nabla\varphi^\epsilon(\cdot - W_r)\rangle \cdot \mathrm{d}W_r + \frac{1}{2}\int_0^t \langle \mu_r, \Delta\varphi^\epsilon(\cdot - W_r)\rangle \mathrm{d}r + \\
&\ - \int_0^t \langle \mu_r, \Delta\varphi^\epsilon(x - W_r)\rangle \mathrm{d}r \\
&= \langle \mu_0, \varphi^\epsilon\rangle - \int_0^t \langle \mu_r, b_r \cdot \nabla\varphi^\epsilon(\cdot - W_r)\rangle \mathrm{d}r.
\end{aligned}$$

Letting $\epsilon \to 0$, since μ, $b\mu$ are assumed to be resp. in $L_t^\infty(\mathcal{M}_x)$ and in $L_t^1(\mathcal{M}_{x,loc})$ for a.e. ω, we have for a.e. (t,ω),

$$\langle \mu_t, \varphi(\cdot - W_t)\rangle = \langle \mu_0, \varphi\rangle - \int_0^t \langle \mu_r, b_r \cdot \nabla\varphi(\cdot - W_r)\rangle \mathrm{d}r.$$

By the change of variable $\tilde{x} = x - W_t$, we end with

$$\langle \tilde{\mu}_t, \varphi\rangle = \langle \mu_0, \varphi\rangle + \int_0^t \langle \tilde{b}\tilde{\mu}, \nabla\varphi\rangle \mathrm{d}r, \tag{3.9}$$

valid for every φ in C_c^∞, for every (t,ω) in a full measure set F_φ, depending on φ. In order to conclude, we need to make the "good" full measure set independent of φ. For this, we use a density argument, similar to that in the proof of Lemma 3.7. Let D be a countable set in C_c^∞, dense in C_b^2, take $F = \cap_{\varphi\in D} F_\varphi$. Then F is a full measure set such that, for every (t,ω) in F, (3.9) holds for every φ in D; we can also assume, possibly passing to a smaller full-measure set F, that $\tilde\mu_t^\omega$ is in $\mathcal{M}_x$ and $|\tilde b^\omega||\tilde\mu^\omega|$ is in $L^1_t(\mathcal{M}_{x,loc})$ for every (t,ω) in F. Now, for a generic φ in $C_{x,c}^\infty$, we take a sequence φ^n in D, satisfying equation (3.9) and converging to φ in C_b^2; by the bounds on $\tilde\mu$ and $|\tilde b||\tilde\mu|$ in F and dominated convergence theorem, we can pass to the limit in the equation, for (t,ω) in F, getting (3.9) for φ. Hence, for a.e. ω, the rCE holds.

It remains to prove the claimed formula (3.8) and the measurability (in (t,x,ω)) of the addends. We start with recalling measurability. All the addends but the stochastic integrals can be interpreted in two ways: as elements in $L^m_{t,x,\omega}$ and, for fixed x, as elements in $L^m_{t,\omega}$; we denote by A^{det} a generic addend in $L^m_{t,x,\omega}$ and $[A^{det}(x)]$ its counterpart in $L^m_{t,\omega}$, for x fixed. As for the stochastic integrals, for fixed x, they are elements in $L^m_{t,\omega}$ (for each x, they are equivalent classes), but stochastic Fubini theorem (Theorem 2.2 in [Ver12]) provides versions of them which are in $L^m_{t,x,\omega}$ (in particular measurable in (t,x,ω)); we also denote by A^{stoch} this version in $L^m_{t,x,\omega}$ and $[A^{stoch}(x)]$ its counterpart in $L^m_{t,\omega}$ (the usual stochastic integral) for x fixed. The formula (3.8) must be intended as equality in $L^m_{t,x,\omega}$, involving the terms A^{det} and A^{stoch}. We prove this equality by proving the corresponding equality in $L^m_{t,\omega}$, among the terms $[A^{det}(x)]$ and $[A^{stoch}(x)]$, for each x. For this, fix x in $\mathbb{R}^d$. By the distributional formulation of the SCE, applied to the test function $\rho_\epsilon(x-\cdot)$, we get the following equation for μ^ϵ:

$$\begin{aligned}&\mu_t^\epsilon(x)\\&= \mu_0^\epsilon(x) - \int_0^t (\mu_r b_r) * \nabla\rho_\epsilon(x)\mathrm{d}r +\\&- \int_0^t \mu_r * \nabla\rho_\epsilon(x)\cdot \mathrm{d}W_r + \frac{1}{2}\int_0^t \mu_r * \Delta\rho_\epsilon(x)\mathrm{d}r.\end{aligned}$$

Applying Itô formula (for a.e. continuous function, Proposition A.12 in the Appendix) to $\mu_t^\epsilon(x)\varphi(x-W_t)$, we get (3.8) for a.e. (t,ω), i.e. the desired equality in $L^m_{t,\omega}$. The proof in the case $s=0$ is complete.

In the general case $s\geq 0$, one can apply the change of variable to the solution $\overrightarrow{\mu}$, starting at 0, defined in Remark 3.3, getting a rCE for $\tilde{\overrightarrow{\mu}}_r = (\cdot - \overrightarrow{W}_r)_\# \overrightarrow{\mu}_r = (\cdot - W_{r+s}+W_s)_\#\mu_{r+s}$. Now notice that $\tilde\mu_t = (\cdot - W_s)_\# \overrightarrow{\mu}_{t-s}$, hence, by a change of variable (notice that, for fixed s and ω, $W_s(\omega)$ is a constant), we get the desired rCE for $\tilde{\mu}$. $\square$

Corollary 3.13. *There exists a version of μ which is weakly-* continuous in time, for a.e. ω.*

Notice that this map is weakly-* measurable in (t, ω) because of Lemma A.7 in the Appendix.

Proof. We take the weakly-* continuous version of $\tilde{\mu}$ (which we still call $\tilde{\mu}$ with abuse of notation), we want to prove that the version of μ defined by $(\cdot + W_t)_\# \tilde{\mu}_t$ (which we still call μ) is weakly-* continuous, i.e., for a.e. ω,

$$t \mapsto \langle \mu_t, \varphi \rangle = \langle \tilde{\mu}_t, \varphi(\cdot + W_t) \rangle \tag{3.10}$$

is continuous for every φ in $C_{x,c}$.

Step 1: It is enough to show this property for every φ in $C^1_{x,c}$. Indeed, for a general φ in $C_{x,0}$, we take ψ in $C^1_{x,c}$ with $\|\varphi - \psi\|_{C_{x,b}} < \epsilon$ and we split

$$\begin{aligned}
&|\langle \tilde{\mu}_t, \varphi(\cdot + W_t) \rangle - \langle \tilde{\mu}_r, \varphi(\cdot + W_r) \rangle| \\
&\leq |\langle \tilde{\mu}_t, \psi(\cdot + W_t) \rangle - \langle \tilde{\mu}_r, \psi(\cdot + W_r) \rangle| + \\
&\quad + \|\tilde{\mu}_t\|_{\mathcal{M}_x} \|\psi(\cdot + W_t) - \varphi(\cdot + W_t)\|_{C_{x,b}} + \|\tilde{\mu}_r\|_{\mathcal{M}_x} \|\psi(\cdot + W_r) - \varphi(\cdot + W_r)\|_{C_{x,b}}.
\end{aligned}$$

The RHS is smaller than 3ϵ for r sufficiently close to t, by the thesis for ψ, the bound $\|\tilde{\mu}_t\|_{\mathcal{M}_x} \leq C$ (which holds for every t, since we are using the continuous version) and the invariance of the $C_{x,b}$ norm under translation.

Step 2: Now we prove continuity of (3.10) for φ in $C^1_{x,c}$. In this case we have

$$\begin{aligned}
&|\langle \tilde{\mu}_t, \varphi(\cdot + W_t) \rangle - \langle \tilde{\mu}_r, \varphi(\cdot + W_r) \rangle| \\
&\leq |\langle \tilde{\mu}_t, \varphi(\cdot + W_t) \rangle - \langle \tilde{\mu}_r, \varphi(\cdot + W_t) \rangle| + \|\tilde{\mu}_r\|_{\mathcal{M}_x} \|\varphi(\cdot + W_t) - \varphi(\cdot + W_r)\|_{C_{x,c}}.
\end{aligned}$$

The first addend tends to 0, as $r \to t$, because of weak-* continuity of $\tilde{\mu}$. For the second addend, we have again $\|\tilde{\mu}_r\|_{\mathcal{M}_x} \leq C$ for every r and also $\|\varphi(\cdot + W_t) - \varphi(\cdot + W_r)\|_{C_{x,c}} \leq \|\varphi\|_{C^1_{x,b}} |W_t - W_r|$, which tends to 0 as $r \to t$. We have proved (3.10) for φ in $C^1_{x,c}$. The proof is complete. $\square$

Similarly one can prove the following link between STE and rTE and the existence of a weakly-* continuous version for any solution to the STE:

Lemma 3.14. *Let v be a distributional, resp. differentiable solution to the backward STE. Then $\tilde{v}$, defined by $\tilde{v}_s = v_s(\cdot - W_s)$, is a distributional, resp. differentiable solution to the backward rTE. The analogous result holds for the forward STE.*

Corollary 3.15. *There exists a version of v which is weakly-* continuous in time, for a.e. ω.*

Remark 3.16. *Notice that the full P-measure set of ω, where the rCE (or the rTE) holds, depends (a priori at least) on the initial time s and datum μ and also on the specific representative of the equivalence class μ (see Remark A.5 in the Appendix). Moreover, if $s \neq 0$, the initial datum $\tilde{\mu}_s$ of the rCE is random, although the dependence on ω is only through W_s, which is independent of the driving Brownian motion $(W_t - W_s)_{t\geq s}$; similarly, the final datum $\tilde{v}$ of the rTE is random, although the dependence on ω is only through W_t.*

3.3 Link ODE-PDEs in the regular setting, stochastic case

Given now all the equations, in this Section and in the next one we give a rigorous link between the ordinary (deterministic or stochastic) differential equation and the associated partial differential equations, the link that was formally given in (3.1) and in (3.2).

We start with the link between SDE and stochastic transport equation in the regular setting. The results together with the proofs are in [Kun84] and [Kun97] (for example, existence, uniqueness and representation formulae of Propositions 3.18 and 3.19 are special cases of Theorem 6.1.9 in [Kun97], Chapter 6); the integrability estimates are proved in [BFGM14], Lemma 12 (for that lemma the coefficients are also regular in time, but the proof can be easily adapted to our case).

Proposition 3.17. *Assume that b is in $L^\infty_t(C^{k,\epsilon}_{x,lin})$ for some integer $k \geq 1$ and some ϵ in $]0,1[$. Then, for every initial time s and datum x, strong existence (i.e. existence of solutions adapted to the Brownian filtration) and pathwise uniqueness hold for the SDE starting at time s and from datum x. Moreover, there exists a modification of the solution which is a stochastic C^k_x flow, namely a map $X : \{0 \leq s \leq t \leq T\} \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d$, such that, for every s, x, $X(s, \cdot, x, \cdot)$ is the adapted solution to the SDE, starting at time s and from datum x, and, for a.e. ω, $X(\cdot, \cdot, \cdot, \omega)$ is a flow of continuous maps, C^k-diffeomorphisms in space.*

Proposition 3.18. *Assume that b is in $L^\infty_t(C^{3,\epsilon}_{x,lin})$, for some $\epsilon > 0$, and the final datum v_t is in $C^\infty_{x,b}$. Then there exists a unique classical solution $v = v^{t,v_t}$ to the backward STE, with final time t and final datum v_t and it holds*

$$v(s, x) = v_t(X_{s,t}(x)).$$

In particular this classical solution is in $C_s(C^2_{x,loc})$ for a.e. ω. Moreover, if v_t and b are compactly supported, then the solution v is in $L^m_{s,\omega}(W^{1,m}_{x,\chi})$ for every finite m and every weight $\chi = \chi_\eta$ of the form $\chi_\eta(x) = (1 + |x|^2)^{\eta/2}$, for every real η.

The main consequence of this result is that one can transfer the a priori estimates from SDE to STE and vice versa.

Proposition 3.19. *Assume that b is in $L^\infty_t(C^{4,\epsilon}_{x,lin})$, for some $\epsilon > 0$, and the initial datum μ_s is in $C^\infty_{x,b} \cap L^1_x$. Then there exists a unique classical solution $\mu = \mu^{s,\mu_s}$ to the SCE, with initial time s and initial datum μ_s and it holds*

$$\mu(t, \cdot)\mathrm{d}x = (X_{s,t})_\#(\mu_s \mathrm{d}x).$$

In particular this classical solution is in $C_t(C^2_{x,loc})$ for a.e. ω. Moreover, if μ_s and b are compactly supported, then the solution μ is in $L^m_{t,\omega}(L^m_{x,\chi})$ for every finite m and every weight $\chi = \chi_\eta$ of the form $\chi_\eta(x) = (1 + |x|^2)^{\eta/2}$, for every real η.

3.4 Link ODE-PDEs in the irregular setting: Lagrangian flows

Now we pass to the case of irregular drifts and we present the main link between ODE and continuity equation, in the deterministic setting, without any regularity assumption on b (apart very mild integrability conditions).

First we discuss a representation formula for solutions to the CE, in terms of solutions to the ODE.

Remark 3.20. *There is an easy way to pass from ODE to CE, even in the irregular case. Even if we do not have a regular flow solution to the ODE, but just existence of at least one solution, we can take a finite signed measure η on $C_t(\mathbb{R}^d)$ which is concentrated on the set of solutions to the ODE. Now, calling $\mu_t = (\pi_t)_\#\eta$ (where $\pi_t(\gamma) := \gamma(t)$ is the evaluation map at time t) the 1-marginals of η, then (μ_t) is a distributional solution to the CE, starting from μ_0. Indeed, for any φ in C^∞_c, we have*

$$\begin{aligned}
&\langle \mu_t, \varphi \rangle = \int_{C_t} \varphi(\gamma_t)\eta(\mathrm{d}\gamma) = \int_{C_t} \varphi(\gamma_s)\eta(\mathrm{d}\gamma) + \int_{C_t}\int_s^t b(\gamma_r) \cdot \nabla\varphi(\gamma_r)\mathrm{d}r\eta(\mathrm{d}\gamma) \\
&= \langle \mu_s, \varphi \rangle + \int_s^t \langle \mu_t, b \cdot \nabla\varphi \rangle \mathrm{d}r,
\end{aligned}$$

where in the second equality we have used that γ satisfies the ODE for η-a.e. γ.

A natural question arises from this Remark: does any solution μ to the CE admit the representation formula above, that is does there exists η finite signed measure on C_t, concentrated on solutions to the ODE and having $(\mu_t)_t$ as 1-marginals? The answer is positive and is contained in the superposition principle ([AC14], Theorem 12). We do not recall the result here, but we mention that this result is the key to build flows solutions to the ODE out of solutions to the CE, in the following existence and uniqueness result 3.24. Notice also that this is a recurrent problem in probability: as in Kolmogorov extension theorem, one is given a family of 1-marginals and wants to find (and, if possible, to identify) a law η having the given 1-marginals and enjoying certain properties. General results in this direction, at least in the SDEs context, are provided by Kurtz and coauthors (see [KO88], [Kur98] and [KN11] among other papers).

Now we discuss the link between CE uniqueness and ODE uniqueness.

Proposition 3.21. *Let x_0 be an initial datum in $\mathbb{R}^d$ (at initial time s) and suppose that uniqueness holds for the CE starting from δ_{x_0} (at s), among distributional solutions in $L^\infty_t(\mathcal{M}_x)$. Then uniqueness holds for the ODE starting from x_0.*

Proof. Let X, Y be two solutions to the ODE, with $X_s = Y_s = x_0$. Then $\mu_t = \delta_{X_t}$, $\nu_t = \delta_{Y_t}$ are two solutions to the CE (in $L^\infty_t(\mathcal{M}_x)$) starting from δ_{x_0}, so they must coincide for a.e. t. This implies that $X_t = Y_t$ for a.e. t, so for every t by continuity. □

Although this last result is valid when b is in $L^1_t(L^1_{x,loc})$, the uniqueness assumption above is very strong and usually does not hold for general b. In many situations, under suitable assumptions on b, one has existence and uniqueness for the CE in a smaller class of solutions, for example solutions which are bounded in time and space. In this case, we cannot expect uniqueness for the ODE among any solutions, but we can have uniqueness among certain kind of generalized flows.

For this, we define Lagrangian flows.

Definition 3.22. *An admissible Lagrangian class $\mathcal{L}$ is a subset of $L^\infty_t(\mathcal{M}_x)$, satisfying the following conditions:*

- *$\mathcal{L}$ is contained in $L^\infty_t(\mathcal{M}_{x,+})$;*
- *$\mathcal{L}$ is convex;*
- *if μ belongs to $\mathcal{L}$ and if μ', element in $L^\infty_t(\mathcal{M}_x)$, with $0 \le \mu'_t \le \mu_t$ for a.e. t, satisfies the CE and the condition $\int_s^T \int_{\mathbb{R}^d} |b|/(1+|x|)\mu'(\mathrm{d}x)\mathrm{d}t < +\infty$, then also μ' belongs to $\mathcal{L}$.*

We use the notation span($\mathcal{L}$) for the vector space generated by $\mathcal{L}$.

Definition 3.23. *An $\mathcal{L}$ Lagrangian flow, solution to the ODE with initial positive finite measure μ_s and initial time s, is a measurable map $X : [s,T] \times \mathbb{R}^d \to \mathbb{R}^d$, such that*

1. *for μ_s-a.e. x, the map $t \mapsto X(t,x)$ solves the ODE starting from x at s;*
2. *the family $(\mu_t^X)_t$, defined by $\mu_t^X := (X(t,\cdot))_\# \mu_s$, belongs to $\mathcal{L}$.*

Reasoning as in Remark 3.20, one can see easily that μ_t^X is a distributional solution to the CE, belonging to $\mathcal{L}$. The main point is that, if uniqueness holds for the CE among solutions in $\mathcal{L}$, then this implies uniqueness among $\mathcal{L}$ Lagrangian flows, and, in this case, existence also transfers from CE to ODE. This result is a consequence of [AC14], Theorems 16 and 19 and relative proofs.

Theorem 3.24. *Let b be in $L^1_t(L^1_{x,loc})$ and let μ_s be in $\mathcal{M}_{x,+}$.*

1. *Assume that uniqueness holds for the CE among $\mathcal{L}$ solutions, starting from s, for any initial nonnegative measure bounded by μ_s. Then uniqueness holds among $\mathcal{L}$ Lagrangian flows, starting from μ_s (and from any initial nonnegative measure bounded by μ_s) at s.*
2. *Assume in addition that existence holds for the CE among $\mathcal{L}$ solutions, for μ_s as initial solution at initial time s. Then existence and uniqueness hold among $\mathcal{L}$ Lagrangian flows, starting from μ_s at s.*
3. *In this case, let μ be the $\mathcal{L}$-solution to the CE (starting from μ_s at s) and let X be the $\mathcal{L}$-Lagrangian flow. Then we have, for every t,*

$$\mu_t = (X_t)_\# \mu_s. \tag{3.11}$$

This completes the link the deterministic case. In order to cover also the stochastic case, we introduce the concept of stochastic Lagrangian flows: at the level of rDE, they are Lagrangian flows solutions of the rDE, indexed by ω, adapted to the proper filtration.

Definition 3.25. *An $\mathcal{L}$ stochastic Lagrangian flow, solution to the SDE with initial positive finite measure μ_s, is a map $X : [s,T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d$, such that*

1. *the map $(t,\omega,x) \mapsto X(t,x,\omega)$ is $\mathcal{L}^1|_{[s,T]} \otimes P \otimes \mu_s$-measurable with respect to $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$;*

2. *for μ_s-a.e. x, the process $t \mapsto X(t,x)$ solves the SDE starting from x;*

3. *for a.e. ω, the family $(\tilde{\mu}_t^{\tilde{X}^\omega})_t$, defined by $\tilde{\mu}_t^{\tilde{X}^\omega} := (\tilde{X}^\omega(t,\cdot))_\# \tilde{\mu}_s^\omega$, belongs to $\mathcal{L}$.*

Remark 3.26. *The condition of measurability with respect to $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$ means that, for every open set A of $\mathbb{R}^d$, $X^{-1}(A)$ is in the completion of $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$ with respect to the measure $\mathcal{L}^1 \otimes P \otimes \mu_s$. It is an adaptability condition (recall that $\mathcal{P}$ is the progressive σ-algebra) and it implies that, for μ_s-a.e. x, the process $t \mapsto X(t,x)$ is progressively measurable.*

The following Lemma, of immediate proof, gives the link between stochastic Lagrangian flows for the SDE and Lagrangian flows associated with the rDE:

Lemma 3.27. *The map X is a stochastic Lagrangian flow if and only if, for a.e. ω, $\tilde{X}^\omega$ is a Lagrangian flow for the rDE and X is $\mathcal{L}^1 \otimes P \otimes \mu_s$-measurable with respect to $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$.*

Remark 3.28. *Existence of a stochastic Lagrangian flow X implies existence of a Lagrangian flow $\tilde{X}$ for the rDE, on a full P-measure set $\Omega_1^{\mu_s}$ in Ω which may depend on the initial measure μ_s. However, if ω is in $\Omega_1^{\mu_s}$ and if ν is another positive measure on $\mathbb{R}^d$ bounded by $\tilde{\mu}_s^\omega$, then the flow $\tilde{X}^\omega$, restricted to the support of ν, is a Lagrangian flow solving the rDE at ω fixed. In particular, if we have existence for μ_s fully supported on $\mathbb{R}^d$ or for a sequence (μ_s^n) of measures whose supports cover $\mathbb{R}^d$, then one can choose Ω_1 independently of μ_s.*

In the following, we will be mainly interested in the path-by-path uniqueness, among single trajectories or among flows.

3.5 Different types of uniqueness

We have seen we can deal with SDE or rDE, and the same for the associated linear PDEs, and among single trajectories or among Lagrangian flows. Connected to each kind of equation and solution, we have a different kind of uniqueness: pathwise uniqueness for the SDE, path-by-path uniqueness for the rDE.

3.5.1 Uniqueness for SDE

Definition 3.29. *We say that the SDE has pathwise uniqueness, starting from x in $\mathbb{R}^d$ at time s, if, for every (countably generated) filtered probability* *space $(\Omega, (\mathcal{F}_{s,t})_t, P)$ and for every Brownian motion with respect to $(\mathcal{F}_{s,t})_t$, the SDE has a unique solution X with $X_s = x$.*

Definition 3.30. *We say that the SDE has path-by-path uniqueness, starting from x in $\mathbb{R}^d$ at time s, if, for a.e. realization $W(\omega)$ of the Brownian motion, the rDE has at most one solution.*

Since any solution of the SDE can be transformed in a solution to the rDE, through Lemma 3.11, we get:

Lemma 3.31. *Path-by-path uniqueness (starting from x at s) implies pathwise uniqueness (starting from x at s).*

Since the rDE brings also the concept of Lagrangian flows, path-by-path uniqueness can be defined also for these flows.

Definition 3.32. *We say that the SDE has path-by-path uniqueness among $\mathcal{L}$ Lagrangian flows, starting from a positive measure $\bar{\mu}_s$ on $\mathbb{R}^d$ at time s, if, for a.e. realization $W(\omega)$ of the Brownian motion, uniqueness among $\mathcal{L}$-Lagrangian flows, starting from $\bar{\mu}_s$ at s, holds for the rDE.*

Path-by-path uniqueness among single solutions is a particular case of this uniqueness, choosing $\mathcal{L} = L^\infty_t(\mathcal{M}_x)$.

One may define also a concept of pathwise uniqueness among stochastic Lagrangian flows, but this concept would be a bit involved (we should take the conditional probability of μ with respect to $W(\omega)$), hence we skip it. Moreover, in many cases where path-by-path uniqueness is available only among Lagrangian flows, it is possible by a modified argument to get pathwise uniqueness even starting from a given point x: for example, existence and path-by-path uniqueness imply existence of a strong (i.e. adapted to the Brownian filtration) solution, for a.e. x; this and uniqueness in law imply pathwise uniqueness (for a.e. x).

3.5.2 Uniqueness for SPDEs

In correspondence with the concepts of pathwise and path-by-path uniqueness for the SDE, we can state also pathwise and path-by-path uniqueness for the SCE and for the STE.

Definition 3.33. *We say that pathwise uniqueness for the SCE holds, in a certain subset $\mathcal{V}$ of $L^\infty_t(\mathcal{M}_x)$, starting from an initial measure μ_s at time s, if, for every (countably generated) filtered probability space $(\Omega, (\mathcal{F}_{s,t})_t, P)$ and for every Brownian motion with respect to $(\mathcal{F}_{s,t})_t$, the SCE has at most* *one solution μ starting from μ_s at s, with μ^ω in $\mathcal{V}$ for a.e. ω. Analogous definition for the (forward or backward, distributional or differentiable) STE, with backward filtration for the backward case.*

Definition 3.34. *We say that path-by-path uniqueness for the SCE holds, in a certain set $\mathcal{V}$, starting from an initial measure $\bar{\mu}_s$ at time s, if, for a.e. realization $W(\omega)$ of the Brownian motion, uniqueness in $\mathcal{V}$, starting from $\bar{\mu}_s$ at s, holds for the rCE. Analogous definition for the (forward or backward, distributional or differentiable) STE.*

By the link between STE and rTE, in Lemma 3.12, we get:

Lemma 3.35. *Assume that $\mathcal{V}$ is invariant under the operation $(\nu_t)_t \mapsto ((\cdot - W_t)_\# \nu_t)_t$. Then path-by-path uniqueness in $\mathcal{V}$ implies pathwise uniqueness in $\mathcal{V}$.*

Remark 3.36. *Stated in this way, the exceptional set in Ω, where uniqueness for the rCE does not hold, may depend on the initial measure μ_s. However, if we prove uniqueness in $\mathcal{L}$ of solutions starting from 0, in a full P-measure set Ω_1, then, by linearity, this uniqueness (in $\mathcal{L}$) holds also for any initial condition in $\mathcal{M}_x$, for every ω in Ω_1 (which is independent of the initial condition).*

Remark 3.37. *Path-by-path uniqueness for the SDE in a class $\mathcal{L}$ and path-by-path uniqueness for the SCE in $\mathcal{L}$ are linked via Theorem 3.24, applied to the rDE and the rCE.*

3.5.3 Wiener uniqueness

When speaking about uniqueness for SCE (and linear SPDEs more in general), there is also another kind of uniqueness: Wiener uniqueness, i.e. uniqueness among solutions to the SCE adapted to the Brownian (completed) filtration.

Definition 3.38. *We say that Wiener uniqueness for the SCE holds, in a certain subset $\bar{V}$ of $L^\infty_{t,\omega}(\mathcal{M}_x)$, starting from an initial measure μ_s at time s, if, given a Brownian motion and its natural completed filtration (starting from s) $(\mathcal{F}_{s,t})_t$, the SCE has at most one solution μ in $\bar{V}$ starting from μ_s at s and adapted to the Brownian completed filtration.*

This is the weaker form of uniqueness for the SCE. This kind of uniqueness should be the translation, at the level of the SCE, of uniqueness in law for the SDE. The intuitive idea behind this correspondence is the following one.

Given any solution X to the SDE, a solution $(\mu_t)_t$ to the continuity equation can be created filtering its law with respect to the Brownian motion and taking the 1-marginal time projections; when the law of X (or, more precisely, the law of (X, W)) is unique, then also (μ_t) is uniquely determined. So, if any solution of the continuity equation, adapted to the Brownian motion, is obtained via this filtering procedure, then uniqueness in law implies Wiener uniqueness.

We do not explore here, in a rigorous way, the link between Wiener uniqueness for the SCE and uniqueness in law for the SDE. However we will show Wiener uniqueness starting from uniqueness of the Fokker-Planck equation associated with the SDE (which is related precisely to uniqueness in law for the SDE): this technique gives a well-posedness result (though weak) with not so much effort.

3.6 Stability and existence for stochastic PDEs via a priori estimates

In this Section we state and prove some stability results for the (stochastic) linear PDEs under convergence of sequences, when uniform bounds for the sequence are given. These results, beside their intrinsic interest, are the basis for the a priori estimates method: we can prove existence of a solution, in the case when the coefficients are irregular, by showing uniform estimates of the solutions of approximating equations with regularized coefficients. All the results are given in the stochastic case, but they are valid also in the deterministic case, with same assumptions and theses but dropping ω (and adaptability).

We start with stochastic transport equation. We state and prove the result only in the forward case, the backward case being completely similar.

Theorem 3.39. *Take $1 < m < +\infty$. Let b^n, n in $\mathbb{N}$, b be vector fields in $L^{m'}_t(L^{m'}_{x,loc})$ such that $(b^n)_n$ converges to b in $L^{m'}_t(L^{m'}_{x,loc})$. Let v^n_0, v^0 be in $C^2_{x,b}$ such that $(v^n_0)_n$ converges weakly to v_0 in $L^m_{x,loc}$. Assume that, for each b^n and v^n_0, there exists v^n differentiable solution to the forward STE driven by the drift b^n, starting from v^n_0. Assume, for every $R > 0$, the uniform bound*

$$\sup_n \|v^n\|_{L^m_{t,\omega}(W^{1,m}_{x,B_R})} + \sup_n \|v^n\|_{L^\infty_{t,x,\omega}} < +\infty. \tag{3.12}$$

Then there exists v in $L^m_{t,\omega}(W^{1,m}_{x,loc})$ differentiable solution to the STE driven by b, starting from v_0, and there exists a subsequence $(v^{n_k})_k$ weakly converging to v in $L^m_{t,\omega}(W^{1,m}_{x,loc})$.

Before going into the proof, we show an elementary fact on conditional expectation, which will be useful to prove adaptedness of v.

Lemma 3.40. *Let $(E,\mathcal{A},Q)$ be a probability space and let $\mathcal{B}$ be a σ-algebra contained in $\mathcal{A}$. Then, for any integrable random variable Z on $(E,\mathcal{A},Q)$, it holds: Z is measurable with respect to $\mathcal{B}$ if and only if, for any bounded random variable F on $(E,\mathcal{A},Q)$,*

$$E[ZF]=E[ZE[F|\mathcal{B}]]. \tag{3.13}$$

Proof. We know that Z is $\mathcal{B}$-measurable if and only if $Z=E[Z|\mathcal{B}]$ a.s., that is, if and only if, for any bounded $\mathcal{A}$-measurable random variable F,

$$E[ZF]=E[E[Z|\mathcal{B}]F].$$

Now, by well-known properties of conditional expectation, we have for the RHS $E[E[Z|\mathcal{B}]F]=E[E[Z|\mathcal{B}]E[F|\mathcal{B}]]=E[ZE[F|\mathcal{B}]]$. The lemma is proved. □

Corollary 3.41. *The space $L^m(E,\mathcal{B},Q)$ is stable under weak $L^m(\mathcal{A})$ convergence, i.e.: if $(Z_n)_n$ is a sequence of random variables in $L^m(\mathcal{B})$, weakly convergent in $L^m(\mathcal{A})$ to Z, then Z is $\mathcal{B}$-measurable.*

Proof. In view of the previous lemma, it is enough to verify that $E[ZF]=E[ZE[F|\mathcal{B}]]$ for any bounded $\mathcal{A}$-measurable F. Since this equality holds for Z_n, we can pass to the limit to get the equality for Z. □

Actually, such a result follows also by an abstract argument in functional analysis (namely: a convex subset of a Banach space is strongly closed if and only if it is weakly closed), but we wanted to remark the Lemma above.

Proof of Theorem 3.39. **First step: existence of a weakly convergent subsequence**. Fix $R>0$. Since $W^{1,m}_{x,B_R}$ is a dual space (see the Appendix, Section A.2), also $L^m_{t,\omega}(W^{1,m}_{x,B_R})$ is a dual space, as a consequence of Proposition A.1 in the Appendix. Hence the uniform bound (3.12) implies by Banach-Alaoglu theorem that there exists a subsequence $(v^{n_k})_k$ converging weakly-* in $L^m_{t,\omega}(W^{1,m}_{x,B_R})$ to a function v. By a diagonal procedure (taking R in $\mathbb{N}$), we can choose the sequence $(n_k)_k$ and the limit v independently of R. We can apply the same argument to the space $L^\infty_{t,x,\omega}$ (which is also a dual space), so we can assume that $(v_{n_k})_k$ converges weakly-* to v in $L^\infty_{t,x,\omega}$.

Second step: verification of adaptedness. We must verify that, for every φ in L^1_x, $\langle v,\varphi\rangle$ is progressively measurable (with respect to $(\mathcal{F}_t)_t$), i.e. it is $\mathcal{P}$-measurable, where $\mathcal{P}$ is the predictable σ-algebra. By a density argument, it is enough to take φ in $C^\infty_{x,c}$. We know that $\langle v^n,\varphi\rangle$ are $\mathcal{P}$-measurable, and they converge weakly to $\langle v,\varphi\rangle$ in $L^m_{t,\omega}$ by the first step. Hence the conclusion follows from Corollary 3.41.

Third step: STE for v. To conclude, we must verify that v satisfies the STE, driven by b, starting from v_0. We know that, for each n, for every φ in C^∞_c,

$$\langle v^n_t,\varphi\rangle = \langle v^n_0,\varphi\rangle + \int_0^t \langle b^n\cdot\nabla v^n,\varphi\rangle \mathrm{d}r + \int_0^t \langle \nabla v^n,\varphi\rangle \mathrm{d}W - \frac{1}{2}\int_0^t \langle \nabla v^n,\nabla\varphi\rangle \mathrm{d}r.$$

We will prove that every term of the formula above converges in a weak sense (against any bounded functions) to the same term without the superscript n, so that v verifies (3.5). For this, let F be a function in $L^\infty_{t,\omega}$, take $R>0$ such that the support of φ is contained in B_R.

It is an immediate consequence of the weak-* convergence of v and of the weak convergence of v_0 that

$$\int_0^T E[\langle v^n_t - v^n_0,\varphi\rangle F_t]\mathrm{d}t \to \int_0^T E[\langle v_t - v_0,\varphi\rangle F_t]\mathrm{d}t.$$

Now we analyze the term with b^n. We split it into

$$\begin{aligned}
&\left|\int_0^T E\left[\int_0^t \langle b^n\cdot\nabla v^n,\varphi\rangle \mathrm{d}r F\right]\mathrm{d}t - \int_0^T E\left[\int_0^t \langle b\cdot\nabla v,\varphi\rangle \mathrm{d}r F\right]\mathrm{d}t\right| \\
&\le \left|\int_0^T E\left[\langle b\cdot(\nabla v^n-\nabla v),\varphi\rangle \int_r^T F\mathrm{d}t\right]\mathrm{d}r\right| + \\
&+ \int_0^T E\left[|b^n-b||\nabla v^n||\varphi|\left|\int_r^T F\mathrm{d}t\right|\right]\mathrm{d}r.
\end{aligned}$$

The first addend converges to 0: indeed v^n converges weakly to v in $L^m_{t,\omega}(W^{1,m}_{x,B_R})$ and the map

$$v \mapsto \int_0^T E\left[\langle b\cdot\nabla v,\varphi\rangle \int_r^T F\mathrm{d}t\right]\mathrm{d}r$$

is a linear continuous functional on $L^m_{t,\omega}(W^{1,m}_{x,B_R})$; here we used that b is in $L^{m'}_t(L^{m'}_{x,loc})$. For the second addend, we use Hölder inequality to get

$$\begin{aligned}
&\int_0^T E\left[|b^n-b||\nabla v^n||\varphi|\left|\int_r^T F\mathrm{d}t\right|\right]\mathrm{d}r \\
&\le \|b^n-b\|_{L^{m'}_t(L^{m'}_{x,B_R})}\|\nabla v^n\|_{L^m_{t,\omega}(L^m_{x,B_R})}T\|F\|_{L^\infty_{t,\omega}}.
\end{aligned}$$

Since b^n converges to b in $L_t^{m'}(L_{x,loc}^{m'})$ and ∇v^n in uniformly bounded in $L_{t,\omega}^m(L_{x,B_R}^m)$, also the second addend goes to 0. Hence the integral with b^n converges to the corresponding integral with b.

For the stochastic integral, we have

$$\int_0^t \langle \nabla v^n, \varphi \rangle \mathrm{d}W \to \int_0^t \langle \nabla v, \varphi \rangle \mathrm{d}W,$$

since the map

$$v \mapsto \int_0^t \langle \nabla v, \varphi \rangle \mathrm{d}W$$

is a linear continuous functional on $L_{t,\omega}^m(W_{x,B_R}^{1,m})$ (by Itô isometry and Burkholder-Davis-Gundy inequality).

With the same reasoning, we also have

$$\int_0^t \langle \nabla v^n, \nabla \varphi \rangle \mathrm{d}r \to \int_0^t \langle \nabla v, \nabla \varphi \rangle \mathrm{d}r.$$

Hence v satisfies (3.5). The proof is complete. □

In many situations, one has uniform bounds in stronger Sobolev-type topology. Precisely, given m as in the previous Theorem, we consider the space

$$LW_{m_t,m_\omega,m_x,R} = L_t^{m_t}(L_\omega^{m_\omega}(W_{x,B_R}^{1,m_x})),$$

with the norm

$$\|f\|_{LW_{m_t,m_\omega,m_x,R}} \leq \|\|\|f\|_{W_{x,B_R}^{1,m_x}}\|_{L_\omega^{m_\omega}}\|_{L_t^{m_t}},$$

where $m \leq m_t, m_\omega, m_x < \infty$, $R > 0$; in the case $R = +\infty$, we replace W_{x,B_R}^{1,m_x} with W_x^{1,m_x} in the definition. We extend this definition to the case when at least one among m_t, m_ω, m_x is $+\infty$: in this case, the space $LW_{m_t,m_\omega,m_x,R}$ is defined as the subspace of $L_t^m(L_\omega^m(W_{x,B_R}^{1,m}))$ with finite LW norm; see the Appendix, Sections A.1 and A.2 for more details.

Proposition 3.42. *Under the assumptions of Theorem 3.39, suppose that, for every $R > 0$ (resp. $R = +\infty$),*

$$\sup_n \|v^n\|_{LW_{m_t,m_\omega,m_x,R}} < +\infty. \tag{3.14}$$

Then v belongs to $LW_{m_t,m_\omega,m_x,R}^\alpha$ for every $R > 0$ (resp. $R = +\infty$) and

$$\sup_n \|v^n\|_{LW_{m_t,m_\omega,m_x,R}} \leq \sup_n \|v^n\|_{LW_{m_t,m_\omega,m_x,R}}. \tag{3.15}$$

Moreover, if m_t, m_ω and m_x are all finite, there exists a subsequence $(v^{n_k})_k$ which converges weakly in $LW_{m_t,m_\omega,m_x,R}$ for every $R > 0$ finite.

Proof. By Theorem 3.39, we can assume (with no loss of generality) that the sequence $(v^n)_n$ converges weakly to v in $L^m_{t,\omega}(W^{1,m}_{x,loc})$. We start with the case of m_t, m_ω, m_x and R all finite (we omit the indices when not necessary). In this case, LW is a reflexive space, as a consequence of reflexivity of W^{1,m_x}_{x,B_R} (see the Appendix, Section A.2) and Proposition A.1 in the Appendix. Hence Banach-Alaoglu theorem applies and gives the existence of an element w in LW and a subsequence $(v^{n_k})_k$ converging weakly to w in LW. Now LW is continuously included in $L^m_{t,\omega}(W^{1,m}_{x,B_R})$, so actually $(v^{n_k})_k$ converges to w also in this space. Hence $w = v$. The bound (3.14) follows from semicontinuity of the norm with respect to weak convergence.

Now we consider the case of $m_x = +\infty$ and m_t, m_ω finite and R finite. For this, recall that, for any f in L^∞_{x,B_R}, as $h \to +\infty$,

$$\frac{1}{C_{h,R}} \|f\|_{L^h_{x,B_R}} \uparrow \|f\|_{L^\infty_{x,B_R}},$$

where $C_{h,R} = |B_R|^{1/h}$ (the fact that the sequence is increasing is a consequence of Hölder inequality). Hence we have, by dominated convergence theorem, for every g in $LW_{m_t,m_\omega,\infty,R}$,

$$\frac{1}{C_{h,R}} \|g\|_{LW_{m_t,m_\omega,h,R}} \uparrow \|g\|_{LW_{m_t,m_\omega,\infty,R}}.$$

Hence (3.15) for finite exponents gives

$$\begin{aligned}
&\|v\|_{LW_{m_t,m_\omega,\infty,R}} = \sup_h \frac{1}{C_{h,R}} \|v\|_{LW_{m_t,m_\omega,h,R}} \\
&\leq \sup_h \sup_n \frac{1}{C_{h,R}} \|v^n\|_{LW_{m_t,m_\omega,h,R}} = \sup_n \frac{1}{C_{h,R}} \|v^n\|_{LW_{m_t,m_\omega,\infty,R}}.
\end{aligned}$$

We pass to the case of $m_\omega = +\infty$, $m \leq m_x \leq +\infty$, m_t and R finite. Again we have, for any G in L^∞_ω, as $h \to +\infty$,

$$\|G\|_{L^h_\omega} \uparrow \|G\|_{L^\infty_\omega},$$

and therefore, by dominated convergence theorem, for every g in $LW_{m_t,\infty,m_x,R}$,

$$\|g\|_{LW_{m_t,h,m_x,R}} \uparrow \|g\|_{LW_{m_t,\infty,m_x,R}}.$$

Hence (3.15) for m_t, m_ω finite and R finite gives

$$\begin{aligned}
&\|v\|_{LW_{m_t,\infty,m_x,R}} = \sup_h \frac{1}{C_{h,R}} \|v\|_{LW_{m_t,h,m_x,R}} \\
&\leq \sup_h \sup_n \frac{1}{C_{h,R}} \|v^n\|_{LW_{m_t,h,m_x,R}} = \sup_n \frac{1}{C_{h,R}} \|v^n\|_{LW_{m_t,\infty,m_x,R}}.
\end{aligned}$$

Iterating again the procedure, we obtain (3.15) in the general case (also m_t possibly $+\infty$), but for R finite.

Finally, for the whole space case ($R = +\infty$), we have

$$\begin{aligned}
&\|v\|_{LW_{m_t,m_\omega,m_x,\infty}} = \sup_{R>0} \|v\|_{LW_{m_t,m_\omega,m_x,R}} \\
&\le \sup_{R>0} \sup_n \|v^n\|_{LW_{m_t,m_\omega,m_x,R}} = \sup_n \|v\|_{LW_{m_t,m_\omega,m_x,\infty}}.
\end{aligned}$$

□

Lemma 3.43. *Assume that $m_t = +\infty$. If we use the weakly-* continuous (in time) version of v, then v satisfies*

$$\|v_t\|_{L^{m_\omega}_\omega(W^{1,m_x}_{x,B_R})} \le \|v\|_{L^\infty_t(L^{m_\omega}_\omega(W^{1,m_x}_{x,B_R}))} \tag{3.16}$$

for every t (not just for a.e. t).

Proof. We start with the case m_ω, m_x and R finite. Take a full-measure (in particular dense) set F in $[0,T]$ such that the bound (3.16) holds for every t in $[0,T]$. Fix t in $[0,T]$, take a sequence $(t_n)_n$ in F converging to t. Since $L^{m_\omega}_\omega(W^{1,m_x}_{x,B_R})$ is a reflexive space, by Banach-Alaoglu theorem there exists a subsequence (t_{n_k}) such that $v_{t_{n_k}}$ converges weakly in $L^{m_\omega}_\omega(W^{1,m_x}_{x,B_R})$ to an element w in $L^{m_\omega}_\omega(W^{1,m_x}_{x,B_R})$ and $\|w\|_{L^{m_\omega}_\omega(W^{1,m_x}_{x,B_R})} \le \|v\|_{L^\infty_t(L^{m_\omega}_\omega(W^{1,m_x}_{x,B_R}))}$ by lower semicontinuity. In particular, for every φ in $C^\infty_{x,c}$, for every G in L^∞_ω,

$$E[\langle w, \varphi\rangle G] = \lim_k E[\langle v_{t_{n_k}}, \varphi\rangle G].$$

On the other hand, weakly-* continuity of v gives (by dominated convergence theorem)

$$E[\langle v_t, \varphi\rangle G] = \lim_k E[\langle v_{t_{n_k}}, \varphi\rangle G].$$

Therefore $v_t = w$ for a.e. (ω, x) and so $\|v_t\|_{L^{m_\omega}_\omega(W^{1,m_x}_{x,B_R})} \le \|v\|_{L^\infty_t(L^{m_\omega}_\omega(W^{1,m_x}_{x,B_R}))}$.

For the case when at least one among m_ω, m_x and R is infinite, we can repeat the procedure in the proof of Proposition 3.42 and get the thesis. □

As an immediate consequence of the stability result, we can infer existence of a differentiable solution to the STE from a priori estimates. Here and in the following, we say that a Banach space V, contained in $L^p_{t,x,loc}$ for some $1 \le p < +\infty$, has $C_t(C^\infty_{x,c})$ as a mildly $L^p_{t,x,loc}$-dense set if $C_t(C^\infty_{x,c})$ is contained in V and, for every f in V, there exists a sequence $(f^n)_n$ in $C_t(C^\infty_{x,c})$ which converges to f in $L^p_{t,x,loc}$ and is bounded in V.

Corollary 3.44. *Fix $1 < m < +\infty$, $\alpha \geq 1$, $m \leq m_t, m_\omega, m_x \leq +\infty$. Let V be a Banach space, contained in $L^{m'}_{t,x,loc}$ and having $C_t(C^\infty_{x,c})$ as a mildly $L^{m'}_{t,x,loc}$-dense subset. Assume that, for every $R > 0$, there exists a locally bounded function $C : [0,+\infty[\times[0,+\infty[\to [0,+\infty[$ such that, for every b in $C_t(C^\infty_{x,c})$ and every initial datum v_0 in $C^\infty_{x,c}$, the corresponding classical solution v to the STE satisfies*

$$\|v\|_{L^m_{t,\omega}(W^{1,m}_{x,B_R})} \leq C(\|b\|_V, \|v_0\|_{C^2_b}). \tag{3.17}$$

Then, for every b in V, for every initial datum v_0 in C^2_b, there exists a differentiable solution v to the corresponding STE, which satisfies the bound (3.17)*. Furthermore, if we also have the uniform bound*

$$\|v\|_{L^{m_t}_t(L^{m_\omega}_\omega(W^{\alpha,m_x}_{x,B_R}))} \leq C(\|b\|_V, \|v_0\|_{C^2_b}), \tag{3.18}$$

then we can choose v satisfying also the bound (3.18)*.*

Proof. Let $(b^n)_n$ be a sequence in $C_t(C^\infty_{x,c})$, converging to b in $L^{m'}_{t,x,loc}$ and bounded in V, and let v^n be the classical solution to the STE driven by b^n (with initial datum v^n_0 being also a proper regularization of v_0). Then the solution v with the estimates (3.17) and (3.18) is obtained as a consequence of the previous Theorem 3.39 and Proposition 3.42, provided we have a uniform bound also in the $L^\infty_{t,x,\omega}$ norm. But this bound holds, since any classical solution v^n verifies $v^n_t = v^n_0(X^{-1}_t)$ and so $\|v^n\|_{L^\infty_{t,x,\omega}} \leq \|v^n_0\|_{L^\infty_x}$. □

A similar result holds for the distributional solution to the STE, we state the result without the proof, which is analogous.

