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# Off-shell construction of some trilinear higher spin gauge field interactions

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Abstract

Several trilinear interactions of higher spin fields involving two equal ($s = s_1 = s_2$) and one higher even ($s_3 > s$) spin are presented. Interactions are constructed on the Lagrangian level using Noether's procedure together with the corresponding next to free level fields of the gauge transformations. In certain cases when the number of derivatives in the transformation is $2s - 1$ the interactions lead to the currents constructed from the generalization of the gravitational Bell-Robinson tensors. In other cases when the number of derivatives in the transformation is more than $2s-1$ we obtain the finite tower of interactions with smaller even spins less than $s_3$ in full agreement with our previous results for the interaction of the higher even spins field with a conformal scalar[1, 2]. The self interacting case is presented as an algorithmic formalism for the construction of the cubic interactions to be applied to spin four and higher.

# 1 Introduction

The construction of interacting *higher spin gauge field* theories (HSF) has always been considered an important task during the last thirty years (See [3]-[8] and ref. there*). The complications and difficulties which accompany any serious attempt to solve the essential problems in this area always attracted interest but activity intensified after discovering the important role HSF plays in $AdS/CFT$ correspondence. Particular attention caused the holographic duality between the $O(N)$ sigma model in three dimensional space and HSF gauge theory living in the four dimensional space with negative constant curvature [9]. This case of holography is singled out by the existence of two conformal points of the boundary theory and the possibility to describe them by the same HSF gauge theory with the help of spontaneously breaking of higher spin gauge symmetry and mass generation by a corresponding Higgs mechanism. All these complicated physical tasks necessitate *quantum loop* calculations for HSF field theory [10]-[16] and therefore information about manifest, off-shell and Lagrangian formulation of possible interactions for HSF. Then after successful calculations on the quantum level the construction can be controlled by comparison with the boundary $O(N)$ model results checking the $AdS/CFT$ correspondence conjecture on the loop level [10], [11], [13].

In this article we continue the construction of possible couplings including different higher spin fields which was started in our previous articles about couplings including HSF and scalar fields [1, 2, 10] and that are important for the Higgs mechanism mentioned above. Here we turn to the trilinear interaction between HSF gauge fields of different spins (s-s-s') in a flat background but the results can be easily generalized to the $AdS$ background. The first three sections are devoted to the development of the idea: how we can apply higher spin gauge symmetry of a spin "s" gauge field to the field with a spin lower than "s". Then getting in this way information about the first order gauge transformation, we can handle Noether's procedure applying this first order transformation to the zero order free Lagrangian and integrating this variation to a first order trilinear interaction. Starting from the construction of the spins 1-1-2 and 1-1-4, we discover in this simple case the same phenomenon as in the previously investigated scalar case [1, 2], namely the appearance of the couplings with all even spins lower than the initial maximal higher spin gauge field involved in the interaction vertex (Section 2). Then we generalize the construction to the more complicated 2-2-4 and then to 2-2-6 where the previously constructed 2-2-4 interaction again appears automatically (Section 3). The next section starts from the description of a technique for working with the HSF fields in Fronsdal's [17] formulation and deWit-Friedman curvatures [18, 19]. In the same section we succeed with the construction of the interaction Lagrangian of spin type s-s-2s together with the first order higher spin gauge transformation. The last section considers a technical setup aimed at an algorithm for the general solution of the selfinteraction problem of a spin "s" gauge field. due to the extreme volume of the linear algebraic task we intend to present the result for the case of spin four in a forthcoming publication [20].

*We do not pretend here for complete quotations and just present some references important for us during this investigation

## 2 Exercises on spin one field couplings with the higher spin gauge fields

We start this section constructing the well known interaction of the electromagnetic field $A_\mu$ in flat $D$ dimensional space-time with the linearized spin two field. Hereby we illustrate how Noether's procedure regulates the relation between gauge symmetries of different spin fields. The standard free Lagrangian of the electromagnetic field is

$$\mathcal{L}_0 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2}(\partial A)^2, \tag{2.1}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad \partial A = \partial_\mu A^\mu. \tag{2.2}$$

To construct the interaction we propose a possible form for the action of the spin two linearized gauge symmetry

$$\delta^0_\varepsilon h^{(2)\mu\nu}(x) = 2\partial^{(\mu}\varepsilon^{\nu)}(x) = \partial^\mu \varepsilon^\nu(x) + \partial^\nu \varepsilon^\mu(x), \tag{2.3}$$

on the spin one gauge field $A_\mu(x)$. Then Noether's procedure fixes this coupling (1-1-2 interaction) of the electromagnetic field with linearized gravity correcting when necessary the proposed transformation.

We start from the following general ansatz for a gauge variation of $A_\mu$ with respect to a spin 2 gauge transformation with vector parameter $\varepsilon^\rho$

$$\delta^1_\varepsilon A_\mu = -\varepsilon^\rho \partial_\rho A_\mu + C\varepsilon^\rho \partial_\mu A_\rho. \tag{2.4}$$

Then we apply this variation (2.4) to (2.1) and after some algebra neglecting total derivatives we obtain †

$$\begin{aligned}\delta^1_\varepsilon \mathcal{L}_0 &= \partial^{(\mu}\varepsilon^{\nu)}\partial_\mu A_\rho \partial_\nu A^\rho - \frac{1}{2}\varepsilon_{(1)}\partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2}\varepsilon_{(1)}(\partial A)^2 + C\partial^{(\mu}\varepsilon^{\nu)}\partial_\rho A_\mu \partial^\rho A_\nu \\ &- 2C\partial^{(\mu}\varepsilon^{\nu)}\partial_\rho A_{(\mu}\partial_{\nu)}A^\rho + \frac{C}{2}\varepsilon_{(1)}\partial_\mu A_\nu \partial^\nu A^\mu - \frac{C}{2}\varepsilon_{(1)}(\partial A)^2 \\ &+ (C-1)(\partial A)\partial^\mu \varepsilon^\nu \partial_\nu A_\mu. \end{aligned} \tag{2.6}$$

Then we have to compensate (or integrate) this variation using the gauge variation of the spin 2 field (2.3) and its trace $\delta^0_\varepsilon h^{(2)\mu}_\mu = 2\varepsilon_{(1)}$ . We see immediately that the last line in (2.6) is irrelevant but can be dropped by choice of the free constant $C = 1$. With this choice we have instead of (2.4)

$$\delta^1_\varepsilon A_\mu = -\varepsilon^\rho \partial_\rho A_\mu + \varepsilon^\rho \partial_\mu A_\rho = \varepsilon^\rho F_{\mu\rho}, \tag{2.7}$$

so that our spin two transformation now is manifestly gauge invariant with respect to the spin one gauge invariance

$$\delta^0_\sigma A_\mu = \partial_\mu \sigma, \tag{2.8}$$

†From now on we will never make a difference between a variation of the Lagrangians or the actions discarding all total derivative terms and admitting partial integration if necessary. For compactness we introduce also shortened notations for divergences of the tensorial symmetry parameters

$$\epsilon^{\mu\nu\dots}_{(1)} = \nabla_\lambda \epsilon^{\lambda\mu\nu\dots}, \quad \epsilon^{\mu\dots}_{(2)} = \nabla_\nu \nabla_\lambda \epsilon^{\nu\lambda\mu\dots}, \quad \dots \tag{2.5}$$

and our spin one gauge invariant free action (2.1) keeps this property also after spin two gauge variation. Namely (2.6) now can be written as

$$\delta^1_\varepsilon \mathcal{L}_0 = \partial^{(\mu}\varepsilon^{\nu)} F_{\mu\rho} F_\nu{}^\rho - \frac{1}{4}\varepsilon_{(1)} F_{\mu\nu} F^{\mu\nu}. \tag{2.9}$$

This variation can be compensated introducing the following 2-1-1 interaction

$$\mathcal{L}_1(A_\mu, h^{(2)}_{\mu\nu}) = \frac{1}{2} h^{(2)\mu\nu} \Psi^{(2)}_{\mu\nu}, \tag{2.10}$$

where

$$\Psi^{(2)}_{\mu\nu} = -F_{\mu\rho} F_\nu{}^\rho + \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}, \tag{2.11}$$

is the well known energy-momentum tensor for the electromagnetic field.

