Scuola Normale Superiore

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PhD Thesis

Some problems in convex analysis across geometry and PDEs

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Contents

Introduction

In this thesis we treat problems which have connections with the study of some geometric properties of the solutions of certain elliptic or parabolic PDEs, such as symmetry, convexity, asymptotic behavior.

The study of the geometric properties of solutions of partial differential equations is an important part of modern analysis; on one hand, answering questions arising in such problems is a stimulating task from a theoretical point of view, on the other hand, this study has several applications in fields such as physical mathematics, engineering and so on.

For instance, it is well known that it is quite impossible to find the explicit expression for the solution of most of differential equations, however in many applications, it may be very helpful to deduce some symmetry results for the solutions, since they allow to significantly simplify the problem.

In some particular situations, symmetry properties are expected by virtue of a physical analysis of the problem. In these cases the study of the sufficient and necessary conditions for the solution to have such properties is fundamental for the consistency of the model.

In those situations it naturally arises the need to study overdetermined problems.

A peculiar feature of this kind of problems is that, as it typically happens, for instance, in shape optimization problems, the domain becomes an unknown variable, so that the problem can also be described in a purely geometric formalism, and symmetrization arguments, comparison estimates and geometric inequalities are fundamental tools adopted to solve these problems.

As we shall see, the first part of this work will be devoted to the study of the geometry of convex bodies and of some problems concerning them. In the second part we treat more directly problems concerning PDEs.

While the literature about overdetermined problems in the elliptic case is boundless (just think of the Serrin's problem concerning the torsion equation, see [Ser], and its several extensions), there are far fewer works dealing with this in the parabolic case, although there are still many stimulating questions to answer.

In this thesis our concern is mainly addressed to evolution equations: one important example of overdetermined problem in the parabolic case is the following conjecture.

Klamkin's conjecture [Kl]. Consider the heat conduction problem for a solid Ω,

$$u_t = \Delta u \quad \text{in} \quad \Omega \times (0, \infty).$$

Initially, $u = 0$. On the boundary $u = 1$. The solution to the problem is well-known for a sphere and, as to be expected, it is radially symmetric. Consequently, the equipotential surfaces do not vary with the time (the temperature on them, of course, varies). It is conjectured for the boundary value problem above, that the sphere is the only bounded solid having the property of invariant equipotential surfaces. If we allow unbounded solids, then another solution is the infinite right circular cylinder which corresponds to the spherical solution in two-dimensions.

L. Zalcman [Za] included this problem in a list of questions about the ball and named it the *Matzoh Ball Soup* problem. For the case of a bounded solid[1], the conjecture was given a positive answer by G. Alessandrini [Al1]: *the ball is the only bounded solid having the property of invariant equipotential surfaces.*

In [MS1], it is shown that, to obtain the spherical symmetry of the solid in Klamkin's setting, it is enough to require that the solution has *only one* invariant equipotential surface (provided this surface is a C^1-regular boundary of domain); to show this, authors proved a formula describing the asymptotic behavior of $u(x, t)$, for $t \to 0^+$. In a subsequent series of papers, the same authors extended their result in several directions: spherical symmetry also holds for certain nonlinear evolution equations ([MS2, MS4, MS6, MS7]); a hyperplane can be characterized as an invariant equipotential surface in the case of an unbounded solid that satisfies suitable sufficient conditions ([MS3, MS5]); spheres, infinite cylinders and planes are characterized as (single) invariant equipotential surfaces in $\mathbb{R}^3$ ([MPeS]); similar symmetry results can also be proven in the sphere and the hyperbolic space ([MS4]).

In [Al2], G. Alessandrini re-considered Klamkin's problem for a bounded domain in the case in which u initially equals any function $u_0 \in L^2(\Omega)$ and is zero on $\partial\Omega$ for all times. He discovered that either u_0 is a Dirichlet eigenfunction or Ω is a ball. A comparable result was obtained by S. Sakaguchi [Sak] when a homogeneous Neumann condition is in force on $\partial\Omega$.

In Chapter 4, we describe some results obtained in [MM3]. There it is shown that Klamkin's property of having invariant equipotential surfaces

[1] By solid we mean, according to [Al1] and [Za], a connected open set that coincides with the interior of its closure and whose complement is connected

characterizes a solution of the heat equation without assuming any whatsoever initial or boundary condition. In particular it is shown that solutions are either isoparametric[2] or split in space-time, see Theorem 4.1. The proof of Theorem 4.1 is based on the ideas exposed in [Al2] and [Sak]; the crucial observation is the following: suppose that $u(\cdot, t)$ is constant on the level sets of $u(\cdot, 0)$, for any fixed time t, then u depends only on the time and on the image of the map $u(\cdot, 0)$, namely there exists a function $\eta : \mathbb{R} \times [0, +\infty)$ such that $u(x, t) = \eta(u(x, 0), t)$. The study of the properties of the function η will lead to a classification of all possible solutions of the heat equation having time-invariant level surfaces (Section 4.1 contains all the details).

The same result extends to a class of quasi-linear parabolic partial differential equations with coefficients which are homogeneous functions of the gradient variable (see Theorem 4.5). This class includes the evolution p-Laplace equation, the normalized evolution p-Laplace equation and the (anisotropic) evolution h-Laplace equation[3].

Besides the Matzoh Ball Soup Problem, it is interesting to consider, as done in [MPS], the following initial value problem

$$\begin{cases} u_t = \Delta u & \text{in } \mathbb{R}^N \times (0, \infty), \\ u = \mathcal{X}_G & \text{in } \mathbb{R}^N \times \{0\}, \end{cases} \tag{1}$$

where G is a measurable subset of $\mathbb{R}^N$.

In [MPS] it is established a characterization of the possible stationary level surfaces of the solution u of (1). Besides the symmetric ones (spheres and cylinders), surprisingly, a helicoid is a possible invariant equipotential surface.

In [MPS] it is shown that if Γ is an invariant equipotential surface if and only if the density function

$$\delta(x, r) = \frac{V(G \cap B(x, r))}{V(B(x, r))}, \tag{2}$$

where V denote the N-dimensional Lebesgue measure, is such that

$$\delta(x, r) = c(r), \quad \text{for } x \in \Gamma. \tag{3}$$

Particularly relevant in [MPS] is the case when Γ is the boundary of G. Sets satisfying (3) for $\Gamma = \partial G$ are called *uniformly dense*, or *B-dense*, and a geometric analysis of their properties reveals fundamental to obtain the characterization of the time invariant equipotential surfaces for the initial-value problem (1).

The study of B-dense sets has been extended (see [ABG, MM1, MM2]) to the case in which, in the definition of the density function, the euclidean

[2]Isoparametric functions are defined in Section 4.1.

[3]Theese equation are defined in Section 4.1.

ball, B, is replaced by an arbitrary convex body K. In such a case these sets are called *K-dense*.

In the spirit of what has been done in [MPS], the study of K-dense sets is motivated by that of time-invariant level surfaces of solutions of the problem

$$\begin{cases} u_t = \Delta_h u & \text{in } \mathbb{R}^N \times (0, \infty), \\ u = \mathcal{X}_G & \text{in } \mathbb{R}^N \times \{0\}, \end{cases} \tag{4}$$

where $\Delta_h u = \mathrm{div}(Dh_K(Du)h_K(Du))$ denotes the Finsler laplacian of u generated by the support function h_K of K.

This problem is more difficult than the one considered in [MPS] –and in fact is still open– because it introduces a further possible unknown: the convex body K. In fact, it is not clear whether a time-invariant level surface may exists for (4) for any choice of K. Thus, the study of K-dense sets appears to be an important testbet to gain some more insight in that problem. Moreover, the study of K-dense sets is also an interesting question in convex geometry, which might have connections with some problems arising in the study of convolution bodies or floating bodies (see for instance [MRS] and [Sta]).

In [ABG] planar K-dense sets have been studied and it is proved that, if G is a C^2-regular convex body and K is C^4-regular, then G is K-dense if and only if G and K are homotetic to the same ellipse. Their proof can be obtained by computing the Taylor expansion for $r \to 0$ of the function $\delta(x, r)$ up to the third order; by imposing that every coefficient in the expansion does not depend on x, for $x \in \partial G$, it is possible to show that $K = G$, up to homotheties, and then, that a local parametrization of the boundary of G must satisfy a certain ordinary differential equation which has solution if and only if G is an ellipse.

In Chapter 3, we summarize all the results obtained in [MM1] and [MM2] about K-dense sets. In [MM1], there is an alternative proof of the characterization theorem of planar K-dense sets in which all the regularity assumptions needed in [ABG] are removed; indeed it is shown that these sets must be necessarily of class C^∞. For our alternative proof only the first two coefficient in the Taylor expansion of the density δ are needed. Indeed that proof combines the local information given by the study of the asymptotic behavior of $V(G \cap (x + rK))$, for small values of r, with some global informations provided by an affine inequality and the Minkowski's first inequality for mixed volume (see Theorem 3.16).

Moreover, in [MM1], are also established some properties of K-dense sets that hold in general dimension: in particular that N-dimensional K-dense sets are strictly convex and $C^{1,1}$-regular.

A geometrical analysis of the computations made in [ABG] allows to calculate in every dimension the first two coefficients in the Taylor expansion of the density function δ; however, as it will be explained in Chapter 3, it is not possible to reproduce the same proof of the characterization theorem when $N > 2$; a change of perspective is needed.

This change is adopted in [MM2], where in every dimension it is proved that, if G is K-dense, then G and K must be homothetic to the same ellipsoid (see Theorem 3.21). To show this, it is necessary to study the asymptotic behavior of the volume of $G \cap (x + rK)$, for large values of r. Sections 3.4 and 3.5 contain all the details of the proof.

Both in the proof of Theorem 3.16 and that of Theorem 3.21 we show that the support function h_K and the Gauss curvature κ of K satisfy a formula, named *Petty's identity*, namely it must hold that

$$h_K^{N+1} = c\kappa, \tag{5}$$

where c is a positive constant.

In [Pe] it is shown that every C^2 convex body that satisfies (5) is an ellipsoid and thus we can prove our characterization, since we were able to show that K-dense sets are sufficiently smooth.

In Chapter 2 there is a short essay about Petty's Theorem. There. we summarize Petty's arguments and in Theorem 2.10 we report [DM, Thm 1.1] which asserts that Petty's identity (5) characterizes ellipsoids without assuming any a priori regularity assumption.

The same statement can also be found in the new edition of [Scn], where there is also a brief outline of the proof of Theorem 2.10 based on Caffarelli's regularity results for the solutions of the Minkowski problem, see the Remark after [Scn, Theorem 10.5.1].

In the last chapter of this thesis we report a comparison theorem that we obtained in [MR] for a class of degenerate elliptic PDEs which allows to estimate the solution (or some symmetrization of it) in terms of a sort of fundamental (family of) solutions.

In the celebrated paper [Ta], G. Talenti established several comparison results between the solutions of the Poisson equation with Dirichlet boundary condition (with suitable data f and E):

$$-\Delta u = f \text{ in } E, \qquad u = 0 \text{ on } \partial E \tag{6}$$

and the solutions of the corresponding problem where f and E are replaced by their spherical rearrangements. Precisely, he proves that if we denote by v the solution of the problem with symmetrized data, then the rearrangement u^* of the (unique) solution u of (6) is pointwise bounded by v. Moreover he shows that the L^q norm of Du is bounded, as well, by the L^q norm of Dv, for $q \in (0, 2]$. The proof of these facts basically relies on two ingredients: the Hardy-Littlewood-Sobolev inequality and the isoperimetric inequality (see [AFP] and [LL] for comprehensive accounts on the subjects).

Later on, following such a scheme, many other works have been developed to prove analogous comparison results related to the solutions of PDEs involving different kind of operators (see for instance [BBMP1, BBMP2, BBMP3, BCM1, BCM2, dB, dBFP, TL] and the references therein). A recurring idea in these works is, roughly speaking, the following: the operator considered is usually linked to a sort of *weighted perimeter*. Thus initially it is necessary to solve a corresponding isoperimetric problem; then the desired comparison results can be obtained following the ideas contained in [Ta].

For example, in [BBMP2], the authors consider a class of weighted perimeters of the form

$$P_w(E) = \int_{\partial E} w(|x|)\, d\mathcal{H}^{N-1}(x),$$

where E is a set with Lipschitz boundary and $w : \mathbb{R} \to [0, \infty)$ is a non-negative function, and prove, under suitable convexity assumptions on the weight w, that the ball centered at the origin is the unique solution of the mixed isoperimetric problem

$$\min\{P_w(E) : V(E) = \text{constant}\}.$$

As a consequence they prove comparison results, analogous to those in [Ta], for the solutions of

$$-\text{div}(w^2 Du) = f \text{ in } E, \qquad u = 0 \text{ on } \partial E.$$

In [BDR], authors proved a quantitative version of the weighted isoperimetric inequality considered in [BBMP1]. Their proof is achieved by means of a sort of *calibration technique*. One advantage of this technique is that it is adaptable to other kind of problems, as that of considering other kind of functions in the weighted perimeter (e.g. Wulff-type weights, see [BF]), or that of considering different measured spaces, as $\mathbb{R}^N$ endowed with the Gauss measure.

In Chapter 5, we summarize the main results contained in [MR]. There it is considered degenerate elliptic equations with Dirichlet boundary condition of the form

$$-\text{div}(w^2\, e^V Du) = f\, e^V \text{ in } E, \qquad u = 0 \text{ on } \partial E$$

where w and V are two given functions, and it is proved analogous comparison results as those in [Ta]. The particular form in which is written the measure e^V is due to the later applications, whose main examples are Gauss-type measures, that is $V(x) = -c|x|^2$.

To obtain the desired comparison result it is solved a class of mixed isoperimetric problems of the form

$$\min\left\{P_{we^V}(E) : \int_E e^V = \text{constant}\right\}.$$

In particular it is proved, by means of a calibration technique reminiscent of that developed in [BDR], that minimizers, under suitable assumptions on V and w, are half-spaces.

Chapter 1

Preliminaries

In this chapter, we recall some of the important properties of convex sets and functions that will be useful in this thesis.

Section 1.1 deals with differential properties of convex functions and introduces the Legendre transformation, which defines an involution playing its role in the set of convex functions just as the polar transformation (defined in Section 1.2) does in the set of convex bodies.

Section 1.3 is devoted to the study of the "smoothing" effects of the so-called Minkowski addition. In Section 1.4 mixed volumes are defined and a number of geometric inequalities which will be used in the following chapters are discussed.

1.1 Fundamental properties of convex functions

This section contains a very short essay about peculiar properties of convex functions. In particular we are interested in the differential theory. We refer to [Ro, Part V] for a very exhaustive exposition of the topic.

A function $u : \Omega \subset \mathbb{R}^N \to [-\infty, +\infty]$ is said to be *convex* if its epigraph

$$\mathrm{epi}(u) = \{(x, t) \in \Omega \times \mathbb{R} \,:\, u(x) \leq t\}$$

is a convex subset of $\mathbb{R}^N \times \mathbb{R}$, moreover u is said to be *proper* if $\{u = -\infty\} = \varnothing$ and $\{u = \infty\} \neq \mathbb{R}^N$, u is said to be *closed* if its epigraph is closed[1]. In particular, it is easy to check that a function $u : \Omega \subset \mathbb{R}^N \to (-\infty, +\infty]$ is convex if and only if Ω is a convex set and

$$u(\lambda x + (1 - \lambda)y) \leq \lambda u(x) + (1 - \lambda)u(y),$$

[1] Closedness is clearly equivalent to lower semi-continuity.

for every $0 \leq \lambda \leq 1$ and every x and $y \in \Omega$. The *effective domain* $\mathrm{dom}(u)$ is the projecton of the epigraph on $\mathbb{R}^N$, namely

$$\mathrm{dom}(u) = \{x \in \Omega \,:\, u(x) < +\infty\}.$$

It is well known that a convex function u is continuous and bounded in its domain (see [Ro, Chapter 10]), moreover it is locally Lipschitz continuous ([Ro, Theorem 10.4]) in the interior of $\mathrm{dom}(u)$, thence, by Rademacher Theorem (see, for instance [EG]), we can say that u is differentiable at x, for almost every point $x \in \mathrm{dom}(u)$. However, it is usefull to introduce a weaker notion of differential, which coincides with the usual one for smooth functions [2].

Let u be a convex function defined on a convex open domain $\Omega \subset \mathbb{R}^N$, the *subdifferential* of u, ∂u, is the multi-valued map given by

$$\partial u(x) = \left\{p \in \mathbb{R}^N \,:\, u(y) \geq u(x) + \langle p, y - x\rangle, \text{ for any } y \in \Omega\right\}. \tag{1.1}$$

With the help of the terminology introduced in Section 1.2, we would say that the graph of the affine function $a(y) = u(x) + \langle p, y - x\rangle$ is a hyperplane supporting $\mathrm{epi}(u)$ at x.

In what follows we will mainly be concerned with proper closed convex functions defined on open convex subsets of $\mathbb{R}^N$. One of the reasons is that this condition ensures the existence of a non vertical tangent hyperplane to every point of the graph. In fact we have that $\partial u(x)$ is a nonempty set, for every x in the interior of dom(u), while for $x \notin \mathrm{dom}(u)$, $\partial u(x) = \varnothing$. A point in which $\partial u \neq \varnothing$ is said a *point of subdifferentiability* of u.

Since the subdifferential consists of solutions of a system of linear inequalities, it is always a convex set. Moreover

$$\Gamma_{\partial u} = \{(x, p) \,:\, p \in \partial u(x)\},$$

the *graph* of the subdifferential, is a closed subset of $\mathbb{R}^N \times \mathbb{R}^N$. The above fact can be used to prove that u is differentiable at x if and only if ∂u is a singleton [Ro, Section 25].

Remark 1.1. For every compact subset $K \subset \Omega$, the set $\partial u(K) = \cup_{x \in K} \partial u(x)$ is compact. Indeed, $\partial u(K)$ is bounded since convex functions are locally Lipschitz continuous, its closedness follows by that of Γ.

[2] The definition of subdifferential can be set, of course, for non-convex functions as well, but, as we should see, many of the "nice" properties of the subdifferential mentioned in this section do not hold true for an arbitrary real-valued function.

For a proper closed convex function $u : \Omega \to (-\infty, +\infty]$ we define the *conjugate* of u, u^* (in literature it is often used the alternative notation $\mathcal{L}u$, where $\mathcal{L}$ stands for the *Legendre-Fenchel transformation*, see [Ro, Sections 12, 26] and [Scn, Section 1.6]), as

$$u^*(p) = \sup_{x\in\Omega} \{\langle p, x\rangle - u(x)\} .$$

This function has a simple geometrical interpretation: we recall that a convex function u is the supremum over all affine functions $h \leq u$, the epigraph of u^* consists of all pairs (p,t) such that

$$\langle x, p\rangle - t \leq u(x),$$

for every $x \in \Omega$.

The conjugate of a proper closed convex function is a convex function itself and the Legendre-Fenchel transformation provides a *one-to-one* involution in the set of proper convex functions.

Remark 1.2.

(i) If Ω is bounded and u is bounded, then u^* is finite.

(ii) The subdifferentials of u and u^* are each one the inverse of one another as multivalued maps, that is $x \in \partial u^*(p)$ if and only if $p \in \partial u(x)$.

Indeed $p \in \partial u(x)$ if and only if the function $y \mapsto \langle p, y\rangle - u(y)$ has a maximum at x, namely $u^*(p) = \langle p, x\rangle - u(x)$. Since $u^{**} = u$, then

$$u^{**}(x) = \langle p, x\rangle - u^*(p),$$

that implies

$$\langle p, x\rangle - u^*(p) \geq \langle q, x\rangle - u^*(q),$$

for all q, namely $x \in \partial u^*(p)$.

In particular, if both u and u^* are differentiable functions, then

$$Du^*(Du(x)) = x. \tag{1.2}$$

In force of the above remark we can prove the following two propositions.

Proposition 1.3. *Let $u : \mathbb{R}^N \to \mathbb{R}$ be a finite convex function, then u is differentiable if and only if u^* is strictly convex.*

Proof. Suppose that a non-vertical relatively open segment, say ℓ, is contained in the graph of u. Let x and p be such that $(x, u(x)) \in \ell$ and $p \in \partial u(x)$, then ℓ is contained in the graph of the function $z \mapsto u(x) - \langle p, z - x\rangle$, and this entails that $p \in \partial u(y)$, for every $y \in \ell$, that is $y \in \partial u^*(p)$, for every y, and thence u^* cannot be differentiable at p. $\square$

Proposition 1.4. *Let Ω be an open convex set and let $u : \Omega \to \mathbb{R}$ be a convex function. Then the set*

$$E = \left\{ p \in \mathbb{R}^N \,:\, p \in \partial u(x) \cap \partial u(y),\, x \neq y \in \Omega \right\}$$

has zero Lebesgue measure.

Proof. By a standard approximation method we can reduce us to consider only bounded sets. In such a case $u^*(p)$ is finite, for every $p \in \mathbb{R}^N$. Let now $p \in E$, then there exist x and $y \in \Omega$, $x \neq y$, such that $x \in \partial u^*(p)$ and $y \in \partial u^*(p)$, that is

$$E \subset \left\{ p \in \mathbb{R}^N \,:\, u^* \text{ is not differentiable at } p \right\}.$$

Since u^* is a finite convex function, then the set on the left-hand side has zero Lebesgue measure. $\square$

1.2 Convex bodies, support functions and duality

Let $\mathcal{K}^N$ denote the set of *convex bodies* of $\mathbb{R}^N$, that is the set of compact convex sets with nonempty interior, and let $\mathcal{K}_0^N$ be the set of convex bodies with the origin lying in their interior.

We say that $H_K(x)$ is a hyperplane *supporting* K at x if $x \in H_K(x)$ and one of the two open half-spaces whose boundary is $H_K(x)$ has empty intersection with K. The boundary of K is differentiable at x if and only if there exists only one hyperplane supporting K at x.

For $x \in \partial K$, let us denote by $N_K(x)$ the *normal cone* of K at x, that is the set of all vectors ω such that

$$\langle \omega, y - x \rangle \leq 0,$$

for every $y \in K$.

If K is differentiable, $N_K(x)$ contains only one ray; in such a case we use to denote by $\nu_K(x)$ the unit vector generating the ray $N_K(x)$. If, for a point x, we can find $u \in N_K(x)$ such that $u \notin N_K(y)$, for any other $y \in \partial K$ we call x an *exposed point* of K. We say that x is an *extremal point* of K if $x \in K$ and $K \setminus x$ is convex.

For $K \in \mathcal{K}_0^N$ we define the *dual body* (or, equivalently, *polar body*) K^* as

$$K^* = \left\{ x \in \mathbb{R}^N \,:\, \langle x, y \rangle \leq 1, \text{ for any } y \in K \right\}.$$

The reason why K^* is called the dual body is the following: K^* canonically corresponds to the subset of the dual of $\mathbb{R}^N$ consisting of the linear applications mapping K into the unit ball.

It is easy to check that also $K^* \in \mathcal{K}_0^N$, moreover the operator *, as an application of $\mathcal{K}_0^N$ in itself, is an involution, namely $K^{**} = K$.

Before establishing a link between the concept of duality for sets and the one for functions we need to provide some further definitions.

In the following sections we will be concerned in describing the geometry of convex bodies in a functional way. To do this, we start from the following straightforward observation: any convex body is the intersection of all its supporting halfspaces, so we can get every significant information about the structure of the body by describing, for any direction $\omega \in \mathbb{S}^{N-1}$, the position of the supporting hyperplane orthogonal to ω. More precisely, given $K \in \mathcal{K}^N$ we define a function $\overline{h}_K : \mathbb{S}^{N-1} \to \mathbb{R}$ as

$$\overline{h}_K(\omega) = \sup \{\langle \omega, x\rangle \, : \, x \in K\} .$$

We denote by h_K the homogeneous (of degree one) extension to all $\mathbb{R}^N$ of the function $\overline{h}_K(\omega)$, namely $h_K(x) = |x|\overline{h}_K(x/|x|)$, and we call it *support function* of the convex body K.

The support function of a convex body is a proper convex function vanishing at the origin, clearly h_K is not differentiable at that point (its epigraph is a convex cone and the origin is its vertex). More precisely, the validity of the condition $\langle y, x\rangle \leq h_K(x)$, for every $y \in K$ and every $x \in \mathbb{R}^N$ entails that

$$\partial h_K(0) = K.$$

As reminded in Section 1.1, convex functions are almost everywhere differentiable, so one may ask whether the support function is differentiable far from the origin. The answer is in general negative, even if we restrict our attention to the set of smooth convex bodies. The obstruction to the regularity of h_K is the possible presence of "flat" parts in the boundary of K in a sense that will be made more precise by the following proposition.

Proposition 1.5. *Let $K \in \mathcal{K}^N$ and $x \in \mathbb{R}^N \setminus \{0\}$. Then*

$$\partial h_K(x) = \{z \in K \, : \, \langle x, z\rangle = h_K(x)\}.$$

In particular, the support function h_K is differentiable at x if and only if the set defined at the right-hand side contains only one elment, say y. In this case we have that

$$Dh_K(x) = y.$$

The above proposition can be read as follows: if no segment lies in the intersection of the support plane orthogonal to $\omega \in \mathbb{S}^{N-1}$ with K, then h_K is differentiable at $\lambda\omega$, for every $\lambda > 0$, and its gradient is the point in the boundary of K, say y, whose outer unit normal, $\nu_K(y)$ equals ω. A trivial corollary of the above proposition is that h_K is differentiable, for every $x \in \mathbb{R}^N \setminus \{0\}$ if and only if K is strictly convex.

There are other functions that can be used to describe a convex body. Let $K \in \mathcal{K}_0^N$, we denote by $\|\cdot\|_K : \mathbb{R}^N \to \mathbb{R}^+$ the function

$$\|x\|_K = \inf\{r > 0 \,:\, x \in rK\}$$

and we call it the *gauge* of the set K.

The gauge of K, just like the support function, is a proper convex function, homogeneous of degree one, and is not differentiable at the origin. The best way to understand the geometrical meaning of such a function is the following: if we denote by $\rho_K(\omega) = \sup\{\lambda \,:\, \lambda\omega \in K\}$ the so called *radial function* of K, then

$$\|x\|_K = \frac{\|x\|}{\rho_K(x/\|x\|)}. \tag{1.3}$$

The above representation of the gauge of K should give a meaning to the notation $\|\cdot\|_K$: indeed when K is centrally symmetric the gauge is a norm and K is the corresponding unit ball. Moreover, (1.3) should suggests that, with the exception of the origin, $\|\cdot\|_K$ has the same regularity of K. In particular $\|\cdot\|_K$ is differentiable at x if and only if the boundary of K is differentiable at the point $x/\|x\|_K$.

There is a deeper link between support and gauge functions, which will justify their introduction, indeed it is just a simple exercise to show that the support function of K^* is nothing else than the gauge of K, namely

$$\|x\|_K = h_{K^*}(x).$$

Moreover, since we have that

$$h_{K^*}(x) = \sup\{\langle x, y\rangle \,:\, h_K(y) \leq 1\} = \sup\{\langle x, y\rangle \,:\, h_K(y) = 1\},$$

we can write:

$$h_{K^*}(x) = \sup_{y \neq 0} \frac{\langle x, y\rangle}{h_K(y)}.$$

Similarly we obtain that

$$\|x\|_{K^*} = \sup_{y \neq 0} \frac{\langle x, y\rangle}{\|x\|_K}.$$

This means that, when K is centrally symmetric, $\|\cdot\|_{K^*} = h_K$ is the dual norm of $\|\cdot\|_K$.