Theorem 3.45. *Let b^n, n in $\mathbb{N}$, b be vector fields in $L^1_t(L^1_{x,loc})$, with* div$b$ *in $L^1_t(L^1_{x,loc})$, such that $(b^n)_n$ and $(\mathrm{div} b^n)_n$ converges resp. to b,* div$b$ *in $L^1_t(L^1_{x,loc})$. Let v^n_0, v^0 be in $C^2_{x,b}$ such that $(v^n_0)_n$ converges weakly to v_0 in $L^1_{x,loc}$. Assume that, for each b^n and v^n_0, there exists v^n distributional solution to the forward STE, driven by the drift b^n, starting from v^n_0. Assume, for every $R > 0$, the uniform bound*

$$\sup_n \|v^n\|_{L^\infty_{t,x,\omega}} < +\infty.$$

Then there exists v in $L^\infty_{t,x,\omega}$ distributional solution to the STE driven by b, starting from v_0, and there exists a subsequence $(n_k)_k$ such that $(v^{n_k})_k$ converges weakly- to v in $L^\infty_{t,x,\omega}$.*

Corollary 3.46. *For every b in $L^1_t(L^1_{x,loc})$ with* div$b$ *in $L^1_t(L^1_{x,loc})$, for every initial datum v_0 in C^2_b, there exists a distributional solution v to the corresponding STE, which satisfies the bound*

$$\|v\|_{L^\infty_{t,x,\omega}} \leq \|v_0\|_{L^\infty_{t,x,\omega}}.$$

Proof. The proof follows again by approximation of b with $C_t(C^\infty_{x,c})$ functions, using then Theorem 3.45 and the bounds for classical solutions $\|v\|_{L^\infty_{t,x,\omega}} \leq \|v_0\|_{L^\infty_x}$. $\square$

Finally we state the stability result for the SCE.

Theorem 3.47. *Take $1 < m < +\infty$. Let b^n, n in $\mathbb{N}$, b be vector fields in $L^{m'}_t(L^{m'}_{x,loc})$ such that $(b^n)_n$ converges to b in $L^{m'}_t(L^{m'}_{x,loc})$. Let μ^n_0, μ^0 be in $L^{m'}_{x,loc} \cap \mathcal{M}_x$ such that $(\mu^n_0)_n$ converges weakly-* to μ_0 in $\mathcal{M}_x$. Assume that, for each b^n and μ^n_0, there exists μ^n distributional solution to the SCE, driven by the drift b^n, starting from μ^n_0. Assume, for every $R > 0$, the uniform bound*

$$\sup_n \|\mu^n\|_{L^m_{t,\omega}(L^m_{x,B_R})} + \sup_n \|\mu^n\|_{L^\infty_{t,\omega}(\mathcal{M}_x)} < +\infty.$$

Then there exists μ in $L^m_{t,\omega}(L^m_{x,loc}) \cap L^\infty_{t,\omega}(\mathcal{M}_x)$ distributional solution to the SCE, driven by b, starting from μ_0, and there exists a subsequence $(n_k)_k$ such that $(\mu^{n_k})_k$ converges weakly to μ in $L^m_{t,\omega}(L^m_{x,loc})$.

Proof. The proof is analogous but for proving that μ is in $L^\infty_{t,\omega}(\mathcal{M}_x)$, for which we cannot use the previous machinery, because this space may not be a dual space. So we prove directly that

$$\|\mu\|_{L^\infty_{t,\omega}(\mathcal{M}_x)} \leq \sup_n \|\mu^n\|_{L^\infty_{t,\omega}(\mathcal{M}_x)}.$$

In order to prove this estimate, it is enough to show the same bound but on a closed ball $\bar{B}_R$ in $\mathbb{R}^d$ (i.e. replacing $\mathcal{M}_x$ with $\mathcal{M}_{x,\bar{B}_R}$), for every $R > 0$. For this, it is enough to prove that the $L^\infty_{t,\omega}(\mathcal{M}_{x,\bar{B}_R})$ norm is lower semicontinuous with respect to the weak convergence in $L^m_{t,\omega}(L^m_{x,\bar{B}_R})$. To prove this, by Lemma A.9, there exists a countable set D in $L^\infty_{t,\omega}(C_{x,\bar{B}_R})$, dense in $L^1_{t,\omega}(C_{x,\bar{B}_R})$, such that

$$\|\mu\|_{L^\infty_t(\mathcal{M}_{x,\bar{B}_R})} = \sup_{G \in D, \|G\|_{L^1_{t,\omega}(C_{x,\bar{B}_R})} \leq 1} \int_0^T E[\langle \mu, G \rangle] \mathrm{d}r$$

Since the map $\nu \mapsto \int_0^T E[\langle \nu, G \rangle] \mathrm{d}r$ is continuous in the weak $L^m_{t,\omega}(L^m_{x,\bar{B}_R})$ topology, the $L^\infty_{t,\omega}(\mathcal{M}_{x,\bar{B}_R})$ norm is lower semicontinuous. The proof is complete. $\square$

In the following, we assume $m \leq m_t, m_\omega, m_x \leq +\infty$; furthermore, for applications, we introduce a weight χ, strictly positive test function in C^∞_x. The definition of the weighted space $L^{m_x}_{x,\chi,B_R}$ is extended to $R = +\infty$ as $L^{m_x}_{x,\chi}$. The proof of the following result is similar to that of Proposition 3.42.

Proposition 3.48. *Under the assumptions of Theorem 3.47, suppose that, for every $R > 0$ (resp. $R = +\infty$),*

$$\sup_n \|v^n\|_{L_t^{m_t}(L_\omega^{m_\omega}(L_{x,\chi,B_R}^{m_x}))} < +\infty.$$

Then μ belongs to $L_{m_t}(L^{m_\omega}(L_{x,\chi,B_R}^{m_x}))$ for every $R > 0$ (resp. $R = +\infty$) and

$$\sup_n \|\mu^n\|_{L_t^{m_t}(L_\omega^{m_\omega}(L_{x,\chi,B_R}^{m_x}))} \le \sup_n \|v^n\|_{L_t^{m_t}(L_\omega^{m_\omega}(L_{x,\chi,B_R}^{m_x}))}.$$

Moreover, if m_t, m_ω and m_x are all finite, there exists a subsequence $(v^{n_k})_k$ which converges weakly in $L_t^{m_t}(L_\omega^{m_\omega}(L_{x,\chi,B_R}^{m_x}))$ for every $R > 0$ finite.

Corollary 3.49. *Fix $1 < m < +\infty$, $m \le m_t, m_\omega, m_x \le +\infty$. Let V be a Banach space, contained in $L_{t,x,loc}^{m'}$ and having $C_t(C_{x,c}^\infty)$ as a mildly $L_{t,x,loc}^{m'}$-dense subset; let V_0 be a Banach space, contained in $L_{x,loc}^{m'} \cap \mathcal{M}_x$ and having $C_{x,c}^\infty$ as a mildly $\mathcal{M}_x$-dense subset. Assume that, for every $R > 0$, there exists a locally bounded function $C : [0,+\infty[\times[0,+\infty[\to [0,+\infty[$ such that, for every b in $C_t(C_{x,c}^\infty)$ and every initial datum μ_0 in $C_{x,c}^\infty$, the corresponding classical solution v to the STE satisfies*

$$\|\mu\|_{L_{t,\omega}^m(L_{x,\chi,B_R}^{m_x})} \le C(\|b\|_V, \|\mu_0\|_{V_0}). \tag{3.19}$$

Then, for every b in V, for every initial datum μ_0 in V_0, there exists a distributional solution μ to the corresponding SCE, which satisfies the bound (3.19)*. If μ_0 is a positive measure, we can choose μ to positive measure valued. Furthermore, if we also have the uniform bound*

$$\|\mu\|_{L_t^{m_t}(L_\omega^{m_\omega}(L_{x,\chi,B_R}^{m_x}))} \le C(\|b\|_V, \|\mu_0\|_{V_0}), \tag{3.20}$$

then we can choose μ satisfying also the bound (3.20)*.*

Proof. The proof follows by approximation of b with $C_t(C_{x,c}^\infty)$ functions, using then Theorem 3.47 and Proposition 3.48 and the bounds for classical solutions $\|\mu\|_{L_{t,\omega}^\infty(\mathcal{M}_x)} \le \|\mu_0\|_{\mathcal{M}_x}$. The positivity preserving property follows from the fact that positivity is preserved in the weak convergence in $L_{t,\omega}^m(L_{x,\chi,B_R}^m)$. $\square$

Chapter 4

Renormalization/duality, uniqueness and regularity

In this chapter we describe the renormalization/duality method: this gives a general way to get uniqueness for the stochastic continuity equation. It is a completely deterministic method, therefore its application at ω fixed to the random equations (rCE and rTE) leads to the path-by-path uniqueness. This method has also another advantage, namely it gives well-posedness as soon as we have the existence of a regular (at least differentiable) solution to the stochastic transport equation, or equivalently of a regular flow solution to the SDE.

The duality argument, as presented in this chapter, is taken mainly from [BFGM14], with some technical differences; see also [Sha14] for a similar argument at the level of flows. Some analogies with the renormalization argument are inspired by [AC14].

4.1 The idea of renormalization/duality

Here we present the idea of the duality argument (the argument will be developed later rigorously). Since the method is deterministic in nature, all the equations are here without noise.

We want to prove uniqueness for the CE

$$\partial_t \mu + \operatorname{div}(b\mu) = 0,$$

with given initial datum μ_0 (and initial time 0 for simplicity), among solutions in an admissible Lagrangian class $\mathcal{L}$ (as in Definition 3.22). [Recall that, by Theorem 3.24, this uniqueness implies uniqueness among $\mathcal{L}$ Lagrangian flow solving the ODE.] By linearity of the CE, it is enough to show that any solution μ in span($\mathcal{L}$) with initial datum $\mu_0 \equiv 0$ is 0.

The main point is the formal duality between CE and backward TE, which we now explain. Let v be an $L^\infty_{t,x}$ solution to the backward TE

$$\partial_s v + b \cdot \nabla v = 0,$$

with final condition v_t in C^∞_c. Formally, multiplying μ and v and integrating by parts, we get

$$\partial_t \int_{\mathbb{R}^d} \mu v \mathrm{d}x = -\int_{\mathbb{R}^d} \mathrm{div}(b\mu) v \mathrm{d}x - \int_{\mathbb{R}^d} \mu b \cdot \nabla v \mathrm{d}x = 0. \tag{4.1}$$

Hence $\langle \mu, v \rangle$ is constant in time, so

$$\langle \mu_t, v_t \rangle = \langle \mu_0, v_0 \rangle = 0. \tag{4.2}$$

Since this is true for every v_t in C^∞_c, then we conclude $\mu_t \equiv 0$.

Of course this argument cannot work, because μ and v are not regular in general and so the computations in (4.1) are not valid. To solve this problem, we look at

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(r,x) v(r,y) \rho_\epsilon(x-y) \mathrm{d}x \mathrm{d}y$$

where $(\rho_\epsilon)_{\epsilon>0}$ is the usual family of mollifiers (with some abuse of notation, we write $\mu(x)\mathrm{d}x$ for $\mu(\mathrm{d}x)$). Then, using the distributional formulation of both CE and TE, we get

$$\begin{aligned} &\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(t,x) v(t,y) \rho_\epsilon(x-y) \mathrm{d}x \mathrm{d}y \\ &= \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(r,x) v(r,y) (b(r,x) - b(r,y)) \cdot \nabla \rho_\epsilon(x-y) \mathrm{d}x \mathrm{d}y \mathrm{d}r + \\ &+ \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(r,x) v(r,y) \mathrm{div} b(r,y) \rho_\epsilon(x-y) \mathrm{d}x \mathrm{d}y \mathrm{d}r \end{aligned} \tag{4.3}$$

(the RHS can have a meaning also when div$b$ is not defined, if we assume more regularity on v, as we will see). So, if we prove that the right-hand side of the above formula tends to 0 (as ϵ goes to 0), we have proved uniqueness.

One may notice the similarity between this idea and the renormalization argument by DiPerna and Lions [DL89] and Ambrosio [Amb04]. They consider mainly the TE, which enjoys the following formal property (renormalization): if v is a solution, then also v^2 is a solution. If this formal fact were true, then one could infer uniqueness easily. In order to make this argument rigorous, one uses approximations u^ϵ of u and ends with controlling a remainder similar to the RHS of (4.3). This control is done, in [DL89] and [Amb04], via a commutator lemma, in the case that b has Sobolev or BV regularity and bounded divergence (actually one requires bounded negative part of the divergence).

Hence in both the duality method and the renormalization method there is a formal argument that gives uniqueness and a rigorous approximation argument, where terms like in (4.3) appear. In view of this similarity, we develop the duality argument in a similar way to the renormalization argument. The duality argument has the advantage to be suitable for several hypothesis: one can require different regularity assumptions on b or on v; it also avoid some technical integrability assumptions (if we had to consider μ^2 for μ solution to the CE, then we should probably require μ in L^2). One should also say that this duality is typical of linear situations, while the renormalization techniques can be applied to nonlinear contexts.

Let us mention that the duality technique has also a counterpart in the method of characteristics. Indeed, at least from a formal point of view, having in mind the representation formula (3.1), the relation (4.2) reads as

$$\langle v_t(X_t), \mu_0 \rangle = \langle v_0, \mu_0 \rangle,$$

which, if true for a dense set of μ_0, leads to

$$v_t(X_t) = v_0,$$

which is the formal relation (3.2) (for the backward TE) used in the method of characteristics.

Finally, there is one way to read and to use the duality method which is purely Lagrangian, without PDEs. Given any solution Y to the ODE starting at 0 and a flow X solution to the ODE, we get formally

$$\frac{\mathrm{d}}{\mathrm{d}r} X_{r,t}(Y_r) = DX_{x,t}(Y_r) b(X_{r,t}(Y_r)) - DX_{x,t}(Y_r) b(X_{r,t}(Y_r)) = 0.$$

where we have used that X, as function of the initial time s and the initial datum x, satisfies the TE. This is the counterpart of the computation in (4.1): notice indeed that, formally, $\mu_t = (Y_t)_\# \mu_0$ and $v_r = v_t(X_{r,t})$ (given by the representation formulae (3.1) and (3.2)) verify

$$\langle \mu_r, v_r \rangle = \langle \mu_0, v_t(X_{r,t}(Y_r)) \rangle$$

and so the scalar product $\langle \mu_r, v_r \rangle$ is read in terms of the composition $X_{r,t}(Y_r)$. If one is interested only in the ODE, one may develop the duality argument only in the Lagrangian context.

4.2 Duality pairs and uniqueness

We saw that uniqueness holds as soon as we have a rigorous duality relation. We formalize this with a definition, in analogy to the definition of renormalized solutions.

Definition 4.1. *Let μ, v be resp. a distributional solution to the CE, with initial time s, and a (distributional or differentiable) solution to the backward TE, with final time t. We say that (μ, v) is a duality pair at times (s, t) if:*

1. *μ_s is in L^1_x and v_t is in $C_{x,b}$;*
2. *it holds*
$$\langle \mu_t, v_t \rangle = \langle \mu_s, v_s \rangle.$$

We say that μ admits a duality pairing if, for every t in $[0, T]$ and every φ in $C^\infty_{x,c}$, there exists a (distributional or differentiable)solution v to the backward TE, with final time t and final datum $v_t = \varphi$, such that (μ, v) is a duality pair at times (s, t).

When not specified, we will assume $s = 0$.

Remark 4.2. *The first condition in the definition above guarantees that $\langle \mu_0, v_0 \rangle$ and $\langle \mu_t, v_t \rangle$ exist. It can be replaced, for example, by requiring v_0 and v_t in $C_{x,b}$, or by requiring μ_0 and μ_t in L^1_x.*

Here is the uniqueness result, again in analogy to uniqueness from renormalized solutions.

Proposition 4.3. *Let $\mathcal{L}$ be a Lagrangian class. Suppose that every solution μ to the CE, in the vector space span$(\mathcal{L})$, starting from $\mu_0 \equiv 0$, admits a duality pairing. Then uniqueness holds for the CE among solutions in span$(\mathcal{L})$.*

Proof. By linearity of the CE and stability of span$(\mathcal{L})$ under difference, it is enough to show that any solution μ to the CE in span$(\mathcal{L})$, starting from 0, is 0. Let μ be such a solution, fix t in $[0, T]$ and φ in C^∞_c. By assumption there exists v solution to the backward TE, with $v_t = \varphi$, such that

$$\langle \mu_t, \varphi \rangle = \langle \mu_0, v_0 \rangle = 0.$$

Since this is true for every t and φ, $\mu \equiv 0$. The proof is complete. □

Remark 4.4. *As one can see from the proof, we can weaken the hypothesis of the previous Theorem, asking that μ has a duality pair (μ, v), at time t and with $v_t = \varphi$, for every t in F and every φ in D, where F, D are countable dense sets of resp. $[0, T]$, $C^\infty_{x,c}$. It is also enough to ask this for measures that are difference of solutions in $\mathcal{L}$ with the same initial datum.*

If uniqueness holds for the CE and there exists a duality pair (μ, v), then we have a useful representation formula for v.

Proposition 4.5. *Assume that existence and uniqueness hold for the CE in the class span($\mathcal{L}$) and let X be the Lagrangian flow solving the ODE. Assume that the set $\{\mu_0|\mu \in span\mathcal{L}\} \cap L^1_x$ is dense in L^1_x and that there exists v, solution to the backward TE with final time t and with v_t in $C_{x,b}$, such that (μ, v) is a duality pair for any solution μ in span($\mathcal{L}$). Then it holds, for a.e. x,*

$$v_0(x) = v_t(X_t(x)). \tag{4.4}$$

Proof. Let μ be a solution to the CE in span($\mathcal{L}$). We know (by formula (3.11)) that μ_t admits the representation formula $\langle \mu_t, \varphi \rangle = \langle \mu_0, \varphi(X_t) \rangle$ for every φ measurable bounded function. This formula, applied to v_t, and the duality pair property, give

$$\langle \mu_0, v_0 \rangle = \langle \mu_t, v_t \rangle = \langle \mu_0, v_t(X_t) \rangle.$$

Since the set of possible μ_0 is dense in L^1_x, we have $v_0 = v_t(X_t)$ for a.e. x. $\square$

We end this section by noticing the symmetric role played by μ and v. Indeed, exchanging μ and v in the proofs above, one get the following result:

Proposition 4.6. *Let U be a vector space, contained in $L^m_{t,\omega}(W^{1,m}_{x,loc})$ (resp. in $L^\infty_{t,x}$). Suppose that every differentiable (resp. distributional) solution v to the backward TE, in the space U, with final time t, admits a duality pairing. Then uniqueness holds for the backward TE among differentiable (resp. distributional) solutions in U.*

Remark 4.7. *As before for the CE, it is possible to weaken the assumptions of the Proposition above, asking that v has a duality pair (μ, v), at times (s, t) and initial measure $\mu_s = \varphi$, for every s in F and every φ in D, where F, D are countable dense sets of resp. $[0, T]$, $C^\infty_{x,c}$.*

4.3 The commutator lemma

Here we develop the duality argument, as in the first Subsection: we regularize the solution to the TE, arriving at (4.3), then we study the RHS of (4.3) and we prove that it goes to 0 if b or v enjoy some regularity assumptions. This leads to uniqueness for the CE under these assumptions.

First we make explicit the computations that bring to (4.3): given μ, v be solutions to resp. the CE and the backward TE (in the distributional or in the differential case), now we get an equation for $\mu\otimes v$, which is the element of $L^\infty_{r,\omega}(\mathcal{M}_{x,y,loc})$ defined by

$$\mu_r\otimes v_r(\mathrm{d}(x,y))=\mu_r(\mathrm{d}x)\otimes v_r(y)\mathrm{d}y.$$

Notice that the map $t\mapsto\mu\otimes v$ is weakly-* measurable as an $\mathcal{M}_{x,y,\bar{B}_R}$-valued map, i.e., for every ψ in $C_{x,y,\bar{B}_R}$, $t\mapsto\langle\mu\otimes v,\psi\rangle$ is measurable (by an argument similar to that below). This measurability property can be extended to Borel functions ψ which are not bounded or are time dependent, provided they satisfy suitable integrability assumptions.

Here, when μ and v appear at fixed time, we use the time weakly-* (with respect to the $\mathcal{M}_x$ topology) continuous version of μ and the time weakly-* (with respect to the L^∞_x topology) continuous version of v. Instead, when μ and v appear as integrands of an integral in time, we use the version which is jointly measurable in (t,x,ω), for which we can use Fubini theorem (mind that this version may not be the weakly-* continuous one).

We start with the case when v is a distributional solution. Let ψ be a C^∞_c function on $\mathbb{R}^{2d}$, then $\varphi:(r,x)\mapsto\int_{\mathbb{R}^d}\psi(x,y)v_r(y)\mathrm{d}y$ verifies the equations, for every (r,x),

$$\begin{aligned}\nabla\varphi(r,x)&=\int_{\mathbb{R}^d}\nabla_x\psi(x,y)v_r(y)\mathrm{d}y,\\ \partial_t\varphi(r,x)&=\int_{\mathbb{R}^d}b(r,y)\cdot\nabla_y\psi(x,y)v_r(y)\mathrm{d}y+\int_{\mathbb{R}^d}\mathrm{div}_yb(r,y)\psi(x,y)v_r(y)\mathrm{d}y;\end{aligned}$$

in particular, φ is in $C_t(C^1_{x,c})$, $\varphi(\cdot,x)$ is in $W^{1,1}_t$ for every x and $\partial_t\varphi$ is in $L^1_t(C_{x,c})$. So, applying φ as time-dependent test function for μ and using Lemma 3.10 and the equalities above, we get the equation for $\langle\mu\otimes v,\psi\rangle=\langle\mu,\varphi\rangle$:

$$\langle\mu_t\otimes v_t,\psi\rangle-\langle\mu_0\otimes v_0,\psi\rangle=\int_0^t\langle\mu\otimes v,b_x\cdot\nabla_x\psi+b_y\cdot\nabla_y\psi\rangle\mathrm{d}r+\int_0^t\langle\mu\otimes v,\mathrm{div}b_y\psi\rangle\mathrm{d}r.$$

In the differentiable case for v, proceeding similarly we arrive at

$$\langle\mu_t\otimes v_t,\psi\rangle-\langle\mu_0\otimes v_0,\psi\rangle=\int_0^t\langle\mu\otimes v,b_x\cdot\nabla_x\psi\rangle\mathrm{d}r-\int_0^t\langle\mu\otimes\nabla_xv,b_y\psi\rangle\mathrm{d}r.$$

Now, in order to arrive to the duality relation between μ and v, we take as test function $\psi(x,y)=\rho^{(2)}_\epsilon(x,y)\chi_R(x)$, where $\rho^{(2)}_\epsilon(x,y):=\rho_\epsilon(x-y)$. Here $(\rho_\epsilon)_\epsilon$ is the usual family of (compactly supported) mollifiers on $\mathbb{R}^d$ and, for $R>0$, χ_R is a smooth function, with $0\le\chi_R\le1$ and $|\nabla\chi_R|\le 2/R$, equal to 1 on B_R and to 0 on B_{2R}^c. Notice that this condition implies (for $R \geq 1$) that

$$|\nabla \chi_R(x)| \leq \frac{8}{1+|x|} 1_{R \leq |x| \leq 2R}.$$

In the distributional case, using that $\nabla_x \rho_\epsilon(x-y) = -\nabla_y \rho_\epsilon(x-y)$, we arrive at

$$\begin{aligned}
&\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} v_t(y) \rho_\epsilon(x-y) \chi_R(x) \mu_t(\mathrm{d}x) \mathrm{d}y - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} v_0(y) \rho_\epsilon(x-y) \chi_R(x) \mu_0(\mathrm{d}x) \mathrm{d}y \qquad (4.5) \\
&= \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} b(r,x) \rho_\epsilon(x-y) \cdot \nabla \chi_R(x) v_r(y) \mu_r(\mathrm{d}x) \mathrm{d}y \mathrm{d}r + \\
&+ \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} v_r(y)(b(r,x) - b(r,y)) \cdot \nabla \rho_\epsilon(x-y) \chi_R(x) \mu_r(\mathrm{d}x) \mathrm{d}y \mathrm{d}r + \\
&+ \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} v_r(y) \mathrm{div} b(r,y) \rho_\epsilon(x-y) \chi_R(x) \mu_r(\mathrm{d}x) \mathrm{d}y \mathrm{d}r.
\end{aligned}$$

For the differentiable case, using again that $\nabla_x \rho_\epsilon(x-y) = -\nabla_y \rho_\epsilon(x-y)$ and bringing the derivative along y from ρ_ϵ to v, we get

$$\begin{aligned}
&\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} v_t(y) \rho_\epsilon(x-y) \chi_R(x) \mu_t(\mathrm{d}x) \mathrm{d}y - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} v_0(y) \rho_\epsilon(x-y) \chi_R(x) \mu_0(\mathrm{d}x) \mathrm{d}y \\
&= \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} b(r,x) \rho_\epsilon(x-y) \cdot \nabla \chi_R(x) v_r(y) \mu_r(\mathrm{d}x) \mathrm{d}y \mathrm{d}r + \\
&+ \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (b(r,x) - b(r,y)) \rho_\epsilon(x-y) \cdot \nabla v_r(y) \chi_R(x) \mu_r(\mathrm{d}x) \mathrm{d}y \mathrm{d}r. \qquad (4.6)
\end{aligned}$$

For simplicity of notation, we sometimes use the first formulation also for the differentiable case, meaning implicitly that the derivative along y must be brought on v.

Proposition 4.8. *Let μ be a solution to the CE starting at time 0 from μ_0 in L^1_x and let v be a solution to the backward TE, with final time t and final condition v_t in $C_{x,b}$. Assume that, for every $R > 0$,*

$$\lim_{\epsilon \to 0} \left| \int_0^t \langle \mu \otimes v, (b_x \cdot \nabla_x \rho_\epsilon^{(2)} + \mathrm{div}_y(b_y \rho_\epsilon^{(2)}))(\chi_R)_x \rangle \mathrm{d}r \right| = 0 \qquad (4.7)$$

Assume also that

$$\lim_{R \to +\infty} \limsup_{\epsilon \to 0} \left| \int_0^t \langle \mu \otimes v, \rho_\epsilon^{(2)} b_x \cdot \nabla_x (\chi_R)_x \rangle \mathrm{d}r \right| = 0. \qquad (4.8)$$

Then (μ, v) is a duality pair (at times $(0,t)$).

The object in the limit in (4.7) is called the commutator.

Remark 4.9. *Condition* (4.7) *on the commutator is the relevant hypothesis in the Proposition above. Condition* (4.8) *is more technical, and it holds easily in many situations, for example when b has at most linear growth outside of a ball, namely b satisfies Condition 2.1.*

Proof. We know that $\langle \mu_r \otimes v_r, \rho_\epsilon^{(2)}(\chi_R)_x\rangle$ satisfies equation (4.5) (or (4.5) in the differentiable case). First we let ϵ go to 0, keeping R fixed. We have that $\langle \mu_r \otimes v_r, \rho_\epsilon^{(2)}(\chi_R)_x\rangle$ tends to $\langle \mu_r, v_r\chi_R\rangle$ for $r = 0, t$, because of the conditions μ_0 in L^1_x, v_0 in L^∞_x, μ_t in $\mathcal{M}_x$, v_t in $C_{x,b}$ (recall we are using the continuous versions of μ and v, so that μ_t is bounded in $\mathcal{M}_x$ for every t and v_s is bounded in L^∞_s for every s). Hence condition 4.7 implies

$$|\langle \mu_t, v_t\chi_R\rangle - \langle \mu_0, v_0\chi_R\rangle| \le \limsup_{\epsilon\to 0}\left|\int_0^t \langle \mu \otimes v, \rho_\epsilon^{(2)} b_x \cdot \nabla_x(\chi_R)_x\rangle \mathrm{d}r\right|.$$

Finally we let R go to $+\infty$. We have that $\langle \mu_r, v_r\chi_R\rangle$ tends to $\langle \mu_r, v_r\rangle$ for $r = 0, t$, because of the global conditions μ_0 in L^1_x, v_0 in L^∞_x, μ_t in $\mathcal{M}_x$, v_t in $C_{x,b}$. So condition 4.8 implies

$$|\langle \mu_t, v_t\rangle - \langle \mu_0, v_0\rangle| = 0,$$

which is the duality relation. □

Now we give some conditions which imply that the commutator is infinitesimal. The first one requires regularity of the drift b.

Lemma 4.10. *Let p, q be in $[1, +\infty[$. Assume that b in $L^q_t(W^{1,p}_{x,loc})$, μ in $L^{q'}_t(L^{p'}_{x,loc}) \cap L^\infty_t(\mathcal{M}_x)$ with $\int_0^T \int_{\mathbb{R}^d} |b|/(1+|x|)\mathrm{d}|\mu| < +\infty$, v in $L^\infty_{t,x}$. Then conditions* (4.7) *and* (4.8) *hold, so, if condition 1 in 4.1 also holds, (μ, v) is a duality pair.*

Before going into the proof, we remind a useful fact concerning the continuity of translations in L^p_{x,B_R} (restricted to a ball B_R): let f be a function in $L^p_{x,loc}$, $1 \le p < +\infty$, then $\|f(\cdot + \epsilon z) - f\|_{L^p_{x,B_R}} \to 0$ (as $\epsilon \to 0$) for every z, for every $R > 0$. To see this, we mimic the proof on the whole $\mathbb{R}^d$: let φ be a continuous bounded function on $\mathbb{R}^d$, such that $\|f - \varphi\|_{L^p_{x,B_{R+1}}} < \eta$ for η small enough, then $\|f - \varphi\|_{L^p_{x,B_R}} < \eta$ and $\|f(\cdot + \epsilon z) - \varphi(\cdot + \epsilon z)\|_{L^p_{x,B_R}} < \eta$ for every $\epsilon < 1$. Therefore, since the thesis is satisfied for φ in place of f, one can conclude as in the proof for the whole $\mathbb{R}^d$.

Proof. We start with the commutator. We write it as

$$\int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(x)v(y)(b(x)-b(y)) \cdot \nabla\rho_\epsilon(x-y)\chi_R(x)\mathrm{d}x\mathrm{d}y\mathrm{d}r$$
$$+ \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(x)v(y)\mathrm{div}b(y)\rho_\epsilon(x-y)\chi_R(x)\mathrm{d}x\mathrm{d}y\mathrm{d}r.$$

Since b is weakly differentiable, it holds for a.e. (x,y)

$$b(x)-b(y) = \int_0^1 Db(y+\xi(x-y))\mathrm{d}\xi(x-y)$$

So the commutator reads

$$\int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(x)v(y) \int_0^1 Db(y+\xi(x-y))\mathrm{d}\xi(x-y) \cdot \nabla\rho_\epsilon(x-y)\chi_R(x)\mathrm{d}x\mathrm{d}y\mathrm{d}r$$
$$+ \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(x)v(y)\mathrm{div}b(y)\rho_\epsilon(x-y)\chi_R(x)\mathrm{d}x\mathrm{d}y\mathrm{d}r,$$

that is, with the change of variable $z = \epsilon^{-1}(x-y)$,

$$\int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(x)v(x-\epsilon z) \int_0^1 Db(x-(1-\xi)\epsilon z)\mathrm{d}\xi z \cdot \nabla\rho(z)\chi_R(x)\mathrm{d}x\mathrm{d}z\mathrm{d}r+$$
$$+ \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(x)v(x-\epsilon z)\mathrm{div}b(x)\rho(z)\chi_R(x)\mathrm{d}x\mathrm{d}z\mathrm{d}r. \tag{4.9}$$

Now we let ϵ go to 0 and we want to show convergence of the formula above to the same expression without ϵ. For the first addend, we split

$$Db(x-(1-\xi)\epsilon z)v(x-\epsilon z) - Db(x)v(x)$$
$$= (Db(x-(1-\xi)\epsilon z) - Db(x))v(x-\epsilon z) + Db(x)(v(x-\epsilon z)-v(x))$$

and we use Hölder inequality, getting

$$\Bigg| \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(x) \cdot$$
$$\cdot \left[v(x-\epsilon z) \int_0^1 Db(x-(1-\xi)\epsilon z)\mathrm{d}\xi - v(x)Db(x) \right] z \cdot \nabla\rho(z)\chi_R(x)\mathrm{d}x\mathrm{d}z\mathrm{d}r \Bigg|$$
$$\leq \int_{\mathbb{R}^d} \int_0^1 \|\mu\|_{L_t^{q'}(L_{x,B_{2R}}^{p'})} \cdot$$
$$\cdot \|v(\cdot-\epsilon z)\|_{L_{t,x}^\infty} \|Db(\cdot-(1-\xi)\epsilon z) - Db\|_{L_t^q(L_{x,B_{2R}}^p)} |z||\nabla\rho(z)|\mathrm{d}\xi\mathrm{d}z+$$
$$+ \left| \int_0^t \int_{\mathbb{R}^d} \chi_R(x)\mu(x)Db(x) \int_{\mathbb{R}^d} (v(x-\epsilon z)-v(x))z \cdot \nabla\rho(z)\mathrm{d}z\mathrm{d}x\mathrm{d}r \right|.$$

For the first term of the RHS above, for every z, ξ, $\|Db(\cdot-(1-\xi)\epsilon z)-Db\|_{L^q_t(L^p_{x,B_{2R}})}$ converges to 0 by continuity of translations in L^p_x and dominated convergence in t ($\|Db(\cdot-(1-\xi)\epsilon z)-Db\|_{L^p_{x,B_{2R}}}$ converges to 0 for a.e. time r and is bounded by $2\|Db\|_{L^p_{x,B_{2R}}}$); so the first term converges to 0 by dominated convergence theorem with respect to z and ξ. For the second term,

$$\int_{\mathbb{R}^d}(v(x-\epsilon z)-v(x))z\cdot\nabla\rho(z)\mathrm{d}z=\int_{\mathbb{R}^d}(v(x-y)-v(x))(y\cdot\nabla\rho(y))_\epsilon\mathrm{d}y$$

goes to 0 for a.e. x, since $(zD\rho(z))_\epsilon$ is a mollifier (up to scaling the L^1 norm) and the convolution of an L^∞ function with any mollifier converges to the function (times the L^1 norm) a.e.. So the second term converges to 0 by dominated convergence theorem ($v(x-\epsilon z)$ is a.e. bounded by $\|v\|_{L^\infty_{t,x}}$). We have proved convergence for the first addend in (4.9). Reasoning similarly for the second addend, we get that the commutator converges to

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^d}\mu(x)v(x)\mathrm{tr}\left[Db(x)\int_{\mathbb{R}^d}zD\rho(z)\mathrm{d}z\right]\chi_R(x)\mathrm{d}x\mathrm{d}r+\\&+\int_0^t\int_{\mathbb{R}^d}\mu(x)v(x)\mathrm{div}b(x)\int_{\mathbb{R}^d}\rho(z)\mathrm{d}z\chi_R(x)\mathrm{d}x\mathrm{d}r.\end{aligned}$$

Integrating by part, we get that $\int_{\mathbb{R}^d}zD\rho(z)\mathrm{d}z=-I\int_{\mathbb{R}^d}\rho(z)\mathrm{d}z=-I$. Hence the limit of the commutator reads

$$\int_0^t\int_{\mathbb{R}^d}\mu(x)v(x)(\mathrm{tr}[-Db(x)]+\mathrm{div}b)\chi_R(x)\mathrm{d}x\mathrm{d}r=0.$$

We have proved condition (4.7) on the commutator.

Condition (4.8) is easily verified. Indeed

$$\left|\int_0^t\langle\mu\otimes v,\rho^{(2)}_\epsilon b_x\cdot\nabla_x(\chi_R)_x\rangle\mathrm{d}r\right|\le 8\|v\|_{L^\infty_{t,x}}\int_0^t\int_{\mathbb{R}^d}\frac{|b||\mu|}{1+|x|}1_{R\le|x|\le 2R}\mathrm{d}x\mathrm{d}r$$

and the RHS is infinitesimal (as $R\to+\infty$) by the condition $\int_0^t\int_{\mathbb{R}^d}\frac{|b||\mu|}{1+|x|}\mathrm{d}x\mathrm{d}r<+\infty$. The proof is complete. □

The second Lemma requires instead regularity on the solution to the TE. For simplicity, we only consider the case b in $L^p_t(L^p_x)$ (without taking different integrability exponents in time and space).

Lemma 4.11. *Assume one of the following conditions:*

- *let p be in $[1,+\infty]$ and let $\tilde{m}$, m be in $[1,+\infty]$ such that $1/m+1/\tilde{m}+1/p\leq 1$, and assume b in $L^p_t(L^p_{x,loc})$, μ in $L^{\tilde{m}}_t(L^{\tilde{m}}_{x,loc})\cap L^\infty_t(\mathcal{M}_x)$ with $\int_0^T\int_{\mathbb{R}^d}|b|/(1+|x|)\mathrm{d}|\mu|<+\infty$, v in $L^m_t(W^{1,m}_{x,loc})\cap L^\infty_{t,x}$;*
- *assume b in $L^\infty_t(C_x)$, μ in $L^\infty_t(\mathcal{M}_x)$ with $\int_0^T\int_{\mathbb{R}^d}|b|/(1+|x|)\mathrm{d}|\mu|<+\infty$, v in $L^1_t(W^{1,\infty}_{x,loc})\cap L^\infty_{t,x}$.*

Then conditions (4.7) *and* (4.8) *hold, so, if condition 1 in 4.1 also holds, (μ,v) is a duality pair.*

Proof. We start with the commutator, first in the case b in $L^p_t(L^p_{x,loc})$, $L^{\tilde{m}}_t(L^{\tilde{m}}_{x,loc})$. Using the regularity of v, we write it as

$$\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\mu(x)(b(x)-b(y))\cdot\nabla v(y)\rho_\epsilon(x-y)\chi_R(x)\mathrm{d}x\mathrm{d}y\mathrm{d}r,$$

that is, with the change of variable $z=\epsilon^{-1}(x-y)$,

$$\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\mu(x)(b(x)-b(x-\epsilon z))\cdot\nabla v(x-\epsilon z)\rho(z)\chi_R(x)\mathrm{d}x\mathrm{d}z\mathrm{d}r.$$

For $p<+\infty$, Hölder inequality gives

$$\left|\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\mu(x)(b(x)-b(x-\epsilon z))\cdot\nabla v(x-\epsilon z)\rho(z)\chi_R(x)\mathrm{d}x\mathrm{d}z\mathrm{d}r\right|$$
$$\leq\int_{\mathbb{R}^d}\|\mu\|_{L^{\tilde{m}}_t(L^{\tilde{m}}_{x,B_{2R}})}\|b(\cdot)-b(\cdot-\epsilon z)\|_{L^p_t(L^p_{x,B_{2R}})}\|\nabla v(\cdot-\epsilon z)\|_{L^m_t(L^m_{x,B_{2R}})}\rho(z)\mathrm{d}z.$$

Since v is in $L^m_t(W^{1,m}_{x,loc})$, by continuity of translations in L^p_x and dominated convergence theorem (in a way similar to the previous proof), this term tends to 0 as $\epsilon\to 0$. For $p=+\infty$ and $\tilde{m}<+\infty$, we split the commutator in

$$\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\mu(x)(b(x)-b(x-\epsilon z))\cdot\nabla v(x-\epsilon z)\rho(z)\chi_R(x)\mathrm{d}x\mathrm{d}z\mathrm{d}r$$
$$=\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\mu(x)(b(x)-b(x-\epsilon z))\cdot\nabla v(x)\rho(z)\chi_R(x)\mathrm{d}x\mathrm{d}z\mathrm{d}r+$$
$$+\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\mu(x)(b(x)-b(x-\epsilon z))\cdot\nabla(v(x-\epsilon z)-v(x))\rho(z)\chi_R(x)\mathrm{d}x\mathrm{d}z\mathrm{d}r$$

For the first addend,

$$\int_{\mathbb{R}^d}(b(x)-b(x-\epsilon z))\rho(z)\mathrm{d}z=\int_{\mathbb{R}^d}(b(x)-b(x-y))\rho_\epsilon(y)\mathrm{d}y$$

goes to 0 for a.e. x, since the convolution of an L^∞ function with any mollifier converges to the function a.e.. So the first attend tends to 0 by dominated convergence theorem. For the second addend, by Hölder inequality we bound it with

$$\left|\int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(x)(b(x) - b(x - \epsilon z)) \cdot \nabla(v(x - \epsilon z) - v(x))\rho(z)\chi_R(x)\mathrm{d}x\mathrm{d}z\mathrm{d}r\right|$$
$$\leq 2 \int_{\mathbb{R}^d} \|\mu\|_{L_t^{\tilde{m}}(L^{\tilde{m}}_{x,B_{2R}})} \|b\|_{L_t^\infty(L^\infty_{x,B_{2R+1}})} \|\nabla v(\cdot - \epsilon z) - \nabla v(\cdot)\|_{L_t^m(L^m_{x,B_{2R}})} \rho(z)\mathrm{d}z,$$

which goes to 0 by continuity of translations in $L^m_{x,B_{2R}}$ and dominated convergence theorem. This proves convergence to 0 of the commutator for $p = +\infty$ and $\tilde{m} < +\infty$. Finally, if $p = +\infty$ and $\tilde{m} = +\infty$ (which forces $m = 1$), we change variable $x' = x - \epsilon z$ and we proceed as in the previous case ($\tilde{m}$ finite) but exchanging the roles of v and $\mu\chi_R$. Condition (4.7) is proved under the first assumption.

In the case b in $L_t^\infty(C_x)$, μ in $L_t^\infty(\mathcal{M}_x)$, proceeding as before (replacing $\mu(x)\mathrm{d}x$ with $\mu(\mathrm{d}x)$), we write the commutator as

$$\int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (b(x) - b(x - \epsilon z)) \cdot \nabla v(x - \epsilon z)\rho(z)\chi_R(x)\mu(\mathrm{d}x)\mathrm{d}z\mathrm{d}r \tag{4.10}$$

Exchanging the order of integration and applying Hölder inequality, we get

$$\left|\int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (b(x) - b(x - \epsilon z)) \cdot \nabla v(x - \epsilon z)\rho(z)\chi_R(x)\mu(\mathrm{d}x)\mathrm{d}z\mathrm{d}r\right|$$
$$\leq \int_0^t \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |b(x) - b(x - \epsilon z)|\rho(z)\mathrm{d}z\right) \|\nabla v(x - \epsilon \cdot)\|_{L_z^\infty} \chi_R(x)|\mu|(\mathrm{d}x)\mathrm{d}r$$
$$\leq \|\nabla v\|_{L^\infty_{x,B_{2R+1}}} \int_0^t \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |b(x) - b(x - \epsilon z)|\rho(z)\mathrm{d}z\right) \chi_R(x)|\mu|(\mathrm{d}x)\mathrm{d}r,$$

where we have used the fact that, for every x, $\|\nabla v(x - \epsilon \cdot)\|_{L_z^\infty}\chi_R(x) \leq \|\nabla v\|_{L^\infty_{x,B_{2R+1}}}$. Since b is in $L_t^\infty(C_x)$, for every t in a full-measure set F (independent of x), for every x, by dominated convergence theorem,

$$\int_{\mathbb{R}^d} |b(x) - b(x - \epsilon z)|\rho(z)\mathrm{d}z \to 0.$$

Since v is in $L_t^1(W^{1,\infty}_{x,loc})$, by dominated convergence theorem (in x before and in t then), we get that (4.10) tends to 0. Condition (4.7) is proved also in this case.

Condition (4.8) is easily verified as in the proof of the previous Lemma. The proof is complete. □

4.4 Regularity implies well-posedness

In the previous section we have seen that the existence of a solution to the TE with Sobolev regularity gives uniqueness for the CE among $L^{\tilde{m}}$ solutions, and that this uniqueness implies uniqueness among Lagrangian flows. We also have seen that such a regular solution to the TE exists if we have suitable a priori estimates, and that these estimates give stability of the solution. These two facts give a strategy to prove well-posedness, starting from Sobolev a priori estimates on the TE. We summarize this strategy in the following theorem. Recall that a Banach space V, contained in some Banach space of functions (or measures) U, has $C^\infty_{x,c}$ (resp. $C_t(C^\infty_{x,c})$) as a U-mildly dense set if $C^\infty_{x,c}$ (resp. $C_t(C^\infty_{x,c})$) is contained in V and, for every f in V, there exists a sequence $(f^n)_n$ in $C^\infty_{x,c}$ (resp. in $C_t(C^\infty_{x,c})$) which converges to f in U and is bounded in V.

From now on, we always assume that the vector field b has at most linear growth, namely Condition 2.1.

Theorem 4.12. *Let p be in $]1,+\infty[$ and let m, $\tilde{m}$ be in $]1,+\infty[$ such that $1/m+1/\tilde{m}+1/p\le 1$. Let V be a Banach space, contained in $L^p_t(L^p_{x,loc})$ and having $C_t(C^\infty_{x,c})$ as a $L^p_t(L^p_{x,loc})$-mildly dense subset. Let V_0 be a normed space contained in $\mathcal{M}_x$, having $C^\infty_{x,c}$ as a $\mathcal{M}_x$-mildly dense subset, stable under the operations $\mu\mapsto\mu^+$, $\mu\mapsto\mu^-$ and with $V_0\cap L^1_x$ dense in L^1_x. Assume the following conditions:*

- *A priori estimates for the STE: for every $R>0$, there exists a locally bounded function $C_{STE}:[0,+\infty[\times[0,+\infty[\to[0,+\infty[$ such that, for every b in $C_t(C^\infty_{x,c})$, for every final time t and every final datum φ in $C^\infty_{x,c}$, the corresponding classical solution $v=v^{b,t,\varphi}$ to the STE satisfies*
$$\sup_{s\in[0,t]} E\|v_s\|^m_{W^{1,m}_{x,B_R}} \le C_{STE}(\|b\|_V,\|\varphi\|_{C^2_b}). \tag{4.11}$$
- *A priori estimates for the SCE: for every $R>0$, there exists a locally bounded function $C_{SCE}:[0,+\infty[\times[0,+\infty[\to[0,+\infty[$ such that, for every b in $C_t(C^\infty_{x,c})$, for every initial time s and every initial datum μ_s in $C^\infty_{x,c}$, the corresponding classical solution $\mu=\mu^{b,s,\mu_s}$ to the SCE satisfies*
$$E\|\mu\|^{\tilde{m}}_{L^{\tilde{m}}_t(L^{\tilde{m}}_{x,B_R})} \le C_{SCE}(\|b\|_V,\|\mu_s\|_{V_0}). \tag{4.12}$$

Then well-posedness holds for b in V, in the following sense:

1. *Well-posedness for the SCE: For every b in V, for every s and every μ_s in V_0, there exists a $\mathcal{F}$-adapted distributional solution $\mu=\mu^{b,s,\mu_s}$*

to the SCE in the class $L^{\tilde m}_{t,\omega}(L^{\tilde m}_{x,loc})$, which is unique in the path-by-path sense in the class $L^{\tilde m}_t(L^{\tilde m}_{x,loc})$. Furthermore, if $(b^n)_n$ converges to b in $L^{\tilde m'}_t(L^{\tilde m'}_{x,loc})$ and is bounded in V and if $(\mu^n_s)_n$ is a sequence in V_0 converging weakly- to μ_s, then $(\mu^{b^n,s,\mu^n_0})_n$ converges weakly to μ^{b,s,μ_s} in $L^{\tilde m}_{t,\omega}(L^{\tilde m}_{x,B_R})$, for every $R>0$.*

2. *Well-posedness and regularity for the STE: For every b in V, for every t and every final φ in C^2_b, there exists a $\mathcal{F}$-adapted differential solution $v = v^{b,t,\varphi}$ to the TE in the class $L^\infty_s(L^m_\omega(W^{1,m}_{x,loc}))$, which is unique in the path-by-path sense in the class $L^m_s(W^{1,m}_{x,loc})$. Furthermore, if $(b^n)_n$ converges to b in $L^{m'}_t(L^{m'}_{x,loc})$ and is bounded in V and if $(\varphi^n)_n$ is a sequence in C^2_b converging weakly to φ_t in $L^m_{x,loc}$, then $(v^{b^n,\varphi^n})_n$ converges weakly to $v^{b,t,\varphi}$ in $L^m_{s,\omega}(W^{1,m}_{x,B_R})$, for every $R>0$.*

3. *Well-posedness and regularity for the SDE: For every b in V, for every s and every μ_s in $V_0\cap\mathcal{M}_{x,+}$, there exists a $L^{\tilde m}_t(L^{\tilde m}_{x,loc})$-Lagrangian flow $X = X^b$ solving the SDE and starting from μ_s; this flow is unique in the path-by-path sense in the class of $L^{\tilde m}_t(L^{\tilde m}_{x,loc})$-Lagrangian flow. This flow is also in $L^\infty_t(L^{\tilde m}_\omega(W^{1,\tilde m}_{x,loc}))$. Furthermore, if $(b^n)_n$ converges to b in $L^p_t(L^p_{x,loc})$ and is bounded in V, then $(X^{b^n})_n$ converges to X^b, in the sense of convergence of $\mu^n_t = (X^{b^n}_{s,t})_\#\mu_s$ to $\mu_t = (X^b_{s,t})_\#\mu_s$ (as solution to the SCE).*

4. *Representation formulae: For every $s<t$, for every φ in C^2_b, for every μ_s in V_0, for a.e. ω, it holds*

$$\mu^{s,\mu_s}_t = (X_{s,t})_\#\mu_s, \tag{4.13}$$
$$v^{t,\varphi}_s = \varphi(X_{s,t}), \tag{4.14}$$

where μ and v are the time weakly- continuous versions (as from 3.13 and 3.15).*

Remark 4.13. *Here and in the following, the a priori Sobolev estimates on the STE (or on the rTE) can be easily replaced by the corresponding Sobolev estimates on the SDE (or on the rDE), by the correspondence between the two equations in the regular setting (Proposition 3.18).*

Proof. The result follows from the previous lemmata. Precisely:

1. Existence of a differentiable solution v for the STE and existence of a distributional solution μ for the SCE follow from the hypotheses (4.12) and (4.11) and the stability results 3.39 and 3.47 (together with 3.42).

