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A closed-formula characterization of the Epps effect

Giuseppe Buccheri*† and Giulia Livieri‡ and Davide Pirino § and Alessandro Pollastri §§

†Scuola Normale Superiore, Piazza dei Cavalieri 7, 56123, Pisa, Italy.
‡Scuola Normale Superiore, Piazza dei Cavalieri 7, 56123, Pisa, Italy.
§ Università degli Studi di Roma Tor Vergata, Dipartimento di Economia e Finanza, Via Columbia 2, 00173, Roma and Scuola Normale Superiore, Piazza dei Cavalieri 7, 56123, Pisa, Italy.
§§Scuola Normale Superiore, Piazza dei Cavalieri 7, 56123, Pisa, Italy.

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In this study we provide an analytical characterization of the impact of zero returns on the popular realized covariance estimator of Barndorff-Nielsen and Shephard (2004). In our framework, efficient price processes evolve as a semimartingale with some likelihood of repeated prices. We show that the standard realized covariance estimator is asymptotically affected by a downward bias, and the size of the bias depends on these likelihoods. We demonstrate that this result can be used to construct a consistent estimator of the integrated covariance of a vector semimartingale. The advantages with respect to other estimators are discussed in data.

Keywords: Epps effect, Realized Covariance, Infill Asymptotic, liquidity.

JEL Classification: G10, C12

1. Introduction

It is well known that the lack of synchronicity in the observation times of stock price returns causes some unwanted features in the inference of their integrated covariance. In particular, non-parametric estimators of the integrated covariance, constructed on artificially synchronized time series by *previous tick* or other interpolation schemes, tend to have an attenuation bias as the sampling interval progressively shrinks. This effect was documented for the first time by Epps (1979) and was named, after him, *Epps effect*. Intuitively, the previous-tick interpolation scheme, which attributes to each instant of the sampling partition the last available observation, generates, at a high-frequency, a large number of zero returns. The latter are known to be the main determinant of the Epps effect (cf. e.g. Hayashi and Yoshida 2005 and references therein). However, the asymptotic bias induced by zero returns on integrated covariance estimators is not analytically known.

Phillips and Yu (2007) study the impact of zero returns or, in their terminology, “flat trading”, on realized volatility and find that the latter remains consistent when trading prices have a nonzero probability of being repeated. This paper investigates the impact of flat trading on the *realized covariance* estimator of Barndorff-Nielsen and Shephard (2004) and provides a closed-form expression for its asymptotic bias, thus leading to an analytical characterization of the Epps effect.

*Corresponding author. Email: giuseppe.buccheri@sns.it.
Email: giulia.livieri@sns.it.
E-mail: davide.pirino@gmail.com.
E-mail: alessandro.pollastri@sns.it

Building on the model[1] of Phillips and Yu (2007), where flat trading is built-in to the data generating process, we show that the standard realized covariance estimator is asymptotically affected by a downward bias that depends on the probabilities of flat trading of the assets.

From an empirical perspective, the Epps effect has been investigated by many scholars over the last decades. Among the numerous references, in the present work we will name just a few of them. The impact of asynchronous data on covariance measurement has been studied in Scholes and Williams (1977) and Lo and MacKinlay (1990). On the other hand, Renò (2003) investigates the relative impact of asynchronous trading and genuine lagged correlations on the Epps effect. Tóth and Kertész (2007b,a) confirm the findings of the previous study in the sense that they conclude that the asynchronous trading affects the Epps effect the most, as a result of the increasing market efficiency. Moreover, the authors put forward the idea that the Epps effect is related also to the typical reaction time of market participants. Münnix et al. (2010) investigate the impact of decimalization of prices and, later on, Münnix et al. (2011) argue that the Epps effect is mainly caused by asynchrony of trades and the tick-size. An empirical study on the effect of asynchronicity and lagged correlations is provided also in Mastromatteo et al. (2011), whereas Saichev and Sornette (2014) propose a simple microstructure return model explaining microstructure noise and the Epps effect. More recently, Gurgul and Machno (2016) and Gurgul and Machno (2017) have investigated the impact of asynchronous trading on the Epps effect in, respectively, the stock exchanges of Vienna and Warsaw. However, less attention has been paid to the problem of finding an analytical characterization of the bias. A notable exception is given by the work of Zhang (2011), who provides an expression for the bias induced by previous-tick synchronization on realized covariance. The main difference with our approach is that, instead of dealing with asynchronous prices, we assume that prices are synchronized but we allow for the possibility of repeated prices. Zero returns naturally arise in this framework as a consequence of illiquidity or, more precisely, of lack of trading activity. We show that the consistency of standard realized covariance can be restored via multiplication by a correction coefficient that is particularly simple to compute in practice.

Researchers have employed several methods to deal with asynchronous data when inferring the integrated covariance of two or more asset prices. Hayashi and Yoshida (2005) provided an estimator based on tick-by-tick data and proved its asymptotic consistency. A similar route has been undertaken by De Jong and Nijman (1997). In contrast, the Multivariate Realized Kernel approach by Barndorff-Nielsen et al. (2011) is based on data synchronized with the refresh-time scheme (introduced, for the first time, by Harris et al. 1995), which prescribes to recursively pick-up the most recent price-update among all the assets under analysis. This methodology has also been employed in Christensen et al. (2010) in addition with a pre-averaging filter introduced to mitigate the effect of market microstructure noise. Mancino and Sanfelici (2011), instead, apply a multivariate Fourier method which does not require any synchronization procedure. Finally, Ait-Sahalia et al. (2010), Corsi et al. (2015), Shephard and Xiu (2017) propose estimators based on quasi-maximum likelihood estimation.

In our approach, we assume that flat trading is a characterizing feature of the data generating process for asset prices. More precisely, we focus on a bivariate process and assume the existence of two correlated latent efficient price processes $\left\{Y_t^{(1)}; t \geqslant 0\right\}$ and $\left\{Y_t^{(2)}; t \geqslant 0\right\}$, each of which follows an Itô-semimartingale. Then, at each sampling time, we assume that for each stock the occurrence of a zero return (or of a repeated price) is driven by a triangular array of i.i.d. Bernoulli random variables. The *observed price* may thus either coincide with the latent efficient price (Bernoulli variate equals to one), or not update and stays constant (Bernoulli variate equals to zero). Using a standard infill asymptotic framework (i.e. processes are assumed to be observed on a fixed time interval with mesh tending to zero) we prove that the realized covariance between the two processes of observed prices converges in probability to the integrated covariance of the two latent prices,

[1]This model appears also in Bandi et al. (2017) and Bandi et al. (2018), where authors prove that zero returns bring insightful economic information. An analogous model is used in Lo and MacKinlay (1990) to show that even in daily data, asynchronicity can cause difficulties.

multiplied by a coefficient that depends on the probabilities of flat trading of the two stocks, i.e. the (uniform) probability that a Bernoulli variate equals one. We find that the multiplying coefficient is always strictly smaller than one, eventually leading to a downward bias of the realized covariance estimator and, as a direct consequence, to the Epps effect. Importantly, the multiplicative bias can be consistently estimated by using recent results in Bandi et al. (2017, 2018). We thus obtain a consistent estimator of the integrated covariance between the efficient prices by correcting the realized covariance estimator for its asymptotic bias.

We conduct a Monte Carlo study in which we confirm the robustness of the proposed correction with respect to three sources of disturbance: microstructure noise, tick size and a time-varying probability of flat trading with a seasonal component.

Our methodology can be used to build an estimator of the integrated covariance of a multivariate semimartingale by applying the asymptotic bias correction to each off-diagonal element of the realized covariance matrix and then regularizing it to obtain a positive-definite matrix. We discuss the advantages of such bias-corrected estimator in an empirical exercise based on minimum variance portfolio (Engle and Colacito 2006, Patton and Sheppard 2009), which proves that the performances of the corrected estimator are, at worst, in line with those of other robust estimators and superior in case of illiquid portfolios. Importantly, our asymptotic correction remains computationally simple even for large dimensions, since it only requires estimates of the probabilities of flat trading. This is a relevant advantage, considering that, with large datasets, methods based on the refresh-time synchronization imply significant data reduction while likelihood-based methods become computationally difficult.

The paper proceeds as follows: in Section 2 we motivate and present our working framework. In Section 3 we provide an asymptotic theory for our Epps-effect corrected estimator of integrated covariance. We discuss the finite sample properties of the proposed estimator via Monte Carlo simulations in Section 4. Section 5 contains the empirical application of our proposed estimator to a portfolio of 10, 20, 30 stocks taken from the universe of the Russel 3000 constituents, and Section 6 concludes. Proofs for all limiting results are presented in the Appendix.

2. Model assumptions

We assume the existence of a filtered probability space $(\Omega, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathcal{P})$ satisfying standard conditions (Protter 1992). We begin with assumptions on the efficient price process and on the volatility process.

Assumption 1 (Efficient price process) *Let $t \in [0,T]$. There exists two real-valued logarithmic efficient price processes, denoted as $Y^{(1)}$ and $Y^{(2)}$, each of which is a Brownian semimartingale*

$$dY_t^{(\ell)} = \mu_t^{(\ell)}\, dt + \sigma_t^{(\ell)}\, dW_t^{(\ell)} \qquad \ell = 1, 2. \tag{1}$$

where $W_t^{(\ell)}$ is a standard Brownian motion, $\mu_t^{(\ell)}$ and $\sigma_t^{(\ell)}$ are predictable and adapted, path-wise Riemann integrable and bounded. The time interval $[0,T]$ can be thought of as representing the trading day. Besides, there exists a process $\rho_t \in (-1,1)$ such that $d\left\langle W^{(1)}, W^{(2)}\right\rangle_t = \rho_t\, dt$. We assume that σ_t has a semimartingale dynamics as well.

Our estimation target is the integrated covariance between the two efficient log-price processes, defined as

$$\mathrm{IC} = \int_0^T \sigma_s^{(1)}\sigma_s^{(2)}\rho_s ds.$$

The typical way of proceeding to estimate IC is to synchronize the prices of the two assets. For instance, in the *previous tick* interpolation scheme, one fixes a regular grid $\Upsilon_n = \{t_{1,n}, \ldots, t_{n,n}\}$ where elements are equally spaced in time and attributes to each instant of the sampling partition the last available observation. Then, the realized covariance estimator of Barndorff-Nielsen and Shephard (2004) is applied to this artificially synchronized time series. If it were possible to directly observe the efficient price processes $Y^{(1)}_{j,n}$ and $Y^{(2)}_{j,n}$ the convergence in probability

$$\left[Y^{(1)}, Y^{(2)}\right]_n \doteq \sum_{j=1}^{n} \left(Y^{(1)}_{j,n} - Y^{(1)}_{j-1,n}\right)\left(Y^{(2)}_{j,n} - Y^{(2)}_{j-1,n}\right) \xrightarrow{p} \mathrm{IC} \tag{2}$$

would appoint $\left[Y^{(1)}, Y^{(2)}\right]_n$ as a consistent estimator of IC. In practice, however, several frictions prevent the observed log-price paths from following the semimartingale dynamics represented by Eq. (1). In fact, it is well known that the estimator represented by the sum in Eq. (2), when implemented on real data, is affected by an attenuation bias that increases with sampling frequency. This problem was first documented by Epps (1979) and was dubbed *Epps effect* after him. The wellspring of this unwanted feature resides in a pretty simple mechanical effect: at a high-frequency, due to the asynchronicity of the trading activity, the previous tick interpolation scheme generates a large number of zero returns[1]. The existence of periods of no-trading activity is the signature left by several illiquidity frictions and translates to a loss of information, which in turn induces the negative bias. Accordingly, this effect is exacerbated for less liquid stocks. In order to analytically assess the impact of no-trading activity on the realized covariance estimator, in our theoretical framework we acknowledge the existence of zero returns as an integral part of the data generating process of asset prices. To this purpose, we build on the model of Phillips and Yu (2007), where the authors study the impact of zero returns on realized measures of volatility. From a microstructure viewpoint, the introduction of a nonzero probability of flat trading can be motivated in a market model with bid-ask spread and asymmetric information, as recently discussed by Bandi et al. (2017) and Bandi et al. (2018).

In what follows, we assume that price processes are observed at $n+1$ non-random times equispaced over the time interval $[0, T]$, namely $0 < t_{0,n} < t_{1,n} < \ldots < t_{n,n} = T$ with $\Delta_n = t_{j,n} - t_{j-1,n}$ for $j \geqslant 1$.

Assumption 2 (Observed price) *The two observed price processes, denoted as $X^{(1)}$ and $X^{(2)}$, are such that, on the time grid $t_j = j\,\Delta_n$, $X^{(\ell)}_{t_{0,n}} = Y^{(\ell)}_{t_{0,n}}$ for $\ell = 1, 2$ while, for $j = 1, \ldots, n$, we have*

$$X^{(\ell)}_{t_{j,n}} = Y^{(\ell)}_{t_{j,n}}\left(1 - B^{(\ell)}_{j,n}\right) + B^{(\ell)}_{j,n} X_{t_{j-1,n}} \tag{3}$$

where $B^{(\ell)}_{j,n}$ with $\ell = 1, 2$ are pairwise-independent triangular arrays of $\mathcal{F}_{t_j}$-measurable i.i.d. Bernoulli variates such that

$$p_{\ell,n} \overset{def}{=} \mathbb{P}\left[B^{(\ell)}_{j,n} = 1\right] = \mathbb{E}\left[B^{(\ell)}_{j,n}\right] \overset{n\to\infty}{\longrightarrow} p_\ell \in (0,1) \quad \textit{and} \;\; n\left(p_{\ell,n} - p_\ell\right) \overset{n\to\infty}{\longrightarrow} 0. \tag{4}$$

Notice that under Assumption 2, we allow for some likelihood of occurrence of zero returns and this likelihood is modeled as being independent across assets and in time-series. Moreover, the probabilities $p_{\ell,n}$, $\ell = 1, 2$, are assumed to be frequency-specific, in agreement with what has been documented on equity stock data (Bandi et al. 2018).