Combining these observations with Proposition 1.5 we infer that, in analogy to what happens to dual functions, K has differentiable boundary if and only if K^* is strictly convex. In particular, if K is differentiable at x, then

$$Dh_{K^*}(x) = \nu_K(x)/h_K(\nu_K(x)),$$

and $\nu_K(x)/h_K(\nu_K(x)) \in \partial K^*$ is an exposed point.

In the following we shall often adopt the notation h_{K^*} to denote the gauge function of K and, if there is no risk of confusion, we shall drop the subscript, simply writing h^*.

The following proposition provides a strong link between duality of functions representing convex sets and duality of convex sets.

Proposition 1.6. *Let K be a convex body and let h and h^* denote its support and gauge function, respectively. Then*

$$\mathcal{L}\left(\frac{1}{2}h^{*2}\right) = \frac{1}{2}h^2. \tag{1.4}$$

Proof. The proof runs by performing a direct computation.

$$\begin{aligned}
\mathcal{L}\left(\frac{1}{2}h^{*2}\right)(x) &= \sup_{y\in\mathbb{R}^N}\left\{\langle x,y\rangle - \frac{1}{2}h^*(y)^2\right\} = \sup_{t>0}\sup_{y\in\partial tK}\left\{\langle x,y\rangle - \frac{1}{2}h^*(y)^2\right\}\\
&= \sup_{t>0}\sup_{y\in\partial tK}\left\{\langle x,y\rangle - \frac{1}{2}t^2\right\} = \sup_{t>0}\left\{\sup_{y\in\partial tK}\langle x,y\rangle - \frac{1}{2}t^2\right\}\\
&= \sup_{t>0}\left\{h(x)t - \frac{1}{2}t^2\right\} = \frac{1}{2}h^2(x).
\end{aligned}$$

In the fifth equality we used the fact that $\sup_{y\in\partial tK}\langle x,y\rangle = \sup_{y\in tK}\langle x,y\rangle$, and the fact that $h_{tK} = th_K$. To get the last equality it is enough to verify that $f(t) = t^2/2$ is a fixed point of the Legendre transformation, namely $\mathcal{L}f = f$. □

As a straightforward corollary we get that, if K is strictly convex and has differentiable boundary, then $\frac{1}{2}Dh^2$ and $\frac{1}{2}Dh^{*2}$ regarded as maps on $\mathbb{R}^N$ into itself are one the inverse of the other. In particular, by recalling equation (1.2), it holds true that

$$h^*(x)h(Dh^*(x))Dh(Dh^*(x)) = x, \tag{1.5}$$

where we used the fact that h and Dh are 1-homogeneous and 0-homogeneous, respectively.

We now consider more regular convex bodies; we say that a convex body K is C^k-regular if its boundary is a k-times continuously differentiable submanifold of $\mathbb{R}^N$, from now on we will simply say that a convex body K is C^k instead of C^k-regular.

Let K be a C^2 convex body; for $x \in \partial K$, let $T_x(K)$ denote the tangent space of ∂K at x and let $\nu : \partial K \to \mathbb{S}^{N-1}$ be the *Gauss map*, that is the application associating each point in the boundary to its outer unit normal. ν is C^1 and its differential, $W_x : T_x(K) \to T_{\nu(x)}(\mathbb{S}^{N-1}) = T_x(K)$, is called the *Weingarten operator*. W_x canonically induces a bilinear form on $T_x(K)$ that we call *shape operator* or *second fundamental form* and denote it by Π_x. When K is strictly convex the Gauss map defines a bijection between ∂K and $\mathbb{S}^{N-1}$; in such a case we can always consider the shape operator as a function of the normals and we define an application S_K by $S_K(\nu(x)) = \Pi_x$

The Weingarten operator contains important information about the shape of the boundary. For instance, it is well known that $W_x v \cdot v$ is the curvature of the planar curve $\partial K \cap \pi$, where π is the 2-plane spanned by $\nu(x)$ and a vector $v \in T_x(K)$. Moreover W_x is symmetric and positive (semi-definite), its eigenvalues are called *principal curvatures* and its eigenvectors *principal directions*. The determinant of W_x, $\kappa(x)$, is called *Gauss curvature*, while the trace is called *mean curvature*.

Remark 1.7. In Chapter 3 we will make use of the following equivalent definition of the shape operator: when K is C^2, the set ∂K is the graph of a C^2-regular convex function f over the tangent space to ∂K at x. The Weingarten operator has a coordinate representation which coincides with the Hessian of f (this can be easily seen by working in local coordinates and using the parametrization $X : T_xK \to \partial K$ given by $X(y) = x+y-f(y)\nu_K(x)$).

We say that K is *strongly convex* (K is C^2_+, for short) if the Gauss map is a diffeomorphism. Evidently strongly convex bodies are also strictly convex, and $S_K(\omega)$ is positive definite everywhere.

Remark 1.8. If K is C^2_+, then K^* is C^2. Indeed, since ν_K and ν_K^{-1} are C^1, again by Proposition 1.5, we have that $Dh_K(\omega) = \nu_K^{-1}(\omega)$, so that h_K is $C^2(\mathbb{R}^N \setminus \{0\})$, and thence K^* is C^2.

It is possible, for C^2_+ bodies, to express the shape operator in terms of the second derivatives of the support function. The following proposition will play a role in the next chapter.

Proposition 1.9. *Let K be a C^2_+ convex body; then it holds true*

$$D^2h(x) = \frac{S_G^{-1}(x/\|x\|)}{\|x\|}, \quad \text{for } x \in \mathbb{R}^N \setminus \{0\}. \tag{1.6}$$

The above equation should be interpreted as follows: for $v, w \in \mathbb{R}^N$ *one has*

$$D^2 h(x) v \cdot w = \frac{1}{\|x\|} S_G^{-1}(x/\|x\|) v' \cdot w',$$

where v' *and* w' *denote the projections of* v *and* w *on* $x^\perp$.

Proof. Set $\omega = x/\|x\|$; by homogeneity, we have to show that

$$D^2 h(\omega) v \cdot w = S_G^{-1}(\omega) v' \cdot w'.$$

Again by homogeneity

$$D^2 h(\omega)\omega = 0,$$

and thence one is left to show that

$$D^2 h(\omega) v \cdot w = S_G^{-1}(\omega) v \cdot w,$$

for $v, w \in \omega^\perp$.

Let σ be the 0-homogeneous extension to $\mathbb{R}^N \setminus \{0\}$ of the inverse Gauss map ν^{-1}. Remark 1.8 gives that

$$D^2 h(\omega) v \cdot w = D_v Dh \cdot w = D_v \sigma \cdot w,$$

where D_v denotes the partial derivative in the v direction. By the homogeneity of σ and since $v \in T_\omega \mathbb{S}^{N-1}$, then

$$D_v \sigma(\omega) = \left(D_\omega \nu^{-1}\right)(v) = S_G^{-1}(\omega) v,$$

that concludes the proof. □

1.3 Minkowski addition and regularity

In this section we introduce a simple operation between sets, which, nevertheless, has some useful and nontrivial properties.

Given subsets of $\mathbb{R}^N$, A and B, we define the *Minkowski addition*, $A+B$ as

$$A + B = \{a + b \,:\, a \in A,\, b \in B\}.$$

If A and B are convex (resp. strictly convex), then $A+B$ is convex (resp. strictly convex); if A and B are compact, then $A+B$ is compact. The set of convex bodies, $\mathcal{K}^N$, together with Minkowski addition in an abelian semigroup with unit $\{0\}$.

Let $K, L \in \mathcal{K}^N$; there is a functional way to describe the set $K + L$: indeed it is easy to show that

$$h_{K+L}(x) = h_K(x) + h_L(x). \tag{1.7}$$

The above equation entails that Minkowski addition fulfills the cancellation rule and in particular it holds true that if $K+M \subset L+M$, then $K \subset L$, for any K, L, $M \in \mathcal{K}^N$.

Minkowski addition can be used to define a distance in $\mathcal{K}^N$, the *Hausdorff distance* δ, which turns $\mathcal{K}^N$ into a complete metric space ([Scn, Theorem 1.8.3]), and for $K, L \in \mathcal{K}^N$ is defined by

$$\delta(K, L) = \min \{\lambda > 0 \, : \, K \subset L + \lambda B \text{ and } L \subset K + \lambda B\},$$

where B is the unit ball.

Remark 1.10. On one hand convergence in the Hausdorff distance may be considered a strong condition since, for instance, it implies that every point of the limit body K is a limit of points in the converging sequence of convex bodies K_i and also that every converging sequence of points $x_i \in K_i$ converges to a point $x \in K$. On the other hand such a notion of convergence is weak enough to allow some important compactness properties: as an instance, the celebrated *Blaschke selection theorem* states that every bounded sequence of convex bodies has a subsequence that converges to a convex body (see [Scn, Theorem 1.8.7]).

Minkowski addition can also be used to introduce a notion of parallelism. We say that a body L is *outer parallel* to a body K if, for some $r > 0$, $L = K + rB$[3].

In the following chapters we will need to investigate the regularity of the Minkowski addition of convex bodies.

Equation (1.7) suggests that $K + L$ might share some of the properties of the bodies K and L, however, as we noticed in the previous section, the regularity of the support function of a set K has connections with strict convexity properties of K itself. As we will see, a condition stronger than strict convexity is indeed necessary to get fine results on the smoothness of the Minkowski sum.

Nevertheless, the following proposition ensures that to get $C^{1,1}$-regularity[4]

[3] Compare with Sections 4.2 and 4.3 where we will introduce the concept of anisotropic parallelism.

[4] We say that a set E is $C^{1,1}$-regular if, for every point x in the boundary there exists a relatively open neighborhood $U_x \subset \partial E$ that is the graph of a differentiable function whose gradient is Lipschitz continuous.

for the Minkowski sum it is enough to impose regularity assumptions on one of the summands.

Proposition 1.11. *Let K and L be convex bodies, suppose that K is $C^{1,1}$, then $K+L$ is $C^{1,1}$.*

The above proposition has a simple geometric interpretation: let $x \in \partial(K+L)$, then $x = y+z$, for some $y \in K$ and $z \in L$. Without loss of generality we can assume that $y \in \partial K$; since K is $C^{1,1}$-regular in a neighborhood of y, then there exists a paraboloid touching the boundary of K from inside. Since the set $K+z$ is contained in $K+L$, then there exists a paraboloid touching the boundary of $K+L$ at the point $y+z$ from inside, and then $\partial(K+L)$ is $C^{1,1}$ at x[5].

As pointed out before, the sum of convex bodies is not, in general more regular than $C^{1,1}$ provided we impose additional assumptions on the summands. In the literature there are famous counterexamples: in [Bo1] it is shown that, in the plane, the sum of C^k bodies is C^k, for $k = 1, 2, 3, 4$, but, for every $\varepsilon > 0$ there exists C^∞-regular convex bodies K and L such that $K+L$ is not $C^{4+\varepsilon}$. In general dimension things are even worse: indeed there exist convex bodies with real analytic boundary whose sum is not C^2 (see [Bo2]).

The following theorem ensures higher differentiability for the set $K+L$ provided that at least one of the two summands is strongly convex.

Theorem 1.12 (Krantz, Parks, [KrPa]). *Let K and L be C^k convex bodies ($k \geq 2$). If K is strongly convex then $K+L$ is C^k as well, and it holds true that*

$$S_{K+L}(x+y) = \left[I + S_K(x)^{-1}S_L(y)\right]^{-1} S_L(y). \tag{1.8}$$

If L is stictly convex the shape operator of $K+L$ as a function of the normals is given by

$$S_{K+L}(u) = \left[I + S_K(u)^{-1}S_L(u)\right]^{-1} S_L(u). \tag{1.9}$$

Proof. Step 1. Let A be a positive definite $(N-1) \times (N-1)$ matrix and let B be a positive semi-definite $(N-1) \times (N-1)$ matrix. Let $M = \{x = (x', x_N) \in \mathbb{R}^N : x_N \geq Ax' \cdot x'\}$, and let $N = \{x \in \mathbb{R}^N : x_N \geq Bx' \cdot x'\}$.

Then $M + N = \{x \in \mathbb{R}^N : x_N \geq Cx' \cdot x'\}$, where

$$C = [I + A^{-1}B]^{-1}B.$$

[5] Compare this remark with the proof of Theorem 3.8.

To show this we choose a point $x \in \partial(M+N)$; there exist $y \in \partial M$ and $z \in \partial N$ such that $x = y + z$. Since M and L are differentiable, then, thanks to (1.7), $\nu_M(y) = \nu_N(z) = \nu_{M+N}(x)$. This condition can be written as

$$Ay' = Bz'.$$

Since A is invertible

$$y' = A^{-1}Bz'.$$

Thus

$$z' = \left(I + A^{-1}B\right)^{-1} x'$$

and

$$y' = A^{-1}B\left(I + A^{-1}B\right)^{-1} x'.$$

By recalling that $y_N = Ay' \cdot y'$ and $z_N = Bz' \cdot z'$, it is easy to compute:

$$x_N = y_N + z_N = [I + A^{-1}B]^{-1}Bx' \cdot x'.$$

Step 2. Let K and L be convex bodies satisfying the assumptions. Then $K + L$ is twice differentiable and (1.8) holds true.

Let $x \in \partial(K+L)$ and let $y \in \partial K$ and $z \in \partial L$ such that $y + z = x$. Again we have

$$\nu_K(y) = \nu_L(z) = \nu_{K+L}(x). \tag{1.10}$$

Without loss of generality we can then assume that $x = y = z = 0$ and that $\nu_{K+L}(x) = -e_N$. For $\varepsilon > 0$ in a small enough neighborhood of the origin we have

$$K_\varepsilon^- \subset K \subset K_\varepsilon^+,$$

where

$$K_\varepsilon^- = \left\{x \,:\, x_N \geq (1+\varepsilon)S_K x' \cdot x'\right\}$$

and

$$K_\varepsilon^+ = \left\{x \,:\, x_N \geq (1+\varepsilon)^{-1}S_K x' \cdot x'\right\}.$$

We can define L_ε^- and L_ε^+ in the same way getting $L_\varepsilon^- \subset L \subset L_\varepsilon^+$.

Thanks to Step 1, the shape operators of $K_\varepsilon^- + L_\varepsilon^-$ and $K_\varepsilon^+ + L_\varepsilon^+$ are given by

$$(1+\varepsilon)\left[I + S_K(x')^{-1}S_L(x')\right]^{-1} S_L(x')$$

and

$$(1+\varepsilon)^{-1}\left[I + S_K(x')^{-1}S_L(x')\right]^{-1} S_L(x')^6,$$

[6] With a slight abuse of notation we write S_K and S_L as functions defined on the tangent space, this should not lead to any possibility of confusion since we are working in a arbitrary small neighborhood of 0 and we can always use the parametrization introduced in Remark 1.7.

respectively.

Moreover

$$K_\varepsilon^- + L_\varepsilon^- \subset K + L \subset K_\varepsilon^+ + L_\varepsilon^+,$$

at least in a small neighborhood of the origin.

In particular, for x' small enough, if we denote by $\varphi(x')$ the convex function such that $(x', \varphi(x')) \in \partial(K+L)$, it holds true that

$$(1+\varepsilon)^{-1}\left[\left(I + S_K^{-1}S_L\right)^{-1} S_L\right] x' \cdot x' \le \varphi(x') \tag{1.11}$$

$$\le (1+\varepsilon)\left[\left(I + S_K^{-1}S_L\right)^{-1} S_L\right] x' \cdot x'. \tag{1.12}$$

The above equation gives the twice differentiability of φ at the point $x' = 0$.

Conclusion. Taking the limit for $\varepsilon \to 0^+$ we get that the shape operator of $K+L$ is actually given by (1.8). When K and L are C^k-regular then S_K and S_L, and thus S_{K+L} are C^{k-2}, and thence $K+L$ is C^k-regular.

Equation (1.9) is then a straightforward consequence of (1.8) and (1.10).

□

1.4 Mixed volumes and affine inequalities

To start with, let us say that a geometric analysis of convex bodies can be carried out by studying inequalities involving some suitable functionals on $\mathcal{K}^N$. Their study has become a key topic in convex analysis: on one hand it has deeply increased our knowledge of the geometry of convex bodies and it allowed to solve a number of long-standing problems, on the other hand it has several applications in very many fields of modern analysis.

In the first part of this section we will see how, combining the concept of volume and Minkowski addition, it is possible to introduce some functionals on convex bodies whose study gave rise to one of the most elegant and inspiring theory in convex geometry: the so called *Brunn-Minkowski Theory.*

Let $V(\cdot)$ denote the N-dimensional Lebesgue measure, and let K and L be convex bodies. The following is the celebrated *Brunn-Minkowski inequality* asserting that the 1-homogeneous volume $V^{1/N}$ is a concave functional on $\mathcal{K}^N$:

$$V((1-\lambda)K + \lambda L)^{1/N} \ge (1-\lambda)V(K)^{1/N} + \lambda V(L)^{1/N}, \quad \lambda \in [0,1], \tag{1.13}$$

with equality if and only if K and L are homothetic, that is, if and only if $K = x + tL$, for some $x \in \mathbb{R}^N$ and $t > 0$.

The above inequality can be extended to bounded measurable sets. It has connections with a large number of other geometrical and functional inequalities. Its history, proof, and applications can be found in the survey by R. Gardner [Ga].

The following theorem, obtained by Minkowski in $\mathbb{R}^3$ in [Mi3] is a generalization of Steiner Formula ([Ste]).

Theorem 1.13 (Minkowski)**.** *Let $K_1, \dots, K_i \in \mathcal{K}^N$ and let $g(\lambda_1, \dots, \lambda_i) = V(\lambda_1 K_1 + \dots + \lambda_i K_i)$, then*

$$g = \sum_{j_1=1}^{i} \dots \sum_{j_N=1}^{i} \lambda_{j_1} \cdots \lambda_{j_N} V(K_{j_1}, \dots, K_{j_N}).$$

In particular, g is a homogeneous polynomial of degree N.

The coefficients of g are called *mixed volumes.*

Mixed volumes are important tools which can be used to investigate the intrinsic geometry of convex bodies. Here we summarize their main properties. We refer to [Scn, Chapter 5] or [S-Y] and the references therein for an interesting and complete dissertation on the topic.

Mixed volumes are positive and symmetric functionals that are linear in each argument, continuous with respect to the Hausdorff convergence (since is so Minkowski addition and the volume functional), and, clearly, it holds that $V(K, \dots, K) = V(K)$.

Moreover, they are increasing in each component, in the sense that, for any choice of convex bodies $K_1, \dots, K_i$, we have that

$$V(K, K_1, \dots, K_i) \leq V(L, K_1, \dots, K_i),$$

whenever $K \subseteq L$.

If we restrict the class of competing arguments in the definition of mixed volumes we find other interesting functionals.

An important instance is the mixed volume

$$V_1(K, L) = V(K, K, \dots, K, L) = \frac{1}{N} \lim_{\varepsilon \to 0^+} \frac{V(K + \varepsilon L) - V(K)}{\varepsilon}. \tag{1.14}$$

In some sense, for any fixed L, $V_1(\cdot, L)$ is a perimeter functional on $\mathcal{K}^N$. Indeed, when L is the unit ball, B, $NV_1(K, B)$ turns out to be the (euclidean) *perimeter* $P(K)$ of K. For general convex bodies K and L, $V_1(K,L)$ represent an anisotripic perimeter having the following integral representation[7]

$$V_1(K,L) = \frac{1}{N}\int_{\partial K} h_L(\nu_K(x))\,d\mathcal{H}^{N-1}(x).$$

Remark 1.14. When K is C^2_+-regular the change of variable $u=\nu_K(x)$ gives

$$V_1(K,L) = \frac{1}{N}\int_{\mathbb{S}^{N-1}} \frac{h_L(u)}{\kappa_K(u)}\,d\sigma,$$

where σ is the Hausdorff measure on the unit sphere and we recall that κ_K is the Gauss curvature of ∂K. The above equation suggests that it is possible to use mixed volumes to get continuous and linear functionals on $C(\mathbb{S}^{N-1})$. Indeed, for each support function h_L, one could set

$$F_K(h_L) = V_1(K,L).$$

It is possible to show that F_K can be (uniquely) extended as a linear and continuous functional on $C(\mathbb{S}^{N-1})$ and then, thanks to Riesz representation theorem, to get a Radon measure μ_K such that

$$V_1(K,L) = \int_{\mathbb{S}^{N-1}} h_L\,d\mu_K.$$

Such a measure is called the *area measure* of K (compare this approach to the one explained in Section 2.2).

From (1.13) it easily follows an isoperimetric-type inequality for convex bodies. We start by setting $\lambda = \varepsilon/(1-\varepsilon)$ in (1.14); we obtain

$$\begin{aligned} NV_1(K,L) &= \lim_{\lambda\to 0^+} \frac{V((1-\lambda)K+\lambda L) - (1-\lambda)^N V((1-\lambda)K)}{\lambda(1-\lambda)^{N-1}} \\ &= \lim_{\lambda\to 0^+} \frac{V((1-\lambda)K+\lambda L) - (1-N\lambda)V(K)}{\lambda} \\ &= \lim_{\lambda\to 0^+} \frac{V((1-\lambda)K+\lambda L) - V(K)}{\lambda} + NV(K). \end{aligned} \tag{1.15}$$

Let g as in Theorem 1.13 and set $f(\lambda) = g(1-\lambda,\lambda)$. We have that

$$f'(0) = \frac{V(K)^{-1+1/N}}{N} \lim_{\lambda\to 0^+} \frac{V((1-\lambda)K+\lambda L) - V(K)}{\lambda}.$$

Thanks to (1.15) we get that

$$f'(0) = \frac{V(K)^{-1+1/N}}{N}(NV_1(K,L) - NV(K)).$$

[7] Compare with the Wulff problem, see, for instance [Ma, Chapter 20].

Equation (1.13) imply that f is a concave function in the λ-variable, in particular $f'(0) \geq f(1) - f(0)$, namely

$$V_1(K, L) \geq V(K)^{\frac{N}{N-1}} V(L)^{\frac{1}{N}}. \tag{1.16}$$

Equality cases can be easily characterized from those of the Brunn-Minkowski inequality: more precisely equality holds in (1.16) if and only if K and L are homothetic.

The above inequality is the so-called *Minkowski's first inequality* and it is a special instance of the *Aleksandrov-Fenchel inequality* (see [Scn, Chapter 7]). When $L = B$ it reduces to the classic isoperimetric inequality for convex bodies, stating that

$$\frac{V(K)}{V(B)} \leq \frac{P(K)^{\frac{N-1}{N}}}{P(B)^{\frac{N-1}{N}}}.$$

Here P denotes the perimeter of K, since $P(K) = NV_1(K, B)$ as already observed.

The *affine surface area* of a C^2_+ convex body is defined as

$$\Omega(K) = \int_{\mathbb{S}^{N-1}} \kappa(u)^{-\frac{N}{N+1}} d\sigma.$$

The *affine isoperimetric inequality* states that

$$\Omega(K)^{N+1} \leq N^{N+1} \omega_N^2 V(K)^{N-1}, \tag{1.17}$$

and equality holds if and only if K is an ellipsoid.

In [Pe] (1.17) has been extended and proved for all convex bodies which possess a positive and continuous curvature function (for its definition, see Section 2.2).

In Chapter 3, we will need the following stronger version of (1.17):

$$\Omega(K)^{N+1} \leq N^{N+1} V(K)^N V(K^*), \tag{1.18}$$

and equality holds if and only if K is an ellipsoid, see [Hu, Lemma 3.7] or [Lu3][8].

[8]There, this equality condition is proved only for C^2 bodies, but, as for the affine isoperimetric inequality, it is possible to show that (1.18) holds true for bodies admitting a continuous and positive curvature function. The equality condition found in those papers is that there exists a positive constant c, such that the curvature function f_K defined in Section 2.2 satisfies $f_K = ch_K^{-N-1}$. In Chapter 2 we show how, without assuming any a priori regularity assumption, it is possible to use such a condition to characterize ellipsoids.

To understand the reason why (1.18) is stronger than (1.17) we must introduce yet another affine inequality, the *Blaschke-Santalò inequality* (1.20) below.

By generalizing the definition given in Section 1.2 we say that the polar of a convex body K with respect to a point p is the set

$$K_p^* = \left\{x \in \mathbb{R}^N \, : \langle x, y - p\rangle \leq 1, \quad \text{for all } y \in K\right\}.$$

One can show that there exists a unique point, p_K, such that

$$V(K_{p_K}^*) = \min\{V(K_p^*) \, : \, p \in \mathbb{R}^N\}; \tag{1.19}$$

p_K is the so-called *Santalò point* of K. The Blaschke-Santalò inequality then states the following estimate for the volume of the polar set $K_{p_K}^*$:

$$V(K)V(K_{p_K}^*) \leq \omega_N^2, \tag{1.20}$$

and equality holds if and only if K (and thus $K_{p_K}^*$) is an ellipsoid.

Clearly, plugging (1.20) into (1.18) immediately gives (1.17).

Chapter 2

Petty's Identity

This chapter is entirely dedicated to the study of a formula, the so-called *Petty's identity*, which characterizes ellipsoids among convex bodies.

In Section 2.1, we provide a proof of Petty's theorem for smooth convex bodies and, in Section 2.2, we define the so-called *area measure*, that we use to get a weaker notion of Gauss curvature which gives rise to an alternative formulation of Petty's identity. In Section 2.4, we prove Petty's theorem without assuming any a priori regularity assumption; to do this, we exploit the regularity theory for *Aleksandrov solutions* to the *Monge-Ampère* equation summarized in Section 2.3.

2.1 Petty's theorem

As stressed in Section 1.4, affine inequalities play a very important role in the study of the geometry of convex bodies and they also find applications in several different fields (*e.g.* ordinary and partial differential equation, functional analysis).

In [Pe] Petty treated three closely related affine problems, namely the Blaschke-Santalò inequality, the affine isoperimetric inequality and the geominimal surface area inequality, and he characterized ellipsoids as the only extremal bodies for these inequalities.

In order to establish this characterization, he proved that if $K \subset \mathbb{R}^N$ is an extremal convex body for these inequalities, then necessarily there must exist a positive constant c_K such that

$$f_K(\omega) = c_K\, h_K^{-N-1}(\omega), \tag{2.1}$$

for every $\omega \in \mathbb{S}^{N-1}$. Here h_K is the support function defined in Section 1.2, and f_K is a continuous and positive function called *curvature function* of K

(see Section 2.2 for the rigorous definition of this function and some more comments on this equation).

Petty was then able to show that (2.1) implies that K is an ellipsoid if $N = 2$. If $N \geq 3$ he obtained the same result only under the assumption that K is a C^2-regular convex body or that K is a body of revolution. In any case this was sufficient to prove that extremal sets for the above mentioned problems are ellipsoids, since symmetrization techniques allow to reduce to the case of axially symmetric sets.

It is however an interesting question to understand to which extent (2.1) characterizes ellipsoids without assuming any a priori regularity assumption on K, see for instance [Lu] on this issue. In [DM] it is shown that every convex body satisfying (2.1) is actually an ellipsoid. That result is reported in Section 2.4, Theorem 2.10.

Besides its own interest, Theorem 2.10 will benefit those problems in which extremal bodies with unknown a priori regularity, are characterized by (2.1), let alone it can provide new shorter proofs of results in which (2.1) appears. As examples let us quote [MRS, Sta, MM1, MM2] concerning, respectively, *convolution bodies*, *floating bodies*, and the so-called *K-dense sets* which are treated in detail in the next chapter.

We now state Petty's theorem.