Moreover, if μ_0 is non-negative, μ takes values in the space of non-negative measures. Thanks to Lemmata 3.14 and 3.12, for fixed initial and final data μ_s and v_t, $\tilde{\mu}$ and $\tilde{v}$ also satisfy resp. the rCE and the rTE for a.e. ω (where the exceptional set in Ω may depend on the initial and final data, on the initial and final times and on the specific version of μ and v chosen).

2. Path-by-path uniqueness for the SCE, i.e. uniqueness (for a.e. ω) of the rCE, follows from existence for the rTE and the duality method. Precisely, fix the initial time s. The estimate (4.11) together with Lemma 3.14 implies the existence of a countable set F dense in $[s,T]$, a countable set D dense in C^2_b and a full-measure set Ω_1 in Ω such that, for every t in F and every φ in D, for every ω in Ω_1, $\tilde{v}^{b,t,\varphi}$ is in $L^m_r(W^{1,m}_{x,loc})$. By Lemma 4.11 applied to the rCE and the rTE, for every ω in Ω_1, for any solution $\tilde{\mu}$ to the rCE at ω fixed, $(\tilde{\mu},\tilde{v})$ is a dual pair. Hence we can apply Proposition 4.3 and Remark 4.4 to the rCE and the rTE (notice that, for fixed ω, $\tilde{v}$ has $\tilde{\varphi}$ as final datum, which however is still in a countable dense set): uniqueness for $\tilde{\mu}$ in $L^{\tilde{m}}_t(L^{\tilde{m}}_{x,loc})$ holds for every ω in Ω_1 and for every initial datum $\bar{\mu}_s$ in $\mathcal{M}_x$.

3. Path-by-path uniqueness for the STE follows similarly, using Proposition 4.6 and Remark 4.7 and the density of $V_0 \cap L^1_x$ in L^1_x.

4. Existence and uniqueness for Lagrangian flows solving the rDE, for a.e. ω, and the representation formula (4.13) for the SCE follow from Theorem 3.24. Precisely, for every μ_s in $\mathcal{M}_{x,+}$, existence of a non-negative solution for the rCE starting from $\tilde{\mu}_s$ and uniqueness from any initial datum, for a.e. ω, imply, via Theorem 3.24, existence and uniqueness for the flow $\tilde{X}^\omega$ solving the rCE and starting from μ_s, for a.e. ω. The representation formula (3.11), applied to the rCE (and extended to initial signed measure), gives the desired formula (4.13). Mind that the family of Lagrangian flows $\tilde{X}^\omega$, built in this way, may not be measurable in ω. For this, we will build a modification which is measurable and adapted.

5. Stability for the SCE and stability for the STE follow from path-by-path uniqueness for the SCE and the STE and the stability results 3.39 and 3.47 (for any limiting point μ of the SCE, then $\tilde{\mu}$ satisfies the rCE and is therefore unique).

6. The representation formula (4.14) for the STE follows the representation formulae (4.4) and (4.13). Indeed, by existence for the rCE, there

exists a countable set D' dense in $C^\infty_{x,c}$ and a full-measure set Ω_1 in Ω such that, for every μ_s in D', $\tilde{\mu}^{b,s,\mu_s}_t$ is in $L^m_s(W^{1,m}_{x,loc})$ and uniqueness holds for the rCE at ω fixed in Ω_1; we can also assume that $\tilde{v}^{b,t,\varphi}_s$ is in $L^m_s(W^{1,m}_{x,loc})$ for every ω in Ω_1. Hence we can apply the representation formula (4.4), which gives the desired formula (4.14) for every ω in Ω_1.

7. Regularity for the Lagrangian flow follow from Sobolev regularity of v_s (which holds at every s because of Lemma 3.43) and the representation formula 4.14. Stability for the flow is an immediate consequence of stability for the SCE.

8. The adaptedness property follows from adaptedness for the STE and the representation formula (4.14). To show this, we fix s and we use v to build a version $\bar{X}$ of X^ω which is measurable in (t,x,ω) and actually adapted. Fix t in a countable dense set F of $[s,T]$, containing the dyadic numbers. The representation formula (4.14) links the map $X_{s,t}$ with the solution $v^{t,\varphi}_s$ to the STE. We claim that this solution is, up to identification, $\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{F}_{s,t}$ measurable. Indeed the map $(r,\omega)\mapsto v^\omega_r$ is weakly-* progressively measurable (as L^∞_x valued map, where we identify the function v^ω_r with its equivalence class), therefore $\omega\mapsto v^\omega_r$ is weakly-* $\mathcal{F}_{r,t}$-measurable for a.e. r and so for every r, since v is the weakly-* continuous version (and the filtration is left continuous in r); in particular, $\omega\mapsto v^\omega_t$ is in $L^2(\Omega,\mathcal{F}_{s,t};L^2_x)$ (we use L^2 since this is a reflexive space). By Lemma A.6, there exists a version of v_t (which we will continue calling it v_t) which is $\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{F}_{s,t}$-measurable and in $L^\infty_{\omega,x}$: we have proved the claim. For this version, the representation formula (4.14) still holds for a.e. ω (the null set possibly depending on φ), for a.e. x (possibly depending on ω). Keeping this version of v_t, we define the version of $X_{s,t}$ as
$$\bar{X}_{s,t} = \lim_n v^{t,\varphi_n}_s$$
for (x,ω) such that the limit exists (and 0 where the limit does not exist), where $(\varphi_n)_n$ is a sequence of $C^2_{x,b}$ functions with $\varphi_n(x)=x$ on B_n. By the representation formula (4.14), the limit exists for every (x,ω) in a full measure set A_t (v_n being definitively constant) and it is equal to X on a full measure subset $B_{t,\omega}$ of $\mathbb{R}^d$, for every ω in a full measure set $\Omega_{1,t}$. By a diagonal argument, we can choose A, Ω_1 and (for ω in Ω_1) B_ω independent of t, for any t in F. Finally, in order to define $\bar{X}$ for any t, we take
$$\bar{X}^n_{s,t}(x,\omega) = \sum_{k=1}^{N(n)} 1_{[t^n_k,t^n_{k+1}[\cap[s,T]}(t)\bar{X}_{s,t^n_k}(x,\omega),$$

where $t_k^n = k2^{-n}$, and we define

$$\bar{X}_{s,t}(x,\omega) = \lim_n \bar{X}^n_{s,t}(x,\omega)$$

for (t,x,ω) such that the limit exists (and 0 where the limit does not exist). Using the fact that $\bar{X}$ is a version of X for every t in F and that, for every ω, for a.e. x, X^ω has continuous trajectories, we get that the limit exists for a.e. (t,x,ω) and is a version of X; moreover it has continuous paths for a.e. (x,ω). It remains to prove adaptability. For t in F, $\bar{X}_{s,t}$ is $\mathcal{L}^d \otimes P$-measurable (and so $\mu_s \otimes P$-measurable) with respect to $\mathcal{B}(\mathbb{R}^d) \times \mathcal{F}_{s,t}$, because pointwise limit of random variables which are measurable with respect to $\mathcal{B}(\mathbb{R}^d) \times \mathcal{F}_{s,t}$. The same holds for a general t, by continuity of the paths of $\bar{X}$ and right continuity of the filtration. This implies adaptability $\bar{X}$ in the sense of Definition 3.25.

□

In the case of a priori estimates in $W^{1,\infty}_x$ on the transport equation, we get a stronger path-by-path uniqueness result. We state this result only for the SDE (for simplicity) but in a strong assumption on the classical estimates on the STE, which hold now for every final datum, every initial time and every final time, independently of ω (outside a P-null set). This assumption will be verified in at least one application.

Theorem 4.14. *Assume that*

- *V, $\bar{V}$ are Banach spaces contained continuously in $C_t(C_{x,lin})$ such that, for every b in V, $\tilde{b}$ is in $\bar{V}$ for a.e. ω. The space $\bar{V}$ has $C_t(C^\infty_{x,c})$ as a $C_t(C_{x,loc})$-mildly dense subset.*

- *for a.e. ω, a priori estimates for the rTE: for every $R > 0$, there exists a locally bounded function $C_{rTE} : [0,+\infty[\times[0,+\infty[\to [0,+\infty[$ such that, for every $\tilde{b}$ in $C_t(C^\infty_{x,c})$, for every final time t and every final datum $\tilde{\varphi}$ in $C^\infty_{x,c}$, the corresponding classical solution $\tilde{v}$ to the rTE satisfies*

$$\sup_{s\in[0,t]} \|\tilde{v}^{\tilde{b},t,\tilde{\varphi}}_s\|_{W^{1,\infty}_{x,B_R}} \leq C_{rTE}(\|\tilde{b}\|_{\bar{V}}, \|\varphi\|_{C^2_b}). \tag{4.15}$$

Then, for every b in V, for every initial time $s \geq 0$, for every initial point x, existence and path-by-path uniqueness hold for the SDE. Moreover there exists a version $X : [s,T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ of the solution which is locally Lipschitz continuous in x.

As we will see from the proof, the set where existence and uniqueness for the rDE hold is actually independent of x.

Proof. Again the result follows from the previous lemmata. Precisely:

1. For uniqueness, fix the full P-measure set Ω_1 such that (4.15) holds. For each ω in Ω_1, the stability results 3.39 and 3.42, adapted to the TE case and applied to the rTE, give the existence of a $L^\infty_t(W^{1,\infty}_{x,loc})$ solution to the rTE (at ω fixed). Then the duality argument gives path-by-path uniqueness for ω in Ω_1.

2. For existence and regularity for the rDE, fix ω in Ω_1. Let $(\tilde{g}^n)_n$ be a sequence in $C_t(C^\infty_{x,c})$ converging to $\tilde{b}$ in $C_t(C_{x,loc})$ and in $\tilde{V}$ (therefore also in $C_t(C_{x,lin})$); let $\tilde{Y}^n$ be the flows solutions to the rDEs driven by $\tilde{g}^n$. The uniform bound (4.15) and the uniform boundedness of $\tilde{b}^n$ in $C_t(C_{x,lin})$ imply that the sequence $(\tilde{Y}^n)_n$ is uniformly bounded in $W^{1,\infty}_{t,x,[0,T]\times B_R}$, for every $R > 0$. Therefore, by Arzerà-Ascoli theorem, $(\tilde{Y}^n)_n$ is precompact in $C_{t,x,[0,T]\times B_R}$. Moreover, since $(\tilde{g}^n)_n$ converges locally uniformly to $\tilde{b}$ and $\tilde{b}$ is continuous, any limit point of $(\tilde{Y}^n)_n$ is a family of solutions (parameterized by the initial datum x) to the rDE driven by $\tilde{b}$. By uniqueness of the rDE (for each x), the whole sequence $(\tilde{Y}^n)_n$ must converge to the solution $\tilde{X}$ to the rDE in $C_{t,x,[0,T]\times B_R}$. The uniform $W^{1,\infty}_{t,x,[0,T]\times B_R}$ bound on $(\tilde{Y}^n)_n$ implies the Lipschitz regularity of $\tilde{X}$.

3. For adaptability, let $(b^n)_n$ be a sequence in $C_t(C^\infty_{x,c})$ converging to b in $C_t(C_{x,loc})$ and bounded in $C_t(C_{x,lin})$ (mind that $\tilde{b}^n$ might be different from $\tilde{g}^n$, which might not be adapted); let X^n be the adapted stochastic flows solutions to the rDEs driven by b^n. Fix the initial datum x in $\mathbb{R}^d$. The uniform boundedness of $\tilde{b}^n$ in $C_t(C_{x,lin})$ implies that, for a.e. ω, $(\tilde{X}^n(x))_n$ is uniformly bounded in $W^{1,\infty}_t$, therefore it is precompact in C_t. Moreover, as before, any limit point of $(\tilde{X}^n(x))_n$ is a solution to the rDE driven by $\tilde{b}$. By uniqueness, the whole sequence $(\tilde{X}^n(x))$ converges to $\tilde{X}(x)$ in C_t, for a.e. ω. So, since $X^n(x)$ is adapted, also $X(x)$ is adapted.

□

4.5 Well-posedness for Sobolev-type drifts

In the case we do not have Sobolev regularity for the solution to the STE, but we do have Sobolev regularity on the drift, we can still conclude a well-posedness result.

Theorem 4.15. *Let p, q be in $[1,+\infty[$ and let $\tilde{m}$ be in $]1,+\infty[$ such that $1/\tilde{m}+1/(p\wedge q)\leq 1$. Let V be a Banach space, contained in $L^{\tilde{m}'}_t(L^{\tilde{m}'}_{x,loc})$ and having $C_t(C^\infty_{x,c})$ as a $L^{\tilde{m}'}_t(L^{\tilde{m}'}_{x,loc})$-mildly dense subset. Let V_0 be a normed space contained in $\mathcal{M}_x$, having $C^\infty_{x,c}$ as a $\mathcal{M}_x$-mildly dense subset, stable under the operations $\mu\mapsto\mu^+$, $\mu\mapsto\mu^-$ and with $V_0\cap L^1_x$ dense in L^1_x. Assume the following conditions:*

- *Sobolev regularity of the drift: V is contained continuously in $L^q_t(W^{1,p}_{x,loc})$.*
- *A priori estimates for the SCE: for every $R>0$, there exists a locally bounded function $C_{SCE}:[0,+\infty[\times[0,+\infty[\to[0,+\infty[$ such that, for every b in $C_t(C^\infty_{x,c})$, for every initial time s and every initial datum μ_s in $C^\infty_{x,c}$, the corresponding classical solution $\mu=\mu^{b,s,\mu_s}$ to the SCE satisfies*
$$E\|\mu\|^{\tilde{m}}_{L^{\tilde{m}}_t(L^{\tilde{m}}_{x,B_R})}\leq C_{SCE}(\|b\|_V,\|\mu_s\|_{V_0}).$$

Then well-posedness holds for b in V, in the following sense:

1. *Well-posedness for the SCE: For every b in V, for every s and every μ_s in V_0, there exists a $\mathcal{F}$-adapted distributional solution $\mu=\mu^{b,s,\mu_s}$ to the SCE in the class $L^{\tilde{m}}_{t,\omega}(L^{\tilde{m}}_{x,loc})$, which is unique in the path-by-path sense in the class $L^{q'}_t(L^{p'}_{x,loc})$. Furthermore, if $(b^n)_n$ converges to b in $L^q_t(L^p_{x,loc})$ and is bounded in V and if $(\mu^n_s)_n$ is a sequence in V_0 converging weakly-* to μ_s in $\mathcal{M}_x$, then $(\mu^{b^n,s,\mu^n_s})_n$ converges weakly to μ^{b,s,μ_s} in $L^{\tilde{m}}_{t,\omega}(L^{\tilde{m}}_{x,B_R})$, for every $R>0$.*

2. *Well-posedness for the STE: For every b in V, for every t and every final φ in C^2_b, there exists a $\mathcal{F}$-adapted distributional solution $v=v^{b,t,\varphi}$ to the TE in the class $L^\infty_{s,x,\omega}$, which is unique in the path-by-path sense in the class $L^\infty_{s,x}$. Furthermore, if $(b^n)_n$ converges to b in $L^q_t(L^p_{x,loc})$ and is bounded in V and if $(\varphi^n)_n$ is a sequence in C^2_b converging weakly to φ_t in $L^\infty_{x,loc}$, then $(v^{b^n,t,\varphi^n})_n$ converges weakly to $v^{b,t,\varphi}$ in $L^\infty_{s,x,\omega}$, for every $R>0$.*

3. *Well-posedness for the SDE: For every b in V, for every s and every μ_s in $V_0\cap\mathcal{M}_{x,+}$, there exists a $L^{\tilde{m}}_t(L^{\tilde{m}}_{x,loc})$-Lagrangian flow $X=X^b$ solving the SDE and starting from μ_s; this flow is unique in the path-by-path sense in the class of $L^{q'}_t(L^{p'}_{x,loc})$-Lagrangian flow. Furthermore, if $(b^n)_n$ converges to b in $L^{\tilde{m}'}_t(L^{\tilde{m}'}_{x,loc})$ and is bounded in V, then $(X^{b^n})_n$ converges to X^b, in the sense of convergence of $\mu^n_t=(X^{b^n}_{s,t})_\#\mu_s$ to $\mu_t=(X^b_{s,t})_\#\mu_s$ (as solution to the SCE).*

4. *Representation formulae: For every $s < t$, for every φ in C_b^2, for a.e. ω,* (4.13) *and* (4.14) *hold (where again μ and v are the time weakly-* continuous versions).*

Proof. The proof is similar to that of Theorem 4.12, using Lemma 4.10 in place of Lemma 4.11 and without regularity for the STE. □

Although uniqueness holds also without noise, let us point out two facts:

1. this "deterministic" uniqueness result translates into a path-by-path uniqueness result;

2. existence holds without requiring boundedness of the divergence of the drift, which usually is assumed in the deterministic case (more precisely, boundedness of the negative part of the divergence is required).

Chapter 5

PDEs: facts and estimates

In this chapter we give some a priori estimates and related results on the parabolic PDEs associated with the SDE. These will be the basis for our analysis on regularization by noise.

The results of Section 5.1 go back at least to Krylov [Kry96] and [Kry08] and are taken, in this form, from [Fla11] and [FF13a]. The content of Sections 5.2 and 5.4 is taken from [MO], the content of Section 5.3 is adapted from [BFGM14].

5.1 A priori estimates, part I

We start with the simplest parabolic equation, namely the backward heat equation on $\mathbb{R}^d$

$$\partial_t v + \frac{1}{2}\Delta v = f, \tag{5.1}$$

with fixed final datum $v_T \equiv 0$ at time T. It is well known that, for f in $C_t(C^\infty_{x,c})$, there exists a bounded solution $v = v^f$ in $C_t(C^\infty_x)$ to the heat equation. The next result gives parabolic estimates on v in terms of f.

Lemma 5.1. *Fix α in $]0,1[$, $\epsilon > 0$ (smaller than α). There exists $C > 0$ such that, for every f in $C_t(C^\infty_{x,c})$, there exists a solution v to the heat equation* (5.1) *(with $v_T \equiv 0$) satisfying*

$$\|v\|_{C_t(C^{2+\alpha-\epsilon}_{x,b})} + \|v\|_{C^{1/2}_t(C^{1+\alpha-\epsilon}_{x,b})} + \|v\|_{C^1_t(C^{\alpha-\epsilon}_{x,b})} \leq C(\|f\|_{C_t(C^\alpha_{x,b})}.$$

Proof. The $C_t(C^{\alpha+2-\epsilon}_{x,b})$ and the $C^1_t(C^{\alpha-\epsilon}_{x,b})$ are for example in [Fla11], Chapter 2 Theorem 2.3. The $C^{1/2}_t(C^{\alpha+1-\epsilon}_{x,b})$ estimate follows by an interpolation argument. Indeed, by Proposition A.11 in the Appendix, we get

$$\begin{aligned}
\|v_t - v_s\|_{C^{1+\alpha-\epsilon}_{x,b}} &\le C\|v_t - v_s\|^{1/2}_{C^{\alpha-\epsilon}_{x,b}}\|v_t - v_s\|^{1/2}_{C^{2+\alpha-\epsilon}_{x,b}} \\
&\le C\|v\|^{1/2}_{C^1_t(C^{\alpha-\epsilon}_{x,b})}\|v\|^{1/2}_{C_t(C^{2+\alpha-\epsilon}_{x,b})}|t-s|^{1/2},
\end{aligned}$$

which gives the desired $C^{1/2}_t(C^{\alpha+1-\epsilon}_{x,b})$ bound. □

In the next result we consider a modified version of the backward Kolmogorov equation associated with the SDE (on $\mathbb{R}^d$), namely

$$\partial_t v + b \cdot \nabla v + \frac{1}{2}\Delta v - \lambda v = f, \tag{5.2}$$

with fixed final datum $v_T \equiv 0$ at time T; here $\lambda > 0$ is a given positive number, we will keep track of the λ dependence by the notation v^λ, when necessary. Again it is well known that, for b, f in $C_t(C^\infty_{x,c})$, there exists a solution $v = v^{b,f}$ in $C^1_t(C^\infty_x)$ to the Kolmogorov equation. Here are the key parabolic estimates.

Lemma 5.2. *Fix p, q satisfying Condition 2.3. Then there exists a locally bounded function $C : [0,+\infty[\times[1,+\infty[\to [0,+\infty[$ such that, for every b, f in $C_t(C^\infty_{x,c})$ and for every $\lambda \ge 1$, there exists a solution v to (5.2) satisfying*

$$\|v^\lambda\|_{L^q_t(W^{2,p}_x)} + \|v^\lambda\|_{C_t(C^1_{x,b})} \le C(\|b\|_{L^q_t(L^p_x)}, \lambda)\|f\|_{L^q_t(L^p_x)}.$$

Moreover we have, for any $c > 0$, as $\lambda \to +\infty$,

$$\sup_{b,f\in C_t(C^\infty_{x,c}),\|b\|_{L^q_t(L^p_x)}\le c,\|f\|_{L^q_t(L^p_x)}\le c} \|\nabla v^\lambda\|_{C_t(C_{x,b})} \to 0.$$

This result is stated and proved, for example, in [FF13a], Lemma 3.2, Theorem 3.3 and Lemma 3.4.

5.2 A priori estimates, part II

In this section and in the next one we provide new a priori estimates on Kolmogorov-like equations. The method (sometimes called energy estimate method) is to estimate the desired norm of the solution v directly: starting from the equation and using the chain rule, we get a PDE for v^m (or for $|\nabla v|^m$), then we take the L^2 norm and use a Gronwall-type argument, taking advantage of the derivative term coming from the Laplacian.

This section is devoted to the case when div$b$ satisfies an integrability assumption; in this case, we provide L^m_x and $W^{1,2}_x$ estimates for a Kolmogorov-like equation.

Precisely, we consider the equation

$$\partial_t v + b \cdot \nabla v + hv = \frac{1}{2}\Delta v, \tag{5.3}$$

with fixed initial datum v_0. This equation covers the cases of Kolmogorov equation ($h = 0$) and Fokker-Planck equation ($h = \mathrm{div} b$).

In the following, we assume for simplicity that the supports of b and h are contained in $[0, T] \times B_R$ for some fixed $R > 0$. We consider the weight $\chi = \chi_{\eta,R}$, strictly positive function in C^∞_x, with $\chi \equiv 1$ on B_R and $\chi(x) = (1 + |x|^2)^{\eta/2}$ on B_{R+1}, for some real number η. This weight satisfies, in particular,

$$|\nabla \chi_{\eta,R}| \leq C_{\eta,R} \frac{\chi_{\eta,R}}{1 + |x|} 1_{B^c_R}.$$

Remark 5.3. *In the case of v_0 in $C^\infty_{x,c}$, b and h in $C_t(C^\infty x, c)$, the equation above admits a solution in $W^{1,m}_t(W^{2,m}_{x,\chi_{\eta,R}})$, for every m in $[1, +\infty]$ and for every weight $\chi_{\eta,R}$, for every real η. The proof of this result is for example in [BFGM14], Lemma 12 (the result is given for a linear stochastic PDE and the coefficients are also regular in time, but the proof can be easily adapted to our case). This justifies the formal computations we will do.*

Theorem 5.4. *Fix p, q satisfying Condition 2.3, fix m positive integer, fix $R > 0$. Then there exists a locally bounded function $C : [0, +\infty[\times[0, +\infty[\to [0, +\infty[$ such that, for every b, h in $C_t(C^\infty_{x,c})$ with support in B_R and for every v_0 in $C^\infty_{x,c}$, it holds*

$$\|v\|_{L^\infty_t(L^{2m}_{x,\chi})} + \|\nabla[v^m]\|_{L^2_t(L^2_{x,\chi})} \leq C(\|\mathrm{div}(b)\|_{L^{q/2}_t(L^{p/2}_{x,\chi})}, \|h\|_{L^{q/2}_t(L^{p/2}_{x,\chi})})\|v_0\|_{L^{2m}_{x,\chi}}.$$

Remark 5.5. *Since in the first step we will compose v with the power function, it is convenient to recall a general key property of the Kolmogorov equation. If v is a regular solution (we always work in the regular setting) and $f : \mathbb{R} \to \mathbb{R}$ is a C^3_b function, then $f(v)$ satisfies the equation*

$$\partial_t f(v) + b \cdot \nabla f(v) + hf'(v)v = \frac{1}{2}\Delta f(v) - \frac{1}{2}f''(v)|\nabla v|^2. \tag{5.4}$$

This fact follows directly from the formulae

$$\partial_t f(v) = f'(v)\partial_t v, \ \nabla f(v) = f'(v)\nabla v, \ \Delta f(v) = f'(v)\Delta v + f''(v)|\nabla v|^2. \tag{5.5}$$

This property is very close to the renormalization property for the transport equation (which is the stability of the TE under composition, see Section 4.1): when the Kolmogorov equation is transformed via composition with f, it keeps its terms and has an additional penalization in the $|\nabla v|^2$ term. The reason for this is that the Kolmogorov equation is the average of the stochastic transport equation, as one can see from the representation formula for the solution. Hence, the stability is due to the renormalization property, and the penalization term is due to the average.

Proof. **Step 1: Parabolic equation for v^m and weighted L^{2m} equality**. In this step we derive an equation for v^m and obtain an L^{2m} equality. Using the formula (5.4) for $f(r) = r^m$, we get

$$\partial_t[v^m] + b \cdot \nabla[v^m] + mhv^m = \frac{1}{2}\Delta[v^m] - \frac{1}{2}m(m-1)v^{m-2}|\nabla v|^2.$$

Multiplying this equation by χv^m (using again (5.5)) and integrating (v is in $L^r_t(L^r_{x,\chi})$ with its first and second derivatives, for every r, by Remark 5.3), we get

$$\begin{aligned}
&\partial_t \int_{\mathbb{R}^d} \chi v^{2m} \mathrm{d}x + \int_{\mathbb{R}^d} \chi b \cdot \nabla[v^{2m}] \mathrm{d}x + m \int_{\mathbb{R}^d} \chi h v^{2m} \mathrm{d}x \\
&= \frac{1}{2} \int_{\mathbb{R}^d} \chi \Delta[v^{2m}] \mathrm{d}x - \frac{m(2m-1)}{m^2} \int_{\mathbb{R}^d} \chi |\nabla[v^m]|^2 \mathrm{d}x.
\end{aligned}$$

Using integration by parts, we get rid of the term with the Laplacian and we bring the derivative of v on b:

$$\begin{aligned}
&\int_{\mathbb{R}^d} \chi v_t^{2m} \mathrm{d}x - \int_{\mathbb{R}^d} \chi v_0^{2m} \mathrm{d}x + \frac{m(2m-1)}{m^2} \int_0^t \int_{\mathbb{R}^d} \chi |\nabla[v^m]|^2 \mathrm{d}x \mathrm{d}r \\
&= \int_0^t \int_{\mathbb{R}^d} \chi \mathrm{div} b v^{2m} \mathrm{d}x \mathrm{d}r - m \int_{\mathbb{R}^d} \chi h v^{2m} \mathrm{d}x + \\
&+ \int_0^t \int_{\mathbb{R}^d} \nabla\chi \cdot b v^{2m} \mathrm{d}x \mathrm{d}r - \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} \nabla\chi \cdot \nabla[v^{2m}] \mathrm{d}x \mathrm{d}r.
\end{aligned}$$

Since χ is 1 on the support of b, we have $\nabla\chi \cdot b \equiv 0$. Therefore

$$\begin{aligned}
&\int_{\mathbb{R}^d} \chi v_t^{2m} \mathrm{d}x - \int_{\mathbb{R}^d} \chi v_0^{2m} \mathrm{d}x + \frac{m(2m-1)}{m^2} \int_0^t \int_{\mathbb{R}^d} \chi |\nabla[v^m]|^2 \mathrm{d}x \mathrm{d}r \\
&= \int_0^t \int_{\mathbb{R}^d} \chi(-mh + \mathrm{div} b) v^{2m} \mathrm{d}x \mathrm{d}r - \int_0^t \int_{\mathbb{R}^d} \nabla\chi v^m \cdot \nabla[v^m] \mathrm{d}x \mathrm{d}r.
\end{aligned} \tag{5.6}$$

The idea is to use now Gronwall lemma to get an estimate of the L^{2m}_x norm of v. For the addend in the RHS with $\nabla\chi$, Young inequality and the fact $|\nabla\chi| \le C\chi$ give, for $\epsilon > 0$ (to be fixed later),

$$\begin{aligned} \left|\int_{\mathbb{R}^d} \nabla\chi v^m \cdot \nabla[v^m]\mathrm{d}x\right| &\le C\int_{\mathbb{R}^d} (\epsilon^{-1/2}\chi^{1/2}v^m)(\epsilon^{1/2}\chi^{1/2}|\nabla[v^m]|)\mathrm{d}x \\ &\le C_\epsilon \int_{\mathbb{R}^d} \chi v^{2m}\mathrm{d}x + \epsilon\int_{\mathbb{R}^d} \chi|\nabla[v^m]|^2\mathrm{d}x. \end{aligned} \tag{5.7}$$

Now we have to estimate the term with $-mh + \mathrm{div}b$ in terms of this norm.

Step 2.1: Estimating the term with $-mh+\mathrm{div}b$, the case $p = +\infty$. For simplicity of notation, we deal only with the term $\mathrm{div}b$, the estimate with mh being analogous. We start with the easy case $\mathrm{div}b$ in L^∞. In this case we obtain

$$\left|\int_{\mathbb{R}^d} \chi \mathrm{div}b v^{2m}\mathrm{d}x\right| \le \|\mathrm{div}b\|_{L^\infty_x}\int_{\mathbb{R}^d} \chi v^{2m}\mathrm{d}x,$$

and so

$$\left|\int_0^t\int_{\mathbb{R}^d} \chi \mathrm{div}b v^{2m}\mathrm{d}x\right| \le \int_0^t \|\mathrm{div}b\|_{L^\infty_x}\int_{\mathbb{R}^d} \chi v^{2m}\mathrm{d}x\mathrm{d}r.$$

Step 2.2: Estimating the term with $-mh+\mathrm{div}b$, the case $p < +\infty$. In the case $p < +\infty$, calling $\tilde{p} = p/2$, Hölder inequality implies

$$\left|\int_{\mathbb{R}^d} \chi \mathrm{div}b v^{2m}\mathrm{d}x\right| \le \|\mathrm{div}b\|_{L^{\tilde{p}}_x}\|\chi^{1/2}v^m\|^2_{L^{2\tilde{p}'}_x},$$

where we have used the fact that $\chi \equiv 1$ on the support on $\mathrm{div}b$. Hence we have to bound $\|\chi v^m\|^2_{L^{2\tilde{p}'}_x}$, which is a stronger norm than the desired $\|\chi^{1/2}v^m\|^2_{L^2_x}$. However, from (5.6), we control also the L^2 norm of $\chi\nabla[v^m]$. So we can use this control, via Sobolev-Gagliardo-Nirenberg inequality, to estimate this higher ($2m\tilde{p}'$) integrability of v.

Lemma 5.6. *For every φ in $C^\infty_x \cap W^{1,2}_x$, it holds*

$$\|\chi^{1/2}\varphi\|_{L^{2\tilde{p}'}_x} \le C\|\chi^{1/2}\varphi\|^{1-a}_{L^2_x}\|\chi^{1/2}\nabla\varphi\|^a_{L^2_x} + C\|\chi^{1/2}\varphi\|_{L^2_x}$$

where $a = d/(2\tilde{p})$.

Proof. The condition $\tilde{p} = p/2 > d/2\vee 1$ (implied by Condition 2.3) guarantees that we can apply Sobolev-Gagliardo-Nirenberg inequality, so that

$$\|\chi^{1/2}\varphi\|_{L^{2\tilde{p}'}_x} \le C\|\chi^{1/2}\varphi\|^{1-a}_{L^2_x}\|\nabla[\chi^{1/2}\varphi]\|^a_{L^2_x},$$

where $0 < a < 1$ is given by

$$\frac{1}{2\tilde{p}'} = a\left(\frac{1}{2} - \frac{1}{d}\right) + (1-a)\frac{1}{2},$$

that is, after easy computations, $a = d/(2\tilde{p})$. Since $\nabla[\chi^{1/2}\varphi] = \nabla[\chi^{1/2}]\varphi + \chi^{1/2}\nabla\varphi$ and $|\nabla[\chi^{1/2}]| = \chi^{-1/2}|\nabla\chi|/2 \leq C\chi^{1/2}$, we get the thesis. $\square$

From this Lemma, we get

$$\begin{aligned}&\left|\int_{\mathbb{R}^d} \chi \mathrm{div} b v^{2m} \mathrm{d}x\right| \\ &\leq C\|\mathrm{div} b\|_{L^{\tilde{p}}_x}\|\chi^{1/2}v^m\|^{2(1-a)}_{L^2_x}\|\chi^{1/2}\nabla[v^m]\|^{2a}_{L^2_x} + C\|\mathrm{div} b\|_{L^{\tilde{p}}_x}\|\chi^{1/2}v^m\|^2_{L^2_x}.\end{aligned}$$

To deal with the second addend in the RHS, we use Young inequality with a penalization term of order ϵ, namely

$$fg \leq C_\epsilon f^{1/(1-a)} + \epsilon g^{1/a},$$

where $C_\epsilon > 0$ is some constant depending on ϵ and a; we will fix ϵ later. Applying this inequality to $f = \|\mathrm{div} b\|_{L^p_x}\|\chi^{1/2}v^m\|^{2(1-a)}_{L^2_x}$, $g = \|\chi^{1/2}\nabla[v^m]\|^{2a}_{L^2_x}$, we get

$$\begin{aligned}&\left|\int_{\mathbb{R}^d} \mathrm{div} b v^{2m} \mathrm{d}x\right| \\ &\leq C_\epsilon\|\mathrm{div} b\|^{1/(1-a)}_{L^{\tilde{p}}_x}\|\chi^{1/2}v^m\|^2_{L^2_x} + \epsilon\|\chi^{1/2}\nabla[v^m]\|^2_{L^2_x} + C\|\mathrm{div} b\|_{L^{\tilde{p}}_x}\|\chi^{1/2}v^m\|^2_{L^2_x}.\end{aligned}$$

Putting all together, we end with

$$\begin{aligned}&\left|\int_0^t\int_{\mathbb{R}^d} \mathrm{div} b v^{2m} \mathrm{d}x\right| \\ &\leq C_\epsilon\int_0^t (\|\mathrm{div} b\|^{p/(p-d)}_{L^{\tilde{p}}_x} + \|\mathrm{div} b\|_{L^{\tilde{p}}_x})\int_{\mathbb{R}^d}\chi v^{2m}\mathrm{d}x\mathrm{d}r + \epsilon\int_0^t\int_{\mathbb{R}^d}\chi|\nabla[v^m]|^2\mathrm{d}x\mathrm{d}r,\end{aligned} \tag{5.8}$$

which includes also the case $p = +\infty$.

Step 3: Conclusion via Gronwall lemma. Putting together (5.6), (5.7) and (5.8), we obtain

$$\begin{aligned}&\left|\int_{\mathbb{R}^d}\chi v_t^{2m}\mathrm{d}x - \int_{\mathbb{R}^d}\chi v_0^{2m}\mathrm{d}x\right| + \frac{m(2m-1)}{m^2}\int_0^t\int_{\mathbb{R}^d}\chi|\nabla[v^m]|^2\mathrm{d}x\mathrm{d}r \\ &\leq C_\epsilon\int_0^t\int_0^t (1 + \|-mh + \mathrm{div} b\|^{p/(p-d)}_{L^{\tilde{p}}_x} + \|-mh + \mathrm{div} b\|_{L^{\tilde{p}}_x})\int_{\mathbb{R}^d}\chi v^{2m}\mathrm{d}x\mathrm{d}r + \\ &+\epsilon\int_0^t\int_{\mathbb{R}^d}\chi|\nabla[v^m]|^2\mathrm{d}x\mathrm{d}r.\end{aligned}$$

Now we choose $\epsilon = 1/2 < 1 \leq m(2m-1)/m^2$. So we have

$$\left|\int_{\mathbb{R}^d} \chi v_t^{2m} \mathrm{d}x - \int_{\mathbb{R}^d} \chi v_0^{2m} \mathrm{d}x\right| + \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} \chi |\nabla[v^m]|^2 \mathrm{d}x\mathrm{d}r$$
$$\leq C\int_0^t (1 + \| - mh + \mathrm{div}b\|_{L_x^{\tilde{p}}}^{p/(p-d)} + \| - mh + \mathrm{div}b\|_{L_x^{\tilde{p}}}) \int_{\mathbb{R}^d} v^{2m} \mathrm{d}x\mathrm{d}r.$$

Condition 2.3 ensures the time integrability of $\|\mathrm{div}b\|_{L_x^p}^{p/(p-d)}$, so we can apply Gronwall lemma and get

$$\int_{\mathbb{R}^d} \chi v_t^{2m} \mathrm{d}x$$
$$\leq \int_{\mathbb{R}^d} \chi v_0^{2m} \mathrm{d}x \exp\left[C\int_0^t (1 + \| - mh + \mathrm{div}b\|_{L_x^{\tilde{p}}}^{p/(p-d)} + \| - mh + \mathrm{div}b\|_{L_x^{\tilde{p}}}) \mathrm{d}r\right]$$

and also

$$\int_0^t \int_{\mathbb{R}^d} \chi |\nabla[v^m]|^2 \mathrm{d}x\mathrm{d}r$$
$$\leq \int_{\mathbb{R}^d} \chi v_0^{2m} \mathrm{d}x \cdot 2C\int_0^t (1 + \| - mh + \mathrm{div}b\|_{L_x^{\tilde{p}}}^{p/(p-d)} + \| - mh + \mathrm{div}b\|_{L_x^{\tilde{p}}}) \mathrm{d}r \cdot$$
$$\cdot \exp\left[C\int_0^t (1 + \| - mh + \mathrm{div}b\|_{L_x^{\tilde{p}}}^{p/(p-d)} + \| - mh + \mathrm{div}b\|_{L_x^{\tilde{p}}}) \mathrm{d}r\right].$$

The proof is complete. □

5.3 A priori estimates, part III

In this section we use a similar method to the previous section, but for a slightly more general Kolmogorov-type equation and with different hypotheses on the coefficients. Before giving the setting and the result, we give a short motivation for this equation. Let v be the solution to the Kolmogorov equation (5.3) and assume for simplicity that we are in (spatial) dimension 1 and that $h = 0$. Then, differentiating the equation in x, we get the following PDE for $\partial_x v$:

$$\partial_t[\partial_x v] + b\partial_x[\partial_x]v + \partial_x b \partial_x v = \frac{1}{2}\partial_x^2[\partial_x v],$$

with fixed initial condition $\partial_x v_0$. Hence we see that $\partial_x v$ satisfies a PDE which is the Kolmogorov equation, plus the term $\partial_x b \partial_x v$. In more than one dimensions, we get a system of PDEs for $\nabla v = (\partial_{x_1} v, \ldots \partial_{x_d} v)$, but the structure is always of a Kolmorogov equation with additional terms of the form $\partial_{x_i} b^j \partial_{x_j} v$. In Chapter 12, we will see that similar equations or systems appear as average of certain SPDEs. The most important example is given by the derivative of the solution to the STE, another example is the stochastic vector advection equation (a linearized version of the 3D stochastic Euler equation with multiplicative noise).

For this reason, we introduce the following Kolmogorov-type equation:

$$\partial_t v + b \cdot \nabla v + (\mathrm{div} g + h) v = \frac{1}{2}\Delta v, \tag{5.9}$$

with given initial datum v_0, and we prove a priori L^m estimates on the solution.

We require the Ladyzhenskaya-Prodi-Serrin integrability assumption on b, as in Condition 2.4, but no differentiability assumptions; the same for g, which should be thought morally as b, and h^2. We also allow the vector fields to have a regular component but possibly with linear growth, in the line of Condition 2.4. We consider a weight χ, strictly positive C^∞_x function, with at most polynomial growth, with the property that, for some $C > 0$,

$$|\nabla \chi(x)| \leq C \frac{\chi(x)}{1+|x|}, \quad \forall x \in \mathbb{R}^d.$$

The typical example of such a χ is

$$\chi_\eta(x) = (1+|x|^2)^{\eta/2}$$

for some real number η.

Theorem 5.7. *Fix p, q satisfying Condition 2.4, fix m positive integer. Write $b = b^{(1)} + b^{(2)}$, $g = g^{(1)} + g^{(2)}$, where all the addends are functions in $C_t(C^\infty_{x,c})$. Then there exists a locally bounded function $C : [0,+\infty[^5 \to [0,+\infty[$ such that, for every $b^{(j)}$, $g^{(j)}$, $j = 1, 2$, h in $C_t(C^\infty_{x,c})$ and for every v_0 in $C^\infty_{x,c}$, it holds*

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^d} \chi(x) v_t^{2m} \mathrm{d}x \leq C \int_{\mathbb{R}^d} \chi(x) v_0^{2m} \mathrm{d}x,$$

where

$$C = C(\|b^{(1)}\|_{L^q_t(L^p_x)}, \|g^{(1)}\|_{L^q_t(L^p_x)}, \|h\|_{L^{q/2}_t(L^{p/2}_x)}, \|b^{(2)}\|_{L^1_t(C^1_{x,lin})}, \|g^{(2)}\|_{L^1_t(C^1_{x,lin})}).$$

Remark 5.8. *The Kolmogorov-type PDE* (5.9) *enjoys a similar property to* (5.4)*, namely*

$$\partial_t f(v) + b \cdot \nabla f(v) + (\mathrm{div} g + h) f'(v) v = \frac{1}{2}\Delta f(v) - \frac{1}{2} f''(v) |\nabla v|^2. \tag{5.10}$$

Moreover, as in the previous Section, in the case of regular compactly supported coefficients and initial datum, the PDE (5.9) *admits a solution in* $W_t^{1,m}(W_{x,\chi_\eta}^{2,m})$*, for every* m *in* $[1,+\infty]$ *and for every weight* χ_η*, for every real* η*, see again [BFGM14], Lemma 12. This justifies the computations below.*

Proof. **Step 1: Parabolic equation for v^m and weighted L^{2m} equality**. We derive an equation for v^m and obtain a weighted L_x^{2m} equality. Using the formula (5.10) for $f(r) = r^m$, we get

$$\partial_t[v^m] + b \cdot \nabla[v^m] + m(\mathrm{div} g + h)v^m = \frac{1}{2}\Delta[v^m] - \frac{1}{2}m(m-1)v^{m-2}|\nabla v|^2.$$

Multiplying this equation by χv^m and integrating, we get

$$\begin{aligned}
&\partial_t \int_{\mathbb{R}^d} \chi v^{2m} \mathrm{d}x + \int_{\mathbb{R}^d} \chi b \cdot \nabla[v^{2m}] \mathrm{d}x + 2m \int_{\mathbb{R}^d} \chi(\mathrm{div} g + h)v^{2m} \mathrm{d}x \\
&= \frac{1}{2}\int_{\mathbb{R}^d} \chi \Delta[v^{2m}] \mathrm{d}x - \frac{m(2m-1)}{m^2} \int_{\mathbb{R}^d} \chi |\nabla[v^m]|^2 \mathrm{d}x.
\end{aligned}$$

Using integration by parts, we bring the derivative of $g^{(1)}$ on v^m and the derivative of v^m on $b^{(2)}$ and on χ:

$$\begin{aligned}
&\int_{\mathbb{R}^d} \chi v_t^{2m} \mathrm{d}x - \int_{\mathbb{R}^d} \chi v_0^{2m} \mathrm{d}x + \frac{m(2m-1)}{m^2} \int_0^t \int_{\mathbb{R}^d} \chi |\nabla[v^m]|^2 \mathrm{d}x \mathrm{d}r \\
&= 2\int_0^t \int_{\mathbb{R}^d} \chi(-b^{(1)} + mg^{(1)})v^m \cdot \nabla[v^m] \mathrm{d}x \mathrm{d}r + \\
&- 2\int_0^t \int_{\mathbb{R}^d} \chi m h v^{2m} \mathrm{d}x \mathrm{d}r + 2\int_0^t \int_{\mathbb{R}^d} \nabla\chi \cdot g^{(1)} v^{2m} \mathrm{d}x \mathrm{d}r + \\
&+ 2\int_0^t \int_{\mathbb{R}^d} \chi(\mathrm{div} b^{(2)} - m \mathrm{div} g^{(2)})v^{2m} \mathrm{d}x \mathrm{d}r + 2\int_0^t \int_{\mathbb{R}^d} \nabla\chi \cdot b^{(2)} v^{2m} \mathrm{d}x \mathrm{d}r + \\
&- \int_0^t \int_{\mathbb{R}^d} \nabla\chi v^m \cdot \nabla[v^m] \mathrm{d}x \mathrm{d}r.
\end{aligned}$$

Since $|\nabla\chi(x)| \le C\chi(x)(1+|x|)^{-1} \le C\chi(x)$, we get

$$\begin{aligned}
&\int_{\mathbb{R}^d} \chi v_t^{2m} \mathrm{d}x - \int_{\mathbb{R}^d} \chi v_0^{2m} \mathrm{d}x + \frac{m(2m-1)}{m^2} \int_0^t \int_{\mathbb{R}^d} \chi |\nabla[v^m]|^2 \mathrm{d}x\mathrm{d}r \qquad (5.11)\\
&\le 2\int_0^t \int_{\mathbb{R}^d} \chi(|b^{(1)}| + m|g^{(1)}|)|v|^m |\nabla[v^m]| \mathrm{d}x\mathrm{d}r+\\
&+ 2\int_0^t \int_{\mathbb{R}^d} \chi(m|h| + |g^{(1)}|) v^{2m} \mathrm{d}x\mathrm{d}r+\\
&+ 2\int_0^t \int_{\mathbb{R}^d} \chi \left(|\mathrm{div} b^{(2)}| + m|\mathrm{div} g^{(2)}| + C\frac{|b^{(2)}|}{1+|x|} \right) v^{2m} \mathrm{d}x\mathrm{d}r+\\
&+ C\int_0^t \int_{\mathbb{R}^d} \chi |v|^m |\nabla[v^m]| \mathrm{d}x\mathrm{d}r.
\end{aligned}$$

The idea is to use now Gronwall lemma to get an estimate of the L^{2m} norm of v. For this we have to estimate the "irregular" terms with $|b^{(1)}| + m|g^{(1)}|$ and $m|h| + |g^{(1)}|$ and the "regular" term with $|\mathrm{div} b^{(2)}| + m|\mathrm{div} g^{(2)}| + m|h| + C|b^{(2)}|/(1+|x|)$.