Thus we solved Noether's equation

$$\delta^1_\varepsilon \mathcal{L}_0(A_\mu) + \delta^0_\varepsilon \mathcal{L}_1(A_\mu, h^{(2)}_{\mu\nu}) = 0 \tag{2.12}$$

in this approximation completely, defining a first order transformation and interaction term at the same time. Finally note that the corrected Noether's procedure spin two transformation of the spin one field (2.7) can be written as a combination of the usual reparametrization for the contravariant vector $A_\mu(x)$ (non invariant with respect to (2.8)) and spin one gauge transformation with the special field dependent choice of the parameter $\sigma(x) = \varepsilon^\rho(x) A_\rho(x)$

$$\delta^1_\varepsilon A_\mu = \varepsilon^\rho F_{\mu\rho} = -\varepsilon^\rho \partial_\rho A_\mu - \partial_\mu \varepsilon^\rho A_\rho + \partial_\mu \left(\varepsilon^\rho(x) A_\rho(x)\right), \tag{2.13}$$

Now we turn to the first nontrivial case of the vector field interaction with a spin four gauge field with the following zero order spin four gauge variation

$$\delta^0_\epsilon h^{\mu\rho\lambda\sigma} = 4\partial^{(\mu}\epsilon^{\rho\lambda\sigma)}, \quad \delta^0_\epsilon h_\rho{}^{\rho\lambda\sigma} = 2\epsilon^{\lambda\sigma}_{(1)}. \tag{2.14}$$

where we have a symmetric and traceless gauge parameter $\epsilon^{\mu\nu\lambda}$ to construct a gauge variation for $A_\mu$. According to the previous lesson we start from a spin one gauge invariant ansatz for the spin four transformation of $A_\mu$ field

$$\delta^1_\epsilon A_\mu = \epsilon^{\rho\lambda\sigma} \partial_\rho \partial_\lambda F_{\mu\sigma}. \tag{2.15}$$

Thus we have now the following variation of $\mathcal{L}_0$

$$\delta^1_\epsilon \mathcal{L}_0 = \delta^1_\epsilon\Big(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}\Big) = (\delta^1_\epsilon A_\nu)\partial_\mu F^{\mu\nu} = -\partial_\mu(\epsilon^{\rho\lambda\sigma}\partial_\rho\partial_\lambda F_{\nu\sigma}) F^{\mu\nu}. \tag{2.16}$$

After some algebra, again neglecting total derivatives and using the Bianchi identity for $F_{\mu\nu}$

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0, \tag{2.17}$$

and taking into account the important relation

$$\begin{aligned} -\partial^\mu \epsilon^{\rho\lambda\sigma} \partial_\rho F_\mu{}^\nu \partial_\lambda F_{\sigma\nu} &= -\partial^{(\mu}\epsilon^{\rho\lambda\sigma)} \partial_{(\rho} F_\mu{}^\nu \partial_\lambda F_{\sigma)\nu} + \frac{1}{4}\epsilon^{\lambda\sigma}_{(1)} \partial^\nu F_{\mu\lambda} \partial^\mu F_{\nu\sigma} \\ &\quad -\frac{1}{2}\partial^\nu \epsilon^{\rho\lambda\sigma} \partial_\lambda F_{\sigma\nu} \partial^\mu F_{\mu\rho} - \frac{1}{4}\epsilon^{\lambda\sigma}_{(1)} \partial^\mu F_{\mu\rho} \partial^\nu F_{\nu\sigma}, \end{aligned} \tag{2.18}$$

we arrive at the following form of the variation convenient for our analysis

$$\begin{aligned}
\delta^1_\epsilon \mathcal{L}_0 &= -\partial^{(\mu}\epsilon^{\rho\lambda\sigma)}\partial_{(\rho}F_\mu{}^\nu\partial_\lambda F_{\sigma)\nu} + \frac{1}{4}\epsilon^{\lambda\sigma}_{(1)}\partial^\nu F_{\mu\lambda}\partial^\mu F_{\nu\sigma} + \frac{1}{4}\epsilon^{\lambda\sigma}_{(1)}\partial_\lambda F_{\mu\nu}\partial_\sigma F^{\mu\nu} \\
&- \partial_\lambda(\epsilon^{\lambda\sigma}_{(1)}F_{\mu\sigma})\partial_\nu F^{\nu\mu} - \frac{1}{4}\epsilon^{\lambda\sigma}_{(1)}\partial^\mu F_{\mu\lambda}\partial^\nu F_{\nu\sigma} - \frac{1}{2}\partial^\rho\epsilon^{\nu\lambda\sigma}\partial_\lambda F_{\sigma\rho}\partial^\mu F_{\mu\nu} \\
&+ \partial^{(\mu}\epsilon^{\nu)}_{(2)}F_{\mu\sigma}F_\nu{}^\sigma - \frac{1}{4}\epsilon_{(3)}F_{\mu\nu}F^{\mu\nu}.
\end{aligned} \tag{2.19}$$

Returning to the gauge variation of the spin four field (2.14) we notice that all terms in the first line of (2.19) and the first two terms in the second line can be integrated to the interaction terms, but the first two terms in the second line are proportional to the equation of motion for the initial Lagrangian (2.1), hence they are not physical and can be removed by the following field redefinition

$$A_\mu \to A_\mu - \partial_\lambda(h_\alpha{}^{\alpha\lambda\sigma}F_{\mu\sigma}) - \frac{1}{4}h^\alpha{}_{\alpha\mu\sigma}\partial_\beta F^{\beta\sigma}. \tag{2.20}$$

The last term in the second line is also proportional to the free field equations but is not integrable, so we can cancel this term only by changing the initial variation of $A_\mu$ (2.15). The modified form of (2.15) is

$$\delta^1_\epsilon A_\mu = \epsilon^{\rho\lambda\sigma}\partial_\rho\partial_\lambda F_{\mu\sigma} + \frac{1}{2}\partial_\rho\epsilon_{\mu\lambda\sigma}\partial^\lambda F^{\sigma\rho}. \tag{2.21}$$

So we can drop the second line of (2.19).

Another novelty in comparison with the previous case is the third line of (2.19). Comparing with (2.9) we see that we can integrate these two terms introducing an additional spin two field coupling and compensate the first and third line introducing the following linearized Lagrangian for the coupling of the electromagnetic field to the spin four and spin two fields

$$\mathcal{L}_1(A_\mu, h^{(2)\mu\nu}, h^{(4)\mu\nu\alpha\beta}) = \frac{1}{4}h^{(4)\mu\nu\alpha\beta}\Psi^{(4)}_{\mu\nu\alpha\beta} + \frac{1}{2}h^{(2)\mu\nu}\Psi^{(2)}_{\mu\nu}, \tag{2.22}$$

where the current $\Psi^{(2)}_{\mu\nu}$ is the same energy-momentum tensor (2.10) and

$$\Psi^{(4)}_{\mu\nu\alpha\beta} = \partial_{(\alpha}F_\mu{}^\rho\partial_\beta F_{\nu)\rho} - \frac{1}{4}g_{(\mu\nu}\partial^\lambda F_{\alpha\sigma}\partial^\sigma F_{\beta)\lambda} + \frac{1}{4}g_{(\mu\nu}\partial_\alpha F^{\sigma\rho}\partial_{\beta)}F_{\sigma\rho}. \tag{2.23}$$

The whole action

$$\mathcal{L}_0(A_\mu) + \mathcal{L}_1(A_\mu, h^{(2)\mu\nu}, h^{(4)\mu\nu\alpha\beta}), \tag{2.24}$$

is invariant with respect to the spin one gauge transformations and the following higher spin transformations

$$\begin{aligned}
&\delta^1 A_\mu = \epsilon^{\rho\lambda\sigma}\partial_\rho\partial_\lambda F_{\mu\sigma} + \frac{1}{2}\partial_\rho\epsilon_{\mu\lambda\sigma}\partial^\lambda F^{\sigma\rho}, \\
&\delta^0 h^{(4)\mu\nu\alpha\beta} = 4\partial^{(\mu}\epsilon^{\nu\alpha\beta)}, \;\; \delta^0_\epsilon h_\mu{}^{\mu\alpha\beta} = 2\epsilon^{\alpha\beta}_{(1)}, \\
&\delta^0 h^{(2)\mu\nu} = 2\partial^{(\mu}\epsilon^{\nu)}_{(2)}, \;\; \delta^0 h^{(2)\mu}_\mu = 2\epsilon_{(3)}.
\end{aligned} \tag{2.25}$$

Therefore we proved that like the previously investigated scalar–higher spin coupling case [2], the interaction with the spin four gauge field leads to the additional interaction with the lower even spin two field.

# 3 Generalization to the 2-2-4 and 2-2-6 interactions

In this section we turn to the spin two field as a lower spin field in the construction of the higher spin gauge invariant interactions with spin 4 and spin 6 gauge potentials. And again we want to keep manifest the lower spin two gauge invariance.

So proceeding similarly as in the previous section we start from the free spin two Pauli-Fierz Lagrangian [21]

$$\mathcal{L}_0(h^{(2)}_{\mu\nu}) = \frac{1}{2}\partial_\mu h^{(2)}_{\alpha\beta}\partial^\mu h^{(2)\alpha\beta} - \partial_\alpha h^{(2)\alpha\beta}\partial_\mu h^{(2)\mu}_{\beta} + \partial_\mu h^{(2)\alpha}_{\alpha}\partial_\beta h^{(2)\beta\mu} - \frac{1}{2}\partial_\mu h^{(2)\alpha}_{\alpha}\partial^\mu h^{(2)\beta}_{\beta}, \quad (3.1)$$

and try to solve the following Noether's equation

$$\delta^1_\varepsilon \mathcal{L}_0(h^{(2)}_{\mu\nu}) + \delta^0_\varepsilon \mathcal{L}_1(h^{(2)}_{\mu\nu}, h^{(4)\alpha\beta\lambda\rho}) = 0. \quad (3.2)$$

For this purpose we introduce the following starting ansatz for the spin four transformation of the spin two field

$$\delta^1_\epsilon h^{(2)}_{\mu\nu} = \epsilon^{\rho\lambda\sigma}\partial_\rho\Gamma_{\lambda\sigma,\mu\nu}, \quad (3.3)$$

where $\Gamma_{\lambda\sigma,\mu\nu}$ is the spin two gauge invariant symmetrized linearized Riemann curvature

$$\Gamma_{\alpha\beta,\mu\nu} = \frac{1}{2}(R_{\alpha\mu,\beta\nu} + R_{\beta\mu,\alpha\nu}), \quad (3.4)$$

$$\Gamma_{(\alpha\beta,\mu)\nu} = 0, \quad (3.5)$$

introduced by de Witt and Freedman for higher spin gauge fields together with the higher spin generalization of the Christoffel symbols [18]. This symmetrized curvature is more convenient for the construction of an interaction with symmetric tensors. The corresponding Ricci tensor (Fronsdal operator for higher spin generalization) and scalar can be defined in the usual manner using traces

$$\mathcal{F}_{\mu\nu} = \Gamma^\lambda_{\mu\nu,\lambda} = \Box h^{(2)}_{\mu\nu} - 2\partial_{(\mu}\partial^\alpha h^{(2)}_{\nu)\alpha} + \partial_\mu\partial_\nu h^{(2)\alpha}_{\alpha}, \quad (3.6)$$

$$\mathcal{F} = \mathcal{F}^\mu_\mu = 2(\Box h^{(2)\mu}_{\mu} - \partial_\mu\partial_\nu h^{(2)\mu\nu}). \quad (3.7)$$

