

REGULARITY AND VARIATIONALITY OF SOLUTIONS  
 TO HAMILTON-JACOBI EQUATIONS.  
 PART I: REGULARITY  
 (ERRATA)

ANDREA C. G. MENNUCCI<sup>1</sup>

**Abstract.** This errata corrects one error in the 2004 version of this paper [Mennucci, *ESAIM: COCV* **10** (2004) 426–451].

**Mathematics Subject Classification.** 49L25, 53C22, 53C60.

Received October 24, 2006.

After the publication of [7] in 2004, it became clear that the regularity of the form  $\alpha$  in Lemma 4.4 had to be related to the regularity of  $K$  and of  $u_0$ ; this influences the minimal regularity of  $K, u_0$ , as needed in hypotheses in Lemma 4.4, in Theorem 4.1, and in many following relevant discussions. This errata corrects that error; to keep the matter short, all material that is unaffected by the error is omitted; whereas care was taken so that results and discussions that are here corrected retain the original numbering as in [7].

#### 4.1. Regularity of conjugate points

We will prove in this section results regarding the set of *focal points*; each following result extends to the set  $\Gamma$  of *conjugate points* that is a subset of the focal points.

**Theorem 4.1.** Assume (CC0,H1,H2). If  $u_0, K, H$  are regular enough, then, by Lemma 4.4, there is a (at most) countable number of  $n - 1$  dimensional submanifolds of  $\mathbb{R} \times O$  that cover all the sets  $G^i$ ; these submanifolds are graphs of functions  $\lambda_{i,h} : A_{i,h} \rightarrow \mathbb{R}$  (for  $h = 1 \dots$ ) where  $A_{i,h} \subset O$  are open sets. The least regular case is  $i = n - 1$ , and the regularity of the  $\lambda$  functions is related to the regularity of  $u_0, K, H$ , and to the dimension  $\dim(M) = n$  as in the following table:

$\dim(M)$	$u_0, K$	$H$	$\lambda$
$n = 2$	$C^{(R+2,\theta)}$	$C^{(R+2,\theta)}$	$C^{(R,\theta)}$
$n \geq 3$	$C^{(R+2,\theta)}$	$C^{(R+n-1,\theta)} \cap C^n$	$C^{(R,\theta)}$

(4.1)

where  $R \in \mathbb{N}, \theta \in [0, 1]$ .

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*Keywords and phrases.* Hamilton-Jacobi equations, cutlocus, conjugate points.

<sup>1</sup> Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa, Italy; a.mennucci@sns.it

We now infer some explanatory results on the regularity of the focal points  $X(\cup_i G^i)$  from the above theorem.

At the lowest regularity, when  $u_0, K \in C^2, H \in C^n$ , we know that  $X \in C^1$  and that the sets  $G^i$  are graphs; we conclude that the set of focal points has measure zero. When  $u_0, K \in C^{(2,\theta)}, H \in C^n \cap C^{(2,\theta)}$ , we know that the dimension of the sets  $G^i$  does not exceed  $n - \theta$ ; so again we conclude that the set of focal points has dimension at most  $n - \theta$ . In the case  $\theta = 1$ , we can obtain the set of all focal points is *rectifiable*; that is, if  $u_0, K \in C^{(2,1)}, H \in C^n \cap C^{(2,1)}$ , then the sets  $G^i$  are covered by Lipschitz graphs, so (by known results in [2]) the set of focal points may be covered by  $(n - 1)$ -dimensional  $C^1$  regular submanifolds of  $M$ , but for a set of Hausdorff  $\mathcal{H}^{n-1}$  measure zero.

When we further raise the regularity, we may suppose that  $u_0, K \in C^{s+3}, H \in C^{s+n}$  (with  $s \in \mathbb{N}$ )<sup>1</sup>; then the sets  $G^i$  are covered by graphs  $(\lambda(y), y)$  inside  $\mathbb{R} \times O$  of regularity  $C^{1+s}$ ; while  $X \in C^{2+s}$  (at least), and we restrict it to those graphs; we can then apply Theorem A.4 to state that the focal points are covered by  $C^{1+s}$  regular submanifolds of  $M$  but for a set of  $\mathcal{H}^\alpha$  measure zero, where  $\alpha \doteq n - 2 + 1/(1 + s)$ .