Step 2: estimating the regular terms. The last term in (5.11) is estimated easily via Young inequality: for every $\epsilon > 0$,

$$\int_0^t \int_{\mathbb{R}^d} \chi |v|^m |\nabla[v^m]| \mathrm{d}x\mathrm{d}r \le C_\epsilon \int_0^t \int_{\mathbb{R}^d} \chi v^{2m} dxdr + \epsilon \int_0^t \int_{\mathbb{R}^d} \chi |\nabla[v^m]|^2 \mathrm{d}x\mathrm{d}r.$$

For the regular term, Hölder inequality gives

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^d} \chi \left(|\mathrm{div} b^{(2)}| + m|\mathrm{div} g^{(2)}| + C\frac{|b^{(2)}|}{1+|x|} \right) v^{2m} \mathrm{d}x\mathrm{d}r\\
&\le \int_0^t \left\| |\mathrm{div} b^{(2)}| + m|\mathrm{div} g^{(2)}| + C\frac{|b^{(2)}|}{1+|x|} \right\|_{L^\infty_x} \int_{\mathbb{R}^d} \chi v^{2m} \mathrm{d}x\mathrm{d}r.
\end{aligned}$$

Step 3.1: estimating the irregular terms, the easy case $p = \infty$. For the irregular terms, we start with the easiest, yet important, case: $p = \infty$. We put $g^{(1)} = 0$ for simplicity of notation, the general case being analogous. Here, for the term with $b^{(1)}$, Hölder inequality gives

$$\int_0^t \int_{\mathbb{R}^d} \chi |b^{(1)}| |v|^m |\nabla[v^m]| \mathrm{d}x\mathrm{d}r \le \int_0^t \|b^{(1)}\|_{L^\infty_x} \|\chi^{1/2} v^m\|_{L^2_x} \|\chi^{1/2} \nabla[v^m]\|_{L^2_x}$$

and then Young inequality gives

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^d}\chi|b^{(1)}||v|^m|\nabla[v^m]|\mathrm{d}x\mathrm{d}r\\&\le C_\epsilon\int_0^t\|b^{(1)}\|_{L^\infty_x}^2\int_{\mathbb{R}^d}\chi v^{2m}\mathrm{d}x\mathrm{d}r+\epsilon\int_0^t\int_{\mathbb{R}^d}\chi|\nabla[v^m]|^2\mathrm{d}x\mathrm{d}r.\end{aligned}$$

For the term with h, we get

$$\int_0^t\int_{\mathbb{R}^d}\chi m|h|v^{2m}\mathrm{d}x\mathrm{d}r\le m\int_0^t\|h\|_{L^\infty_x}\int_{\mathbb{R}^d}\chi v^{2m}\mathrm{d}x\mathrm{d}r.$$

Step 3.2: estimating the irregular terms, the general case. In the case $p<+\infty$, for the term with $b^{(1)}$ (again we put $g^{(1)}=0$ for simplicity), Hölder inequality (applied twice) implies

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^d}\chi|b^{(1)}||v|^m|\nabla[v^m]|\mathrm{d}x\mathrm{d}r\le\int_0^t\|\chi^{1/2}|b^{(1)}||v|^m\|_{L^2_x}\|\chi^{1/2}|\nabla[v^m]|\|_{L^2_x}\mathrm{d}r\\&\le\int_0^t\|b^{(1)}\|_{L^p_x}\|\chi^{1/2}v^m\|_{L^{2\tilde p'}_x}\|\chi^{1/2}|\nabla[v^m]|\|_{L^2_x}\mathrm{d}r,\end{aligned}\tag{5.12}$$

where $\tilde p=p/2$. Hence we have to bound $\|\chi^{1/2}|v|^m\|_{L^{\tilde p'}_x}$, a stronger norm than the desired $\|\chi^{1/2}v^m\|^2_{L^2_x}$. As in the previous Section, from (5.6), we control also the L^2 norm of $\chi\nabla[v^m]$ and we can use this control, via Sobolev-Gagliardo-Nirenberg inequality, to estimate this higher $\tilde p'$ integrability of $|v|^m$.

Indeed, we can Lemma 5.6, which can be extended to the case $p=d$. We get from (5.12)

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^d}\chi|b^{(1)}||v|^m|\nabla[v^m]|\mathrm{d}x\mathrm{d}r\\&\le C\int_0^t\|b^{(1)}\|_{L^p_x}\|\chi^{1/2}v^m\|_{L^2_x}^{1-d/p}\|\chi^{1/2}|\nabla[v^m]|\|_{L^2_x}^{1+d/p}\mathrm{d}r+\\&\quad+C\int_0^t\|b^{(1)}\|_{L^p_x}\|\chi^{1/2}v^m\|_{L^2_x}\|\chi^{1/2}|\nabla[v^m]|\|_{L^2_x}\mathrm{d}r.\end{aligned}$$

First suppose $p>d$, i.e. $a=d/p<1$. Young inequality applied to the first addend of the RHS, with exponents $2/(1-a)$, $2/(1+a)$, gives, for every $\epsilon>0$,

$$\begin{aligned}&\int_0^t\|b^{(1)}\|_{L^p_x}\|\chi^{1/2}v^m\|_{L^2_x}^{1-d/p}\|\chi^{1/2}|\nabla[v^m]|\|_{L^2_x}^{1+d/p}\mathrm{d}r\\&\le C_\epsilon\int_0^t\|b^{(1)}\|_{L^p_x}^{2p/(p-d)}\int_{\mathbb{R}^d}\chi v^{2m}\mathrm{d}x\mathrm{d}r+\epsilon\int_0^t\int_{\mathbb{R}^d}\chi|\nabla[v^m]|^2\mathrm{d}x\mathrm{d}r.\end{aligned}$$

Similarly, Young inequality applied to the second addend, with exponents 2, 2, gives

$$\begin{aligned}&\int_0^t \|b^{(1)}\|_{L^p_x}\|\chi^{1/2}v^m\|_{L^2_x}\|\chi^{1/2}|\nabla[v^m]|\|_{L^2_x}\mathrm{d}r\\&\le C_\epsilon\int_0^t \|b^{(1)}\|^2_{L^p_x}\int_{\mathbb{R}^d}\chi v^{2m}\mathrm{d}x\mathrm{d}r+\epsilon\int_0^t\int_{\mathbb{R}^d}\chi|\nabla[v^m]|^2\mathrm{d}x\mathrm{d}r.\end{aligned}$$

We end (in the case $p>d$) with

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^d}\chi|b^{(1)}||v|^m|\nabla[v^m]|\mathrm{d}x\mathrm{d}r\\&\le C_\epsilon\int_0^t(\|b^{(1)}\|^{2p/(p-d)}_{L^p_x}+\|b^{(1)}\|^2_{L^p_x})\int_{\mathbb{R}^d}\chi v^{2m}\mathrm{d}x\mathrm{d}r+\epsilon\int_0^t\int_{\mathbb{R}^d}\chi|\nabla[v^m]|^2\mathrm{d}x\mathrm{d}r.\end{aligned}$$

For $p=d\ge 3$ (which implies $a=1$), we cannot use Young inequality for the first addend, since the term $\|\chi^{1/2}z_m\|$ does not appear. In this case we have by Hölder inequality

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^d}\chi|b^{(1)}||v|^m|\nabla[v^m]|\mathrm{d}x\mathrm{d}r\\&\le C(\|b^{(1)}\|_{L^\infty_t(L^p_x)}+\epsilon)\int_0^t\int_{\mathbb{R}^d}\chi|\nabla[v^m]|^2\mathrm{d}x\mathrm{d}r+\\&\quad+C_\epsilon\|b^{(1)}\|^2_{L^\infty(L^p_x)}\int_0^t\int_{\mathbb{R}^d}\chi v^{2m}\mathrm{d}x\mathrm{d}r.\end{aligned}$$

For the term with $mh+|g^{(1)}|$, we put again $g^{(1)}=0$: the estimate with h can be applied also to $g^{(1)}$, replacing p with $2p$ and q with $2q$ (since $2p$ and $2q$ also satisfy Condition 2.4). Hölder inequality gives

$$\int_0^t\int_{\mathbb{R}^d}\chi|h|v^{2m}\mathrm{d}x\mathrm{d}r\le\int_0^t\|h\|_{L^{\tilde p}_x}\|\chi^{1/2}v^m\|^2_{L^{2\tilde p'}_x}\mathrm{d}r,$$

where $\tilde p=p/2$. Applying again Lemma 5.6, we get

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^d}\chi|h|v^{2m}\mathrm{d}x\mathrm{d}r\\&\le\int_0^t\|h\|_{L^{p/2}_x}\|\chi^{1/2}v^m\|^{2(1-d/p)}_{L^2_x}\|\chi^{1/2}|\nabla[v^m]|\|^{2d/p}_{L^2_x}\mathrm{d}r+\int_0^t\|h\|_{L^{p/2}_x}\|\chi^{1/2}v^m\|^2_{L^2_x}\mathrm{d}r.\end{aligned}$$

First suppose $p > d$ (i.e. $a = d/p < 1$). Young inequality applied to the first addend of the RHS, with exponents $1/(1-a)$, $1/a$, gives, for every $\epsilon > 0$,

$$\begin{aligned}&\int_0^t \|h\|_{L^{p/2}_x} \|\chi^{1/2} v^m\|_{L^2_x}^{2(1-d/p)} \|\chi^{1/2}|\nabla[v^m]|\|_{L^2_x}^{2d/p} \mathrm{d}r \\ &\le C_\epsilon \int_0^t \|h\|_{L^{p/2}_x}^{p/(p-d)} \int_{\mathbb{R}^d} \chi v^{2m} \mathrm{d}x\mathrm{d}r + \epsilon \int_0^t \int_{\mathbb{R}^d} \chi |\nabla[v^m]|^2 \mathrm{d}x\mathrm{d}r.\end{aligned}$$

We end (in the case $p > d$) with

$$\begin{aligned}&\int_0^t \int_{\mathbb{R}^d} \chi |h| v^{2m} \mathrm{d}x\mathrm{d}r \\ &\le C_\epsilon \int_0^t (\|h\|_{L^{p/2}_x}^{p/(p-d)} + \|h\|_{L^{p/2}_x}) \int_{\mathbb{R}^d} \chi v^{2m} \mathrm{d}x\mathrm{d}r + \epsilon \int_0^t \int_{\mathbb{R}^d} \chi |\nabla[v^m]|^2 \mathrm{d}x\mathrm{d}r.\end{aligned}$$

For $p = d \ge 3$ (which implies $a = 1$), we have by Hölder inequality

$$\begin{aligned}&\int_0^t \int_{\mathbb{R}^d} \chi |h| v^{2m} \mathrm{d}x\mathrm{d}r \\ &\le C\|h\|_{L^\infty_t(L^{p/2}_x)} \int_0^t \int_{\mathbb{R}^d} \chi |\nabla[v^m]|^2 \mathrm{d}x\mathrm{d}r + C\|h\|_{L^\infty_t(L^{p/2}_x)} \int_0^t \int_{\mathbb{R}^d} \chi v^{2m} \mathrm{d}x\mathrm{d}r.\end{aligned}$$

Step 4: conclusion via Gronwall lemma. In the case $d < p \le +\infty$, putting together all these terms, we find, for every $\epsilon > 0$,

$$\begin{aligned}&\left|\int_{\mathbb{R}^d} \chi v_t^{2m} \mathrm{d}x - \int_{\mathbb{R}^d} \chi v_0^{2m} \mathrm{d}x\right| + \frac{m(2m-1)}{m^2} \int_0^t \int_{\mathbb{R}^d} \chi |\nabla[v^m]|^2 \mathrm{d}x\mathrm{d}r \\ &\le C_\epsilon \int_0^t \rho(r) \int_{\mathbb{R}^d} \chi v^{2m} \mathrm{d}x\mathrm{d}r + C\epsilon \int_0^t \int_{\mathbb{R}^d} \chi |\nabla[v^m]|^2 \mathrm{d}x\mathrm{d}r.\end{aligned} \tag{5.13}$$

where

$$\begin{aligned}\rho(r) = 1 &+ \|b^{(1)}\|_{L^p_x}^{2p/(p-d)} + \|b^{(1)}\|_{L^p_x}^2 + \|g^{(1)}\|_{L^p_x}^{2p/(p-d)} + \|g^{(1)}\|_{L^p_x}^2 + \\ &+ \|h\|_{L^{p/2}_x}^{p/(p-d)} + \|h\|_{L^{p/2}_x} + \|g\|_{L^p_x}^{2p/(2p-d)} + \|g\|_{L^p_x} + \\ &+ \|\mathrm{div} b^{(2)}\|_{L^\infty_x} + \|\mathrm{div} g^{(2)}\|_{L^\infty_x} + \|b^{(2)}\|_{C_{x,lin}}.\end{aligned}$$

Choosing ϵ small enough, we get

$$\int_{\mathbb{R}^d} \chi v(t)^{2m} \mathrm{d}x - \int_{\mathbb{R}^d} \chi v(0)^{2m} \mathrm{d}x \le C \int_0^t \rho(r) \left(\int_{\mathbb{R}^d} \chi v(r)^{2m} \mathrm{d}x\right) \mathrm{d}r.$$

By Condition 2.4, ρ is in $L^1([0,T])$. Hence we can apply Gronwall lemma, obtaining

$$\int_{\mathbb{R}^d} \chi v(t)^{2m} \mathrm{d}x \leq e^{C\int_0^t \rho(r)\mathrm{d}r} \left(\int_{\mathbb{R}^d} \chi v(0)^{2m} \mathrm{d}x \right).$$

The thesis is reached in the case $p > d$.

When $p = d \geq 3$, we get

$$\begin{aligned} &\left| \int_{\mathbb{R}^d} \chi v_t^{2m} \mathrm{d}x - \int_{\mathbb{R}^d} \chi v_0^{2m} \mathrm{d}x \right| + \frac{m(2m-1)}{m^2} \int_0^t \int_{\mathbb{R}^d} \chi |\nabla[v^m]|^2 \mathrm{d}x \mathrm{d}r \\ &\leq C_\epsilon \rho \int_0^t \int_{\mathbb{R}^d} \chi v^{2m} \mathrm{d}x \mathrm{d}r + C(\psi + \epsilon) \int_0^t \int_{\mathbb{R}^d} \chi |\nabla[v^m]|^2 \mathrm{d}x \mathrm{d}r. \end{aligned} \tag{5.14}$$

where

$$\begin{aligned} &\rho = 1 + \|b^{(1)}\|^2_{L_t^\infty(L_x^p)} + \|g^{(1)}\|^2_{L_t^\infty(L_x^p)} + \|h\|_{L_t^\infty(L_x^{p/2})} + \\ &\quad + \|g^{(1)}\|^2_{L_t^\infty(L_x^p)} + \|g^{(1)}\|_{L_t^\infty(L_x^p)} + \|\mathrm{div} b^{(2)}\|_{L_x^\infty} + \|\mathrm{div} g^{(2)}\|_{L_x^\infty} + \|b^{(2)}\|_{C_{x,lin}}, \\ &\psi = \|b^{(1)}\|^2_{L_t^\infty(L_x^p)} + \|g^{(1)}\|^2_{L_t^\infty(L_x^p)} + \|h\|_{L_t^\infty(L_x^{p/2})}. \end{aligned}$$

By Condition 2.4, both ρ and ψ are finite and we can assume ψ small enough (recall Remark 2.5). Hence, choosing ϵ also small, we obtain

$$\int_{\mathbb{R}^d} \chi v(t)^{2m} \mathrm{d}x - \int_{\mathbb{R}^d} \chi v(0)^{2m} \mathrm{d}x \leq C\rho \int_0^t \left(\int_{\mathbb{R}^d} \chi v(r)^{2m} \mathrm{d}x \right) \mathrm{d}r$$

and, as before by Gronwall lemma, we find

$$\int_{\mathbb{R}^d} \chi v(t)^{2m} \mathrm{d}x \leq e^{C\rho t} \left(\int_{\mathbb{R}^d} \chi v(0)^{2m} \mathrm{d}x \right).$$

The proof is complete. □

Remark 5.9. *As one can see from the proof, the result is valid under a more general assumption on $b^{(1)}$, $g^{(1)}$ and h, namely: $b^{(1)} = \sum_{k=1}^n b^{(1),k}$ and all the estimates are in terms of $\|b^{(1),k}\|_{L_t^{q_k}(L_x^{p_k})}$ for p_k, q_k satisfying Condition 2.4; similarly for g with $\|g^{(1),k}\|_{L_t^{q_k}(L_x^{p_k})}$ and h with $\|h^{(1),k}\|_{L_t^{q_k/2}(L_x^{p_k/2})}$.*

5.4 Uniqueness by duality

In this section we use the previous a priori estimates to obtain uniqueness (among distributional solutions) for the Fokker-Planck equation

$$\partial_t u + \mathrm{div}[bu] = \frac{1}{2}\Delta u \tag{5.15}$$

with given initial datum u_0.

Notice also that this equation is obtained by averaging over ω the stochastic continuity equation, as the Kolmogorov equation is obtained by averaging the stochastic transport equation. Having in mind the correspondence between Fokker-Planck equation and SCE and between Kolmogorov equation and STE, one can provide existence results via stability and uniqueness results via duality (between Fokker-Planck equation and Kolmogorov equation), in the same way one does with (S)CE and (S)TE.

Here are the precise definitions of solutions for the Fokker-Planck equation and the Kolmogorov equation. Concerning the Fokker-Planck equation, for simplicity, we deal only with $L^\infty_t(L^2_{x,loc})$ solutions (starting at time 0) and with vector fields in $L^2_t(L^2_{x,loc})$. Moreover, for technical reasons (that will be clear in Chapter 7), we do not ask the solution u to be in $L^\infty_t(\mathcal{M}_x)$.

Definition 5.10. *Let b be in $L^2_t(L^2_{x,loc})$ and let u_0 be in $L^2_{x,loc}$. A distributional solution to the Fokker-Planck equation is a measurable map $u:[0,T]\times\mathbb{R}^d\to\mathbb{R}$, in $L^2_t(L^2_{x,loc})$, such that, for every φ in $C^\infty_{x,c}$, it holds*

$$\langle u_t,\varphi\rangle = \langle u_0,\varphi\rangle + \int_0^t \langle u_r, b_r\cdot\nabla\varphi\rangle \mathrm{d}r + \frac{1}{2}\int_0^t \langle u_r,\cdot\Delta\varphi\rangle \mathrm{d}r. \tag{5.16}$$

Definition 5.11. *Let b be in $L^2_t(L^2_{x,loc})$ and let v_t be in $L^2_{x,loc}$. A differentiable solution to the backward Kolmogorov equation is a measurable map $v:[0,t]\times\mathbb{R}^d\to\mathbb{R}$, in $L^\infty_{s,x}\cap L^2_s(W^{1,2}_{x,loc})$, such that, for every φ in $C^\infty_{x,c}$, it holds*

$$\langle v_s,\varphi\rangle = \langle v_t,\varphi\rangle + \int_s^t \langle b_r\cdot\nabla v_r,\varphi\rangle \mathrm{d}r - \frac{1}{2}\int_s^t \langle \nabla v_r,\nabla\varphi\rangle \mathrm{d}r.$$

Here is the existence and uniqueness result. The strategy, as said, is by stability and duality. In particular, the stability results given for SCE and STE are valid also in this context, replacing the (S)CE with the Fokker-Planck and the (S)TE with the Kolmogorov equation, and without the probabilistic datum ω (this even simplifies the proof); the assumptions are the same (up to removing ω when necessary), apart for the global bound $L^\infty_t(\mathcal{M}_x)$, which is missing here. The only problem, of technical nature, is that the missing L^∞ bound in time does not allow u (solution to the Fokker-Planck equation) to have a L^2_x-valued weakly continuous (in time) version, but this missing continuity property can be replaced by weak continuity in negative order Sobolev spaces. For the duality result, we repeat the computations in Proposition 4.8 and Lemma 4.11; the main difference is in some additional global conditions one has to impose (as known, the heat equation has more than one solution without global assumptions). In the following, $\chi=\chi_{\eta,R}$

is the weight defined before Theorem 5.4, i.e. a strictly positive function in C_x^∞, with $\chi = 1$ on B_R and $\chi(x) = (1+|x|^2)^{\eta/2}$ on B_{R+1}, where $R > 0$ is such that B_R contains the support of b.

Theorem 5.12. *Fix p, q satisfying Condition 2.3 and fix m positive integer, $R_0 > 0$ and $\eta > d(m-1)$, $\chi = \chi_{\eta,R_0}$. Let $\bar{m}$ be in $[2,+\infty]$ such that $1/2 + 1/2m + 1/\bar{m} \le 1$ and assume that b is in $L_t^{\bar{m}}(L_{x,loc}^{\bar{m}})$ with compact support in B_{R_0} and that $\mathrm{div} b$ is in $L_t^{q/2}(L_x^{p/2})$. Assume also that u_0 is in L_x^∞ with compact support and that v_T is in $C_{x,c}^2$. Then there exists a differentiable solution v to the backward Kolmogorov equation, in the class $L_{t,x}^\infty \cap L_t^2(W_x^{1,2})$, and the Fokker-Planck equation* (5.15) *admits a unique solution in the class $L_t^{2m}(L_{x,\chi}^{2m})$.*

Proof. **Existence for Kolmogorov equation**. Existence follows by a priori estimates in 5.4 (applied in the case of L^2 estimates, with weight $\equiv 1$), via the stability results 3.39 and 3.48, adapted to the Kolmogorov equation.

Existence for the Fokker-Planck equation. Again existence follows by a priori estimates in 5.4 (for any integer m), via the stability results 3.47 and 3.48, adapted to the Fokker-Planck equation and without the global bound $L_t^\infty(\mathcal{M}_x)$ here (and also without the L^2-valued weak continuity property, which is not needed for stability).

Uniqueness for the Fokker-Planck equation. The proof of uniqueness is by duality, exploiting the existence of v solution to Kolmogorov equation in $L_t^2(W_x^{1,2})$ and a suitable adaptation of the arguments in Proposition 4.8 and Lemma 4.11, with Condition 1. We sketch the main steps of the proof. By linearity, it is enough to show that, for any solution u to the Fokker-Planck equation in $L_t^{2m}(L_{x,\chi}^{2m})$ starting from 0, for any solution v to the Kolmogorov equation in $L_{t,x}^\infty \cap L_t^2(W_{x,\chi}^{1,2})$ with final time t and final datum v_t in $C_{x,c}^\infty$, we get $\langle u_t, v_t\rangle = \langle u_0, v_0\rangle = 0$.

Step zero: One can choose a version of the solution u such that $t \mapsto u_t \in (W_{x,B_R,0}^{2,k})^*$ is weakly-* continuous, for every $R > 0$, provided k is a finite sufficiently large number (such that $W_{x,B_R}^{2,k}$ is embedded into $W_{x,B_R}^{1,\infty}$). Indeed all the addends in the RHS of (5.16) are bounded, uniformly in t, by $C(\|\varphi\|_{W_{x,B_R}^{2,k}} + \|\varphi\|_{W_{x,B_R}^{1,\infty}})$ (where C is a constant dependent possibly on R); this bound implies, as in the proof of Lemma 3.7, the existence of the weakly-* continuous version. Using this version, we can extend the distributional formulation of the solution to time-dependent test functions φ, provided these are regular enough, namely φ in $C_t(C_{x,c}^2)$, with $\varphi(\cdot,x)$ in $W_t^{1,1}$ for every x and $\partial_t\varphi$ in $L_t^1(C_{x,c})$; the proof is analogous to the proof of Lemma 3.10.

First step: equation satisfied by $u \otimes v$. Repeating the steps at the beginning of Section 4.3 and using the $(W_{x,B_R,0}^{2,k})^*$-valued weak-* continuous version of u, we find, for every φ in $C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, the equality

$$\langle u_t \otimes v_t \rangle = \int_0^t \langle u \otimes v, b_x \cdot \nabla_x \varphi + \mathrm{div}_y(b_y \cdot \nabla_y \varphi) \rangle \mathrm{d}r + \frac{1}{2} \int_0^t \langle u \otimes v, \Delta_x \varphi - \Delta_y \varphi \rangle \mathrm{d}r.$$

Second step: expression for the commutator. We take ρ_ϵ, χ_R as in Section 4.3, with the additional condition (compatible with the other conditions) that $|\Delta \chi_R| \leq C$ for some C independent on R; we also take R such that the support of b is contained in B_R (and so $b \cdot \nabla \chi_R = 0$). Inserting $\varphi(x, y) = \rho_\epsilon(x - y)\chi_R(x)$ in the equation above, we get

$$\begin{aligned} &\langle u_t \otimes v_t, \rho_\epsilon^{(2)} (\chi_R)_x \rangle \qquad (5.17) \\ &= \int_0^t \langle u \otimes v, (\chi_R)_x (b_x \cdot \nabla_x \rho_\epsilon^{(2)} + \mathrm{div}_y(b_y \cdot \nabla_y \rho_\epsilon^{(2)}) \rangle \mathrm{d}r + \\ &+ \frac{1}{2} \int_0^t \langle u \otimes v, \rho_\epsilon^{(2)} \Delta(\chi_R)_x - 2\nabla_y \rho_\epsilon^{(2)} \cdot \nabla(\chi_R)_x \rangle \mathrm{d}r. \end{aligned}$$

Third step: controlling the commutator. We first let ϵ go to 0, keeping R fixed. Using the $W_x^{1,2}$ regularity of v and proceeding as in Lemma 4.11, Condition 1, we get that

$$\int_0^t \langle u \otimes v, (\chi_R)_x (b_x \cdot \nabla_x \rho_\epsilon^{(2)} + \mathrm{div}_y(b_y \cdot \nabla_y \rho_\epsilon^{(2)}) \rangle \mathrm{d}r \to 0.$$

As for the other addend in (5.17), using integration by parts in y (at R fixed, so that we can use the compact support of $\rho_\epsilon(x - y)\chi_R(x)$), we rewrite this addend as

$$\frac{1}{2} \int_0^t \langle u \otimes v, \rho_\epsilon^{(2)} \Delta(\chi_R)_x \rangle \mathrm{d}r + \int_0^t \langle u \otimes \nabla_y v, \rho_\epsilon^{(2)} \nabla(\chi_R)_x \rangle \mathrm{d}r.$$

From this formula and the fact that u, v and ∇v are in $L_t^2(L_{x,loc}^2)$, we get convergence of this term to

$$\frac{1}{2} \int_0^t \langle uv, \Delta \chi_R \rangle \mathrm{d}r + \int_0^t \langle u \nabla v, \nabla \chi_R \rangle \mathrm{d}r.$$

Putting all together, we find

$$\langle u_t v_t, \chi_R \rangle = \frac{1}{2} \int_0^t \langle uv, \Delta \chi_R \rangle \mathrm{d}r + \int_0^t \langle u \nabla v, \nabla \chi_R \rangle \mathrm{d}r.$$

Fourth step: control at ∞ and conclusion. Now we let $R \to +\infty$. We need that u is in L^2_x globally. For this, we have by Hölder inequality

$$\int_{\mathbb{R}^d} |u|^2 \mathrm{d}x \le C \int_{\mathbb{R}^d} \chi_{-\eta/m,R} \chi_{\eta/m,R} |u|^2 \mathrm{d}x$$
$$\le C \left(\int_{\mathbb{R}^d} \chi_{-\eta/(m-1),R} \mathrm{d}x \right)^{1-1/m} \left(\int_{\mathbb{R}^d} \chi_{\eta,R} |u|^{2m} \mathrm{d}x \right)^{1/m}.$$

Since $\eta > d(m-1)$, $\chi_{-\eta/(m-1),R}$ is integrable and so u is in $L^2_t(L^2_x)$. Using this, the fact that v and ∇v are in $L^2_t(L^2_x)$ (globally in space) and the fact that $|\nabla \chi_R|$ and $|\Delta \chi_R|$ are uniformly bounded and supported on B^c_R, we can pass to the limit and get finally $\langle u_t, v_t \rangle = 0$. $\square$

Chapter 6

Existence for stochastic continuity equation

We start the main part of the thesis by showing regularization by noise in an existence result, for the SCE, under mild assumptions on the drift or on its divergence; such a result is false without noise. If we also require the drift to be in $W^{1,1}_x$, but with no boundedness assumption on the divergence, then the theory in the deterministic case gives also (path-by-path) uniqueness; we then conclude existence and uniqueness of suitable Lagrangian flows.

Throughout all this chapter, we assume b with at most linear growth outside of a ball, namely Condition 2.1.

6.1 A priori estimates

The existence results are based on the a priori estimates. Here is the first one. In the following, we fix $R > 0$ such that the support of b is in B_R and we consider the weight $\chi = \chi_{\eta,R}$, strictly positive function in C^∞_x, with $\chi = 1$ on B_R and $\chi(x) = (1 + |x|^2)^{\eta/2}$ on B_{R+1}, for some real number η.

Theorem 6.1. *Fix p, q satisfying* (2.3)*, fix m positive integer, fix $R > 0$ and η real number. Then there exists a locally bounded function $C : [0, +\infty[\to [0, +\infty[$ such that, for every b in $C_t(C^\infty_{x,c})$ with support in B_R and for every μ_0 in $C^\infty_{x,c}$, it holds*

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^d} \chi_{\eta,R} E[\mu_t^m]^2 \mathrm{d}x \le C(\|\mathrm{div} b\|_{L^{q/2}_t(L^{p/2}_x)}) \int_{\mathbb{R}^d} \chi_{\eta,R} |\mu_0|^{2m} \mathrm{d}x.$$

The method is as follows: we write the equation satisfied by $E\mu_t^m$, which turns to be a parabolic equation, and then we use the parabolic estimates in

Chapter 5. We will comment later, in Chapter 12, on this method, which in that Chapter will be used to get a priori estimates on the derivative of the transport equation. We only mention that, in order to get an equation for $E\mu_t^m$, we use the renormalization property for transport-like equations and the zero expectation of the Itô integral.

Notice that, since we deal with regular compactly supported initial datum and coefficients, there exists a unique classical solution to the SCE, which is integrable in the sense of Proposition 3.19; this justifies the computations below.

Proof. **Step 1: the parabolic equation for $E[\mu^m]$.** Applying Itô formula to the SCE and using chain rule ($\partial_t[\mu^m] = m\mu^{m-1}\partial_t\mu$, $\nabla[\mu^m] = m\mu^{m-1}\nabla\mu$), we get

$$\partial_t[\mu^m] + b \cdot \nabla[\mu^m] + m\mathrm{div}b\mu^m + \nabla[\mu^m] \circ \dot{W} = 0,$$

which reads with Itô integral

$$\partial_t[\mu^m] + b \cdot \nabla[\mu^m] + m\mathrm{div}b\mu^m + \nabla[\mu^m]\dot{W} = \frac{1}{2}\Delta[\mu^m].$$

Taking expectation, we obtain

$$\partial_t E[\mu^m] + b \cdot \nabla E[\mu^m] + m\mathrm{div}bE[\mu^m] + \nabla E[\mu^m]\dot{W} = \frac{1}{2}\Delta E[\mu^m].$$

This is the parabolic PDE (5.3), with $h = m\mathrm{div}b$.

Step 2: conclusion. From Theorem 5.4 (applied just for L^2 estimates), we conclude immediately the thesis. $\square$

The second a priori estimate is the following one. Here we consider the weight χ, strictly positive C_x^∞ function, with at most polynomial growth, satisfying, for every x in $\mathbb{R}^d$,

$$|\nabla\chi(x)| \leq C\frac{\chi(x)}{1+|x|}. \tag{6.1}$$

Theorem 6.2. *Fix p, q satisfying Condition 2.4, fix m positive integer. Write $b = b^{(1)} + b^{(2)}$, where the addends are functions in $C_t(C_{x,c}^\infty)$. Then there exists a locally bounded function $C : [0, +\infty[^2 \to [0, +\infty[$ such that, for every $b^{(j)}$, $j = 1, 2$, in $C_t(C_{x,c}^\infty)$ and for every μ_0 in $C_{x,c}^\infty$, it holds*

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^d} \chi(x)E[\mu_t^m]^2 \mathrm{d}x \leq C(\|b^{(1)}\|_{L_t^q(L_x^p)}, \|b^{(2)}\|_{L_t^1(C_{x,lin}^1)}) \int_{\mathbb{R}^d} \chi(x)|\mu_0|^{2m}\mathrm{d}x.$$

Proof. The strategy of proof is identical to the previous proof: we write the equation for $E[\mu^m]$, which in this case we identify with the parabolic PDE 5.9, with $g = mb$, $h = 0$; then we apply the a priori estimates in Theorem 5.7. $\square$

6.2 The existence results

From the a priori estimates before, one can deduce almost immediately existence results, using Theorem 3.47. We emphasize that these results do not hold in the deterministic case: one usually has to require boundedness of the divergence (more precisely, of its negative part), see for example [AC14].

Theorem 6.3. *Fix m positive even integer. Let μ_0 be in L^∞_x with compact support. Assume one of the following assumptions:*

- *b belongs to $L^{m'}_t(L^{m'}_{x,loc})$ with support in B_R and $\mathrm{div} b$ belongs to $L^{q/2}_t(L^{p/2}_x)$, where p, q satisfy (2.3); the weight χ is of the form $\chi = \chi_{\eta,R}$ (defined in the previous Section), for some real η;*

- *b belongs to the class 2.4; the weight χ is of the form $\chi(x) = \chi_\eta(x) = (1+|x|^2)^{\eta/2}$.*

Then there exists a distributional solution μ to the SCE driven by b and starting from μ_0, in the class $L^m_{t,\omega}(L^m_{x,\chi})$.

Proof. The a priori estimate provided by Theorem 6.1 (for the first hypothesis) and Theorem 6.2 (for the second hypothesis), applied for m even (so that $\mu^m = |\mu|^m$), and the stability result 3.47 (where we use the integrability hypothesis on b) give the existence of a solution in the class $L^m_{t,\omega}(L^m_{x,loc})$. To get the bound with the weight, notice that

$$\begin{aligned}\int_{\mathbb{R}^d} \chi_\eta E[\mu^m]\mathrm{d}x &= \int_{\mathbb{R}^d} \chi_{\eta+d/2+1} E[\mu^m]\chi_{-d/2-1}\mathrm{d}x \\ &\le \left(\int_{\mathbb{R}^d} \chi_{2\eta+d+2} E[\mu^m]^2\mathrm{d}x\right)^{1/2} \left(\int_{\mathbb{R}^d} \chi_{-d-2}\mathrm{d}x\right)^{1/2},\end{aligned}$$

and similarly for $\chi_{\eta,R}$; since χ_{-d-2} is in L^1_x, we obtain by Theorem 6.1 and Theorem 6.2 an a priori bound on the $L^m_{t,\omega}(L^m_{x,\chi})$ norm. Hence the stability result 3.48 implies the desired $L^m_{t,\omega}(L^m_{x,\chi})$ bound. □

6.3 Path-by-path uniqueness under Sobolev assumptions

Recall that, in the deterministic context, uniqueness holds for the SCE if b has some Sobolev regularity. Then, putting together this uniqueness result and the previous existence result, we get a well-posedness result for the SCE, in a path-by-path way, which implies well-posedness at the level of Lagrangian flows. In particular, we get well-posedness for the class of drifts 2.6.

Theorem 6.4. *Assume the hypotheses of Theorem 6.3. Assume also that b is in $L^{m'}_t(W^{1,m'}_{x,loc})$. Then existence, path-by-path uniqueness and stability hold, in the class $L^m_{t,\omega}(L^m_{x,\chi})$, starting from initial datum in L^∞_x compactly supported, in the sense of Theorem 4.15.*

Proof. The result is a consequence of Theorem 6.3, via Theorem 4.15. $\square$

Chapter 7

Wiener uniqueness for stochastic continuity equation

In this chapter we aim to prove uniqueness for the SCE in the class of Wiener solutions: these are the solutions which are adapted to the Brownian filtration; hence we refer to this kind of uniqueness as Wiener uniqueness. The main tool is Wiener chaos decomposition: this allows to reduce Wiener uniqueness to uniqueness for the associated Fokker-Planck equation.

The result presented here is taken from [MO]. The method is slightly different and inspired to [Mau11].

7.1 Wiener chaos decomposition

We recall here some facts about Wiener chaos decomposition. In the following H is a separable Hilbert space, $(\Omega, \mathcal{A}, P)$ is a probability space (countably generated), W is a d-dimensional Brownian motion on $(\Omega, \mathcal{A}, P)$ and $(\mathcal{F}_t^W)_t$ is the associated Brownian (completed) filtration, i.e. the filtration generated by W completed with the P-null events. [Notice that any function from Ω to H is weakly measurable if and only if it is strongly measurable if and only if it is measurable as a function $(\Omega, \mathcal{F}_T^W, P) \to (H, \mathcal{B}(H))$; the same holds for maps from $[0,T] \times \Omega$ to H and when H is replaced by a separable Banach space, see Remark A.2 in the Appendix.]

For n in $\mathbb{N}$, we call $\Delta_n(T) = \{(t_1, \dots t_n) | 0 \le t_1 \le \dots t_n \le T\}$. For f in $L^2(\Delta_n(T); H)^{nd}$, we define its n-iterated stochastic integral as

$$\int_0^T f(r) \mathrm{d}^n W(r) = \sum_{k_1, \dots k_n = 1}^{d} \int_0^T \dots \int_0^{r_2} f_{k_1, \dots k_n}(r_1, \dots r_n) \mathrm{d}W_{r_1}^{k_1} \dots \mathrm{d}W_{r_n}^{k_n}.$$

The map $f \mapsto \int_0^T f(r)\mathrm{d}^n W(r)$ is an isometry between $L^2(\Delta_n(T); H)^{nd}$, with the norm $\|f\|_{L^2}^2 = \sum_{k_1,\dots k_n} \|f_{k_1,\dots k_n}\|_{L^2}^2$, and $L^2(\Omega, \mathcal{F}_T^W, P; H)$.

Definition 7.1. *For n in $\mathbb{N}$, the n-th Wiener chaos $E_n = E_n(T)$ is the close subspace of $L^2(\Omega, \mathcal{F}_T^W, P; H)$ given by all r.v. Y such that*

$$Y = \int_0^T f(r)\mathrm{d}^n W(r)$$

for some f in $L^2(\Delta_n(T); H)^{nd}$. We call $\Pi_n = \Pi_n(T)$ the orthogonal projector (in $L^2(\Omega, \mathcal{F}_T^W, P; H)$) on E_n.

It is easy to see that the spaces E_n are orthogonal one to each other. The next result, stated for example in [BH91], Chapter 2 Proposition 2.12 (for $H = \mathbb{R}$, extended to general H in Chapter 3), states that this collection is complete.

Theorem 7.2. *Let $(\mathcal{F}_t^W)_t$ the Brownian completed filtration. Then the space $L^2(\Omega, \mathcal{F}_T^W, P; H)$ has the following orthogonal decomposition:*

$$L^2(\Omega, \mathcal{F}_T^W, P; H) = \oplus_{n=0}^\infty E_n.$$

Remark 7.3. *The end time T is not so relevant (and we will omit it when not necessary): indeed, for a r.v. Z adapted to F_t, $t \le T$, $\Pi_n(T)Z = \Pi_n(t)Z$.*

The following shift property is the main tool in the proof of Wiener uniqueness. Here and in the following, we set $E_{-1} = \{0\}$, $\Pi_{-1} \equiv 0$; we extend the definition of Π_n for Y r.v. on H^m, defining $\Pi_n Y$ componentwise.

Lemma 7.4 (Shift property)**.** *Let Y be in $L_{t,\omega}^2(H)^d$ adapted to $(\mathcal{F}_t^W)_t$. Then, for any n in $\mathbb{N}$, it holds*

$$\Pi_n \int_0^T Y_r \mathrm{d}W_r = \int_0^T \Pi_{n-1} Y_r \mathrm{d}W_r.$$

Proof. We must verity that $\int_0^T \Pi_{n-1} Y_r \mathrm{d}W_r$ belongs to E_n and that, for any r.v. F in E_n,

$$E\left[F \int_0^T \Pi_{n-1} Y_r \mathrm{d}W_r\right] = E\left[F \int_0^T Y_r \mathrm{d}W_r\right]. \tag{7.1}$$

The term $\int_0^T \Pi_{n-1} Y_r \mathrm{d}W_r$ belongs to E_n because it is the stochastic integral of an $n-1$-iterated stochastic integral, so it is an n-iterated stochastic integral. As for (7.1), there are two cases. In the case $n = 0$, (7.1) holds because both the LHS and the RHS are zero: for $n = 0$, F is a constant and the stochastic integral has zero mean. In the case $n > 0$, since F belongs to E_n, there exists G in $L^2_t(E_{n-1})^d$ adapted to $(\mathcal{F}^W_t)_t$ such that $F = \int_0^T G_r \mathrm{d}W_r$. So, by Itô isometry and the property of projection (applied to the product $E[G_r \Pi_{n-1} Y_r]$), we have

$$\begin{aligned} E\left[F \int_0^T \Pi_{n-1} Y_r \mathrm{d}W_r\right] &= \int_0^T E\left[G_r \Pi_{n-1} Y_r\right] \mathrm{d}r \\ = \int_0^T E\left[G_r Y_r\right] \mathrm{d}r &= E\left[F \int_0^T Y_r \mathrm{d}W_r\right]. \end{aligned}$$

The lemma is proved. □

Before ending the section, we state some other easy properties that will be useful in the proof of the main result. With a small abuse of notation, we use Π_n without distinguishing the underlying Hilbert space H.

Remark 7.5. *We have*

1. *for any Y in $L^2(\Omega, \mathcal{F}^W_T, P; H)$ and any φ in H, $\Pi_n \langle Y, \varphi \rangle = \langle \Pi_n Y, \varphi \rangle$;*
2. *for any Y in $L^2_{t,\omega}(\mathbb{R})$, adapted to $(\mathcal{F}^W_t)_t$, $\Pi_n \int_0^T Y_r \mathrm{d}r = \int_0^T \Pi_n Y_r \mathrm{d}r$.*

We omit the proof of these two facts, which is similar to the proof of the previous lemma: for example, for the first point, one shows that $\langle \Pi_n Y, \varphi \rangle$ belongs to E_n and that $E[F\langle \Pi_n Y, \varphi \rangle] = E[F\langle Y, \varphi \rangle]$ for any F in E_n.

Finally, we recall that, given a measurable map Y in $L^2_{t,\omega}(H)$, measurable with respect to $\mathcal{B}([0,T]) \times \mathcal{F}^W_T$, then $\Pi_n Y$ is well-defined as a map in $L^2_t(L^2_\omega(H))$ (it is measurable with respect to t, since Π_n is a continuous functional on $L^2_\omega(H)$); by Lemma A.6 in the Appendix, it can be identified with a map in $L^2_{t,\omega}(H)$; we will use implicitly this identification in what follows.

7.2 Wiener uniqueness

Now we state the main result. For simplicity, we set the initial time s to 0.

Theorem 7.6. *Assume that b is in $L^2_t(L^2_{x,loc})$. Suppose that Fokker-Planck equation has uniqueness property in $L^2_t(L^2_x)$, starting from initial datum in a certain vector space V_0. Then the SCE has Wiener uniqueness property in $L^2_t(L^2_x)$, starting from initial datum in V_0.*

The idea of the proof is simple and based on Lemma 7.4. By completeness of Wiener chaos decomposition (here we take $H = L^2_x$), uniqueness for a solution μ to the CE reduces to uniqueness of $\Pi_n\mu$. Projecting the SCE on E_n, Lemma 7.4 gives

$$\mathrm{d}\Pi_n\mu + b\cdot\nabla\Pi_n\mu\mathrm{d}t = \frac{1}{2}\Delta\Pi_n\mu\mathrm{d}t - \nabla\Pi_{n-1}\mu\mathrm{d}W,$$

so $\Pi_n\mu$ satisfies the Fokker-Planck equation with a stochastic source term, driven by $\Pi_{n-1}\mu$. Hence, by induction, uniqueness for $\Pi_n\mu$ reduces to uniqueness for Fokker-Planck equation.

Proof. By linearity of the SCE, it is enough to prove that $\mu_0 = 0$ implies $\mu \equiv 0$ for any Wiener solution μ to the SCE starting from μ_0. Since μ is adapted to the Brownian filtration, by Theorem 7.2 (in the case $H = L^2_x$) it is enough to show that $\Pi_n\mu \equiv 0$ for any n in $\mathbb{N}$. We prove this by induction.

For $n = 0$, projecting the SCE on the 0-th Wiener chaos (i.e. taking expectation), we get, for every φ in $C^\infty_{x,c}$, for a.e. t,

$$\langle\Pi_0\mu_t,\varphi\rangle = \int_0^t \langle\Pi_0\mu_r, b_r\cdot\nabla\varphi\rangle\mathrm{d}r + \int_0^t \langle\Pi_0\mu_r, \frac{1}{2}\Delta\varphi\rangle\mathrm{d}r.$$

So $\Pi_0\mu$ satisfies Fokker-Planck equation and is in $L^2_t(L^2_x)$. Hence, by uniqueness for Fokker-Planck equation, $\Pi_0\mu = 0$.

For the inductive step, from $n-1$ to n, projecting the SCE on the n-th Wiener chaos, by Lemma 7.4 and Remark 7.5 we get, for every φ in $C^\infty_{x,c}$, for a.e. t, for a.e. ω,

$$\begin{aligned}&\langle\Pi_n\mu_t,\varphi\rangle \qquad (7.2)\\ &= \int_0^t \langle\Pi_n\mu_r, b_r\cdot\nabla\varphi\rangle\mathrm{d}r + \int_0^t \langle\Pi_n\mu_r, \frac{1}{2}\Delta\varphi\rangle\mathrm{d}r + \sum_k \int_0^t \langle\Pi_{n-1}\mu_r, \partial_{x_k}\varphi\rangle\mathrm{d}W^k_r.\end{aligned}$$

By the inductive hypothesis, $\Pi_{n-1}\mu = 0$, so the stochastic integral is 0 a.s.: for every φ in $C^\infty_{x,c}$, for a.e. (t,ω), $\langle\Pi_n\mu,\varphi\rangle$ satisfies (7.2) without the stochastic integral, which is the weak formulation of the Fokker-Planck equation.