In terms of these objects the Bianchi identities can be written as

$$\partial_\lambda\Gamma_{\mu\nu,\alpha\beta} = \partial_{(\mu}\Gamma_{\nu)\lambda,\alpha\beta} + \partial_{(\alpha}\Gamma_{\beta)\lambda,\mu\nu}, \quad (3.8)$$

$$\partial_\lambda\mathcal{F}_{\alpha\beta} = \partial^\mu\Gamma_{\mu\lambda,\alpha\beta} + \partial_{(\alpha}\mathcal{F}_{\beta)\lambda}, \quad (3.9)$$

$$\partial^\lambda\mathcal{F}_{\lambda\mu} = \frac{1}{2}\partial_\mu\mathcal{F}^\alpha_\alpha. \quad (3.10)$$

Then a variation of (3.1) with respect to (3.3) is

$$\delta^1_\epsilon\mathcal{L}_0(h^{(2)}_{\mu\nu}) = \frac{\delta\mathcal{L}_0}{\delta h^{(2)}_{\mu\nu}}\delta^1_\epsilon h^{(2)}_{\mu\nu} = -(\mathcal{F}^{\mu\nu} - \frac{1}{2}g^{\mu\nu}\mathcal{F})\epsilon^{\rho\lambda\sigma}\partial_\rho\Gamma_{\lambda\sigma,\mu\nu}. \quad (3.11)$$

To integrate it and solve the equation (3.2) we submit to the following strategy:

1) First we perform a partial integration and use the Bianchi identity (3.9) to lift the variation to a curvature square term.

2) Then we make a partial integration again and rearrange indices using (3.5) and (3.8) to extract an integrable part.

3) Symmetrizing expressions in this way we classify terms as

- integrable
- integrable and subjected to field redefinition (proportional to the free field equation of motion)
- non integrable but reducible by deformation of the initial ansatz for the gauge transformation (again proportional to the free field equation of motion)

Then if no other terms remain we can construct our interaction together with the corrected first order transformation. Following this strategy after some fight with formulas we win the battle obtaining the following expression

$$\begin{aligned}\delta^1_\epsilon\mathcal{L}_0(h^{(2)}_{\mu\nu}) &= -\partial^{(\alpha}\epsilon^{\beta\mu\nu)}(\Psi^{(4)}_{(\Gamma)\alpha\beta\mu\nu}-\Psi^{(4)}_{(\mathcal{F})\alpha\beta\mu\nu})\\ &- \epsilon^{\mu\nu}_{(1)}\Gamma_{\mu\nu,\alpha\beta}\frac{\delta\mathcal{L}_0}{\delta h^{(2)}_{\alpha\beta}}+\partial^\rho\epsilon_\alpha{}^{\mu\nu}\Gamma_{\beta\rho,\mu\nu}\frac{\delta\mathcal{L}_0}{\delta h^{(2)}_{\alpha\beta}},\end{aligned} \tag{3.12}$$

where

$$\Psi^{(4)}_{(\Gamma)\alpha\beta\mu\nu}=\Gamma_{(\alpha\beta,}{}^{\rho\sigma}\Gamma_{\mu\nu),\rho\sigma}-\frac{2}{3}g_{(\alpha\beta}\Gamma_\mu{}^{\rho,\sigma\lambda}\Gamma_{\nu)\rho,\sigma\lambda}, \tag{3.13}$$

$$\Psi^{(4)}_{(\mathcal{F})\alpha\beta\mu\nu}=\mathcal{F}_{(\alpha\beta}\mathcal{F}_{\mu\nu)}-g_{(\alpha\beta}\mathcal{F}^\sigma_\mu\mathcal{F}_{\nu)\sigma}=-\frac{\delta\mathcal{L}_0}{\delta h^{(2)(\alpha\beta}}\mathcal{F}_{\mu\nu)}+g_{(\alpha\beta}\frac{\delta\mathcal{L}_0}{\delta h^{(2)\mu}_\sigma}\mathcal{F}_{\nu)\sigma}, \tag{3.14}$$

$$\frac{\delta\mathcal{L}_0}{\delta h^{(2)\alpha\beta}}=-\mathcal{F}_{\alpha\beta}+\frac{1}{2}g_{\alpha\beta}\mathcal{F}. \tag{3.15}$$

So we see immediately that in (3.12) only the last term of the second line is not integrable but proportional to the equation of motion and can be dropped by the correction to the initial gauge transformation (3.3). On the other hand taking into account (2.14) and (3.13)-(3.15) we can compensate $\Psi^{(4)}_{(\mathcal{F})}$ and the first term in the second line of (3.12) by the following field redefinition

$$h^{(2)}_{\mu\nu}\to h^{(2)}_{\mu\nu}-\frac{1}{2}h^{(4)\alpha\lambda\sigma}_\alpha\Gamma_{\lambda\sigma,\mu\nu}-\frac{1}{4}h^{(4)\alpha\lambda}_{\mu\nu}\mathcal{F}_{\alpha\lambda}+\frac{1}{4}h^{(4)\alpha\lambda}{}_{\alpha(\mu}\mathcal{F}_{\nu)\lambda}. \tag{3.16}$$

Thus after field redefinition we arrive at the 4-2-2 gauge invariant interaction

$$\begin{aligned}\mathcal{L}_1(h^{(2)}_{\mu\nu},h^{(4)}_{\alpha\beta\mu\nu}) &= \frac{1}{4}h^{(4)\alpha\beta\mu\nu}\Psi^{(4)}_{(\Gamma)\alpha\beta\mu\nu}(h^{(2)}_{\mu\nu})\\ &= \frac{1}{4}h^{(4)\alpha\beta\mu\nu}\Gamma_{\alpha\beta,\rho\sigma}\Gamma_{\mu\nu,}{}^{\rho\sigma}-\frac{1}{6}h^{(4)\alpha\mu\nu}_\alpha\Gamma_\mu{}^{\rho,\sigma\lambda}\Gamma_{\nu\rho,\sigma\lambda},\end{aligned} \tag{3.17}$$

with the following gauge transformations

$$\delta_\epsilon h^{(2)}_{\mu\nu}=\epsilon^{\rho\lambda\sigma}\partial_\rho\Gamma_{\lambda\sigma,\mu\nu}-\partial_\rho\epsilon_{\lambda\sigma(\mu}\Gamma^{\rho,\lambda\sigma}_{\nu)}, \tag{3.18}$$

$$\delta^0_\epsilon h^{(4)\mu\rho\lambda\sigma}=4\partial^{(\mu}\epsilon^{\rho\lambda\sigma)},\quad \delta^0_\epsilon h^{(4)\rho\lambda\sigma}_\rho=2\epsilon^{\lambda\sigma}_{(1)}. \tag{3.19}$$

Now in possession of knowledge about the 2-2-4 interaction we start to construct the most nontrivial interaction in this article between spin 2 and spin 6 gauge fields. We would like to check the appearance of the 2-2-4 coupling during the construction of 2-2-6 which we expect from the analogy with the scalar case considered in [1, 2] and the 1-1-4 case considered in the previous section. To proceed we have to solve the following Noether's equation

$$\delta^1_\varepsilon\mathcal{L}_0(h^{(2)}_{\mu\nu})+\delta^0_\varepsilon\mathcal{L}_1(h^{(2)}_{\mu\nu},h^{(6)}_{\alpha\beta\lambda\rho\sigma\delta})=0, \tag{3.20}$$

with a starting ansatz for the spin 6 first order gauge transformation for the spin 2 field:

$$\delta^1_\epsilon h^{(2)}_{\mu\nu}(x) = \epsilon^{\alpha\beta\rho\lambda\sigma}(x)\partial_\alpha\partial_\beta\partial_\rho\Gamma_{\lambda\sigma,\mu\nu}(x), \tag{3.21}$$

and the standard zero order gauge transformation for the spin 6 gauge field

$$\delta^0_\epsilon h^{(6)\mu\nu\alpha\beta\sigma\rho} = 6\partial^{(\mu}\epsilon^{\nu\alpha\beta\sigma\rho)}(x), \tag{3.22}$$

$$\delta^0_\epsilon h^{(6)\mu\alpha\beta\sigma\rho}_\mu = 2\epsilon^{\alpha\beta\sigma\rho}_{(1)}. \tag{3.23}$$