Theorem 2.1 ([Pe])**.** *Let K be a C^2-regular convex body and suppose that there exists a constant $c_K > 0$ such that*

$$\kappa(\omega) = c_K h_K^{N+1}(\omega), \tag{2.2}$$

for every $\omega \in \mathbb{S}^{N-1}$. Then K is an ellispoid.

Remark 2.2. Here κ is the determinant of the shape operator of K. Since K satisfies equation (2.2), it follows that the shape operator is positive definite everywhere, and then K is C^2_+. In particular, K is also strictly convex and we can always think S_K, and thence κ, as functions of the normals, thus giving sense to equation (2.2).

The following proposition is the main step for the proof of Theorem 2.1.

Proposition 2.3. *Let K a C^2_+-regular convex body. Then*

$$\det\left[D^2\left(\frac{1}{2}h^2\right)(\omega)\right] = h^{N+1}(\omega)\kappa(\omega)^{-1}. \tag{2.3}$$

Proof. First, we have that

$$D^2\left(\frac{1}{2}h^2\right)(\omega) = Dh(\omega) \otimes Dh(\omega) + h(\omega)D^2h(\omega).$$

We can choose a coordinate system in which $e_N = \omega$, and $e_1, \dots, e_{N-1}$ are the eigenvectors of $S_K(\omega)$. In such coordinates, by recalling Proposition 1.9, we know that $D^2 h$ is a diagonal matrix and its diagonal entries are given by: $\left(D^2 h\right)_{N,N} = 0$, $\left(D^2 h\right)_{i,i} = k_i^{-1}$, for $i = 1, \dots, N-1$, where k_i is the i-th principal curvature of ∂K at $Dh(\omega)$.

Let now M be the $N \times N$ matrix such that $MDh = e_N$ and $Me_i = e_i$ for every $i = 1, \dots, N-1$. Notice that, by homogeneity, we have that

$$Dh \cdot e_N = h(e_N) > 0,$$

that means that M exists and it is an isomorphism. Moreover the columns M^i of M can be explicitly computed: in fact, for $i = 1, \dots, N-1$ $M^i = e_i$ and

$$M^N = \frac{1}{Dh \cdot e_N} e_N - \sum_{i=1}^{N-1} \frac{Dh \cdot e_i}{Dh \cdot e_N} e_i.$$

In particular, we get that $\det M = (Dh \cdot e_N)^{-1} = h(e_N)^{-1}$.

Straightforward but tedious computations give that $M\left(Dh \otimes Dh + hD^2 h\right)$ is the same matrix as $hD^2 h$ except the N-th row, which is Dh. Being thus $M\left(Dh \otimes Dh + hD^2 h\right)$ a triangular matrix, its determinant is the product of the diagonal elements, namely

$$\det\left[M\left(Dh \otimes Dh + hD^2 h\right)\right] = \frac{h^{N-1} Dh \cdot e_N}{k_1 \cdots k_{N-1}}.$$

Thus, finally

$$\det\left[Dh \otimes Dh + hD^2 h\right] = \frac{h^N}{\kappa} \det\left(M^{-1}\right) = \frac{h^{N+1}}{\kappa}.$$

□

Proposition 2.3 together with Remark 2.2 yield the following corollary.

Corollary 2.4. *Let K be a C^2 convex body satisfying* (2.2)*. Then, for every $x \in \mathbb{R}^N \setminus \{0\}$ the support function h satisfies the following equation:*

$$\det \frac{1}{2} D^2 h^2(x) = c. \tag{2.4}$$

Equation (2.4) is a *Monge-Ampère* equation. We refer to Section 2.3 and the references therein for a short summary of the main properties of its solutions.

To prove Theorem 2.1, Petty relied on regularity results for the Monge-Ampère equation and in particular, thanks to [Po2, Chapter 5, Theorem 5], he was able to show that h is analytic in $\mathbb{R}^N \setminus \{0\}$ and then, thanks to [Br], to conclude that D^2h^2 is a constant matrix.

In Section 2.4 we closely follow Petty's strategy. By an approximation procedure we show that if a convex set K satisfies (2.1), then its support function still satisfies (2.4) in the Aleksandrov sense.

By relying on standard techniques one can then show that any Aleksandrov solution of (2.4) is smooth and hence, by applying classical results due to Pogorelov [Po1, Po2, Po3], a quadratic polynomial.

2.2 Curvature functions

There are several almost equivalent ways to introduce the concept of surface area measure and curvature functions leading to interesting geometrical interpretations. Here we follow the ideas of Aleksandrov [Alek].

We can associate a convex body K with a measure μ_K supported on the unit sphere, called the *surface area measure*, with the property that, for every Borel set $A \subset \mathbb{S}^{N-1}$, $\mu_K(A)$ is the $(N-1)$-dimensional Hausdorff measure of the set of the points in the boundary of K whose normal cone has nonempty intersection with A. More precisely, if for $x \in \partial K$, we define a possibly multivalued map by setting

$$\overline{N}_K(x) = \big\{\omega \in \mathbb{S}^{N-1} : \omega \cdot (y - x) \leq 0 \ \text{ for all } y \in K\big\},$$

then

$$\mu_K(A) = \mathcal{H}^{N-1}(\overline{N}_K^{-1}(A)).$$

This definition coincides with the one suggested in Remark 1.14 by virtue of [Scn, Theorem 4.2.3].

It is possible to show (see [Scn2, Proposition 4.10]) that such a measure is continuous in the K-variable with respect to the Hausdorff convergence. Namely,

$$\lim_i \int_{\mathbb{S}^{N-1}} \varphi \, d\mu_{K_i} = \int_{\mathbb{S}^{N-1}} \varphi \, d\mu_K, \tag{2.5}$$

for every $\varphi \in C(\mathbb{S}^{N-1})$, whenever $\delta(K_i, K) \to 0$, as defined in Section 1.3.

When K is C^2_+, as observed in Remark 1.14, the surface area measure is absolutely continuous with respect to the Hausdorff measure $\mathcal{H}^{N-1} \llcorner \mathbb{S}^{N-1}$ and its density is given by κ^{-1} (as usual, when dealing with C^2_+ bodies we denote, with a slight abuse of notation, by κ the Gauss curvature as a function of the normals).

A convex body K is said to possess a *curvature function* provided there exists a positive and continuous function $f_K : \mathbb{S}^{N-1} \to \mathbb{R}$ such that

$$\mu_K = f_K \mathcal{H}^{N-1} \llcorner \mathbb{S}^{N-1}.$$

Conversely, given a positive and continuous function $f : \mathbb{S}^{N-1} \to \mathbb{R}$, Minkowski's existence and uniqueness theorem (see [CY, Mi1, Mi2, Ni, Po2]) asserts that, provided f fulfills the necessary condition

$$\int_{\mathbb{S}^{N-1}} \omega f(\omega)\, d\mathcal{H}^{N-1}(\omega) = 0\,, \tag{2.6}$$

then there exists a unique (up to translation) convex body K whose curvature function equals f.

Let us go back for a moment to Equation (2.1) and notice that, while the left-hand side of (2.1) is invariant under translations of K, the right-hand is affected by translations. However, we shall now show that, for the Santalò point p_K, h_{K-pK}^{-N-1} is a curvature function. It is well-known that the polar reciprocal of a convex body with respect to its Santalò point has its center of mass at the origin (see [San]), this implies that

$$\int_{\mathbb{S}^{N-1}} \omega \rho_{K^*}^{N+1}(\omega)\, d\mathcal{H}^{N-1}(\omega) = 0,$$

where ρ_{K^*} denotes the radial function of the convex set $K^*_{p_K}$.

We note that $h_{K-pK}(\omega) = \rho_{K^*}(\omega)^{-1}$, for any for $\omega \in \mathbb{S}^{N-1}$; then $h_{K-p_K}^{-N-1}$ satisfies condition (2.6) and hence, by Minkowski's Theorem, for every $K \in \mathcal{K}^N$, there exists a body K' such that $f_{K'} = h_{K-p_K}^{-N-1}$.

From these considerations we note that, if we define a map Λ, from the set of convex bodies whose Santalò point is at the origin, that associates to each convex body K the solution of the Minkoski problem with data h_K^{-N-1}, then K is a solution of (2.1) if and only if its image $\Lambda(K)$ is a dilation of K. We refer the reader to [Lu] for more details.

2.3 Aleksandrov solutions of the Monge-Ampère equation

In this section we recall the notion of Aleksandrov solutions of the Monge-Ampère equation and we summarize the properties that we will need in the sequel (see [DF, Gu] for a more detailed exposition).

The following theorem is an easy consequence of Remark 1.1 and Proposition 1.4.

Theorem 2.5. *[Gu, Theorem 1,2] Let u be a convex function defined on a convex open domain $\Omega \subset \mathbb{R}^N$, then the family*

$$\mathcal{M} = \{E \subset \Omega \, : \, \partial u(E) \text{ is Lebesgue measurable}\}$$

is a σ-algebra containing all the Borel subsets of Ω. Moreover,

$$\nu_u(A) = V(\partial u(A)) = V\Big(\bigcup_{x \in A} \partial u(x)\Big) \tag{2.7}$$

defines a measure which is finite on compact subsets of Ω. We call ν_u the Monge-Ampère measure of u.

Note that if $u \in C^2$, the change of variable formula gives that $d\nu_u = \det D^2 u \, dx$. We then call u an *Alexandrov solution* of the equation

$$\det D^2 u = f \tag{2.8}$$

provided $\nu_u = f\,dx$. Among the various properties of Aleksandrov solutions we are going to use one that concerns their stability under uniform limit (see [Gu, Lemma 1.2.3] for a proof).

Lemma 2.6. *If u_k are convex functions defined on an open set Ω and $u_k \to u$ uniformly, then*

$$\nu_{u_k} \overset{*}{\rightharpoonup} \nu_u$$

as Radon measures in Ω, that is

$$\int \varphi \, d\nu_{u_k} \to \int \varphi \, d\nu_u \quad \text{for every} \quad \varphi \in C^0_c(\Omega).$$

By relying on the uniqueness of the Aleksandrov solutions to the Dirichlet problem for the Monge-Ampère equation, [Gu, Corollary 1.4.7], and on their stability under uniform limits, one can prove the following classical theorem. For the sake of completeness, we sketch the main steps of its proof (see also [DF, Section 2] for a more detailed account).

Theorem 2.7. *Let u be a* strictly *convex function defined on a convex set Ω and satisfying*

$$\nu_u = f\,dx \qquad \text{in } \Omega.$$

If $f \in C^\infty(\Omega)$ and $\lambda \le f \le \Lambda$ for some $\lambda, \Lambda > 0$, then $u \in C^\infty(\Omega')$, for every open set Ω' such that its closure is compact and is contained in Ω.

Proof. Fix $x_0 \in \Omega'$, $p \in \partial u(x_0)$, and consider the *section* of u at height t defined by

$$S(x, p, t) = \big\{ y \in \Omega \, : \, u(y) \le u(x) + p \cdot (y - x) + t \big\}.$$

Since u is strictly convex we can choose $t > 0$ small enough so that the closure of $S(x_0, p, t)$ is compact and contained in Ω'. Then we consider a sequence of smooth uniformly convex sets S_i, converging to $S(x_0, p, t)$ and we apply classical continuity methods in order find a function $v_i \in C^\infty(S_\varepsilon)$ solving the problem

$$\begin{cases} \det D^2 v_i = f * \varrho_{\varepsilon_i} & \text{in } S_i \\ v_i = 0 & \text{on } \partial S_i, \end{cases}$$

where ϱ_{ϵ_i} is a sequence of mollifying kernels (see [DF, Theorem 2.11] and [GT, Chapter 17]). We apply to v_i Pogorelov estimates (see for instance [DF, Theorem 2.12]) to infer that

$$|D^2 v_i| \leq C \qquad \text{in } S(x_0, p, t/2) \subset S(x_0, p, t),$$

for some constant C independent on $i \in \mathbb{N}$. Since $S_i \to S(x_0, p, t)$ and $u(x) = u(x_0) + p \cdot x + t$ on $\partial S(x_0, p, t)$, by stability and uniqueness of weak solutions we deduce that $v_i + u(x_0) + p \cdot x + t \to u$ uniformly as $i \to \infty$; hence $|D^2 u| \leq C$ in $S(x_0, p, t/2)$. This makes the Monge-Ampère equation uniformly elliptic and hence Evans-Krylov's theorem and Schauder's theory imply that $u \in C^\infty(S(x_0, p, t/4))$ (see [GT, Chapter 17]). By the arbitrariness of x_0 we obtain that $u \in C^\infty(\Omega')$, as desired. □

By a well-known example, strict convexity of u is necessary in order to prove the above theorem. The following result, due to Caffarelli, implies that the obstruction to strict convexity can only arise from the boundary behavior. In particular every entire solution has to be strictly convex. We recall that x is an extremal point of a convex set K if $x \in \overline{K}$ and $K \setminus \{x\}$ is convex.

Theorem 2.8 ([Ca]). *Let Ω be an open convex set, and let u be a convex function such that*

$$\lambda\, dx \leq \nu_u \leq \Lambda\, dx$$

for some $\lambda, \Lambda > 0$. For every $x \in \Omega$ and $p \in \partial u(x)$, if the set

$$\Gamma_{x,p} = \{y \in \Omega \,:\, u(y) = u(x) + p \cdot (y - x)\}$$

contains more than one point, then it has no extremal points in Ω.

An easy corollary of Theorem 2.8 is the following:

Corollary 2.9. *Let $u : \mathbb{R}^N \to \mathbb{R}$ be a convex function such that*

$$\lambda\, dx \leq \nu_u \leq \Lambda\, dx \tag{2.9}$$

for some $\lambda, \Lambda > 0$. Then u is strictly convex.

Proof. Let us assume by contradiction that for some $x_0 \in \mathbb{R}^N$ and $p_0 \in \partial u(x_0)$ the set Γ_{x_0,p_0} contains more than one point; then according to Theorem 2.8 it must contain a line. Up to subtracting a linear function and to change the coordinates, we can then assume that $u \geq 0$ and $u = 0$ on the line

$$\ell = \{x \in \mathbb{R}^N : x = (x_1, 0, \ldots, 0)\}.$$

This easily implies that $\partial u(\mathbb{R}^N) \subset e_1^\perp$ and hence that $\nu_u = 0$, contradicting (2.9). □

2.4 A proof of Petty's theorem under natural regularity assumptions

In this section we prove the following extension of Petty's theorem,

Theorem 2.10 ([DM])**.** *Let K be a convex body which possesses a curvature function f_K; if equation* (2.1) *is satisfied for some positive constant c_K, then K is an ellipsoid.*

The argument is based on an approximation procedure that shows that, for a convex body satisfying (2.1), $h_K^2/2$ is an Aleksandrov solution of (2.4). Consequently, we can apply Corollary 2.9 and Theorem 2.7 to show that h_k^2 is smooth and hence the classical Pogorelov's argument can be applied. Theorem 2.10 will be a consequence of the following result.

Theorem 2.11. *Let K be a convex body that possesses a curvature function f_K and let h_K be its support function. Then*

$$\det D^2\left(\frac{1}{2}h_K^2\right) = f_K\Big(\frac{x}{|x|}\Big)h_K^{N+1}\Big(\frac{x}{|x|}\Big)\,dx \quad on \quad \mathbb{R}^N \tag{2.10}$$

in the Aleksandrov sense.

In order to prove Theorem 2.11 we need to approximate, in the Hausdorff topology, a convex body with C^2_+ bodies, for which we know that (2.10) holds true at least in $\mathbb{R}^N \setminus \{0\}$, thanks to Proposition 2.3.

We know from an old theorem by Minkowski that convex sets with analytic boundary are dense in $\mathcal{K}^N$. Several years later Schmuckenschläger (see [Sc]) gave a simpler proof of that theorem and showed that one can explicitly write down the relevant approximating sequence under further additional properties.

Theorem 2.12 ([Sc])**.** *Let K be a convex body. There exists an increasing sequence $\{K_i\}_{i\in\mathbb{N}}$ such that*

- *K_i and K_i^* have real analytic boundaries;*
- *The Gauss curvature of both K and K^* is strictly positive;*
- *$\delta(K_i, K) \to 0$ as $i \to \infty$.*

Proof of Theorem 2.11. We divide the proof in three steps.

• *Step 1: Equation* (2.10) *holds true if $K \in C^2_+$.* Let $K \in C^2_+$; then, by Remark 1.8, $h_K \in C^2(\mathbb{R}^N \setminus \{0\})$. Thus Proposition 2.3 yields

$$\det \frac{1}{2} D^2 h_K^2(x) = f_K\Big(\frac{x}{|x|}\Big) h_K^{N+1}\Big(\frac{x}{|x|}\Big) \quad \text{for every} \quad x \in \mathbb{R}^N \setminus \{0\}.$$

In particular, by the change of variable formula, if we denote by ν_K the Monge-Ampère measure of h_K^2 we have that

$$\nu_K = f_K\Big(\frac{x}{|x|}\Big) h_K^{N+1}\Big(\frac{x}{|x|}\Big)\, dx$$

in the sense of the Radon measures on $\mathbb{R}^N \setminus \{0\}$. Moreover, since h_K^2 is homogeneous of degree two, it is differentiable at 0 and $\partial h_K^2(0) = \{0\}$. By recalling the definition of the Monge-Ampère measure (2.7), we then see that for every Borel set $A \subset \mathbb{R}^n$

$$\begin{aligned} \nu_K(A) &= \nu_K(A \setminus \{0\}) + \nu_K(\{0\}) \\ &= \nu_K(A \setminus \{0\}) + V(\{0\}) = \nu_K(A \setminus \{0\}). \end{aligned}$$

Hence (2.12) is valid (as equality between measures) also in $\mathbb{R}^N$.

• *Step 2: Equation* (2.10) *is closed in the topology of Hausdorff distance.*

Let K, K_i $i = 1, 2, \ldots$, be convex bodies such that $\delta(K, K_i) \to 0$ as $i \to \infty$; then $h_{K_i} \to h_K$ uniformly on $\mathbb{S}^{N-1}$ and $h_{K_i}^2 \to h_K^2$ locally uniformly in $\mathbb{R}^N$. According to Lemma 2.6 it is enough to show that

$$\nu_{K_i} = f_{K_i}\Big(\frac{x}{|x|}\Big) h_{K_i}^{N+1}\Big(\frac{x}{|x|}\Big)\, dx \overset{*}{\rightharpoonup} f_K\Big(\frac{x}{|x|}\Big) h_K^{N+1}\Big(\frac{x}{|x|}\Big)\, dx, \tag{2.11}$$

as Radon measures in $\mathbb{R}^N$. To this end, let $\varphi \in C^0_c(\mathbb{R}^N)$ and note that for every $\varrho \in [0, +\infty)$ the function $\mathbb{S}^{N-1} \ni \omega \mapsto \varphi(\varrho\,\omega)$ is continuous. Since $h_{K_i} \to h_K$ uniformly on $\mathbb{S}^{n-1}$ and

$$\int \varphi\, d\nu_{K_i} = \int_0^\infty \varrho^{N-1} \int_{\mathbb{S}^{N-1}} \varphi(\varrho\,\omega) f_{K_i}(\omega) h_{K_i}(\omega) d\mathcal{H}^{N-1}(\omega)\,,$$

an application of Lebesgue Dominated Convergence Theorem (recall that φ is compactly supported) shows that (2.11) will be a consequence of the fact that

$$f_{K_i}(\omega)\, d\mathcal{H}^{N-1} \overset{*}{\rightharpoonup} f_K(\omega)\, d\mathcal{H}^{N-1}$$

as Radon measures on $\mathbb{S}^{N-1}$. This convergence however follows from (2.5), that is from the continuity of curvature measures under the Hausdorff convergence, (2.5).

• *Step 3: Conclusion.* If K is a convex body admitting a curvature function we can apply Theorem 2.12 to approximate it with a sequence of convex bodies $K_i \in C^2_+$; by Step 1 the conclusion of the Theorem holds true for K_i and hence by Step 2 also for K. $\square$

Proof of Theorem 2.10. According to Theorem 2.11, if K is a convex body satisfying (2.1), then

$$\det \frac{1}{2} D^2 h_K^2 = c_K \, dx \qquad \text{on } \mathbb{R}^N$$

in the Aleksandrov sense. By Corollary 2.9, h_K^2 is strictly convex and by Theorem 2.7, $h_K^2 \in C^\infty(\mathbb{R}^N)$. By applying the classical Pogorelov argument (see [Gu, Theorem 4.3.1] for a proof), $h_K^2(x) = Ax \cdot x$ for some positive symmetric matrix A, which immediately implies that K is an ellipsoid. $\square$

Chapter 3

Characterization of ellipsoids as K-dense sets

In this chapter we provide a characterization of ellipsoids of $\mathbb{R}^N$ as K-dense sets, *i.e.* domains which satisfy an identity involving the volume of the intersection with a given convex body K.

In Section 3.1, we start investigating some basic properties of K-dense sets, in particular we show that they are strictly convex and $C^{1,1}$.

In Section 3.2, we study the asymptotic behavior, for small values of the parameter r, of the volume of $G \cap B_K(x,r)$, where G is the K-dense set and $B_K(x,r)$ is the K-ball centered at x with radius r. We deduce a symmetry property for the set K and an equation involving a weighted mean curvature of G (where the weights depend on some moment of inertia of K) which, at least in the planar case, is enough to deduce that K-dense sets are ellipsoids.

The proof of this characterization is given in Section 3.3, separately from that in general dimension.

The proof of the characterization in general dimension is given in Sections 3.4 and 3.5. In Section 3.4, we study the volume of $G \cap B_K(x,r)$ for "large" K-balls, in order to show that K-dense bodies are strongly convex. This information allows us to use Theorem 1.12 and infer a strong symmetry result for K-dense sets, that leads us to conclude, in Section 3.5, that they must be ellipsoids.

3.1 Properties of K-dense sets

Let $K \in \mathcal{K}^N$ and G be a measurable subset of $\mathbb{R}^N$ with positive Lebesgue measure $V(G)$. For each fixed $r > 0$, we define a density function

$\delta_K : \mathbb{R}^N \times (0, \infty) \to \mathbb{R}$ as follows:

$$\delta_K(x, r) = \frac{V(G \cap (x + r\,K))}{V(r\,K)}, \ x \in \mathbb{R}^N. \tag{3.1}$$

We say that G is *uniformly K-dense,* or *K-dense* for short, if there is a function $c : (0, \infty) \to (0, \infty)$ such that

$$\delta_K(x, r) = c(r) \ \text{ for every } \ (x, r) \in \partial G \times (0, \infty), \tag{3.2}$$

where ∂G denotes the topological boundary of the set G.

As remarked in the introduction, when K is the euclidean unit ball B of $\mathbb{R}^N$, K-dense domains have been studied in [MPS] in connection with the so-called *stationary isothermic surfaces* — the time-invariant level surfaces of solutions of the heat equation. There, it is shown that the only B-dense sets with finite volume of $\mathbb{R}^N$ are balls; moreover, it observed in that, if E is an ellipsoid, then bounded E-dense domains must be ellipsoids homothetic to E.

As already said, in this chapter we show that if G is K-dense, then the only possibility is that K (and hence G) is an ellipsoid (see Theorem 3.21).

Here we study some general properties of K-dense sets that will be useful to understand the proof of Theorem 3.21.

Given a measure μ on $\mathbb{R}^+$ and set $\phi(t) = \mu([0, t))$, we define a function $f^\phi : \mathbb{R}^N \to \mathbb{R}$ as follows:

$$f^\phi(x) = \int_G \phi(\|y - x\|_K)\,dy = \int_G \phi(\|x - y\|_{-K})\,dy; \tag{3.3}$$

f^ϕ is thus the convolution of the characteristic function $\mathcal{X}_G$ and the composition of ϕ with the gauge of $-K$ defined in Section 1.2.

If μ is a Borel and locally finite measure, we can use the *layer-cake* representation theorem (see [LL] for instance) in order to write:

$$\begin{aligned} f^\phi(x) &= \int_0^{+\infty} V(G \cap \{y : \|y - x\|_K > t\})\,d\mu \\ &= \int_0^{+\infty} V(G \setminus B_K(x, t))\,d\mu, \end{aligned} \tag{3.4}$$

where $B_K(x, r)$ is the interior of the set $x + rK$.

If G is K-dense, the last integral does not depend on x, for $x \in \partial G$. Conversely, if $f^\phi(x)$ is constant on ∂G for every choice of the measure μ, then for each given $r > 0$ we can set $\mu = \delta_r$ (the Dirac's delta measure centered at r) in (3.4) and obtain that $f^\phi(x) = V(G \setminus B_K(x, r))$. When G has finite measure, the assumption on f^ϕ and the fact that r is arbitrary imply that G must be K-dense. Thus, we can state the following characterization.

Theorem 3.1. *Let G be a bounded*[1]*, measurable subset of $\mathbb{R}^N$ of positive volume. Then the following conditions are equivalent:*

(i) G is K-dense;

(ii) for every Borel, locally finite measure μ on $\mathbb{R}^+$, the function f^ϕ defined in (3.3) *does not depend on x, for $x \in \partial G$.*

The following lemma is instrumental to prove the convexity of G; its proof is straightforward.

Lemma 3.2. *Let the function $\phi(t) = \mu([0,t))$ be convex, increasing and non-constant, and let f^ϕ be the function defined in* (3.3)*. Then:*

(i) f^ϕ is convex and hence, in particular, continuous;

(ii) f^ϕ is coercive, that is $f^\phi \to +\infty$ as $|x| \to \infty$.

Theorem 3.3 ([MM1])**.** *Let G be a bounded K-dense set; then G is strictly convex.*

Moreover, if the function $\phi(t) = \mu([0,t))$ is convex and strictly increasing, then G is a regular level set for f^ϕ.

Proof. First, we show that, if ϕ satisfies the assumptions, then f^ϕ cannot be constant on a segment whose middle point belongs to $\overline{G}$.

By contradiction, let x and y be the endpoints of a segment on which f^ϕ is constant and suppose the midpoint $\frac{1}{2}(x+y) \in \overline{G}$; then

$$\int_G \{\phi(\|z-x\|_K)/2 + \phi(\|z-y\|_K)/2 - \phi\left(\|z-(x+y)/2\|_K\right)\}\, dz = 0.$$

Since the integrand is always non-negative, we get that

$$2\,\phi\left(\|z-(x+y)/2\|_K\right) = \phi(\|z-x\|_K) + \phi(\|z-y\|_K)$$

for every $z \in \overline{G}$, since both ϕ and $\|\cdot\|_K$ are continuous[2]. Thus, if we choose $z = \frac{1}{2}(x+y)$ we get a contradiction.

Now we observe that there exists a segment in the convex hull of ∂G whose middle point belongs to $\overline{G}$. Indeed, consider a line, say ℓ, containing at least three points of G, say x, y and z, with $y \in]xz[$[3]; then, being G bounded, ∂G intersects every connected component $\ell \setminus]xz[$ and thus every point of $]xz[$

[1]It is possible to replace this assumption by asking that $V(G) < \infty$; however, it turns out that there not exists any unbounded K-dense set of finite measure.

[2]This is clear when G is connected. Otherwise, it is sufficient that, for each $x \in \overline{G}$, every neighborhood of x has intersection with G of positive measure. This is guaranteed by the fact that G is K-dense.

[3]We denote by $]xz[$ the relatively open segment from x to z.

belongs to the convex hull of ∂G; if $\varepsilon > 0$ is such that $[y-\varepsilon, y+\varepsilon] \subset]x, z[$ then also $[y-\varepsilon, y+\varepsilon]$ is contained in the convex hull of ∂G, and y, its middle point, belongs to $\overline{G}$.