In order to conclude, we need to get the exceptional set of ω independent of φ. For this, we proceed as usual by a density argument. Let D be a countable set in $C^\infty_{x,c}$, dense in $C^2_{x,c}$ (in the sense that it is dense in C^2_{x,B_R} for every R); let A be a full-measure set in $[0,T]\times\Omega$ where the equality (7.2) (without the stochastic integral) holds for every φ in D and where $\|\Pi_n\mu_t(\omega)\|_{L^2_x}$ and $\|\Pi_n\mu(\omega)\|_{L^2_t(L^2_x)}$ are finite. Then, using the density of D, one can extend the formula (7.2) to every φ in C^∞_c, for all (t,ω) in A. Hence, for a.e. ω, $\Pi_n\mu$ satisfies Fokker-Planck equation and is in $L^2_t(L^2_x)$. So $\Pi_n\mu = 0$ for a.e. ω and the thesis is proved. $\square$

Remark 7.7. *Here we see the reason why, in Definition 5.10 of solution for the Fokker-Planck equation, we have not required the solution to stay in* $L^\infty_{t,\omega}(\mathcal{M}_x)$*: this property may not be preserved by the projections* Π_n*, hence* $\Pi_n\mu$ *may not satisfy it.*

7.3 Extension to weighted Lebesgue spaces

In this section we want to extend our uniqueness result to the space $L^m_{x,\chi}$, for $2 \le m < +\infty$, with weight χ of the form $\chi(x) = \chi_\eta(x) = (1+|x|^2)^{\eta/2}$. The main result of this section is the following:

Theorem 7.8. *Assume that* b *is in* $L^2_t(L^2_{x,loc})$*. Suppose that Fokker-Planck equation has uniqueness property in* $L^m_t(L^m_{x,\chi})$*, starting from initial datum in a certain vector space* V_0*. Then the SCE has Wiener uniqueness property in* $L^m_t(L^m_{x,\chi})$*, starting from initial datum in* V_0*.*

The problem in the proof of this result is that $L^m_{x,\chi}$ is not an Hilbert space in general (it is a separable Banach space), so we cannot apply directly the theory of Wiener chaos taking $H = L^m_{x,\chi}$. Hence we have to adapt the technique to our case.

For this, for $R > 0$, we define Π^R_n as the projector Π_n associated with the Hilbert space $H = L^2_{x,B_R}$. We start with the following properties of Π_n.

Remark 7.9. *1. For any* $+\infty \ge R' > R$*, for any* f *in* $L^2(\Omega, \mathcal{F}^W_T, P; L^2_{x,B_{R'}})$*, we have* $\Pi^{R'}_n f|_{B_R} = \Pi^R_n(f|_{B_R})$*, hence we can omit* R *and define* $\Pi_n f$ *for* f *in* $L^2(\Omega, \mathcal{F}^W_T, P; L^2_{x,loc})$*.*

2. For any f *in* $L^2(\Omega, \mathcal{F}^W_T, P; L^2_{x,loc})$*, for any* ψ *in* $C^\infty(\mathbb{R}^d)$*,* $\Pi_n[\psi f] = \psi \Pi_n f$*.*

The proof of these two facts is similar to the proof of Lemma 7.4.

The key fact for L^m_ω bounds (in ω) is that these are preserved by each projection Π_n (although not uniformly in n): we have the following result, which follows for example from [Maa10], Proposition 3.1.

Proposition 7.10. *Take* $H = \mathbb{R}$*. For every* $2 \le m < +\infty$*, for every* n *in* $\mathbb{N}$*, there exists* $C_{m,n} > 0$ *such that, for every real-valued random variable* Y *in* $L^2(\Omega, \mathcal{F}^W_T, P)$*,*

$$E[|\Pi_n Y|^m] \le C_{m,n} E[|Y|^m].$$

As a consequence of this result, we can extend each projection Π_n to the space $L^m_{x,\chi}$.

Corollary 7.11. *Fix $2 \leq m < +\infty$, n in $\mathbb{N}$. Then Π_n is a bounded operator from $L^m(\Omega, \mathcal{F}_T^W, P; L^m_{x,\chi})$ into itself.*

Proof. By a density argument, it is enough to prove that, for every f in $L^m(\Omega, \mathcal{F}_T^W, P; C_{x,c})$, for some constant $C > 0$ (independent on φ but possibly dependent on n, m), it holds

$$E[\|\Pi_n f\|^m_{L^m_{x,\chi}}] \leq C E[\|f\|^m_{L^m_{x,\chi}}]. \tag{7.3}$$

For fixed x in $\mathbb{R}^d$, the previous Proposition gives

$$E[\chi(x)|\Pi_n f(x)|^m] \leq C_{m,n} E[\chi(x)|f(x)|^m].$$

Integrating in x this inequality, we get (7.3). □

Proof of Theorem 7.8. The proof is as the proof of Theorem 7.6, with Π_n defined on $L^m(\Omega, \mathcal{F}_T^W, P; L^m_{x,\chi})$ as before and replacing the fact that $\Pi_n \mu$ is in $L^2_t(L^2_x)$ a.s. with the fact that $\Pi_n \mu$ is in $L^m_t(L^m_{x,\chi})$ a.s., which is a consequence of the previous Corollary. □

7.4 Application

As an application of Theorem 7.8, we have Wiener uniqueness for the SCE, in the class of drifts 2.7.

Theorem 7.12. *Fix p, q satisfying Condition 2.3 and fix m positive integer, $R > 0$ and $\eta > d(m-1)$, $\chi = \chi_{\eta,R}$. Let $\bar{m}$ be in $[2, +\infty]$ such that $1/2 + 1/2m + 1/\bar{m} \leq 1$ and assume that b is in $L^{\bar{m}}_t(L^{\bar{m}}_{x,loc})$ with compact support in B_R and that $\mathrm{div} b$ is in $L^{q/2}_t(L^{p/2}_x)$. Assume also that u_0 is in L^∞_x with compact support. Then existence and Wiener uniqueness hold for the SCE, in the class $L^{2m}_{t,\omega}(L^{2m}_{x,\chi}) \cap L^\infty_{t,\omega}(\mathcal{M}_x)$, starting from μ_0.*

Proof. Existence has been proved in Theorem 6.3 (with the first hypothesis, replacing m with $2m$). Uniqueness follows from Theorem 5.12, via Theorem 7.8, applied with the weight $\chi = \chi_{\eta,R}$. □

Chapter 8

The pathwise Young argument: the Lagrangian approach

In this Chapter we show regularization by noise for the SDE by a pathwise argument, mainly based on a regularization of the transformed drift $\tilde{b}$ of the associated random ODE (rDE) and on Young estimates. We consider a drift b in $C_t^{0+}(C_{x,b}^{1/2+})$ and we work on the rDE, which we recall here:

$$\mathrm{d}\tilde{X} = \tilde{b}(\tilde{X})\mathrm{d}t. \tag{8.1}$$

We prove a priori Lipschitz estimates on the flow solution to the rDE, for a.e. ω. The argument is in two parts. In the first part we show that, for b in $C_t^{0+}(C_{x,b}^{1/2+})$, $\tilde{b}$ is in in $C_t^{-1/2+}(C_x^{3/2+})$: this means that $\tilde{b}$ has more regularity in space, at the price of losing regularity in time. In the second part we prove that, for $\tilde{b}$ bounded and in $C_t^{-1/2+}(C_x^{3/2+})$, the flow solution has uniform estimates in $W_{x,loc}^{1,\infty}$. Well-posedness follows then by the duality argument.

The content of this Chapter is based on the argument by Catellier and Gubinelli [CG12] for the idea and the second part (although we use linear Young theory on Banach spaces). The first part is proved instead via the Itô-Tanaka trick, see for example [Fla11]. The Itô-Tanaka trick is a martingale-based argument, but there is another proof of the first part (with slightly different assumptions) in [CG12], which does not use martingales or PDEs and can be generalized to more general driving noises (for example fractional Brownian motion).

8.1 The main result

Here is the main well-posedness result, in the class of drifts 2.2 with 1/2+-Hölder regularity in space.

Theorem 8.1. *Assume that b is in $C_t(C^\alpha_{x,b})$ for some $\alpha > 1/2$. Then well-posedness and Lipschitz regularity hold for the SDE, in the sense of Theorem 4.14.*

The proof of this result is at the end of the Chapter.

8.2 Improved space regularity for the drift

In this section we address at the first part of the proof, the improved space regularity of $\tilde{b}$. For any f regular compactly supported function on $\mathbb{R}^d$, we define

$$\tilde{f}(t,x) = f(r, x + W_r).$$

We then call $F : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ the function $F(t,x) = \int_0^t \tilde{f}(r,x)\mathrm{d}r$, so that $F(t,x) - F(s,x) = \int_s^t \tilde{f}(r,x)\mathrm{d}r$.

Theorem 8.2. *Fix $R > 0$, $\alpha > 0$, $\epsilon > 0$ (small enough) and $m \geq 1$ finite. There exists a constant $C > 0$ such that, for every f in $C_t(C^\infty_{x,c})$, for every s, t in $[0,T]$, for every x, y in B_R, it holds*

$$E\left|\int_s^t [\nabla\tilde{f}(r,x) - \nabla\tilde{f}(r,y)]\mathrm{d}r\right|^m \leq C\|f\|^m_{C^\beta_t(C^\alpha_{x,b})}|x-y|^{(\alpha-\epsilon)m}|t-s|^{m/2} \tag{8.2}$$

$$E\left|\int_s^t \tilde{f}(r,x)\mathrm{d}r\right|^m + E\left|\int_s^t \nabla\tilde{f}(r,x)\mathrm{d}r\right|^m \leq C\|f\|^m_{C^\beta_t(C^\alpha_{x,b})}|t-s|^{m/2} \tag{8.3}$$

In particular, we have

$$E\|F_t - F_s\|^m_{W^{1+\alpha-2\epsilon,m}_{x,B_R}} \leq C\|f\|^m_{C^\beta_t(C^\alpha_{x,b})}|t-s|^{m/2}, \tag{8.4}$$

$$E[\|F\|^m_{W^{1/2-\epsilon,m}_t(W^{1+\alpha-2\epsilon,m}_{x,B_R})}] \leq C\|f\|^m_{C^\beta_t(C^\alpha_{x,b})}. \tag{8.5}$$

The proof is based on the estimates on the heat equation in Lemma 5.1 and on the Itô-Tanaka trick. We recall the backward heat equation on $[0,T] \times \mathbb{R}^d$ with source term f:

$$\partial_t v + \frac{1}{2}\Delta v = f \tag{8.6}$$

with final condition $v_T \equiv 0$.

Lemma 8.3 (Itô-Tanaka trick). *For every t in $[0,T]$, for every x in $\mathbb{R}^d$, we have a.s.*

$$\int_0^t \tilde{f}(r,x)\mathrm{d}r = v(t, x+W_t) - v(0,x) - \int_0^t \nabla v(r, x+W_r)\cdot \mathrm{d}W_r \tag{8.7}$$

Proof. By Lemma 5.1, v is in $C^1_t(C_x) \cap C_t(C^2_x)$, so we can apply Itô formula and get

$$\begin{aligned} & v(t, x+W_t) - v(0,x) \\ & = \int_0^t \nabla v(r, x+W_r) \cdot \mathrm{d}W_r + \int_0^t [\partial_t v(r, x+W_r) + \frac{1}{2}\Delta v(r, x+W_r)]\mathrm{d}r. \end{aligned}$$

Since v satisfies the backward heat equation (8.6), the thesis follows immediately. $\square$

The idea of the proof of Theorem 8.2 is the following one. By (8.7), $\int_0^t \tilde{f}(r,x)\mathrm{d}r$ is the sum of three terms which have more regularity in space: the "worse" term (as for regularity), the stochastic integral, is expected to be in $C^{1+\alpha-\epsilon}_x$ in space, since this is the space regularity of ∇u; we pay the price of less regularity in time, since the stochastic integral, even for f time-independent, is no better that $1/2-$-Hölder continuous in time.

Proof of Theorem 8.2. For the first part, we only show (8.2), the proof of (8.3) being similar and easier. By the Itô-Tanaka trick, we write $\int_s^t [\nabla \tilde{f}(r,x) - \nabla \tilde{f}(r,y)]\mathrm{d}r$ in terms of v, namely

$$\begin{aligned} & \int_s^t [\nabla \tilde{f}(r,x) - \nabla \tilde{f}(r,y)]\mathrm{d}r \\ & = \nabla v(t, x+W_t) - \nabla v(t, y+W_t) - \nabla v(s, x+W_s) + \nabla v(s, y+W_s) + \\ & \quad - \int_s^t [D^2 v(r, x+W_r) - D^2 v(r, y+W_r)] \cdot \mathrm{d}W_r \end{aligned}$$

By Burkholder inequality we get

$$\begin{aligned} & E\left|\int_s^t [\nabla \tilde{f}(r,x) - \nabla \tilde{f}(r,y)]\mathrm{d}r\right|^m \\ & \leq CE|\nabla v(t, x+W_t) - \nabla v(t, y+W_t) - \nabla v(s, x+W_s) + \nabla v(s, y+W_s)|^m + \\ & \quad + CE\left|\int_s^t |D^2 v(r, x+W_r) - D^2 v(r, y+W_r)|^2 \mathrm{d}r\right|^{m/2} \end{aligned}$$

We analyze the two addends on the RHS separately. We start from the stochastic integral (which seems the most relevant one). We have

$$E\left|\int_s^t |D^2v(r,x+W_r)-D^2v(r,y+W_r)|^2\mathrm{d}r\right|^{m/2} \tag{8.8}$$
$$\leq E\left|\int_s^t \|D^2v(r,\cdot+W_r)\|^2_{C^{\alpha-\epsilon}_{x,b}}|x-y|^{2(\alpha-\epsilon)}\mathrm{d}r\right|^{m/2}$$
$$\leq \|D^2v\|^m_{C_t(C^{\alpha-\epsilon}_{x,b})}|x-y|^{(\alpha-\epsilon)m}|t-s|^{m/2}.$$

As for the first addend, we write it as

$$\begin{aligned}&E|\nabla v(t,x+W_t)-\nabla v(t,y+W_t)-\nabla v(s,x+W_s)+\nabla v(s,y+W_s)|^m\\ &\leq CE|\nabla v(t,x+W_t)-\nabla v(t,y+W_t)-\nabla v(s,x+W_t)+\nabla v(s,y+W_t)|^m+\\ &+CE|\nabla v(s,x+W_t)-\nabla v(s,y+W_t)-\nabla v(s,x+W_s)+\nabla v(s,y+W_s)|^m.\end{aligned}$$

So we have (we call $\Delta_{x,y}D^2v_s = D^2v(s,x)-D^2v(s,y)$)

$$\begin{aligned}&E|\nabla v(t,x+W_t)-\nabla v(t,y+W_t)-\nabla v(s,x+W_s)+\nabla v(s,y+W_s)|^m\\ &\leq CE\|\nabla v(t,\cdot+W_t)-\nabla v(s,\cdot+W_t)\|^m_{C^{\alpha-\epsilon}_{x,b}}|x-y|^{(\alpha-\epsilon)m}+\\ &+CE\left|\int_0^1 \Delta_{x+W_s+\xi(W_t-W_s),y+W_s+\xi(W_t-W_s)}D^2v_s\mathrm{d}\xi\cdot(W_t-W_s)\right|^m\\ &\leq C\|\nabla v\|^m_{C_t^{1/2}(C^{\alpha-\epsilon}_{x,b})}|x-y|^{(\alpha-\epsilon)m}|t-s|^{m/2}+\\ &+C\|D^2v\|^m_{C_t(C^{\alpha-\epsilon}_{x,b})}|x-y|^{(\alpha-\epsilon)m}|t-s|^{m/2}.\end{aligned} \tag{8.9}$$

Putting together (8.8) and (8.9), we obtain

$$\begin{aligned}&E\left|\int_s^t[\nabla\tilde f(r,x)-\nabla\tilde f(r,y)]\mathrm{d}r\right|^m\\ &\leq c(\|v\|^m_{C_t^{1/2}(C^{1+\alpha-\epsilon}_{x,b})}+\|v\|^m_{C_t(C^{2+\alpha-\epsilon}_{x,b})})|x-y|^{(\alpha-\epsilon)m}|t-s|^{m/2}.\end{aligned}$$

Now Lemma 5.1 gives (8.2).

Now we prove (8.4) and (8.5). Notice that

$$
\begin{aligned}
&\|F_t - F_s\|^m_{W^{1+\alpha-2\epsilon,m}_{x,B_R}} \\
&= \|F_t - F_s\|^m_{L^m_{x,B_R}} + \|DF_t - DF_s\|^m_{L^m_{x,B_R}} + \\
&+ \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|DF(t,x) - DF(s,x) - DF(t,y) + DF(s,y)|^m}{|x-y|^{d+(\alpha-2\epsilon)m}} \mathrm{d}x\mathrm{d}y \\
&= \int_{\mathbb{R}^d} \left| \int_s^t \tilde{f}(r,x)\mathrm{d}r \right|^m \mathrm{d}x + \int_{\mathbb{R}^d} \left| \int_s^t \nabla \tilde{f}(r,x)\mathrm{d}r \right|^m \mathrm{d}x + \\
&+ \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|\int_s^t [D\tilde{f}(r,x) - D\tilde{f}(r,y)]\mathrm{d}r|^m}{|x-y|^{d+(\alpha-2\epsilon)m}} \mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

Hence (8.2) and (8.3) give (8.4), by Hölder inequality, taking into account that $|x-y|^{-d+\epsilon m}$ is an integrable function. The proof of (8.5) is a bit more lengthy but on the same line, we omit it. □

The following Corollary gives the desired improved regularity for $\tilde{f}$.

Corollary 8.4. *Fix $R > 0$, $\alpha > 0$, $\epsilon > 0$ and $m \geq 1$ finite large enough. There exists a constant $C > 0$ such that, for every f in $C_t(C^\alpha_{x,b})$, it holds*

$$
E[\|\tilde{f}\|^m_{C_t^{-1/2+\epsilon}(C^{1+\alpha-6\epsilon}_{x,B_R})}] \leq C\|f\|^m_{C^\beta_t(C^\alpha_{x,b})}
$$

Proof. First step: extension of the bound (8.4) to f in $C_t(C^\alpha_{x,b})$. Fix $s < t$. For f in $C_t(C^\alpha_{x,b})$, we take a sequence $(f^n)_n$ in $C_t(C^\infty_{x,c})$, bounded in $C_t(C^\alpha_{x,b})$, converging to f in $C_t(C^{\alpha'}_{x,B_R})$ for some $\alpha' < \alpha$ and for every $R > 0$; we call $F^n(t,x) = \int_0^t \tilde{f}^n(r,x)\mathrm{d}r$. The bound (8.4) applied to F^n gives that $(F^n_t - F^n_s)_n$ is a bounded sequence in $L^m_\omega(W^{1+\alpha-\epsilon,m}_{x,B_R})$. Now $W^{1+\alpha-\epsilon,m}_{x,B_R}$ is a reflexive space, so, by Proposition A.1 in the Appendix, $L^m_\omega(W^{1+\alpha-\epsilon,m}_{x,B_R})$. Therefore, by Banach-Alaoglu theorem, there exists a subsequence $(F^{n_k})_k$ converging weakly to some element G in $L^m_\omega(W^{1+\alpha-\epsilon,m}_{x,B_R})$. On the other hand $(F^n)_n$ converges to F in $L^\infty_\omega(C_t(C_x))$. Therefore $G = F$ for a.e. ω and, by lower semicontinuity of the $L^m_\omega(W^{1+\alpha-\epsilon,m}_{x,B_R})$ norm, (8.4) holds for F.

Second step: conclusion. Choosing m large enough in the bound (8.5), we can apply Sobolev embedding in space (on B_R), losing ϵ in the space exponent:

$$
E\|F_t - F_s\|^m_{C^{1+\alpha-3\epsilon}_{x,B_R}} \leq C\|f\|^m_{C^\beta_t(C^\alpha_{x,b})}|t-s|^{m/2}
$$

(more precisely, the bound above is satisfied by the space continuous version of F, which however is F). On the other hand, we have the trivial estimate

$$
\|F_t - F_s\|^m_{C^{\alpha-3\epsilon}_{x,B_R}} \leq C\|F_t - F_s\|^m_{C^\alpha_{x,B_R}} \leq C\|f\|^m_{C^\beta_t(C^\alpha_{x,b})}|t-s|^m.
$$

which holds for every ω, for a constant C independent of ω. These two bounds give, by the interpolation inequality (Proposition A.11 in the Appendix), for $0 < \rho < 1$,

$$
\begin{aligned}
&E\|F_t - F_s\|^m_{C^{\alpha-3\epsilon+\rho}_{x,B_R}} \\
&\leq E\left[\|F_t - F_s\|^{(1-\rho)m}_{C^{\alpha-3\epsilon}_{x,B_R}} \|F_t - F_s\|^{\rho m}_{C^{1+\alpha-3\epsilon}_{x,B_R}}\right] \leq C\|f\|^m_{C^\beta_t(C^\alpha_{x,b})}|t-s|^{(2-\rho)m/2}.
\end{aligned}
$$

Now we choose $\rho = 1-3\epsilon$. Again for m large enough, we use Kolmogorov continuity criterion (for $C^\alpha_{x,b}$-valued functions, see for example [Kun97], Chapter 1 Theorem 1.4.1), losing $\epsilon/2$ in the time exponent, and so we get the thesis (again for the $C^\alpha_{x,b}$-valued continuous version of F, which must be F itself by continuity of f). □

8.3 Regularity estimate for the random ODE

We have shown in the previous section the improved space regularity of $\tilde{b}$. Here we go on with the second part of the proof, namely we prove that, for such a vector field $\tilde{b}$, when $\alpha > 1/2$, the flow solving (8.1) is Lipschitz continuous. Notice that this part is completely decoupled from the previous one: whenever one has a vector field $\tilde{b}$ with certain regularity properties, then the flow is Lipschitz continuous. This result can therefore be applied also for other cases, for example when the Brownian motion is replaced by a fractional Brownian motion, provided one proves the required regularity for $\tilde{b}$; this has been done in [CG12].

Given a regular bounded vector field $\tilde{b}$, we use $\tilde{X}^{\tilde{b},s}(t,x)$ to denote the flow solution to the ODE (8.1) starting from s (omitting the superscripts $\tilde{b}$ or s when not necessary).

Theorem 8.5. *Fix $\epsilon > 0$, $R > 0$. There exist $R' > 0$ (locally bounded function of $\|\tilde{b}\|_{L^\infty_t(C_{x,b})}$) and a locally bounded functions $C : [0,+\infty[\to [0,+\infty[$, such that, for every vector field $\tilde{b}$ in $C_t(C^\infty_{x,c})$, for every $s \geq 0$, the corresponding flow $\tilde{X}^{\tilde{b},s}$ satisfies*

$$
\|\tilde{X}^{\tilde{b}}\|_{W^{1,\infty}_t(L^\infty_{x,B_R})} + \|D\tilde{X}^{\tilde{b}}\|_{C^{1/2+\epsilon}_t(L^\infty_{x,B_R})} \leq C(\|\tilde{b}\|_{L^\infty_t(C_{x,b})} + \|\tilde{b}\|_{C^{-1/2+\epsilon}_t(C^{3/2+\epsilon}_{x,B_{R'}})}).
$$

For simplicity, we fix $s = 0$ in the proof: it is easy to see that all the estimates hold for any s, uniformly in s (in $[0,T]$).

The first remark for the proof is that the derivative $D\tilde{X}$ of the flow, which we want to estimate, satisfies the linear equation

$$
\partial_t D\tilde{X} = D\tilde{b}(\tilde{X})D\tilde{X}.
$$

Since $D\tilde{b}$ has C^α_x space regularity but negative order time regularity, one hopes to apply Young integration theory to derive estimates. However $D\tilde{b}$ appears composed with the flow $\tilde{X}$, so we cannot apply the estimates for $D\tilde{b}$ directly (and in fact they do not hold in general for the composition).

The idea of the proof is in two steps. In the first step we analyze the regularity properties of $D\tilde{b}(\tilde{X})$. For this analysis, the following remark is crucial: the composition (with $\tilde{X}_t$) operator S_t, namely $S_t\varphi(x) = \varphi(\tilde{X}(t,x))$, is linear, so $D\tilde{b}_t(\tilde{X}_t) = S_t D\tilde{b}_t$ is a "product" on a Banach space; hence it can be estimated through infinite-dimensional Young integration theory (which comes with no differences from the finite-dimensional theory, see the Appendix, Section A.3). First we give bounds on S_t in terms of the flow $\tilde{X}$, and then give an estimate of $D\tilde{b}(\tilde{X})$ in terms of these bounds on S_t.

In the second step, using Young integration theory, we estimates $D\tilde{X}$ in terms of the regularity of $D\tilde{b}(\tilde{X})$. The first step then allows to conclude.

Notice that some of the partial results we will give have their own interest and may be used in other contexts (see the beginning of the next Chapter).

8.4 Estimates on the composition

We start with an easy Lipschitz in time estimate on $\tilde{X}$.

Lemma 8.6. *It holds*

$$\sup_{0\le s<t\le T} \sup_{x\in\mathbb{R}^d} |\tilde{X}(t,x) - \tilde{X}(s,x)| \le \|\tilde{b}\|_{L^\infty_t(C_{x,b})} |t-s|. \tag{8.10}$$

In particular

$$|\tilde{X}(t,x) - x| \le \|\tilde{b}\|_{L^\infty_t(C_{x,b})} T. \tag{8.11}$$

Proof. The proof is immediate from the expression

$$\tilde{X}(t,x) - \tilde{X}(s,x) = \int_s^t \tilde{b}(r, \tilde{X}(r,x))\mathrm{d}r.$$

□

For the next lemma, we introduce the following (crucial) notation: given a Borel bounded function $\varphi : \mathbb{R}^d \to \mathbb{R}$, we call

$$S_t\varphi(x) = \varphi(\tilde{X}(t,x))$$

For every t, S_t is a linear operator from BB_x to BB_x (where BB_x is the set of Borel bounded functions on $\mathbb{R}^d$). This is relevant because it transforms the composition, a nonlinear operation in x, into a linear operation, at the price of working with the infinite-dimensional space of functions. The next Lemma specifies in which spaces S_t is bounded, and which regularity it has in time.

Lemma 8.7. *Call $R' = R + \|\tilde{b}\|_{L^\infty_t(C_{x,b})}T$. It holds for every $0 < \alpha' < 1$, for every φ in $C^\infty_{x,c}$, for every $s < t$ in $[0, T]$,*

$$\|S_t\varphi - S_s\varphi\|_{L^\infty_{x,B_R}} \leq \|\tilde{b}\|^{\alpha'}_{L^\infty_t(C_{x,b})}\|\varphi\|_{C^{\alpha'}_{x,B_{R'}}}|t-s|^{\alpha'},$$

that is

$$\|S\|_{C^{\alpha'}_t(\mathrm{Lin}(C^{\alpha'}_{x,B_{R'}};L^\infty_{x,B_R}))} \leq 1 + \|\tilde{b}\|^{\alpha'}_{L^\infty_t(C_{x,b})}.$$

Proof. Fix x in B_R. By the previous Lemma, $\tilde{X}(t,x)$ and $\tilde{X}(s,x)$ live in $B_{R'}$, with $R' = R + \|\tilde{b}\|_{L^\infty_t(C_{x,b})}T$. Hence we have

$$\begin{aligned} |S_t\varphi(x) - S_s\varphi(x)| &= |\varphi(\tilde{X}(t,x)) - \varphi(\tilde{X}(s,x))| \\ &\leq \|\varphi\|_{C^{\alpha'}_{x,B_{R'}}}|\tilde{X}(t,x) - \tilde{X}(s,x)|^{\alpha'} \leq \|\varphi\|_{C^{\alpha'}_{x,B_{R'}}}\|\tilde{b}\|^{\alpha'}_{L^\infty_t(C_{x,b})}|t-s|^{\alpha'}, \end{aligned} \tag{8.12}$$

where the last inequality uses the previous Lemma again. This proves the first inequality. The second one follows from the first inequality and from the trivial bound $\|S_t\varphi\|_{L^\infty_{x,B_R}} \leq \|\varphi\|_{L^\infty_{x,B_R}}$ for every t. □

The next lemma deals with the regularity property of $D\tilde{b}(\tilde{X})$. It is the key lemma.

Lemma 8.8. *For any $\alpha' > 0$, $\beta' > 0$ such that $\alpha' + \beta' > 1$, it holds*

$$\|D\tilde{b}(\tilde{X})\|_{C^{\beta'-1}_t(L^\infty_{x,B_R})} \leq C\|\tilde{b}\|_{C^{\beta'-1}_t(C^{1+\alpha'}_{x,B_{R'}})}(1 + \|\tilde{b}\|^{\alpha'}_{L^\infty_t(C_{x,b})}). \tag{8.13}$$

Proof. We recall the following Young inequality in infinite dimension (here we use $\alpha' + \beta' > 1$), a particular case of Theorem A.13 in the Appendix: for any φ in $C_t(C^\infty_{x,c})$,

$$\|S\varphi\|_{C^{\beta'-1}_t(L^\infty_{x,B_R})} \leq C\|\varphi\|_{C^{\beta'-1}_t(C^{\alpha'}_{x,B_{R'}})}\|S\|_{C^{\alpha'}_t(\mathrm{Lin}(C^{\alpha'}_{x,B_{R'}};L^\infty_{x,B_R}))}. \tag{8.14}$$

Applying this inequality to $\varphi = D\tilde{b}$ (precisely, to each of its components) and using the previous Lemma, we get the thesis. □

8.5 Estimates on the flow derivative

Now we can analyze the derivative of the flow $D\tilde{X}$. As already written, this derivative satisfies the linear equation

$$D\tilde{X}_t(x) = I + \int_0^t D\tilde{b}_r(\tilde{X}_r(x))D\tilde{X}_r(x)\mathrm{d}r. \tag{8.15}$$

For x fixed, this equation can be interpreted as a linear Young ODE in $D\tilde{X}_r(x)$. We can therefore use Young integration theory to provide estimates in the uniform (in space) topology.

Lemma 8.9. *For any $\alpha' > 0$, $\beta' > 1/2$ with $\alpha' + \beta' > 1$, we have*

$$\|D\tilde{X}\|_{C_t^{\beta'}(L^\infty_{x,B_R})} \le C\exp[C(1 + A^{1/\beta'})](1 + A) \tag{8.16}$$

where $A = \|\tilde{b}\|_{C_t^{\beta'-1}(C^{1+\alpha'}_{x,B_{R'}})}(1 + \|\tilde{b}\|^{\alpha'}_{L^\infty_t(C_{x,b})})$.

Proof. Fix x in B_R. By Young ODE a priori estimates applied to (8.15) (see Lemma A.14 in the Appendix, here we need $\beta' > 1/2$), we get

$$\begin{aligned} &\|D\tilde{X}(x)\|_{C_t^{\beta'}} \\ &\le C\exp[C(1 + \|D\tilde{b}(\tilde{X}(x))\|^{1/\beta'}_{C_t^{\beta'-1}})](1 + \|D\tilde{b}(\tilde{X}(x))\|_{C_t^{\beta'-1}}), \end{aligned} \tag{8.17}$$

so, taking the supremum over x in B_R,

$$\|D\tilde{X}\|_{C_t^{\beta'}(L^\infty_{x,B_R})} \le C\exp[C(1+\|D\tilde{b}(\tilde{X})\|^{1/\beta'}_{C_t^{\beta'-1}(L^\infty_{x,B_R})})](1+\|D\tilde{b}(\tilde{X})\|_{C_t^{\beta'-1}(L^\infty_{x,B_R})}).$$

Combining this estimate with (8.13), we get the thesis. $\square$

Proof of Theorem 8.5. The thesis follows from (8.16), applied with $\alpha' = 1/2 + \epsilon$, $\beta' = 1/2 + \epsilon$, and from (8.11). $\square$

Remark 8.10. *Notice that all the proof was essentially the consequence of these four inequalities, each of them elementary (assuming Young theory) and independent from the other ones:* (8.10), (8.12), (8.14) *and* (8.17).

We can finally prove our main result.

Proof of Theorem 8.1. The result follows, via Theorem 4.14 and Remark 4.13, from the a priori estimates in 8.5 and from the fact that $\tilde{b}$ is in $C_t^{-1/2+\epsilon}(C^{1+\alpha-6\epsilon}_{x,loc})$ for a.e. ω, by Corollary 8.4. $\square$

Remark 8.11. *After using Young integration theory, a natural question arises: is it possible to go beyond the case* b *in* $C_x^{1/2+}$ *(that is* $D\tilde{b}$ *in* $C_t^{-1/2+}(C_x^{1/2+})$*), via a pathwise approach but exploiting rough paths? Indeed, rough paths theory can be viewed (also) as the extension of Young integration theory beyond the case of* $C_t^{1/2+}$ *integrators. The main point seems that we need an additional regularity property on the transformed drift* $\tilde{b}$ *(in order to have the second-order iterated integral in rough paths theory). This possible extension has to be investigated.*

Chapter 9

The pathwise Young argument: the Eulerian approach

In this Chapter we keep the pathwise point of view but we analyze the linear random PDEs (continuity and transport equations) rather than the ODE, proving uniqueness directly by Young methods. Precisely, we consider again a drift b in $C_t(C^{1/2+}_{x,b})$ (for simplicity we suppose it compactly supported and divergence free) and we work on the random continuity equation (rCE) associated with the random ODE, which we recall here:

$$\partial_t \tilde{u} + \operatorname{div}(\tilde{b}\tilde{u}) = 0, \tag{9.1}$$

with given initial datum $\tilde{u}_0$. We want to get (path-by-path) uniqueness for the rCE, in the class $L^\infty_t(\mathcal{M}_x)$, directly (without using the uniqueness result for the rDE in the previous Chapter). The idea is again to use that $\tilde{b}$, the transformed drift, is a.s. in $C^{-1/2+}_t(C^{3/2+}_x)$ and to apply a suitable adaptation of the duality/renormalization method with Young integration techniques.

Although uniqueness for the rCE itself follows from uniqueness for the rDE, this method has the peculiarity of being only PDE-based and we hope to apply it also to more general vector fields of the form b in $C_t(W^{1/2,p}_x)$ for some finite p (possibly 1) and get uniqueness for the CE in the L^∞ class. This extension would be the analogue of the works by DiPerna and Lyons [DL89] and Ambrosio [Amb04]: they prove, in the deterministic context, well-posedness for the continuity/transport equation beyond the Lipschitz assumptions on the drift, allowing a $W^{1,1}_x$ or even BV drift with bounded divergence.

The argument here is based on [GM]. The proof uses ideas and techniques from the paper by Bailleul and Gubinelli [BG15], which deal with vector fields (rough drivers) coming from rough paths theory. We adapt some ideas to the Young case (we do not need rough paths), but we modify them to take into account both the Young-type assumption b in $C_t^{-1/2+}(C_x^{3/2+})$ and the uniform bound b in $L^\infty_{t,x}$.

9.1 The result and the strategy

The main result of the Chapter is the following:

Theorem 9.1. *Let $\tilde b$ be in $C_t(C_{x,b}) \cap C_t^{\beta-1}(C_{x,b}^{1,\alpha})$ for some $\beta > 1/2$, $\alpha > 1/2$ and assume that* $\mathrm{div} b = 0$*. Then uniqueness holds for the CE 9.1 in the class $L_t^\infty(\mathcal{M}_x)$, starting from any initial measure μ_s and any time s.*

We emphasize that this result is completely deterministic (there is no ω involved), we kept the notation with "tilde" only to remember the application to regularization by noise, but $\tilde b$ does not need to be $b(t, x + W_t)$. We will take $s = 0$ in the following, for simplicity of notation.

As an application of this result, we have path-by-path uniqueness for the SCE:

Corollary 9.2. *Assume that b is in $C_t(C_{x,c}^\alpha)$ and that* $\mathrm{div} b = 0$*. Then path-by-path uniqueness holds for the SCE in the class $L_t^\infty(\mathcal{M}_x)$.*

Proof. By 8.2, the transformed drift $\tilde b$ satisfies the assumptions of the Theorem above, replacing α with $\alpha - \epsilon$: the compact support assumption guarantees that the local $C_t^{\beta-1}(C_x^{1,\alpha})$ bound can be extended globally. Hence the thesis follows from the Theorem above applied to the rCE. □

In the following, we change the notation in this way: we call T (instead of t) the final time involved in the duality method, while $s < t$ denote two times in $[0, T]$ (the initial time is fixed to 0).

Here is the idea of the proof of the main result. We consider the commutator appearing in Lemma 4.8 (using the notation $\mu(x)\mathrm{d}x$ for $\mu(\mathrm{d}x)$):

$$\left| \int_0^T \langle \tilde\mu \otimes \tilde v, (\tilde b_x \cdot \nabla_x \rho_\epsilon^{(2)} + \mathrm{div}_y(\tilde b_y \rho_\epsilon^{(2)}))(\chi_R)_x \rangle \mathrm{d}r \right|,$$

which reads (at least formally), with the divergence-free condition of $\tilde b$,

$$\begin{aligned} &\left| \int_0^T \langle \tilde\mu(x)\tilde v(y), (\tilde b(x) - \tilde b(y)) \cdot \nabla \rho_\epsilon(x-y)\chi_R(x) \rangle \mathrm{d}r \right| \\ &= \left| \int_0^1 \int_0^T \langle \tilde\mu(x)\tilde v(y), Db(y + \xi(x-y))(x-y) \cdot \nabla\rho_\epsilon(x-y)\chi_R(x) \rangle \mathrm{d}r\mathrm{d}\xi \right|. \end{aligned}$$

Since $(x-y)D\rho_\epsilon(x-y)$ has derivatives which explode in 0 when $\epsilon \to 0$, it is convenient to make the change of variable $z = (y-x)/\epsilon$ and remove the singularity from the ρ:

$$\left| \int_0^1 \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tilde{\mu}(x)\tilde{v}(x+\epsilon z) Db(x+\epsilon z) z \cdot \nabla\rho(-z)\chi_R(x) \mathrm{d}x\mathrm{d}z\mathrm{d}r\mathrm{d}\xi \right|.$$

From this formula, we would like to bound the commutator using the regularity assumption on b: Db is in $C_t^{\beta-1}(C_x^\alpha)$, which invites to exploit Young integration. To do so, heuristically, we need $\tilde{\mu}(x)\tilde{v}(x+\epsilon z)$ to stay in $C_t^\beta((C_x^\alpha)^*)$ or in $C_t^\beta(U^*)$, where U is some suitable Banach space containing $Db(x+\epsilon z)$. Hence we need a priori estimates on the negative Sobolev norm of $\tilde{\mu}(x)\tilde{v}(x+\epsilon z)$.

For this, we start with the simple estimate, for every φ in $C_{x,c}^\infty$, for every $s < t$.

$$|\langle \tilde{\mu}_t - \tilde{\mu}_s, \varphi\rangle| \le \int_s^t |\tilde{b}||\nabla\varphi|\mathrm{d}|\tilde{\mu}| \le |t-s| \|\tilde{b}\|_{C_{t,x}} \|\varphi\|_{C^1_{x,c}} \|\tilde{\mu}\|_{L_t^\infty(\mathcal{M}_x)},$$

which implies that $\tilde{\mu}$ is in $C_t^1((C_{x,c}^1)^*)$. Interpolating this result with the bound in $L_t^\infty(\mathcal{M}_x) = L_t^\infty(C_{x,c}^*)$, we get (up to details) $\tilde{\mu}$ in $C_t^{1/2+}((C_{x,c}^{1/2+})^*)$. Unfortunately, when we consider $\tilde{\mu}(x)\tilde{v}(x+\epsilon z)$, such a simple reasoning does not bring any useful estimate: the strategy of interpolating after estimating does not work we can no more get a $C_t^1((C_{x,c}^1)^*)$ estimate. Hence we have to estimate the $C_t^{1/2+}((C_{x,c}^{1/2+})^*)$ norm, or the $C_t^{1/2+}(U^*)$ for a suitable U, directly. For this, we use the idea proposed by Bailleul and Gubinelli [BG15], where, in some sense, we interpolate before estimating, bringing the interpolation on the test function. Proceeding in this way, we get that $\tilde{\mu}(x)\tilde{v}(x+\epsilon z)$ is "almost" in $C_t^\beta(U^*)$, where U is morally $L_z^1(W_x^{\alpha',\infty}) \cap L_x^\infty(W_z^{1,1})$ ("almost" because actually $\tilde{\mu}(x)\tilde{v}^\delta(x+\epsilon z)$ is in $C_t^\beta(U^*)$, see below). Notice that $Db(x+\epsilon z)$ "almost" belongs to this space, in the sense that $Db(x)$ is in the space. Forgetting about the "almost" problems, we could conclude.

The true strategy actually is based not directly on duality but on approximate duality: we replace $\tilde{v}$ by $\tilde{v}^\delta$ solution to the backward transport equation driven by the vector field $\tilde{b}^\delta$; this $\tilde{b}^\delta$ obtained by convolution of b with the usual mollifier ρ_δ, where δ is to be chosen. The reason for this approximated duality is to deal with the z dependence of $Db(x+\epsilon z)$: this is problematic because this field is not differentiable in z, but on the other hand we would like to take advantage of the ϵ factor in front of z. One may introduce definitions of approximating duality pairing and other details, but here we just adapt the strategy of duality to our context. First we get an equation for $\tilde{\mu}(x)\tilde{v}(x+\epsilon z)$ and we use this equation to get uniform (in δ and ϵ) $C_t^{1/2+}(U^*)$ bounds on $\tilde{\mu}(x)\tilde{v}(x+\epsilon z)$, provided we tune δ accordingly to ϵ. Finally we use these bounds as in the commutator Lemma 4.10, to show that the approximated commutator is infinitesimal in ϵ and conclude.

9.2 First estimates

In the following, ρ is a nonnegative even function in $C^\infty_{x,c}$, normalized to have $\|\rho\|_{L^1_x}=1$, and, for $\epsilon>0$, we define $\rho_\epsilon(x)=\epsilon^{-d}\rho(\epsilon^{-1}x)$ (the usual mollifier); for a function f on $\mathbb{R}^d$, we define $f^\rho=f*\rho$, and similarly for a measure ν.

Let $\tilde{\mu}$ be a solution of the CE. Let $\tilde{\varphi}$ be a function in $C^\infty_{x,c}$ and, for any $\delta>0$, let $\tilde{v}^\delta$ be the solution to the backward TE driven by $\tilde{b}^\delta=\tilde{b}*\rho_\delta$, with final time T and final datum $\tilde{\varphi}$:

$$\partial_s\tilde{v}^\delta+\tilde{b}^\delta\cdot\nabla\tilde{v}^\delta=0.$$

For both $\tilde{\mu}$ and $\tilde{v}$, we take the time weakly-* continuous versions.

First we get an equation for $\tilde{\mu}(x)\tilde{v}^\delta(x+\epsilon z)$, for ϵ, δ fixed (where, with some abuse of notation, we use $\tilde{\mu}(x)\mathrm{d}x$ for $\tilde{\mu}(\mathrm{d}x)$). Notice that $\tilde{\mu}(x)\tilde{v}^\delta(x+\epsilon z)$ is an element of $L^\infty_t(\mathcal{M}_{x,z,B_R})$ (for every $R>0$) and, for every t, we have $\langle\tilde{\mu}_t(x)\tilde{v}_t(x+\epsilon z),\psi(x,z)\rangle=\langle\tilde{\mu}_t(x)\tilde{v}_t(y),\epsilon^{-d}\psi(x,\epsilon^{-1}(y-x))\rangle$ for every ψ in $C^\infty_{x,z,c}$. Since $\tilde{v}^\delta$ is continuous in space (at every time), for every time t we can repeat the computations at the beginning of Section 4.3, replacing $\tilde{v}$ with $\tilde{v}^\delta$, and get, for every ψ in $C^\infty_{x,z,c}$,

$$\langle\tilde{\mu}_t\otimes\tilde{v}_t,\psi\rangle-\langle\tilde{\mu}_0\otimes\tilde{v}_0,\psi\rangle=\int_0^t\langle\mu\otimes\tilde{v},\tilde{b}_x\cdot\nabla_x\psi+\tilde{b}^\delta_y\cdot\nabla_y\psi\rangle\mathrm{d}r+\int_0^t\langle\tilde{\mu}\otimes v,\mathrm{div}\tilde{b}^\delta_y\psi\rangle\mathrm{d}r.$$

Replacing ψ with $\epsilon^{-d}\psi(x,\epsilon^{-1}(y-x))$, using the change of variable $z=(y-x)/\epsilon$ and the condition $\mathrm{div}b=0$ (which stays true for $\tilde{b}^\delta$), we arrive at the equation

$$\begin{aligned}
&\langle\tilde{\mu}_t(x)\tilde{v}_t(x+\epsilon z)-\tilde{\mu}_0(x)\tilde{v}_0(x+\epsilon z),\psi(x,z)\rangle \\
&=\int_0^t\langle\tilde{\mu}(x)\tilde{v}^\delta(x+\epsilon z),\tilde{b}(x)\cdot\nabla_x\psi(x,z)\rangle\mathrm{d}r+ \\
&\quad-\int_0^t\langle\tilde{\mu}(x)\tilde{v}^\delta(x+\epsilon z),\frac{1}{\epsilon}(\tilde{b}^\delta(x+\epsilon z)-\tilde{b}(x))\cdot\nabla_z\psi(x,z)\rangle\mathrm{d}r.
\end{aligned}\tag{9.2}$$

Now we introduce the space where we want $\tilde{\mu}(x)\tilde{v}(x+\epsilon z)$ to be Hölder continuous in time: this is the space U^* dual of the space U. Here U is defined as the closure of the functions in $C^\infty_{x,z,c}$ under the norm

$$\|\psi\|_U:=\|\psi\|_{W^{\alpha',\infty}_x(L^1_z)}+\|\psi\|_{L^\infty_x(W^{1,1}_z)},$$

for some $1/2 < \alpha' < \alpha$.

From the formula above, it seems relevant to control $\frac{1}{\epsilon}(\tilde{b}^\delta(x+\epsilon z) - \tilde{b}(x)) \cdot \nabla_z \psi(x,z)$ in the U norm. Hence we aim now at such an estimate.

The following interpolation bounds will be crucial. We recall that ρ is a non-negative even $C^\infty_{x,c}$ function, $\rho_\delta(x) = \delta^{-d}\rho(\delta^{-1}x)$ for $\delta > 0$ and, for f measurable locally integrable function, $f^\eta = f * \rho_\delta$. We also recall the following easy but relevant property of the approximate identities: for every k in $\mathbb{N}$, for every f in $L^1_{x,loc}$, $D^k(f^\epsilon) = \epsilon^{-k} f * (D^k\rho)_\epsilon$.