[1]To have an intuition on the number of zero returns in high-frequency data, see Table (4) in the empirical application in Section 5.

3. Limiting properties: an analytical characterization of the Epps effect

In order to develop our limit theory, we need to restrict the class of triangular arrays $B_{j,n}$ with the following additional assumption:

Assumption 3 *For all $j = 1, \ldots, n$ and $\ell = 1, 2$ define the number of consecutive flat trades for asset ℓ before instant $t_{j,n}$ as*

$$K_{j,n}^{(\ell)} = \min\left\{k \in \{0, \ldots, j\} \,|\, B_{j,n}^{(\ell)} = 1, B_{j-1,n}^{(\ell)} = 1, \ldots, B_{j-k+1,n}^{(\ell)} = 1, B_{j-k,n}^{(\ell)} = 0\right\}. \tag{5}$$

We assume that the maximum $K_n^{(\ell)} = \max_{j=1,\ldots,n} K_{j,n}^{(\ell)}$ is such that

$$\frac{K_n^{(\ell)} \log(n)}{n} \xrightarrow{p} 0, \text{ as } n \to \infty.$$

The theorem that follows constitutes the main theoretical result of the paper.

Theorem 3.1 *Let $\left\{Y_t^{(\ell)}; t \geqslant 0\right\}$ and $\left\{X_t^{(\ell)}; t \geqslant 0\right\}$, with $\ell = 1, 2$, be as in Assumption 1 and 2, with the triangular arrays of Bernoulli $B_{j,n}^{(\ell)}$ satisfying Assumption 3. Then, the Realized Covariance estimator RC_n, defined as*

$$\mathsf{RC}_n \doteq \sum_{j=1}^{n} \left(X_{j,n}^{(1)} - X_{j-1,n}^{(1)}\right)\left(X_{j,n}^{(2)} - X_{j-1,n}^{(2)}\right). \tag{6}$$

has the following limit in probability

$$\mathsf{RC}_n \xrightarrow{p} \frac{(1-p_1)(1-p_2)}{(1-p_1\,p_2)} \int_0^1 \sigma_s^{(1)} \sigma_s^{(2)} \rho_s \, ds, \text{ as } n \to \infty. \tag{7}$$

Proof. See Appendix A.

As a result of our limit theory, RC_n converges to the true integrated covariance, up to a multiplicative bias depending on the two asymptotic probabilities of flat trading p_1 and p_2. Note that the bias is the ratio between the probability that both asset prices are updated and the probability that at least one asset price is updated. Thus, apart from the trivial case $p_1 = p_2 = 0$, the multiplicative coefficient is lower than one and the estimated covariance is always smaller than the true integrated covariance. We define our bias-corrected realized covariance estimator as:

$$\mathsf{RC}_n^{\star} \doteq \frac{(1-\widehat{p}_{n,1}\,\widehat{p}_{n,2})}{(1-\widehat{p}_{n,1})(1-\widehat{p}_{n,2})} \mathsf{RC}_n, \tag{8}$$

where, for $\ell = 1, 2$, $\widehat{p}_{n,\ell}$ is the estimator defined as

$$\widehat{p}_{n,\ell} = \frac{1}{n} \sum_{j=1}^{n} \mathbb{1}_{\left\{X_{j,n}^{(\ell)} - X_{j-1,n}^{(\ell)} = 0\right\}} \xrightarrow{p} p_\ell \tag{9}$$

and where the last convergence in probability holds under Assumption 2 and 3 as proved in Bandi et al. (2017).

As a direct consequence of Theorem 3.1 and Eqs. (9) we have

$$\mathsf{RC}_n^{\star} \xrightarrow{p} \int_0^1 \sigma_s^{(1)} \sigma_s^{(2)} \rho_s \, ds,$$

i.e. $\mathsf{RC}_n^{\star}$ is a consistent estimator of the integrated covariance under the presence of flat trading. It is immediate to see that, in case the efficient price process $Y_{t,n}^{(\ell)}$ is contaminated by an uncorrelated noise term, $\mathsf{RC}_n^{\star}$ remains asymptotically unbiased but has an inflated variance depending on the variance of the noise. Note that the same happens to RC_n in absence of flat trading. The effect of microstructure noise will be discussed in the simulation study in the next section.

4. Monte Carlo Simulations

Here, we assess the finite sample accuracy of the bias-corrected RC estimator proposed in Section 3. We study the impact of three (independent) sources of disturbance: micro-structural noise, price rounding and intra-day effect in the probability of flat trading. Finally, we investigate how the variance of RC is affected by the asymptotic probabilities of flat trading.

For these purposes, we simulate the efficient log-prices $Y_t^{(\ell)}$, $\ell = 1, 2$, following the model by Barndorff-Nielsen et al. (2011)

$$dY_t^{(\ell)} = \mu_t^{(\ell)} \, dt + \rho \, \sigma_t^{(\ell)} \, dZ_t^{(\ell)} + \sqrt{1-\rho^2} \, \sigma_t^{(\ell)} \, dW_t, \quad \ell = 1, 2, \tag{10}$$

where $Z_t^{(\ell)}$ and $W_t^{(\ell)}$ are two independent Brownian motions. The stochastic volatility $\sigma_t^{(\ell)}$ is simulated according to the following SDE

$$\begin{aligned} \sigma_t^{(\ell)} &= \exp\left(\beta_0^{(\ell)} + \beta_1^{(\ell)} \, f_t^{(\ell)}\right), \\ df_t^{(\ell)} &= \alpha^{(\ell)} \, f_t^{(\ell)} \, dt + dZ_t^{(\ell)}. \end{aligned}$$

The value chosen for the model parameters are $\left(\mu^{(\ell)}, \alpha^{(\ell)}, \beta_0^{(\ell)}, \beta_1^{(\ell)}\right) = (0.03, -1/40, -5/16, 1/8)$, for $\ell = 1, 2$ and $\rho = -0.3$. This set of parameters ensures that $\mathbb{E}\left[\int_0^1 \left(\sigma^{(\ell)}\right)^2 ds\right] = 1$. We simulate[1] 1000 replications of a trading day of 6.5 hours on a time-grid of one second, for a total of 6.5×60×60 steps.

High-frequency noise. To assess how the estimator $\mathsf{RC}_n^{\star}$ is affected by micro-structural noise, we sample every second the efficient price paths simulated according to Eq. (10). The sampled prices, $Y_{j,n}^{(\ell)}$ with $j = 1, ..., n$ and $n = 6.5 \times 60 \times 60$, are then contaminated through an additive noise in the following way

$$\widetilde{Y}_{j,n}^{(\ell)} = Y_{j,n}^{(\ell)} + \eta_{j,n}^{(\ell)}$$

where $\eta_{j,n}^{(\ell)} \overset{d}{\sim} \mathcal{N}\left(0, \omega^2\right)$ and $\omega^2 = \xi^2 \sqrt{\frac{1}{n} \sum_{j=1}^n \left(\sigma_{j,n}^{(\ell)}\right)^4}$. The tuning parameter ξ^2 determines the impact of the noise on the efficient price and it is inversely proportional to the signal-to-noise ratio[2]. Once simulated, the (contaminated) efficient price is re-sampled every 1 second, 30 seconds,

[1]Numerical integration of the SDEs is performed on a one-second time grid via the Euler scheme.
[2]The larger the ξ the larger the impact of the noise and, hence, the lower the signal-to-noise ratio.

1 minute, 5 minutes and 10 minutes. On the coarser time-grids we construct the observed price process $X_{j,m}^{(\ell)}$, for $\ell = 1, 2$, following the recursive equation

$$\begin{cases} X_{0,m}^{(\ell)} = \widetilde{Y}_{0,m}^{(\ell)} \\ X_{j,m}^{(\ell)} = \left(1 - \mathbb{B}_{j,m}^{(\ell)}\right) \widetilde{Y}_{j,m}^{(\ell)} + \mathbb{B}_{j,m}^{(\ell)} X_{j-1,m}^{(\ell)}, \quad j = 1, ..., m, \end{cases}$$

where m is the number of points in the time-grid (hence $m = n = 6.5 \times 60 \times 60$ for the one-second time grid, $m = 6.5 \times 60 \times 60/30$ for the 30-second time grid, and so on and so forth), $\widetilde{Y}_{0,m}^{(\ell)} = \log(P_0)$ with $P_0 = 100$ and $\mathbb{B}_{j,m}^{(\ell)}$ are independent Bernoulli random variables with $\mathbb{E}\left[\mathbb{B}_{j,m}^{(\ell)}\right] = p_{\ell,m}$, $\ell = 1, 2$. We input the following scaling law for the probabilities of flat trading,

$$p_{\ell,m} = p_\ell \left(1 - \exp\left(-0.001\, m\right)\right), \tag{11}$$

where the parameter p_ℓ determines the asymptotic probability of flat trading. Note that, the scaling law in Eq. (11) is compatibile with the requirement imposed by Eq. (4) of Assumption 2. As an explicative example, at the sampling frequency of one-minute we have

$$m = 390 \quad \Rightarrow \quad p_{\ell,m} \approx 0.32 \times p_\ell,$$

that is, the probability of flat trading is 32% of the asymptotic probability.

We consider two different scenarios. In the first, the two stocks feature the same level of illiquidity (i.e. $p_1 = p_2$ with $p_1 \in \{0.20, 0.40, 0.60\}$). In the second, they present different levels of illiquidity (i.e. $p_1 \neq p_2$, with $p_1 \in \{0.20, 0.40, 0.60\}$ and $p_2 = 0.80$). Table 1 (resp. Table 2) reports the average, across all replications, of the relative[1] percentage bias in the first (resp. second) case. Both tables entail different values for ξ^2 (the aggressiveness of the noise) and Δ (the sampling frequency).

The results in Table 1 indicate that the magnitude of the bias of both RC and $\mathsf{RC}^\star$ is weakly affected by the entity of microstructure noise. Concerning RC, for a fixed level of ξ^2 and p_1, the bias increases (in absolute value) with the sampling frequency, because of the Epps effect. Moreover, consistently with the result in Theorem 3.1, the probability p_1 plays a crucial role. For instance, when $p_1 = 0.6$ and at the sampling frequency of $\Delta = 1$ second, the conventional realized covariance estimator has a downward bias of roughly 75%, compared with a downward bias of roughly 33% when $p_1 = 0.2$. This bias is attenuated at lower frequencies. At the sampling frequency of $\Delta = 10$ minutes, for example, the bias of RC_n varies from (approximately) 3% to 6% when $p_1 = 0.2$ and $p_1 = 0.6$, respectively. On the other hand, the bias of RC^* is lower than 3% in all the scenarios. The increase of the bias of RC^* with Δ is a finite sample effect.

The results of Table 2 can be summarized in the following two points: 1) the bias of RC is mainly driven by the most sluggish asset and 2) (the relative) performances of both RC and $\mathsf{RC}^\star$ are weakly affected by the presence of the noise.

Finally, Figure 2 shows the kernel densities of the relative bias of both RC and $\mathsf{RC}^\star$ (vertical axes are reported in logarithmic scale) in the scenario $p_1 = p_2 = 0.20$, for different values of the sampling frequency. The figure confirms the consistency of our bias-corrected estimator. Furthermore, Figure 2 confirms that the aggressiveness of the noise ξ solely impacts the variance of the estimator, without affecting the bias, and it plays a significant role only at very high sampling frequencies (such as one second).

[1] We consider the difference between the estimator and the true value, in units of the true value. Accordingly, the relative bias of RC_m is computed as

$$100 \times \frac{\mathsf{RC}_m - \mathrm{IC}}{\mathrm{IC}},$$

where IC is the true integrated covariance defined in Eq. (2). An identical definition holds for $\mathsf{RC}_m^\star$.

Tick size. The finite sample performances of the bias-corrected estimator RC^* may be undermined by stock tick size. In fact transaction prices are, for institutional settings, rounded at one cent. Hence, a fraction of the observed zero returns cannot be inputed to the data generating process in Assumption 2, which is designed to describe the lack of price adjustment induced by absence of trading activity.

To assess the impact of tick size on the bias-corrected estimator RC^* we explicitly accommodate rounding of the simulated prices. We consider the case of two stocks with an equal level of illiquidity. To isolate the role of the tick size, in this paragraph, we simulate the model in Eq. (10) with $P_0 = 50$ (instead of $P_0 = 100$) for 1000 replications, 6.5 hours on a time-grid of one second and $\xi = 0$ (no micro-structural noise). Then we round prices to the nearest cent (\$0.01), as imposed by the actual settings of electronic financial markets. Since, in these settings, the estimator of the probability of zero return in Eq. (9) is highly biased, we adopt the finite-sample correction proposed by Bandi et al. (2018). Here, the authors develop an estimator of the probability of zero return robust to the presence of rounding. Table 3, second two columns, collects the results. We see that the bias of RC^* is, for all the scenarios considered, remarkably smaller than that of RC.