Therefore, we can claim that the function f^ϕ does not reach its minimum on the boundary of G, otherwise f^ϕ would be constant on the convex hull of ∂G and, in particular, on $[y-\varepsilon, y+\varepsilon]$, that is impossible.

Hence, there exists a positive number s such that the set A where $f^\phi < s$ is open, bounded and convex; also, $\partial G \subseteq \partial A = \{x \in \mathbb{R}^N : f^\phi(x) = s\}$.

Since G is bounded, it is contained in the convex hull of ∂G, and thence in the convex hull of ∂A, $\overline{A}$. Thus $G \subseteq \overline{A}$. It is also possible to check that $A \subseteq G$. Indeed, suppose by contradiction that there exists a point $x \in G \setminus A$; in particular $x \notin \partial G$, then every open segment between and x and ∂A must have empty intersection with G, otherwise it would contain also a point belonging to ∂G and this contradicts the fact that $\partial G \subseteq \partial A$. But if every open segment between and x and ∂A does not intersect G, then G is contained in the complementary of A, and, since $G \in \overline{A}$, then we would have that $G \subseteq \partial A$, that contradicts the fact that $V(G) > 0$.

We than have that $A \subseteq G \subseteq \overline{A}$, and, in particular, that G is convex and hence strictly convex.

$\square$

The following statement is straightforward but, as it will be clear, it will play an important role in the proof of Theorem 3.8, which is fundamental in our further analysis.

Corollary 3.4. *Let G be a K-dense body; then the function*

$$x \mapsto \max_{y \in G} \|y - x\|_K$$

is constant on ∂G.

Proof. Let x and $z \in \partial G$ and suppose by contradiction that

$$d_1 = \max_{y \in G} \|y - x\|_K < \max_{y \in G} \|y - z\|_K = d_2.$$

Then $G \setminus B_K(z, d_1) \neq \varnothing$ and hence $V(G \setminus B_K(z, d_1)) > 0$, being G a body and $B_K(z, d_1)$ open; thus,

$$\begin{gathered} V(G \cap B_K(x, d_1)) = V(G) \\ = V(G \setminus B_K(z, d_1)) + V(G \cap B_K(z, d_1)) > V(G \cap B_K(z, d_1)). \end{gathered}$$

$\square$

We now study the regularity of K-dense sets.

Theorem 3.5 ([MM1])**.** *Let G be a K-dense body; then ∂G is of class $C^{1,1}$*

Proof. Set $f = f^\phi$ with $\phi(t) = t$. By Theorem 3.3, it is sufficient to show that $f \in C^{1,1}$.

Consider the incremental ratio of f at x in a canonical direction e_i :

$$\frac{f(x+te_i) - f(x)}{t} = \int_G \frac{\|x-z+te_i\|_{-K} - \|x-z\|_{-K}}{t}\, dz.$$

Since $\|\cdot\|_{-K}$ is almost everywhere differentiable and its gradient is a bounded map over $\mathbb{R}^N$, by the dominated convergence theorem, we obtain that the partial derivative $\partial_{x_i} f(x)$ exists and equals

$$\int_G \frac{\partial}{\partial x_i}\|x-z\|_{-K}\, dz = \int_{\mathbb{R}^N} \mathcal{X}_G(x-z)\,\frac{\partial}{\partial z_i}\|z\|_{-K}\, dz,$$

and the second factor in the integrand is bounded almost everywhere by a constant, say, L. Thus, for $x, y \in \mathbb{R}^N$, we obtain the estimate:

$$\begin{aligned} |\partial_{x_i} f(x) - \partial_{x_i} f(y)| \;\; &\leq \;\; L\int_{\mathbb{R}^N} |\mathcal{X}_G(x-z) - \mathcal{X}_G(y-z)|\, dz \leq \\ & \;\;\;\;\; L\,\mathcal{H}^{N-1}(\partial G)\,\|x-y\|, \end{aligned}$$

since G is convex and bounded.

Therefore, f is differentiable and has Lipschitz continuous partial derivatives. □

Since the function $\|\cdot\|_K$ has the same regularity as ∂K at all points of $\mathbb{R}^N$ except the origin, then if $\partial K \in C^{m,1}$ for some integer m, by the same arguments used in the proof of Theorem 3.5, we can easily prove the following result.

Theorem 3.6. *Let G be a bounded K-dense set, and let $\partial K \in C^{m,1}$ for some integer m. Then $\partial G \in C^{m+1,1}$.*

Corollary 3.7. *Let G be a bounded K-dense set. If the class of homothetical images of K contains G, then $\partial G \in C^\infty$.*

Proof. We show that $G \in \mathcal{C}^{m,1}$ for every $m \in \mathbb{N}$ by induction on m. The base step is exhibited in Theorem 3.5; the inductive step is the subject of Theorem 3.6. □

The following result shows that, surprisingly, at least when K is centrally symmetric, the existence of a K-dense set implies some regularity of K itself.

Theorem 3.8 ([MM1])**.** *Let K be a convex body symmetric with respect to the origin of $\mathbb{R}^N$, and let G be a K-dense body. Then it holds that*

(a) $K = G - G$*, up to homotheties;*

(b) K is strictly convex;

(c) ∂K and ∂G are respectively $C^{1,1}$-smooth and $C^{2,1}$-smooth.

Proof. (a) Without loss of generality, thanks to Corollary 3.4, let us suppose that

$$\max_{y\in G} \|y-x\|_K = 1, \quad \text{for every} \quad x \in \partial G.$$

We have that

$$\max_{y\in G-x} \|y\|_K = 1,$$

and hence $G - x \subseteq K$ for every $x \in \partial G$. It follows that $G - G \subseteq K$.

Indeed, if $z \in G - G$, then $z = x - y$ for some points $x, y \in G$; since G is convex, there are points x_1 and x_2 in ∂G and a number $0 \le \lambda \le 1$ such that $x = \lambda x_1 + (1-\lambda)x_2$. Hence,

$$z = \lambda\,(y - x_1) + (1-\lambda)(y - x_2).$$

Since K is convex and contains both $y - x_1$ and $y - x_2$, we get that $z \in K$.

Viceversa, let x be an exposed point of ∂K and let $u \in \mathbb{S}^{N-1}$ be such that H_u is the supporting hyperplane which intersects K only at the point x.

Next, choose $y \in \partial G$ such that the unit normal to ∂G at y, $\nu_G(y)$, coincides with u (it exists since we already know that G is smooth and strictly convex). Also, pick a point $z \in \partial G$ that maximizes the K-distance from y, that is, such that $\|y - z\|_K = 1$. Note that $y - z \in (G - z) \cap \partial K$ and, since $G - z \subseteq K$, we get the following reverse inclusion for the normal cones:

$$N_K(y-z) \cap \mathbb{S}^{N-1} \subseteq \{-\nu_{G-z}(y-z)\} = \{-\nu_G(y)\} = \{u\}.$$

Hence, our choice of x and u allows us to write $x = y - z$. Thus, $G - G$ contains all the exposed points of ∂K, and being $G - G$ a convex set it contains the convex hull of them. It is easy to show that a convex body is the closure of the convex hull of its exposed points, thence $G - G \supseteq K$.

(b) It follows from (a) and Theorem 3.3.

(c) From (a) and Theorem 3.5, it follows that ∂K is $C^{1,1}$-smooth, since the Minkowski sum of $C^{1,1}$ sets is $C^{1,1}$, see Theorem 1.12. Theorem 3.6 then implies that ∂G is $C^{2,1}$-smooth. □

Remark 3.9. In the next section, we will prove the central symmetry of K, so that the above theorem will be a substantial progress for the proof of Theorem 3.21. Indeed, Theorem 3.8 suggests that a step toward a proof of our characterization should be to show that G is centrally symmetric as well. As a matter of fact, once we get the symmetry of G we immediately deduce that $K = G$ and, in force of Corollary 3.7, that ∂G (and ∂K) is C^∞-smooth.

3.2 Asymptotics as $r \to 0^+$ and symmetry

In [ABG] a proof of Theorem 3.21 is given in the planar case under suitable regularity assumptions. There, by calculating for a fixed $x \in G$, the Taylor expansion of the function δ_K in (3.1) as $r \to 0^+$ up to the third order, it is proved that the radial function of K satisfies an ordinary differential equation which is fulfilled only by ellipses.

It seems difficult to extend the analysis employed in [ABG] to the case $N \geq 3$ for several reasons. The most apparent one is that, in general dimension, it is very difficult to compute the higher-order terms in the Taylor expansion of $\delta(x, r)$. Nevertheless, the asymptotic analysis as r tends to 0 will give us some useful geometric informations about K-dense bodies.

We first settle on a convenient notation. Given a unit vector $u \in \mathbb{S}^{N-1}$, we write $H_u^+ = \{x \in \mathbb{R}^N : \langle x, u\rangle \geq 0\}$ and $H_u^- = H_{-u}^+$.

Theorem 3.10 ([MM1]). *Let G and K be convex bodies and suppose that ∂G is differentiable at x. Then*

$$\delta_0(x) = \lim_{r \to 0^+} \delta_K(x, r) = \frac{V(K \cap H_{\nu(x)}^-)}{V(K)}.$$

In particular, if G is K-dense, then

$$V(K \cap H_u^-) = \frac{1}{2} V(K) \quad \text{for all } u \in \mathbb{S}^{N-1}. \tag{3.5}$$

Proof. For $r > 0$ we have:

$$r^{-N} V(G \cap (x + r\,K)) = V\left(\frac{G - x}{r} \cap K\right). \tag{3.6}$$

Since ∂G is differentiable at x, as r decreases to 0, $\dfrac{G - x}{r} \cap K$ increases to $H_{\nu(x)}^+ \cap K$. The first claim of the theorem then follows from the monotone convergence theorem.

Now, suppose that G is K-dense. Since G is strictly convex differentiable and bounded, then the Gauss map, $\nu_G(x)$ is a bijection. Hence, for every $u \in \mathbb{S}^{N-1}$, there exist $x, x' \in \partial G$ such that $u = \nu(x) = -\nu(x')$.

Since G is K-dense, then the quantity $V(K \cap H^+_{\nu(x)})$ does not depend on x, for $x \in \partial G$. Thus, our choice of x and x' enables us to write that

$$V(K \cap H^-_{\nu(x)}) = V(K \cap H^-_{\nu(x')}) = V(K \cap H^+_{\nu(x)}).$$

Since $V(K \cap H^-_{\nu(x)}) + V(K \cap H^+_{\nu(x)}) = V(K)$, then we find that

$$V(K \cap H^-_{\nu(x)}) = \frac{1}{2} V(K).$$

□

Remark 3.11. When $N = 2$, it is not difficult to show that (3.5) implies that K is centrally symmetric, see [MM1]; indeed, that is also true for $N \geq 3$, by a non-trivial result of Schneider [Scn1]. Then, as already stressed in Remark 3.9 the conclusion of Theorem 3.8 is valid; more precisely we have the following corollary.

Corollary 3.12. *Let G be a K-dense body, then $G \in C^{2,1}$.*

The emphasis given to this observation is justified by the fact that the twice differentiability of G allows us to compute the second term in the Taylor expansion of δ_K, in the following theorem we will show that δ can be written as

$$\delta_K(x, r) = \delta_0(x) - \delta_1(x)\, r + o(r) \quad \text{as} \quad r \to 0^+, \tag{3.7}$$

and we compute $\delta_1(x)$.

Theorem 3.13 ([MM1])**.** *Let G be a convex body with C^2-smooth boundary, let $x \in \partial G$ and denote by $\kappa_1(x), \dots, \kappa_{N-1}(x)$ the principal curvatures of ∂G at x with respect to the inward normal unit vector.*

Then, we have the formula:

$$\lim_{r \to 0^+} \frac{\delta_K(x, r) - \delta_0(x)}{r} = -\frac{1}{2\, V(K)} \sum_{i=1}^{N-1} m_i(x)\, \kappa_i(x), \tag{3.8}$$

where the coefficients m_i are given by

$$m_i(x) = \int_{K \cap H_{\nu(x)}} \langle \xi, e_i(x) \rangle^2 \, d\mathcal{H}^{N-1}, \; i = 1, \dots, N-1; \tag{3.9}$$

here, again, H_ν denotes the hyperplane $\{\langle y, \nu \rangle = 0\}$.

Therefore, (3.7) *holds.*

Proof. We choose a coordinate system $\{e_1, \dots, e_{N-1}, \nu\}$ around the point $x \in \partial G$ such that e_i, for $i = 1, \dots, N-1$, is the i-th principal direction of ∂G at x and $\nu = \nu(x)$ is the normal.

In these coordinates $B_K(x,r)$ can be written as

$$B_K(x,r) = \left\{ x + \sum_{i=1}^{N-1} z_i e_i + z_N \nu : z \in \mathbb{R}^N, \ \left\| \sum_{i=1}^{N-1} z_i e_i + z_N \nu \right\|_K \leq r \right\}.$$

Also, in these same coordinates, ∂G can be locally parametrized by a convex function $\psi \in C^2$ and, clearly, $\psi(0) = 0$ and $D\psi(0) = 0$. Furthermore, our choice of the axes $e_1, \dots, e_N$ allow us to write that

$$\psi(z') = \frac{1}{2} \sum_{i=1}^{N-1} \kappa_i(x) z_i^2 + o(\|z'\|^2),$$

for $z' = (z_1, \dots, z_{N-1}) \in \mathbb{R}^{N-1}$ in a sufficiently small neighborhood of 0.

We need to estimate the measure of the remainder set

$$R(x,r) = B_K(x,r) \cap H^+_{\nu(x)} \setminus G;$$

for sufficiently small $r > 0$, $R(x,r)$ can be written as

$$\left\{ x + \sum_{i=1}^{N-1} z_i e_i - z_N \nu : \left\| \sum_{i=1}^{N-1} z_i e_i - z_N \nu \right\|_K \leq r, \, 0 \leq z_N \leq \psi(z'), z' \in V \right\},$$

where V is some neighborhood of 0 in $\mathbb{R}^{N-1}$.

Next, we make the following change of variables: $z_i = r\xi_i$, for $i = 1, \dots, N-1$ and $z_N = r^2 \xi_N$; since $\| \cdot \|_K$ is positively homogeneous, we get that

$$V(R(x,r)) = r^{N+1} V(S_r),$$

where S_r is the set

$$\left\{ \xi \in \mathbb{R}^N : \, \xi' \in r^{-1} V, \, \left\| \sum_{i=1}^{N-1} \xi_i e_i + r \xi_N \nu \right\|_K \leq 1; \, 0 \leq \xi_N \leq \frac{\psi(r\xi_1, \dots, r\xi_{N-1})}{r^2} \right\}.$$

Now, if we define the set

$$S_0 = \left\{ \xi \in \mathbb{R}^N : \left\| \sum_{i=1}^{N-1} \xi_i e_i(x) \right\|_K < 1; 0 \leq \xi_N < \frac{1}{2} \sum_{i=1}^{N-1} \kappa_i(x) \, \xi_i^2 \right\},$$

we easily check that

$$S_0 \subseteq \bigcup_{r>0} \Big(\bigcap_{0<t<r} S_t \Big) \subseteq \bigcap_{r>0} \Big(\bigcup_{0<t<r} S_t \Big) \subseteq \overline{S}_0.$$

Since $V(S_0) = V(\overline{S}_0)$, the smoothness assumptions on ∂G give the sufficient uniform boundedness to infer that

$$\lim_{r\to 0^+} \frac{V(R(x,r))}{r^{N+1}} = V(S_0).$$

By the definition of S_0, $V(S_0)$ is easily computed as

$$V(S_0) = \int_{K\cap\pi_{\nu(x)}} \frac{1}{2}\sum_{i=1}^{N-1} \kappa_i(x)\, \xi_i^2\, d\xi = \frac{1}{2}\sum_{i=1}^{N-1} m_i(x)\, \kappa_i(x),$$

that implies the desired formula (3.8).

□

Corollary 3.14. *Let G be K-dense, then there exists a constant $c > 0$ such that*

$$\sum_{i=1}^{N-1} m_i(x)\, \kappa_i(x) = c\, V(K), \quad x \in \partial G, \tag{3.10}$$

Proof. The fact that the right-hand side of (3.10) does not depend on x for $x \in \partial G$ clearly follows from (3.2) and (3.7). Since K is a convex body, then the $m_i(x)$'s are all positive; if c were zero, then all the curvatures would be zero for every $x \in \partial G$ and this is impossible, since G is a convex body. □

3.3 Characterization of ellipses in the two-dimensional case

In this section we provide a proof of the characterization theorem (Theorem 3.16) for planar K-dense sets. We give it separately from that of Theorem 3.21, mainly because the planar proof is very different from that for the case $N \geq 3$ and that given in [ABG], and the techniques used here may be of some interest. We also stress the fact that, besides dropping the smoothness assumptions needed in [ABG], our proof only needs the pointwise information given by (3.10). However, it should be noticed that the proof of [ABG] works even if we restrict the validity of (3.2) to small values of the parameter r.

We start analyzing Equation (3.9). The only moment of inertia $m = m_1$ can be easily computed and, by setting $u = \nu(x)$, re-defined as a function on $\mathbb{S}^1$ as

$$m(u) = \frac{2}{3}\, \rho_K(\hat{u})^3, \;\; u \in \mathbb{S}^1, \tag{3.11}$$

where ρ_K denotes, as usual, the radial function of K and $\hat{u}$ is the unit vector obtained from u by a clockwise rotation of 90 degrees.

In the following theorem we show how, by exploiting (3.8), one can get the central symmetry of G.

Theorem 3.15 ([MM1])**.** *Let $K \subset \mathbb{R}^2$ be a convex body. If $G \subset \mathbb{R}^2$ is a K-dense body, then G and K are homothetic and both ∂K and ∂G are C^∞-smooth.*

Proof. C^2_+-regularity easily follows from Corollary 3.14.

Plugging (3.11) into (3.8), and in view of the geometric meaning of the curvature function, can be written as

$$\rho_K(\hat{u})^3 = c\,V(K)\,f_G(u),\ \ u \in \mathbb{S}^1, \tag{3.12}$$

where c is some positive constant. Also, being K centrally symmetric, $\rho_K(-\hat{u}) = \rho_K(\hat{u})$ and hence $f_G(-u) = f_G(u)$ for every $u \in \mathbb{S}^1$; this means that also G is centrally symmetric.

Thus, by Remark 3.11 and Theorem 3.8 K and G differ by a homothety and both ∂K and ∂G are C^∞-smooth in force of Corollary 3.7. □

Before giving the proof of Theorem 3.16 we recall that in two dimensions the Minkowski's first inequality for mixed volumes (1.16) writes

$$V(K,G) \geq \sqrt{V(K)\,V(G)}, \tag{3.13}$$

and the affine inequality (1.18) reduces to

$$\Omega(K)^3 \leq 8\,V(K)^2 V(K^*), \tag{3.14}$$

where the affine area is given by

$$\Omega(K) = \int_{\mathbb{S}^1} f_K(u)^{2/3}\,du. \tag{3.15}$$

As remarked in Section 1.4, the sign of equality holds if and only if K is an ellipse.

We are now ready to prove the following

Theorem 3.16 ([MM1])**.** *Let $K \subset \mathbb{R}^2$ be a convex body and let G be a bounded measurable set in $\mathbb{R}^2$.*

If G is K-dense, then K and G are ellipses that differ from one another by a homothety.

Proof. In view of Theorem 3.15, we know that G and K have smooth boundaries and only differ by a homothety; without loss of generality, we shall assume that $G = K$. Thus, (3.12) reads:

$$\rho_K(\hat{u})^3 = c\,V(K)\,f_K(u),\ \ u \in \mathbb{S}^1. \tag{3.16}$$

Our goal is to show that (3.16) leads inequality (3.14) into an equality; then we shall conclude that K is an ellipse.

By a well-known formula, we then compute:

$$2V(K) = \int_{\mathbb{S}^1} \rho_K(u)^2 du = \int_{\mathbb{S}^1} \rho_K(\hat{u})^2 du =$$
$$[c\,V(K)]^{2/3} \int_{\mathbb{S}^1} f_K(u)^{2/3}\,du = [c\,V(K)]^{2/3}\,\Omega(K),$$

that gives:

$$c^{-2} = \frac{\Omega(K)^3}{8V(K)}.$$

On the other hand, (3.16) also gives:

$$V(K, \hat{K}^*) = \frac{1}{2}\int_{\mathbb{S}^1} f_K(u)\,h_{K^*}(\hat{u})\,du = \frac{1}{2}\int_{\mathbb{S}^1} \frac{f_K(u)}{\rho_K(\hat{u})}\,du =$$
$$[c\,V(K)]^{-1}\,\frac{1}{2}\int_{\mathbb{S}^1} \rho_K(\hat{u})^2\,du = c^{-1},$$

here we denoted by $\hat{K}^*$ the body obtained by rotating the set K^* of 90 degrees and we have used that $h_{K^*} = 1/\rho_K$.

Therefore, by applying (3.13) and (3.14) successively, we obtain that

$$\frac{\Omega(K)^3}{8V(K)} = c^{-2} = V(K, \hat{K}^*)^2 \geq V(K)\,V(\hat{K}^*)$$
$$= V(K)\,V(K^*) \geq \frac{\Omega(K)^3}{8V(K)},$$

that is inequality (3.14) holds with the sign of equality. This concludes the proof.

□

3.4 Asymptotics as $r \to 1$ and strong convexity

So far, we are not able to reproduce the argument used in the proof of Theorem 3.16 in general dimension.

To succeed in our purpose we change strategy: we give up the asymptotic expansion for $r \to 0^+$ in favor of an expansion like

$$V(G \cap (x + r\,K)) = V(G) + W(x)\,(r_G - r)^{\frac{N+1}{2}} + o\left((r_G - r)^{\frac{N+1}{2}}\right) \quad (3.17)$$

as $r \to r_G^-$, where

$$r_G = \inf\{r > 0 : G \subseteq x + r\,K\},\ x \in \partial G.$$

We recall that, thanks to Corollary 3.4, r_G is independent on $x \in \partial G$; since our problem is invariant with respect to dilations of K, we shall assume, again, that $r_G = 1$.

As we will see in Theorem 3.22, the expression for $W(x)$ involves the support function of K and the shape operators of both K and G. The crucial step is to prove the symmetry of G; to do this, we will need to write the shape operator of K in terms of that of G.

In force of Remark 3.11, Theorem 3.8 holds true and thence $K = G - G$. Thus, our aim is now to show that K-dense bodies are strongly convex; then, by Theorem 1.12, we will gain the necessary regularity of K that gives a meaning to (1.9) with $K = G$ and $L = -G$.

The study of the asymptotic behavior of $V(G \backslash (x+rK))$ as $r \to 1^-$ carried out in Lemma 3.19 will show that K-dense bodies are strongly convex (see Corollary 3.20). However, before starting our analysis, we must consider that, if we want to express $V(G \setminus (x+rK))$ in terms of the shape operator of ∂G at some point $\overline{x} \in \partial G$, it is important to make sure that G shares with the boundary of $x + K$ only one point, since the shape operator carries only local information about the boundary.

Remark 3.17. We observe that this is not always the case: indeed, consider the Releaux triangle[4] as the set G and let x denote one of its vertices; then, $K = G - G$ is a ball and $G \cap (x + K)$ is one of the arcs constituting the triangle's boundary; hence, so to speak, $G \setminus (x + rK)$ can not be localized around any point of ∂G.

Notice that such a G is strictly convex, but ∂G is not differentiable at all points. Likewise, if we consider differentiable bodies which are not strictly convex, we can still provide an example of the same phenomenon: in fact, it is enough to set $G = B + Q$, where B is the unit ball and Q is the unit square (see figure 3.1).

The following lemma shows that we can get the desired result, if we assume that G is both differentiable and strictly convex.

Lemma 3.18. *Let G be a strictly convex body with differentiable boundary and set $K = G - G$, then for each $x \in \partial G$ the set $\partial(x + K) \cap G$ consists of only one point $\overline{x} \in \partial G$ characterized by $\nu_K(\overline{x} - x) = -\nu_G(x)$.*

[4] As is well-known, the Releaux triangle is the simplest example of a body of constant width different from the disk.

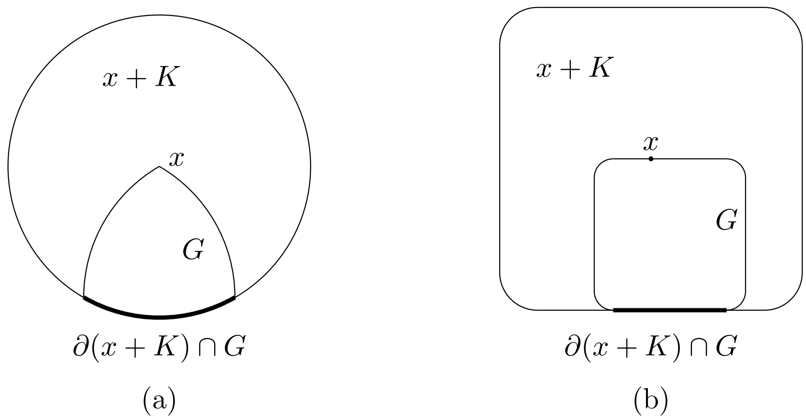

Figure 3.1: In case (a) G is a Releaux triangle and x is one of the vertices, in case (b) G is the sum of a square and a ball.

Proof. Let $z \in \partial K \cap (G-x)$ and let $u = \nu_K(z)$. Clearly $z+x \in \partial G$ and, since $G-x$ is contained in K and touches K at z from inside, then $\nu_G(z+x) = u$. Since $K = G - G$, we have

$$\begin{aligned} h_G(u) + h_G(-u) = h_K(u) &= \langle z, u\rangle = \langle z+x, u\rangle + \langle x, -u\rangle \\ &= h_G(u) + \langle x, -u\rangle. \end{aligned}$$

Thus, $h_G(-u) = \langle x, -u\rangle$, that is $\nu_G(x) = -u$. It is then enough to set $\overline{x} = z + x$.

Now, suppose that there exists another point z' such that $z' \in \partial K \cap (G-x)$ and set $u' = \nu_K(z')$; by the same argument as above, we get that $\nu_G(x) = -u'$, and hence $u = u'$. Since K is strictly convex (being G so), we finally find $z = z'$. □

Lemma 3.19. *Let G be a strictly convex body with boundary of class C^2 and let $K = G - G$. For $x \in \partial G$ and $\overline{x} \in \partial G$ such that $u = \nu_G(\overline{x}) = -\nu_G(x)$, It holds:*

(i) if $\kappa_G(u) = 0$, then

$$\liminf_{r\to 1^-} \frac{V(G \setminus (x + rK)}{(1-r)^{\frac{N+1}{2}}} = +\infty;$$

(ii) if $\kappa_G(u) > 0$, then

$$\limsup_{r\to 1^-} \frac{V(G \setminus (x + rK)}{(1-r)^{\frac{N+1}{2}}} \le \frac{2^{\frac{N+1}{2}}\omega_{N-1}}{N+1}\, \kappa_G(u)\, h_K(u)^{\frac{N+1}{2}} (1+\Lambda)^{\frac{N-1}{2}},$$

where Λ is the maximal principal curvature of ∂G at x.

Proof. First, notice that, by the above lemma, our choice of x and $\overline{x}$ ensures that $\{\overline{x}\} = \partial(x+K) \cap G$. Without loss of generality, we can always assume that $\overline{x} = 0$ and that $u = (0, 0, \dots, -1)$; then, in a neighborhood of $\overline{x}$, ∂G can be parametrized by

$$y_N = \frac{1}{2}\langle S_G(u)\, y, y\rangle + o(|y|^2) \quad \text{as} \quad |y| \to 0, \tag{3.18}$$

where $y = (y_1, \dots, y_{N-1})$ ranges in the tangent space to ∂G at $\overline{x}$.