[... unchanged material deleted ...]

The main tool is this lemma; the complete proof of the lemma is in Section 6.

**Lemma 4.4.** *We assume that the hypotheses (CC0,H1,H2) hold.*

*We set the regularity of the data  $u_0, K, H$  by defining parameters  $R, R' \in \mathbb{N}, \theta, \theta' \in [0, 1]$ , and assuming that*

$$u_0 \in C^{(R'+2,\theta')}, \quad K \in C^{(R'+2,\theta')}, \quad H \in C^{(R+2,\theta)};$$

*by Proposition 3.7, the flow  $\Phi = (X, P)$  is  $C^{(R+1,\theta)}$  regular; and  $O$  is a  $C^{(R'+1,\theta')} \cup C^{(R+2,\theta)}$  manifold (that is, the least regular of the two).*

*Lets fix  $i \geq 1, i \leq n - 1$ , and fix a point  $(s', y') \in \mathbb{R} \times O$ , such that  $(s', y') \in G^{(i)}$ .*

*Let  $\mathcal{U}$  be a neighbourhood of 0 in  $\mathbb{R}^{n-1}$  and let  $\phi : \mathcal{U} \rightarrow O$  be a local chart to the neighbourhood  $\phi(\mathcal{U})$  of  $y' = \phi(0)$ . The map  $\phi$  has regularity  $C^{(R'+1,\theta')} \cup C^{(R+2,\theta)}$ . In the following,  $y$  will be a point in  $\phi(\mathcal{U})$ .*

*To study  $G^{(i)}$ , we should study the rank of the Jacobian of the map  $(t, x) \mapsto X(t, \phi(x))$ ; since the regularity of  $X$  is related only to the regularity of  $H$ , it will be useful to decouple this Jacobian in two parts. To this end, we define a  $n$ -form  $\alpha$  on  $\mathbb{R} \times O$ , with requirement that  $\alpha(t, y) = \alpha(y)$  (that is,  $\alpha$  does not depend on  $t$ ).*

*Writing  $X^{(t,y)}$  for  $X(t, y)$ , let*

$$X^{(t,y)*} \alpha$$

*be the push-forward of  $\alpha$  along  $X$ ;  $X^{(t,y)*} \alpha$  is then a tangent form defined on  $T_{X(t,y)}M$ ; it will be precisely defined in equation (6.2). We remark that  $X^{(t,y)*} \alpha = 0$  iff  $(t, y) \in \cup_j G^j$ . Note that the pushforward  $X^{(t,y)*}$  is  $C^{(R,\theta)}$  regular, while the form  $\alpha$  is as regular as  $TO$ , that is,  $\alpha$  is  $C^{(R',\theta')} \cup C^{(R+1,\theta)}$ .*

*Note that, since  $X$  solves an O.D.E., then  $X$  and  $\frac{\partial}{\partial t} X$  have the same regularity; note moreover that*

$$\frac{\partial^j}{\partial t^j} \left( X^{(s',y')*} \alpha \right) = \left( \frac{\partial^j}{\partial t^j} X \right)^{(s',y')*} \alpha$$

*since  $\alpha$  does not depend on  $t$ . So, by hypotheses and by the definition (6.2) of  $X^{(t,y)*} \alpha$ , the forms  $X^{(t,y)*} \alpha$  and  $\frac{\partial}{\partial t} (X^{(t,y)*} \alpha)$  have regularity  $C^{(R,\theta)} \cap C^{(R',\theta')}$  (see also Eq. (6.3)); the derivatives  $\frac{\partial^j}{\partial t^j} X^{(s',y')*} \alpha$  with  $j \geq 1$  have regularity  $C^{(R-j+1,\theta)} \cup C^{(R',\theta')}$ .*

*Then, when  $R + 1 \geq i$ , we prove (in Sect. 6) that*

$$X^{(s',y')*} \alpha = 0, \quad \frac{\partial}{\partial t} X^{(s',y')*} \alpha = 0, \quad \dots \quad \frac{\partial^{i-1}}{\partial t^{i-1}} X^{(s',y')*} \alpha = 0$$