Time-varying probability of flat trading. Here, we evaluate the impact of a time-varying probability of zero returns on the covariance estimators RC^* and RC. For this purpose, we generate 1000 replications of the price process in Eq. (4) (using, again, an initial price of $P_0 = 100$, a time grid of $m = n = 6.5 \times 60 \times 60$ and $\xi = 0$) but with a time-dependent specification of the probability of flat trading. For this purpose, we need the following additional notation: let $p^{(\ell)}_{j,n}$ be the probability that the Bernoulli random variable $B^{(\ell)}_{j,n}$, appearing in Eq. (4), is equal to one, i.e. $p^{(\ell)}_{j,n} = \mathbb{E}\left[B^{(\ell)}_{j,n}\right]$. We construct $p^{(\ell)}_{j,n}$ in the following way. Mirroring a standard approach in the high-frequency volatility literature (see, among many others, Engle and Sokalska 2012), we assume that $p^{(\ell)}_{j,n} = \bar{p}^{(\ell)}_{j,n}\,\vartheta^{(\ell)}_{j,n}$, where $\bar{p}^{(\ell)}_{j,n}$ and $\vartheta^{(\ell)}_{j,n}$ represent two distinct sources of time-variation in the probability of observing a zero return. The first factor, i.e. $\bar{p}^{(\ell)}_{j,n}$, denotes the stochastic component of $p^{(\ell)}_{j,n}$. The second factor, i.e. $\vartheta^{(\ell)}_{j,n}$, is the deterministic seasonal component that represents the diurnal pattern. To ensure identification, we impose that $(1/n)\sum_{j=1}^{n}\vartheta^{(\ell)}_{j,n} = 1$. In order to generate $\bar{p}^{(\ell)}_{j,n}$ we proceed in the following way (cfr. also Kolokolov et al. 2018). For each replication, we construct a trajectory of a latent stochastic process u with the following integration scheme:

$$\begin{cases} u^{(\ell)}_{0,n} = F^{-1}\left(p_\ell\right), \\ u^{(\ell)}_{j,n} = u^{(\ell)}_{j-1,n} + \left(F^{-1}\left(p_\ell\right) - u^{(\ell)}_{j-1,n}\right)/n + \sigma_u \varepsilon^{(\ell)}_{j,n}/\sqrt{n}, \end{cases}$$

where $j = 1, \ldots, n$, $p_\ell \in \{0.20, 0.40, 0.60\}$, and where $F^{-1}(x)$ is the inverse of the cumulative distribution function of a standard Gaussian variable. In addition, the $\varepsilon_{j,n}$'s are *i.i.d.* standard Gaussian shocks, and $\sigma_u = 0.5$ is a tuning parameter. Then, we set $\bar{p}^{(\ell)}_{j,n} = F\left(u^{(\ell)}_{j,n}\right)$; in this way the stochastic component of the probability of observing a zero return is mean-reverting around p_ℓ. The seasonal pattern $\vartheta^{(\ell)}_{j,n}$ is estimated directly on the available real data using a non-parametric estimator. Let then $X^{(\ell)}_{j,n,t}$ be the observed price process $X^{(\ell)}_{j,n}$ in the t-th day of the sample. We first compute the average

$$\widehat{\vartheta}^{(\ell),\star}_{j,n} \doteq \frac{1}{T}\sum_{t=1}^{T} \mathbb{1}_{\left\{X^{(\ell)}_{j,n,t} - X^{(\ell)}_{j-1,n,t} = 0\right\}},$$

where T is the number of days in the sample. Then, we normalize $\widehat{\vartheta}^{(\ell),\star}_{j,n}$ to obtain the desired quantity. For all the simulations of this paragraph, we adopt the seasonal pattern produced by the estimator (4) computed on the transaction history of GE (General Electric). Since the time resolution of our data is one-minute (see Section 5), we implement a linear interpolation to obtain the diurnal pattern for higher frequencies. Figure 1 reports (continuous red line) the seasonal pattern used for the simulation, together with an example of the simulated trajectory of the time-varying probability $p_{j,n}$ (blue line with crosses). Note that the intraday pattern has an inverse U-shape, which reflects the fact that zero returns are more concentrated in the middle of the day, mirroring the U-shaped profile of intraday volatility. Table 3, first two columns, displays the results. We compare them with the first two columns in Table 1, where a constant probability is considered. As expected, the performance of $\mathsf{RC}^\star$ deteriorates at high frequencies ($\Delta = 1$ second and $\Delta = 30$ seconds), whereas RC_n is unaffected by the presence of a time-of-the-day dependent probability of zero returns. Nonetheless, even in the worst case scenario ($\Delta = 1$ second and $p_1 = 0.60$) the bias of the $\mathsf{RC}_n^\star$ is $+2.92\%$ against a -73.72% of RC.

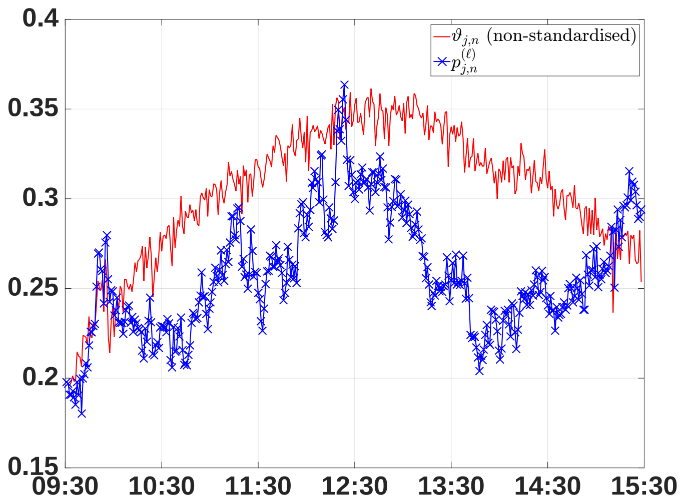

Figure 1.: Example of a non-standardised diurnal pattern $\vartheta_{j,n}$ (in red) together with a typical trajectory of the time-varying probability $p_{j,n}$ (in blue)

Asymptotic variance of the bias-corrected estimator. The asymptotic variance of the bias-corrected estimator $\mathsf{RC}^\star$ may be affected by the asymptotic probabilities of flat trading defined in Eq. (4) of Assumption 2. Since the derivation of a central limit theorem for $\mathsf{RC}^\star$ is beyond the scope of this paper, we investigate this aspect through Monte Carlo simulations. To do that, we produce 10^4 replications of two price processes under the frictional dynamics described by Assumption 2, where efficient prices are simulated according to the model in Eq. (10). Figure 3 reports the variance of $\mathsf{RC}^\star$ computed at the frequency of one second, hence with $6.5 \times 60 \times 60 = 23400$ observations. The two horizontal axes report the values of the probabilities of flat trading that have been inputed in simulating the recursive equation (3), hence they must be interpreted as probabilities of flat trading at one second. The vertical axis reports the percentage variance of $\mathsf{RC}^\star$ which is, notably, almost unaffected by the choice of p_1 and p_2, unless the two probabilities are simultaneously close to one. In summary, apart from the case of two highly illiquid assets, the precision of the bias-corrected estimator does not depart much from the ideal case $p_1 = p_2 = 0$ of two perfectly liquid assets.

5. Empirical assessment of the bias-corrected estimator

In this section we prove empirically that the performances of the bias-corrected estimator $\mathsf{RC}^\star$ are generally comparable with those of commonly used integrated covariance estimators but remarkably

		ξ^2					
p_1	Freq.(sec)	0		0.0001		0.0005	
		$\mathsf{RC}^{\star}$	RC	$\mathsf{RC}^{\star}$	RC	$\mathsf{RC}^{\star}$	RC
	1	-0.2089	-33.4646	-0.1481	-33.4135	0.3174	-33.1220
	30	-0.2083	-19.7722	-0.5220	-19.9147	-0.3229	-19.8248
0.20	60	-0.4244	-12.6576	-0.5537	-12.6153	-1.2145	-13.2915
	300	-1.2053	-4.1650	-1.3661	-4.2597	-2.7168	-5.6878
	600	-3.3331	-4.7520	-2.6566	-4.0619	-4.0480	-5.4835
	1	-0.2914	-57.2765	-0.3836	-57.3072	-0.5998	-57.4007
	30	-0.2161	-35.8139	0.4310	-35.4004	0.5382	-35.2916
0.40	60	-0.4645	-23.3011	0.1741	-22.7971	-0.5998	-23.3209
	300	-1.8349	-7.4120	-0.8162	-6.5223	-1.4490	-7.2148
	600	-2.7332	-5.6796	-1.0361	-3.9416	-4.3822	-7.2807
	1	-0.3292	-75.0813	-0.2192	-75.0418	0.5513	-74.8715
	30	-0.4685	-49.3759	-0.8919	-49.5189	-0.7147	-49.3824
0.60	60	-0.4878	-32.7471	-0.8243	-33.0470	1.1100	-31.8801
	300	-1.7899	-10.3175	-1.7988	-10.1267	-3.3381	-11.4537
	600	-1.5240	-6.2752	-2.7887	-7.2564	-3.1937	-7.4286

Table 1.: Relative biases in percentage, for different values of the sampling frequency and of the parameter ξ^2, of the estimators RC^* and RC when $p_1 = p_2 \in \{0.20, 0.40, 0.60\}$.

		ξ^2					
p_1	Freq.(sec)	0		0.0001		0.0005	
		$\mathsf{RC}^{\star}$	RC	$\mathsf{RC}^{\star}$	RC	$\mathsf{RC}^{\star}$	RC
	1	-0.1239	-80.9760	-0.5174	-81.0509	2.0211	-80.5674
	30	-0.4140	-47.1973	-0.2231	-47.0961	-0.7550	-47.3781
0.20	60	-0.6337	-29.8954	-0.5343	-29.8253	-1.4998	-30.5065
	300	-0.7756	-8.0488	-1.9026	-9.0931	-1.2924	-8.5277
	600	-3.0459	-6.7098	-4.1530	-7.7750	-2.3958	-6.0843
	1	-0.0330	-82.3588	0.3222	-82.2961	-3.2452	-82.9256
	30	-0.4828	-51.2427	-0.6279	-51.3138	-0.7975	-51.3969
0.40	60	0.1730	-33.0704	-1.2493	-34.0207	0.8851	-32.5946
	300	-1.8595	-10.3580	-1.5207	-10.0486	-1.0047	-9.5772
	600	-2.6301	-7.0102	-3.1146	-7.4729	-2.2761	-6.6721
	1	-0.6269	-84.7118	-0.2296	-84.6507	0.1582	-84.5910
	30	-0.9231	-55.8857	-1.5412	-56.1608	-0.6926	-55.7830
0.60	60	-1.0472	-37.7141	-0.2797	-37.2310	-1.4429	-37.9632
	300	-2.9917	-12.6843	-1.3131	-11.1733	-0.2304	-10.1989
	600	-3.7065	-8.7312	-1.2934	-6.4440	-3.1883	-8.2400

Table 2.: Relative biases in percentage, for different values of the sampling frequency and of the parameter ξ^2, of the estimators RC^* and RC when $p_1 \neq p_2$. We set $p_1 \in \{0.20, 0.40, 0.60\}$ and $p_2 = 0.8$.

p_1	Freq.(sec)	*Intraday Effects*		*Effect of Rounding*	
		RC*	RC	RC*	RC
	1	0.3978	-33.9595	-0.4815	-33.5203
	30	-0.2273	-20.9712	0.5927	-19.5979
0.20	60	-0.5715	-13.5051	0.2380	-12.3971
	300	-0.9797	-4.5269	-0.4773	-3.6810
	600	-3.3701	-5.2793	-1.7587	-3.4271
	1	1.4317	-56.3516	0.4024	-57.2634
	30	0.7505	-35.2118	0.2097	-35.9009
0.40	60	0.2386	-23.5389	-0.1514	-23.5474
	300	-0.4127	-6.7385	-1.5869	-7.5290
	600	-0.5073	-3.9314	-2.9282	-6.0529
	1	2.9288	-73.7234	-0.8021	-75.0454
	30	1.3802	-48.8772	0.6840	-48.9279
0.60	60	0.4521	-32.7392	0.3702	-32.5286
	300	-1.2073	-10.5503	-0.1117	-8.9239
	600	-1.5775	-7.0337	-2.7468	-7.3826

Table 3.: Relative biases in percentage for different values of the sampling frequency of the estimators RC* and RC in case of a time-dependent probability of observing a zero return (*Intraday Effects*) and in the case of rounded observed prices (*Effect of Rounding*). In the first case we input a time-dependent probability which is mean-reverting around $p_1 = p_2 \in \{0.20, 0.40, 0.60\}$. In the second case the probabilities are constant with $p_1 = p_2 \in \{0.20, 0.40, 0.60\}$. In the *Effect of Rounding* case we set $P_0 = 50$ instead of $P_0 = 100$ to magnify the impact of tick size.