Lemma 9.3. *Fix* $0 \leq \gamma \leq 1$. *We have, for every* $\delta > 0$, *for every* f *in* $W^{\gamma,\infty}_x$, *for every* g *in* $W^{1+\gamma,\infty}_x$,

$$\begin{aligned}
&\|\nabla f^\delta\|_{L^\infty_x} \leq C\delta^{-(1-\gamma)}\|f\|_{W^{\gamma,\infty}_x},\\
&\|f - f^\delta\|_{L^\infty_x} \leq C\delta^\gamma \|f\|_{W^{\gamma,\infty}_x},\\
&\|g - g^\delta\|_{W^\gamma_x(L^1_z)} \leq C\delta\|g\|_{W^{1+\gamma,\infty}_x}.
\end{aligned}$$

Proof. We take the continuous versions of f and g. For the first statement, notice that

$$\nabla f * \rho^\delta(x) = \int_{\mathbb{R}^d} (f(x) - f(x'))\nabla\rho_\delta(x - x')\mathrm{d}x',$$

because $\int_{\mathbb{R}^d} f(x)\nabla\rho(x - x')\mathrm{d}x' = 0$. Hence

$$\begin{aligned}
&\|\nabla f^\delta\|_{L^\infty_x} = \sup_x \left| \int_{\mathbb{R}^d} (f(x) - f(x'))\nabla\rho_\delta(x - x')\mathrm{d}x' \right|\\
&\leq \sup_x \int_{\mathbb{R}^d} |f(x) - f(x')||\nabla\rho_\delta(x - x')|\mathrm{d}x'\\
&= \delta^{-1} \sup_x \int_{\mathbb{R}^d} |f(x - \delta y) - f(x)||\nabla\rho(y)|\mathrm{d}y\\
&\leq \delta^{-1}\delta^\gamma \|f\|_{W^{\gamma,\infty}_x} \int_{\mathbb{R}^d} |y|^\gamma |\nabla\rho(y)|\mathrm{d}y.
\end{aligned}$$

This proves the first statement.

For the second statement, we have similarly

$$\begin{aligned}
&\|f - f^\delta\|_{L^\infty_x} = \sup_x \left| \int_{\mathbb{R}^d} (f(x) - f(x'))\rho_\delta(x - x')\mathrm{d}x' \right|\\
&\leq \sup_x \int_{\mathbb{R}^d} |f(x) - f(x')|\rho_\delta(x - x')\mathrm{d}x'\\
&= \sup_x \int_{\mathbb{R}^d} |f(x - \delta y) - f(x)|\rho(y)\mathrm{d}y\\
&\leq \delta^\gamma \|f\|_{W^{\gamma,\infty}_x}.
\end{aligned}$$

For the third statement, we have, for every x, y,

$$
\begin{aligned}
&|g(x) - g^\delta(x) - g(y) + g^\delta(y)| \\
&= \left| \int_{\mathbb{R}^d} [g(x) - g(x - \delta w) - g(y) + g(y - \delta w)] \rho_\delta(w) \mathrm{d}w \right| \\
&= \left| \int_{\mathbb{R}^d} \int_0^1 [\nabla g(x - \xi\delta w) - \nabla g(y - \xi\delta w)] \cdot \delta w \mathrm{d}\xi \rho(w) \mathrm{d}w \right| \\
&\leq \delta |x - y|^\gamma \|\nabla g\|_{W^{\gamma,\infty}_x} \int_{\mathbb{R}^d} |w| \rho(w) \mathrm{d}w.
\end{aligned}
$$

From this the third statement follows. The proof is complete. □

Now we can prove the following estimate on $(\tilde{b}^\delta(x + \epsilon z) - \tilde{b}(x))/\epsilon$:

Lemma 9.4. *For every $\beta > 1/2$, $\alpha > 1/2$ (and actually for every positive β, α), there exists a constant $C > 0$ such that, for every b in $C^{\beta-1}_t(C^\alpha_{x,b})$,*

$$
\begin{aligned}
&\left\| \int_s^t \frac{1}{\epsilon} (\tilde{b}^\delta(x + \epsilon z) - \tilde{b}(x)) \mathrm{d}r \right\|_{W^{\alpha',\infty}_x(L^\infty_z) \cap L^\infty_x(W^{1,\infty}_z)} \\
&\leq C(1 + \epsilon\delta^{-(1-\alpha)} + \epsilon^{-1}\delta) \|\tilde{b}\|_{C^{\beta-1}_t(C^{1,\alpha}_{x,b})} |t - s|^\beta.
\end{aligned}
$$

Proof. We split the term with $\tilde{b}$ into two terms, which will be estimates separately:

$$
\frac{1}{\epsilon}(\tilde{b}^\delta(x + \epsilon z) - \tilde{b}(x)) = \frac{1}{\epsilon}(\tilde{b}^\delta(x + \epsilon z) - \tilde{b}^\delta(x)) + \frac{1}{\epsilon}(\tilde{b}^\delta(x) - \tilde{b}(x)).
$$

First term. We estimate the term with $(\tilde{b}^\delta(x + \epsilon z) - \tilde{b}^\delta(x))/\epsilon$, which we write as

$$
\int_s^t \frac{1}{\epsilon} (\tilde{b}^\delta(x + \epsilon z) - \tilde{b}(x)) \mathrm{d}r = \int_0^1 D \int_s^t \tilde{b}^\delta(x + \epsilon\xi z) z \mathrm{d}r \mathrm{d}\xi. \tag{9.3}
$$

We start with the estimate in the $W^{\alpha',\infty}_x(L^\infty_z)$ norm. Since $D\tilde{b}$ is in $C^{\beta-1}_t(C^\alpha_{x,b})$, we have

$$
\sup_{x \neq y} \sup_z \frac{\left| D \int_s^t \tilde{b}^\delta(x + \epsilon\xi z) - D \int_s^t \tilde{b}^\delta(x + \epsilon\xi z) \right|}{|x - y|^\alpha} \leq \|D\tilde{b}\|_{C^{\beta-1}_t(C^\alpha_{x,b})} |t - s|
$$

and so

$$
\left\| \int_0^1 D \int_s^t \tilde{b}^\delta(x + \epsilon\xi z) z \mathrm{d}r \mathrm{d}\xi \right\|_{W^{\alpha',\infty}_x(L^\infty_z)} \leq \|D\tilde{b}\|_{C^{\beta-1}_t(C^\alpha_{x,b})} |t - s|^\beta.
$$

Now we estimate the $L^\infty_x(W^{1,\infty}_z)$ norm, which reduces to the $L^\infty_{x,z}$ norm for the derivative of (9.3) in the z direction (since the control of the $L^\infty_{x,z}$ norm of (9.3) contained in the $L^\infty_z(W^{\alpha',\infty}_x)$ bound). Here we see the usefulness of the approximation with b^δ. Indeed, since $D\tilde{b}$ is in $C^{\beta-1}_t(C^\alpha_{x,b})$, by the interpolation Lemma (9.3) we have

$$\|D^2\int_s^t \tilde{b}^\delta \mathrm{d}r\|_{L^\infty_x} \le C\delta^{-(1-\alpha)}\|D\int_s^t \tilde{b}\mathrm{d}r\|_{W^{\alpha,\infty}_x} \le \delta^{-(1-\alpha)}\|D\tilde{b}\|_{C^{\beta-1}_t(C^\alpha_x)}|t-s|^\beta$$

and so

$$\begin{aligned}&\left\|D_z\int_0^1 D\int_s^t \tilde{b}^\delta(x+\epsilon\xi z)z\mathrm{d}r\mathrm{d}\xi\right\|_{L^\infty_{x,z}} = \left\|\int_0^1 \epsilon\xi D^2\int_s^t \tilde{b}^\delta(x+\epsilon\xi z)\mathrm{d}r\mathrm{d}\xi\right\|_{L^\infty_{x,z}}\\ &\le C\epsilon\delta^{-(1-\alpha)}|t-s|^\beta.\end{aligned}$$

Without δ, we could have not differentiated this term in the z direction. This concludes the bound for the term with $(\tilde{b}^\delta(x+\epsilon z)-\tilde{b}^\delta(x))/\epsilon$.

Second term. We estimate the term with $(\tilde{b}^\delta(x)-\tilde{b}(x))/\epsilon$. In this case it is enough to bound the $W^{\alpha',\infty}_x$ norm, the term being independent of z. Again by the interpolation Lemma 9.3, we have

$$\begin{aligned}&\left\|\int_s^t \frac{1}{\epsilon}(\tilde{b}^\delta(x)-\tilde{b}(x))\mathrm{d}r\right\|_{W^{\alpha',\infty}_x} \\ &\le C\epsilon^{-1}\delta\|\int_s^t \tilde{b}\mathrm{d}r\|_{W^{1+\alpha,\infty}_x} \le \epsilon^{-1}\delta\|D\tilde{b}\|_{C^{\beta-1}_t(C^\alpha_x)}|t-s|^\beta.\end{aligned} \tag{9.4}$$

This proves the bound for this term. Putting together the two bounds, we find the thesis. $\square$

Corollary 9.5. *For every $\beta>1/2$, $\alpha>1/2$ (and actually for every positive β, α), there exists a constant $C>0$ such that, for every b in $C^{\beta-1}_t(C^\alpha_{x,b})$, for every ψ in U, it holds*

$$\begin{aligned}&\left\|\int_s^t \frac{1}{\epsilon}(\tilde{b}^\delta(x+\epsilon z)-\tilde{b}(x))\cdot\nabla_z\psi\mathrm{d}r\right\|_U\\ &\le C(1+\epsilon\delta^{-(1-\alpha)}+\epsilon^{-1}\delta)\|b\|_{C^{\beta-1}_t(C^{1,\alpha}_{x,b})}\|\nabla_z\psi\|_U|t-s|^\beta.\end{aligned}$$

In the proof we use the following bound: for any functions f in $C^\infty_{x,z}$, g in $C^\infty_{x,z,c}$, it holds

$$\|fg\|_U \le 4\|f\|_{W^{\alpha',\infty}_x(L^\infty_z)\cap L^\infty_x(W^{1,\infty}_z)}\|g\|_U. \tag{9.5}$$

This follows from the following inequalities, of easy proof:

$$\|fg\|_{W_x^{\alpha',\infty}(L_z^1)} = \sup_{x\neq y} \frac{\|f(x)g(x) - f(y)g(y)\|_{L_z^1}}{|x-y|^\alpha}$$
$$\leq \|f\|_{W_x^{\alpha',\infty}(L_z^\infty)}\|g\|_{L_x^\infty(L_z^1)} + \|f\|_{L_x^\infty(L_z^\infty)}\|g\|_{W_x^{\alpha',\infty}(L_z^1)},$$
$$\|fg\|_{L_x^\infty(W_z^{1,1})} \leq \|\|f\|_{W_z^{1,\infty}}\|g\|_{L_z^1}\|_{L_x^\infty} + \|\|f\|_{L_z^\infty}\|g\|_{W_z^{1,1}}\|_{L_x^\infty}$$
$$\leq \|f\|_{L_x^\infty(W_z^{1,\infty})}\|g\|_{L_x^\infty(L_z^1)} + \|f\|_{L_x^\infty(L_z^\infty)}\|g\|_{L_x^\infty(W_z^{1,1})}.$$

The bound (9.5) can be extended to all functions f and g with f in the closure of $C^\infty_{x,z}$ functions with respect to the $W_x^{\alpha',\infty}(L_z^\infty)\cap L_x^\infty(W_z^{1,\infty})$ norm and g in U.

Proof. First, notice that $C^\alpha_{x,b}$ is contained in the closure of C^∞_x with respect to the $W_x^{\alpha',\infty}$ norm, since $\alpha' < \alpha$ (see Remark A.10 in the Appendix). So the function $(x,z)\mapsto \int_s^t \tilde b(x)\mathrm{d}r$ is in the closure of $C^\infty_{x,z}$ with respect to the $W_x^{\alpha',\infty}(L_z^\infty)\cap L_x^\infty(W_z^{1,\infty})$ norm. Moreover, since $\tilde b$ is globally bounded, $(x,z)\mapsto \int_s^t \tilde b^\delta(x+\epsilon z)\mathrm{d}r$ is regular with bounded derivatives of all orders; hence (again by Remark A.10) it is also in the closure of $C^\infty_{x,z}$ with respect to the $W^{1,\infty}_{x,z}$ norm and in particular it is in the closure of $C^\infty_{x,z}$ with respect to the $W_x^{\alpha',\infty}(L_z^\infty)\cap L_x^\infty(W_z^{1,\infty})$ norm. Hence we can apply the previous inequality to $f(x,z) = f^{\tilde b}(x,z) = \epsilon^{-1}\int_s^t(\tilde b^\delta(x+\epsilon z) - \tilde b(x))\mathrm{d}r$ and $g(x,z) = \nabla_z\psi(x,z)$ and get

$$\left\|\int_s^t \frac{1}{\epsilon}(\tilde b^\delta(x+\epsilon z) - \tilde b(x))\cdot\nabla_z\psi\mathrm{d}r\right\|_U \leq 4\|f^{\tilde b}\|_{W_x^{\alpha',\infty}(L_z^\infty)\cap L_x^\infty(W_z^{1,\infty})}\|\nabla_z\psi\|_U.$$

We conclude by Lemma 9.4. □

For this reason, we choose from now on $\delta = \epsilon^{(2-\alpha)/(2-2\alpha)}$; in fact any choice between ϵ and $\epsilon^{1/(1-\alpha)}$ works as well, apart for $\delta=\epsilon$ (since later we will need $\epsilon^{-1}\delta\to 0$).

9.3 Estimates for the duality pair

Here we give estimates on the approximated duality pair, in the space $C^\beta_t(U^*)$, using the equation (9.2).

Proposition 9.6. *Assume that* $1/2<\beta<\alpha'<\alpha$. *There exists* $C>0$ *such that*

$$\|\mu(x)v^\delta(x+\epsilon z)\|_{C^\beta_t(U^*)} \leq C.$$

Remark 9.7. *For every $\delta > 0$, $\epsilon > 0$, the norm above is finite. Indeed one can show that $\|\mu\|_{C_t^\beta(U^*)}$ is finite, by reasoning for example as in Section 9.1 (we omit some details, which are in the line of the arguments below); this implies that $\|\mu(x)v^\delta(x+\epsilon z)\|_{C_t^\beta(U^*)}$ is finite since v^δ is regular.*

Let ρ be a non-negative even $C^\infty_{x,c}$ function; for $\eta = (\eta_x, \eta_z)$ with $\eta_x, \eta_z > 0$, define the anisotropic mollifier $\bar\rho_\eta(x,z) = \eta_x^{-d}\eta_z^{-d}\rho(\eta_x^{-1}x)\rho(\eta_z^{-1}z)$. Define $\psi^\eta = \psi * \bar\rho_\eta$ (the convolution being on the (x,z) variable). The following bounds are similar in spirit to the interpolation Lemma 9.3.

Lemma 9.8. *We have*

$$\|\nabla_x\psi^\eta\|_{L^\infty_x(L^1_z)} \le C\eta_x^{-(1-\alpha')}\|\psi\|_U, \tag{9.6}$$

$$\|\nabla_z\psi^\eta\|_U \le C\eta_z^{-1}\|\psi\|_U, \tag{9.7}$$

$$\|\psi-\psi^\eta\|_{L^\infty_x(L^1_z)} \le C(\eta_x^{\alpha'}+\eta_z)\|\psi\|_U. \tag{9.8}$$

Proof. For the first bound, we adapt the proof of Lemma 9.3. Recall that, for each (x,z),

$$\nabla_x\psi * \bar\rho^\eta(x,z) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(\psi(x',z')-\psi(x,z'))\nabla_x\bar\rho_\eta(x-x',z-z')\mathrm{d}x'\mathrm{d}z',$$

because $\int_{\mathbb{R}^d}\psi(x,z')\nabla_x\bar\rho(x-x',z-z')\mathrm{d}x' = 0$. Hence

$$\begin{aligned}
&\|\nabla_x\psi^\eta\|_{L^\infty_x(L^1_z)}\\
&= \sup_x\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(\psi(x',z')-\psi(x,z'))\nabla_x\bar\rho_\eta(x-x',z-z')\mathrm{d}x'\mathrm{d}z'\right|\mathrm{d}z\\
&\le \sup_x\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|\psi(x',z')-\psi(x,z')||\nabla\rho_{\eta_x}(x-x')|\rho_{\eta_z}(z-z')\mathrm{d}z'\mathrm{d}z\mathrm{d}x'\\
&\le \sup_x\int_{\mathbb{R}^d}\|\psi(x',\cdot)-\psi(x,\cdot)\|_{L^1_z}|\nabla\rho_{\eta_x}(x-x')|\mathrm{d}x'\\
&= \eta_x^{-1}\sup_x\int_{\mathbb{R}^d}\|\psi(x-\eta_x y,\cdot)-\psi(x,\cdot)\|_{L^1_z}|\nabla\rho(y)|\mathrm{d}y\\
&\le \eta_x^{-1}\eta_x^{\alpha'}\|\psi\|_{W^{\alpha',\infty}_x(L^1_z)}\int_{\mathbb{R}^d}|y|^{\alpha'}|\nabla\rho(y)|\mathrm{d}y,
\end{aligned}$$

where, in the second inequality, we used Young inequality for the convolution in z. The first bound is proved.

The second bound follows from the fact that $\nabla_z\psi^\eta = \eta_z^{-1}\psi * (\nabla_z\bar\rho)_\eta$ (where $(\nabla_z\bar\rho)_\eta = \eta_x^{-d}\eta_z^{-d}\rho(\eta_x^{-1}x)\nabla\rho(\eta_z^{-1}z)$) and that $\|\psi*(\nabla_z\bar\rho)_\eta\|_U \le C\|\psi\|_U$ ($(\nabla_z\rho)_{\eta_z}$ being bounded in L^1_z).

For the third bound, we write $\psi - \psi^{\eta} = (\psi - \psi^{(\eta_x,0)}) + (\psi^{(\eta_x,0)} - \psi^{\eta})$, where $\psi^{(\eta_x,0)}$ is obtained convolving ψ with ρ_{η_x} only in the x variable. For the first addend, we have, proceeding as before,

$$
\begin{aligned}
&\|\psi - \psi^{(\eta_x,0)}\|_{L^\infty_x(L^1_z)} = \sup_x \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (\psi(x',z) - \psi(x,z))\rho_{\eta_x}(x-x')\mathrm{d}x' \right| \mathrm{d}z \\
&\le \eta_x^{\alpha'} \|\psi\|_{W^{\alpha',\infty}_x(L^1_z)} \int_{\mathbb{R}^d} |y|^{\alpha'} \rho(y)\mathrm{d}y.
\end{aligned}
$$

For the second addend, we have

$$
\begin{aligned}
&\|\psi^{(\eta_x,0)} - \psi^{\eta}\|_{L^\infty_x(L^1_z)} = \sup_x \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} [\psi^{(\eta_x,0)}(x,z) - \psi^{(\eta_x,0)})(x,z')]\rho_{\eta_z}(z')\mathrm{d}z' \right| \mathrm{d}z \\
&\le C \sup_x \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\psi^{(\eta_x,0)}(x,z) - \psi^{(\eta_x,0)})(x, z - \epsilon_z y)| \rho(y)\mathrm{d}y\mathrm{d}z \\
&\le C\eta_z \sup_x \int_0^1 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\nabla_z \psi^{(\eta_x,0)}(x, z - \xi\epsilon_z y)|\mathrm{d}z|y|\rho(y)\mathrm{d}y\mathrm{d}\xi \\
&\le C\eta_z \|\nabla_z \psi\|_{L^\infty_x(L^1_z)} \int_{\mathbb{R}^d} |y|\rho(y)\mathrm{d}y,
\end{aligned}
$$

where we have used that $\|\nabla_z \psi^{(\eta_x,0)}\|_{L^\infty_x(L^1_z)} \le \|\nabla_z \psi\|_{L^\infty_x(L^1_z)}$. Putting together this two estimates, we find the third bound. The proof is complete. $\square$

Given the previous Lemma, the idea of the proof is as follows. We take $\Delta_{s,t} := \tilde{\mu}_t(x)\tilde{v}_t(x+\epsilon z) - \tilde{\mu}_s(x)\tilde{v}_s(x+\epsilon z)$ and test it against a function in U (we want a Hölder bound in time). We split the test function $\psi = (\psi - \psi^{\eta}) + \psi^{\eta}$. When testing $\Delta_{s,t}$ against $\psi - \psi^{\eta}$, we only use the $L^\infty_x(L^1_z)$ bound before and gain a factor $\eta_x^{\alpha'} + \eta_z$ (with no Hölder type term like $|t-s|^{\gamma}$). When testing $\Delta_{s,t}$ against ψ^{η}, we use the equation (9.2) with Young integration bounds and get factors like $\eta_x^{-(1-\alpha')}|t-s|^{\gamma_x}$, $\eta_z^{-1}|t-s|^{\gamma_z}$. At this point we tune η_x, η_y proportional to a suitable power of $|t-s|$ and obtain the desired Hölder bound in time.

Proof of Proposition 9.6. We have to estimate

$$
\begin{aligned}
&\|\tilde{\mu}_t(x)\tilde{v}^\delta_t(x+\epsilon z) - \tilde{\mu}_s(x)\tilde{v}^\delta_s(x+\epsilon z)\|_{U^*} \\
&= \sup_{\psi \in C^\infty_{x,z,c}, \|\psi\|_U \le 1} |\langle \tilde{\mu}_t(x)\tilde{v}^\delta_t(x+\epsilon z) - \tilde{\mu}_s(x)\tilde{v}^\delta_s(x+\epsilon z), \psi \rangle|.
\end{aligned}
$$

For this, fix $s<t$ and ψ as above. We split $\langle \mu_t(x)v^\delta_t(x+\epsilon z) - \mu_s(x)v^\delta_s(x+$

$\epsilon z), \psi\rangle$ in

$$\begin{aligned}&\langle\mu_t(x)v_t^\delta(x+\epsilon z)-\mu_s(x)v_s^\delta(x+\epsilon z),\psi\rangle\\&=\langle\mu_t(x)v_t^\delta(x+\epsilon z)-\mu_s(x)v_s^\delta(x+\epsilon z),\psi-\psi^\eta\rangle+\\&\ +\langle\mu_t(x)v_t^\delta(x+\epsilon z)-\mu_s(x)v_s^\delta(x+\epsilon z),\psi^\eta\rangle=:I+II.\end{aligned}$$

The η is fixed and will be determined later (dependently on $t-s$).

For the term I with $\psi-\psi^\eta$, by (9.8) we get, via Hölder inequality (in x first and in z then),

$$\begin{aligned}|I|&\le 2\sup_t\|\tilde{\mu}(x)\|_{\mathcal{M}_x}\|\langle\tilde{v}^\delta(x+\epsilon z),\psi-\psi^\eta\rangle_z\|_{C_{x,b}}\\&\le 2\sup_t\|\tilde{\mu}(x)\|_{\mathcal{M}_x}\|\tilde{v}^\delta(x+\epsilon z)\|_{L^\infty_x(L^\infty_z)}\|\psi-\psi^\eta\|_{L^\infty_x(L^1_z)}\\&\le C(\eta_x^{\alpha'}+\eta_z)\|\tilde{\mu}\|_{L^\infty_t(\mathcal{M}_x)}\|\tilde{v}^\delta\|_{L^\infty_{t,x}}\|\psi\|_U\end{aligned}$$

(notice that $x\mapsto\langle\tilde{v}^\delta(x+\epsilon z),\psi-\psi^\eta\rangle_z$ is continuous, so its L^∞_x norm coincides with the $C_{x,b}$ norm; similarly for other maps below).

For the term II with ψ^η, we use equation (9.2): we have

$$\begin{aligned}II=&\int_s^t\langle\tilde{\mu}(x)\tilde{v}^\delta(x+\epsilon z),\tilde{b}(x)\cdot\nabla_x\psi^\eta(x,z)\rangle\mathrm{d}r+\\&-\int_s^t\langle\tilde{\mu}(x)\tilde{v}^\delta(x+\epsilon z),\frac{1}{\epsilon}(\tilde{b}^\delta(x+\epsilon z)-\tilde{b}(x))\cdot\nabla_z\psi^\eta(x,z)\rangle\mathrm{d}r=:II.1+II.2.\end{aligned}$$

We estimate the addend $II.1$ by (9.6) and the fact that b is in $C_t(C_{x,b})$, obtaining, via Hölder inequality (in x first and in z and t then),

$$\begin{aligned}|II.1|&\le\int_s^t\|\tilde{\mu}(x)\|_{\mathcal{M}_x}\|\langle\tilde{v}^\delta(x+\epsilon z),\tilde{b}(x)\cdot\nabla_x\psi^\eta(x,z)\rangle_z\|_{C_{x,b}}\mathrm{d}r\\&\le|t-s|\|\tilde{\mu}\|_{L^\infty_t(\mathcal{M}_x)}\|\tilde{v}^\delta\|_{L^\infty_{t,x}}\|\tilde{b}\|_{C_t(C_{x,b})}\|\nabla_z\psi^\eta\|_{L^\infty_x(L^1_z)}\\&\le C\eta_x^{-(1-\alpha')}|t-s|\|\tilde{\mu}\|_{L^\infty_t(\mathcal{M}_x)}\|\tilde{v}^\delta\|_{L^\infty_{t,x}}\|\tilde{b}\|_{L^\infty_t(C_{x,b})}\|\psi\|_U.\end{aligned}$$

We use Young estimates (on Banach spaces, Section A.3 in the Appendix) for the addend $II.2$: by Corollary 9.5 and (9.7) we have

$$\begin{aligned}&\left|II.2-\langle\tilde{\mu}_s(x)\tilde{v}_s^\delta(x+\epsilon z),\frac{1}{\epsilon}\int_s^t(\tilde{b}^\delta(x+\epsilon z)-\tilde{b}(x))\cdot\nabla_z\psi^\eta(x,z)\mathrm{d}r\rangle\right|\\&\le C|t-s|^{2\beta}\|\tilde{\mu}(x)\tilde{v}^\delta(x+\epsilon z)\|_{U^*}\left\|\frac{1}{\epsilon}\int_s^t(\tilde{b}^\delta(x+\epsilon z)-\tilde{b}(x))\cdot\nabla_z\psi^\eta(x,z)\right\|_U\\&\le C\eta_z^{-1}|t-s|^{2\beta}\|\tilde{\mu}(x)\tilde{v}(x+\epsilon z)\|_{C_t^\beta(U^*)}\|\tilde{b}\|_{C_t^{\beta-1}(C^{1,\alpha}_{x,b})}\|\psi\|_U.\end{aligned}$$

Moreover, using (9.3) and (9.4), we get, via Hölder inequality,

$$\begin{aligned}
&\left|\langle \tilde\mu_s(x)\tilde v^\delta_s(x+\epsilon z), \frac{1}{\epsilon}\int_s^t (\tilde b^\delta(x+\epsilon z)-\tilde b(x))\cdot\nabla_z\psi^\eta(x,z)\mathrm{d}r\rangle\right|\\
&\le \|\tilde\mu_s\|_{\mathcal{M}_x}\left\|\langle \tilde v^\delta_s(x+\epsilon z), \frac{1}{\epsilon}\int_s^t (\tilde b^\delta(x+\epsilon z)-\tilde b(x))\mathrm{d}r\cdot\nabla_z\psi^\eta(x,z)\rangle_z\right\|_{C_{x,b}}\\
&\le \|\tilde\mu_s\|_{\mathcal{M}_x}\|\tilde v^\delta_s\|_{L^\infty_x}\left\|\frac{1}{\epsilon}\int_s^t (\tilde b^\delta(x+\epsilon z)-\tilde b(x))\mathrm{d}r\right\|_{L^\infty_x(L^\infty_z)}\|\nabla_z\psi^\eta(x,z)\|_{L^\infty_x(L^1_z)}\\
&\le C|t-s|^\beta\|\tilde\mu\|_{L^\infty_t(\mathcal{M}_x)}\|\tilde v^\delta\|_{L^\infty_{t,x}}\|\tilde b\|_{C^{\beta-1}_t(C^{1,\alpha}_{x,b})}\|\psi\|_U.
\end{aligned}$$

So we obtain

$$\begin{aligned}
&|II.2|\\
&\le C|t-s|^\beta\|\tilde\mu\|_{L^\infty_t(\mathcal{M}_x)}\|\tilde v^\delta\|_{L^\infty_{t,x}}\|\tilde b\|_{C^{\beta-1}_t(C^{1,\alpha}_{x,b})}\|\psi\|_U+\\
&+C\eta_z^{-1}|t-s|^{2\beta}\|\tilde\mu(x)\tilde v(x+\epsilon z)\|_{C^\beta_t(U^*)}\|\tilde b\|_{C^{\beta-1}_t(C^{1,\alpha}_{x,b})}\|\psi\|_U.
\end{aligned}$$

Putting all together, we end with

$$\begin{aligned}
&|\langle\tilde\mu_t(x)\tilde v^\delta_t(x+\epsilon z)-\tilde\mu_s(x)\tilde v^\delta_s(x+\epsilon z),\psi\rangle|\\
&\le C(\eta_x^{\alpha'}+\eta_z)\|\tilde\mu\|_{L^\infty_t(\mathcal{M}_x)}\|\tilde v^\delta\|_{L^\infty_{t,x}}\|\psi\|_U+\\
&+C\eta_x^{-(1-\alpha')}|t-s|\|\tilde\mu\|_{L^\infty_t(\mathcal{M}_x)}\|\tilde v^\delta\|_{L^\infty_{t,x}}\|\tilde b\|_{L^\infty_t(C_{x,b})}\|\psi\|_U+\\
&+C|t-s|^\beta\|\tilde\mu\|_{L^\infty_t(\mathcal{M}_x)}\|\tilde v^\delta\|_{L^\infty_{t,x}}\|\tilde b\|_{C^{\beta-1}_t(C^{1,\alpha}_{x,b})}\|\psi\|_U+\\
&+C\eta_z^{-1}|t-s|^{2\beta}\|\tilde\mu(x)\tilde v(x+\epsilon z)\|_{C^\beta_t(U^*)}\|\tilde b\|_{C^{\beta-1}_t(C^{1,\alpha}_{x,b})}\|\psi\|_U.
\end{aligned}$$

Since we want to control the $\|\mu(x)v(x+\epsilon z)\|_{C^\beta_t(V^*_R)}$ norm, we choose now η such that $\eta_z=\eta_x^{\alpha'}=c|t-s|^\beta$, where $c>0$ is a constant to be fixed later. The condition $\beta<\alpha'$ ensures that $\eta_x^{-(1-\alpha')}|t-s|\le c^{-(1-\alpha')/\alpha'}C|t-s|^\beta$ (for every s,t in $[0,T]$). Hence the previous estimate becomes

$$\begin{aligned}
&|\langle\tilde\mu_t(x)\tilde v^\delta_t(x+\epsilon z)-\tilde\mu_s(x)\tilde v^\delta_s(x+\epsilon z),\psi\rangle|\\
&\le C|t-s|^\beta\|\tilde\mu\|_{L^\infty_t(\mathcal{M}_x)}\|\tilde v^\delta\|_{L^\infty_{t,x}}(c^{-1}+c^{-(1-\alpha')/\alpha'}\|\tilde b\|_{L^\infty_t(C_{x,b})}+\|\tilde b\|_{C^{\beta-1}_t(C^{1,\alpha}_{x,b})})\|\psi\|_U\\
&+cC|t-s|^\beta\|\tilde\mu(x)\tilde v(x+\epsilon z)\|_{C^\beta_t(U^*)}\|\tilde b\|_{C^{\beta-1}_t(C^{1,\alpha}_{x,b})}\|\psi\|_U.
\end{aligned}$$

Taking the supremum over ψ with $\|\psi\|_U \leq 1$ and over $s < t$, we get

$$\begin{aligned}
&\|\tilde{\mu}(x)\tilde{v}(x+\epsilon z)\|_{C_t^\beta(U^*)} \leq C\|\tilde{\mu}_0\|_{\mathcal{M}_x}\|\tilde{v}^\delta\|_{L^\infty_{t,x}} + \\
&\quad + C\|\tilde{\mu}\|_{L_t^\infty(\mathcal{M}_x)}\|\tilde{v}^\delta\|_{L^\infty_{t,x}}(c^{-1} + c^{-(1-\alpha')/\alpha'}\|\tilde{b}\|_{L_t^\infty(C_{x,b})} + \|\tilde{b}\|_{C_t^{\beta-1}(C^{1,\alpha}_{x,b})})\|+ \\
&\quad + cC\|\tilde{\mu}(x)\tilde{v}(x+\epsilon z)\|_{C_t^\beta(U^*)}\|\tilde{b}\|_{C_t^{\beta-1}(C^{1,\alpha}_{x,b})},
\end{aligned}$$

where $C\|\tilde{\mu}_0\|_{\mathcal{M}_x}\|\tilde{v}^\delta\|_{L^\infty_{t,x}}$ takes into account the U^* norm of $\tilde{\mu}_0\tilde{v}^\delta_0$ (which enters the $C_t^\beta(U^*)$ norm).

Finally we choose $c > 0$ so that $cC\|\tilde{b}\|_{C_t^{\beta-1}(C^{1,\alpha}_{x,b})} \leq 1/2$ (i.e. c^{-1} proportional to $\|\tilde{b}\|_{C_t^{\beta-1}(C^{1,\alpha}_{x,b})}$). With this choice (and since $\|\tilde{\mu}(x)\tilde{v}(x+\epsilon z)\|_{C_t^\beta(U^*)}$ is finite) we get

$$\begin{aligned}
&\|\tilde{\mu}(x)\tilde{v}(x+\epsilon z)\|_{C_t^\beta(U^*)} \\
&\leq C\|\tilde{\mu}_0\|_{\mathcal{M}_x}\|\tilde{v}^\delta\|_{L^\infty_{t,x}} + C\|\tilde{\mu}\|_{L_t^\infty(\mathcal{M}_x)}\|\tilde{v}^\delta\|_{L^\infty_{t,x}} \cdot \\
&\quad \cdot (\|\tilde{b}\|_{C_t^{\beta-1}(C^{1,\alpha}_{x,b})} + \|\tilde{b}\|^{(1-\alpha')/\alpha'}_{C_t^{\beta-1}(C^{1,\alpha}_{x,b})}\|\tilde{b}\|_{L_t^\infty(C_{x,b})} + \|\tilde{b}\|^2_{C_t^{\beta-1}(C^{1,\alpha}_{x,b})}).
\end{aligned}$$

The proof is complete. □

Remark 9.9. *In the proof we used Young estimates for a Lebesgue integral, the term $II.2$. More in general, given ν in $L_t^\infty(\mathcal{M}_{x,z,B_R}) \cap C_t^\beta(U^*)$, g in $L_t^\infty(C_{x,z}) \cap C_t^{\beta-1}(U)$ with support on $[0,T] \times B_R$, an integral of the form*

$$\int_0^t \langle \nu, g\rangle \mathrm{d}r,$$

can be interpreted both as Lebesgue integral (when the duality product is between $\mathcal{M}_{x,z,B_R}$ and $C_{x,z}$) and as Young integral (when the duality product is between U^* and U). The two integrals coincide, so we can use any tool from Lebesgue and Young integrals. Indeed, when g is in $C_t^1(U)$, then the Young integral reduces to a Riemann integral and therefore coincides with the Lebesgue integral. For a general g in $L_t^\infty(C_{x,z}) \cap C_t^{\beta-1}(U)$, both the Lebesgue integral and the Young integral can be approximated by $\int_0^t \langle \nu, g^n\rangle \mathrm{d}r$, where $(g^n)_n$ is a sequence in $L_t^\infty(C_{x,z}) \cap C_t^1(U)$ converging to g in $L_t^1(C_{x,z}) \cap C_t^{\beta'-1}(U)$, for any $\beta' < \beta$ (for example, $g^n(t) = \fint_{t-1/n}^{t+1/n} g(r)\mathrm{d}r$).

9.4 The commutator lemma

Now we take $\psi(x,z) = \rho(z)\chi_R(x)$ as test function for equation (9.2), where ρ is a nonnegative even function in $C^\infty_{x,c}$ (for simplicity, the ρ considered before) and, for $R > 0$, χ_R is a smooth function, with $0 \le \chi_R \le 1$ and $|\nabla \chi_R| \le 2/R$, equal to 1 on B_R and to 0 on B_{2R}^c. As in Section 4.3, we find a commutator. We show now that this commutator is infinitesimal in ϵ.

Lemma 9.10. *We have, as $\epsilon \to 0$,*

$$\int_0^T \langle \tilde{\mu}(x)\tilde{v}^\delta(x+\epsilon z), \frac{1}{\epsilon}(\tilde{b}^\delta(x+\epsilon z) - \tilde{b}(x)) \cdot \nabla\rho(z)\chi_R(x)\rangle \mathrm{d}r \to 0.$$

Proof. In view of Young integration theory, we split the integrals on intervals $[t_j, t_{j+1}]$, for $t_j = jh \wedge T$ $(t_N = T)$, $h > 0$:

$$\int_0^T \langle \tilde{\mu}(x)\tilde{v}^\delta(x+\epsilon z), \frac{1}{\epsilon}(\tilde{b}^\delta(x+\epsilon z) - \tilde{b}(x)) \cdot \nabla\rho(z)\chi_R(x)\rangle \mathrm{d}r = \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \dots.$$

For each integral, we use Corollary 9.5 and, by Young integration, we get

$$\begin{aligned}
&\left| \int_{t_j}^{t_{j+1}} \langle \tilde{\mu}(x)\tilde{v}^\delta(x+\epsilon z), \frac{1}{\epsilon}(\tilde{b}^\delta(x+\epsilon z) - \tilde{b}(x)) \cdot \nabla\rho(z)\chi_R(x)\rangle \mathrm{d}r \right| \qquad (9.9)\\
&\le \left| \langle \tilde{\mu}_{t_j}(x)\tilde{v}_{t_j}^\delta(x+\epsilon z), \frac{1}{\epsilon}\int_{t_j}^{t_{j+1}} (\tilde{b}^\delta(x+\epsilon z) - \tilde{b}(x)) \cdot \nabla\rho(z)\chi_R(x)\mathrm{d}r\rangle \right|\\
&\quad + Ch^{2\beta}\|\tilde{\mu}(x)v(x+\epsilon z)\|_{C_t^\beta(U^*)}\|\tilde{b}\|_{C_t^{\beta-1}(C_{x,b}^{1,\alpha})}.
\end{aligned}$$

Now we let $\epsilon \to 0$ (and so $\delta = \epsilon^{(2-\alpha)/(2-2\alpha)} \to 0$), keeping $h > 0$ fixed. We start by showing that the first addend in the RHS above tends to 0. For this, we first prove the convergence of $\int_s^t \frac{1}{\epsilon}(\tilde{b}^\delta(x+\epsilon z) - \tilde{b}(x))\mathrm{d}r$ to $D\int_s^t \tilde{b}(x)\mathrm{d}r$. We split

$$\begin{aligned}
&\left| \int_s^t \frac{1}{\epsilon}(\tilde{b}^\delta(x+\epsilon z) - \tilde{b}(x))\mathrm{d}r - D\int_s^t \tilde{b}(x)\mathrm{d}r \right|\\
&\le \left| \frac{1}{\epsilon}(\tilde{b}^\delta(x+\epsilon z) - \tilde{b}^\delta(x)) - D\int_s^t \tilde{b}^\delta(x)\mathrm{d}r\mathrm{d}\xi \right| +\\
&\quad + \left| D\int_s^t \tilde{b}^\delta(x)\mathrm{d}r - D\int_s^t \tilde{b}(x)\mathrm{d}r \right| + \left| \int_s^t \frac{1}{\epsilon}(\tilde{b}^\delta(x) - \tilde{b}(x))\mathrm{d}r \right|\\
&= \left| \int_0^1 \left(D\int_s^t \tilde{b}^\delta(x+\epsilon\xi z)\mathrm{d}r - D\int_s^t \tilde{b}^\delta(x)\mathrm{d}r \right) \mathrm{d}\xi \right| +\\
&\quad + \left| D\int_s^t \tilde{b}^\delta(x)\mathrm{d}r - D\int_s^t \tilde{b}(x)\mathrm{d}r \right| + \left| \int_s^t \frac{1}{\epsilon}(\tilde{b}^\delta(x) - \tilde{b}(x))\mathrm{d}r \right|
\end{aligned}$$

Using this splitting, the Hölder continuity properties of $\tilde{b}$ and the interpolation Lemma 9.3, we get, for every (x, z),

$$\left| \int_s^t \frac{1}{\epsilon}(\tilde{b}^\delta(x+\epsilon z) - \tilde{b}(x))\mathrm{d}r - D\int_s^t \tilde{b}(x)\mathrm{d}r \right| \le C|t-s|^\beta(\epsilon^\alpha + \delta^\alpha + \epsilon^{-1}\delta).$$

By our choice of δ, $\epsilon^{-1}\delta \to 0$. Hence, for ϵ, δ going to 0, we get

$$\left\| \int_s^t \frac{1}{\epsilon}(\tilde{b}^\delta(x+\epsilon z) - \tilde{b}(x))\mathrm{d}r - D\int_s^t \tilde{b}(x)\mathrm{d}r \right\|_{L^\infty_{x,z}} \to 0.$$

Putting this into the first addend of the RHS of (9.9), by Hölder inequality (using the $L^\infty_t(\mathcal{M}_x)$ norm of μ and the uniform $L^\infty_{t,x}$ bound of v^δ), we obtain the convergence

$$\begin{aligned}
&\langle \tilde{\mu}_{t_j}(x)\tilde{v}^\delta_{t_j}(x+\epsilon z), \left(\frac{1}{\epsilon}\int_{t_j}^{t_{j+1}} (\tilde{b}^\delta(x+\epsilon z) - \tilde{b}(x))\mathrm{d}r\right) \cdot \nabla\rho(z)\chi_R(x)\rangle \\
&\to \langle \tilde{\mu}_{t_j}(x)\tilde{v}^\delta_{t_j}(x+\epsilon z), \left(D\int_{t_j}^{t_{j+1}} \tilde{b}(x)\mathrm{d}r\right) z\cdot\nabla\rho(z)\chi_R(x)\rangle.
\end{aligned}$$

Since b is divergence-free, $D\int_{t_j}^{t_{j+1}} \tilde{b}(x)\mathrm{d}r z\cdot\nabla\rho(z) = 0$ and so the first addend of the RHS in (9.9) is infinitesimal, as desired.

Therefore for every $h > 0$ we have, first passing to the limit $\epsilon \to 0$ in (9.9) and then summing over j,

$$\begin{aligned}
&\limsup_{\epsilon\to 0} \left| \int_{t_j}^{t_{j+1}} \langle \tilde{\mu}(x)\tilde{v}^\delta(x+\epsilon z), \frac{1}{\epsilon}(\tilde{b}^\delta(x+\epsilon z) - \tilde{b}(x))\nabla\rho(z)\chi_R(x)\rangle \mathrm{d}r \right| \\
&\le Ch^{-1}h^{2\beta}\limsup_{\epsilon\to 0} \|\tilde{\mu}(x)\tilde{v}(x+\epsilon z)\|_{C^\beta_t(U^*)}\|\tilde{b}\|_{C^{\beta-1}_t(C^{1,\alpha}_{x,b})}.
\end{aligned}$$

Now the RHS is infinitesimal in h, because $\beta > 1/2$. Hence, by arbitrariness of h, we conclude that the LHS is 0. The proof is complete. □

We are ready to prove uniqueness from approximated duality.

Proof of Theorem 9.1. Equation (9.2), applied to the CE with $\tilde{\mu}_0 \equiv 0$, with test function $\psi(x, z) = \rho(z)\chi_R(x)$, gives that

$$\begin{aligned}
&\langle \tilde{\mu}_T(x)\tilde{\varphi}(x+\epsilon z), \rho(z)\chi_R(x)\rangle \\
&= -\int_0^T \langle \tilde{\mu}(x)\tilde{v}^\delta(x+\epsilon z), \frac{1}{\epsilon}(\tilde{b}^\delta(x+\epsilon z) - \tilde{b}(x))\nabla\rho(z)\chi_R(x)\rangle \mathrm{d}r+ \\
&\quad + \int_0^T \langle \tilde{\mu}(x)\tilde{v}^\delta(x+\epsilon z), \tilde{b}(x)\rho(z)\cdot\nabla\chi_R(x)\rangle \mathrm{d}r.
\end{aligned}$$

We let $\epsilon \to 0$: using the commutator lemma 9.10 (and the uniform $L^\infty_{t,x}$ bound on v^δ in terms of the final datum φ), we get

$$|\langle \tilde{\mu}_T, \tilde{\varphi}\chi_R\rangle| \le \limsup_{\epsilon\to 0} \left| \int_0^T \langle \tilde{\mu}(x)\tilde{v}^\delta(x+\epsilon z), \tilde{b}(x)\rho(z)\cdot\nabla\chi_R(x)\rangle \mathrm{d}r \right|.$$

Since $|\nabla\chi_R(x)| \le 8/(1+|x|)1_{R\le|x|\le 2R}$, we get

$$|\langle \tilde{\mu}_T, \tilde{\varphi}\chi_R\rangle| \le C\frac{1}{R}\|\tilde{\mu}\|_{L^\infty_t(\mathcal{M}_x)}\|\tilde{\varphi}\|_{L^\infty_x}\|\tilde{b}\|_{C_t(C_x)}.$$

Letting $R \to +\infty$, we finally obtain $\langle \tilde{\mu}_T, \tilde{\varphi}\rangle = 0$. By the arbitrariness of T and $\tilde{\varphi}$, we conclude $\mu \equiv 0$. The proof of uniqueness is complete. $\square$

Chapter 10

The Girsanov argument

In this chapter we consider again the random ODE

$$\mathrm{d}\tilde{X} = \tilde{b}(\tilde{X})\mathrm{d}t,$$

but under weaker regularity assumption on b that $C_x^{1/2+}$, using more the stochastic nature. The starting point is the following remark: by the bound (8.17) in Chapter 9, a bound on the $C_t^{1/2+}(L_x^\infty)$ norm of $D\tilde{b}(\tilde{X})$ implies a bound on the Lipschitz norm of $D\tilde{X}$. Now it may happen that $D\tilde{b}(\tilde{X})$ is bounded in $C_t^{1/2+}(L_x^\infty)$, even when $\tilde{b}$ is not regular enough to apply Young integration techniques for the composition $D\tilde{b}(\tilde{X})$.

This is the case, for example, when we have Girsanov theorem. Indeed, for fixed x, this theorem implies that $X(x)$ and $x+W$ are equivalent in law, so that $D\tilde{b}(\tilde{X}(x)) = Db(X(x))$ and $D\tilde{b}(x) = Db(x+W)$ are equivalent as well. Hence, roughly speaking, the bounds on $D\tilde{b}(X)$ can be reduced to those on $D\tilde{b}$.

Unfortunately this is not true at the level of L_x^∞ bounds, since the equivalence above holds only for the law at x fixed. However, we can still work with L^m bounds for m finite.

The content of this Chapter is inspired again by the work [CG12] and also by the approach by Proske and coauthors ([MPMBN$^+$13] for example, see also [Rez14]).

10.1 The result

Here is the main result, in the class of drifts 2.3:

Theorem 10.1. *Assume that b is in the class 2.3 (and satisfies Condition 2.1). Fix $2 \le m < +\infty$, $2 \le \tilde{m} < +\infty$ such that $1/m + 1/\tilde{m} + 1/(p \wedge q) \le 1$.*

Then existence, path-by-path uniqueness and stability hold in the sense of Theorem 4.12.

The proof of this result is at the end of the Chapter.

10.2 Digression on exponentials

In what follows, we will often meet solutions A to the matrix-valued differential equation

$$\partial_t A = M(Y)A$$

where M is a certain matrix field and Y is a stochastic process, usually the solution an SDE. The most important example is given by the space derivative DX of a solution X to the SDE

$$dX = b(X)dt + dW,$$

in this case $M = Db$ and $Y = X$. Another important example, in view of Girsanov transform, is the case when Y is a Brownian motion. Notice that, in dimension one, A ha the explicit form $A_t = A_0 \exp[\int_0^t M_r(Y_r)\mathrm{d}r]$, that is why we speak of exponentials.