(i) Set $\varepsilon = 1 - r$. Let ε_n be an infinitesimal sequence of positive numbers such that

$$\liminf_{r \to 1^-} \frac{V(G \setminus (x + rK))}{(1-r)^{\frac{N+1}{2}}} = \lim_{n \to \infty} \frac{V(G_n)}{\varepsilon_n^{\frac{N+1}{2}}},$$

where $G_n := G \setminus (x + (1 - \varepsilon_n)K)$; then (3.18) suggests that, by possibly extracting a subsequence from ε_n, we can fit in G_n the set E_n bounded by the paraboloid

$$y_N = \frac{1}{2}\langle S_G(u)\, y, y\rangle + \frac{1}{n}\,|y|^2$$

and the hyperplane $\varepsilon_n\, h_K(u)\, u + u^\perp$ supporting the set $x + (1 - \varepsilon_n)K$ at the point whose outer unit normal coincides with u. In our coordinates,

$$E_n = \{(y, y_N) : \frac{1}{2}\langle S_G(u)\, y, y\rangle + \frac{1}{n}\,|y|^2 < y_N < \varepsilon_n\, h_K(u)\}$$

and $E_n \subseteq G_n$.

Thus, by Fubini's theorem and some calculations, we get:

$$V(G_n) \geq V(E_n) = \int_0^{\varepsilon_n h_K(u)} \mathcal{H}^{N-1}\left(\left\{y : \langle\left[\frac{S_G(u)}{2} + \frac{1}{n}\, I\right]\; y, y\rangle \leq t\right\}\right) dt =$$

$$\frac{\omega_{N-1}}{\det\left[\frac{S_G(u)}{2} + \frac{1}{n}\, I\right]^{1/2}} \int_0^{\varepsilon_n h_K(u)} t^{\frac{N-1}{2}}\, dt = \frac{2\omega_{N-1}\, \varepsilon_n^{\frac{N+1}{2}}\; h_K(u)^{\frac{N+1}{2}}}{(N+1)\det\left[\frac{S_G(u)}{2} + \frac{1}{n}\, I\right]^{1/2}}.$$

Therefore,

$$\liminf_{\varepsilon \to 0^+} \frac{V(G \setminus (x + (1-\varepsilon)K)}{\varepsilon^{\frac{N+1}{2}}} = \lim_{n \to \infty} \varepsilon_n^{-\frac{N+1}{2}}\, V(G_n) \geq$$

$$\lim_{n \to \infty} \frac{2\,\omega_{N-1}\, h_K(u)^{\frac{N+1}{2}}}{(N+1)\sqrt{\det\left[\frac{S_G(u)}{2} + \frac{1}{n}\, I\right]}} = +\infty,$$

since $\det S_G(u) = \kappa_G(u) = 0$.

(ii) We shall obtain the desired inequality by observing that the domain $G \setminus (x + (1 - \varepsilon)K)$ can be contained in the region $F_{\varepsilon,\delta}$ bounded by two paraboloids: one outside G and tangent to ∂G at $\overline{x}$, the other one tangent to the boundary of $x + (1-\varepsilon)K$ from inside. In order to show it, we assume as before that $\overline{x} = 0$ and $u = -e_N$ and, moreover, that $S_G(u) = I$ (this can be done since the affine tranformation $S_G(u)$ is invertible, being $\det S_G(u) = \kappa_G(u) > 0$): the desired formula will then be obtained by multiplying the right-hand side of (3.19) by the factor $\kappa_G(u)$.

We proceed to contruct $F_{\varepsilon,\delta}$. We choose any number $\lambda > 0$ such that $\lambda\, I > S_G(-u)$[5], that is such that $\lambda > \Lambda$. Since $\kappa_G(u) > 0$, Theorem 1.12 and the following remark imply that ∂K is twice differentiable at $\overline{x} - x$; moreover equation (1.9) turns into

$$S_K(u) < \frac{\lambda}{1+\lambda}\, I;$$

hence,

$$S_{(1-\varepsilon)K}(u) < \frac{\lambda}{(1+\lambda)(1-\varepsilon)}\, I.$$

For $\varepsilon > 0$ sufficiently small, we define $F_{\varepsilon,\delta}$ as

$$F_{\varepsilon,\delta} = \left\{ (y, y_N) : \frac{\delta}{2}\, |y|^2 \le y_N \le \varepsilon\, h_K(u) + \frac{\lambda}{2(1+\lambda)(1-\varepsilon)}\, |y - \varepsilon x_*|^2 \right\},$$

where δ is chosen in the interval $(\frac{\lambda}{(1+\lambda)(1-\varepsilon)}, 1)$ and x_* is the projection of x on the tangent space to ∂G at $\overline{x}$; in this way,

$$G \setminus (x + (1-\varepsilon)K) \subset F_{\varepsilon,\delta}.$$

Indeed, equation (3.18) guarantees that the above inclusion holds, at least inside a small neighborhood of $\overline{x}$; however, by Lemma 3.18, we know that $G \setminus (x + (1-\varepsilon)K)$ is contained in a ball B_r around $\overline{x}$ whose radius $r = r(\varepsilon)$ tends to 0 as $\varepsilon \to 0$.

By using the rescaling $(y, y_N) = (\sqrt{\varepsilon}\, \xi, \varepsilon\, \xi_N)$, we obtain that $V(F_{\varepsilon,\delta}) = \varepsilon^{\frac{N+1}{2}}\, V(F'_{\varepsilon,\delta})$, where

$$F'_{\varepsilon,\delta} = \left\{ (\xi, \xi_N) : \frac{\delta}{2}\, |\xi|^2 \le \xi_N \le h_K(u) + \frac{\lambda}{2(1+\lambda)(1-\varepsilon)}\, |\xi - \sqrt{\varepsilon}\, x_*|^2 \right\},$$

and it is easy to show that $V(F'_{\varepsilon,\delta}) \to V(F'_{0,\delta})$.

By a straightforward computation of $V(F'_{0,\delta})$, we get that

$$\limsup_{\varepsilon \to 0} \frac{V(G \setminus (x + (1-\varepsilon)K))}{\varepsilon^{\frac{N+1}{2}}} \le \frac{\omega_{N-1}}{N+1} \times \frac{2^{\frac{N+1}{2}}\, h_K(u)^{\frac{N+1}{2}}}{(\delta - \frac{\lambda}{1+\lambda})^{\frac{N-1}{2}}},$$

[5] By $A > B$ we mean that the matrix $A - B$ is positive definite.

and minimizing the right-hand side of this formula for $\lambda/(1+\lambda) < \delta < 1$ and $\lambda > \Lambda$ then gives:

$$\limsup_{\varepsilon\to 0} \frac{V(G \setminus (x+(1-\varepsilon)K))}{\varepsilon^{\frac{N+1}{2}}} \leq \frac{\omega_{N-1}}{N+1} \times 2^{\frac{N+1}{2}} h_K(u)^{\frac{N+1}{2}} (1+\Lambda)^{\frac{N-1}{2}}. \tag{3.19}$$

□

Corollary 3.20. *If G is K-dense, then ∂K is of class C^2 and every point of ∂G is a point of strong convexity. The latter conditions and the fact that $K = G - G$ allow us to write, thanks to Theorem 1.12,*

$$S_K(u) = \left[I + S_G(u)^{-1} S_G(-u)\right]^{-1} S_G(-u). \tag{3.20}$$

Proof. Since G is K-dense, then the limits in items (i) and (ii) in Lemma 3.19 do not depend on the particular point $x \in \partial G$; in other words, they must be constant functions on ∂G. Since G is a convex body and ∂G is of class C^2, then κ_G is not identically zero; hence, the limit in item (ii) of Lemma 3.19 is a finite constant. As a consequence, item (i) of the same lemma implies that $\kappa_G > 0$ (and hence $S_G > 0$) on ∂G. Formula (3.20) is then a straightforward consequence of Theorem 1.12. □

3.5 Simmetry of G and characterization of ellipsoids

In this section we prove the following characterization theorem.

Theorem 3.21 ([MM2])**.** *Let $K \subset \mathbb{R}^N$ be a convex body and assume that there is a set $G \subset \mathbb{R}^N$ of finite positive measure such that* (3.2) *holds.*

Then, both K and G must be homothetic to the same ellipsoid.

We start by explicitly computing the coefficient $W(x)$ of formula (3.17).

Theorem 3.22 ([MM2])**.** *Let G be a strongly convex body with boundary of class C^2 and set $K = G - G$. Chose x, u and $\overline{x}$ as in Lemma 3.19; then*

$$\lim_{r\to 1^-} \frac{V(G \setminus (x+rK))}{(1-r)^{\frac{N+1}{2}}} = \frac{2\omega_{N-1}\, h_K(u)^{\frac{N+1}{2}}}{(N+1)\det\left[\frac{S_G(u)}{2} - \frac{S_K(u)}{2}\right]^{1/2}}.$$

Proof. Again we set $\varepsilon = 1 - r$. We begin by showing that

$$\limsup_{\varepsilon\to 0^+} \frac{V(G \setminus (x+(1-\varepsilon)K))}{\varepsilon^{\frac{N+1}{2}}} \leq \frac{2\omega_{N-1}\, h_K(u)^{\frac{N+1}{2}}}{(N+1)\det\left[\frac{S_G(u)}{2} - \frac{S_K(u)}{2}\right]^{1/2}}. \tag{3.21}$$

As in the proof of Lemma 3.19, without loss of generality, we can set $u = -e_N$ and $\overline{x} = 0$.

We recall that $0 = \overline{x} \in \partial(x + K)$, thus $-x \in \partial K$ and $-(1-\varepsilon)x \in \partial((1-\varepsilon)K)$; namely $\varepsilon x \in \partial(x + (1-\varepsilon)K)$ and u is the unit normal to $\partial(x + (1-\varepsilon)K)$ at that point.

By a scaling argument, we know that

$$S_{x+(1-\varepsilon)K}(u) = \frac{S_K(u)}{1-\varepsilon}.$$

Notice that formula (3.20) implies that $S_G(u) > S_K(u)$; hence, we can choose $\overline{n} \in \mathbb{N}$ such that

$$\frac{S_G(u) - S_K(u)}{4} > \frac{I}{\overline{n}}. \tag{3.22}$$

In order to get an estimate from above for $V(G \setminus (x + (1-\varepsilon)K))$ we construct a set $C_{\varepsilon,n}$ containing $G \setminus (x + (1-\varepsilon)K)$. In fact, for $n > \overline{n}$ we set

$$\begin{aligned} C_{\varepsilon,n} = \Big\{ (y, y_N) : \langle \left(\frac{S_G(u)}{2} - \frac{1}{n} I \right) y, y \rangle < y_N < \\ \varepsilon h_K(u) + \langle \left[\frac{S_K(u)}{2(1-\varepsilon)} + \frac{1}{n} I \right] (y - \varepsilon x_*), (y - \varepsilon x_*) \rangle \Big\}, \end{aligned}$$

where x_* denotes the projection of x on $u^\perp$; $C_{\varepsilon,n}$ is the region bounded by two paraboloids, one touching ∂G at $\overline{x}$ from below, the other one touching $\partial(x + (1-\varepsilon)K)$ at εx from above and, for ε small enough, we have:

$$G \setminus (x + (1-\varepsilon)K) \subset C_{\varepsilon,n}.$$

Also, condition (3.22) guarantees that

$$\frac{S_G(u)}{2} - \frac{I}{n} > \frac{S_K(u)}{2(1-\varepsilon)} + \frac{I}{n} > 0,$$

for ε small enough, thus forcing $C_{\varepsilon,n}$ to be bounded.

The usual change of variables $(y, y_N) = (\sqrt{\varepsilon}\,\xi, \varepsilon\,\xi_N)$ gives that $V(C_{\varepsilon,n}) = \varepsilon^{\frac{N+1}{2}} V(C'_{\varepsilon,n})$, where

$$\begin{aligned} C'_{\varepsilon,n} = \Big\{ (\xi, \xi_N) : \langle \left[\frac{S_G(u)}{2} - \frac{1}{n} I \right] \xi, \xi \rangle < \xi_N < h_K(u) + \\ \langle \left[\frac{S_K(u)}{2(1-\varepsilon)} + \frac{1}{n} I \right] (\xi - \sqrt{\varepsilon} x_*), (\xi - \sqrt{\varepsilon} x_*) \rangle \Big\}. \end{aligned}$$

Since clearly $V(C'_{\varepsilon,n}) \to V(C'_{0,n})$ as $\varepsilon \to 0$, a straightforward computation gives:

$$\limsup_{\varepsilon\to 0^+} \frac{V(G \setminus (x + (1-\varepsilon)K))}{\varepsilon^{\frac{N+1}{2}}} \le V(C'_{0,n}) = \frac{2\omega_{N-1}\, h_K(u)^{\frac{N+1}{2}}}{(N+1)\det\left[\frac{S_G(u)}{2} - \frac{S_K(u)}{2} - \frac{2}{n}\, I\right]^{\frac{1}{2}}}. \tag{3.23}$$

Since (3.23) holds for all n large enough, (3.21) follows at once by taking the limit for $n \to \infty$.

The converse inequality,

$$\liminf_{\varepsilon\to 0^+} \frac{V(G \setminus (x + (1-\varepsilon)K))}{\varepsilon^{\frac{N+1}{2}}} \ge \frac{2\omega_{N-1}\, h_K(u)^{\frac{N+1}{2}}}{(N+1)\det\left[\frac{S_G(u)}{2} - \frac{S_K(u)}{2}\right]^{\frac{1}{2}}},$$

is proved by using the same strategy used for (3.21): we choose $\overline{n}$ such that

$$S_G(u) > \frac{I}{\overline{n}}$$

and then we construct, for $n > \overline{n}$ and ε small, a set $D_{\varepsilon,n} \subseteq G \setminus (x + (1-\varepsilon)K)$:

$$D_{\varepsilon,n} = \Big\{ (y, y_N) : \langle \left(\frac{S_G(u)}{2} + \frac{1}{n} I \right) y, y \rangle < y_N < \\ < \varepsilon h_K(u) + \langle \left[\frac{S_K(u)}{2(1-\varepsilon)} - \frac{1}{n} I \right] (y - \varepsilon x_*), (y - \varepsilon x_*) \rangle \Big\}.$$

As before, the usual rescaling gives

$$\liminf_{\varepsilon\to 0^+} \frac{V(G \setminus (x + (1-\varepsilon)K))}{\varepsilon^{\frac{N+1}{2}}} \ge \frac{2\omega_{N-1}\, h_K(u)^{\frac{N+1}{2}}}{(N+1)\det\left[\frac{S_G(u)}{2} - \frac{S_K(u)}{2} + \frac{2}{n} I\right]^{\frac{1}{2}}}.$$

Again, we conclude by taking the limit for $n \to \infty$. $\square$

Corollary 3.23. *Let G be a K-dense body, then (3.17) holds with the coefficient $W(x)$ given by*

$$W(x) = -\frac{2\omega_{N-1}\, h_K(u)^{\frac{N+1}{2}}}{(N+1)\ \det\left[\frac{S_G(u)}{2} - \frac{S_K(u)}{2}\right]^{1/2}} \quad \textit{with} \quad u = \nu(\overline{x}). \tag{3.24}$$

Moreover, the function defined by

$$u \mapsto \frac{h_K(u)^{\frac{N+1}{2}}}{\det[S_G(u) - S_K(u)]^{\frac{1}{2}}}, \quad u \in \mathbb{S}^{N-1} \tag{3.25}$$

is constant.

Proof. Corollary 3.20 ensures that G satisfies the assumptions of Theorem 3.22. Since $V(G\cap(x+r\,K)) = V(G)-V(G\setminus(x+r\,K))$, then for the function given in (3.24) the following equality holds

$$W(x) = -\lim_{r\to 1^-}\frac{V(G\setminus(x+r\,K))}{(1-r)^{\frac{N+1}{2}}} = \lim_{r\to 1^-}\frac{V(G\cap(x+rK))-V(G)}{(1-r)^{\frac{N+1}{2}}} \tag{3.26}$$

hence (3.17). Observe that $W|_{\partial G}$ has to be constant, by formula (3.26) and the K-density assumption. Finally, since G is strictly convex, the last assertion follows from the suriectivity of the Gauss map. $\square$

Now, we are going to show that if G is K-dense, then G and K must be equal up to homotheties.

Proposition 3.24. *Let G be a K-dense body, then $\kappa_G(u) = \kappa_G(-u)$.*

Proof. Let $u \in \mathbb{S}^{N-1}$ and $L = L_u$ be a linear map of $\mathbb{R}^N$ in itself, which leaves unchanged the unit vector u and whose restriction to $u^\perp$ equals $S_G(u)^{-\frac{1}{2}}$.

First, notice that, as an easy consequence of (3.2), the set LG is LK-dense, so that Corollary 3.23 holds for this set; in particular, (3.25) implies:

$$h_{LK}(-u)^{\frac{N+1}{2}}\{\det[S_{LG}(-u)-S_{LK}(-u)]\}^{-\frac{1}{2}} = h_{LK}(u)^{\frac{N+1}{2}}\{\det[S_{LG}(u)-S_{LK}(u)]\}^{-\frac{1}{2}}. \tag{3.27}$$

Secondly, we know that K is centrally symmetric, and so must be LK; then, $S_{LK}(u) = S_{LK}(-u)$ and $h_{LK}(u) = h_{LK}(-u)$. Hence, by (3.27):

$$\det[S_{LG}(-u)-S_{LK}(u)] = \det[S_{LG}(u)-S_{LK}(u)]. \tag{3.28}$$

As we shall see, this condition together with equation (1.9) is enough to prove that

$$\det[S_{LG}(u)] = \det[S_{LG}(-u)].$$

Indeed, by plugging (1.9) into (3.28) we get

$$\det\Big(S_{LG}(-u) - \left[I + S_{LG}(u)^{-1}S_{LG}(-u)\right]^{-1}S_{LG}(-u)\Big) = \det\Big(S_{LG}(u) - \left[I + S_{LG}(u)^{-1}S_{LG}(-u)\right]^{-1}S_{LG}(-u)\Big)\,; \tag{3.29}$$

furthermore, our chioice of the affine transformation L ensures that

$$S_{LG}(u) = I,$$

and

$$S_{LG}(-u) = S_G(u)^{-\frac{1}{2}}S_G(-u)S_G(u)^{-\frac{1}{2}}. \tag{3.30}$$

Equation (3.29) then turns into

$$\det\Big(S_{LG}(-u) - [I + S_{LG}(-u)]^{-1}\, S_{LG}(-u)\Big) = \det\Big(I - [I + S_{LG}(-u)]^{-1}\, S_{LG}(-u)\Big)\,; \quad (3.31)$$

by multiplying both sides of (3.31) by $\det[I + S_{LG}(-u)]$ and using Binet's identity, we get

$$\det[S_{LG}(-u)^2] = 1.$$

Hence (3.30) yields $\det[S_G(u)] = \det[S_G(-u)]$, that is $\kappa_G(u) = \kappa_G(-u)$. □

Corollary 3.25. *Let G be K-dense. Then G is symmetric and $K = 2G$.*

Proof. The two bodies $G - G$ and $2G$ have the same Gaussian curvature as a function on $\mathbb{S}^{N-1}$; thus, they only differ by a translation. □

The following theorem together with Petty's characterization of ellipsoids complete the proof of Theorem 3.21.

Theorem 3.26 ([MM2]). *Let G be a K-dense set. Then, for every $x \in \partial G$ it holds that*

$$\lim_{r \to 1^-} \frac{V(G \setminus (x + rK))}{(1-r)^{\frac{N+1}{2}}} = \frac{2^N \omega_{N-1}\, h_K(u)^{\frac{N+1}{2}}}{(N+1)\, \det[S_G(u)]^{\frac{1}{2}}} \quad \textit{with} \quad u = \nu(\overline{x})$$

and $\{\overline{x}\} = \partial G \cap (x + K)$.

In particular, there exists a positive constant c such that

$$\kappa_G(u) = c\, h_G(u)^{N+1} \quad \textit{for every} \quad u \in \mathbb{S}^{N-1}.$$

Therefore, G must be an ellipsoid.

Chapter 4

Stationary isothermic surfaces

In this chapter we study the solutions of some evolution equations that have Klamikin's property of having invariant level surfaces.

In Section 4.1, we characterize all the solutions of the heat equation that have all their (spatial) equipotential surfaces which do not vary with the time (see Theorem 4.1). In particular we show that such solutions are either isoparametric or split in space-time. We then extend that result to a class of quasi-linear parabolic equations (see Theorem 4.5).

In Section 4.2, we study the geometry of the level sets of K-isoparametric functions.

In Section 4.3, we start investigating the properties of the solutions of the h-Laplace evolution equation with only one invariant equipotential surface. We study the asymptotic behavior of the solutions of a family of elliptic problems which have, at least in the linear case, a strong connection with the solutions of the heat equation.

4.1 The Matzoh Ball Soup Problem

Let Ω be a domain and let u be the solution of the following initial-Dirichlet problem

$$\begin{cases} u_t = \Delta u & \text{in } \Omega \times (0, \infty) \\ u = 0 & \text{in } \Omega \times \{0\} \\ u = 1 & \text{on } \partial\Omega \times (0, \infty) \end{cases} \tag{4.1}$$

It is well known that, if Ω is the euclidean ball, then the solution of (4.1) is radially symmetric. Consequently, its level surfaces do not vary with the time, since all of them are concentric spheres.

In this section we completely characterize all the solutions of the heat equation

$$u_t = \Delta u \quad \text{in} \quad \Omega \times (0, \infty), \tag{4.2}$$

having the Klamkin's property of time-invariant level surfaces (Theorem 4.1). Then we extend our analysis to a class of quasi-linear evolution equation (Theorem 4.5). All these results can be found in the recent paper [MM3].

Theorem 4.1 ([MM3]). *Let $\Omega \subseteq \mathbb{R}^N$ be a domain and let u be a solution of equation* (4.2)

Assume that there exists a number $\tau > 0$ such that, for every $t > \tau$, $u(\cdot, t)$ is constant on the level surfaces of $u(\cdot, \tau)$ and $Du(\cdot, \tau) \neq 0$ in Ω.

Then one of the following occurrences holds:

(i) the function $\varphi = u(\cdot, \tau)$ (and hence u) is isoparametric, that is there exist two real-valued functions f and g such that φ is a solution of the following system of equations:

$$|D\varphi|^2 = f(\varphi) \quad \textit{and} \quad \Delta\varphi = g(\varphi) \quad \textit{in} \quad \Omega;$$

(ii) there exist two real numbers λ, μ such that

$$u(x, t) = e^{-\lambda t}\phi_\lambda(x) + \mu, \quad (x, t) \in \Omega \times [\tau, \infty),$$

where

$$\Delta\phi_\lambda + \lambda\,\phi_\lambda = 0 \quad \textit{in} \quad \Omega;$$

(iii) there exists a real number γ such that

$$u(x, t) = \gamma\, t + w(x), \quad (x, t) \in \Omega \times [\tau, \infty),$$

where

$$\Delta w = \gamma \quad \textit{in} \quad \Omega.$$

Proof. As originally observed in [Al1], the assumption on the level surfaces of u implies that, if we set $\varphi(x) = u(x, \tau)$, then there exist a number $T > \tau$ and a function $\eta : \mathbb{R} \times [\tau, T) \to \mathbb{R}$, with

$$\eta(s, \tau) = s, \quad s \in \mathbb{R}, \tag{4.3}$$

such that

$$u(x, t) = \eta(\varphi(x), t) \tag{4.4}$$

for every $(x, t) \in \overline{\Omega} \times [\tau, T)$; thus, (4.2) gives that

$$\eta_{ss}(\varphi, t)\,|D\varphi|^2 + \eta_s(\varphi, t)\,\Delta\varphi = \eta_t(\varphi, t) \quad \text{in } \Omega \times [\tau, T).$$

By differentiating in t this identity, we obtain that φ must satisfy in Ω the following system of equations:

$$\begin{aligned} \eta_s(\varphi,t)\,\Delta\varphi + \eta_{ss}(\varphi,t)\,|D\varphi|^2 = \eta_t(\varphi,t),\\ \eta_{st}(\varphi,t)\,\Delta\varphi + \eta_{sst}(\varphi,t)\,|D\varphi|^2 = \eta_{tt}(\varphi,t), \end{aligned} \tag{4.5}$$

for $t \in [\tau, T)$.

Notice that the necessary smoothness of the function η can be proved by a standard finite difference argument: in fact, one can prove that, since $D\varphi \neq 0$, then $\eta \in C^\infty(I \times [\tau, T))$, where $I = (\inf_\Omega \varphi, \sup_\Omega \varphi)$ and $\eta_s > 0$ on $I \times [\tau, T)$ (see [Al1, Lemma 1], [Al2, Lemma 2.1] or [Sak, Lemma 2.1] for details).

As observed in [Sak], for the system (4.5), it is enough to consider the alternative cases in which the determinant

$$D(s,t) = \eta_s\eta_{sst} - \eta_{st}\eta_{ss}$$

is zero or not zero. In fact, if $D(s,t) \neq 0$ at some $(s,t) \in I \times [\tau, T)$, then $D \neq 0$ in an open neighborhood, say $U \times V \subset I \times [\tau, T)$, of (s,t); (4.5) then implies that

$$|D\varphi|^2 = f(\varphi) \ \text{ and } \ \Delta\varphi = g(\varphi)$$

at least in a subdomain Ω' of Ω, and the expressions of f and g are given by the formulas

$$f = \frac{\eta_s\eta_{tt} - \eta_{st}\eta_t}{\eta_s\eta_{sst} - \eta_{st}\eta_{ss}}, \quad g = \frac{\eta_t\eta_{sst} - \eta_{tt}\eta_{ss}}{\eta_s\eta_{sst} - \eta_{st}\eta_{ss}};$$

clearly, f and g are analytic functions. Thus $u(\cdot, t)$ (and hence φ) is isoparametric in an open subdomain of Ω; by using the classification result for isoparametric functions by Levi-Civita and Segre and the analiticity of φ, we can then conclude that the function φ is isoparametric in the whole Ω.

Otherwise, we have $D = 0$ in $I \times [\tau, T)$. Thus, since

$$\frac{\partial^2}{\partial s \partial t} \log(\eta_s) = (\eta_s)^{-2}\,(\eta_s\eta_{sst} - \eta_{st}\eta_{ss}) = 0 \ \text{ in } \ I \times [\tau, T),$$

we have that $\log \eta_s(s,t)$ splits up into the sum of a function of s plus a function of t; (4.3) then implies that $\eta_s(s,t)$ only depends on t, and hence it is easy to conclude that

$$\eta(s,t) = a(t)\,s + b(t), \quad (s,t) \in I \times [\tau, T),$$

for some smooth functions a and b such that

$$a(\tau) = 1 \ \text{ and } \ b(\tau) = 0.$$

Now, we now know that

$$u(x,t) = a(t)\,\varphi(x) + b(t)$$

is a solution of (4.2), thus (4.5) can be written as a linear system of equations:

$$\begin{aligned} a'(t)\,\varphi(x) - a(t)\,\Delta\varphi &= -b'(t),\\ a''(t)\,\varphi(x) - a'(t)\,\Delta\varphi &= -b''(t). \end{aligned} \tag{4.6}$$

The determinant of this system must be zero, otherwise φ would be constant (in fact, it would be that $\varphi(x)$ is a function of t); thus,

$$a(t)a''(t) - a'(t)^2 = 0 \quad \text{for} \quad t \in [\tau, \infty), \;\; a(\tau) = 1.$$

All solutions of this problem can be written as $a(t) = e^{-\lambda(t-\tau)}$ for $\lambda \in \mathbb{R}$ and, by going back to (4.6), we obtain that

$$\Delta\varphi(x) + \lambda\,\varphi(x) = b'(t)\,e^{\lambda(t-\tau)} = \gamma,$$

for some constant γ. Since $b(\tau) = 0$, we have:

$$b(t) = \gamma\,\frac{1 - e^{-\lambda(t-\tau)}}{\lambda} \quad \text{if} \;\; \lambda \neq 0 \qquad \text{and} \qquad b(t) = \gamma\,(t-\tau) \;\; \text{if} \;\; \lambda = 0.$$

Therefore, we have obtained:

$$\begin{aligned} u(x,t) &= e^{-\lambda(t-\tau)}\,[\varphi(x) - \gamma/\lambda] + \gamma/\lambda \;\; \text{if} \;\; \lambda \neq 0,\\ u(x,t) &= \varphi(x) + \gamma(t-\tau) \;\; \text{if} \;\; \lambda = 0. \end{aligned}$$

In conclusion, by setting $\phi_\lambda = e^{\lambda\tau}[\varphi - \gamma/\lambda]$ and $\mu = \gamma/\lambda$ for $\lambda \neq 0$, we get case (ii), while setting $w = \varphi - \gamma\tau$ for $\lambda = 0$ yields case (iii).