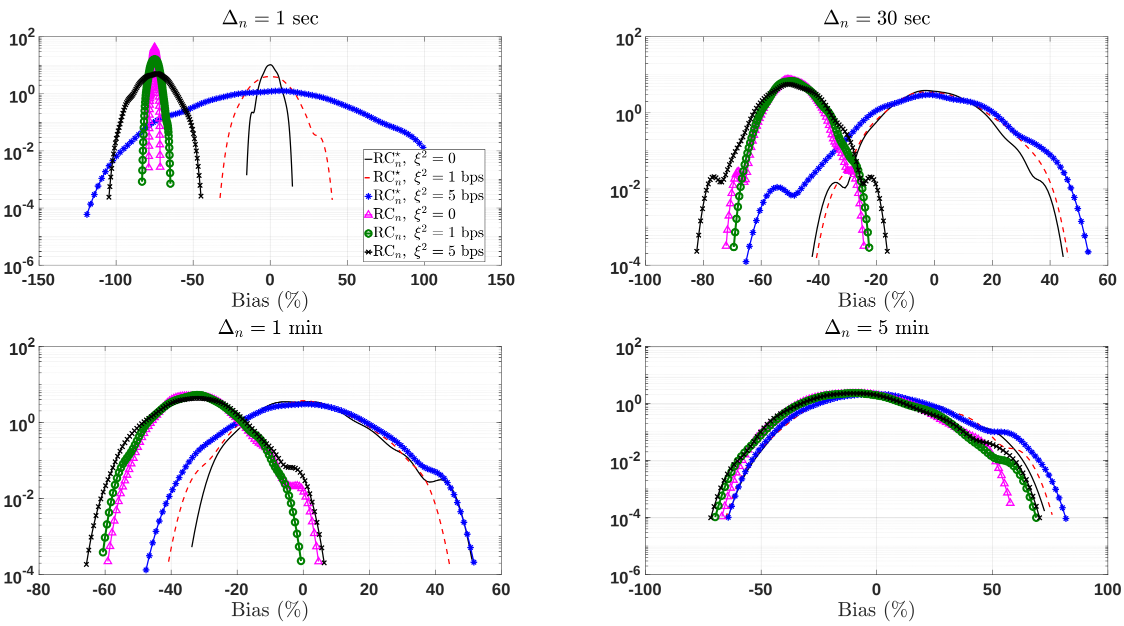

Figure 2.: Kernel density of the relative bias (in percentage) of the standard realized covariance estimator and our bias-corrected estimator. Different sampling frequencies Δ and levels of noise ξ^2 are considered.

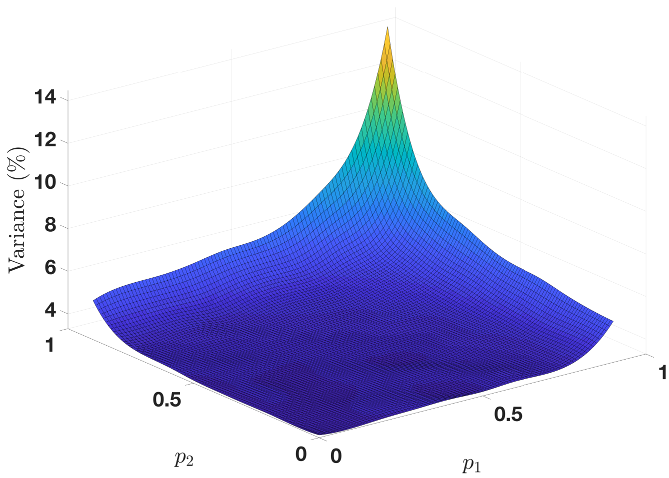

Figure 3.: Variance of the estimator $\mathsf{RC}^{\star}$ computed over 10^4 replications of two observed price processes under the frictional dynamics postulated by Assumption 2. The value of the probabilities of flat trading p_1 and p_2 are reported in the two horizontal axes. The vertical axis reports the percentage variance.

superior in case of portfolios with low liquidity. We perform a horse-racing exercise comparing the performances of different estimators through the minimum variance portfolio criterion (for an exhaustive discussion on the topic see Engle and Colacito 2006, Patton and Sheppard 2009, amongst others). This method adopts the realized (ex-post) variance of the (ex-ante) minimum variance portfolio as a loss measure. In particular, given a group of N assets and a forecast $\widehat{\Sigma}_t$ of their variance-covariance matrix for day t, the criterion requires to solve the the global minimum variance portfolio (GMV) problem

$$\begin{aligned} \widehat{w}_t = & \underset{w_t \in \mathbb{R}^N}{\text{argmin}} \quad w_t \, \widehat{\Sigma}_t \, w_t \\ & \text{subject to} \quad w_t \, \iota = 1, \end{aligned} \tag{12}$$

where $\iota \in \mathbb{R}^N$ is a vector of ones and $w_t \in \mathbb{R}^N$ are portfolio weights. The loss function is defined as the norm

$$d_t = \widehat{w}_t \, r_t \, r_t' \widehat{w}_t \tag{13}$$

where $r_t \in \mathbb{R}^N$ is the vector of daily open-to-close returns and $\widehat{w}_t$ is the solution of the GMV problem (12). The criterion selects the best covariance estimator as that with the lowest portfolio variance, computed as the daily average of d_t. In our analysis, one-day-ahead covariance forecasts are computed through the HAR-DRD model of Oh and Patton (2016) fitted on times series of competing realized covariance estimators.

When $N > 2$, the asymptotic bias correction in Eq.(8) can be applied to any off-diagonal element of the realized covariance matrix. Therefore, the construction of the new estimator requires estimates of the probabilities of flat trading for each of the N assets in the portfolio. These probabilities can be estimated by solely using transaction data, as described by Bandi et al. (2017) and Bandi et al. (2018). As anticipated in Eq. (9), we follow Bandi et al. (2018) and we estimate $p_{n,\ell}$, $\ell = 1, \dots, N$, as the daily fraction of zero returns, at a given sampling frequency. The resulting estimator is thus computationally simple. However, it is not necessarily positive definite and needs to be regularized in order to be employed in the portfolio optimization problem. As a regularization method, we simply replace negative eigenvalues with the smallest positive eigenvalue. We verify that this approach leads to well conditioned covariance matrices and stable portfolio weights. Other more sophisticated techniques are possible, such as those based on random matrix theory (Hautsch et al. 2012).

Our dataset[1] consists of unbalanced one-minute transaction data of Russel 3000 constituents over the period from 18-11-1999 to 27-09-2013. The total number of assets is 4166. In order to avoid discontinuities due to changes on index composition, we restrict the analysis to the subsample comprising the last $T = 2000$ days, that is all data from 27-09-2005 to 27-09-2013. For each day in the sample we consider trades from 9:30 to 16:00, leading to 390 one-minute timestamps per day. As a further cleaning procedure, we select assets having at least ten trades per day. This choice removes from the sample extremely illiquid assets, whose presence can cause the occurrence of poor and ill-conditioned covariance estimates. After this filtering, we are left with $N = 984$ assets. Within this universe, we form five *liquidity portfolios* made of $N_p = 10, 20, 30$ assets and built using the following procedure:

(i) we compute, for all assets, the average probability of flat trading $\overline{p}_\ell$, $\ell = 1, ..., N$, using all data at one-minute frequency.
(ii) We compute the quartile separators Q_j, $j = 1, 2, 3$ of the empirical distribution of $(\overline{p}_1, \cdots, \overline{p}_N)$.
(iii) For a given value of N_p, the first, second, third and fourth portfolios are obtained by picking up, randomly, N_p assets from the N available with, respectively, $\overline{p}_\ell < Q_1$, $Q_1 \leqslant \overline{p}_\ell < Q_2$, $Q_2 \leqslant \overline{p}_\ell < Q_3$ and $Q_3 \leqslant \overline{p}_\ell$. These portfolios are called, in order, *very high*, *high*, *medium*, *low* liquidity portfolios.
(iv) The N_p constituents of the fifth portfolio are randomly selected within the whole sample of 984 assets and, therefore, this portfolio includes assets with different levels of liquidity. Accordingly, we refer to it as the *mixed* liquidity portfolios.

Min	Q_1	Q_2	Q_3	Max
0.0258	0.2847	0.3991	0.5633	0.9246

Table 4.: Minimum, maximum and quartiles of one-minute average probabilities of flat trading of the $N = 984$ assets in the dataset.

On the aforementioned portfolios we compute nine different variance-covariance matrix estimators. The first six estimators are the realized covariance RC defined in Eq. (6) and our bias-corrected $\mathsf{RC}^\star$ defined in Eq. (8), both computed at one-, five- and ten-minute sampling frequencies. Along with these estimators, we consider the Multivariate Realized Kernel (MRK) of Barndorff-Nielsen et al. (2011), which is a kernel estimator applied to one-minute synchronized data. We implement the MRK with a Parzen kernel and, following the standard practice, we use the refresh time as a synchronization scheme. To avoid excessive data reduction due to refresh-time synchronization, we compute the MRK separately for each couple of assets. Similarly to our $\mathsf{RC}^\star$ estimators, the full covariance matrix of N_p assets is regularized in order to guarantee positive-definiteness. We also consider the covariance matrix constructed through the following method: the off-diagonal elements of the realized covariance are replaced by the pairwise estimator of Hayashi and Yoshida (2005); the diagonal elements are set equal to those of standard realized covariance. We denote this estimator as HY. One advantage of the HY is that it uses all available data when constructing covariances. Similarly to $\mathsf{RC}^\star_n$ and MRK, HY is regularized to guarantee positive-definiteness. Finally, we consider the quasi-maximum likelihood (QMLE) estimator of Corsi et al. (2015) and Shephard and Xiu (2017). This estimator does not require synchronization and, as well as the MRK, is robust

[1]The data that support the findings of this study are available from Kibot (`http://www.kibot.com/`). Restrictions apply to the availability of these data, which were used under license for this study.

to microstructure noise.

The nine estimators described thus far are used to construct daily time series of covariance estimates. The sample of $T = 2000$ days is divided in two parts of equal length. The former is used to estimate the HAR-DRD model, whereas the latter employs the forecasts $\widehat{\Sigma}_t$ (derived using the OLS estimates of the first part) to obtain, solving the minimization problem (12), a sequence of daily portfolio weights $\widehat{w}_t$. In order to assess if the variance of a portfolio is significantly lower than the variances of other portfolios, we use the model confidence set (MCS) of Hansen et al. (2011). In particular, we consider a model confidence set at the 90% confidence level, denoted by $\mathcal{M}_{90\%}$.

Fig. 4 shows averages of one-step-ahead covariance forecasts obtained through the HAR-DRD fitted on both standard (RC) and bias-corrected ($\mathsf{RC}^{\star}$) realized covariances. We consider the GMV portfolio with $N_p = 10$ assets belonging to the group with low liquidity. Note that, as the sampling frequency increases, covariance forecasts obtained through standard realized covariance are largely downward biased. In contrast, covariance forecasts constructed through the bias-corrected estimator $\mathsf{RC}^{\star}$ are very similar across sampling frequencies, an empirical result that confirms the robustness to price flatness of the corrected estimator. At ten minutes, standard realized covariance forecasts are close to those obtained from bias-corrected realized covariances, consistently with the fact that the probability of flat trading decreases as the sampling frequency decreases. Thus, the Epps effect can be seen as a direct consequence of ignoring price flatness when computing realized covariance.

Tables 5, 6 and 7 show the results of the analysis, for, respectively $N_p = 10, 20, 30$. We first note that the variances of portfolios constructed through $\mathsf{RC}^{\star}$ are significantly lower than those of portfolios built through RC. In particular, larger differences are observed at high sampling frequencies (e.g. one minute) and for "low" liquidity portfolios. This is related to the fact that the probability of flat trading is larger in these cases and, consequently, realized covariance estimates are extremely biased. Note also that, in these cases, the MCS tends to exclude the standard realized covariance, while it includes $\mathsf{RC}^{\star}$.

Significant portfolio variance reduction is also observed in the "mixed" liquidity portfolio. For instance, the variance of the $N_p = 30$ portfolio constructed through $\mathsf{RC}^{\star}$ at one-minute frequency is $\approx 17\%$ lower than that of the portfolio constructed through RC at the same frequency. Lower differences are instead observed for portfolios of assets with "medium" and "high" liquidity, consistently with the fact that the average probability of flat trading is smaller in these portfolios. However, the bias-correction leads to significant portfolio variance reduction. For instance, the $\mathsf{RC}^{\star}$ portfolio of $N_p = 10$ assets shows the lowest variance in the "high liquidity" case.

In the "very high liquidity" portfolio, probabilities of flat trading are extremely low and, consequently, the Epps effect is less relevant. In this case, the use of our asymptotic bias correction does not lead to significant variance reduction, except for the portfolio with $N_p = 20$ assets, where the MCS selects five-minutes $\mathsf{RC}^{\star}$ and excludes all portfolios built through RC. Note also that, in this scenario, the performances of all estimators are generally closer among each other, with the MCS including all of them in the portfolio with $N_p = 30$ assets.

As a final remark, we note that the performances of the bias-corrected estimators are comparable to those of MRK, HY and QMLE. In particular, all the portfolios built with $\mathsf{RC}^{\star}$ feature the lowest variance in the "low liquidity" case. Besides, for this liquidity scenario and for $N_p = 20$, the estimator $\mathsf{RC}^{\star}$ implemented at ten-minute frequency is the only estimator included in the MCS.