There are (at least) two ways to deal with these objects. The first way consists of writing the explicit formula for A, namely

$$\begin{aligned} A_t & \\ &= A_0 + \sum_{n=1}^{\infty} \int_0^t \int_0^{t_1} \dots \int_0^{t_{n-1}} M_{t_1}(Y_{t_1}) M_{t_2}(Y_{t_2}) \dots M_{t_n}(Y_{t_n}) \mathrm{d}t_n \dots \mathrm{d}t_2 \mathrm{d}t_1, \end{aligned} \tag{10.1}$$

and to estimate every term of the sum. This approach requires to have estimates on the marginal laws of Y, it can be lengthy but does not require a priori a martingale structure. This is the method followed by Proske and coauthors, in [MPMBN$^+$13] and other papers, and extended in [Rez14] to the case of b in the class 2.3.

The second way is to write an SPDE for A and to use a priori estimates for that SPDE. This is somehow a quicker method, but requires the martingale structure. Precisely, we take the solution $Y = Y(s,t,x) = Y_{s,t}(x)$ to the following SDE on $\mathbb{R}^d$

$$\mathrm{d}Y = B(Y)\mathrm{d}t + \mathrm{d}W, \quad Y_s = x,$$

and we assume that $A = A(s,t,x) = A_{s,t}(x)$ satisfies the linear SDE on $\mathbb{R}^{n \times n}$

$$\mathrm{d}A = M(Y)A\mathrm{d}t, \quad A_s = I,$$

where $B : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$, $M : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{n\times n}$ are in $C_t(C^\infty_{x,c})$ (and W is a d-dimensional Brownian motion as usual). Then we have the following representation result:

Proposition 10.2. *Fix t in $[0,T]$ and U_t in $(C^\infty_{x,c})^n$. Then*

$$U(s,x) = A(s,t,x)^* U_t(Y(s,t,x))$$

is in $C_s(C^2_{x,loc})$ and satisfies the following linear backward SPDE:

$$\partial_s U + (B\cdot\nabla)U + M^* U + \sum_k \partial_{x_k} U \circ \dot{W}^k = 0. \tag{10.2}$$

Proof. For a vector v in $\mathbb{R}^n$, call $V(x,v) = A(x)v$. This process solves the same SDE of A, but with initial datum v. Now the link (3.2) between SDEs and backward STEs (rigorously, Theorem 3.18, which can be extended also to the case of general $C_t(C^4_{x,lin})$ drifts and diffusion coefficients) applies to the couple (Y,V) and gives that $\bar{U}(s,x,v) = U(s,x)\cdot v = U_t(Y(s,t,x))\cdot V(s,t,x,v)$ satisfies the following backward STE on $[0,t]\times\mathbb{R}^d\times\mathbb{R}^n \ni (s,x,v)$:

$$\partial_s \bar{U}(x,v) + B(x)\cdot\nabla_x \bar{U}(x,v) + M(x)v\cdot\nabla_v \bar{U}(x,v) + \nabla_x \bar{U}(x,v)\circ\dot{W} = 0.$$

But we also know that $\bar{U}$ is linear in the v component, precisely $\bar{U} = U\cdot v$. Hence the product $(M(x)v\cdot\nabla_v)\bar{U}(x,v)$ is actually $U(x)\cdot M(x)v$. Hence the equation above reads

$$\partial_s U(x)\cdot v + (B(x)\cdot\nabla_x)U(x)\cdot v + M(x)^* U(x)\cdot v + \nabla_x(U(x)\cdot v)\circ\dot{W} = 0.$$

Since this is true for every vector v, we get the thesis. $\square$

Remark 10.3. *The key idea in this Proposition is to interpret the term M^*U, roughly speaking, as a transport-type term of the form $Mv\cdot\nabla_v[U\cdot v]$, where v is an additional free variable, so that the SPDE is a transport equation with respect to (x,v).*

10.3 Application of Girsanov transform

In this Section we prove the Sobolev (in space) a priori estimates on the flow.

Theorem 10.4. *Fix p, q satisfying Condition 2.3, fix $R > 0$ and an even integer $m \geq 2$. Then there exists a locally bounded function $C : [0,+\infty[\to [0,+\infty[$ such that, for any vector field b in $C_t(C^\infty_{x,c})$, for every $s \geq 0$, it holds*

$$\sup_{t\in[s,T]} \int_{B_R} E\left[|X(t,x)|^m + |DX(t,x)|^m\right] \mathrm{d}x \leq C(\|b\|_{L^q_t(L^p_x)}).$$

For simplicity, we fix $s = 0$ in the following: it is easy to see that all the estimates hold for any s, uniformly in s (in $[0, T]$).

The following two lemmata provide the key estimates. The first one is related to Novikov condition.

Lemma 10.5. *Fix $R > 0$ and an even integer $m \geq 2$. Then it holds*

$$\sup_{t\in[0,T]} \int_{B_R} E \exp\left[m \int_s^t |b(r, x + W_r - W_s)|^2 \mathrm{d}r\right] \mathrm{d}x \leq C(\|b\|_{L_t^q(L_x^p)}). \quad (10.3)$$

For the second one, given a continuous trajectory $\gamma : [0, T] \to \mathbb{R}^d$, call $A^{(\gamma)}$ the solution to the linear ODE

$$\partial_t A^{(\gamma)} = D\tilde{b}(\gamma)A^{(\gamma)}, \quad A_0^{(\gamma)} = I.$$

Notice that A is a bounded function of γ (and it is measurable as a function $(C_t, \mathcal{B}(C_t)) \to (C_t, \mathcal{B}(C_t)))$. Then call $A(x) = A^{(x+W)}$.

Lemma 10.6. *Fix $R > 0$ and an even integer $m \geq 2$. Then it holds*

$$\sup_{t\in[0,T]} \int_{B_R} E[|A|^m]\mathrm{d}x \leq C(\|b\|_{L_t^q(L_x^p)}). \quad (10.4)$$

For the proof of these two lemmata, notice that they are both particular cases of the situation in the previous paragraph, precisely: for 10.5, take $Y = x + W$, $n = 1$, $M = |b|^2$; for 10.6, take $Y = x + W$, $n = d$, $M = Db$. Hence they can be proved with the two techniques described there (the one based on the expansion (10.1) and the one based on the SPDE (10.2)).

Here we prove the lemmata only using the SPDE technique. The proof uses the techniques developed for SPDEs in Chapter 12, in particular Theorem 12.5 in that Chapter. We give a proof only for 10.6 (the proof of 10.5 being similar).

Proof of 10.6. By Proposition 10.2, applied with $U_t = e_i$ (the i-th vector of the canonical basis of $\mathbb{R}^d$), we get that A satisfies the SPDE

$$\partial_s A + ADb + \sum_{k=1}^d \partial_{x_k} A \circ \dot{W}^k = 0.$$

From Theorem 12.5, we get (10.4). □

Proof of Theorem 10.4. First step: application of Girsanov theorem. Fix x in B_R. Since b is in $C_t(C^\infty_{x,c})$ (we are dealing with a priori estimates), Novikov condition holds and we can apply Girsanov theorem: the law of $X(x)$, which we denote by $P\circ X(x)^{-1}$, is absolutely continuous with respect to the Wiener measure starting at x, with density given by

$$\rho_T(x) = \exp\left[\int_0^T b(x+W_r)\mathrm{d}W_r - \frac{1}{2}\int_0^T |b(x+W_r)|^2\mathrm{d}r\right].$$

Notice that, for every $\eta\ge 1$, by Hölder inequality and the martingale property of the exponentials,

$$\begin{aligned} &E\rho_T(x)^\eta \qquad (10.5)\\ &\le E\left[\exp\left[\eta^2\int_0^T b(x+W_r)\mathrm{d}W_r - \frac{\eta^2}{2}\int_0^T |b(x+W_r)|^2\mathrm{d}r\right]\right]^{1/\eta}\cdot\\ &\quad\cdot E\left[\exp\left[\frac{\eta^2(\eta+1)}{2}\int_0^T |b(x+W_r)|^2\mathrm{d}r\right]\right]^{1-1/\eta}\\ &= E\left[\exp\left[\frac{\eta^2(\eta+1)}{2}\int_0^T |b(x+W_r)|^2\mathrm{d}r\right]\right]^{1-1/\eta}. \end{aligned}$$

Now $D\tilde{X}(x)$ satisfies the ODE

$$\partial_t D\tilde{X}(x) = D\tilde{b}(\tilde{X}(x))D\tilde{X}(x),\quad D\tilde{X}_0 = I,$$

and so the process $D\tilde{X}(x)$ is a (measurable bounded) function of $X(x)$, precisely $DX(x) = A^{(X(x))}$. Hence Girsanov theorem (together with (10.5)) gives

$$\begin{aligned} &E\left[|X(t,x)|^m + |DX(t,x)|^m\right] = E\left[\left(|x+W_t|^m + |A(t,x)|^m\right)\rho_T\right] \qquad (10.6)\\ &\le 2^{1/2}E\left[|x+W_t|^{2m} + |A(t,x)|^{2m}\right]^{1/2} E[\rho_T^2]^{1/2}\\ &= 2^{1/2}E\left[|x+W_t|^{2m} + |A(t,x)|^{2m}\right]^{1/2} E\left[\exp\left[6\int_0^T |b(x+W_r)|^2\mathrm{d}r\right]\right]^{1/4}. \end{aligned}$$

Second step: conclusion. Having the estimate above, we can conclude thanks to the bounds (10.3) and (10.4) and Hölder inequality. □

Proof of Theorem 10.1. The result follows, via Theorem 4.12 and Remark 4.13, from the a priori estimates in 10.4 (for the STE) and the estimates in 6.2 (for the SCE). □

10.4 Final remarks

Looking more carefully at this proof, we see two advantages and two disadvantages.

The first advantage is of course the more general result with respect to the previous Chapter. The second, maybe even more important, advantage is that the strategy still keeps two arguments separated, namely the Girsanov argument and the STE argument. This can be relevant in contexts where there is still a Gaussian structure but the STE analysis is more complicated or not available (as for example when the Brownian motion is replaced by the fractional Brownian motion): one can still rely on some Girsanov theorem, and then use other arguments (for example controlling every term in (10.1)) for the estimates on A. This has been done for the fractional Brownian motion in [CG12] and in [BNP15].

The first disadvantage, which comes together with the more general result, is the more strict link with the stochastic nature of the problem, which for example does not allow to identify clearly the P-null set where "things could go wrong", that is where the flow may not be regular (Lipschitz or Sobolev). The second disadvantage is that it seems not a flexible argument for Hölder and fractional Sobolev estimates: indeed the Girsanov equivalence between $X(x)$ and $x+W$ is limited to a fixed x and is false in general for the joint laws of $(X(x), X(y))$ and $(x+W, y+W)$, while the Hölder and fractional Sobolev estimates usually require a control on the difference $X(x)-X(y)$.

Chapter 11

The martingale argument: the Lagrangian approach

In this chapter we show a priori estimates on the derivative of the flow by an elegant argument which relies on Itô formula and regularity theory for the associated second-order PDE, hence on the martingale structure of the problem.

The content of this Chapter is mainly taken from [FF11], [FF13a], [FF13b].

11.1 The main result

The following main result, in the class of drifts 2.3, is the same of the previous chapter. The method for the a priori estimates is different.

Theorem 11.1. *Assume that b is in the class 2.3 (and satisfies Condition 2.1). Fix $2 \leq m < +\infty$, $2 \leq \tilde{m} < +\infty$ such that $1/m + 1/\tilde{m} + 1/(p \wedge q) \leq 1$. Then existence, path-by-path uniqueness and stability hold in the sense of Theorem 4.12.*

The proof is at the end of this Chapter.

11.2 A transformation of the SDE

For simplicity, we fix $s = 0$ in the following: it is easy to see that all the estimates hold for any s, uniformly in s (in $[0, T]$).

We want to derive information on the SDE from information on the PDE. For this, we have seen, at the end of Section 4.1, the duality relation between the flow X and the solution of the backward stochastic transport equation

v: formally, $\mathrm{d}[v_r(X_r)] = 0$. The idea is to replace v by its average, or, more precisely, by the solution $\bar{\Psi}$ of the backward Kolmorogov equation

$$\partial_t \bar{\Psi} + b \cdot \nabla \bar{\Psi} + \frac{1}{2}\Delta \bar{\Psi} = 0, \quad \bar{\Psi}_T = id.$$

Applying Itô formula, we get the following equation for $Y = \bar{\Psi}(X)$:

$$dY = D\bar{\Psi}(X)dW = D\bar{\Psi}(\bar{\Psi}^{-1}(Y))dW. \tag{11.1}$$

Now we can expect $\bar{\Psi}$ to be at least twice weakly differentiable, from parabolic regularity theory, and from this we hope to get regularity for $D\bar{\Psi}(\bar{\Psi}^{-1})$ and deduce regularity of Y and then on X.

We start defining precisely the transformation of the SDE and proving its regularity. One problem in the transformation above is that the invertibility of $\bar{\Psi}$ (which is required for formula (11.1)) does not hold in general. Actually, $\bar{\Psi}$ is invertible on $[0,T]$ for $T \leq T_0$, for a suitable T_0. To see this invertibility property up to T_0, we write $\bar{\Psi} = id + \bar{\psi}$ and we notice that $\bar{\psi}$ satisfies the PDE

$$\partial_t \bar{\psi} + b \cdot \nabla \bar{\psi} + \frac{1}{2}\Delta \bar{\psi} = b, \quad \bar{\psi}_T = 0.$$

Adapting the result 5.2 to this parabolic PDEs, one gets that $\|D\bar{\psi}\|_{C^\beta_t(L^\infty_x)}$ is bounded by $C(\|b\|_{L^q_t(L^p_x)})$, where C is as usual a locally bounded function; this fact and the fact that $\bar{\psi}_T = 0$ imply that $\|D\bar{\psi}_s\|_{L^\infty_x} \leq C(\|b\|_{L^q_t(L^p_x)})|T-t|^\beta$. So there exists $t_0 < T$ (possibly negative), with $2(T-t_0)^{-\beta} < C(\|b\|_{L^q_t(L^p_x)})$, such that $\|D\bar{\psi}_t\|_{L^\infty_x} < 1/2$ for every t in $[t_0,T]$. This implies invertibility (in x) of $\bar{\Psi}$ on $[t_0,T]$. In particular, if T is such that $t_0 \leq 0$, then we get the desired invertibility on $[0,T]$.

We could just keep the transformation $\bar{\Psi}$, restricting the attention first on an interval $[0,T_0]$ where invertibility holds and then iterating to reach every time. However we use a slightly modification of $\bar{\Psi}$, which is invertible for every time and is still related to a PDE similar to the backward Kolmogorov equation above, so that we can transform the original SDE into a SDE with more regular coefficients. We consider $\Psi^\lambda = id + \psi^\lambda$, with ψ^λ satisfying the PDE

$$\partial_t \psi^\lambda + b \cdot \nabla \psi^\lambda + \frac{1}{2}\Delta \psi^\lambda + \lambda \psi^\lambda = -b, \quad \psi^\lambda_T = 0.$$

The estimates in 5.2 then gives the following:

Corollary 11.2. *There exist locally bounded functions C, $\lambda = \lambda(\|b\|_{L^q_t(L^p_x)})$ such that, for any vector field b in $C_t(C^\infty_{x,c})$, for every t in $[0,T]$, Ψ^λ_t is a C^1*

diffeomorphism and it holds

$$\|\Psi^\lambda\|_{C_t(C_{x,lin})} + \|D\Psi^\lambda\|_{C_{t,x}} + \|(\Psi^\lambda)^{-1}\|_{C_t(C_{x,lin})} + \|D[(\Psi^\lambda)^{-1}]\|_{C_{t,x}} + \\ + \|D^2\Psi^\lambda\|_{L^q_t(L^p_x)} \leq C(\|b\|_{L^q_t(L^p_x)}).$$

Proof. The estimate on Ψ and its space derivatives are a consequence of Theorem (5.2) and of the definition of Ψ^λ (actually, for every $\lambda > 0$).

As for the invertibility of the map $\Psi_t : \mathbb{R}^d \to \mathbb{R}^d$, Theorem (5.2) guarantee the existence of $\lambda = \lambda(\|b\|_{L^q_t(L^p_x)})$ (locally bounded function) such that $\|D\psi^\lambda\|_{C_{t,x}} < 1/2$. Moreover, for every t, $|\Psi_t(x)| \to +\infty$ as $|x| \to +\infty$, as ψ^λ is bounded (again by Theorem 5.2). Hence, by Hadamard theorem (see for example [Pro04], Theorem 59), Ψ_t is a C^1_x diffeomorphism. Moreover the formula

$$D\Psi_t^{-1}(x) = (D\Psi_t(\Psi_t^{-1}(x)))^{-1} = \sum_{n=0}^{\infty}(-D\psi_t(\Psi_t^{-1}(x)))^n$$

and the bound $\|D\psi_t^\lambda\|_{C_{t,x}} < 1/2$ imply that $D[\Psi_t^{-1}]$ is (continuous and) uniformly bounded in (t,x) (over all $[0,T] \times \mathbb{R}^d$). The estimate on Ψ_t^{-1} and its space derivative follows easily. The proof is complete. □

Now we fix λ such that the above result holds (we can choose the same λ for all b with bounded $\|b\|_{L^q_t(L^p_x)}$) and we call

$$Y = \Psi^\lambda(X).$$

Lemma 11.3. *Y satisfies the SDE*

$$\mathrm{d}Y = -\lambda\Psi^\lambda(X)\mathrm{d}t + D\Psi^\lambda(X)\mathrm{d}W \tag{11.2}$$

Proof. The lemma is a consequence of Itô formula. □

From now on, we omit the symbol λ.

11.3 The estimates on the transformed SDE

We are ready to give the $W^{1,m}_x$ bound on the flow X, using the flow Y. Looking at the transformed SDE (11.2), the coefficients have at least Sobolev regularity in space $W^{1,p}_x$, for finite p; this indicates already a regularization.

If we had also $W^{1,\infty}_x$ bounds, then we could conclude. However, since we do not have such bounds (even when $p = \infty$), it is not clear how such a Sobolev estimate implies the Sobolev regularity: indeed, replacing for a moment $\mathrm{d}W$ with $\mathrm{d}t$ (just to have an idea of what is happening), the usual DiPerna-Lions and Ambrosio results would just give a Lagrangian flow, without any L^m estimates on the derivative. The reason for such an improvement is again in Girsanov theorem: the law of X, equivalent to the Wiener measure, regularizes the term $\int_0^t |D^2\Psi_r(X_r(x))|^2 \mathrm{d}r$.

Theorem 11.4. *Fix p, q satisfying Condition 2.3. Fix an even integer $m \geq 2$, $R > 0$. Then there exists a locally bounded function $C : [0, +\infty[\to [0, +\infty[$ such that, for any vector field b in $C_t(C^\infty_{x,c})$, it holds*

$$\sup_{t\in[0,T]} \sup_{x\in B_R} E\left[|DX(t,x)|^m\right] \leq C(\|b\|_{L^q_t(L^p_x)}).$$

Proof. By Corollary 11.2, it is enough to prove the same estimate on DY.

The flow derivative DY satisfies the equation

$$\mathrm{d}DY = -\lambda D\Psi(X)DX\mathrm{d}t + D^2\Psi(X)DX\mathrm{d}W.$$

Now we want to apply Itô formula for $|DY|^m$, for m even integer. We start with the one-dimensional case: here we get simply

$$\begin{aligned}
&\mathrm{d}[|DY|^m] \\
&= -\lambda m D\Psi(X)DXDY^{m-1}\mathrm{d}t + mD^2\Psi(X)DXDY^{m-1}\mathrm{d}W+ \\
&+ \frac{1}{2}m(m-1)|D^2\Psi(X)DX|^2DY^{m-2}\mathrm{d}t.
\end{aligned}$$

We would like then to take expectation, but then we could not apply Gronwall lemma, since the term $D\Psi(X)$, $D^2\Psi(X)^2$ would be unbounded term inside the time integral. Hence we do as follow. Define A as the process

$$\begin{aligned}
&A(t,x) = A^{(X,m)}(t,x) \\
&= (C+C^2)\int_0^t \left[\lambda m|D\Psi(r,X_r(x))| + \frac{1}{2}m(m-1)|D^2\Psi(r,X_r(x))|^2\right]\mathrm{d}r.
\end{aligned}$$

where C is a constant such that $|DX| \leq C|DY|$ (it exists thanks to Corollary 11.2). Applying again Itô formula to $e^{-A}DY^m$ we get rid of the deterministic integral, more precisely

$$\begin{aligned}
&\mathrm{d}[e^{-A}|DY|^m] \\
&= \left[-A - \lambda m D\Psi(X)DXDY^{m-1} + \frac{1}{2}m(m-1)|D^2\Psi(X)DX|^2DY^{m-2}\right]\mathrm{d}t+ \\
&+ e^A mD^2\Psi(X)DXDY^{m-1}\mathrm{d}W
\end{aligned}$$

and the integrand in the deterministic integral is nonpositive. So taking expectation, we get

$$\partial_t E[e^{-A}|DY|^m] \leq 0. \tag{11.3}$$

In the case of general dimension d, we should use Itô formula for $|DY|^m = (\mathrm{tr}[DY DY^*])^{m/2}$. The notation then becomes more involved, but the strategy remains the same, the only change is to define A as

$$\begin{aligned} A(t,x) &= A^{(X,m)}(t,x) \\ &= c(C+C^2)\int_0^t \left[\lambda m|D\Psi(r,X_r(x))| + \frac{1}{2}m(m-1)|D^2\Psi(r,X_r(x))|^2\right]\mathrm{d}r. \end{aligned}$$

for some constant c to be fixed later, depending only on the dimension d and (possibly) on m. Then we use Itô formula to $e^{-A}|DY|^m$ (for m even integer) and we fix c so that the deterministic integrand in the Itô differential of $e^{-A}|DY|^m$ is nonpositive. In this way we still get (11.3).

The estimate (11.3) gives

$$\begin{aligned} E[|DY|^m] &= E[e^{A^{(2m)}/2}e^{-A^{(2m)}/2}|DY|^m] \\ &\leq E[e^{A^{(2m)}}]^{1/2}E[e^{-A^{(2m)}}|DY|^{2m}]^{1/2} \leq E[e^{A^{(2m)}}]^{1/2}. \end{aligned}$$

Hence it is enough to get, for every m, a uniform (in b) estimate on $E[e^{A^{(X,m)}}]$.

This can be done again by Girsanov theorem. Precisely, calling again

$$\rho_T(x) = \exp\left[\int_0^T b(x+W_r)\mathrm{d}W_r - \frac{1}{2}\int_0^T |b(x+W_r)|^2\mathrm{d}r\right],$$

we have (recall the estimate of $E[\rho_T^2]$ in (10.6))

$$\begin{aligned} E\left[e^{A^{(X,m)}}\right] &= E\left[e^{A^{(W,m)}}\rho_T\right] \leq E\left[e^{2A^{(W,m)}}\right]^{1/2} E[\rho_T^2]^{1/2} \\ &\leq E\left[e^{2A^{(W,m)}}\right]^{1/2} E\left[\exp\left[6\int_0^T |b(x+W_r)|^2\mathrm{d}r\right]\right]^{1/4}, \end{aligned}$$

where $A^{(W,m)}$ is defined as $A^{(X,m)}$, replacing $X(x)$ with $x+W$. Now Lemma 10.5 in Chapter 10 gives that

$$E\left[\exp\left[6\int_0^T |b(x+W_r)|^2\mathrm{d}r\right]\right] \leq C(\|b\|_{L_t^q(L_x^p)}).$$

Also Lemma 10.6 in Chapter 10 and the bounds on Ψ in Corollary 11.2 give that

$$E\left[e^{2A^{(W,m)}}\right] \le C(\|D\Psi\|_{L^\infty_t(L^\infty_x)} + \|D^2\Psi\|_{L^q_t(L^p_x)}) \le C(\|b\|_{L^q_t(L^p_x)}).$$

Putting together these two bounds, we get that $E[e^{A^{(m)}}]$ and so $E[|DY|^m]$ is controlled by $C(\|b\|_{L^q_t(L^p_x)})$. The proof is complete. □

Proof of Theorem 11.1. The result follows, via Theorem 4.12 and Remark 4.13, from the a priori estimates in 11.4 (for the STE) and the estimates in 6.2 (for the SCE). □

11.4 The result for Hölder continuous coefficients

The same strategy can be used to prove Lipschitz estimates for the flow, in the case of α-Hölder (in space) continuous drifts, for $\alpha > 0$.

Theorem 11.5. *Assume that b is in $C_t(C^\alpha_{x,b})$ for some $\alpha > 0$. Fix $0 < \alpha' < \alpha$, $2 \le m < +\infty$, $R > 0$. Then there exists a locally bounded function $C : [0, +\infty[\to [0, +\infty[$ such that, for any vector field b in $C_t(C^\infty_{x,c})$, it holds*

$$\sup_{t\in[0,T]} E\left[\|DX_t\|^m_{C^{\alpha'}_{x,B_R}}\right] \le C(\|b\|_{C_t(C^\alpha_{x,b})}).$$

We do not give a proof of this result, which is in [FGP10], Theorem 5. The strategy is in the same line as above (using Kolmogorov continuity criterion and not using Girsanov theorem): we transform the SDE through Ψ^λ, whose $L^\infty_t(C^{2+\alpha'+}_{x,lin})$ norm is controlled by the $C_t(C^\alpha_{x,b})$ norm of b; the new SDE has then $L^\infty_t(C^{1+\alpha'+}_{x,lin})$ coefficients, therefore the $C^{1+\alpha'}_x$ norm of the solution X can be estimated classically, via Kolmogorov continuity criterion.

Starting from this result and applying the machinery in Chapter 4, one can prove existence, path-by-path uniqueness (among single solutions), starting from every point x and existence of a stochastic flow which is Lipschitz continuous in x (see [BFGM14] and [Sha14]).

Chapter 12

The martingale argument: the Eulerian approach

In this chapter we show Sobolev-type a priori estimates on the stochastic transport equation. We use a direct energy method for the derivative of the solution, which satisfies a similar linear SPDE because of the renormalization property of the STE. This method exploits Itô formula and the zero expectation of the stochastic integral and ends in bounds on a Kolmogorov-type equation (hence we use again the martingale structure). The same argument works also for a more general type of first order linear vector-valued SPDEs, like the stochastic vector advection equation (a linearized version of the stochastic three-dimensional Euler equations with multiplicative noise).

The content of this Chapter is mainly based on the paper [BFGM14]. A similar result on the stochastic vector advection equation, but with Hölder continuous vector fields and based on the method of characteristics, is in [FMN14]. Another application in fluid dynamics, to the Navier-Stokes equation, is in [Rez14], where the vorticity formulation of the equation is used (as here for the vector advection equation), although with different scopes and techniques.

12.1 The main result

The following main result, in the class of drifts 2.4, is a slight extension of the previous chapter. The method for the a priori estimates is purely based on PDEs arguments.

Theorem 12.1. *Assume that b is in the class 2.4. Fix $2 \leq m < +\infty$, $2 \leq \tilde{m} < +\infty$ such that $1/m + 1/\tilde{m} + 1/(p \wedge q) \leq 1$. Then existence, path-by-path uniqueness and stability hold in the sense of Theorem 4.12.*

The proof is at the end of Section 12.4.

12.2 A stochastic PDE for the derivative of the STE

Before stating and proving the result, we focus on two key properties of the STE: the derivative of its solution satisfies also a stochastic transport-like linear PDE and the same holds also for its powers. The last property is linked to the renormalization property of the transport equation, we investigate this in the next section.

In order to deal with Sobolev estimates for the STE (we use the forward formulation, the backward one being completely similar), we derive a SPDE for the derivative of the solution v. We start from (spatial) dimension 1. Differentiating the STE with respect to x, we get

$$\partial_t[\partial_x v] + b\partial_x[\partial_x v] + \partial_x b \partial_x v + \partial_x[\partial_x v] \circ \dot{W} = 0,$$

with fixed initial condition $\partial_x v_0$. As for the Kolmorogov equation, we see that $\partial_x v$ satisfies a SPDE which is given by the stochastic transport equation plus the term $\partial_x b \partial_x v$.

In more than one dimension, we get a system of SPDEs for the vector field $\nabla v = (\partial_{x_1} v, \dots \partial_{x_d} v)$, precisely

$$\partial_t[\partial_{x_i} v] + b \cdot \nabla[\partial_{x_i} v] + \sum_{j=1}^{d} \partial_{x_i} b^j \partial_{x_j} v + \nabla[\partial_{x_i} v] \circ \dot{W} = 0, \quad i = 1, \dots d,$$

or equivalently, in matrix notation (where $D_x v$ is a row vector and $\nabla v = D_x v^*$ is its transpose vector),

$$\partial_t D_x v + (b \cdot \nabla) D_x v + D_x v Db + \nabla D_x v \circ \dot{W} = 0,$$

with fixed initial condition $\partial_x v_0$. Although we have more than one equation, again the structure is always of a stochastic transport equation with additional terms of the form $\partial_{x_i} b^j \partial_{x_j} v$.

12.3 The renormalization property

The renormalization property for the stochastic transport equation is the following fact: if v is a regular solution to the STE and $f : \mathbb{R} \to \mathbb{R}$ is a C_b^2 function, then $f(v)$ is also a regular solution to the STE. In the context of regular coefficients and solution, this fact is an easy consequence of the chain rule for the space derivative, namely

$$\nabla f(v) = f'(v)\nabla v,$$

and of Itô formula (for Stratonovich integral) for the time differential.

Since ∇v satisfies a system of transport-like equations, we expect it to enjoy the renormalization property which is typical of this kind of equation. Again we start with (spatial) dimension 1. In this case, using the chain rule (for the space derivative) and Itô formula (for the time differential), we get, for any f in C_b^2, the following equation for $f(\partial_x v)$:

$$\partial_t[f(\partial_x v)] + b\partial_x[f(\partial_x v)] + \partial_x b f'(\partial_x v)\partial_x v + \partial_x[f(\partial_x v)] \circ \dot{W} = 0.$$

Again this equation has the form of the stochastic transport equation, with the additional term $\partial_x b f'(\partial_x v)\partial_x v$. Applying this formula to $f(r) = r^m$, for m positive integer, we obtain the following equation for $(\partial_x v)^m$:

$$\partial_t[(\partial_x v)^m] + b\partial_x[(\partial_x v)^m] + m\partial_x b(\partial_x v)^m + \partial_x[(\partial_x v)^m] \circ \dot{W} = 0.$$

Now we come to the case of general dimension, where we have a system of equations. In analogy to the one dimensional case, we would like to have an equation for $|\nabla v|^m$, or a system of equations for $(\partial_{x_i} v)^m$. Unfortunately, in both the cases we do not obtain a close equation or system, because of the terms $\partial_{x_i} b^j \partial_{x_j} v$ in the system for ∇v: indeed, take for example $m = 2$, then the equation for $(\partial_{x_i} v)^2$ becomes

$$\partial_t[(\partial_{x_i} v)^2] + b \cdot \nabla[(\partial_{x_i} v)^2] + 2\sum_{j=1}^{d} \partial_{x_i} b^j \partial_{x_i} v \partial_{x_j} v + \nabla[(\partial_{x_i} v)^2] \circ \dot{W} = 0,$$

hence the mixed products $\partial_{x_i} v \partial_{x_j} v$ appear and the system is not closed. The solution to this problem is to consider, for $m = 2$, a system with $\binom{d+1}{2}$ equations, having all the products $\partial_{x_i} v \partial_{x_j} v$ as unknown variables. For general m, this means to consider a system with $\binom{d+m-1}{m}$ equations, having as unknown variables the products

$$w_I = \prod_{i \in I} \partial_{x_i} v, \tag{12.1}$$

for any multi-index I in $\{1, \dots d\}^m$ (see below for the precise notation). This needs more effort of notation and makes the computations a bit lengthy.

As for notation, we denote by $I = (I_1, \dots I_m)$ a multi-index in $\{1, \dots d\}^m$ and, with a little abuse of notation, we identify I and J multi-indices when there exists a permutation σ of $\{1,\dots m\}$ such that $J_{\sigma(k)} = I_k$ for every k. Again with abuse of notation, we write $i \in I$ to mean that there exists k such that $I_k = i$. We also write

$$\prod_{i\in I} \partial_{x_i} v = \prod_{k=1}^m \partial_{x_{I_k}} v, \quad \sum_{i\in I} \partial_{x_i} v = \sum_{k=1}^m \partial_{x_{I_k}} v, \quad \dots.$$

Finally, given a multi-index I and two indices i, j in $\{1,\dots d\}$ with $i \in I$, we call $(I \setminus i) \cup j$ the multi-index obtained by I replacing one i in I with j; more precisely, taking a k such that $I_k = i$, we impose $((I \setminus i) \cup j)_k = j$ and $((I \setminus i) \cup j)_h = I_h$ for all $h \neq k$ (by our identification of multi-indices, the choice of k is not relevant).

With this notation, we can state the system of PDEs for $(w_I)_I$:

$$\partial_t w_I + b \cdot \nabla w_I + \sum_{i\in I} \sum_{j=1}^d \partial_{x_i} b^j w_{(I\setminus i)\cup j} + \nabla w_I \circ \dot{W} = 0, \quad I \text{ multiindex.} \tag{12.2}$$

We see again that this system is made of transport-like equations. Indeed the computations for the multidimensional case do not have any substantial difference from those in the one dimensional case.

12.4 The Sobolev estimates on the STE

Given the equation satisfied by w_I, we are ready to prove the a priori Sobolev estimates on the STE.

Theorem 12.2. *Fix p, q satisfying Condition 2.4. Fix $m \geq 2$ even integer. Write $b = b^{(1)} + b^{(2)}$, where the addends are functions in $C_t(C^\infty_{x,c})$. Then there exists a locally bounded function $C : [0,+\infty[^2 \to [0,+\infty[$ such that, for every $b^{(j)}$, $j = 1,2$, in $C_t(C^\infty_{x,c})$, for every $s \geq 0$ and every v_s in $C^\infty_{x,c}$, it holds*

$$\begin{aligned}&\sup_{t\in[s,T]} \int_{\mathbb{R}^d} \chi(x) E[|\nabla v_t|^m]^2 \mathrm{d}x \\ &\leq C(\|b^{(1)}\|_{L^q_t(L^p_x)}, \|b^{(2)}\|_{L^1_t(C^1_{x,lin})}) \int_{\mathbb{R}^d} \chi(x) |\nabla v_s|^{2m} \mathrm{d}x.\end{aligned}$$

Again, for simplicity, we fix $s = 0$ in the proof: it is easy to see that all the estimates hold for any s, uniformly in s (in $[0,T]$).

In the case of v_0 in $C^\infty_{x,c}$, b in $C_t(C^\infty_{x,c})$, Proposition 3.18 ensures that the regular solution of this equation, given by Proposition 3.18, is in $L^m_\omega(W^{1,m}_t(W^{2,m}_{x,\chi_\eta}))$, for every m in $[1,+\infty[$ and for every weight $\chi_\eta(x) = (1+|x|^2)^{\eta/2}$, for every real η. This justifies the computations below.

Proof. **Step 1: the parabolic system for $E[w_I]$.** We write the system for $z_I = E[w_I]$, for I multi-index of length m. Equation (12.2) reads with Itô integral

$$\partial_t w_I + b\cdot\nabla w_I + \sum_{i\in I}\sum_{j=1}^{d}\partial_{x_i}b^j w_{(I\setminus i)\cup j} + \nabla w_I\dot{W} = \frac{1}{2}\Delta w_I, \quad I \text{ multiindex}$$

Taking expectation, we obtain

$$\partial_t z_I + b\cdot\nabla z_I + \sum_{i\in I}\sum_{j=1}^{d}\partial_{x_i}b^j z_{(I\setminus i)\cup j} = \frac{1}{2}\Delta z_I, \quad I \text{ multiindex}$$

This is a parabolic system, whose form is similar to the parabolic equation (5.9) in Theorem 5.7, by putting $g = b$, $h = 0$.

Continuation. The rest of the proof goes on in analogy to Step 2, 3, 4 of the proof of Theorem 5.7. The only relevant difference is that, before Step 4, we sum over all multi-indices I with length m: we get an inequality similar to (5.13) (or (5.14) for $p = d \geq 3$), with $\theta_m(t,x)^2 = \chi(x)\sum_{|I|=m} z_I(t,x)^2$ replacing $\chi(x)v^{2m}(t,x)$ and $\rho_m(t,x)^2 = \chi(x)\sum_{|I|=m}|\nabla z_I(t,x)|^2$ replacing $\chi(x)|\nabla[v^m]|^2$; see [BFGM14] for the computations in full detail. $\square$

Proof of Theorem 12.1. The result follows, via Theorem 4.12, from the a priori estimates in 12.2 (for the STE) and the estimates in 6.2 (for the SCE). $\square$

12.5 Existence for the stochastic vector advection equation and other linear SPDEs

The method above, based on a priori estimates, can in principle be applied to a vector valued stochastic linear transport-like PDE of the form

$$\partial_s v + (b\cdot\nabla)v + (Nv\cdot\nabla)g + hv + \nabla v\circ\dot{W} = 0, \tag{12.3}$$

where $v : [0,t]\times\mathbb{R}^d\times\Omega \to \mathbb{R}^n$ is the (n-dimensional) solution, $b : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$, $g : [0,T]\times\mathbb{R}^d \to \mathbb{R}^n$, $h : [0,T]\times\mathbb{R}^d \to \mathbb{R}$ are given (deterministic) fields, N in $\mathbb{R}^{d\times n}$ is a given matrix and W is a d-dimensional Brownian motion (with the assumptions we had so far).

We first state the definition of classical solution and the existence and the representation formula, when the coefficients are regular.

Definition 12.3. *Assume that all the coefficients are in $C_t(C^\infty_{x,c})$ and that the final datum v_t is in $C^\infty_{x,c}$. A classical solution to the SPDE* (12.3) *is a measurable function $v : [0,T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}^n$, which is in $C_s(C^2_{x,loc})$ for a.e. ω and satisfies, for a.e. ω,*

$$\begin{aligned}
&v(s,x)\\
&= v(t,x) + \int_s^t (b(r,x)\cdot\nabla)v(r,x)\mathrm{d}r + \int_s^t (Nv(r,x)\cdot\nabla)g(r,x)\mathrm{d}r +\\
&+ \int_s^t h(r,x)v(r,x)\mathrm{d}r + \sum_{k=1}^d \int_s^t \partial_{x_k} v(r,x)\circ \mathrm{d}W^k_r + \frac{1}{2}\int_s^t \Delta v(r,x)\mathrm{d}r, \quad \forall (s,x).
\end{aligned}$$

Proposition 12.4. *Assume that all the coefficients are in $C_t(C^\infty_{x,c})$ and that the final datum v_t is in $C^\infty_{x,c}$. Then there exists a classical solution to* (12.3) *and is given by*

$$v(s,x) = A^s(t,x)^* v_t(X^s(t,x)),$$

where (A,X) satisfies the system of SDEs

$$\begin{aligned}
&\mathrm{d}X^s(t,x) = b(t,X^s(t,x))\mathrm{d}t + \mathrm{d}W_t,\\
&\mathrm{d}A^s(t,x) = (N^* Dg(t,X^s(t,x))^* + h(t,X^s(t,x))I)A^s(t,x)\mathrm{d}t,
\end{aligned}$$

with initial data $X^s(s,x) = x$, $A^s(s,x) = I$.

Proof. The thesis is a consequence of Proposition 10.2, taking $B = b$ and $M = N^* Dg^* + hI$. $\square$

Here are the a priori estimates. As usual, we state the estimates for the forward equation, but the analogous result holds for the backward case.

Theorem 12.5. *Fix p, q satisfying Condition 2.4. Fix m positive even integer. Write $b = b^{(1)}+b^{(2)}$, $g = g^{(1)}+g^{(2)}$, where all the addends are functions in $C_t(C^\infty_{x,c})$. Then there exists a locally bounded function $C : [0,+\infty[^5 \to [0,+\infty[$ such that, for every $b^{(j)}$, $g^{(j)}$, $j = 1,2$, h in $C_t(C^\infty_{x,c})$ and for every v_0 in $C^\infty_{x,c}$, it holds*

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^d} \chi(x) E[|v_t|^m]^2 \mathrm{d}x \le C \int_{\mathbb{R}^d} \chi(x) v_0^{2m} \mathrm{d}x.$$

where

$$C = C(\|b^{(1)}\|_{L^q_t(L^p_x)}, \|g^{(1)}\|_{L^q_t(L^p_x)}, \|h\|_{L^{q/2}_t(L^{p/2}_x)}, \|b^{(2)}\|_{L^1_t(C^1_{x,lin})}, \|g^{(2)}\|_{L^1_t(C^1_{x,lin})}).$$

The proof is in the same line of Theorem 12.2. From the point of view of notation, we denote now: $w_I = \prod_{i\in I} v^i$ (where v^i is the i-th component of v), for I multi-index in $\{1,\dots n\}$. Also in this case, the solution is in $L^m_\omega(W^{1,m}_t(W^{2,m}_{x,\chi_\eta}))$, for every m in $[1,+\infty[$ and for every weight $\chi_\eta(x) = (1+|x|^2)^{\eta/2}$, for every real η. This fact can be proved adapting the proof in [BFGM14], Lemma 12. This justifies the computations in the proof below.

Proof. **Step 1: the parabolic system for $E[w_I]$.** We write the system for $z_I = E[w_I]$, for I multi-index of length m. Using Itô integral as before, we get

$$\partial_t w_I + b\cdot\nabla w_I + \sum_{i\in I}\sum_{j=1}^n N_j\cdot\nabla g^i w_{(I\setminus i)\cup j} + h w_I + \nabla w_I \dot{W} = \frac{1}{2}\Delta w_I, \quad I \text{ multiindex}$$

Taking expectation, we obtain

$$\partial_t(z_I) + b\cdot\nabla(z_I) + \sum_{i\in I}\sum_{j=1}^n N_j\cdot\nabla g^i z_{(I\setminus i)\cup j} + h z_I = \frac{1}{2}\Delta z_I, \quad I \text{ multiindex}$$

This is a parabolic system, whose form is similar to the parabolic equation (5.9) in Theorem 5.7.

Continuation. The rest of the proof goes on in analogy to Step 2, 3, 4 of the proof of Theorem 5.7. The only relevant difference is again in replacing $\chi(x)v^{2m}(t,x)$ with $\theta_m(t,x)^2 = \chi(x)\sum_{|I|=m} z_I(t,x)^2$ and $\chi(x)|\nabla[v^m]|^2$ with $\rho_m(t,x)^2 = \chi(x)\sum_{|I|=m}|\nabla z_I(t,x)|^2$. □

We apply this result in two examples, which are relevant for us. The first one is the proof of Lemma 10.6, involving the exponential $A(x)$. For each $i = 1,\dots d$, $v = A(x)^i$ satisfies the SPDE 12.3 in the case $b = 0$, $h = 0$ and $v_T = e^i$ (the i-th vector of the canonical basis of $\mathbb{R}^d$). Hence Lemma 10.6 follows from Theorem 12.5. Analogously one can prove Lemma 10.5.

The second example is the (forward) stochastic vector advection equation on $\mathbb{R}^3$, namely

$$\begin{aligned}&\partial_t v + (Dv)b - (Db)v + \nabla v\circ\dot{W} = 0,\\ &\mathrm{div} v = 0,\end{aligned}$$

where b is assumed divergence-free. This example is a linearized version of the three-dimensional stochastic Euler equation, in vorticity form (see [FMN14] for more details). In the regular case, applying Proposition 12.4 to the first equation, after some passages that we omit here (see [FMN14] for another proof), we get the representation formula

$$v(t,x) = DX_t(x)v_0(X_t^{-1}(x)). \tag{12.4}$$

In particular, since b and v_0 is divergence-free, this formula implies that v is divergence-free, so that the two equations above are compatible and we get existence in the regular case.

In the non-regular case (if b is non Lipschitz), in the deterministic setting (i.e. without noise), the solution can explode at some time, as one can expect from the representation formula (12.4); an example of explosion at one point (where the solution is no more in L^m for high m too), for a class of bounded b, is given in [FMN14]. On the contrary, in the stochastic case, Theorem 12.5, applied with $d = 3$, $g = -b$, $h = 0$, gives a priori estimates which will imply the existence of an "almost bounded" solution.

We first give the definition of (distributional) solution, for simplicity in the forward case. Notice that, thanks to the divergence-free assumptions on v and b, we can bring all the derivatives on the test function.

Definition 12.6. *Fix $1 < m < +\infty$. Let b be in $L_t^{m'}(L_{x,loc}^{m'})$ with $\mathrm{div} b \equiv 0$ (in the sense of distribution) and let v_0 be in $L_{x,loc}^1$ also with $\mathrm{div} v_0 \equiv 0$. A solution v to the stochastic vector advection equation is a map $v : [0,T] \times \mathbb{R}^3 \times \Omega \to \mathbb{R}^3$, in $L_{t,\omega}^m(L_{x,loc}^m)^3$, with $\mathrm{div} v \equiv 0$, weakly progressively measurable, such that, for every φ in $C_c^\infty(\mathbb{R}^3;\mathbb{R}^3)$, it holds, for a.e. (t,ω),*

$$\begin{aligned} &\langle v_t, \varphi\rangle \\ &= \langle v_0, \varphi\rangle + \int_0^t \langle v, \mathit{Anti}(D\varphi)b\rangle \mathrm{d}r + \sum \int_0^t \langle v, \partial_{x_k}\varphi\rangle \mathrm{d}W_r^k + \frac{1}{2}\int_0^t \langle v, \Delta\varphi\rangle \mathrm{d}r, \end{aligned} \tag{12.5}$$

where $\mathit{Anti}(D\varphi)$ denotes the antisymmetric part of the matrix $D\varphi$.

Here is the existence result:

Theorem 12.7. *Assume that b is in the class 2.4 with $\mathrm{div} b \equiv 0$ and the initial datum v_0 is in L_x^∞ with compact support and with $\mathrm{div} v_0 \equiv 0$. Then there exists a solution v to the stochastic vector advection equation, which is in $L_t^\infty(L_\omega^m(L_{x,loc}^m))$ for every finite m.*

Proof. The proof is in the line of the stability result for the STE 3.39. We start with a family of b^n in $C_t(C_{x,c}^\infty)$, divergence-free, converging to b in $L_t^{m'}(L_{x,loc}^{m'})$ and with bounded norm in the class 2.4 (i.e. $b^n = b^{n,(1)} + b^{n,(2)}$, where $b^{n,(1)}$ have bounded $L_t^q(L_x^p)$ norm and $b^{n,(2)}$ have bounded $L_t^1(C_{x,lin}^1)$ norm); similarly we take a family of v_0^n in $C_{x,c}^\infty$ converging to v in $L_{x,loc}^1$, with bounded L^∞_x norm and with all the support contained in a ball B_{R_0} independent of n; we call v^n the solutions to the corresponding stochastic vector advection equation. Theorem 12.5 provides uniform bounds in $L^\infty_t(L^m_\omega(L^m_{x,B_R}))$. As in the proof of 3.39, first one proves weak compactness of $(v^n)_n$ in the space $L^m_{t,\omega}(L^m_{x,B_R})$, for every $R>0$, using the uniform bounds; then one proves that any limit point v (in the weak topology) is weakly progressively measurable and satisfies equation 12.5. We are left to prove that v is divergence-free. But this follows by passing to the limit in the equality

$$\int_0^T E[F\langle v^n_t, \nabla\psi_t\rangle]\mathrm{d}r = 0,$$

which holds for any ψ in $C_t(C^\infty_{x,c})$ and any F in L^∞_ω, because of the divergence-free property of v^n. The proof is complete. $\square$

Appendix A

Technical facts and Young integration

A.1 Some facts on measurability

Here we recall some facts on measurability and L^p spaces of Banach-valued functions. We need this in order to work with spaces like $L^m_{t,\omega}(L^m_x)$ or $L^\infty_{t,\omega}(\mathcal{M}_x)$.