First of all we have to transform the variation

$$\delta^1_\varepsilon\mathcal{L}_0(h^{(2)}_{\mu\nu}) = -(\mathcal{F}^{\mu\nu} - \frac{1}{2}g^{\mu\nu}\mathcal{F})\epsilon^{\alpha\beta\rho\lambda\sigma}\partial_\alpha\partial_\beta\partial_\rho\Gamma_{\lambda\sigma,\mu\nu}, \tag{3.24}$$

into a form convenient for integration. Following the same strategy as before in the 2-2-4 case, using many times partial integration and Bianchi identities (3.5), (3.8)-(3.10), we obtain after tedious but straightforward calculations

$$\begin{aligned}
\delta^1_\varepsilon\mathcal{L}_0(h^{(2)}_{\mu\nu}) &= \partial^{(\alpha}\epsilon^{\beta\mu\nu\lambda\rho)}\Psi^{(6)}_{(\Gamma)\alpha\beta\mu\nu\lambda\rho} - \partial^{(\alpha}\epsilon^{\beta\mu\nu)}\Psi^{(4)}_{(\Gamma)\alpha\beta\mu\nu}\\
&+\frac{4}{3}\partial^\rho\epsilon_\alpha{}^{\mu\nu\lambda\sigma}\partial_\lambda\partial_\sigma\Gamma_{\beta\rho,\mu\nu}\frac{\delta\mathcal{L}_0}{\delta h^{(2)}_{\alpha\beta}} - \frac{1}{3}\partial^\rho\partial^\lambda\epsilon_{\alpha\beta}{}^{\mu\nu\sigma}\partial_\sigma\Gamma_{\rho\lambda,\mu\nu}\frac{\delta\mathcal{L}_0}{\delta h^{(2)}_{\alpha\beta}}\\
&-R^{\mu\nu}_{int}(\Gamma,\mathcal{F})\frac{\delta\mathcal{L}_0}{\delta h^{(2)}_{\mu\nu}},
\end{aligned} \tag{3.25}$$

where

$$\begin{aligned}
\Psi^{(6)}_{(\Gamma)\alpha\beta\mu\nu\lambda\rho} &= \partial_{(\alpha}\Gamma_{\beta\mu,}{}^{\sigma\delta}\partial_\nu\Gamma_{\lambda\rho),\sigma\delta} - g_{(\alpha\beta}\partial_\mu\Gamma_\nu{}^{\kappa,\sigma\delta}\partial_\lambda\Gamma_{\rho)\kappa,\sigma\delta}\\
&-\frac{1}{2}g_{(\alpha\beta}\partial^\kappa\Gamma_{\mu\nu,}{}^{\sigma\delta}\partial_\sigma\Gamma_{\lambda\rho),\kappa\delta},
\end{aligned} \tag{3.26}$$

$$\Psi^{(4)}_{(\Gamma)\alpha\beta\mu\nu} = \Gamma_{(\alpha\beta,}{}^{\rho\sigma}\Gamma_{\mu\nu),\rho\sigma} - \frac{2}{3}g_{(\alpha\beta}\Gamma_\mu{}^{\rho,\sigma\lambda}\Gamma_{\nu)\rho,\sigma\lambda}, \tag{3.27}$$

and $R^{\mu\nu}_{int}(\Gamma,\mathcal{F})\frac{\delta\mathcal{L}_0}{\delta h^{(2)}_{\mu\nu}}$ are remaining integrable terms proportional to the equation of motion. Indeed the symmetric tensor $R^{\mu\nu}_{int}(\Gamma,\mathcal{F})$ is expressed through the only integrable combinations of derivatives of gauge parameter

$$\begin{aligned}
R^{\mu\nu}_{int}(\Gamma,\mathcal{F},\epsilon) \;=\;& \epsilon^{\alpha\beta\lambda\delta}_{(1)}\partial_\alpha\partial_\beta\Gamma_{\lambda\delta,}{}^{\mu\nu} - \frac{1}{3}\partial^\lambda\epsilon^{\alpha\beta\delta(\mu}_{(1)}\partial_\alpha\Gamma^{\nu)}{}_{\lambda,\beta\delta} + \partial_\lambda\left[\partial^{(\lambda}\epsilon^{\alpha\beta\delta\mu\nu)}\partial_\alpha\mathcal{F}_{\beta\delta}\right]\\
-\;& \frac{2}{3}\partial_\lambda\left[\epsilon^{\lambda\alpha\mu\nu}_{(1)}\partial_\alpha\mathcal{F}\right] + \frac{1}{6}\epsilon^{\alpha\beta\mu\nu}_{(1)}\partial_\alpha\partial_\beta\mathcal{F} + \partial^{(\alpha}\epsilon^{\beta\mu\nu)}_{(2)}\mathcal{F}_{\alpha\beta} + \frac{5}{3}\partial^\alpha\epsilon^{\beta\lambda\mu\nu)}_{(1)}\partial_\lambda\mathcal{F}_{\alpha\beta}\\
-\;& \frac{5}{3}\partial_\lambda\left[\epsilon^{\lambda\alpha\beta(\mu}_{(1)}\partial_\alpha\mathcal{F}^{\nu)}_\beta\right] + \frac{1}{6}\Box\epsilon^{\alpha\beta\mu\nu}_{(1)}\mathcal{F}_{\alpha\beta} - \frac{1}{6}\partial^\lambda\epsilon^{\alpha\beta\mu\nu}_{(1)}\partial_\lambda\mathcal{F}_{\alpha\beta} - \frac{1}{2}\epsilon^{\alpha(\mu}_{(3)}\mathcal{F}^{\nu)}_\alpha.
\end{aligned} \tag{3.28}$$

Substituting into this expression $\partial^{(\lambda}\epsilon^{\alpha\beta\delta\mu\nu)}$ with $\frac{1}{6}h^{(6)\lambda\alpha\beta\delta\mu\nu}$ , $\partial^{(\alpha}\epsilon^{\beta\mu\nu)}_{(2)}$ with $\frac{1}{4}h^{(4)\alpha\beta\mu\nu}$, and correspondingly $2\epsilon^{\alpha\beta\mu\nu}_{(1)}$ and $2\epsilon^{\alpha\beta}_{(3)}$ with their traces, we define a field redefinition for $h^{(2)\mu\nu}$

$$h^{(2)\mu\nu} \to h^{(2)\mu\nu} + R^{\mu\nu}_{int}(\Gamma,\mathcal{F},h^{(6)},h^{(4)}), \tag{3.29}$$

using which we can drop the third line in (3.25). The second line in (3.25) can be cancelled by the following deformation of the initial ansatz for the transformation (3.21)

$$\delta^1_\epsilon h^{(2)}_{\alpha\beta} = \epsilon^{\mu\nu\rho\lambda\sigma}\partial_\mu\partial_\nu\partial_\rho\Gamma_{\lambda\sigma,\alpha\beta} - \frac{4}{3}\partial^\rho\epsilon_\alpha{}^{\mu\nu\lambda\sigma}\partial_\lambda\partial_\sigma\Gamma_{\beta\rho,\mu\nu} + \frac{1}{3}\partial^\rho\partial^\lambda\epsilon_{\alpha\beta}{}^{\mu\nu\sigma}\partial_\sigma\Gamma_{\rho\lambda,\mu\nu}. \tag{3.30}$$

Thus we arrive at the promised result that the 2-2-6 interaction automatically includes also the 2-2-4 interaction constructed above, and the corresponding trilinear interaction Lagrangian is

$$\mathcal{L}_1(h^{(2)},h^{(4)},h^{(6)}) = -\frac{1}{6}h^{(6)\alpha\beta\mu\nu\lambda\rho}\Psi^{(6)}_{(\Gamma)\alpha\beta\mu\nu\lambda\rho} + \frac{1}{4}h^{(4)\alpha\beta\mu\nu}\Psi^{(4)}_{(\Gamma)\alpha\beta\mu\nu}$$
$$-\frac{1}{6}h^{(6)\alpha\beta\mu\nu\lambda\rho}\partial_\alpha\Gamma_{\beta\mu,}{}^{\sigma\delta}\partial_\nu\Gamma_{\lambda\rho,\sigma\delta} + \frac{1}{6}h^{(6)\alpha\mu\nu\lambda\rho}_{\alpha}\partial_\mu\Gamma_{\nu}{}^{\kappa,\sigma\delta}\partial_\lambda\Gamma_{\rho\kappa,\sigma\delta}$$
$$+\frac{1}{12}h^{(6)\alpha\mu\nu\lambda\rho}_{\alpha}\partial^\kappa\Gamma_{\mu\nu,}{}^{\sigma\delta}\partial_\sigma\Gamma_{\lambda\rho),\kappa\delta} + \frac{1}{4}h^{(4)\alpha\beta\mu\nu}\Gamma_{\alpha\beta,\rho\sigma}\Gamma_{\mu\nu,}{}^{\rho\sigma} - \frac{1}{6}h^{(4)\alpha\mu\nu}_{\alpha}\Gamma_{\mu}{}^{\rho,\sigma\lambda}\Gamma_{\nu\rho,\sigma\lambda}. \quad (3.31)$$

This formula together with the corrected gauge transformation (3.30) solves completely Noether's equation (3.20).