□

Remark 4.2. The proof of Theorem 4.1 relies on and completes those contained in [Al2] and [Sak]: there, option (iii) and the fact that initial and boundary conditions are unnecessary were overlooked.

Remark 4.3. For the sake of simplicity, in Theorem 4.1 we assumed that $Du \neq 0$ in the whole Ω, in order to be able to deduce the necessary regularity for η. Here, we will show how that assumption can be removed.

The only possible obstruction to the regularity of η is the presence of level sets of φ whose points are all critical. Indeed η is always smooth in the t variable and is smooth in the s variable, for every semi-regular value s of φ (*i.e.* a value that is the image of at least one regular point). For every open subset of Ω of regular points for φ corresponding to a semiregular value of φ, the above theorem still holds true even if we remove the assumption on the gradient of the solution u.

We now show that, in any case, the classification holds true globally in Ω. For simplicity we assume that there exists only one critical level set.

Let then s be a value in the range of φ such that $D\varphi(x) = 0$, for every $x \in \varphi^{-1}(s)$ and set $\Omega^+ = \{x \in \Omega : \varphi(x) > s\}$, and $\Omega^- = \{x \in \Omega : \varphi(x) <$

$s\}$. Clearly the above classification holds true separately in Ω^+ and in Ω^-. Being cases (i), (ii) and (iii) closed relations involving continuous functions, if the same case occours both in Ω^+ and Ω^-, then it holds in the whole domain Ω. Thus, we are left with all the cases in which in Ω^+ and Ω^- the solution assumes two different representations of those given in Theorem 4.1.

We proceed by direct ispection. If case (i) occurs in Ω^+ or in Ω^-, as it has been already remarked, then φ extends to an isoparametric function in an open domain containing Ω.

We then have only to study the case in which instances (ii) and (iii) are in force in Ω^- and Ω^+, respectively; by contradiction, we shall see that it is not possible to have a critical level set. In fact, up to sum a constant, we can assume without loss of generality that $s \neq 0$. As shown in [Al2, Lemma 2.2], the presence of a critical level set implies that $\Delta\varphi(x) = 0$, for every $x \in \overline{\Omega^+} \cap \overline{\Omega^-}$. Then, according to the previous computations, we get that $0 = \Delta\varphi(x) = \lambda\varphi(x)$, and, being $\lambda \neq 0$, we have that $\varphi(x) = 0$ and thence $s = 0$, that is a contradiction.

Remark 4.4. *Isoparametric functions* are well-known in the literature; accordingly, their level surfaces are called *isoparametric surfaces* and can also be characterized as those surfaces whose principal curvatures are all constant. The classical results of T. Levi-Civita [Le] and B. Segre [Se] classify isoparametric functions in $\mathbb{R}^N$ by their level surfaces: they can be either concentric spheres, co-axial spherical cylinders (that is cartesian products of an M-dimensional sphere by an $(N-M-1)$-dimensional euclidean space, $0 \le M \le N-2$), or parallel hyperplanes (affine spaces of co-dimension 1). By using this fact, one can conclude fairly easily that, in Klamkin's setting, that is when (i) of Theorem 4.1 holds and u is constant on $\partial\Omega$, then the possible shapes of a domain Ω can be one of the following: a ball, its exterior or a spherical annulus; a spherical cylinder, its exterior or a cylindrical annulus; a half-space or an infinite strip (see [Al2] for the solid case). Analogous results can be drawn in the case we impose on u a homogeneous Neumann condition (see [Sak]),

$$u_\nu = 0 \quad \text{on} \quad \partial\Omega \times (0, \infty). \tag{4.7}$$

Thus, a caloric function that has invariant equipotential surfaces enjoys some *splitting property in space-time*, since it is always separable (either with respect to addition or to multiplication). We point out that this behaviour is not restricted to the case of the heat equation, since it also occurs for other *linear* evolution equations, such as the *wave equation*, the *Schrödinger equation* or any partial differential equation connected to the heat equation by some integral transform.

In the last part of this section we show that a similar behaviour holds for the following class of quasi-linear evolution equations:

$$u_t = \mathcal{Q}\,u \quad \text{in} \quad \Omega \times (0, \infty), \tag{4.8}$$

where the operator

$$\mathcal{Q}\,u = \sum_{i,j=1}^{N} a_{ij}(Du)\,u_{x_i x_j} \tag{4.9}$$

is elliptic and the coefficients $a_{ij}(\xi)$ are sufficiently smooth α-homogeneous functions of $\xi \in \mathbb{R}^N$, $\alpha > -1$, that is such that

$$a_{ij}(\sigma\,\xi) = \sigma^{\alpha} a_{ij}(\xi) \ \text{ for every } \ \xi \in \mathbb{R}^N, \ \sigma > 0 \ \text{ and } \ i, j = 1, \dots, N; \tag{4.10}$$

Important instances of (4.8) are the *evolution p-Laplace* equation,

$$u_t = \operatorname{div}\{|Du|^{p-2}\,Du\} \ \text{ in } \ \Omega \times (0, \infty); \tag{4.11}$$

the *normalized evolution p-Laplace* equation,

$$u_t = |Du|^{2-p}\operatorname{div}\{|Du|^{p-2}\,Du\} \ \text{ in } \ \Omega \times (0, \infty); \tag{4.12}$$

the (anisotropic) *evolution h-Laplace equation*,

$$u_t = \Delta_h u \ \text{ in } \ \Omega \times (0, \infty); \tag{4.13}$$

here,

$$\Delta_h u = \operatorname{div}\{h(Du)\,Dh(Du)\} \tag{4.14}$$

is the so-called *anisotropic* h-laplacian, or *Finsler* laplacian, where h is the support function of a C^2_+ convex body (for more detail see Section 4.2).

Before stating the theorem characterizing all possible solutions of (4.8) we define a generalized gradient operator:

$$\mathcal{G}u = \sum_{i,j=1}^{N} a_{ij}(Du)\,u_{x_i} u_{x_j}. \tag{4.15}$$

Theorem 4.5. *[MM3] Let $\Omega \subseteq \mathbb{R}^N$ be a domain and let $u \in C^1((0,\infty); C^2(\Omega))$ be a solution of equation* (4.8)*,*

$$u_t = \mathcal{Q}\,u \ \text{ in } \ \Omega \times (0, \infty),$$

where the operator $\mathcal{Q}$, given in (4.9)*, is elliptic with coefficients $a_{ij}(\xi)$ that satisfy* (4.10)*.*

Assume that there exists a $\tau > 0$ such that, for every $t > \tau$, $u(\cdot, t)$ is constant on the level surfaces of $u(\cdot, \tau)$ and $Du(\cdot, \tau) \neq 0$ in Ω.

Then the there exists a countable set of values $a_1 < a_2 < \dots a_i < \dots$, such that

$$\Omega = \bigcup_{i \in \mathbb{N}} \varphi^{-1}([a_i, a_{i+1}])$$

and, for every subdomain $\Omega' \subset \varphi^{-1}([a_i, a_{i+1}])$ one of the following cases occurs:

(i) *there exist two real-valued functions f and g such that $\varphi = u(\cdot, \tau)$ is a solution of the following system of equations:*

$$\mathcal{G}\varphi = f(\varphi) \quad \text{and} \quad \mathcal{Q}\varphi = g(\varphi) \quad \text{in} \quad \Omega';$$

(ii) *there exist two real numbers λ, μ such that*

$$u(x,t) = [1 + \lambda\,(t-\tau)]^{-1/\alpha}\phi_\lambda(x) + \mu, \quad (x,t) \in \Omega' \times [\tau, \infty),$$

with

$$\mathcal{Q}\phi_\lambda + \frac{\lambda}{\alpha}\,\phi_\lambda = 0 \quad \text{in} \quad \Omega',$$

if $\alpha \neq 0$;

$$u(x,t) = e^{-\lambda(t-\tau)}\phi_\lambda(x) + \mu, \quad (x,t) \in \Omega' \times [\tau, \infty),$$

with

$$\mathcal{Q}\phi_\lambda + \lambda\,\phi_\lambda = 0 \quad \text{in} \quad \Omega',$$

if $\alpha = 0$;

(iii) *there exists a real number γ such that*

$$u(x,t) = \gamma\,(t-\tau) + w(x), \quad (x,t) \in \Omega' \times [\tau, \infty),$$

where

$$\mathcal{Q}w = \gamma \quad \text{in} \quad \Omega'.$$

Proof. The proof runs similarly to that of Theorem 4.1. We still begin by setting $\varphi(x) = u(x,\tau)$ and $u(x,t) = \eta(\varphi(x),t)$, where η satisfies (4.3). Since $Du \neq 0$ equation (4.8) is uniformly parabolic, by standard parabolic regularity (see, for instance [LU]), we have the necessary regularity to give sense to the following computations.

By arguing as in the proof of Theorem 4.1, we obtain the system of equations:

$$\begin{aligned} \xi(\varphi,\tau)\,\mathcal{Q}\varphi + \xi_s(\varphi,\tau)\,\mathcal{G}\varphi &= \eta_t(\varphi,\tau), \\ \xi_t(\varphi,\tau)\,\mathcal{Q}\varphi + \xi_{st}(\varphi,\tau)\,\mathcal{G}\varphi &= \eta_{tt}(\varphi,\tau), \end{aligned} \tag{4.16}$$

where $\xi = (\eta_s)^{\alpha+1}$.

At this point, the proof is slightly different from that of Theorem 4.1. If there exists (s,t) such that $D(s,t) = \xi\,\xi_{st} - \xi_s\,\xi_t \neq 0$, then $D(u(x,t),t) \neq 0$ for $x \in \Omega' = \{x \in \Omega : u(x,t) \in (s-\delta, s+\delta)\}$ for some $\delta > 0$. By setting

$$\overline{f} = \frac{\xi_s\,\eta_{tt} - \xi_t\,\eta_t}{\xi\,\xi_{st} - \xi_s\,\xi_t}, \quad \overline{g} = \frac{\xi_{st}\,\eta_t - \xi_s\,\eta_{tt}}{\xi\,\xi_{st} - \xi_s\,\xi_t},$$

case (i) holds true for the function $u(\cdot,t)$. The same fact holds true for φ, since $u(x,t) = \eta(\varphi(x),t)$ and hence φ and $u(x,t)$ share the same level sets;

thus, there exist functions f and g such that $\mathcal{G}\varphi = f(\varphi)$ and $\mathcal{Q}\varphi = g(\varphi)$ in Ω'.

Now, we define J as the closure of the complement in the image of u of the closure of the set $\{s : D(s,t) \neq 0, \text{ for some } t\}$; J is a countable union of disjoint intervals. Let $[p,q] \subseteq J$; in $\Omega' = \varphi^{-1}([p,q])$, we get

$$\begin{aligned}
&\frac{\partial^2}{\partial s \partial t} \log(\xi) = (\xi)^{-2} (\xi\, \xi_{st} - \xi_t\, \xi_s) = 0 \;\text{ in }\; [p,q] \times [\tau, T),\\
&\xi(s,\tau) = 1 \;\text{ for }\; s \in [p,q].
\end{aligned}$$

As before, we obtain

$$\eta(s,t) = a(t)\, s + b(t),$$

with $a(\tau) = 1$ and $b(\tau) = 0$. Proceeding as before with $u(x,t) = a(t)\,\varphi(x) + b(t)$ gives the system:

$$\begin{aligned}
&a'(t)\,\varphi(x) - a(t)^{\alpha+1}\,\mathcal{Q}\varphi = -b'(t),\\
&a''(t)\,\varphi(x) - (\alpha+1)\,a(t)^{\alpha} a'(t)\,\mathcal{Q}\varphi = -b''(t).
\end{aligned} \tag{4.17}$$

The determinant of this system must be zero, otherwise φ would be constant; thus, a must satisfy the problem

$$a^{\alpha+1} a'' - (\alpha+1)\, a^{\alpha} (a')^2 = 0 \;\text{ in }\; [\tau, T), \quad a(\tau) = 1.$$

The solutions of this problem are for $t \in [\tau, T)$

$$\begin{aligned}
&a(t) = [1 + \lambda\,(t-\tau)]^{-1/\alpha} \;\text{ if }\; \alpha \neq 0,\\
&a(t) = e^{-\lambda\,(t-\tau)} \;\text{ if }\; \alpha = 0,
\end{aligned}$$

for some $\lambda \in \mathbb{R}$, and going back to the first equation in (4.17) gives

$$\begin{aligned}
&\mathcal{Q}\varphi + \frac{\lambda}{\alpha}\,\varphi(x) = b'(t)\, a(t)^{-(\alpha+1)} = \gamma \;\text{ if }\; \alpha \neq 0,\\
&\mathcal{Q}\varphi + \lambda\,\varphi(x) = b'(t)\, a(t)^{-1} = \gamma \;\text{ if }\; \alpha = 0.
\end{aligned}$$

Thus, we have that

$$\begin{aligned}
&b(t) = \frac{\gamma\alpha}{\lambda}\,\{1 - [1 + \lambda\,(t-\tau)]^{-1/\alpha}\} \;\text{ if }\; \alpha \neq 0,\\
&b(t) = \gamma\,\frac{1 - e^{-\lambda\,(t-\tau)}}{\lambda} \;\text{ if }\; \alpha = 0,
\end{aligned}$$

for $\lambda \neq 0$,

$$b(t) = \gamma\,(t-\tau),$$

for $\lambda = 0$ and any $\alpha \geq 0$.

Therefore, (ii) follows when $\lambda \neq 0$, by setting $\phi_\lambda = \varphi - \gamma\alpha/\lambda$ and $\mu = \gamma\alpha/\lambda$ for $\alpha \neq 0$ and $\phi_\lambda = \varphi - \gamma/\lambda$ and $\mu = \gamma/\alpha$ for $\alpha = 0$; (iii) follows when $\lambda = 0$, by choosing $w = \varphi$.

□

4.2 K-isoparametric functions

The analysis performed in the study of the heat equation suggests that case (i) of Theorem 4.5, should entail some symmetry result for the domain, at least when suitable initial and boundary conditions are given. However it seems difficult even to imagine, given an arbitrary operator of the form (4.9), the shape of the level surfaces of functions φ for which case (i) of Theorem 4.5 occurs.

In the special case when the differential operator defined in (4.9) is given by (4.14) it is possible to give a catalog of all possible level surfaces of such functions.

Let K be a C^2_+ convex body; by setting

$$H = \frac{1}{2}\, h^2, \tag{4.18}$$

the system of equations that appears in item (i) of Theorem 4.5 can be conveniently re-written as

$$DH(D\varphi) \cdot D\varphi = f(\varphi) \ \text{ in } \Omega, \tag{4.19}$$

$$\Delta_h \varphi = g(\varphi) \ \text{ in } \ \Omega. \tag{4.20}$$

We also notice that, since H is 2-homogeneous, by Euler's identity (4.19) can be re-written as

$$2\, H(D\varphi) = f(\varphi) \ \text{ in } \ \Omega. \tag{4.21}$$

We shall say that a function φ is *K-isoparametric* if it is a solution of (4.19)-(4.20); accordingly, its level surfaces will be called *K-isoparametric surfaces.*

Theorem 4.6 (K-isoparametric functions)**.** *Let $K \subset \mathbb{R}^N$ be a convex body of class C^2_+ and let h denote its support function.*

Let $\varphi \in C^2$ be a K-isoparametric function. Then its level surfaces are of the form

$$Dh(\mathbb{S}^M) \times \mathbb{R}^{N-1-M}, \quad M = 0, \ldots, N-1. \tag{4.22}$$

Remark 4.7. Equation (4.22) should be interpreted as follows: $Dh(\mathbb{S}^M)$ is an M-dimensional submanifold of $Dh(\mathbb{S}^{N-1}) = \partial K$ and the vector-valued function $\psi : Dh(\mathbb{S}^M) \times \mathbb{R}^{N-1-M} \to \mathbb{R}^N$ defined by $\psi(Dh(\nu), y) = Dh(\nu) + j(y)$, where j is the natural inclusion of $\mathbb{R}^{N-1-M}$ in $\mathbb{R}^N$, defines an embedding and its image coincides, up to homoteties, with a level surface of φ.

The proof of the Theorem 4.6 relies on the results obtained in [GM] and [HLMG], which generalize the classical ones of Levi-Civita and Segre ([Le],

[Se]). In this new setting, the metric defined on the submanifolds (the level sets of the function φ) is given by an anisotropic (non constant) operator. In [GM], the authors prove a classification theorem for hypersurfaces with constant anisotropic principal curvatures; the proof is mainly based on a Cartan-type identity which forces the K-isoparametric surface to admit at most two different values for principal curvatures.

Proof. Let $\varphi(x)$ be a regular value of the function φ, Σ the level surface $\{y \in \Omega : \varphi(y) = \varphi(x)\}$ and let $T_x(\Sigma)$ be the tangent space to Σ at x. We also introduce the *K-anisotropic Weingarten operator*,

$$W = D^2 h \left(\frac{D\varphi}{|D\varphi|} \right) \frac{D^2\varphi}{|D\varphi|}, \tag{4.23}$$

and the *K-anisotropic mean curvature*,

$$M = \frac{1}{h(D\varphi)} \left\{ \Delta_h \varphi - \frac{D\varphi \cdot [D^2 H(D\varphi)]\,[D^2\varphi]\,D\varphi}{|D\varphi|^2} \right\}. \tag{4.24}$$

We differentiate (4.19) and (4.21) and obtain the identities (by the square brackets, we denote matrices):

$$\begin{aligned} &[D^2 H(D\varphi)][D^2\varphi]\,D\varphi + [D^2\varphi]\,DH(D\varphi) = f'(\varphi)\,D\varphi, \\ &2\,[D^2\varphi]\,DH(D\varphi) = f'(\varphi)\,D\varphi, \\ &[D^2 H(D\varphi)]\,[D^2\varphi]\,D\varphi = [D^2\varphi]\,DH(D\varphi). \end{aligned} \tag{4.25}$$

After straightforward computations, from the definition (4.24), the identities (4.20), (4.21) and (4.25) imply that

$$M = \frac{g(\varphi) - f'(\varphi)/2}{\sqrt{f(\varphi)}},$$

that means that M is constant on Σ.

We are now going to show that M is actually the trace of the K-anisotropic Weingarten operator; to do this we prove the following identity

$$W(x) = \frac{D^2 H(D\varphi(x)) D^2\varphi(x)}{h(D\varphi(x))} \quad \text{on} \quad T_x(\Sigma); \tag{4.26}$$

in other words, we show that the two matrices coincide as bilinear forms on $T_x(\Sigma)$.

In fact, (4.18) and the homogeneities of h, DH and D^2H imply that

$$\begin{aligned} \frac{[D^2 h(\nu)][D^2\varphi]}{|D\varphi|} &= \frac{[D^2 H(\nu)][D^2\varphi]}{h(\nu)\,|D\varphi|} - \frac{[DH(\nu) \otimes DH(\nu)][D^2\varphi]}{h(\nu)^3\,|D\varphi|} = \\ &\frac{[D^2 H(D\varphi)][D^2\varphi]}{h(D\varphi)} - \frac{[DH(D\varphi) \otimes DH(D\varphi)][D^2\varphi]}{h(D\varphi)^3} = \\ &= \frac{[D^2 H(D\varphi)][D^2\varphi]}{h(D\varphi)} - \frac{1}{2}\,g'(\varphi)\,\frac{[DH(D\varphi) \otimes D\varphi]}{h(D\varphi)^3}, \end{aligned}$$

where, in the last equality, we used the second identity in (4.25). The desired formula (4.26) is then obtained by noticing that $T_x(\Sigma)$ lies in the kernel of $[DH(D\varphi(x)) \otimes D\varphi(x)]$, being orthogonal to $D\varphi(x)$.

Notice that M only depends on the geometry of the level surface; indeed, $\nu(x) = D\varphi(x)/|D\varphi(x)|$ is the normal unit vector to Σ at x and the restriction of $-[D^2\varphi(x)]/|D\varphi(x)|$ to $T_x(\Sigma)$ is the shape operator of Σ.

Now, we claim that there exist a relatively compact neighborhood $U_x \subset \Sigma$ of x and a number $\delta > 0$ such that, for any $y \in U_x$, it holds that

$$\varphi(y + \tau\, DH(D\varphi(y))) = \varphi(x + \tau\, DH(D\varphi(x))), \tag{4.27}$$

for every $0 < \tau < \delta$.

Without loss of generality we can assume that $f(\varphi) = 1$; indeed, since Σ is a regular level surface for φ, then $f(\varphi) > 0$; by taking $\psi = F(\varphi)$ with F such that $(F')^2 = f$, it is easy to show that ψ is another isoparametric function, with the same level surfaces of φ, and such that $H(D\psi) = 1$.

To prove our claim, we first have to show that the integral curves of $DH(D\varphi)$ are geodesics. Let $U_x \subset \Sigma$ be a relatively compact neighborhood of x (of course, $D\varphi$ does not vanish on U_x). For every $y \in U_x$, let $\gamma_y(\tau)$ be the solution of the Cauchy problem

$$\gamma_y'(\tau) = DH(D\varphi(\gamma_y(\tau))), \quad \gamma_y(0) = y,$$

and let δ be such that, for every $y \in U_x$, γ_y remains regular on $[0, \delta]$. Then we have that

$$\begin{aligned}\varphi(\gamma_y(\tau)) - \varphi(y) = \int_0^\delta D\varphi(\gamma_y(\sigma)) \cdot \gamma_y'(\sigma)\, d\sigma = \\ \int_0^\delta D\varphi(\gamma_y(\sigma)) \cdot DH(D\varphi(\gamma_y(\sigma)))\, d\sigma = 2\int_0^\delta H(\varphi(\gamma_y(\sigma)))\, d\sigma = 2\delta,\end{aligned}$$

where we used Euler identity for H and the fact that we are assuming that $f = 1$.

Moreover, we compute that

$$\begin{aligned}\gamma_y''(\tau) &= [D^2H(D\varphi(\gamma_y(\tau)))][D^2\varphi(\gamma_y(\tau))]\,\gamma_y'(\tau) = \\ &[D^2H(D\varphi(\gamma_y(\tau)))][D^2\varphi(\gamma_y(\tau))]\, DH(D\varphi(\gamma_y(\tau))) = \\ &= \frac{1}{2} f'(\varphi(\gamma_y(\tau)))\,[D^2H(D\varphi(\gamma_y(\tau)))]\, D\varphi(\gamma_y(\tau)) = 0,\end{aligned}$$

where we used (4.25) and the fact that we are assuming that $f = 1$.

Thus,

$$\gamma_y(\tau) = y + \tau\, DH(D\varphi(y)) \ \text{ for } \ 0 \le \tau < \delta,$$

and hence

$$\varphi(y + \tau\, DH(D\varphi(y))) - \varphi(y) = 2\delta \ \text{ for } \ 0 \le \tau < \delta,$$

which means that (4.27) holds.

Therefore, we have proved that, for every $\tau \in [0, \delta)$, the surfaces

$$\Sigma_\tau = \{y : \varphi(y) = \varphi(\gamma_x(\tau))\}$$

are parallel with respect to the anisotropic metric induced by K; also, every Σ_τ has constant K-anisotropic mean curvature.

Thus, (4.22) follows the results [GM, Theorem 2.1] and [GM, Theorem 1.1] recalled in remark 4.7.

□

4.3 Anisotropic Matzoh Ball Soup Problem

As already stressed, in [MS1] it is stidued the following initial-Dirichlet problem

$$\begin{cases} u_t = \Delta u & \text{in } \Omega \\ u = 0 & \text{in } \Omega \times \{0\} \\ u = 1 & \text{on } \partial\Omega \times (0, +\infty). \end{cases} \tag{4.28}$$

Under the assumption of the existence of a sufficiently smooth subdomain G, such that $u(x,t) = c(t)$, for every $x \in \partial G$, it is shown that:

(i) $\partial\Omega$ and ∂G are parallel,

(ii) both Ω and G must be balls.

The arguments exploited to prove this result are widely different from those used in [Al2] and [Sak]. Indeed, to prove (i), it is studied the asymptotic behavior of $u(x,t)$, as $t \to 0$. More precisely it is considered the function

$$W(x,s) = s \int_0^\infty u(x,t) e^{-st}\, dt, \tag{4.29}$$

and it is proved that $s^{-1} \log W(x,s)$ converges to the function $\mathrm{dist}(x, \partial\Omega)$, uniformly on $\overline{\Omega}$, as $s \to \infty$ (see also [Va]). Then, since ∂G is a level set of u, for any $t > 0$, it must also be a level set of W, for any $s > 0$, and thus a level set for the function $\mathrm{dist}(x, \partial\Omega)$.

Notice that, the functions $W(\cdot, s)$ are solutions of the following family of elliptic problems.

$$\begin{cases} \Delta W = sW & \text{in } \Omega \\ W = 1 & \text{on } \partial\Omega. \end{cases} \tag{4.30}$$

Moreover ∂G is a level set for every solution of (4.30).

It would be interesting to extend the result of [MS1] to the the anisotropic case, that is when the usual Laplace operator is replaced by the h-Laplace operator. In this section we study the behavior of the solutions of the family of elliptic problems given by

$$\begin{cases} \varepsilon\Delta_h u^\varepsilon = u^\varepsilon & \text{in } \Omega \\ u^\varepsilon = 1 & \text{on } \partial\Omega, \end{cases} \tag{4.31}$$

for $\varepsilon \to 0$. In particular we shall prove that, if there exists a surface Γ which is a level set of every u^ε, then Γ has to be parallel to the boundary of Ω in the anisotropic sense (see Theorem 4.10).

It is important to stress that, in this case, if we apply the transformation given by (4.29) to the solutions of the h-Laplace evolution equation, we will not find a solution of (4.31); however we consider this problem an important preliminary step to solve the corresponding overdetermined parabolic problem, and it might help to understand if (and how) some of the techniques adopted in the linear case can be extended to treat this kind of operator.

Throughout this section we consider an open and bounded convex subset Ω of $\mathbb{R}^N$, a C^2_+ convex body K, and we denote $H = h_K^2$.

Let u^ε be a solution of (4.31); by setting

$$v^\varepsilon = \varepsilon \ln u^\varepsilon, \tag{4.32}$$

we obtain a solution of the following equation.

$$\begin{cases} H(Dv^\varepsilon) - \varepsilon\Delta_h v^\varepsilon = 1 & \text{in } \Omega \\ v^\varepsilon = 0 & \text{on } \partial\Omega \end{cases} \tag{4.33}$$

The following lemma provides some fundamental bounds for such solutions.