In the following, $(E, \mathcal{E}, \lambda)$ is finite measure space; we always assume that such $\mathcal{E}$ is countably generated, up to λ-null set, i.e. it is given by the completion (with respect to λ) of the σ-algebra generated by a countable subset $\mathcal{C}$ of $\mathcal{E}$. This is needed to have $L^p(E;\mathbb{R})$ separable for finite p and it is crucial at least in Proposition A.1. The assumption that λ is finite can be easily replaced by σ-finite hypothesis, with the natural changes in statements and proofs. When we work with more variables x, y, ..., we use the notation $(E_x, \mathcal{E}_x, \lambda_x)$, $(E_y, \mathcal{E}_y, \lambda_y)$, ... for the spaces where these variables live.

We say that a real-valued function $f : E \to \mathbb{R}$ is measurable (or, more precisely, λ-measurable), if, for every open set A in $\mathbb{R}$, $f^{-1}(A)$ is in $\bar{\mathcal{E}}^\lambda$, the σ-algebra obtained by completion of $\mathcal{E}$ with respect to λ. This is equivalent to say that there exists $\tilde{f} : E \to \mathbb{R}$, strictly measurable (i.e., for every open set A in $\mathbb{R}$, $\tilde{f}^{-1}(A)$ is in $\mathcal{E}$), which coincides λ-a.e. with f. This definition can be extended to $[-\infty, +\infty]$-valued functions and to $\mathbb{R}^d$-valued functions, componentwise. Notice that, with this definition, a composition $g(f)$ of a measurable function f with a Borel function $g : \mathbb{R} \to \mathbb{R}$ is still measurable, but this is no more true when g is only measurable.

We say that a property $P = P(x)$ holds true a.e. if there exists a full-measure set A such that P is true on A. Mind that we do not require, a priori, that the set $\{x|P(x) \text{ holds }\}$ is measurable (it is measurable a posteriori, since it contains a full-measure set).

Let V be a Banach space and let K be a subspace of V^*. We say that a function $f : E \to V$ is K-weakly measurable if, for every φ in K, the real-valued map $x \mapsto \langle f(x), \varphi \rangle$ is measurable (in the sense above): when $K = V^*$, we just say weakly measurable; when $V = W^*$ for some Banach space W and $K = W$, we say weakly-* measurable. We say that $f : E \to V$ is strongly measurable if there exists a sequence of simple functions $f^n : E \to V$ converging pointwise λ-a.e. to f. A strongly measurable function is always weakly measurable. We denote by

$$\mathcal{L}^0(E;V) := \mathcal{L}^0(E,\mathcal{E},\lambda,K;V) := \{f : E \to V \ K\text{-weakly measurable}\},$$
$$L^0(E;V) := L^0(E,\mathcal{E},\lambda,K;V) := \{f : E \to V \ K\text{-weakly measurable}\}/\sim$$

the sets of resp. all K-weakly measurable functions and all equivalence classes of K-weakly measurable functions, where $\sim$ is the usual equivalence relation defined by: $f \sim g$ if $f = g$ λ-a.e.. They are both vector spaces.

From now on, we assume, unless differently specified, that the K is a closed subspace of V^* and has a countable subset D such that, for every v in V,

$$\|v\|_V = \sup_{\varphi \in D, \|\varphi\|_{V^*} \le 1} |\langle v, \varphi \rangle|. \tag{A.1}$$

This is the case when at least one of these two conditions is satisfied:

- V is a separable Banach space and $K = V^*$: the weak-* topology closed unit ball $\bar{B}_1^{V^*}$ in V^* is sequentially compact (by Banach-Alaoglu theorem) and metrizable (by separability of V), therefore it is separable; taking as D a countable weakly-* dense set in $\bar{B}_1^{V^*}$, we find (A.1).

- $V = W^*$ for a separable Banach space W and $K = W$, taking as D a countable dense set in W.

Under this assumption, for every K-weakly measurable function f, $\|f\|$ is measurable. Therefore we can define, for $1 \le p \le +\infty$, the vector spaces

$$\begin{aligned}
&\mathcal{L}^p(E;V) := \mathcal{L}^p(E,\mathcal{E},\lambda,K;V) := \\
&\quad = \{f : E \to V \ K\text{-weakly measurable} \mid \|f\| \in L^p(E,\mathcal{E},\lambda)\}, \\
&L^p(E;V) := L^p(E,\mathcal{E},\lambda,K;V) := \\
&\quad = \{f : E \to V \ K\text{-weakly measurable} \mid \|f\| \in L^p(E,\mathcal{E},\lambda)\}/\sim .
\end{aligned}$$

The latter space is a normed vector space with the norm

$$\|f\|_{L^p(E;V)} = \|\|f\|_V\|_{L^p(E)}.$$

For an element $[f]$ in $L^p(E;V)$, we can define the K^*-valued integral

$$\int_E f \mathrm{d}\lambda$$

as the unique linear continuous functional on K such that, for every φ in K,

$$\langle \int_E f \mathrm{d}\lambda, \varphi \rangle = \int_E \langle f, \varphi \rangle \mathrm{d}\lambda.$$

In the case $V = K^*$, this integral is V-valued.

In the case of V separable and $K = V^*$, the space $L^p(E;V)$ has nice properties:

Proposition A.1. *Assume that V is a separable Banach space and that $K = V^*$. Then the following properties hold.*

- *A function $f : E \to V$ is weakly measurable if and only if it is strongly measurable.*
- *The space $L^p(E;V)$ is complete for every $1 \le p \le +\infty$ and separable for every $1 \le p < +\infty$.*
- *If moreover V is reflexive and $1 \le p < +\infty$, then it holds $L^p(E;V)^* = L^{p'}(E;V^*)$, in particular $L^p(E;V)$ is reflexive for $1 < p < +\infty$.*

Proof. The first statement is the well-known Pettis measurability theorem, see for example Theorem 2 in [DU77], Chapter II. Concerning separability, notice that simple functions are dense in $L^p(V)$. Moreover, there exists a countable dense set F in V (by separability of V) and there exists a countable set $\mathcal{C}$ which generates $\mathcal{E}$ (up to λ-null sets), since we assumed $\mathcal{E}$ countably generated. Therefore any simple function can be approximated, in $L^p(V)$, by functions of the form $\sum_{j=1}^n a_j 1_{C_j}$, for a_j in F, C_j in $\mathcal{C}$. Hence the set of such functions is countable and dense in $L^p(V)$. Completeness and (for V reflexive) duality and reflexivity are again in [DU77], Chapter IV Section 1. □

Remark A.2. *When V is a separable space, weak measurability and strong measurability coincide with measurability with values in $(V, \mathcal{B}(V))$, the measurable space of Borel (in the strong topology) sets of V: a map $f : E \to V$ is weakly or equivalently strongly measurable if and only if, for every open set A of V, $f^{-1}(A)$ is in $\bar{\mathcal{E}}^\lambda$. This follows from the fact that any open set in V can be obtained as countable union of balls (by separability) and that, for any f weakly measurable, $f^{-1}(B_R(x)) = \{\|f - x\|_V \le R\}$ is measurable.*

Remark A.3. *Given W Banach space, $1 \le p \le +\infty$, for any functions f in $\mathcal{L}^p(E, W; W^*)$, g in $\mathcal{L}^{p'}(E, W^*; W)$, the map $x \mapsto \langle f(x), h(x)\rangle$ is measurable: indeed, g is strongly measurable by the previous Proposition, so there exist g^n simple functions converging to g a.e. and in L^p, hence $(x \mapsto \langle f(x), g^n(x)\rangle)_n$ is a sequence of measurable maps converging a.e. to $\langle f, g\rangle$, which is therefore measurable. Moreover $\|\langle f, g\rangle\|_{L^1_x} \le \|f\|_{L^p(W^*)}\|g\|_{L^{p'}(W)}$. It follows that $L^{p'}(W)$ is continuously embedded in $(L^p(W^*))^*$ and $L^p(W^*)$ is continuously embedded in $(L^{p'}(W))^*$.*

Now we come to the functions defined on a product space $E_x \times E_y$ with values in V. We assume here that $V = W^*$ for a separable space W and $K = W$ or that V is separable and $K = V^*$. The space

$$L^p_{x,y}(V) = L^p(E_x \times E_y, \mathcal{E}_x \otimes \mathcal{E}_y, \lambda_x \otimes \lambda_y, K; V).$$

is well-defined; similarly for $\mathcal{L}^p_{x,y}$. Since $L^{p'}(K)$ can be embedded in $(L^p(V))^*$, we can also define the space

$$L^0_x(L^p_y(V)) = L^0(E_x, \mathcal{E}_x, \lambda_x, L^{p'}(E_y, \mathcal{E}_y, \lambda_y, V; K); L^p(E_y, \mathcal{E}_y, \lambda_y, K; V));$$

similarly for $\mathcal{L}^0_x(L^p_y(V))$. Notice that, in general, we do not know whether, for any f in $\mathcal{L}^0_x(L^p_y(V))$, $x \mapsto \|f(x)\|_{L^p_y(V)}$ is measurable. For this reason, we do not define $L^p_x(L^p_y(V))$ for a general V.

Remark A.4. *We keep assuming that $V = W^*$ for a separable space W and $K = W$ or that V is separable and $K = V^*$. Given a function f in $\mathcal{L}^p_{x,y}(V)$, then, for λ_x-a.e. x,*

$$y \mapsto f(x, y)$$

is a function in $\mathcal{L}^p_y(V)$, as a consequence of Fubini theorem. Moreover, the map $x \mapsto [y \mapsto f(x, y)]$ is $L^{p'}_y(K)$-weak measurability, i.e.

$$x \mapsto \int_{E_y} \langle f(x, y), \psi(y)\rangle \lambda_y(\mathrm{d}y)$$

is measurable for any ψ in $L^{p'}(K)$, as follows from measurability of $(x, y) \mapsto \langle f(x, y), \psi(y)\rangle$ and Fubini theorem. Furthermore, if g is equivalent to f in $\mathcal{L}^p_{x,y}(V)$ (i.e. they coincide $\lambda_x \otimes \lambda_y$-a.e.), then, for λ_x-a.e. x, $y \mapsto f(x, y)$ and $y \mapsto g(x, y)$ are equivalent in $\mathcal{L}^p_y(V)$. It follows that the map

$$L^p_{x,y}(V) \ni [f] \mapsto [x \mapsto [y \mapsto f(x, y)]] \in L^0_x(L^p_y(V)).$$

is well-defined. It is actually an injective map: for $[f] \neq [g]$ in $L^p_{x,y}(V)$, chosen two representatives f, g, by Fubini theorem, there exists a positive-measure set A in E_x, such that, for each x in A, there exists a positive-measure set B_x in E_y with $f(x, y) \neq g(x, y)$ for y in B_y; that is, for each x in A, $[y \mapsto f(x, y)] \neq [y \mapsto g(x, y)]$.

Remark A.5. *At this point, we remark one fact concerning null-measure sets. When we say that, given* $[f]$ *in* $L^0(V)$*, a property* $P = P^f(x)$ *(depending on* f*) holds true a.e., we should notice that, in general, the exceptional set where* P^f *may not hold depends on the representative of* f*. In particular, when we say that, given* $[f]$ *in* $L^p_{x,y}(V)$*, a property* $P^{x\mapsto[f(x,\cdot)]}$ *holds true for a.e.* x*, the exceptional set does depend on the choice of the representative of* $[x \mapsto [f(x,\cdot)]]$*. With a small abuse of notation, we still keep the notation* $P^f(x)$*,* $P^{x\mapsto[f(x,\cdot)]}$*, meaning implicitly that we have chosen a representative.*

Next we move to the case of V separable reflexive space and $K = V^*$, $1 < p < +\infty$. In this case, we can define the Banach space

$$L^p_x(L^p_y(V)) = L^p(E_x, \mathcal{E}_x, \lambda_x, L^{p'}(E_y, \mathcal{E}_y, \lambda_y, V; K); L^p(E_y, \mathcal{E}_y, \lambda_y, K; V)).$$

Proposition A.6. *Assume that* V *is a separable reflexive space and* $K = V^*$*, take* $1 < p < +\infty$*. Then* $L^p_{x,y}(V)$ *and* $L^p_x(L^p_y(V))$ *can be identified, in the sense that the map*

$$L^p_{x,y}(V) \ni [f] \mapsto [x \mapsto [y \mapsto f(x,y)]] \in L^p_x(L^p_y(V))$$

is a well-defined, bijective isometry between Banach spaces.

Proof. We only have to prove the surjectivity. For this, let $[F]$ be a map in $L^p_x(L^p_y(V))$ and choose a (weakly measurable) representative F; we must find a representative of $[F]$ which is jointly measurable in (x, y). By Proposition A.1, F is also strongly measurable, i.e. there exists a sequence $(F_n)_n$ of simple functions in $\mathcal{L}^p_x(L^p_y(V))$ which converges to F in $L^p_y(V)$ for a.e. x and, without loss of generality, in $L^p_x(L^p_y(V))$. We can write F_n as

$$F_n(x) = \sum_{k=1}^{N(n)} [F_{n,k}] 1_{A_{n,k}}(x)$$

for some measurable $A_{n,k}$ and some element $[F_{n,k}]$ in $L^p_y(V)$. Now we define, for each n, the map $G_n : E_x \times E_y \to V$ given by

$$G_n(x, y) = \sum_{k=1}^{N(n)} G_{n,k}(y) 1_{A_{n,k}}(x),$$

where $G_{n,k}$ is a representative of $[F_{n,k}]$. The function G_n is measurable, as sum of tensor products of measurable functions. Moreover, since $\|G_n - G_m\|_{L^p_{x,y}(V)} = \|F_n - F_m\|_{L^p_x(L^p_y(V))}$, the sequence $([G_n])_n$ is Cauchy in $L^p_{x,y}(V)$, therefore it converges to some $[G]$ in $L^p_{x,y}(V)$ (this space being complete by Proposition A.1). This implies that $[x \mapsto [y \mapsto G_n(x,y)]]$ converges to $[x \mapsto [y \mapsto G(x,y)]]$ in $L^p_x(L^p_y(V))$; since $[y \mapsto G_n(x,y)]$ is a representative of $F_n(x)$ and $[F_n]$ converges to $[F]$ in $L^p_x(L^p_y(V))$, it follows that $[F] = [x \mapsto [y \mapsto G(x,y)]]$. Hence G is the desired representative of F. $\square$

Now we come to a measurability problem on the product space. We assume that $V = K^*$, K is a separable Banach space and $1 < p < +\infty$. We take E_y as a "nice" (possibly unbounded) domain of $\mathbb{R}^e$ (for example balls, rectangles, stripes).

Lemma A.7. *Let f be a function in $\mathcal{L}^p_{x,y}(V)$ and assume that, for λ_x-a.e. x, $y \mapsto f(x,y)$ admits a weakly-* continuous version (with values in V); then there exists $\tilde{f} : E_x \times E_y \to V$, weakly-* measurable, representative of f, such that $\tilde{f}(x,\cdot)$ is weakly-* continuous for λ_x-a.e. x. In fact, any weakly-* continuous (in y) version $\tilde{f}$ of f is weakly-* measurable in (x,y).*

Proof. For n in $\mathbb{N}$, k in $\mathbb{Z}^e$, we take $y^n_k = 2^{-n}k$ and $r_n = 2^{-n-1}$; we define $Q_{r_n}(y^n_k)$ as the intersection between E_y and the cube of $\mathbb{R}^e$ centered in y^n_k and with edges of length $2r_n$ (we assume that $Q_{r_n}(y^n_k)$ form a disjoint partition of E into positive measure sets; this can be always done for nice domains, in the worst case via a slight modification in the definition of $Q_{r_n}(y^n_k)$). We define f^n as

$$f^n(x,y) = \sum_{k\in\mathbb{Z}^e, |k|\le n} 1_{Q_{r_n}(y^n_k)}(y) \fint_{Q_{r_n}(y^n_k)} f(x,y')\lambda_y(\mathrm{d}y')$$

This is a weakly-* measurable map. By assumption, there exists a full-λ_x measure set A in E_x such that, for all x in A, $y \mapsto f(x,y)$ admits a weakly-* continuous version. Then, for every (x,y) in $A \times E_y$ (calling $y^n_k(y)$ the unique point y^n_k such that y belongs to $Q_{r_n}(y^n_k)$), for every φ in K,

$$\langle f_n(x,y) - \tilde{f}(x,y), \varphi\rangle = \fint_{Q_{r_n}(y^n_k(y))} \langle f(x,y') - \tilde{f}(x,y), \varphi\rangle \lambda_y(\mathrm{d}y') \to 0$$

since $\tilde{f}$ is the continuous version of f. Then $\tilde{f}$ is the weakly-* pointwise limit of the weakly-* measurable maps f^n, on the measurable set $A \times E_y$. Therefore $\tilde{f}$ is weakly-* measurable in (x,y). $\square$

A.2 Spaces of functions and interpolation

Here we define the space of functions that we use in the thesis, according to the previous section. We also give an interpolation lemma and a statement on Itô formula.

Before going on, we recall a fact that we use in at least two arguments: every subset A of a separable metric space (X, d) is separable as well. Indeed, if D is a countable dense set in X, then the set $D_A = \{y(x,n) | x \in D, n \in \mathbb{N}\}$ is dense in A, where $y(x,n)$ is a point in A with $d(y(x,n), x) \leq (1/n) \vee \inf_{z\in A}(d(z,x)+1)$.

The countably generated probability space. We assume on $(\Omega, \mathcal{F}_{0,T}, P)$ Condition 3.1: $(\Omega, \mathcal{F}_{0,T})$ is a countably generated space, up to P-null sets. It implies:

Lemma A.8. *For all $s < t$, the σ-algebra $\mathcal{F}_{s,t}$ is countably generated. The progressive σ-algebra $\mathcal{P}$ is also countably generated.*

Proof. Fix $s < t$. The space S of indicator functions (precisely, classes of equivalence) on Ω adapted to $\mathcal{F}_{s,t}$ is a subset of $L^2(\Omega, \mathcal{F}_{0,T}, P)$, which is separable (since $\mathcal{F}_{0,T}$ is countably generated), so S must be also separable. Therefore there exists a countable set $\mathcal{C}$ in $\mathcal{F}_{s,t}$ such that every indicator function can be approximated in L^2 by an indicator function of a set in $\mathcal{C}$, this implies that $\mathcal{C}$ generates $\mathcal{F}_{s,t}$ up to P-null sets.

Similarly one proves that $\mathcal{P}$ is countably generated: $\mathcal{P}$ is a sub-σ algebra of $\mathcal{B}([0,T]) \times \mathcal{F}_{0,T}$, which is countably generated. □

Therefore we can apply the results of the previous Section to L^p_ω, whatever σ-algebra is involved.

The space $L^\infty_{t,\omega}(\mathcal{M}_x)$. The space $\mathcal{M}_x$ is the dual space of the separable space $C_{x,0}$ of continuous functions vanishing at infinity. Therefore we can define the space $L^\infty_{t,\omega}(\mathcal{M}_x)$, where the solutions to the SCE live, as the space of weakly-* measurable functions from $[0,T]\times\Omega$ with values in $\mathcal{M}_x$. Similarly in the deterministic context (without the variable ω) and for $\mathcal{M}_{x,\bar{B}_R}$. Notice that, when μ^ω_t has a density in L^1_x, then this density (which we still denote by μ) is in $L^\infty_{t,\omega}(L^1_x)$, the space of L^1_x-valued weakly measurable functions with L^1_x norm essentially bounded: in particular, μ can be identified with a measurable function in (t,x,ω).

On this space, we need the following:

Lemma A.9. *Fix $R > 0$. There exists a countable set D in $L^\infty_{t,\omega}(C_{x,\bar{B}_R})$, dense in $L^1_{t,\omega}(C_{x,\bar{B}_R})$, such that*

$$\|\mu\|_{L^\infty_t(\mathcal{M}_{x,\bar{B}_R})} = \sup_{G\in D, \|G\|_{L^1_{t,\omega}(C_{x,\bar{B}_R})}\leq 1} \int_0^T E[\langle \mu, G\rangle] \mathrm{d}r \tag{A.2}$$

Proof. First step: it holds

$$\|\mu\|_{L^\infty_t(\mathcal{M}_{x,\bar{B}_R})} = \sup_{G\in L^\infty_{t,\omega}(C_{x,\bar{B}_R}), \|G\|_{L^1_{t,\omega}(C_{x,\bar{B}_R})}\leq 1} \int_0^T E[\langle \mu, G\rangle] \mathrm{d}r. \tag{A.3}$$

To see this, it is enough to show the $\geq$ inequality (the $\leq$ inequality is a consequence of Hölder inequality). Fix μ in $L_t^\infty(\mathcal{M}_x)$. Let D_C be a countable set of $C_{x,\bar{B}_R}$, dense in $\{\varphi \in C_{x,\bar{B}_R} | \|\psi\|_{C_{x,\bar{B}_R}} \leq 1\}$; let $D_{t,\omega}$ be a countable set in $L^\infty_{t,\omega}$, dense in $\{\psi \in L^1_t | \|\psi\|_{L^1_{t,\omega}} \leq 1\}$. We order $\varphi_1, \varphi_2, \dots$ the elements in D_C. Fix n in $\mathbb{N}$, define A^n_k, $k = 1 \dots n$, by induction on k: $A^n_0 = \emptyset$,

$$
\begin{aligned}
&A^n_k \\
&= \{(t,\omega) \in [0,T] \times \Omega | \min_{h=1,\dots n} \|\mu^\omega_t\|_{\mathcal{M}_{x,B_R}} - \langle \mu^\omega_t, \varphi_h \rangle = \|\mu^\omega_t\|_{\mathcal{M}_{x,B_R}} - \langle \mu^\omega_t, \varphi_k \rangle\} \setminus \\
&\quad \setminus \cup_{m=1,\dots k-1} A^n_m.
\end{aligned}
$$

The family A^n_k forms a partition of $[0,T] \times \Omega$ into measurable disjoint sets. Finally, for n in $\mathbb{N}$, ψ in $D_{t,\omega}$, define

$$
G^{n,\psi}(t,\omega,x) = T^{-1}\psi(t,\omega) \sum_{k=1}^n 1_{A^n_k}(t,\omega)\varphi_k(x)
$$

For each n and ψ, $G^{n,\psi}$ is in $L^\infty_{t,\omega}(C_{x,\bar{B}_R})$ and

$$
\|G^{n,\psi}\|_{L^1_{t,\omega}(C_{x,\bar{B}_R})} \leq \|\psi\|_{L^1_{t,\omega}} \sup_k \|\varphi_k\|_{C_{x,\bar{B}_R}} \leq 1.
$$

Now, by the definition of A^n_k, for each (t,ω), $(\|\mu^\omega_t\|_{\mathcal{M}_{x,B_R}} - \langle \mu_t, \sum_{k=1}^n 1_{A^n_k}(t,\omega)\varphi_k \rangle)_n$ is non-negative non-increasing, bounded by $2\|\mu^\omega_t\|_{\mathcal{M}_{x,B_R}}$, converging to 0 as $n \to +\infty$. Therefore, for each ψ,

$$
\begin{aligned}
&\left| \int_0^T E[\|\mu\|_{\mathcal{M}_{x,\bar{B}_R}} \psi - \langle \mu, G^{n,\psi} \rangle] \mathrm{d}r \right| \\
&\leq \int_0^T E[|\psi|(\|\mu\|_{\mathcal{M}_{x,\bar{B}_R}} - \langle \mu, \sum_{k=1}^n 1_{A^n_k}(t,\omega)\varphi_k \rangle)] \mathrm{d}r \to 0.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|\mu\|_{L^\infty_t(\mathcal{M}_{x,\bar{B}_R})} &= \sup_{\psi \in D_{t,\omega}} \left| \int_0^T E[\|\mu\|_{\mathcal{M}_{x,\bar{B}_R}} \psi - \langle \mu, G^{n,\psi} \rangle] \mathrm{d}r \right| \\
&\leq \sup_{\psi \in D_{t,\omega}, n \in \mathbb{N}} \left| \int_0^T E[\langle \mu, G^{n,\psi} \rangle] \mathrm{d}r \right|.
\end{aligned}
$$

This proves (A.3). Notice that we obtained the norm $\|\mu\|_{L^\infty_t(\mathcal{M}_{x,\bar{B}_R})}$ as the supremum of a countable set, but this set depends on μ through A^n_k.

Second step: conclusion. Recall that $L^1_{t,\omega}(C_{x,\bar{B}_R})$ is a separable space (by Proposition A.1, since $C_{x,\bar{B}_R}$ is separable). Therefore the subspace of functions in $L^\infty_{t,\omega}(C_{x,\bar{B}_R})$ is also separable in the $L^1_{t,\omega}(C_{x,\bar{B}_R})$ topology. Hence (A.3) implies (A.2), taking as D a countable set in $L^\infty_{t,\omega}$, dense in $L^\infty_{t,\omega}(C_{x,\bar{B}_R})$ (and then in $L^1_{t,\omega}(C_{x,\bar{B}_R})$) with respect to the $L^1_{t,\omega}(C_{x,\bar{B}_R})$ topology. The proof is complete. □

The space $L^\infty_{t,\omega}(L^\infty_x)$. The space L^∞_x is the dual space of L^1_x. Therefore we can define the space $L^\infty_{t,\omega}(L^\infty_x)$, where the solutions to the STE live, as the space of weakly-* measurable functions from $[0,T]\times\Omega$ with values in L^∞_x. This space can be identified with the space $L^\infty_{t,x,\omega}$ by a suitable extension of Proposition A.6: every element v in $L^\infty_{t,\omega}(L^\infty_x)$ is also in $L^2_{t,\omega}(L^2_{x,B_R})$, for every R, hence it can be seen as an element in L^2_{t,x,ω,B_R} and, by the L^∞ bound, actually in $L^\infty_{t,x,\omega}$. Similarly in the deterministic context and for L^∞_{x,B_R}.

The spaces $L^p_{t,\omega}(L^p_{x,B_R,\chi})$ and $L^{p_t}_t(L^{p_\omega}_\omega(L^{p_x}_{x,B_R,\chi}))$. Given $1<p<+\infty$, $R>0$ (possibly $R=+\infty$ replacing B_R with $\mathbb{R}^d$), χ strictly positive function in C^∞_x, the space $L^p_{x,B_R,\chi}$ is defined as the space of measurable functions (precisely, classes of equivalence) f on B_R such that

$$\|f\|_{L^p_{x,B_R,\chi}} := \|\chi^{1/p}f\|_{L^p_{x,B_R}} = \left(\int_{B_R}\chi|f|^p\mathrm{d}x\right)^{1/p} < +\infty.$$

It is a separable reflexive space: indeed it is isomorphic, via the map $f\mapsto f\chi^{1/p}$, to the space L^p_{x,B_R}, which is separable and reflexive. Therefore we can define the space $L^p_{t,\omega}(L^p_{x,B_R,\chi})$, which can be identified with the space $L^p_{t,x,\omega}$, by Proposition A.6.

We can also define, for all finite exponents >1, $L^{p_t}_t(L^{p_\omega}_\omega(L^{p_x}_{x,B_R,\chi}))$, which are separable reflexive space by Proposition A.1.

In Proposition 3.48 we have also used the $L^{p_t}_t(L^{p_\omega}_\omega(L^{p_x}_{x,B_R,\chi}))$ norm for infinite exponents (only the norm, not the space): notice that, for any f in $L^2_{t,\omega}(L^2x,B_R)$, the norm $\|f\|_{L^{p_t}_t(L^{p_\omega}_\omega(L^{p_x}_{x,B_R,\chi}))}$ is well defined (possibly $+\infty$).

The spaces $W^{k,p}_{x,B_R}$ and $W^{k,p}_{x,B_R,0}$. Given a centered ball B_R in $\mathbb{R}^d$ (or, for $R=+\infty$, in $B_\infty:=\mathbb{R}^d$), given a C^∞_x strictly positive weight χ, given $1\le p\le+\infty$, the space $W^{1,p}_{x,B_R,\chi}$ is defined as the space of functions on B_R whose distributional derivative, in the domain B_R, lies in $L^p_{x,B_R,\chi}$. Precisely, a function f is defined to be in $W^{1,p}_{x,B_R,\chi}$ if, for every $i=1,\dots d$, there exists g_i in $L^p_{x,B_R,\chi}$ such that, for every φ in $C^\infty_{x,c}$ with support in B_R, it holds

$$\langle f,\partial_{x_i}\varphi\rangle = -\langle g_i,\varphi\rangle.$$

The function g_i is denoted by $\partial_{x_i}f$. The space $W^{1,p}_{x,B_R,\chi}$ is a Banach space with the norm $\|f\|_{W^{1,p}_{x,B_R,\chi}} = \|f\|_{L^p_{x,B_R,\chi}} + \|\nabla f\|_{L^p_{x,B_R,\chi}}$. The space $W^{1,p}_{x,B_R,\chi,0}$ is the closure (in $W^{1,p}_{x,B_R,\chi}$) of the $C^\infty_{x,c}$ functions with support in B_R; it is strictly contained in $W^{1,p}_{x,\chi}$ if $R < +\infty$ and it coincides with $W^{1,p}_{x,\chi}$ for $R = +\infty$.

The spaces $W^{1,p}_{x,B_R,\chi}$ and $W^{1,p}_{x,B_R,\chi,0}$ are isomorphic, via the map $f \mapsto (f, Df)$, to closed subspaces of $L^p_{x,B_R,\chi} \times L^p_{x,B_R,\chi}$, which is a separable space for $1 \le p < +\infty$ and reflexive for $1 < p < +\infty$; hence $W^{1,p}_{x,B_R,\chi}$ and $W^{1,p}_{x,B_R,\chi,0}$ are separable for $1 \le p < +\infty$ and reflexive for $1 < p < +\infty$ (recall that closed subspaces of reflexives spaces are reflexive).

These definitions and facts can be extended to $W^{k,p}_{x,B_R}$ and $W^{k,p}_{x,B_R,0}$, for k integer: $W^{k,p}_{x,B_R,\chi}$ and $W^{k,p}_{x,B_R,\chi,0}$ are separable for $1 \le p < +\infty$ and reflexive for $1 < p < +\infty$.

The spaces $L^p_{t,\omega}(W^{1,p}_{x,B_R,\chi})$ and $L^{p_t}_t(L^{p_\omega}_\omega(W^{1,p_x}_{x,B_R,\chi}))$. Given $1 < p < +\infty$, $R > 0$ (possibly $+\infty$ replacing B_R with $\mathbb{R}^d$), χ strictly positive function in C^∞_x, the space $W^{1,p}_{x,B_R,\chi}$ is a separable reflexive space. Therefore we can define the spaces $L^p_{t,\omega}(W^{1,p}_{x,B_R,\chi})$ and, for all finite exponents > 1, $L^{p_t}_t(L^{p_\omega}_\omega(W^{1,p_x}_{x,B_R,\chi}))$, which are separable reflexive space by Proposition A.1. In Proposition 3.42 we have also used the $L^{p_t}_t(L^{p_\omega}_\omega(W^{1,p_x}_{x,B_R,\chi}))$ norm for infinite exponents (only the norm, not the space): notice that, for any f in $L^2_{t,\omega}(L^2x, B_R)$, the norm $\|f\|_{L^{p_t}_t(L^{p_\omega}_\omega(W^{1,p_x}_{x,B_R,\chi}))}$ is well defined (possibly $+\infty$).

Fractional Sobolev norm. Given a (possibly unbounded) domain A of $\mathbb{R}^n$, a Banach space V, the fractional Sobolev norm $W^{\alpha,p}(V)$, $0 < \alpha \le 1$, $1 < p < +\infty$, of a function $f : A \to V$ is defined as

$$\|f\|_{W^{\alpha,p}_{x,A}(V)} = \|f\|_{L^p_{x,A}} + \left(\int_A \int_A \frac{\|f(x) - f(y)\|^p_V}{|x-y|^{d+\alpha p}} \mathrm{d}x\mathrm{d}y\right)^{1/p}.$$

This definition is extended to $p = +\infty$, $0 \le \alpha \le 1$, as the norm

$$\|f\|_{W^{\alpha,\infty}_{x,A}(V)} = \|f\|_{L^\infty_{x,A}} + \operatorname*{ess\text{-}sup}_{x \ne y} \frac{\|f(x) - f(y)\|_V}{|x-y|^\alpha}.$$

and it coincides with the $C^\alpha_{x,A}(V)$ norm when f has a continuous representative (which is the case at least for $V = \mathbb{R}$, by Sobolev embedding).

Fractional Sobolev space. When $V = \mathbb{R}$, $1 < p < +\infty$, the space $W^{\alpha,p}_{x,A} = W^{\alpha,p}_{x,A}(\mathbb{R})$ (defined by the measurable classes of equivalence f with finite $W^{\alpha,p}_{x,A}$ norm) is separable and reflexive: indeed it is isomorphic, via the map $f \mapsto (f, |f(x) - f(y)|/|x-y|^{\alpha + d/p})$, to a closed subspace of $L^p_x \otimes L^p_{x,y}$, which is separable and reflexive. Therefore we can define the space $L^p_\omega(W^{\alpha,\infty}_{x,A}$, which is separable and reflexive.

Analogous results hold when $W^{\alpha,p}_{x,A}$ is replaced by $W^{k+\alpha,p}_{x,A}$, for k integer, where the $W^{k+\alpha,p}_{x,A}$ norm is defined by

$$\|f\|_{W^{k+\alpha,p}_{x,A}(V)} = \|f\|_{W^{k,p}_{x,A}} + \left(\int_A \int_A \frac{\|D^k f(x) - D^k f(y)\|_V^p}{|x-y|^{d+\alpha p}} \mathrm{d}x\mathrm{d}y\right)^{1/p}.$$

The spaces C^α and $L^p_t(C^\alpha_x)$. For each integer k (possibly 0), for $R > 0$ finite, the space $C^k_{x,\bar{B}_R}$ is a separable space, hence we can define the spaces $L^p_t(C^k_{x,\bar{B}_R})$, $L^p_{t,\omega}(C^k_{x,\bar{B}_R})$; similarly for $C_{x,0}$. We also use the norms $L^p_t(C^\alpha_{x,B_R})$, $L^p_t(C^\alpha_{x,b})$, $L^p_t(C^k_{x,b})$, $L^p_t(C^k_{x,lin})$, as norms for Borel functions from $[0,T] \times \mathbb{R}^d$ to $\mathbb{R}$ (mind however that C^α_{x,B_R}, $C^\alpha_{x,b}$, ... are not separable).

Remark A.10. *For $0 \le \alpha' < \alpha < +\infty$, $W^{\alpha,\infty}_x$ (which can be identified with $C^\alpha_{x,b}$ when α is not an integer) is in the closure of C^∞_x with respect to the $W^{\alpha',\infty}_x$ norm. To see this, first notice the following interpolation inequality, of easy proof: $\|g\|_{W^{\alpha',\infty}_x} \le C\|g\|^{(1-\alpha+\alpha')/(\alpha-k)}_{W^{\alpha,\infty}_x}\|g\|^{(\alpha-\alpha')/(\alpha-k)}_{W^{k,\infty}_x}$, for every g in $W^{\alpha,\infty}_x$. Now, given f in $C^\alpha_{x,b}$ and the approximations f^ϵ given by the convolution with a standard mollifier, then $\|f - f^\epsilon\|_{W^{\alpha',\infty}_x} \le C\epsilon^{\alpha-\alpha'}\|f\|_{W^{\alpha,\infty}_x}$, as follows from the previous bound with $g = f - f^\epsilon$, the bound $\|f - f^\epsilon\|_{W^{\alpha,\infty}_x} \le C\|f\|_{W^{\alpha,\infty}_x}$ and the bound on $\|f - f^\epsilon\|_{W^{k,\infty}_x}$ coming from Lemma 9.3. Therefore f can be approximated in $W^{\alpha',\infty}_x$ by the C^∞_x functions f^ϵ.*

The $C^{\beta-1}_t$ norm. Given a function f in $C_{t,x}$, when we write f in $C^{\beta-1}_t(V)$ for some vector space V, we mean that F is in $C^\beta_t(V)$, where $F(t,x) = \int_0^t f(r,x)\mathrm{d}r$.

Localization. Here and in the following, we use the notation with *loc*. For example, when we say that a certain property holds for a function f with values in $L^p_{x,loc}$, we mean that the property holds for f restricted to B_R, as a L^p_{x,B_R}-valued function, for every $R > 0$; similarly for other spaces of functions.

The space $L^\infty_t(\mathcal{M}_{x,y,loc})$. The notation "$\mu \otimes v$ is in $L^\infty_t(\mathcal{M}_{x,y,loc})$" means that, for every $R > 0$, $\mu \otimes v$ is in $L^\infty_t(\mathcal{M}_{x,y,B_R})$.

Bounded-weak (progressive) measurability and the $L^1_t(\mathcal{M}_{x,loc})$ condition. For the continuity equation, we need to define two additional conditions. The first one is the bounded-weak (progressive) measurability: this means that, for every φ in BB_x (bounded Borel functions on $\mathbb{R}^d$), $(t,\omega) \mapsto \langle \mu_t, \varphi\rangle$ is (progressively) measurable.

The second condition is $|b||\mu|$ in $L^1_t(\mathcal{M}_{x,loc})$ for a.e. ω. This means that, for fixed ω (outside a P-null set), for every $R > 0$, for a.e. t, $1_{B_R}|b_t|$ is integrable with respect to $|\mu_t|$ (in particular, $1_{B_R}|b_t||\mu_t|$ is a well-defined signed measure on $\mathbb{R}^d$) and the map $t \mapsto \|1_{B_R}|b_t||\mu_t|\|_{\mathcal{M}_x}$ is in L^1_t. Notice that $t \mapsto \|1_{B_R}|b_t||\mu_t|\|_{\mathcal{M}_x}$ is measurable because, for every t,

$$\|1_{B_R}|b_t||\mu_t|\|_{\mathcal{M}_x} = \sup_{\varphi \in D, \|\varphi\|_{C_{x,b}} \leq 1} |\langle \mu_t, 1_{B_R} b_t \varphi \rangle|$$

for a countable dense set D in $C_{x,b}$ and $t \mapsto \langle \mu_t, 1_{B_R} b_t \varphi \rangle$ is measurable because of the bounded-weak measurability assumption.

Convolutions. Given μ in $L^\infty_{t,\omega}(\mathcal{M}_x)$ and a test function φ in $C^\infty_{x,c}$, the convolution map $[0,T] \times \mathbb{R}^d \times \Omega \ni (t,y,\omega) \mapsto \mu^\omega_t * \varphi(y) \in \mathbb{R}$ is measurable. Indeed we can add the variable y and consider μ as a map in $L^\infty_{t,y,\omega}(\mathcal{M}_x)$; so the convolution map, which reads as $(t,y,\omega) \mapsto \langle \varphi(y-\cdot), \mu^\omega_t \rangle$, is measurable by Remark A.3. Moreover, any continuous version (when it exists), with respect to t or y or (t,y), is measurable by Lemma A.7. Similarly for convolutions with functions in $L^p_{t,\omega}(L^p_{x,loc})$, $1 \leq p \leq +\infty$.

An interpolation lemma. Here we give a classical interpolation result, see for example in [Tri78], Section 2.7.2 Remark 1 for the $\mathbb{R}^d$ case and Section 4.5.2, Remarks 2 and 3 for the B_R case.

Proposition A.11. *Let $0 < \alpha < \beta < +\infty$, $0 < \rho < 1$; assume that $\alpha + \rho(\beta - \alpha)$ is not an integer. Then there exists $C > 0$ such that, for every f in $C^\beta_{x,b}$, it holds*

$$\|f\|_{C^{\alpha+\rho(\beta-\alpha)}_{x,b}} \leq C \|f\|^{1-\rho}_{C^\alpha_{x,b}} \|f\|^\rho_{C^\beta_{x,b}}.$$

The same result holds replacing the Hölder norms on $\mathbb{R}^d$ with the Hölder norms on B_R, for $R > 0$.

Itô formula for classes of equivalence. We conclude with a version of the Itô formula for $L^p_{t,\omega}$ classes of equivalence.

Proposition A.12. *Let $[X]$ be an element of $L^p_{t,\omega}$, $1 \leq p \leq +\infty$. Assume that there exists $[B]$, $[G]$ in $L^2_{t,\omega}$, progressively measurable, such that $[X]$ satisfies, for a.e. (t,ω)*

$$X_t = X_0 + \int_0^t B_r \mathrm{d}r + \int_0^t G_r \mathrm{d}W_r \tag{A.4}$$

(more precisely, the equality above holds as equality in $L^2_{t,\omega}$, i.e. between classes of equivalence). Let $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ be a function in $C_t(C^2_{x,b}) \cap$

$C_{x,b}(C_t^1)$. *Then it holds, for a.e.* (t,ω),

$$\begin{aligned} &f_t(X_t) - f_0(X_0) = \\ &= \int_0^t \partial_t f_r(X_r)\mathrm{d}r + \int_0^t \nabla f_r(X_r)\cdot B_r \mathrm{d}r + \\ &+ \int_0^t \nabla f_r(X_r)\cdot G_r \mathrm{d}W_r + \frac{1}{2}\int_0^t \mathrm{tr}[D^2 f_r(X_r) G_r G_r^*]\mathrm{d}r \end{aligned}$$

(more precisely, the equality above holds as equality in $L^2_{t,\omega}$*).*

Proof. By formula (A.4), there exists a version X of $[X]$ with continuous trajectories and satisfying (A.4) for every t, for a.e. ω (with the exceptional ω-set independent of t): indeed, we simply take X defined for every t by formula (A.4), taking the continuous versions of the integrals. Applying Itô formula for this version X, we find the thesis. □

A.3 Young integration theory

We review some facts on Young integration theory, which is made to give sense of integrals where the integrator path is not of BV type, but satisfies assumptions like Hölder continuity.

Given U, V Banach spaces, we call $\mathrm{Lin}(V,U)$ the set of linear continuous functions from V to U; it is a Banach space with the operator norm. The following result is at the basis of Young integration theory. It is due to Young [You36] for $\mathbb{R}$-valued functions. The extension on Banach space is taken from [FH14]: the statement below is a consequence of the Sewing Lemma 4.2 in [FH14], Chapter 4 (we do not need assumptions on $\mathrm{Lin}(V,\mathrm{Lin}(V,U))$ as in [FH14], since these are needed for the rough paths case).

Theorem A.13. *Fix* $0<\alpha<1$, $0<\beta<1$ *such that* $\alpha+\beta>1$. *Let* U, V *be Banach spaces. Then, for every* $0\le s\le t\le T$, *for every* $g:[0,T]\to V$ *in* $C_t^\beta(V)$, *for every* $f:[0,T]\to Lin(V,U)$ *in* $C_t^\alpha(Lin(V,U))$, *there exists the* U*-valued limit*

$$\int_s^t f_r \mathrm{d}g_r := \lim_n \sum_{[s',t']\in\Pi_n} f_{s'}(g_{t'}-g_{s'})$$

for every sequence $(\Pi_n)_n$ *of finite partitions of* $[s,t]$ *with infinitesimal size* $|\Pi_n| = \sup_{[s',t']\in\Pi_n}(t'-s')$*; the limit is independent of the choice of sequence* $(\Pi_n)_n$. *Moreover it holds*

$$\left|\int_s^t f_r \mathrm{d}g_r - f_s(g_t-g_s)\right| \le C|t-s|^{\alpha+\beta}\|f\|_{C_t^\alpha(Lin(V,U))}\|g\|_{C_t^\beta(V)}$$

for a constant C independent of s, t (in $[0,T]$) and of f and g. In particular

$$\|\int_0^t f_r \mathrm{d}g_r\|_{C_t^\beta(U)} \le \|f\|_{C_t^\alpha(Lin(V,U))}\|g\|_{C_t^\beta(V)}.$$

The integral above is called Young integral.

We conclude the section with an a priori estimate on the solution to linear Young differential equations.

Lemma A.14. *Fix $\beta > 1/2$. Let X be in C_t^1 and let Y be the solution to the linear equation*

$$\dot{Y}_t = Y_t \dot{X}_t.$$

Then it holds

$$\|Y\|_{C_t^\beta} \le C|Y_0| \exp[C(1+\|X\|_{C_t^\beta}^{1/\beta})](1+\|X\|_{C_t^\beta}).$$

For the proof, we use the notation

$$\|f\|_{C^\beta_{t,[u,v]}} := |f_u| + \sup_{u\le s<t\le v} \frac{|f_t - f_s|}{|t-s|^\beta},$$
$$\|f\|_{C_{t,[u,v]}} := \sup_{u\le t\le v} |f_t|.$$

Proof. We take $h>0$ to be fixed later and call $t_j = jh \wedge T$, $j = 0 \dots N$. By Young estimate, we have, for any j, for any $t_j \le s < t \le t_{j+1}$,

$$|Y_t - Y_s| \le |Y_s||X_t - X_s| + C|t-s|^{2\beta}\|Y\|_{C^\beta_{t,[t_j,t_{j+1}]}}\|X\|_{C_t^\beta}.$$

Dividing by $|t-s|^\beta$ (and adding the initial datum Y_{t_j}), we get

$$\|Y\|_{C^\beta_{t,[t_j,t_{j+1}]}} \le |Y_{t_j}| + \|Y\|_{C_{t,[t_j,t_{j+1}]}}\|X\|_{C_t^\beta} + C|t-s|^\beta\|Y\|_{C^\beta_{t,[t_j,t_{j+1}]}}\|X\|_{C_t^\beta}$$

and, using the inequality $\|Y\|_{C_{t,[t_j,t_{j+1}]}} \le |Y_{t_j}| + h^\beta\|Y\|_{C^\beta_{t,[t_j,t_{j+1}]}}$, we obtain

$$\|Y\|_{C^\beta_{t,[t_j,t_{j+1}]}} \le |Y_{t_j}|(1+\|X\|_{C_t^\beta}) + Ch^\beta\|Y\|_{C^\beta_{t,[t_j,t_{j+1}]}}\|X\|_{C_t^\beta}.$$

Now we fix h such that $Ch^\beta\|X\|_{C_t^\beta} < 1/2$, for example $h^\beta = 1/(2C\|X\|_{C_t^\beta}) \wedge 1$. We get

$$\|Y\|_{C^\beta_{t,[t_j,t_{j+1}]}} \le 2|Y_{t_j}|(1+\|X\|_{C_t^\beta})$$

and so

$$\|Y\|_{C_{t,[t_j,t_{j+1}]}} \le |Y_{t_j}| + h^\beta\|Y\|_{C^\beta_{t,[t_j,t_{j+1}]}} \le C|Y_{t_j}|.$$

Iterating this inequality in j, we get the global exponential bound

$$\|Y\|_{C_t} \leq |Y_0| \exp[CT/h] \leq |Y_0| \exp[CT(1 + \|X\|_{C_t^\beta}^{1/\beta})].$$

Finally we obtain

$$\begin{aligned}\|Y\|_{C_t^\beta} &\leq \|Y\|_{C_t} + \sup_j \|Y\|_{C^\beta_{t,[t_j,t_{j+1}]}} + \|Y\|_{C_t} h^{-\beta} \\ &\leq C|Y_0| \exp[CT(1 + \|X\|_{C_t^\beta}^{1/\beta})](1 + \|X\|_{C_t^\beta}).\end{aligned}$$

The proof is complete. $\square$

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