<sup>1</sup>A similar result may be obtained when  $u_0, K \in C^{(s+3,\theta)}, H \in C^{(s+n,\theta)}$ .

whereas

$$\frac{\partial^i}{\partial t^i} X^{(s',y')^*} \alpha \neq 0.$$

We define eventually the map  $F : \mathbb{R} \times \mathbb{R}^{n-1} \rightarrow \mathbb{R}$  given by

$$F(t, x) = \frac{\partial^{i-1}}{\partial t^{i-1}} X^{(t,\phi(x))^*} \alpha;$$

since

$$\frac{\partial}{\partial t} F(t, x) \stackrel{\text{def}}{=} \frac{\partial^i}{\partial t^i} X^{(t,\phi(x))^*} \alpha \neq 0$$

the above Dini lemma implies that the set  $G^{(i)}$  is locally covered by the graph of a function  $\lambda_i$  defined on a open subset of  $O$ ;  $\lambda$  has the same regularity of  $F$ , so, if  $i = 1$  then  $\lambda$  is in  $C^{R,\theta} \cup C^{(R',\theta')}$  while for  $i \geq 2$  it is  $C^{(R-i+2,\theta)} \cup C^{(R',\theta')}$ .

The above directly implies Theorem 4.1.  
 [... all other results are unchanged ...]

## 5. APPLICATIONS

### 5.1. The Cauchy problem

We show now how the above theorems may be used for the Cauchy problem (1.2)

$$\begin{cases} \frac{\partial}{\partial t} w(t, x') + H'(t, x', \frac{\partial}{\partial x'} w(t, x')) = 0 & \text{for } t > 0, x' \in M' \\ w(0, x') = w_0(x') & \forall x' \in M'. \end{cases} \tag{1.2}$$

[... the preliminary discussion is unchanged ...]

This improves the results of 4.10, 4.12 and 4.17 in [1]; to provide for an easy comparison, we summarize these results

- if  $n' = \dim(M')$ ,  $n = n' + 1$ , if  $H' \in C^s$  with  $s = n \vee 3$  and  $w_0 \in C^2$ , then the set  $\Gamma$  has measure zero, so the set  $\overline{\Sigma}_u = \Sigma \cup \Gamma$  has measure zero;
- if  $H, w_0 \in C^{(2,1)}$ , then the set  $\Gamma$  is rectifiable, so the set  $\overline{\Sigma}_u = \Sigma \cup \Gamma$  is rectifiable;
- and when  $H' \in C^{R+1,\theta}$ ,  $w_0 \in C^{R+1,\theta}$ ,  $R \geq 2$ ,  $w$  is continuous, we prove that the Hausdorff dimension of  $\Gamma \setminus \Sigma$  is at most  $\beta$ , and moreover  $\mathcal{H}^\beta(\Gamma \setminus \Sigma) = 0$  if  $\theta = 0$ , where  $\beta = n' - 1 + 2/(R + \theta)$ .

In the counterexample in Section 4.4 in [1],  $w_0$  is  $C^{1,1}(M')$  and not  $C^2(M')$ ; so our results close the gap between the counterexample, where  $w_0$  is  $C^{1,1}(M')$ , and the theorem, where  $w_0$  is  $C^2(M')$ ; and actually, studying the counterexample, it is quite clear that, if  $w_0$  is smoothed to become a  $C^2(M')$  function, then the counterexample would not work.

### 5.2. Eikonal equation and cutlocus

As in Section 3.5, consider a smooth Riemannian manifold  $M$ , and a closed set  $K \subset M$  and let  $d_K(x) = d(x, K)$  be the distance to  $K$ . We set  $u_0 = 0$ : then  $O$  is the bundle of unit covectors that are normal to  $TK$ , and  $d_K(x)$  coincides with the *min* solution  $u(x)$ .

We define

$$\Sigma_{d_K} \stackrel{\text{def}}{=} \{x \mid \# \nabla d_K(x)\}$$

If  $K$  is  $C^1$ , then  $\Sigma_{d_K}$  coincides with  $\Sigma$  as defined in (4.1).

Since  $d_K$  is semiconcave in  $M \setminus K$ ,  $\Sigma_{d_K}$  is always rectifiable.