In summary, the empirical advantages of the bias-corrected estimator $\mathsf{RC}^{\star}$ are independent of the number N_p of assets held in the portfolio. Besides, $\mathsf{RC}^{\star}$ requires few additional computations with respect to RC since flat trading probabilities are estimated through the Eq. (9), hence as simple means.

		Portfolio liquidity, $N_p = 10$.				
Estimator	Freq. (min)	Very high	High	Medium	Low	Mixed
RC	1	1.7157*	1.4254	0.9673	1.3918	0.9695
		(0.754)	(0.012)	(0.000)	(0.053)	(0.001)
	5	1.7370*	1.3151*	0.9036*	1.3148	0.8918
		(0.408)	(0.718)	(0.100)	(0.053)	(0.064)
	10	1.7563*	1.3090*	0.9044*	1.2887*	0.8945
		(0.408)	(0.806)	(0.100)	(0.932)	(0.064)
RC*	1	1.8410	1.3568	0.9221	1.2909*	0.8667*
		1.0730	*0.9519*	*0.9533*	*0.9275*	*0.8940*
		(0.061)	(0.046)	(0.031)	(0.932)	(0.581)
	5	1.7698*	**1.2711***	0.8984*	1.2944*	0.8487*
		1.0189	*0.9665*	*0.9942*	*0.9845*	*0.9517*
		(0.135)	(1.000)	(0.100)	(0.902)	(0.900)
	10	1.7466*	1.2834*	0.8917*	**1.2828***	0.8726
		0.9945	*0.9804*	*0.9859*	*0.9954*	*0.9755*
		(0.408)	(0.806)	(0.100)	(1.000)	(0.088)
MRK	-	1.7106*	1.3714*	0.9261	1.2830*	0.8898
		(0.754)	(0.354)	(0.031)	(0.932)	(0.088)
HY	-	1.7379*	1.2896*	**0.8655***	1.3107*	0.8549*
		(0.408)	(0.806)	(1.000)	(0.795)	(0.771)
QMLE	-	**1.6929***	1.3163*	0.8790*	1.3129*	**0.8470***
		(1.000)	(0.806)	(0.364)	(0.795)	(1.000)

Table 5.: Ex-post average variances ($\times 10^4$) of the five randomly selected portfolios of $N_p = 10$ assets. Numbers in parentheses denote p-values of the MCS test. The asterisk implies that the estimator is included in $\mathcal{M}_{90\%}$. For the three RC* estimators, we report *in italics* the ratio between the variance obtained through the bias-corrected estimator and that obtained through the standard realized covariance estimator computed at the same sampling frequency. Bold numbers denote portfolios with lowest absolute variance.

6. Conclusion

Studies of correlation estimated from *asynchronous* stock price returns documented the Epps effect (Epps 1979), i.e. a progressive increase of the (negative) bias in estimating the integrated covariance as the sampling frequency increases. Since Epps (1979), researchers have been trying to mitigate the impact of this unwanted feature. In this paper we provide an analytical characterization of the Epps effect by showing how zero returns, which naturally arise in previous-tick interpolated data due to asynchronicity in the trading activity, induce a negative bias in the estimation of the integrated covariance. In particular, we prove that the realized covariance estimator of Barndorff-Nielsen and Shephard (2004) of a two-dimensional vector semimartingale with some asset-specific likelihood of repeated prices is asymptotically downward biased, with the bias depending only on the probabilities of repeated price of the two assets. Since these likelihoods can be consistently estimated from transaction prices, a consistent estimator is constructed by adjusting realized covariance for its asymptotic bias.

The finite sample properties of the proposed estimator are assessed through Monte Carlo simulations. In the presence of microstructure noise, the estimator remains unbiased but it is affected by an inflated variance depending on the signal-to-noise ratio. In the presence of rounded prices or a time-varying probability of zero returns, the performances of the proposed estimator slightly deteriorates at high frequency. Nonetheless, even in the worst case scenario, the relative bias of the corrected estimator is orders of magnitudes smaller than that of the standard realized covariance.

		Portfolio liquidity, $N_p = 20$.				
Estimator	Freq. (min)	Very high	High	Medium	Low	Mixed
RC	1	1.1864	1.2130	0.6919	1.5975	0.9037
		(0.025)	(0.045)	(0.019)	(0.007)	(0.000)
	5	1.1998	1.1115*	0.6549*	1.6264	0.8015
		(0.025)	(0.122)	(0.769)	(0.007)	(0.012)
	10	1.2150	1.1165	0.6626*	1.6061	0.7853*
		(0.025)	(0.051)	(0.528)	(0.023)	(0.179)
RC*	1	1.1798	1.1228	0.6769*	1.4435	0.7891*
		0.9944	*0.9257*	*0.9782*	*0.9036*	*0.8732*
		(0.031)	(0.097)	(0.351)	(0.023)	(0.179)
	5	1.1752*	1.0678*	0.6455*	1.4116	0.7729*
		0.9796	*0.9606*	*0.9856*	*0.8679*	*0.9644*
		(0.160)	(0.914)	(0.769)	(0.023)	(0.416)
	10	1.1939	1.0904*	0.6526*	**1.3432***	0.7809*
		0.9827	*0.9767*	*0.9849*	*0.8363*	*0.9943*
		(0.025)	(0.234)	(0.769)	(1.000)	(0.267)
MRK	-	1.1562*	1.1476	0.6720	1.5988	**0.7432***
		(0.207)	(0.051)	(0.066)	(0.023)	(1.000)
HY	-	1.1335*	**1.0637***	0.6502*	1.4090	0.7795*
		(0.272)	(1.000)	(0.769)	(0.034)	(0.383)
QMLE	-	**1.1131***	1.0758*	**0.6411***	1.7431	0.7619*
		(1.000)	(0.914)	(1.000)	(0.007)	(0.542)

Table 6.: Ex-post average variances ($\times 10^4$) of the five randomly selected portfolios of $N_p = 20$ assets. Numbers in parentheses denote p-values of the MCS test. The asterisk implies that the estimator is included in $\mathcal{M}_{90\%}$. For the three RC* estimators, we report *in italics* the ratio between the variance obtained through the bias-corrected estimator and that obtained through the standard realized covariance estimator computed at the same sampling frequency. Bold numbers denote portfolios with lowest absolute variance.

Finally, a horse-race exercise based on minimum variance portfolios, proves empirically that the proposed estimator reduces significantly the ex-post portfolio variance and that its performances are, typically, in line with those of other robust estimators and, in particular, superior in case of illiquid portfolios. Remarkably, the empirical advantages of the bias-corrected estimator are independent of the number of assets held in the portfolio and, although its computational complexity grows as the square of the number of assets, the additional operations involved are simple arithmetic means. Hence, the proposed bias-correction can be applied even in the case of very large datasets, where other methods imply significant data reduction or become computationally cumbersome.

		Portfolio liquidity, $N_p = 30$.				
Estimator	Freq. (min)	Very high	High	Medium	Low	Mixed
RC_n	1	0.5042*	0.7437	0.4882	1.2053	0.7115
		(1.000)	(0.003)	(0.068)	(0.014)	(0.000)
	5	0.5332*	0.7191	0.4597*	1.2427	0.6088*
		(1.000)	(0.098)	(0.740)	(0.014)	(0.108)
	10	0.5347*	0.7149	0.4613*	1.2569*	0.6020*
		(1.000)	(0.084)	(0.740)	(0.387)	(0.369)
$\mathsf{RC}_n^\star$	1	0.5168*	0.7352	0.5100	**1.0251***	0.5925*
		1.0250	*0.9886*	*1.0445*	*0.8505*	*0.8328*
		(1.000)	(0.006)	(0.002)	(1.000)	(0.715)
	5	0.5243*	0.7008*	0.4654*	1.0343*	0.5875*
		0.9832	*0.9747*	*1.0124*	*0.8323*	*0.9650*
		(1.000)	(0.140)	(0.283)	(0.748)	(0.792)
	10	0.5421*	0.7171	0.4575*	1.0263*	0.5960*
		1.0139	*1.0032*	*0.9918*	*0.8166*	*0.9900*
		(1.000)	(0.082)	(0.740)	(0.978)	(0.657)
MRK	-	0.5007*	0.7139	0.4701*	1.0693*	**0.5721***
		(1.000)	(0.098)	(0.283)	(0.746)	(1.000)
HY	-	0.4891*	0.6814*	0.4551*	1.1399*	0.6051*
		(1.000)	(0.250)	(0.740)	(0.387)	(0.413)
QMLE	-	**0.4835***	**0.6638***	**0.4506***	1.2321	0.5805*
		(1.000)	(1.000)	(1.000)	(0.014)	(0.889)

Table 7.: Ex-post average variances ($\times 10^4$) of the five randomly selected portfolios of $N_p = 30$ assets. Numbers in parentheses denote p-values of the MCS test. The asterisk implies that the estimator is included in $\mathcal{M}_{90\%}$. For the three $\mathsf{RC}^\star$ estimators, we report *in italics* the ratio between the variance obtained through the bias-corrected estimator and that obtained through the standard realized covariance estimator computed at the same sampling frequency. Bold numbers denote portfolios with lowest absolute variance.

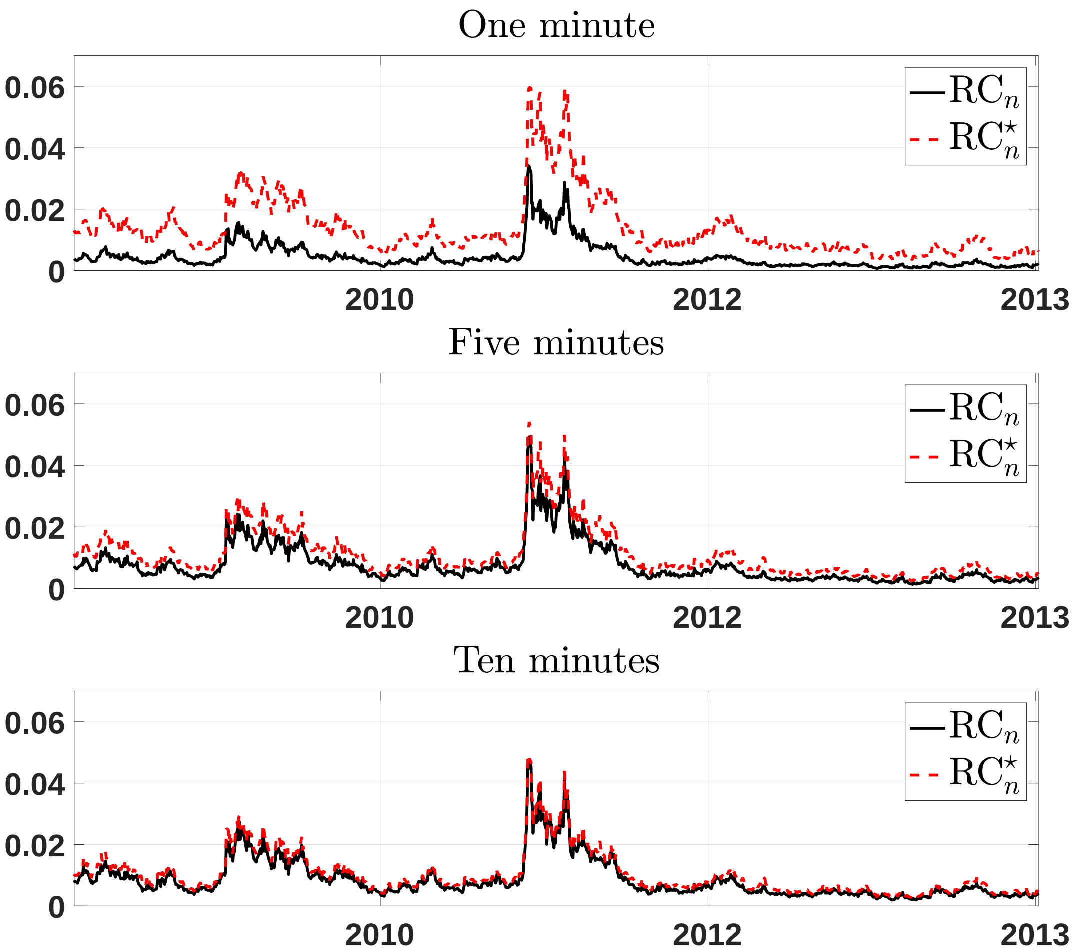

Figure 4.: Averages of off-diagonal elements (reported in percentage) of one-step-ahead covariance matrix forecasts for the portfolio with $N_p = 10$ assets belonging to the group with low liquidity. Black lines are standard realized covariance forecasts while dotted lines are bias-corrected forecasts.

APPENDIX

The Appendix is divided into two parts. Section A introduces the notation and collects auxiliary lemmas. Section B is dedicated to the proof of the main theorem.