Lemma 4.8. *Let v^ε be a solution of (4.33), then $0 \le v^\varepsilon \le \sup_{x\in\Omega} \operatorname{dist}_K(x,\partial\Omega)$, where $\operatorname{dist}_K(x,\partial\Omega) = \inf_{y\in\partial\Omega} \|x-y\|_K$, and $H(Dv^\varepsilon(x)) \le 1$, for every $v^\varepsilon \in \partial\Omega$.*

Proof. Let $d_K(x) = \operatorname{dist}_K(x,\partial\Omega)$, d is a viscosity solution of

$$\begin{cases} H(Dv) = 1 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.34}$$

It is easy to check that, since Ω is a convex set, d_K is a concave function, and then $\Delta_H d_K \le 0$. This means that d_K is a viscosity supersolution of (4.33). Thus $v^\varepsilon - d$ cannot have any local maximum inside Ω (see, for instance, [CIL]), namely $v^\varepsilon \le d_K$, for every x in Ω. Analogously the constant function 0, being a viscosity subsolution of (4.33), is a lower bound for v^ε.

We are left to show the equiboundedness of the gradients of v^ε on the boundary of Ω. Since v^ε is constant on $\partial\Omega$, then

$$H(Dv^\varepsilon(x)) = (\partial_\nu v^\varepsilon)^2 H(\nu),$$

where ν is the normal unit to $\partial\Omega$ at x. The considerations above will lead to conclude that $|\partial_\nu v^\varepsilon| \leq |\partial_\nu d_K|$, and then $H(Dv^\varepsilon) \leq H(Dd_K) = 1$. $\square$

Lemma 4.9. *Let $\{v^\varepsilon\}_{\varepsilon>0}$ be the family of solutions of* (4.33)*, then there exists a subsequence $\{v^{\varepsilon_n}\}_{n\in\mathbb{N}}$, $\varepsilon_n \to 0$ as $n \to \infty$, that converges uniformly in the the closure of Ω.*

Proof. We have already shown that the functions v^ε are equibounded, we are now going to show that their gradient are equibounded as well. The desired result will follow thanks to Ascoli-Arzelà's theorem.

We set $\psi^\varepsilon(x) = H(Dv^\varepsilon) + \alpha v^\varepsilon$, $\alpha > 0$. Suppose now, by contradiction that, for every $k > 0$, there exist ε_k and x_k such that $H(Dv^\varepsilon(x_k)) > k$. Since the functions v^ε are equibounded in the closure of Ω, and $H(Dv^\varepsilon)$ are equibounded on $\partial\Omega$, then, when k is sufficiently large, we can always assume that ψ^{ε_k} reach its maximum inside Ω. We are going to show that this leads to a contradiction.

We first compute the gradient and the hessian of ψ^ε, we get

$$\partial_i\psi^\varepsilon = \partial_k H \partial_{ki} v^\varepsilon + \alpha \partial_i v^\varepsilon, \tag{4.35}$$

and

$$\partial_{ij}\psi^\varepsilon = \partial_k H \partial_{kij} v^\varepsilon + \partial_{kl} H \partial_{kj} v^\varepsilon \partial_{li} v^\varepsilon + \alpha \partial_{ij} v^\varepsilon. \tag{4.36}$$

Differentiating with respect to x the first equation in (4.33) we get

$$\partial_i H \partial_{ik} v^\varepsilon - \varepsilon \left[\partial_{ijl} H \partial_{lk} v^\varepsilon \partial_{ij} v^\varepsilon + \partial_{ij} H \partial_{kij} v^\varepsilon\right] = 0. \tag{4.37}$$

We now suppose, by contradiction, that there exists $x_0 \in \Omega$ such that $\psi^\varepsilon(x_0) \geq \psi^\varepsilon(x)$, for every $x \in \overline{\Omega}$. We will have $D\psi^\varepsilon(x_0) = 0$, and $D^2\psi^\varepsilon(x_0) \leq 0$. At $x = x_0$ it holds true that

$$-\alpha \partial_k v^\varepsilon + \varepsilon \partial_{ijl} H \partial_{lk} v^\varepsilon \partial_{ij} v^\varepsilon + \varepsilon \partial_{ij} H \partial_{kij} v^\varepsilon = 0. \tag{4.38}$$

By multiplying (4.38) by $\partial_k H(Dv^\varepsilon)$ and by taking the sum over k, we obtain

$$-\alpha \partial_k H \partial_k v^\varepsilon + \varepsilon \partial_k H \partial_{ijl} H \partial_{lk} v^\varepsilon \partial_{ij} v^\varepsilon + \varepsilon \partial_k H \partial_{ij} H \partial_{kij} v^\varepsilon = 0. \tag{4.39}$$

We now study every summand; by Euler's laws we have that

$$-\alpha \partial_k H \partial_k v^\varepsilon = -2\alpha H.$$

Since $\partial_k H \partial_{lk} v^\varepsilon = -\alpha \partial_l v^\varepsilon$, then

$$\partial_k H \partial_{ijl} H \partial_{lk} v^\varepsilon = -\alpha \partial_{ijl} H \partial_l v^\varepsilon,$$

then, again by using Euler's law

$$\partial_k H \partial_{ijl} H \partial_{lk} v^\varepsilon \partial_{ij} v^\varepsilon = 0.$$

Finally, by recalling (4.36) we have that

$$\partial_k H \partial_{ij} H \partial_{kij} v^\varepsilon = \partial_{ij} H \partial_{ij} \psi^\varepsilon - \partial_{ij} H \partial_{kl} H \partial_{kj} v^\varepsilon \partial_{li} v^\varepsilon - \alpha \Delta_h v^\varepsilon.$$

Going back to (4.39) we find that

$$-2\alpha H + \varepsilon \partial_{ij} H \partial_{ij} \psi^\varepsilon - \varepsilon \partial_{ij} H \partial_{kl} H \partial_{kj} v^\varepsilon \partial_{li} v^\varepsilon - \alpha\varepsilon \Delta_h v^\varepsilon = 0.$$

We now observe that, at $x = x_0$ we will have $\partial_{ij} H \partial_{ij} \psi^\varepsilon \leq 0$, since it is the trace of the product of a positive matrix and a negative semi-definite matrix. Moreover $\partial_{ij} H \partial_{kl} H \partial_{kj} v^\varepsilon \partial_{li} v^\varepsilon \geq 0$, since is the trace of a square of a symmetric matrix. By recalling (4.33) we can infer that

$$0 \leq -2\alpha H + \alpha H - \alpha,$$

that is impossible since $\alpha > 0$. □

We are now going to prove the following

Theorem 4.10. *Let u^ε be solutions of* (4.31) *and let $\Gamma \subset \Omega$ be a level surface of u^ε, for every $\varepsilon > 0$. Then Γ is parallel to $\partial\Omega$, namely $d_K(x)$ is constant on Γ.*

Proof. Let v^ε be defined by (4.32); Γ is a level surface for every v^ε. By Lemmas 4.8 and 4.9 we know that v^ε converges uniformly to a function v that is a viscosity solution of (4.34) (see again [CIL]). Since (4.34) admits only one solution (see [BC]), then $v = d_K$.

For every $\delta > 0$ there exists a $\varepsilon > 0$ such that $|v^\varepsilon(x) - d_K(x)| \leq \delta$, for every $x \in \overline{\Omega}$. Then, for x and y in Γ we can compute

$$|d_K(x) - d_K(y)| \leq |v^\varepsilon(x) - d_K(x)| + |v^\varepsilon(y) - d_K(y)| \leq 2\delta,$$

namely d_K is constant on Γ. □

Chapter 5

A comparison result for the solutions of degenerate elliptic equations

In this last chapter we establish some comparison results between the solutions of a family of degenerate elliptic equations of the form

$$-\operatorname{div}(w^2\, e^V Du) = f\, e^V \text{ in } E, \qquad u = 0 \text{ on } \partial E, \tag{5.1}$$

and the solutions of the corresponding problem where the data f and the domain E are replaced by their right rearrangement.

Section 5.1 contains the main definitions concerning the concept of rearrangement and symmetrization.

In Section 5.2, we solve a family of mixed isoperimetric problems of the form

$$\min\left\{P_{we^V}(E) : \int_E e^V = \text{constant}\right\},$$

and, in Section 5.3 we use the fact that half-spaces are the only minimizers for such problems to prove our main result, Theorem 5.7.

5.1 Symmetrization and rearrangement inequalities

In this section we introduce the main definitions and properties about the concept of symmetrization and rearrangement we shall make use of. For the definition and main properties of the *spherical rearrangement* we refer to [LL, Chapter 3].

Let μ be a finite Radon measure on $\mathbb{R}^N$, a *right rearrangement* with respect to μ is defined, for any Borel set A, as

$$R_A^\mu = \{(x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1} \,:\, x_1 > t_A\},$$

where $t_A = \inf\big\{t \,:\, \mu(A) = \mu(\{(x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1} \,:\, x_1 > t\})\big\}$. Notice that if $d\mu = f dx$, for some positive and measurable function f, then the value of t is uniquely determined.

Given a non-negative Borel function $f : \mathbb{R}^N \to [0, +\infty)$, we call *right increasing rearrangement* of f the function $f^{*\mu}$ given by

$$f^{*\mu}(x) = \int_0^{+\infty} \mathcal{X}_{R^\mu_{\{f>t\}}}(x)\, dt$$

As an aside we notice that the right increasing rearrangement of the characteristic function of a Borel set A coincides with the characteristic function of R_A^μ. Clearly $f^{*\mu}$ is non-negative, increasing with respect to the first variable x_1, and constant on the sets $\{(x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1} \,:\, x_1 = t\}$, for $t \in \mathbb{R}$. Moreover f and $f^{*\mu}$ share the same distribution function, namely

$$\mu_f(t) = \mu(\{f > t\}) = \mu(\{f^{*\mu} > t\}) = \mu_{f^{*\mu}}(t).$$

We furthermore define $f^{\star\mu} : \mathbb{R}^+ \to \mathbb{R}^+$ as the smallest decreasing function satisfying $f^{\star\mu}(\mu_f(t)) \geq t$; in other words

$$f^{\star\mu}(s) = \inf\{t > 0 \,:\, \mu_f(t) < s\}.$$

It is useful to bear in mind that $\{s : f^{\star\mu}(s) > t\} = [0, \mu_f(t)]$, so that by the Layer-Cake Representation Theorem we have

$$\int_0^{\mu(\{x_1>t\})} f^{\star\mu}(s)\, ds = \int_t^{\infty} \mu_f(s)\, ds = \int_{\{x_1>t\}} f^{*\mu}(x)\, dx. \tag{5.2}$$

We conclude this section by proving the *Hardy-Littlewood* rearrangement inequality related to the right symmetrization.

Lemma 5.1 (Hardy-Littlewood rearrangement inequality)**.** *Let f and g be non-negative Borel functions from $\mathbb{R}^N$ to $\mathbb{R}$. Then for any non-negative Borel measure μ we have*

$$\int_{\mathbb{R}^N} f\, g\, d\mu \leq \int_{\mathbb{R}^N} f^{*\mu} g^{*\mu} d\mu.$$

Proof. We have

$$
\begin{aligned}
\int_{\mathbb{R}^N} f\, g\, d\mu &= \int_{\mathbb{R}^N} \int_0^\infty \int_0^\infty \mathcal{X}_{\{f>t\}}(x)\mathcal{X}_{\{g>s\}}(x)\, dt\, ds\, d\mu(x) \\
&= \int_0^\infty \int_0^\infty \int_{\mathbb{R}^N} \mathcal{X}_{\{f>t\}\cap\{g>s\}}(x)\, d\mu(x)\, dt\, ds \\
&= \int_0^\infty \int_0^\infty \mu(\{f>t\}\cap\{g>s\})\, dt\, ds \\
&\le \int_0^\infty \int_0^\infty \min(\mu(\{f>t\}),\, \mu(\{g>s\}))\, dt\, ds \\
&= \int_0^\infty \int_0^\infty \min(\mu(\{f^{*\mu}>t\}),\, \mu(\{g^{*\mu}>s\}))\, dt\, ds \\
&= \int_0^\infty \int_0^\infty \mu(\{f^{*\mu}>t\}\cap\{g^{*\mu}>s\})\, dt\, ds = \int_{\mathbb{R}^N} f^{*\mu}\, g^{*\mu}\, d\mu,
\end{aligned}
$$

where we used the fact that $\{f^{*\mu} > t\}$ and $\{g^{*\mu} > s\}$ are half-spaces of the form $\{(x_1, x') \in \mathbb{R}\times\mathbb{R}^{N-1} : x_1 > r\}$ for some $r \in \mathbb{R}$ and so

$$
\min(\mu(\{f^{*\mu}>t\}),\, \mu(\{g^{*\mu}>s\})) = \mu(\{f^{*\mu}>t\}\cap\{g^{*\mu}>s\}).
$$

□

Remark. Setting $g = \mathcal{X}_A$ in Lemma 5.1 and thanks to (5.2) we get

$$
\int_A f\, dx \le \int_{R_A^\mu} f^{*\mu}(x)\, dx = \int_0^{\mu(A)} f^{\star\mu}(s)\, ds. \tag{5.3}
$$

5.2 A class of weighted Gauss-type isoperimetric inequalities

Given a measurable function $V : \mathbb{R}^N \to \mathbb{R}$ we denote by $\mu[V]$ the absolutely continuous measure whose density equals e^V, that is, for any measurable set $E \subset \mathbb{R}^N$

$$
\mu[V](E) = \int_E e^{V(x)} dx;
$$

in what follows, with the scope of simplifying the notation, and if there is no risk of confusion, we will drop the dependence of V, writing μ instead of $\mu[V]$. Moreover we will often adopt the notation $x = (x_1, x') \in \mathbb{R}\times\mathbb{R}^{N-1}$ and denote by R_A instead of $R_A^{\mu[V]}$ the right rearrangement of A with respect to the measure $\mu[V]$.

Given a Borel *weight* function $w : \mathbb{R} \to [0, +\infty]$ we define, for any open set A with Lipschitz boundary, the following *weighted perimeter*:

$$P_{w,V}(A) = \int_{\partial A} w(x_1) e^{V(x)} d\mathcal{H}^{N-1}(x).$$

In the following proposition we show that, under suitable conditions on w and V, the half-spaces of the form $\{(x_1, x') : x_1 > t\}$ are the only minimizers of the weighted perimeter among the sets of fixed volume with respect to the measure $\mu[V]$.

Proposition 5.2 ([MR]). *Let $A \subset \mathbb{R}^N$ be a set with Lipschitz boundary. Suppose that $w : \mathbb{R} \to \mathbb{R}^+$ and $V : \mathbb{R}^N \to \mathbb{R}$ are C^1-regular functions satisfying the following assumptions:*

(i) $\mu(A) = \mu(R_A) < +\infty$,

(ii) the function $\partial_1 V(x)$ depends only on x_1 and $g(x) := -w'(x_1) - w(x_1)\partial_1 V(x)$ is a non-negative decreasing function on the real line.

Then

$$P_{w,V}(A) \geq P_{w,V}(R_A). \tag{5.4}$$

Proof. We start by noticing that if $P_{w,V}(A) = +\infty$ there is nothing to prove. Hence we can suppose that

$$P_{w,V}(A) < +\infty. \tag{5.5}$$

Let $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^N$ and consider the vector field $-e_1 w(x_1) e^{V(x)}$. Its divergence is given by

$$\operatorname{div}(-e_1 w(x_1) e^V(x)) = (-w'(x_1) - w(x_1)\partial_1 V(x)) e^{V(x)} = g(x) e^{V(x)}.$$

By an application of the Divergence theorem we have

$$\begin{aligned} \int_A g(x) d\mu(x) &= \int_A \operatorname{div}(-e_1 w(x_1) e^{V(x)}) dx \\ &= \int_{\partial A} w(x_1) e^{V(x)} \langle \nu_A(x), -e_1 \rangle d\mathcal{H}^{N-1}(x) \\ &\leq \int_{\partial A} w(x_1) e^{V(x)} d\mathcal{H}^{d-1}(x) = P_{w,V}(A). \end{aligned} \tag{5.6}$$

Let t_A be a real number such that the right half-space $R_A = \{(x_1, x') : x_1 \geq t_A\}$ satisfies $\mu(R_A) = \mu(A)$. Then, since the outer normal of R_A is the constant vector field $-e_1$, the inequality in (5.6) turns into an equality if we replace A with R_A. Notice that by condition (ii) and (5.6) we have

$$P_{w,V}(R_A) = \int_{R_A \setminus A} g \, d\mu + \int_{R_A \cap A} g \, d\mu \leq g(t_A)\mu(A) + P_{w,V}(A).$$

Thanks to assumption (i) and (5.5) such quantities are finite and so we get

$$P_{w,V}(A) - P_{w,V}(R_A) \geq \int_A g(x)d\mu(x) - \int_{R_A} g(x)d\mu(x).$$

Since, by definition, $\mu(A) = \mu(R_A) < +\infty$ again by condition (i) we obtain $\mu(A \setminus R_A) = \mu(R_A \setminus A) < +\infty$. Thus

$$\begin{aligned}\int_A g(x)d\mu(x) - \int_{R_A} g(x)d\mu(x) &= \int_{A\setminus R_A} g(x)d\mu(x) - \int_{R_A\setminus A} g(x)d\mu(x)\\ &= \int_{A\setminus R_A} (g(x) - g(t_A e_1))d\mu(x) - \int_{R_A\setminus A} (g(x) - g(t_A e_1))d\mu(x).\end{aligned} \tag{5.7}$$

Since every $x \in A \setminus R_A$ (respectively $x \in R_A \setminus A$) satisfies $\langle x, e_1\rangle < t_A$ (respectively $\langle x, e_1\rangle > t_A$), by condition *(ii)* we deduce

$$\begin{aligned}P_{w,V}(A) - P_{w,V}(R_A) &\geq \int_{A\setminus R_A} |g(x) - g(t_A e_1)|d\mu(x) + \int_{R_A\setminus A} |g(x) - g(t_A e_1)|d\mu(x)\\ &= \int_{A\Delta R_A} |g(x) - g(t_A e_1)|d\mu \geq 0,\end{aligned} \tag{5.8}$$

where $A\Delta R_A = (A \setminus R_A) \cup (R_A \setminus A)$ stands for the symmetric difference between A and R_A. This concludes the proof. □

Remark 5.3 (Necessity of the assumptions). We stress that the integrability condition (i) is necessary to formulas (5.6) and (5.7) (and thus to our proof) to work.

Concerning condition (ii), we note that it is needed just for technical reasons. Nonetheless we stress that our proof offers a slightly stronger inequality than (5.4). Indeed the right-hand side of (5.8) may be seen as a modulus of continuity of the L^1 distance between A and R_A. Thus it would be interesting to understand how much our hypotheses are far from optimality (compare also with [BDR, Remark 2.3]).

Remark 5.4 (Equality cases). An inspection of the proof of Proposition 5.2, and in particular of inequality (5.6), shows that if $w > 0$, then we have equality in (5.4) only if A is equal to the half space R_A, up to set of zero N-dimensional Lebesgue measure. On the other hand, if the set $\{w = 0\}$ has positive Lebesgue measure, we can not expect any kind of uniqueness for the equality cases of such an inequality.

Example. A non-trivial example fulfilling condition (ii) of Proposition 5.2 is the following

$$V(x_1, x') = -c(x_1|x_1| + |x'|^2), \quad w(x_1) = e^{-ax_1},$$

with $a, c > 0$ constants satisfying $a^2 - 2c \geq 0$. To prove this fact we initially observe that if $x_1 \neq 0$ such a condition is equivalent to require that

$$w''(x_1) + V_1''(x_1)w(x_1) + V_1'(x_1)w'(x_1) \geq 0 \tag{5.9}$$

which turns out to be equivalent, in our example, to

$$a^2 - 2c + 2ac|x_1| \geq 0.$$

Then, since $-w'(x_1) - w(x_1)\partial_1 V(x_1)$ is continuous in $x_1 = 0$, condition (ii) is satisfied everywhere.

To transform inequality (5.4) into a well posed isoperimetric problem, it would be more advisable to eliminate the integrability hypothesis (i) in Proposition 5.2 by requiring that the measure $\mu(\mathbb{R}^N) < +\infty$. This fact, together with ordinary differential inequality required in assumption (ii), is seldom satisfied.

Hence, to get other instances of functions which fulfill inequality (5.9) together with the integrability property *(i)* of Proposition 5.2 it is worth restricting our attention to the half-space

$$\mathbb{R}^N_+ = \{(x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1} : x_1 > 0\}.$$

As an immediate corollary of Proposition 5.2 we get that the solution of the problem

$$\min\left\{P_{w,V}(A) : A \subseteq \mathbb{R}^N_+,\ \mu(A) = c,\ \partial A \text{ Lipschitz}\right\} \tag{5.10}$$

is given by $R_c = \{x_1 \geq t_c\}$ where t_c is such that $\mu(R_c) = c$.

Remark 5.5. Notice that the non-mixed Gauss case, w constant and $V(x) = -c|x|^2$, is not covered by our hypotheses.

Nevertheless in this case examples of functions w which satisfy the hypotheses of Proposition 5.2 are given by $w(t) = t^{-a}$ with $a \geq 1$ or $w(t) = b + e^{-at}$, with $a, b \geq 0$ such that $a^2 - 2c(1 + b) > 0$ (as can be easily seen reasoning as in the previous example). In the latter case at least if $b = 0$ we have that

$$we^V = e^{a^2/(4c)} \exp\left(-c\left|x + \mathbf{e_1}\frac{a}{2c}\right|^2\right),$$

where $\mathbf{e_1} = (1, 0, \ldots, 0) \in \mathbb{R}^N$, which can be rephrased as the fact that the solutions of the isoperimetric problem in the half-space $\mathbb{R}^N_+$ with (suitable) mixed Gaussian conditions

$$\min\left\{P_{\gamma_{\sigma,\eta}}(E) : \gamma_{\sigma,0}(E) = \text{constant},\ E \subseteq \mathbb{R}^N_+,\ \partial E \text{ Lipschitz}\right\}$$

are right-half spaces. Here we denoted by $\gamma_{\sigma,\eta}$ the normal distribution whose covariance matrix is $\sigma\mathrm{Id}$ and whose mean vector η is given by $\eta = -\frac{a}{2c}\mathbf{e_1}$. If $b \neq 0$ the unique change is that the perimeter is weighted by means of the sum of two Gaussian measures. We recall that similar problems related to the Gauss measure are considered in [BBMP3, BCM1, dB, dBFP, TL].

Notice that we defined the perimeter $P_{w,V}$ only for sets with Lipschitz boundary, but for our later applications it will be useful to have a definition of perimeter which comprehends also less regular subsets of $\mathbb{R}^N$. A measurable set A is said to have locally finite (Euclidean) perimeter (we refer to [Ma] for a complete overview on the subject) if there exists a vector-valued Radon measure ν_A called *Gauss–Green measure* of the set A such that, for every $T \in C^1_c(\mathbb{R}^N;\mathbb{R}^N)$, it holds true that

$$\int_A \operatorname{div} T = \int_{\mathbb{R}^N} \langle T, d\nu_A \rangle.$$

The perimeter of A is defined in terms of the total variation of the Gauss–Green measure of A as $P(A) = |\nu_A|(\mathbb{R}^N)$. For any set A of locally finite perimeter we then define the *weighted perimeter* $P_{w,V}$ by

$$P_{w,V}(A) = we^V |\nu_A|(\mathbb{R}^N).$$

Since when A has Lipschitz boundary $|\nu_A| = \mathcal{H}^{N-1} \llcorner \partial A$, the above definition is coherent with the one given at the beginning of this section on such sets.

Theorem 5.6. *[MR] Let w and V non-negative and C^1-regular functions satisfying condition (ii) of Proposition 5.2. Suppose moreover that $\mu(\mathbb{R}^N_+) < +\infty$; then the problem*

$$\min\left\{P_{w,V}(A) : A \subseteq \mathbb{R}^N_+,\ \mu(A) = c\right\}$$

admits a solution, and this solution coincides with the one of (5.10).

Proof. Let A be a measurable set of locally finite perimeter and suppose, by contraddiction, that $P_{w,V}(A) < P_{w,V}(R_A)$. We start by noticing that $P_{w,V}(R_A) < +\infty$, indeed, recalling (5.6) we have that

$$P_{w,V}(R_A) = \int_{R_A} g(x)\, d\mu(x) \leq g(0)\mu(A).$$

By [Ma, Theorem II.2.8] we can find a sequence of sets A_n with smooth boundary such that $\mathcal{X}_{A_n} \to \mathcal{X}_A$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and $|\nu_{A_n}| \rightharpoonup^* |\nu_A|$, where $\rightharpoonup^*$ indicates the weak* convergence of Radon measures. Since $\mu(\mathbb{R}^N_+) < +\infty$, we also have that

$$\mathcal{X}_{A_n} \to \mathcal{X}_A \quad \text{in } L^1(\mathbb{R}^N, \mu) \tag{5.11}$$

and, since we^V is a continuous function

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} we^V|\nu_{A_n}| = \int_{\mathbb{R}^N} we^V|\nu_A|. \tag{5.12}$$

Thanks to (5.12) and Proposition 5.2 we get

$$P_{w,V}(A) = \lim_{n\to\infty} P_{w,V}(A_n) \geq \lim_{n\to\infty} P_{w,V}(R_{A_n}).$$

We are left to show that $\lim_{n\to\infty} P_{w,V}(R_{A_n}) = P_{w,V}(R_A)$, but

$$|P_{w,V}(R_A) - P_{w,V}(R_{A_n})| \leq g(0)|\mu(A) - \mu(A_n)|,$$

and we can conclude thanks to (5.11) and the fact that $\mu(\mathbb{R}^N_+) < +\infty$. $\square$

5.3 A comparison result for the solutions of some degenerate elliptic equations

In this section we consider sets $E \subseteq \mathbb{R}^N_+$ and we define $d\mu = e^V\,dx$, $R_E = \{x_1 > t_E\}$ where $t_E \in \mathbb{R}$ is such that $\mu(R_E) = \mu(E)$ and $f^* = f^{*\mu}$ the right rearrangement of a function f with respect to μ. In what follows we consider problems of the form

$$\begin{cases} -\mathrm{div}(w^2\, e^V Du) = f\, e^V & \text{in } E \\ u = 0 & \text{on } \partial E \end{cases} \tag{5.13}$$

which must be intended in weak sense. Precisely, a solution of (5.13) is a function $u \in H^1_0(e^V, w^2e^V, E)$, defined as the closure of $C^\infty_0(E)$ with respect to the norm

$$\|u\|_{H^1_0(e^V,w^2e^V,E)} = \left(\int_E u^2e^V\,dx + \int_E |Du|^2we^V\,dx\right)^{1/2},$$

and which satisfies

$$\int_E \langle Du, D\phi\rangle w^2e^V\,dx = \int_E f\,\phi\,e^V\,dx \tag{5.14}$$

for any $\phi \in H^1_0(e^V, w^2e^V, E)$.

The main scope of this section is to prove *a priori* estimates for the solutions of problem (5.13). For this reason we shall always consider that a solution u exists. Clearly this requirement depends on the choice of w, V and f. General instances of such functions for which the existence of a solution for problem (5.13) is guaranteed, can be found in [Tr] (see also [dB, TL, BBMP3, dBFP]). Here we limit ourselves to state that most of the examples considered in Remark 5.5, as the *mixed-Gaussian case* $V(x) = -c|x|^2$, $w(t) = b + e^{-at}$ with $a^2 - 2c(1+b) > 0$ and b strictly positive, are covered by the cases considered in [Tr], whenever $f \in L^2(E, e^V)$.