# 4 2s-s-s interaction Lagrangian

The most elegant and convenient way of handling symmetric tensors such as $h^{(s)}_{\mu_1\mu_2\ldots\mu_s}(z)$ is by contracting it with the $s$'th tensorial power of a vector $a^\mu$ of the tangential space at the base point $z$ [12]-[16]

$$h^{(s)}(z;a) = \sum_{\mu_i}(\prod_{i=1}^{s} a^{\mu_i})h^{(s)}_{\mu_1\mu_2\ldots\mu_s}(z). \quad (4.1)$$

In this way we obtain a homogeneous polynomial in the vector $a^\mu$ of degree $s$. In this formalism the symmetrized gradient, trace and divergence are[‡]

$$Grad: h^{(s)}(z;a) \Rightarrow Grad h^{(s+1)}(z;a) = (a\nabla)h^{(s)}(z;a), \quad (4.2)$$
$$Tr: h^{(s)}(z;a) \Rightarrow Tr h^{(s-2)}(z;a) = \frac{1}{s(s-1)}\Box_a h^{(s)}(z;a), \quad (4.3)$$
$$Div: h^{(s)}(z;a) \Rightarrow Div h^{(s-1)}(z;a) = \frac{1}{s}(\nabla\partial_a)h^{(s)}(z;a). \quad (4.4)$$

The gauge variation of a spin $s$ field is

$$\delta h^{(s)}(z;a) = s(a\nabla)\epsilon^{(s-1)}(z;a), \quad (4.5)$$

with traceless gauge parameter

$$\Box_a\epsilon^{(s-1)}(z;a) = 0, \quad (4.6)$$

for the double traceless gauge field

$$\Box_a^2 h^{(s)}(z;a) = 0. \quad (4.7)$$

We will use the deWit-Freedman curvature and Cristoffel symbols [18, 19]. We contract them with the degree $s$ tensorial power of one tangential vector $a^\mu$ in the first set of s

[‡]To distinguish easily between "a" and "z" spaces we introduce for space-time derivatives $\frac{\partial}{\partial z^\mu}$ the notation $\nabla_\mu$ and as before we will admit integration everywhere where it is necessary (we work with a Lagrangian as with an action) and therefore we will neglect all space-time total derivatives when making a partial integration

indices and with a similar tensorial power of another tangential vector $b^\nu$ in its second set. The deWit-Freedman curvature and n-th Cristoffel symbol are then written as

$$\Gamma^{(s)}(z;b,a): \qquad \Gamma^{(s)}(z;b,\lambda a)=\Gamma^{(s)}(z;\lambda b,a)=\lambda^s\Gamma^{(s)}(z;b,a), \tag{4.8}$$

$$\Gamma^{(s)}_{(n)}(z;b,a): \qquad \Gamma^{(s)}_{(n)}(z;b,\lambda a)=\lambda^s\Gamma^{(s)}_{(n)}(z;b,a), \tag{4.9}$$

$$\Gamma^{(s)}_{(n)}(z;\lambda b,a)=\lambda^n\Gamma^{(s)}_{(n)}(z;b,a), \tag{4.10}$$

$$\Gamma^{(s)}(z;b,a)=\Gamma^{(s)}_{(n)}(z;b,a)|_{n=s}. \tag{4.11}$$

Next we introduce the notation $*_a, *_b$ for a contraction in the symmetric spaces of indices $a$ or $b$

$$*_a \quad = \quad \frac{1}{(s!)^2}\prod_{i=1}^{s}\overleftarrow{\partial}^{\mu_i}_{a}\overrightarrow{\partial}^{a}_{\mu_i}. \tag{4.12}$$

To manipulate reshuffling of different sets of indices we employ other differentials with respect to $a$ and $b$, e.g.

$$A_b=(a\partial_b), \tag{4.13}$$

$$B_a=(b\partial_a). \tag{4.14}$$

Then we see that operators $A_b, a^2, b^2$ are dual (or adjoint) to $B_a, \Box_a, \Box_b$ with respect to the "star" product (4.12)

$$A_b f(a,b) *_{a,b} g(a,b) \quad = \quad f(a,b) *_{a,b} B_a g(a,b), \tag{4.15}$$

$$\begin{pmatrix} a^2 \\ b^2 \end{pmatrix} f(a,b) *_{a,b} g(a,b) \quad = \quad f(a,b) *_{a,b} \begin{pmatrix} \Box_a \\ \Box_b \end{pmatrix} g(a,b). \tag{4.16}$$

In the same fashion gradients and divergences are dual with respect to the full scalar product in the space $(z,a,b)$

$$\begin{matrix}(a\nabla)\\(b\nabla)\end{matrix} f(z;a,b) *_{a,b} g(z;a,b) \quad = \quad -f(z;a,b) *_{a,b} \begin{matrix}(\nabla\partial_a)\\(\nabla\partial_b)\end{matrix} g(z;a,b). \tag{4.17}$$

Now one can prove that [18, 16]:

$$A_b\Gamma^{(s)}(z;a,b)=B_a\Gamma^{(s)}(z;a,b)=0. \tag{4.18}$$

These "primary Bianchi identities" are manifestations of the hidden antisymmetry. The n-th deWit-Freedman-Cristoffel symbol is

$$\Gamma^{(s)}_{(n)}(z;b,a)\equiv\Gamma^{(s)}_{(n)\rho_1\ldots\rho_n,\mu_1\ldots\mu_\ell}b^{\rho_1}\ldots b^{\rho_n}a^{\mu_1}\ldots a^{\mu_\ell}=[(b\nabla)-\frac{1}{n}(a\nabla)B_a]\Gamma^{(s)}_{(n-1)}(z;b,a), \tag{4.19}$$

or in another way

$$\Gamma^{(s)}_{(n)}(z;b,a)=(\prod_{k=1}^{s}[(b\nabla)-\frac{1}{k}(a\nabla)B_a])h^{(s)}(z;a). \tag{4.20}$$

Using the following commutation relations

$$
\begin{aligned}
&[B_a,(a\nabla)] = (b\nabla), &(4.21)\\
&[B_a^k,(a\nabla)] = kB_a^{k-1}(b\nabla), &(4.22)\\
&[B_a,(a\nabla)^k] = k(b\nabla)(a\nabla)^{k-1}, &(4.23)\\
&\Box_b(b\nabla)^i = i(i-1)(b\nabla)^{i-2}\Box, &(4.24)\\
&\partial_\mu^b(b\nabla)^i\partial_b^\mu B_a^j = ij(b\nabla)^{i-1}B_a^{j-1}(\nabla\partial_a), &(4.25)\\
&\Box_b B_a^j = j(j-1)B_a^{j-2}\Box_a, &(4.26)
\end{aligned}
$$

and mathematical induction we can prove that

$$\Gamma^{(s)}_{(n)}(z;b,a) = \sum_{k=0}^{n}\frac{(-1)^k}{k!}(b\nabla)^{n-k}(a\nabla)^k B_a^k h^{(s)}(z;a). \tag{4.27}$$

The gauge variation of n-th Cristoffel symbol is

$$\delta\Gamma^{(s)}_{(n)}(z;b,a) = \frac{(-1)^n}{n!}(a\nabla)^{n+1}B_a^n\epsilon^{(s-1)}(z;a), \tag{4.28}$$

putting here $n = s$ we obtain gauge invariance for the curvature

$$\delta\Gamma^{(s)}_{(s)}(z;b,a) = 0. \tag{4.29}$$

Tracelessness of the gauge parameter (4.6) implies that b-traces of all Cristoffel symbols are gauge invariant

$$\Box_b\delta\Gamma^{(s)}_{(n)}(z;b,a) = \frac{(-1)^n}{(n-2)!}(a\nabla)^{n+1}B_a^{n-2}\Box_a\epsilon^{(s-1)}(z;a) = 0. \tag{4.30}$$

Thus for the second order gauge invariant field equation we can use the trace of the second Cristoffel symbol, the so called Fronsdal tensor:

$$
\begin{aligned}
\mathcal{F}^{(s)}(z;a) &= \frac{1}{2}\Box_b\Gamma^{(s)}_{(2)}(z;b,a)\\
&= \Box h^{(s)}(z;a) - (a\nabla)(\nabla\partial_a)h^{(s)}(z;a) + \frac{1}{2}(a\nabla)^2\Box_a h^{(s)}(z;a). \qquad (4.31)
\end{aligned}
$$

Using equation (4.27) for Cristoffel symbols and after long calculations we obtain the following expression

$$\Box_b\Gamma^{(s)}_{(n)}(z;b,a) = \sum_{k=0}^{n-2}\frac{(-1)^k}{k!}(n-k)(n-k-1)(b\nabla)^{n-k-2}(a\nabla)^k B_a^k\mathcal{F}^{(s)}(z;a). \tag{4.32}$$

We have expressed the b-trace of any $\Gamma^{(s)}_{(n)}$ through the Fronsdal tensor or the b-trace of the second Cristoffel symbol, but this is not the whole story. Using mathematical induction and (4.21)-(4.26) again we can show that

$$
\begin{aligned}
&\sum_{k=0}^{n-2}\frac{(-1)^k}{k!}(n-k)(n-k-1)(b\nabla)^{n-k-2}(a\nabla)^k B_a^k\mathcal{F}^{(s)}(z;a)\\
&\qquad = n(n-1)\Big(\prod_{k=3}^{n}[(b\nabla)-\frac{1}{k}(a\nabla)B_a]\Big)\mathcal{F}^{(s)}(z;a). \qquad (4.33)
\end{aligned}
$$

In particular for the trace of the curvature we can write

$$\Box_b \Gamma^{(s)}(z;b,a) = s(s-1)\mathcal{U}(a,b,3,s)\mathcal{F}^{(s)}(z;a), \tag{4.34}$$

where we introduced an operator mapping the Fronsdal tensor on the trace of the curvature

$$\mathcal{U}(a,b,3,s) = \prod_{k=3}^{s}[(b\nabla) - \frac{1}{k}(a\nabla)B_a]. \tag{4.35}$$