Appendix A: Auxiliary Lemmas

In what follows, for a generic index j with $j \in \{1, \ldots, n\}$, we denote by $t_{j,n} = j/n$ the deterministic equispaced partition of the interval $[0, 1]$ and with $\Delta_{j,n} = \Delta_n = t_{j,n} - t_{j-1,n}$ the distance between two consecutive points of the partition. For any stochastic process X we denote by $\Delta_j^n X$ the difference process $\Delta_j^n X \doteq X_{t_{j,n}} - X_{t_{j-1,n}}$. In subsequent statements and proofs, to save upon notation, we write $X_{j,n}$ in place of $X_{t_{j,n}}$ whenever this does not cause any ambiguity.
Additionally, for a generic index j with $j \in \{1, \ldots, n\}$, we denote by $\mathbb{P}_j[\,\cdot\,]$, $\mathbb{E}_j[\,\cdot\,]$, and $\mathbb{V}_j[\,\cdot\,]$ the conditional probability, the conditional expectation, and the conditional variance with respect to a suitable filtration $\mathcal{F}_{t_{j,n}}$. Finally, $\xrightarrow{p}$ denotes the convergence in probability.
First, we provide lemmas regarding some statistics and asymptotic results for $K_{j,n}^{(\ell)}$, $\ell \in \{1, 2\}$.

Lemma 1 *For each $n \in \mathbb{N}$ and each $j \in \{1, \ldots, n\}$, let $K_{j,n}^{(1)}$ and $K_{j,n}^{(2)}$ be defined as in Definition (3), and let $\bar{B}_{j,n}^{(\ell)} \doteq \left(1 - B_{j,n}^{(\ell)}\right)$ and $p_{n,\ell} = \mathbb{E}\left[B_{j,n}^{(\ell)}\right]$, $\ell \in \{1, 2\}$. Then, the following equality hold true:*

$$\mu_{j,n}^{(\ell)} \doteq \mathbb{E}\left[K_{j,n}^{(\ell)}\right] = \frac{p_{n,\ell}\left(1 - p_{n,\ell}^j\right)}{1 - p_{n,\ell}},$$

$$\nu_{j,n}^{(\ell)} \doteq \mathbb{E}\left[\left(K_{j,n}^{(\ell)}\right)^2\right] = \frac{p_{n,\ell}\left(1 + p_{n,\ell} - \left(2\,j\,\left(1 - p_{n,\ell}\right) + 1 + p_{n,\ell}\right) p_{n,\ell}^j\right)}{\left(1 - p_{n,\ell}\right)^2},$$

$$\Phi_{j,n}^{(1,2)} \doteq \mathbb{E}\left[K_{j,n}^{(1)} \mathbb{1}_{\left\{K_{j,n}^{(1)} \leqslant K_{j,n}^{(2)}\right\}}\right] = \frac{p_{n,1}\left(p_{n,\star}^j\left(j\left(1 - p_{n,2}\right)\left(1 - p_{n,\star}\right) - \left(1 - p_{n,1}\right) p_{n,2}\right) - p_{n,\star} + p_{n,2}\right)}{\left(1 - p_{n,1}p_{n,2}\right)^2},$$

$$\Phi_{j,n}^{(1,2,2)} \doteq \mathbb{E}\left[\left(K_{j,n}^{(1)}\right)^2 \mathbb{1}_{\left\{K_{j,n}^{(1)} \leqslant K_{j,n}^{(2)}\right\}}\right] = \frac{p_{n,1}\left((p_{n,1} - 1)p_{n,2}(p_{n,\star} + 1)\right)}{(p_{n,\star} - 1)^3},$$

$$+ \frac{p_{n,1}\left(p_{n,\star}^j\left(j^2(p_{n,2} - 1)(p_{n,\star} - 1)^2 + 2j(p_{n,1} - 1)p_{n,2}(p_{n,\star} - 1) - (p_{n,1} - 1)p_{n,2}(p_{n,\star} + 1)\right)\right)}{(p_{n,\star} - 1)^3},$$

where $p_{n,\star} \doteq p_{n,1}p_{n,2}$.

Proof. First, notice that since $B_{0,n}^{(1)} \equiv 0$, i.e. at $t_{0,n}$ the observed price coincides with the efficient one, we have that $\mathbb{P}\left[K_{j,n}^{(1)} = j\right] = (p_{n,1})^j$ and, besides, that asymptotic results are not influenced by the initial condition. Therefore, we have:

$$K_{j,n}^{(1)} = \begin{cases} 0 & \text{with probability } 1 - p_{n,1} \\ 1 & \text{with probability } \left(1 - p_{n,1}\right) p_{n,1} \\ 2 & \text{with probability } \left(1 - p_{n,1}\right) \left(p_{n,1}\right)^2 \\ \vdots & \vdots \\ j - 1 & \text{with probability } \left(1 - p_{n,1}\right) \left(p_{n,1}\right)^{j-1} \\ j & \text{with probability } \left(p_{n,1}\right)^j . \end{cases}.$$

At this point it is a matter of elementary calculations to compute the desired quantities.

$$\begin{aligned}\mu_{j,n} &\doteq \mathbb{E}\left[K^{(1)}_{j,n}\right] = 0\cdot(1-p_{n,1}) + 1\cdot(1-p_{n,1})\,p_{n,1} + 2\,(1-p_{n,1})\,p^2_{n,1} + \ldots + (j-1)\,(1-p_{n,1})\,p^{j-1}_{n,1} + j\,p^j_{n,1}\\ &= \frac{p_{n,1}\left(1-p^j_{n,1}\right)}{1-p_{n,1}}.\end{aligned}$$

$$\begin{aligned}\nu_{j,n} &\doteq \mathbb{E}\left[\left(K^{(1)}_{j,n}\right)^2\right] = 0\cdot(1-p_{n,1}) + 1\,(1-p_{n,1})\,p_{n,1} + 2^2\,(1-p_{n,1})\,p^2_{n,1} + \ldots + (j-1)^2\,(1-p_{n,1})\,p^{j-1}_{n,1} + j^2\,p^j_{n,1}\\ &= \frac{p_{n,1}\,\left(1+p_{n,1}-\left(2j\,(1-p_{n,1})+1+p_{n,1}\right)x^j_n\right)}{(1-p_{n,1})^2},\end{aligned}$$

$$\begin{aligned}\Phi^{(1,2)}_{j,n} &= \mathbb{E}\left[\mathbb{E}\left[K^{(1)}_{j,n}\,\mathbb{1}_{\left\{K^{(1)}_{j,n}\leqslant K^{(2)}_{j,n}\right\}}\,\middle|\,K^{(1)}_{j,n}\right]\right] = \mathbb{E}\left[K^{(1)}_{j,n}\left((1-p_{n,2})\sum_{q=K^{(1)}_{j,n}}^{j-1}p^q_{n,2}+p^j_{n,2}\right)\right]\\ &= \mathbb{E}\left[K^{(1)}_{j,n}\left(p^{K^{(1)}_{j,n}}_{n,2}-p^j_{n,2}+p^j_{n,2}\right)\right]\\ &= \mathbb{E}\left[K^{(1)}_{j,n}\,p^{K^{(1)}_{j,n}}_{n,2}\right] = \frac{p_{n,1}\left(p^j_{n,1}p^j_{n,2}(j(1-p_{n,2})(1-p_{n,1}p_{n,2})-(1-p_{n,2})p_{n,2})-p_{n,2}p_{n,2}+p_{n,2}\right)}{(1-p_{n,1}p_{n,2})^2},\end{aligned}$$

and, by using the same strategy one can easily derives an expression for $\Phi^{(1,2)}_{j,n}$. $\square$

Lemma 2 *For each $n\in\mathbb{N}$ and each $j\in\{1,\ldots,n\}$, let $K^{(1)}_{j,n}$ and $K^{(2)}_{j,n}$ be defined as in Definition* (3). *Let η_t be any bounded stochastic process path-wise Riemann integrable and independent of $K^{(1)}_{j,n}$, and let $\bar{B}^{(1)}_{j,n}=\left(1-B^{(2)}_{j,n}\right)$ and $\bar{B}^{(2)}_{j,n}$. We have:*

$$\sum_{j=0}^{n-1}\bar{B}^{(1)}_{j+1,n}\bar{B}^{(2)}_{j+1,n}\,\eta_{j,n}\,\Delta_n\,K^{(1)}_{j,n}\xrightarrow{p}\mu_K\int_0^1\eta_s\,ds, \tag{A1}$$

where the constant μ_K is given by

$$\mu_K=\lim_{j\to\infty,\ j\leqslant n}\mathbb{E}\left[\bar{B}^{(1)}_{j+1,n}\,\bar{B}^{(2)}_{j+1,n}\,K^{(1)}_{j,n}\right]=p_1\,(1-p_2)\,, \tag{A2}$$

and

$$\sum_{j=0}^{n-1}\bar{B}^{(1)}_{j+1,n}\bar{B}^{(2)}_{j+1,n}\,K^{(1)}_{j,n}\eta_{j,n}\,\Delta_n\,\mathbb{1}_{\left\{K^{(1)}_{j,n}\leqslant K^{(2)}_{j,n}\right\}}\xrightarrow{p}\bar{\mu}_K\int_0^1\eta_s\,ds, \tag{A3}$$

where the constant $\bar{\mu}_K$ is given by

$$\bar{\mu}_K=\lim_{j\to\infty,\ j\leqslant n}\mathbb{E}\left[\bar{B}^{(1)}_{j+1,n}\bar{B}^{(2)}_{j+1,n}\,K^{(1)}_{j,n}\,\mathbb{1}_{\left\{K^{(1)}_{j,n}\leqslant K^{(2)}_{j,n}\right\}}\right]=\frac{(1-p_1)^2\,(1-p_2)\,p_1\,p_2}{(1-p_1\,p_2)^2}. \tag{A4}$$

Moreover, if the indicator function in Eq.(A3) *is replaced with $\mathbb{1}_{\left\{K^{(1)}_{j,n}=K^{(2)}_{j,n}\right\}}$, then the constant $\bar{\mu}_K$*

is given by

$$\bar{\mu}_K = \frac{(1-p_1)^2\,(1-p_2)^2\,p_1\,p_2}{(1-p_1\,p_2)^2}. \tag{A5}$$

Before proceeding with the proof, we need the following result.

LEMMA 3 *Let $f\,:[0,1]\to\mathbb{R}$ any Riemann-integrable bounded function. For all n let $s_{j,n}\geqslant 0$ with $j=0,\ldots,n$ a positive sequence of real numbers such that $s_{j,n}\to S$ when $j\to\infty$ and*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n}|S-s_{j,n}|=0.$$

Therefore

$$\sum_{j=0}^{n} f\left(t_{j,n}\right)\,\Delta_{j,n}\,s_{j,n}\to S\int_0^1 f\left(t\right)\,dt.$$

Proof. Immediate:

$$\frac{1}{n}\sum_{j=0}^{n} f\left(t_{j,n}\right)s_{j,n} = S\underbrace{\sum_{j=0}^{n} f\left(t_{j,n}\right)\,\Delta_{j,n}}_{\mathrm{A}_n}-\underbrace{\sum_{j=0}^{n} f\left(t_{j,n}\right)\,\Delta_{j,n}\,(S-s_{j,n})}_{\mathrm{B}_n},$$

since $\mathrm{A}_n\to\int_0^1 f\left(t\right)\,dt$ using the Riemann-integrability and $\mathrm{B}_n\to 0$ is such that:

$$|\mathrm{B}_n|\leqslant\sum_{j=0}^{n}|f\left(t_{j,n}\right)\,\Delta_{j,n}|\;|S-s_{j,n}|\leqslant C\,\bar{\Delta}_n\sum_{j=0}^{n}|S-s_{j,n}|\to 0,$$

where $\bar{\Delta}_n=\max_{j=1,\ldots,n}\Delta_{j,n}$ and $C=\sup_{t\in[0,1]}|f\left(t\right)|<\infty$. □

Proof of Lemma 2. We write $Z_{j,n}\doteq\bar{B}^{(1)}_{j+1,n}\bar{B}^{(2)}_{j+1,n}K^{(1)}_{j,n}$ and $L_{j,n}=\bar{B}^{(1)}_{j+1,n}\bar{B}^{(2)}_{j+1,n}K^{(1)}_{j,n}\mathbb{1}_{\left\{K^{(1)}_{j,n}\leqslant K^{(2)}_{j,n}\right\}}$ and we notice that they form a triangular array of *dependent* random variable. Thus, to prove the convergence in probability of Riemman sums of the type $\mathrm{R}_n=\sum_{j=0}^{n}\eta_{j,n}\,\Delta_n\,Z_{j,n}$, we have to show that the following two conditions hold:

i) $\mathbb{V}\mathrm{ar}\left[Z_{j,n}\right]\doteq\mathbb{E}\left[\left(Z_{j,n}-\mathbb{E}\left[Z_{j,n}\right]\right)^2\right]\xrightarrow{j\to\infty}\sigma^2_\infty<\infty$

ii) $\frac{1}{n^2}\sum_{j=1}^{n}\sum_{k=1}^{n-1}\mathbb{C}\mathrm{ov}\left[Z_{j,n},Z_{j+k,n}\right]\xrightarrow{n\to\infty}0$.