This primal problem is a good test bed to discuss the differences and synergies of the results in this paper and the results in Itoh and Tanaka [4] and Li and Nirenberg [5].

- In the example in Section 3 in [6], there is a curve  $K \subset \mathbb{R}^2$ ,  $K \in C^{1,1}$  such that  $\overline{\Sigma}_{d_K}$  has positive Lebesgue measure. Note that in this example  $\overline{\Sigma}_{d_K} \neq \text{Cut}(K) = \Sigma_{d_K}$ , so the cutlocus  $\text{Cut}(K)$  is rectifiable (but not closed).

We do not know if there is a curve  $K \in C^{1,1}$  such that  $\text{Cut}(K)$  is not rectifiable. (We recall that, by Prop. 14 in [3],  $\text{Cut}(K)$  has always measure zero).

- Theorem 4.1 states that if  $K$  is  $C^2$ , then  $\Gamma$  has measure zero, so by (1.4) and 4.11.4, we obtain that  $\overline{\Sigma}_{d_K} = \text{Cut}(K)$  has measure zero; so Theorem 4.1 closes the gap between the counterexample in Section 3 [6] and the previous available results.
- In example in Remark 1.1 in [5], for all  $\theta \in (0, 1)$  there is a compact curve  $K \in C^{2,\theta}$  such that the distance to the cutlocus is not locally Lipschitz; by Theorem 4.1, the cutlocus has dimension at most  $n - \theta$ .

We do not know if there exists an example of a compact curve  $K \in C^{2,\theta}$  such that  $\mathcal{H}^{n-1}(\text{Cut}(K)) = \infty$

- By the results in Itoh and Tanaka [4] and Li and Nirenberg [5], when  $K \in C^3$ , the distance to the cutlocus is locally Lipschitz and the cutlocus is rectifiable, and moreover (by Cor 1.1 in [5]), for any  $B$  bounded  $\mathcal{H}^{n-1}(\text{Cut}(K) \cap B) < \infty$ . By Theorem 4.1, the set of (non optimal) focal points is rectifiable as well.

5.2.1. *Improvements*

[... the discussion is unchanged ...]

**Corollary 5.1.** *Consider a 2-dimensional smooth Riemannian manifold  $M$ ; suppose that  $K$  is a compact  $C^{3+s}$  embedded submanifold.*

*Then, for any open bounded set  $A \subset M$ , the set  $A \cap \Gamma$  is  $C^{s+1}$ - $M^{1/(s+1)}$ -rectifiable: that is, it can be covered by at most countably many  $C^{s+1}$  curves, but for a set  $E$  such that  $\mathcal{M}^{1/(s+1)}(E) = 0$ .*

6. PROOF OF 4.4

[... the two lemma are unchanged ...]

Now we prove Lemma 4.4.

We want to define the  $n$  form  $\alpha$  so that  $\alpha$  does not depend on  $t$ ; and so that  $\alpha = e_1 \wedge \dots \wedge e_n$  where the vectors fields  $e_{n-i+1} \dots e_n$  span the kernel of  $\frac{\partial}{\partial \bar{x}} X$  at the point  $(s', y')$  (kernel that we will call  $V$ ) while  $\frac{\partial}{\partial \bar{x}} X$  is full rank on  $e_1 \dots e_{n-i}$  (that generate the space  $W$ ).

One possible way to this is to fix the local chart  $\phi : \mathcal{U} \subset \mathbb{R}^{n-1} \rightarrow O$ , define

$$\hat{e}_1 \stackrel{\text{def}}{=} \phi^* \frac{\partial}{\partial x_1}, \dots, \hat{e}_{n-1} \stackrel{\text{def}}{=} \phi^* \frac{\partial}{\partial x_{n-1}}, \hat{e}_n \stackrel{\text{def}}{=} \frac{\partial}{\partial t}$$

and then choose a  $n \times n$  constant matrix  $A$ , so that

$$e_h \stackrel{\text{def}}{=} \sum_k A_{h,k} \hat{e}_k$$

satisfy the requirements.

[... the rest of the proof is unchanged ...]

*Acknowledgements.* The author thanks Prof. Graziano Crasta for spotting the error that is corrected in this errata.

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