Now let us consider this curvature in more detail. First we have the symmetry under exchange of $a$ and $b$

$$\Gamma^{(s)}(z;a,b) = \Gamma^{(s)}(z;b,a). \tag{4.36}$$

Therefore the operation "$a$-trace" can be defined by (4.34) with exchange of $a$ and $b$ at the end. The mixed trace of the curvature can be expressed through the $a$ or $b$ traces using "primary Bianchi identities" (4.18)

$$(\partial_a\partial_b)\Gamma^{(s)}(z;b,a) = -\frac{1}{2}B_a\Box_b\Gamma^{(s)}(z;b,a) = -\frac{1}{2}A_b\Box_a\Gamma^{(s)}(z;b,a). \tag{4.37}$$

The next interesting properties of the higher spin curvature and corresponding Ricci tensors are so called generalized secondary or differential Bianchi identities. We can formulate these identities in our notation in the following compressed form ([...] denotes antisymmetrization )

$$\frac{\partial}{\partial a^{[\mu}}\frac{\partial}{\partial b^{\nu}}\nabla_{\lambda]}\Gamma^{(s)}(z;a,b) = 0. \tag{4.38}$$

This relation can be checked directly from representation (4.27). Then contracting with $a^\mu$ and $b^\nu$ we get a symmetrized form of (4.38)

$$s\nabla_\mu\Gamma^{(s)}(z;a,b) = (a\nabla)\partial^a_\mu\Gamma^{(s)}(z;a,b) + (b\nabla)\partial^b_\mu\Gamma^{(s)}(z;a,b). \tag{4.39}$$

Now we can contract (4.39) with a $\partial_b^\mu$ and using (4.37) obtain a connection between the divergence and the trace of the curvature

$$(s-1)(\nabla\partial_b)\Gamma^{(s)}(z;a,b) = [(b\nabla) - \frac{1}{2}(a\nabla)B_a]\Box_b\Gamma^{(s)}(z;a,b). \tag{4.40}$$

These two identities with a similar identity for the Fronsdal tensor

$$(\nabla\partial_a)\mathcal{F}^{(s)}(z;a) = \frac{1}{2}(a\nabla)\Box_a\mathcal{F}^{(s)}(z;a), \tag{4.41}$$

play an important role for the construction of the interaction Lagrangian. To complete the free field information we present here Fronsdal's Lagrangian in terms of our quantities:

$$\begin{aligned}\mathcal{L}_0(h^{(s)}(a)) &= \frac{1}{2}\nabla_\mu h^{(s)}(a) *_a \nabla^\mu h^{(s)}(a) - \frac{1}{2}(\nabla\partial_a)h^{(s)}(a) *_a (\nabla\partial_a)h^{(s)}(a)\\ &+\frac{1}{2}(\nabla\partial_a)h^{(s)}(a) *_a (a\nabla)\Box_a h^{(s)}(a) - \frac{1}{4}\nabla_\mu\Box_a h^{(s)}(a) *_a \nabla^\mu\Box_a h^{(s)}(a)\\ &-\frac{1}{8}(\nabla\partial_a)\Box_a h^{(s)}(a) *_a (\nabla\partial_a)\Box_a h^{(s)}(a).\end{aligned} \tag{4.42}$$

The same Lagrangian can be written in the following compact form

$$\mathcal{L}_0(h^{(s)}(a)) = -\frac{1}{2}h^{(s)}(a) *_a \mathcal{F}^{(s)}(a) + \frac{1}{8}\Box_a h^{(s)}(a) *_a \Box_a\mathcal{F}^{(s)}(a). \tag{4.43}$$

To obtain the equation of motion we vary (4.42) or (4.43) and obtain

$$\delta\mathcal{L}_0(h^{(s)}(a)) = -(\mathcal{F}^{(s)}(a) - \frac{a^2}{4}\Box_a\mathcal{F}^{(s)}(a)) *_a \delta h^{(s)}(a). \tag{4.44}$$

Zero order gauge invariance can be checked easily by substitution of (4.5) into this variation and use of the duality relation (4.17) and identity (4.41) taking into account tracelessness of the gauge parameter (4.6). Now we turn to the generalization of the Noether procedure of the 2-2-4 case to the general s-s-2s interaction construction. So we must propose a first order variation of the spin s field with respect to a spin 2s gauge transformation. Remembering that Fronsdal's higher spin gauge potential is double traceless, we must make sure that the same holds for the variation. Expanding the general variation in powers of $a^2$

$$\delta h^{(s)}(a) = \delta h^{(s)}_{(1)}(a) + a^2\delta h^{(s-2)}(a) + (a^2)^2\delta h^{(s-4)}(a) + \dots, \tag{4.45}$$

we see that the double tracelessness condition $\Box^2_a\delta h^{(s)}(a) = 0$ expresses the third and higher terms of the expansion (4.45) through the first two free parameters $\delta h^{(s)}_{(1)}(a)$ and $\delta h^{(s-2)}(a)$[§]. From the other hand Fronsdal's tensor is double traceless by definition and therefore all these $O(a^4)$ terms are unimportant because they do not contribute to (4.44). This leaves us freedom in the choice of $\delta h^{(s-2)}(a)$. Substituting (4.45) in (4.44) we discover that the following choice of $\delta h^{(s-2)}(a)$

$$\delta h^{(s-2)}(a) = \frac{1}{2(D+2s-2)}\Box_a\delta h^{(s)}_{(1)}(a), \tag{4.46}$$

reduces our variation (4.44) to

$$\delta_{(1)}\mathcal{L}_0(h^{(s)}(a)) = -\mathcal{F}^{(s)}(a) *_a \delta h^{(s)}_{(1)}(a). \tag{4.47}$$

Then we propose the following spin 2s transformation of the spin s potential

$$\delta h^{(s)}_{(1)}(a) = 2s\tilde{\mathcal{U}}(b,a,2,s)\epsilon^{2s-1}(z;b) *_b \Gamma^{(s)}(z;b,a), \tag{4.48}$$

where

$$\tilde{\mathcal{U}}(b,a,2,s) = \prod_{k=2}^{s}\left[(\nabla\partial_b) - \frac{1}{k}A_b(\nabla\partial_a)\right], \tag{4.49}$$

is operator dual to

$$[(b\nabla) - \frac{1}{2}(a\nabla)B_a]\mathcal{U}(b,a,3,s) = \prod_{k=2}^{s}[(b\nabla) - \frac{1}{k}(a\nabla)B_a], \tag{4.50}$$

[§]For completeness we present here the solution for $\delta h^{(s-4)}(a)$ following from the double tracelessness condition

$$\delta h^{(s-4)}(a) = -\frac{1}{8\alpha_1\alpha_2}\left[\Box^2_a\delta h^{(s)}_{(1)}(a) + 4\alpha_1\Box_a\delta h^{(s-2)}(a)\right],$$
$$\alpha_k = D + 2s - (4+2k), \quad k \in \{1,2\}.$$

with respect to the $*_{a,b}$ contraction product. Taking into account (4.34) and Bianchi identities (4.40) we get

$$\begin{aligned}
&\delta_{(1)}\mathcal{L}_0(h^{(s)}(a)) = 2\epsilon^{2s-1}(z;b) *_b \Gamma^{(s)}(z;b,a) *_a s[(b\nabla) - \frac{1}{2}(a\nabla)B_a]\mathcal{U}(b,a,3,s)\mathcal{F}^{(s)}(z;a)\\
&= 2\epsilon^{2s-1}(z;b) *_b \Gamma^{(s)}(z;b,a) *_a \frac{1}{s-1}[(b\nabla) - \frac{1}{2}(a\nabla)B_a]\Box_b\Gamma^{(s)}(z;b,a)\\
&= 2\epsilon^{2s-1}(z;b) *_b \Gamma^{(s)}(z;b,a) *_a (\nabla\partial_b)\Gamma^{(s)}(z;b,a)\\
&= -(b\nabla)\epsilon^{2s-1}(b) *_b \Gamma^{(s)}(b,a) *_a \Gamma^{(s)}(b,a) - 2\epsilon^{2s-1}(b) *_b \nabla_\mu\Gamma^{(s)}(b,a) *_a \partial_b^\mu\Gamma^{(s)}(b,a). \qquad (4.51)
\end{aligned}$$

Then using a secondary Bianchi identity (4.39) and a primary one (4.18) one can show that

$$\begin{aligned}
&-2\epsilon^{2s-1}(b) *_b \nabla_\mu\Gamma^{(s)}(b,a) *_a \partial_b^\mu\Gamma^{(s)}(b,a)\\
&\qquad = \frac{1}{s+1}(\nabla\partial_b)\epsilon^{2s-1}(b) *_b \partial_\mu^b\Gamma^{(s)}(b,a) *_a \partial_b^\mu\Gamma^{(s)}(b,a). \qquad (4.52)
\end{aligned}$$

Putting all together we see that the integrated first order interaction Lagrangian

$$\mathcal{L}_1(h^{(s)}(a), h^{(2s)}(b)) = \frac{1}{2s}h^{(2s)}(z;b) *_b \Psi^{(2s)}_{(\Gamma)}(z;b), \qquad (4.53)$$

$$\Psi^{(2s)}_{(\Gamma)}(z;b) = \Gamma^{(s)}(b,a) *_a \Gamma^{(s)}(b,a) - \frac{a^2}{2(s+1)}\partial_\mu^b\Gamma^{(s)}(b,a) *_a \partial_b^\mu\Gamma^{(s)}(b,a). \quad (4.54)$$

supplemented with transformation (4.48) for $h^{(s)}(a)$ and the standard zero order for $h^{(2s)}(a)$

$$\delta_0 h^{(2s)}(z;b) = 2s(b\nabla)\epsilon^{(2s-1)}(z;b), \qquad (4.55)$$

$$\delta_0\Box_b h^{(2s)}(z;b) = 4s(\nabla\partial_b)\epsilon^{(2s-1)}(z;b), \qquad (4.56)$$

completely solves Noether's equation

$$\delta_{(1)}\mathcal{L}_0(h^{(s)}(a)) + \delta_0\mathcal{L}_1(h^{(s)}(a), h^{(2s)}(b)) = 0. \qquad (4.57)$$

Note that here just as in the 2-2-4 case we did not obtain an interaction with lower spins because all derivatives included in the ansatz were used for the lifting to the second curvature.