The same reasoning applies to the triangular array $(L_{j,n})$. In what follows, we prove results in (A1) and (A3). The proof of the result in (A5) is omitted since it can be easily obtained from that of (A3) with minor changes. Additionally, for sake of convention, we use the following notation: $x_n=p_{n,1}$ and $x=p_1$, so that $x_n\to x$ when $n\to\infty$.
We start from the result in Eq.(A1) and in particular from the first condition i). By using Lemma

(1), the variance of $Z_{j,n}$ is given by:

$$\begin{aligned}\mathbb{V}\mathrm{ar}\left[Z_{j,n}\right] &= \mathbb{E}\left[Z_{j,n}^2\right] - \mathbb{E}\left[Z_{j,n}\right]^2 = \mathbb{E}\left[\left(\bar{B}^{(1)}_{j+1,n}\,\bar{B}^{(2)}_{j+1,n}\,K^{(1)}_{j,n}\right)^2\right] - \mathbb{E}\left[\bar{B}^{(1)}_{j+1,n}\,\bar{B}^{(2)}_{j+1,n}\,K^{(1)}_{j,n}\right]^2\\
&= \mathbb{E}\left[\bar{B}^{(1)}_{j+1,n}\right]\mathbb{E}\left[\bar{B}^{(2)}_{j+1,n}\right]\mathbb{E}\left[\left(K^{(1)}_{j,n}\right)^2\right] - \mathbb{E}\left[\bar{B}^{(1)}_{j+1,n}\right]^2\mathbb{E}\left[\bar{B}^{(2)}_{j+1,n}\right]^2\mathbb{E}\left[\left(K^{(1)}_{j,n}\right)\right]^2\\
&= (1-x_n)\,(1-y_n)\,\nu_{j,n} - (1-x_n)^2\,(1-y_n)^2\,\mu_{j,n}^2 \xrightarrow{j\to\infty} \frac{x(y-1)\left(x^2(y-1)-xy-1\right)}{1-x} < \infty\end{aligned}$$

Regarding the computation of the covariance between $Z_{j,n}$ and $Z_{j+k,n}$, we first notice that – similar expressions holds for $K^{(2)}_{j+k,n}$:

$$K^{(1)}_{j+1,n} = B^{(1)}_{j+1,n} + B^{(1)}_{j+1,n}\,K^{(1)}_{j,n}.$$

Therefore:

$$K^{(1)}_{j+k,n} = B^{(1)}_{j+k,n} + B^{(1)}_{j+k,n}\,B^{(1)}_{j+k-1,n} + B^{(1)}_{j+k,n}\,B^{(1)}_{j+k-1,n}B^{(1)}_{j+k-2,n} + ... + B^{(1)}_{j+k,n}\cdots B^{(1)}_{j+1,n}K^{(1)}_{j,n},$$

and, as a consequence we have:

$$\begin{aligned}&\mathbb{E}\left[Z_{j,n}\,Z_{j+k,n}\right] = \mathbb{E}\left[K^{(1)}_{j,n}\,\bar{B}^{(1)}_{j+1,n}\,\bar{B}^{(2)}_{j+1,n}\,K^{(1)}_{j+k,n}\,\bar{B}^{(1)}_{j+k+1,n}\,\bar{B}^{(2)}_{j+k+1,n}\right] = \mathbb{E}\left[\bar{B}^{(2)}_{j+k+1,n}\,\bar{B}^{(2)}_{j+1,n}\,\bar{B}^{(1)}_{j+k+1,n}\right]\times\\
&\times\mathbb{E}\left[K^{(1)}_{j,n}\left(B^{(1)}_{j+k,n} + B^{(1)}_{j+k,n}\,B^{(1)}_{j+k-1,n} + B^{(1)}_{j+k,n}\,B^{(1)}_{j+k-1,n}B^{(1)}_{j+k-2,n} + ... + B^{(1)}_{j+k,n}\cdots B^{(1)}_{j+1,n}K^{(1)}_{j,n}\right)\bar{B}^{(1)}_{j+1,n}\right]\\
&= (1-y_n)^2\,(1-x_n)\times\\
&\times\mathbb{E}\left[K^{(1)}_{j,n}\left(B^{(1)}_{j+k,n} + B^{(1)}_{j+k,n}\,B^{(1)}_{j+k-1,n} + B^{(1)}_{j+k,n}\,B^{(1)}_{j+k-1,n}B^{(1)}_{j+k-2,n} + ... + B^{(1)}_{j+k,n}\cdots B^{(1)}_{j+2,n}\right)\left(1-B^{(1)}_{j+1,n}\right)\right]\\
&= (1-y_n)^2\,(1-x_n)\,\mathbb{E}\left[K^{(1)}_{j,n}\right]\left(x_n + x_n^2 + ... + x_n^{k-1}\right)(1-x_n) = x_n\,\left(1-x_n^{k-1}\right)(1-y_n)^2\,(1-x_n)\,\mu_{j,n}.\end{aligned}$$

The auto-covariance at lag k of $(Z_{j,n})$ is given by:

$$\begin{aligned}\mathbb{C}\mathrm{ov}\left[Z_{j,n}, Z_{j+k,n}\right] &= \mathbb{E}\left[Z_{j,n}Z_{j+k,n}\right] - \mathbb{E}\left[Z_{j,n}\right]\mathbb{E}\left[Z_{j+k,n}\right]\\
&= x_n\left(1-x_n^{k-1}\right)(1-y_n)^2\,(1-x_n)\,\mu_{j,n} - (1-x_n)^2\,(1-y_n)^2\,\mu_{j,n}\,\mu_{j+k,n}\\
&= -(1-y_n)^2\left(1-x_n^j\right)\left(1-x_n^{j+1}\right)x_n^{k+1}.\end{aligned}$$

In particular, it holds that :

$$\frac{1}{n^2}\sum_{j=1}^{n}\sum_{k=1}^{n-j}\mathbb{C}\mathrm{ov}\left[Z_{j,n}, Z_{j+k,n}\right] = \frac{1}{n^2}\frac{x^2(y-1)^2\left(n\left(x^2-1\right)\left((x+1)x^n+1\right)-\left(x^n-1\right)\left(x\left(x\left(x^n+2\right)+2\right)+1\right)\right)}{(x-1)^2\,(x+1)} \xrightarrow{n\to\infty} 0,$$

i.e. both conditions i) and ii) are satisfied. The conclusion readily derives by applying Lemma 3 to $s_{j,n} = \mathbb{E}\left[Z_{j,n}\right]$ with $S = x(1-y)$. Therefore:

$$\frac{1}{n}\sum_{j=0}^{n}\eta_{t_{j,n}}\,\mathbb{E}\left[\bar{B}^{(1)}_{j+1,n}\,\bar{B}^{(2)}_{j+1,n}\,K^{(1)}_{j,n}\right] \xrightarrow{p} p_1\,(1-p_2)\int_0^1\eta_s\,ds$$

path-wise on Ω.
We proceed with the result in Eq.(A3). Again, we need to show that conditions i) and ii) are

satisfied. By using Lemma (1), the variance of $L_{j,n}$ is given by:

$$
\begin{aligned}
\mathbb{V}\mathrm{ar}\left[L_{j,n}\right] &= \mathbb{E}\left[L_{j,n}^2\right] - \mathbb{E}\left[L_{j,n}\right]^2 \\
&= \mathbb{E}\left[\left(\bar{B}^{(1)}_{j+1,n}\,\bar{B}^{(2)}_{j+1,n}\,K^{(1)}_{j,n}\mathbb{1}_{\left\{K^{(1)}_{j,n}\leqslant K^{(2)}_{j,n}\right\}}\right)^2\right] - \mathbb{E}\left[\bar{B}^{(1)}_{j+1,n}\,\bar{B}^{(2)}_{j+1,n}\,K^{(1)}_{j,n}\mathbb{1}_{\left\{K^{(1)}_{j,n}\leqslant K^{(2)}_{j,n}\right\}}\right]^2 \\
&= \mathbb{E}\left[\bar{B}^{(1)}_{j+1,n}\right]\mathbb{E}\left[\bar{B}^{(2)}_{j+1,n}\right]\mathbb{E}\left[\left(K^{(1)}_{j,n}\right)^2\mathbb{1}_{\left\{K^{(1)}_{j,n}\leqslant K^{(2)}_{j,n}\right\}}\right] - \mathbb{E}\left[\bar{B}^{(1)}_{j+1,n}\right]^2\mathbb{E}\left[\bar{B}^{(2)}_{j+1,n}\right]^2\mathbb{E}\left[\left(K^{(1)}_{j,n}\right)^2\mathbb{1}_{\left\{K^{(1)}_{j,n}\leqslant K^{(2)}_{j,n}\right\}}\right]^2 \\
&= (1-x_n)(1-y_n)\Phi^{(1,2,2)}_{j,n} - (1-x_n)^2(1-y_n)^2\left(\Phi^{(1,2,2)}_{j,n}\right)^2 \\
&\xrightarrow{j\to\infty} \frac{(1-x)^2x(1-y)y(xy(x(x(y-1)-3y+2)+y-1)+1)}{(1-xy)^4} < \infty.
\end{aligned}
$$

To prove ii), we need some additional steps. Let $\mathrm{I}_{j+k,n}$ be defined as $\mathrm{I}_{j+k,n} \doteq \mathbb{1}_{\left\{K^{(1)}_{j+k,n}\leqslant K^{(2)}_{j+k,n}\right\}}$. Then:

$$
\mathrm{I}_{j+k,n} = \left(1 - B^{(1)}_{j+k,n}\right) + B^{(1)}_{j+k,n}\,B^{(2)}_{j+k,n}\,\mathbb{1}_{\left\{K^{(1)}_{j+k-1,n}\leqslant K^{(2)}_{j+k-1,n}\right\}} \doteq \bar{B}^{(1)}_{j+k,n} + C_{j+k,n}\mathrm{I}_{j+k-1,n},
$$

where $C_{j+k,n} = B^{(1)}_{j+k,n}B^{(1)}_{j+k,n}$. The solution of the recursion in the previous equation is given by:

$$
\begin{aligned}
\mathrm{I}_{j+k,n} = {} & \bar{B}^{(1)}_{j+k,n,n} + C_{j+k,n}\,\bar{B}^{(1)}_{j+k-1,n,n} + C_{j+k,n}\,C_{j+k-1,n}\,\bar{B}^{(1)}_{j+k-2,n,n} \\
& + \ldots + C_{j+k,n}\,C_{j+k-1,n}\,\cdots\,C_{j+2,n}\,\bar{B}^{(1)}_{j+1,n,n} + C_{j+k,n}\cdots C_{j+1,n}\,\mathrm{I}_{j,n}.
\end{aligned}
$$

By using the property that $\mathrm{I}^2_{j,n} = \mathrm{I}_{j,n}$ and $C_{j+1,n}\,\bar{B}^{(1)}_{j+1,n,n}\,\bar{B}^{(2)}_{j+1,n,n} \equiv 0$, we write

$$
\begin{aligned}
& \mathrm{I}_{j+k,n}\,\mathrm{I}_{j,n}\,\bar{B}^{(1)}_{j+1,n,n}\,\bar{B}^{(2)}_{j+1,n,n} = \\
& \mathrm{I}_{j,n}\,\bar{B}^{(1)}_{j+1,n,n}\,\bar{B}^{(2)}_{j+1,n,n}\left(\bar{B}^{(1)}_{j+k,n,n} + C_{j+k,n}\,\bar{B}^{(1)}_{j+k-1,n,n} + \right. \\
& \left. + C_{j+k,n}\,C_{j+k-1,n}\,\bar{B}^{(1)}_{j+k-2,n,n} + \ldots + C_{j+k,n}\,C_{j+k-1,n}\,\cdots\,C_{j+2,n}\,\bar{B}^{(1)}_{j+1,n,n}\right) + \\
& + C_{j+k,n}\cdots C_{j+1,n}\,\bar{B}^{(1)}_{j+1,n,n}\,\bar{B}^{(2)}_{j+1,n,n}\mathrm{I}_{j,n}. \\
= {} & \mathrm{I}_{j,n}\,\bar{B}^{(1)}_{j+1,n,n}\,\bar{B}^{(2)}_{j+1,n,n}\left(\bar{B}^{(1)}_{j+k,n,n} + C_{j+k,n}\,\bar{B}^{(1)}_{j+k-1,n,n} + \right. \\
& \left. + C_{j+k,n}\,C_{j+k-1,n}\,\bar{B}^{(1)}_{j+k-2,n,n} + \ldots + C_{j+k,n}\,C_{j+k-1,n}\,\cdots\,C_{j+2,n}\,\bar{B}^{(1)}_{j+1,n}\right).
\end{aligned}
$$

Since, for any $q \leqslant j+k$, it holds that

$$
\begin{aligned}
\mathbb{E}\left[C_{q,n}\,\middle|\,K^{(1)}_{j+k,n}\right] = B^{(1)}_{q,n}\,y_n,\ \mathbb{E}&\left[\bar{B}^{(1)}_{j+1,n}\,\bar{B}^{(2)}_{j+1,n}\,\bar{B}^{(1)}_{q,n}\,\middle|\,K^{(1)}_{j+k,n}\right] \\
&= (1-y_n)\ \bar{B}^{(1)}_{j+1,n}\,\bar{B}^{(1)}_{q,n}
\end{aligned} \tag{A6}
$$

and $\mathbb{E}\left[\mathrm{I}_{j,n} \;\middle|\; K^{(1)}_{j+k,n}\right] = y_n^{K^{(1)}_{j,n}}$, we get

$$\mathbb{E}\left[\mathrm{I}_{j+k,n}\,\mathrm{I}_{j,n}\,\bar{B}^{(1)}_{j+1,n,n}\,\bar{B}^{(2)}_{j+1,n,n} \;\middle|\; K^{(1)}_{j+k,n}\right] = (1-y_n)\; y_n^{K^{(1)}_{j,n}}\,\bar{B}^{(1)}_{j+1,n}\left(\bar{B}^{(1)}_{j+k,n} + y_n\,B^{(1)}_{j+k,n}\,\bar{B}^{(1)}_{j+k-1,n} + \right.$$
$$\left. + y_n^2\,B^{(1)}_{j+k,n}\,B^{(1)}_{j+k-1,n}\,\bar{B}^{(1)}_{j+k-2,n} + \ldots + y_n^{k-1}\,B^{(1)}_{j+k,n}\,B^{(1)}_{j+k-1,n}\cdots B^{(1)}_{j+2,n}\,\bar{B}^{(1)}_{j+1,n}\right).$$