Theorem 5.7 ([MR])**.** *Suppose that the set $E \subset \mathbb{R}^N_+ = \{(x_1, x') : x_1 > 0\}$ and the functions $w : [0, +\infty] \to (0, +\infty]$ and $V : \mathbb{R}^N \to \mathbb{R}$ satisfy the hypotheses of Proposition 5.2. Consider the two problems*

$$\begin{cases} -\mathrm{div}(w^2\, e^V Du) = f\, e^V & \text{in } E \\ u = 0 & \text{on } \partial E \end{cases} \tag{5.15}$$

and

$$\begin{cases} -\mathrm{div}(w^2\, e^V Dv) = f^* e^V & \text{in } R_E \\ v = 0 & \text{on } \partial R_E \end{cases} \tag{5.16}$$

where $0 < f \in L^2(\mathbb{R}^N_+, \mu)$. Then the problem (5.16) *has as solution the one variable function $v(z)$ given by*

$$v((z, z')) = v(z) = \int_{\mu(\{x_1 \geq z\})}^{\mu(R_E)} \frac{1}{h^2(s)} \left(\int_0^s f^*(\xi)\, d\xi \right) ds, \tag{5.17}$$

where

$$h(m) = w(\Phi^{-1}(m)) \int_{\mathbb{R}^{N-1}} \mu(\Phi^{-1}(m), x')\, dx', \tag{5.18}$$

being $\Phi(t) = \mu(\{x_1 > t\})$. Moreover, for any solution u of the problem (5.15)*, we have*

$$u^*(x) \leq v(x), \tag{5.19}$$

and, for any $q \in (0, 2]$,

$$\int_E |Du|^q w^q\, d\mu \leq \int_{R_E} |Dv|^q w^q\, d\mu \tag{5.20}$$

Proof. Let us suppose for the moment that the function v given in (5.17) is a solution for the problem (5.16). To prove (5.19) and (5.20) we consider the functions ϕ_h defined as

$$\phi_h(x) = \begin{cases} \text{sign }(u) & \text{if } |u| > t + h \\ \frac{u(x) - t\text{sign } u(x)}{h} & \text{if } |u| \in [t, t+h) \\ 0 & \text{if } |u| < t, \end{cases}$$

where $0 \leq t < \operatorname{ess\,sup}|u|$ and $h > 0$. Notice that, for every $h > 0$, ϕ_h is an admissible test function, since the solution u belongs to the space $H^1_0(e^V, w^2 e^V, E)$. Then (5.14) turns into

$$\frac{1}{h} \int_{\{|u| \in [t, t+h)\}} \langle Du, Du \rangle w^2\, d\mu = \frac{1}{h} \int_{\{|u| \in [t, t+h)\}} f\, (u - t\frac{u}{|u|}) d\mu + \int_{\{|u| > t+h\}} f \text{ sign } (u)\, d\mu.$$

Taking the limit for $h \to 0$, we get

$$-\frac{d}{dt}\int_{\{|u|>t\}} |Du|^2 w^2 \, d\mu = \int_{\{|u|>t\}} f \, d\mu. \tag{5.21}$$

Let us analyze the left-hand side of equation (5.21). We claim that the following inequality holds true for almost every t:

$$-\frac{d}{dt}\int_{\{|u|>t\}} |Du|^2 w^2 \, d\mu \geq \frac{\left(-\frac{d}{dt}\int_{\{|u|>t\}} |Du| w \, d\mu\right)^2}{-\mu_u'(t)}, \tag{5.22}$$

where $\mu_u(t)$ is the distribution function of u introduced in the Section 5.1. Indeed $\mu_u(t)$ is a decreasing function and thence it is derivable for almost every t, thanks to the Hölder inequality we get

$$\begin{aligned}
-\frac{d}{dt}\int_{\{|u|>t\}} |Du| w \, d\mu &= \lim_{h\to 0}\frac{1}{h}\int_{t<|u|<t+h} |Du| w \, d\mu \\
&\leq \lim_{h\to 0}\left(\int_{\{t<|u|<t+h\}} |Du|^2 w^2 \, d\mu\right)^{1/2}\left(\int_{\{t<|u|<t+h\}} \frac{1}{h^2} \, d\mu\right)^{1/2} \\
&= \lim_{h\to 0}\left(\frac{1}{h}\int_{\{t<|u|<t+h\}} |Du|^2 w^2 \, d\mu\right)^{1/2}\left(\frac{1}{h}\int_{\{t<|u|<t+h\}} 1 \, d\mu\right)^{1/2} \\
&= \left(-\frac{d}{dt}\int_{\{|u|>t\}} |Du|^2 w^2 \, d\mu\right)^{1/2} \left(-\mu_u'(t)\right)^{1/2}
\end{aligned}$$

By the Co-Area formula and the fact that w is strictly positive and C^1, we easily get that the set $\{u > t\}$ is a set of locally finite (Euclidean) perimeter. Thus, thanks to Proposition 5.2 and Theorem 5.6 we get

$$-\frac{d}{dt}\int_{\{|u|>t\}} |Du| w \, d\mu = \int_{\{|u|=t\}} w \, d\mu = P_{w,V}(\{|u| > t\}) \geq P_{w,V}(\{u^* > t\}). \tag{5.23}$$

We introduce the function

$$\Phi(t) = \mu(\{x_1 > t\}). \tag{5.24}$$

We recall that the weight function w is constant on the boundary of the super level sets of u^*, so that the perimeter of $\{u^* > t\}$ can be written as

$$P_{w,V}(\{u^* > t\}) = w(\tau)\int_{\mathbb{R}^{N-1}} \mu(\tau, x') \, dx'.$$

Moreover $\tau \in \mathbb{R}$ satisfies $\mu_{u^*}(t) = \Phi(\tau)$ that is $\tau = \Phi^{-1}(\mu_{u^*}(t))$ (notice that Φ is a strictly decreasing function and thus invertible) so that we can write the previous formula as

$$P_{w,V}(\{u^* > t\}) = w(\Phi^{-1}(\mu_{u^*}(t))) \int_{\mathbb{R}^{N-1}} \mu(\Phi^{-1}(\mu_{u^*}(t)), x')\,dx' := h(\mu_{u^*}(t)). \tag{5.25}$$

Plugging (5.23) in (5.22), and recalling (5.25) we get that

$$-\frac{d}{dt}\int_{\{|u|>t\}} |Du|^2 w^2\,d\mu \geq \frac{h(\mu_{u^*}(t))^2}{-\mu'_{u^*}(t)}. \tag{5.26}$$

We pass now to estimate the right-hand side of (5.21): equation (5.3) with $A = \{|u| > t\}$ turns into

$$\int_{\{|u|>t\}} f\,d\mu \leq \int_{\{|u^*|>t\}} f^*\,d\mu = \int_0^{\mu_{u^*}(t)} f^\star(s)\,ds. \tag{5.27}$$

Combining (5.27) and (5.26) we get

$$\frac{\left(\int_0^{\mu_{u^*}(t)} f^\star(s)\,ds\right)\mu'_{u^*}(t)}{h^2(\mu_{u^*}(t))} \leq -1. \tag{5.28}$$

Reasoning analogously for the function v, we easily see that, since v is constant on every set $\{x_1 = t\}$ and since $v = v^*$, (5.28) holds for v as an equality. Consider now the real function

$$F(r) = \frac{\int_0^r f(s)\,ds}{h(r)^2},$$

and let G be a primitive of F. Since $F \geq 0$, we have that G is increasing. Moreover by our previous analysis we have that

$$F(\mu_{u^*}(t))\mu'_{u^*}(t) \leq -1 = F(\mu_v(t))\mu'_v(t).$$

We recall that here $\mu'_{u^*}(t)$ denotes the derivative almost everywhere of the function $\mu_{u^*}(t)$. Moreover $t \mapsto G(\mu_{u^*}(t))$ is a monotone non-increasing function which satisfies the chain rule in any point of differentiability of μ_{u^*}, so that, by [AFP, Corollary 3.29], we get that

$$G(\mu_{u^*}(t)) \leq G(\mu_{u^*}(0)) + \int_0^t F(\mu_{u^*}(\tau))\mu'_{u^*}(\tau)\,d\tau. \tag{5.29}$$

On the other hand, being $\mu_v(t)$ an absolutely continuous function (since v is a C^1 with positive derivative one variable function) we have

$$G(\mu_v(t)) = G(\mu_{u^*}(0)) + \int_0^t F(\mu_v(\tau))\mu'_v(\tau)\,d\tau, \tag{5.30}$$

so that, since $G(\mu_v(0)) = G(\mu_{u^*}(0))$, we get that $G(\mu_{u^*}(t)) \leq G(\mu_v(t))$. This implies that $\mu_{u^*}(t) \leq \mu_v(t)$ for any t and hence that $u^* \leq v$, since u^* and v depends only on x_1 and are increasing functions of such a variable.

We pass now to the proof of (5.20). Using the Hölder inequality and reasoning as before we obtain, for $0 < q \leq 2$,

$$\begin{aligned} -\frac{d}{dt}\int_{\{|u|>t\}} |Du|^q w^q \, d\mu &= \lim_{h\to 0} \frac{1}{h}\int_{t<|u|<t+h} |Du|^q w^q \, d\mu \\ &\leq \lim_{h\to 0} \left(\frac{1}{h}\int_{\{t<|u|<t+h\}} |Du|^2 w^2 \, d\mu\right)^{q/2} \left(\frac{1}{h}\int_{\{t<|u|<t+h\}} d\mu\right)^{1-q/2} \\ &= \left(-\frac{d}{dt}\int_{\{|u|>t\}} |Du|^2 w^2 \, d\mu\right)^{q/2} (-\mu_u'(t))^{1-q/2}. \end{aligned}$$

Recalling (5.21) and (5.27) we have

$$-\frac{d}{dt}\int_{\{|u|>t\}} |Du|^2 w^2 \, d\mu \leq \int_0^{\mu_{u^*}(t)} f^*(s)\, ds,$$

thus

$$-\frac{d}{dt}\int_{\{|u|>t\}} |Du|^q w^q \, d\mu \leq \left(\int_0^{\mu_{u^*}(t)} f^\star(s)\, ds\right)^{q/2} (-\mu_u'(t))^{1-q/2}. \quad (5.31)$$

Combining (5.31) and (5.28) we finally get

$$-\frac{d}{dt}\int_{\{|u|>t\}} |Du|^q w^q \, d\mu \leq (-\mu_{u^*}'(t)) \left(h(\mu_{u^*}(t))^{-1}\int_0^{\mu_{u^*}(t)} f^\star(s)\, ds\right)^q.$$

By integrating on both side between 0 and $+\infty$, we get

$$\int_E |Du|^q w^q \, d\mu \leq \int_0^\infty (-\mu_{u^*}'(t)) \left(h(\mu_{u^*}(t))^{-1}\int_0^{\mu_{u^*}(t)} f^\star(s)\, ds\right)^q dt.$$

We perform the change of variables $r = \mu_{u^*}(t)$, so that the above equation turns into

$$\int_E |Du|^q w^q \, d\mu \leq \int_0^{\mu(E)} \left(h(r)^{-1}\int_0^r f^\star(s)\, ds\right)^q dr.$$

By a straightforward inspection of those steps we notice that v satisfies

$$\int_{R_E} |Dv|^q w^q \, d\mu = \int_0^\infty (-\mu_v'(t)) \left(h(\mu_v(t))^{-1}\int_0^{\mu_v(t)} f^\star(s)\, ds\right)^q dt;$$

By performing the change of variables $r = \mu_v(t)$ we find

$$\int_{R_E} |Dv|^q w^q \, d\mu = \int_0^{\mu(R_E)} \left(h(r)^{-1} \int_0^r f^\star(s)\, ds \right)^q dr.$$

Since $\mu(E) = \mu(R_E)$ we get the desired result.

We are left to prove that the function v given by (5.17) is a solution of problem (5.16). We start by noticing that equation (5.28) suggests how to derive (5.17): indeed, as we pointed out, any solution v of (5.16) such that $v = v^*$ satisfies

$$\frac{\int_0^{\mu_v(t)} f^\star(s)\, ds}{h^2(\mu_v(t))} \mu_v'(t) = -1.$$

By integrating both sides between 0 and r we obtain

$$\int_0^r \frac{\int_0^{\mu_v(t)} f^\star(s)\, ds}{h^2(\mu_v(t))} \mu_v'(t)\, dt = -r.$$

so that, by performing the change of variables $m = \mu_v(t)$, we get

$$\int_{\mu_v(r)}^{\mu(R_E)} \frac{\int_0^m f^\star(s)\, ds}{h^2(m)} dm = r$$

which is equivalent to

$$v(z, z') = \int_{\mu\{x_1 > z\}}^{\mu(R_E)} \frac{\int_0^m f^\star(s)\, ds}{h^2(m)} dm,$$

that is (5.17). Notice that v is strictly decreasing and belongs to $C^{1,1}_{\text{loc}}(R_E)$. Indeed, recalling (5.18) one can explicitly compute

$$Dv(z, z') = e_1 \frac{\partial v}{\partial z}(z, z') = -e_1 \frac{\int_0^{\mu\{x_1 > z\}} f^\star(s)\, ds}{w^2(z) \int_{R^{d-1}} e^{V(z,x')}\, dx'},$$

where $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^N$. Since $f^\star$ is a decreasing and locally integrable function, then $f^\star \in L^\infty_{\text{loc}}(\mathbb{R})$; thus, being $z \mapsto \mu(\{x_1 > z\})$ C^1-regular, we get that $\int_0^{\mu\{x_1 > z\}} f^\star(s)\, ds$ is a locally Lipschitz function. Moreover the denominator is locally Lipschitz as well, and locally bounded away from zero. Hence we have that Dv is locally Lipschitz. Thus, recalling that $\partial_1 V$ depends only on the first variable x_1 it is possible to explicitly compute the divergence of $w^2 Dv\, e^V$ and check that it satisfies (5.16). This concludes the proof of the theorem. □

Bibliography

[Al1] G. Alessandrini: *Matzoh ball soup: a symmetry result for the heat equation*, J. Analyse Math., **54** (1990), 229–236.

[Al2] G. Alessandrini: *Characterizing spheres by functional relations on solutions of elliptic and parabolic equations*, Applicable Anal., **40** (1991), 251–261.

Math., **11** (1956),vvv 5–17.

58 (1962), 303–315.

[Alek] A. D. Aleksandrov: *Selected Works. Part I* in Selected Scientific Papers. (Yu. G. Reshetnyak, S. S. Kutateladze, eds), trans. from the Russian by P. S. Naidu. Classics of Soviet Mathematics, 4. Gordon and Breach, Amsterdam, 1996.

[ABG] M. Amar, L.R. Berrone, R. Gianni: *A non local quantitative characterization of ellipses leading to a solvable differential relation*, J. Inequal. in Pure & Appl. Math., **9** (2008), 14 pp.

[AFP] L. Ambrosio, N. Fusco, D. Pallara: *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 2000.

[BC] M. Bardi, I. Capuzzo-Dolcetta: *Optimal Control and Viscosity Solutions of HamiltonJacobi-Bellman Equations*, Birkhäuser, Boston, 1997.

[BBMP1] M.F. Betta, F. Brock, A. Mercaldo, M. R. Posteraro: *A comparison result related to Gauss measure*, C. R. Acad. Sci. Paris, **334** (2002), 451–456.

[BBMP2] M.F. Betta, F. Brock, A. Mercaldo, M.R. Posteraro, *A weighted isoperimetric inequality and applications to symmetrization*, J. of lnequal. Appl., **4** (1999), 215–240.

[BBMP3] M. F. Betta, F. Brock, A. Mercaldo, M.R. Posteraro: *A comparison result related to Gauss measure*, Comptes Rendus Mathematique, **334** (2002), 451–456.

[Bo1] J. Booman: *The sum of two plane convex C^∞ sets is not always C^5*, Math. Scand., **66** (1990), 216–224.

[Bo2] J. Booman: *Smoothness of sums of convex sets with real analytic boundaries*, Math. Scand., **66** (1990), 225–230.

[BDR] L. Brasco, G. De Philippis, B. Ruffini: *Spectral optimization for the Stekloff-Laplacian: the stability issue*, J. Funct. Anal., **262** (2012), 4675–4710.

[BF] L. Brasco, G. Franzina: *An anisotropic eigenvalue problem of Stekloff type and weighted Wulff inequalities*, Nonlinear Differential Equations and Applications, **6** (2013), 1795–1830.

[Br] F. Brickel: *A new proof of Deike's theorem on homogeneous functions*, Proc. Amer. Math. Soc., **16** (1965), 190–191.

[BCM1] F. Brock, F. Chiacchio, A. Mercaldo: *A class of degenerate elliptic equations and a Dido's problem with respect to a measure*, J. Math. Anal. Appl., **348** (2008), 356–365.

[BCM2] F. Brock, F. Chiacchio, A. Mercaldo: *Weighted isoperimetric inequalities in cones and applications*, Nonlinear Analysis: Theory, Methods and Applications, **75** (2012), 5737–5755.

[Ca] L. A. Caffarelli: *A localization property of viscosity solutions to the Monge-Ampère equation and their strict convexity*, Ann. of Math., (2) **131** (1990), 129–134.

[CIL] M. G. Crandall, H. Ishii, P.-L. Lions: *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc., **27** (1992), 1–67.

[CY] S. Y. Cheng, S. T. Yau: *On the regularity of the solution of the n-dimensional Minkowski problem*, Comm. Pure Appl. Math., **29** (1976), 495–516.

345 (2009), 859–881.

[DF] G. De Philippis, A. Figalli: *The Monge-Ampère equation and its link to optimal transportation*, Bull. Amer. Math Soc., **51** (2014), 527–580.

[DM] G. De Philippis, M.Marini: *A note on Petty's Theorem*, Kodai Math. J., **37** (2014), 586–594.

[dB] G. Di Blasio: *Linear elliptic equations and Gauss measure*, J. Inequal. Pure Appl. Math., **4** (2003), 1–11.

[dBFP] G. Di Blasio, F. Feo, M. R. Posteraro: *Regularity results for degenerate elliptic equations related to Gauss measure*, Mathematical Inequalities and Applications, **10** (2007), 771–797.

[EG] L. C. Evans, R. F. Gariepy: *Measure Theory and Fine Properties of Functions.* Studies in Advanced Mathematics., CRC Press, Boca Raton, FL, 1992.

[Ga] R. J. Gardner: *The Brunn-Minkowski inequality*, Bull. Amer. Math., Soc. **39** (2002), 355–405.

[GM] J. Q. Ge & H. Ma: *Anisotropic isoparametric hypersurfaces in Euclidean spaces*, Ann. Global Anal. Geom. 41 (2012), 347–355.

[GT] D. Gilbarg, N.S. Trudinger: *Elliptic Partial Differential Equations of Second Order.* Springer-Verlag, New York (1983).

[Gu] C. Gutierrez: *The Monge-Ampère Equation*, Progress in Nonlinear Differential Equations and their Applications, 44. Birkhäuser Boston, Inc., Boston, MA, 2001.

[HLMG] Y. J. He, H. Z. Li, H. Ma & J. Q. Ge: *Compact embedded hypersurfaces with constant higher order anisotropic mean curvatures*, Indiana Univ. Math. J., **58** (2009), 853–868.

[Hu] D. Hug: *Contributions to affine surface area*, Manuscripta Math., **91** (1996), 283–301.

[Kl] M. S. Klamkin: *A physical characterization of a sphere*, (Problem 64-5*) SIAM Review **6** (1964), 61; also in Problems in Applied Mathematics. Selection from SIAM Review. Edited by M.S. Klamkin. SIAM, Philadelphia, PA. 1990.

[KrPa] S. Krantz, H. Parks: *On the vector sum of two convex sets in space*, Can. J. Math., **43** (1991), 347–355.

[LU] O.A. Ladyzenskaya, N. N. Uraltseva: *Linear and Quasilinear Elliptic Equations*, Academic Press, New York, 1968.

[Le] T. Levi-Civita: *Famiglie di superficie isoparametriche nell'ordinario spazio euclideo,* Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., **26** (1937), 355–362.

[LL] E. Lieb, M. Loss: *Analysis,* Vol 14 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 1997.

[Lu] E. Lutwak: *Selected affine isoperimetric inequalities*, in P.M. Gruber, J.M. Wills, eds., Handbook of convex geometry, North-Holland, Amsterdam, 1993, 151–176.

[Lu3] E. Lutwak: *The Brunn-Minkowski-Firey Theory II: Affine and Geominimal Surface Area*, Adv Math., **118** (1996), 244–294.

[Ma] F. Maggi: *Sets of Finite Perimeter and Geometric Variational Problems*,Cambridge University Press, Cambridge, 2012.

[MM1] R. Magnanini, M. Marini: *Characterization of ellipses as uniformly dense sets with respect to a family of convex bodies* , Ann. Mat. Pura Appl., **193** (2014), 1383–1395.

[MM2] R. Magnanini, M. Marini: *Characterization of ellipsoids as K-dense sets* , Proc. Roy. Edin. Soc. A: to appear.

[MM3] R. Magnanini, M. Marini: *The Matzoh Ball Soup Problem: a complete characterization*, Nonlinear Anal.-Theor., (2015), doi:10.1016/j.na.2015.06.022.

[MPeS] R. Magnanini, D. Peralta-Salas, S. Sakaguchi: *Stationary isothermic surfaces in Euclidean 3-space*, to appear in Math. Ann., DOI:10.1007/s00208-015-1212-1.

[MPS] R. Magnanini, J. Prajapat, S. Sakaguchi: *Stationary isothermic surfaces and uniformly dense domains*, Trans. Am. Math. Soc. **385** (2006), 4821–4841.

[MS1] R. Magnanini, S. Sakaguchi: *Matzoh ball soup: heat conductors with a stationary isothermic surface*, Ann. Math. **156** (2002), 931–946.

[MS2] R. Magnanini, S. Sakaguchi: *Interaction between degenerate diffusion and shape of domain*, Proc. Roy. Soc. Edin. **137**A (2007), 373–388.

[MS3] R. Magnanini, S. Sakaguchi: *Stationary isothermic surfaces for unbounded domains*, Indiana Univ. Math. Journ., **56** no. 6 (2007), 2723–2738.

[MS4] R. Magnanini, S. Sakaguchi: *Nonlinear diffusion with a bounded stationary level surface*, Ann. Inst. Henri Poincaré Analyse Nonlinéaire, **27** (2010), 937–952.

[MS5] R. Magnanini, S. Sakaguchi: *Stationary isothermic surfaces and some characterizations of the hyperplane in N-dimensional space*, J. Diff. Eqs. **248** (2010), 1112–1119.

[MS6] R. Magnanini, S. Sakaguchi: *Interaction between nonlinear diffusion and geometry of domain*, J. Diff. Eqs., **252** (2012), 236–257.

[MS7] R. Magnanini, S. Sakaguchi: *Matzoh ball soup revisited: the boundary regularity issue,* Math. Meth. Appl. Sci., **36** (2013), 2023–2032.

[MR] M. Marini, B. Ruffini: *On a class of weighted Gauss-type isoperimetric inequalities and applications to symmetrization*, Rend. Sem. Mat. Univ. Padova, **133** (2014), 197–214.

[MRS] M. Meyer, S. Reisner, M. Schmuckenschläger: *The volume of the intersection of a convex body with its translates*, Mathematika **40** (1993), 278–289.

[Mi1] H. Minkowski: *Allgemeine lehrsätze über die konvexen polyeder*, Nachr. Ges. Wiss., Göttingen (1887), 103–121.

[Mi2] H. Minkowski: *Volumen und oberfläche*, Math. Ann., **57** (1903), 447–495.

[Mi3] H. Minkowski: *Theorie der konvexen Körper, insbesondere Begründung ihres Oberflächenbegriffs*, Gesammelte Abhandlungen, Vol. II (Teubner, Leipzig) (1911), 131–229.

[Ni] L. Nirenberg: *The Weyl and Minkowski problems in differential geometry in the large*, Comm. Pure App. Math. 6 (3) (1953), 337–394.

[Pe] C. M. Petty: *Affine isoperimetric problems*, Ann. N.Y. Acad. Sci., **440** (1985), 113–127.

[Po1] A. V. Pogorelov: *On the improper convex affine hyperspheres*, Geometriae Dedicata, **1** (1972), no. 1, 33–46.

[Po2] A. V. Pogorelov: *The Minkowski multidimensional problem*, Washington: Scripta (1979), 97 p.

[Po3] A. V. Pogorelov: *Regularity of generalized solutions of the equation* $\det(u_{ij})\theta(Du,u,x)=\varphi(x)$. Dokl. Akad. Nauk SSSR, **275** (1984), no. 1, 26–28.

[Ro] R.T. Rockafellar: *Convex Analisys*, Reprint of the 1970 original. Princeton Landmarks in Mathematics. Princeton Paperbacks, Princeton University press, Princeton, NJ, 1997.

[Sak] S. SAKAGUCHI: *When are the spatial level surfaces of solutions of diffusion equations invariant with respect to the time variable?*, J. Analyse Math., **78** (1999), 219–243.

[S-Y] J.R. SANGWINE-YAGER: *Mixed Volumes*, in P.M. Gruber, J.M. Wills, eds., Handbook of convex geometry, North-Holland, Amsterdam, 1993, 43–71.

[San] A. SANTALÒ: *Un invariante afin para los cuerpos convexos del espacio de n dimensiones*, Portugal. Math., **8** (1949), 155–161.

[Sc] M. SCHMUCKENSCHLÄGER: *A simple proof of an approximation theorem of H. Minkowski*, Geom. Dedicata, **48** (1993), 319–324.

[Scn1] R. SCHNEIDER: *Uber eine integralgleichung in der theorie der konvexen körper*, Math. Nachr., **44** (1970), 55–75.

[Scn2] R. SCHNEIDER: *Curvature measures of convex bodies*, Ann. Mat. Pura Appl., **116** (1978), 101–134.

[Scn] R. SCHNEIDER: *Convex bodies: the Brunn-Minkovski theory*, Second Edition, Cambridge University Press, Cambridge, 2014.

[Se] B. SEGRE: *Famiglie di ipersuperficie isoparametriche negli spazi euclidei ad un qualunque numero di dimensioni*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., **27** (1938), 203–207.

[Ser] J SERRIN: *A symmetry problem in potential theory*, Arch. Ration. Mech. Anal., **43** (1971), 304–318.

[Sta] A. STANCU: *The floating body problem*, Bull. London Math. Soc., **38** (2006), 839–846.

[Ste] J. STEINER: *Über paralelle Flächen*, Jbr. Preuss. Akad. Wiss., (1940), 114–118.

[Ta] G. TALENTI: *Elliptic equations and rearrangements*, Ann. Scuola Norm. Sup. Pisa Cl. Sci., **4** (1976), 697–718.

[TL] Y. J. TIAN, F. Q. LI: *On the Case of Equalities in Comparison Results for Elliptic Equations Related to Gauss Measure*, Journal of Mathematical Research Exposition, **30** (2010), 761–774.

[Tr] N. S. TRUDINGER: *Linear elliptic operators with measurable coefficients*, Annali della Scuola Normale Superiore di Pisa-Classe di Scienze, **27** (1973), 265–308.

[Va] S. R. S. VARADHAN: *On the behavior of the fundamental solution of the heat equation with variable coefficients*, Comm. Pure Appl. Math., **20** (1967), 431–455.

[Za] L. Zalcman, *Some inverse problems of potential theory*, Cont. Math., **63** (1987), 337–350.