# 5 Cubic self-interaction of even higher spin fields

Though it is desirable to use curvatures and Christoffel symbols to express also local manifestly covariant self-interactions of higher spin fields, such ansatz has not yet been successful. Of course cubic self-interactions can be built from three curvatures by tensorial contractions, but this ansatz is in this context considered to be trivial because of the great number of derivatives (3s for spin s) it contains. A minimally improved ansatz with 3s-2 derivatives has been shown to exist for all s [22] by deriving it by a QFT calculation. Extending known examples for spin 2 and 3 [3], we would like to construct the interaction with s derivatives for spin s. To construct such interaction we have developed an algorithm displayed in the remainder of this article.

As in the classical work [3] the presumed interaction Lagrangian $\mathcal{L}_1$ gives by variation the current

$$J(z;a) = \frac{\delta}{\delta h(z;a)}\mathcal{L}_1, \tag{5.1}$$

and this current is as a consequence of gauge invariance required to be conserved "on-shell" modulo $a^2$

$$(\nabla \cdot \partial_a)J(z;a) = j_0(z;a) + a^2 j_1(z;a) + O((a^2)^2), \tag{5.2}$$

$$j_0(z;a) = \hat{\mathcal{Y}}(a,b;\nabla) *_b \hat{\mathcal{F}^{(s)}}(z;b) + \mathcal{R}(z;a), \tag{5.3}$$

and $\hat{\mathcal{F}^{(s)}}$ is the first variation of the Lagrangian $\mathcal{L}_0$

$$\hat{\mathcal{F}^{(s)}}(z;b) = \mathcal{F}^{(s)}(z;b) - \frac{1}{4}b^2\mathcal{F}^{(s)}(z;b), \tag{5.4}$$

where $\mathcal{F}^{(s)}(z;b)$ is Fronsdal's operator (4.31). Thus the remainder $\mathcal{R}(z;a)$ of the coset decomposition of $j_0(z;a)$ is postulated to vanish.

There are two provisos to be made. We will in this analysis use deDonder gauge throughout. Use of free gauge did not allow us to produce a recursive algorithm. Moreover we will make the coset decomposition with respect to Fronsdal's $\mathcal{F}^{(s)}$ indeed, it is later possible to correct for this technical defect with not much effort. Let us now describe the basic concepts of the algorithm.

We consider one specific term in the large number of cubic interactions of supposedly general form

$$\begin{aligned}
\mathcal{L}_{1,specific} \quad &= A\int dz_1dz_2dz_3\delta(z_1-z_2)\delta(z_1-z_3)(\partial_a\partial_b)^{Q_{12}}(\partial_a\partial_c)^{Q_{13}}(\partial_b\partial_c)^{Q_{23}} \\
&\Box_a^{\delta_1}\Box_b^{\delta_2}\Box_c^{\delta_3}(\nabla_1\nabla_2)^{\omega_{12}}(\nabla_1\nabla_3)^{\omega_{13}}(\nabla_2\nabla_3)^{\omega_{23}} \\
&(\partial_a\nabla_2)^{\alpha_2}(\partial_a\nabla_3)^{\alpha_3}(\partial_b\nabla_2)^{\beta_2}(\partial_b\nabla_3)^{\beta_3}(\partial_c\nabla_2)^{\gamma_2}(\partial_c\nabla_3)^{\beta_3} \\
&h(z_1;a)h(z_2;b)h(z_3;c).
\end{aligned} \tag{5.5}$$

All space derivatives can be applied to the second and third field factors if $\nabla_1$ is replaced by $-\nabla_2 - \nabla_3$. The factor $A$ is a kind of a coupling constant, it can be characterized as a function of all exponents

$$A = A(\alpha_2,\alpha_3;\beta_2,\beta_3;\gamma_2,\gamma_3 \mid \delta_1,\delta_2,\delta_3 \mid \omega_{12},\omega_{13},\omega_{23}). \tag{5.6}$$

The domain on which these functions are defined is the first topic to be studied.

First we have

$$\delta_{1,2,3} \in \{0,1\}, \tag{5.7}$$

since the double trace of the fields $h$ vanishes. Moreover we use the shorthands

$$\alpha = \alpha_2 + \alpha_3; \quad \beta = \beta_2 + \beta_3, \quad \gamma = \gamma_2 + \gamma_3; \tag{5.8}$$

$$\omega = \omega_{12} + \omega_{13} + \omega_{23}; \tag{5.9}$$

and denote the fixed degree in the space derivatives by $\Delta$. All exponents are obviously nonnegative.

The balancing equations for these exponents are

$$\Delta = \alpha+\beta+\gamma+2\omega, \tag{5.10}$$

$$s = Q_{12} + Q_{13} + \alpha + 2\delta_1, \tag{5.11}$$

$$s = Q_{12} + Q_{23} + \beta + 2\delta_2, \tag{5.12}$$

$$s = Q_{13} + Q_{23} + \gamma + 2\delta_3, \tag{5.13}$$

Introducing once again shorthands by

$$\begin{aligned}\sigma_1 &= s - 2\delta_1 - \alpha \geq 0, &(5.14)\\ \sigma_2 &= s - 2\delta_2 - \beta \geq 0, &(5.15)\\ \sigma_3 &= s - 2\delta_3 - \gamma \geq 0, &(5.16)\end{aligned}$$

we can express the exponents $Q_{ij}$ by

$$\begin{aligned}Q_{12} &= 1/2(\sigma_1 + \sigma_2 - \sigma_3), &(5.17)\\ Q_{13} &= 1/2(\sigma_1 + \sigma_3 - \sigma_2), &(5.18)\\ Q_{23} &= 1/2(\sigma_2 + \sigma_3 - \sigma_1). &(5.19)\end{aligned}$$

These $Q_{ij}$ must be positive which entails triangular inequalities for the $\sigma_{1,2,3}$. Inserting then (5.14) - (5.16) into (5.17) - (5.19) and identifying

$$\Delta = s, \tag{5.20}$$

we obtain the explicit formulae

$$\begin{aligned}Q_{12} &= \omega + \gamma - \delta_{12,3}, &(5.21)\\ Q_{13} &= \omega + \beta - \delta_{13,2}, &(5.22)\\ Q_{23} &= \omega + \alpha - \delta_{23,1}, &(5.23)\end{aligned}$$

where e.g.

$$\delta_{12,3} = \delta_1 + \delta_2 - \delta_3, \tag{5.24}$$

which takes values between $-1$ and $+2$. We obtain

$$\begin{aligned}\alpha &\geq max\{0, \delta_{23,1} - \omega\}, &(5.25)\\ \beta &\geq max\{0, \delta_{13,2} - \omega\}, label5.26 &(5.26)\\ \gamma &\geq max\{0, \delta_{12,3} - \omega\}. &(5.27)\end{aligned}$$

The identification (5.20) is tentative and the proof that a cubic invariant form does exist under this condition is one issue of this investigation.

When we take the variation of $\mathcal{L}_{1,specific}$ with respect to the field $h$ we get three terms. The first is

$$\begin{aligned}&X(\alpha_2, \alpha_3; \beta_2, \beta_3; \gamma_2, \gamma_3 \mid \delta_1, \delta_2, \delta_3 \mid \omega_{12}, \omega_{13}, \omega_{23}) = (a^2)^{\delta_1} \Box_b^{\delta_2} \Box_c^{\delta_3} (a\partial_b)^{Q_{12}} (a\partial_c)^{Q_{13}}\\ &\times (\partial_b \partial_c)^{Q_{23}} (a\partial_2)^{\alpha_2} (a\partial_3)^{\alpha_3} (\partial_b \partial_2)^{\beta_2} (\partial_b \partial_3)^{\beta_3} (\partial_c \partial_2)^{\gamma_2} (\partial_c \partial_3)^{\gamma_3} (-\Box_2 - \partial_2 \cdot \partial_3)^{\omega_{12}}\\ &\times (-\Box_3 - \partial_2 \cdot \partial_3)^{\omega_{13}} (\partial_2 \partial_3)^{\omega_{23}} h(z_2; b) h(z_3; c) \mid_{z_2 = z_3 = z} \quad . \end{aligned} \tag{5.28}$$

This being the variation with respect to the first factor $h$, we take next the variation with respect to the second factor renaming the internal ("dull") variables $a, b, c$ and $z_1, z_2, z_3$ and performing the necessary partial integrations to get rid of the space derivatives on the field which is varied

$$c \to b \to a \to c; \quad 3 \to 2 \to 1 \to 3, \tag{5.29}$$

but maintain all exponents $\alpha, \beta, \gamma, \delta, \omega$. Finally we do the same with the third factor in the Lagrangian applying the inverse (equal the square) of the permutation (5.29). Together we obtain the sum of three expressions $X(..;..;..\mid ...\mid ...)$

$$
\begin{aligned}
& X(\alpha_2,\alpha_3;\beta_2,\beta_3;\gamma_2,\gamma_3\mid\delta_1,\delta_2,\delta_3\mid\omega_{12},\omega_{13},\omega_{23}) \\
& +(-1)^{\alpha_2+\beta_2+\gamma_2}\sum_{k_\alpha,k_\beta,k_\gamma}\binom{\alpha_2}{k_\alpha}\binom{\beta_2}{k_\beta}\binom{\gamma_2}{k_\gamma}X(\beta-k_\beta,k_\beta;\gamma-k_\gamma,k_\gamma;\alpha-k_\alpha,k_\alpha \\
& \mid\delta_3,\delta_1,\delta_2\mid\omega_{23},\omega_{12},\omega_{13})+(-1)^{\alpha_3+\beta_3+\gamma_3}\sum_{k_\alpha,k_\beta,k_\gamma}\binom{\alpha_3}{k_\alpha}\binom{\beta_3}{k_\beta}\binom{\gamma_3}{k_\gamma} \\
& X(k_\gamma,\gamma-k_\gamma;k_\alpha,\alpha-k_\alpha;k_\beta,\beta-k_\beta\mid\delta_2,\delta_3,\delta_1\mid\omega_{13},\omega_{23},\omega_{12}).
\end{aligned} \tag{5.30}
$$

If we return to the current we can expand it as

$$
\begin{aligned}
J(z;a) = & \sum_{domain(A)} A(\alpha_2,\alpha_3;\beta_2,\beta_3;\gamma_2,\gamma_3\mid\delta_1,\delta_2,\delta_3\mid\omega_{12},\omega_{13},\omega_{23}) \\
& X(\alpha_2,\alpha_3;\beta_2,\beta_3;\gamma_2,\gamma_3\mid\delta_1,\delta_2,\delta_3\mid\omega_{12},\omega_{13},\omega_{23}),
\end{aligned} \tag{5.31}
$$

and the question arises under which condition this current can be integrated over $h$ to obtain a Lagrangian.

The answer to this question is simple: the integrability conditions are dual to the relation (5.30) between the basis elements. In explicit terms the integrability necessitates that amplitudes form triplets and the relations valid inside the triplets are

$$
\begin{aligned}
& A(\beta-m_\beta,m_\beta;\gamma-m_\gamma,m_\gamma;\alpha-m_\alpha,m_\alpha\mid\delta_3,\delta_1,\delta_2\mid\omega_{23},\omega_{12},\omega_{13}) \\
& =\sum_{n_\alpha,n_\beta,n_\gamma}(-1)^{n_\alpha+n_\beta+n_\gamma}\binom{n_\alpha}{m_\alpha}\binom{n_\beta}{m_\beta}\binom{n_\gamma}{m_\gamma} \\
& A(n_\alpha,\alpha-n_\alpha;n_\beta,\beta-n_\beta;n_\gamma,\gamma-n_\gamma\mid\delta_1,\delta_2,\delta_3\mid\omega_{12},\omega_{13},\omega_{23}).
\end{aligned} \tag{5.32}
$$

A second relation connects two $A$ with the inverse permutation of arguments.

The triangular matrix used in (5.32)

$$
P_{m,n}=(-1)^n\binom{n}{m}, 0\le m\le n\le N, \tag{5.33}
$$

has the property

$$
P^2=\mathbf{1}, \tag{5.34}
$$

and therefore typically represents a transposition (or reflection). To the integrability constraints belongs also an almost trivial exchange symmetry between the two fields contained in the current ("$2,3$-symmetry"). It has the form

$$
\begin{aligned}
& A(\alpha_2,\alpha_3;\beta_2,\beta_3;\gamma_2,\gamma_3\mid\delta_1,\delta_2,\delta_3\mid\omega_1,\omega_2,\omega_3) \\
& =A(\alpha_3,\alpha_2;\gamma_3,\gamma_2;\beta_3,\beta_2\mid\delta_1,\delta_3,\delta_2\mid\omega_1,\omega_3,\omega_2).
\end{aligned} \tag{5.35}
$$

Using the basis (5.28) on shell, the product over the derivatives $(\nabla_i\nabla_j)^{\omega_{ij}}$ degenerates into $(\nabla_2\nabla_3)^\omega$ (5.13). So in the recursion equations only this exponent $\omega$ appears. If we reconstruct the interaction Lagrangian at the end we face the ambiguity under

$$
(\nabla_2\nabla_3)^\omega\to\prod_{ij}(\nabla_i\nabla_j)^{\omega_{ij}}. \tag{5.36}
$$

However the Laplacians appearing are irrelevant since they can be (and should be) removed by field redefinitions.

Requiring $\mathcal{R}(z;a)$ to vanish on shell we use the basis (5.28) with the simplification $\{\omega_{ij}\} \to \omega$, this we will address as the "canonical" basis. Then quartets of labels $(\alpha, \beta, \gamma \mid \omega)$ with (5.10)

$$\alpha + \beta + \gamma + 2\omega = \Delta + 1, \tag{5.37}$$

characterize each recursion equation. These recursion equations are

$$\begin{aligned}
&\alpha_2 A(\alpha_2 + 1, \alpha_3; \beta_2, \beta_3; \gamma_2, \gamma_3 \mid \delta_1, \delta_2, \delta_3 \mid \omega - 1) \\
&+\alpha_3 A(\alpha_2, \alpha_3 + 1; \beta_2, \beta_3; \gamma_2, \gamma_3 \mid \delta_1, \delta_2, \delta_3 \mid \omega - 1) \\
&+Q_{12}[A(\alpha_2, \alpha_3; \beta_2 - 1, \beta_3; \gamma_2, \gamma_3 \mid \delta_1, \delta_2, \delta_3 \mid \omega) \\
&+A(\alpha_2, \alpha_3; \beta_2, \beta_3 - 1; \gamma_2, \gamma_3 \mid \delta_1, \delta_2, \delta_3 \mid \omega)] \\
&+Q_{13}[A(\alpha_2, \alpha_3; \beta_2, \beta_3; \gamma_2 - 1, \gamma_3 \mid \delta_1, \delta_2, \delta_3 \mid \omega) \\
&+A(\alpha_2, \alpha_3; \beta_2, \beta_3; \gamma_2, \gamma_3 - 1 \mid \delta_1, \delta_2, \delta_3 \mid \omega)] \\
&+2(\delta_1 + 1)[A(\alpha_2 - 1, \alpha_3; \beta_2, \beta_3; \gamma_2, \gamma_3 \mid \delta_1 + 1, \delta_2 \delta_3 \mid \omega) \\
&+A(\alpha_2, \alpha_3 - 1; \beta_2, \beta_3; \gamma_2, \gamma_3 \mid \delta_1 + 1, \delta_2, \delta_3 \mid \omega)] = 0.
\end{aligned} \tag{5.38}$$

It is obvious that the variables $\delta_2, \delta_3$ are fixed in all entries of this equation. Let us assume them to be fixed now. Then the function $A$ with prescribed arguments $\alpha, \beta, \gamma, \delta_1, \omega$ is regarded as an element of a vector space $B(\alpha, \beta, \gamma \mid \delta_1, \omega)$. Since $\alpha_2, \beta_2, \gamma_2$ vary and enumerate the components of the vectors of this space, it has dimension $(\alpha+1)(\beta+1)(\gamma+1)$. The recursion defines linear relations between the vectors of a quartet of spaces

$$\begin{aligned}
&B(\alpha + 1, \beta, \gamma \mid \delta_1, \omega - 1), B(\alpha, \beta - 1, \gamma \mid \delta_1, \omega), \\
&B(\alpha, \beta, \gamma - 1 \mid \delta_1, \omega), B(\alpha - 1, \beta, \gamma \mid \delta_1 + 1, \omega).
\end{aligned} \tag{5.39}$$

The domain $\mathcal{D}[\delta_2, \delta_3]$ is defined to consist of those points (base points) on which a vector space (fibre) $B(\alpha, \beta, \gamma \mid \delta_1, \omega)$ exists. Two base points can be connected by a line if both appear in the same recursion equation. This decomposes the domain $\mathcal{D}[\delta_2, \delta_3]$ into several connected graphs. Actually some base points may not appear in any recursion equation. To each connected graph we have a linear homogeneous equation system for the vectors of its spaces. Therefore for each graph there is a linear space of solutions. Using the canonical basis the direct sum of all these can be projected on four subspaces with fixed $n$

$$\delta_1 + \delta_2 + \delta_3 = n \in \{0, 1, 2, 3\}, \tag{5.40}$$

and on each of these four spaces the integrability conditions act again as a linear system of equations. The vector space of these solutions defines the interaction Lagrangian.

Finally the first order correction $\delta_{(1)}^{(s)}$ to the gauge transformation is derived from Noether's equation

$$[\delta_{(1)}^{(s)}(z;a) - \epsilon(z;b) *_b \hat{\mathcal{Y}}(a, b; -\overleftarrow{\nabla})] *_a \hat{\mathcal{F}}(z;a) = 0, \tag{5.41}$$

where $\hat{\mathcal{Y}}$ is taken from (5.2).

# 6 Conclusions

We presented interaction Lagrangians for triplets of higher spin fields, a pair of which has equal spin $s_1$ whereas the third has spin $s_2 > s_1$. Besides the Lagrangians the next-to-leading order of the gauge transformations is given. The fields of smaller spins appear combined into currents of the Bell-Robinson form [3]. Remarkable is that for one such spin $s_2$ the interaction implies the existence of a whole ladder of interactions for smaller spins $s_2 - 2n > s_1$. In the case of three equal spins we presented only the formalism which can be transcribed into an algorithm. The applications of this algorithm will be presented in a forthcoming article [20].


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