At this point, we can compute the following expected value:

$$\begin{aligned}\mathbb{E}\left[L_{j,n}L_{j+k,n}\right] &= \mathbb{E}\left[K^{(1)}_{j,n}\,K^{(1)}_{j+k,n}\,\mathrm{I}_{j+k,n}\,\mathrm{I}_{j,n}\,\bar{B}^{(1)}_{j+k+1,n}\,\bar{B}^{(1)}_{j+1,n}\,\bar{B}^{(2)}_{j+k+1,n}\,\bar{B}^{(2)}_{j+1,n}\right]\\ &= (1-y_n)\;(1-x_n)\;\left(y_n^0\,A_1 + y_n^1\,A_2 + ... + y_n^{k-1}\,A_k\right),\end{aligned}$$

where the quantities $A_1, A_2, ..., A_k$ are computed as:

$$\begin{aligned}A_1 &= \mathbb{E}\left[y^{K^{(1)}_{j,n}}\,\bar{B}^{(1)}_{j+k,n}\,\bar{B}^{(1)}_{j+1,n,n}\,\bar{B}^{(2)}_{j+1,n,n}\,K_{j,n}\,K_{j+k,n}\right]\\ &= \mathbb{E}\left[y^{K^{(1)}_{j,n}}\,\bar{B}^{(1)}_{j+k,n}\,K_j\;\left(B_{j+k,n} + B_{j+k,n}\,K_{j+k-1,n}\right)\;\bar{B}^{(1)}_{j+1,n,n}\,\bar{B}^{(2)}_{j+1,n,n}\right] = 0\\ A_2 &= \mathbb{E}\left[y^{K^{(1)}_{j}}\,B_{j+k,n}\,\bar{B}_{j+k-1,n}\,K_j\,K_{j+k,n}\,\bar{B}^{(1)}_{j+1,n,n}\,\bar{B}^{(2)}_{j+1,n,n}\right]\\ &= \mathbb{E}\left[y^{K^{(1)}_{j}}\,B_{j+k,n}\,\bar{B}_{j+k-1,n}\,K_j\;\left(B_{j+k,n} + B_{j+k,n}\,K_{j+k-1,n}\right)\;\bar{B}^{(1)}_{j+1,n,n}\,\bar{B}^{(2)}_{j+1,n,n}\right]\\ &= \mathbb{E}\left[y^{K^{(1)}_{j}}\,B_{j+k,n}\,\bar{B}_{j+k-1,n}\,K_j\,B_{j+k,n}\,\bar{B}^{(1)}_{j+1,n,n}\,\bar{B}^{(2)}_{j+1,n,n}\right]\\ &= \mathbb{E}\left[y^{K^{(1)}_{j}}\,B_{j+k,n}\,\bar{B}_{j+k-1,n}\,K_j\,\bar{B}^{(1)}_{j+1,n,n}\,\bar{B}^{(2)}_{j+1,n,n}\right] = x\;(1-x)^2\;(1-y)\;\Phi^{(1,2)}_{j,n}\\ A_3 &= \mathbb{E}\left[y^{K^{(1)}_{j}}\,B_{j+k,n}\,B_{j+k-1,n}\,\bar{B}^{(1)}_{j+k-2,n}\,K_{j,n}\,K_{j+k,n}\,\bar{B}^{(1)}_{j+1,n,n}\,\bar{B}^{(2)}_{j+1,n,n}\right]\\ &= 2\,\mathbb{E}\left[y^{K^{(1)}_{j}}\,B_{j+k,n}\,B_{j+k-1,n}\,\bar{B}^{(1)}_{j+k-2,n}\,K_{j,n}\,\bar{B}^{(1)}_{j+1,n,n}\,\bar{B}^{(2)}_{j+1,n,n}\right]\\ &= 2\,x_n^2\;(1-x_n)^2\;(1-y_n)\;\Phi^{(1,2)}_{j,n}\end{aligned}$$

and so on and so forth. In summary, for a generic $q \leqslant k-1$, the corresponding quantity A_q results:

$$\begin{aligned}A_q &= \mathbb{E}\left[y^{K^{(1)}_{j,n}}\,B^{(1)}_{j+k,n}\,B^{(1)}_{j+k-1,n}\cdots B^{(1)}_{j+k-(q-2),n}\bar{B}^{(1)}_{j+k-(q-1),n}\,K^{(1)}_{j,n}\,K^{(1)}_{j+k,n}\,\bar{B}^{(1)}_{j+1,n,n}\,\bar{B}^{(2)}_{j+1,n}\right]\\ &= \mathbb{E}\Bigg[y^{K^{(1)}_{j,n}}\,B^{(1)}_{j+k,n}\,B^{(1)}_{j+k-1,n}\cdots B^{(1)}_{j+k-(q-2),n}\bar{B}^{(1)}_{j+k-(q-1),n}\,K^{(1)}_{j,n}\times\\ &\quad\times\Bigg(B^{(1)}_{j+k,n} + B^{(1)}_{j+k,n}\,B^{(1)}_{j+k-1,n} + \ldots + B^{(1)}_{j+k,n}\,B^{(1)}_{j+k-1,n}\cdots B^{(1)}_{j+k-(q-2),n} +\\ &+ B^{(1)}_{j+k,n}\,B^{(1)}_{j+k-1,n}\cdots B^{(1)}_{j+k-(q-2),n}\,K^{(1)}_{j+k-(q-1),n}\times\bar{B}^{(1)}_{j+1,n,n}\;\bar{B}^{(2)}_{j+1,n,n}\Bigg)\Bigg]\\ &= (q-1)\,\mathbb{E}\left[y^{K^{(1)}_{j,n}}\,B^{(1)}_{j+k,n}\,B^{(1)}_{j+k-1,n}\cdots B^{(1)}_{j+k-(q-2),n}\bar{B}^{(1)}_{j+k-(q-1),n}\,K^{(1)}_{j,n}\;\bar{B}^{(1)}_{j+1,n}\;\bar{B}^{(2)}_{j+1,n}\right]\\ &= (q-1)\,x_n^{q-1}\;(1-x_n)^2\;(1-y_n)\;\Phi^{(1,2)}_{j,n},\end{aligned}$$

whereas for $q = k$ we have:

$$
\begin{aligned}
A_k &= \mathbb{E}\left[y^{K^{(1)}_{j,n}}\, B^{(1)}_{j+k,n}\, B^{(1)}_{j+k-1,n} \cdots B^{(1)}_{j+2,n} \bar{B}^{(1)}_{j+1,n}\, K^{(1)}_{j,n}\, K^{(1)}_{j+k,n}\, \bar{B}^{(1)}_{j+1,n,n}\, \bar{B}^{(2)}_{j+1,n}\right] \\
&= \mathbb{E}\Bigg[y^{K^{(1)}_{j,n}}\, B^{(1)}_{j+k,n}\, B^{(1)}_{j+k-1,n} \cdots B^{(1)}_{j+2,n} \bar{B}^{(1)}_{j+1,n}\, K^{(1)}_{j,n} \times \\
&\quad \times \Bigg(B^{(1)}_{j+k,n} + B^{(1)}_{j+k,n}\, B^{(1)}_{j+k-1,n} + \ldots + B^{(1)}_{j+k,n}\, B^{(1)}_{j+k-1,n} \cdots B^{(1)}_{j+2,n} + \\
&\quad + B^{(1)}_{j+k,n}\, B^{(1)}_{j+k-1,n} \cdots B^{(1)}_{j+2,n}\, K^{(1)}_{j+1,n}\Bigg)\, \bar{B}^{(2)}_{j+1,n,n}\Bigg] \\
&= (k-1)\, \mathbb{E}\left[y^{K^{(1)}_{j,n}}\, B^{(1)}_{j+k,n}\, B^{(1)}_{j+k-1,n} \cdots B^{(1)}_{j+2,n} \bar{B}^{(1)}_{j+1,n}\, K^{(1)}_{j,n}\; \bar{B}^{(2)}_{j+1,n,n}\right] \\
&= (k-1)\, x_n^{k-1}\, (1-x_n)\, (1-y_n)\, \Phi^{(1,2)}_{j,n}.
\end{aligned}
$$

In particular:

$$
\mathbb{E}\left[L_{j,n} L_{j+k,n}\right] = (1-x_n)^3\, (1-y_n)^2\, \Phi^{(1,2)}_{j,n} \sum_{q=1}^{k-1} y_n^{q-1}\, (q-1)\, x_n^{q-1} (k-1)\, y_n^{k-1}\, x_n^{k-1}\, (1-x_n)^2\, (1-y_n)^2\, \Phi^{(1,2)}_{j,n},
$$

and:

$$
\begin{aligned}
\mathbb{C}\mathrm{ov}\left[L_{j,n}, L_{j+k,n}\right] &= \mathbb{E}\left[L_{j,n} L_{j+k,n}\right] - \mathbb{E}\left[L_{j,n}\right] \mathbb{E}\left[L_{j+k,n}\right] \\
&= (1-x_n)^3\, (1-y_n)^2\, \Phi^{(1,2)}_{j,n} \sum_{q=1}^{k-1} y_n^{q-1}\, (q-1)\, x_n^{q-1} + (k-1)\, y_n^{k-1}\, x_n^{k-1}\, (1-x_n)^2\, (1-y_n)^2\, \Phi^{(1,2)}_{j,n} \\
&\quad - (1-x_n)^2\, (1-y_n)^2\, \Phi^{(1,2)}_{j,n}\, \Phi^{(1,2)}_{j+k,n}.
\end{aligned}
$$

In particular, it holds that:

$$
\lim_{n\to\infty} \frac{1}{n^2} \sum_{j=1}^{n} \sum_{k=1}^{n-j} \mathbb{C}\mathrm{ov}\left[L_{j,n}, L_{j+k,n}\right] = 0.
$$

The conclusion readily derives by applying Lemma 3 to $s_{j,n} = \mathbb{E}\left[L_{j,n}\right]$ with:

$$
S = \frac{(1-x)^2\,(1-y)\,x\,y}{(1-x\,y)}.
$$

Therefore, path-wise on Ω,

$$
\frac{1}{n} \sum_{j=0}^{n} \eta_{t_{j,n}}\, \mathbb{E}\left[\bar{B}^{(1)}_{j+1,n}\, \bar{B}^{(2)}_{j+1,n}\, K^{(1)}_{j,n}\, \mathbb{1}_{\left\{K^{(1)}_{j,n} \leqslant K^{(2)}_{j,n}\right\}}\right] \xrightarrow{p} \frac{(1-p_1)^2\, (1-p_2)\, p_1 p_2}{(1-p_1\, p_2)^2} \int_0^1 \eta_s\, ds.
$$

□

Lemma 4 *Let $K^{(1)}_{j,n}$ and $K^{(2)}_{j,n}$ be defined as in Definition 3. Let η_s be any bounded stochastic process path-wise Riemann integrable and independent from $K^{(1)}_{j,n}$. Then*

$$
\sum_{j=0}^{n} \bar{B}^{(1)}_{j+1,n}\, \bar{B}^{(2)}_{j+1,n}\, \eta_{t_{j,n}} \left(W^{(1)}_{j,n} - W^{(1)}_{j-K^{(1)}_{j,n},n}\right)^2 \xrightarrow{p} \mu_K \int_0^1 \eta_s\, ds. \tag{A7}
$$

where the constant μ_K is given in Eq.(A2) *and*

$$\sum_{j=0}^{n} \bar{B}^{(1)}_{j+1,n}\, \bar{B}^{(2)}_{j+1,n}\, \eta_{t_{j,n}} \left(W^{(1)}_{j,n} - W^{(1)}_{j-K^{(1)}_{j,n},n} \right)^2 \mathbb{1}_{\{K^{(1)}_{j,n} \leqslant K^{(2)}_{j,n}\}} \xrightarrow{p} \bar{\mu}_K \int_0^1 \eta_s\, ds, \tag{A8}$$

where the constant $\bar{\mu}_K$ is given in Eq.(A4)*. Moreover, if the indicator function in Eq.*(A7) *is replaced with $\mathbb{1}_{\{K^{(1)}_{j,n} = K^{(2)}_{j,n}\}}$, then the constant $\bar{\mu}_K$ is given in Eq.*(A5)*. Finally*

$$\sum_{j=0}^{n-1} \bar{B}^{(1)}_{j+1,n}\, \bar{B}^{(2)}_{j+1,n}\, \eta_{t_{j,n}} \left(W^{(1)}_{j,n} - W^{(1)}_{j-K^{(1)}_{j,n},n} \right) \left(W^{(1)}_{j-K^{(1)}_{j,n},n} - W^{(1)}_{j-K^{(2)}_{j,n},n} \right) \mathbb{1}_{\{K^{(1)}_{j,n} \leqslant K^{(2)}_{j,n}\}} \xrightarrow{p} 0. \tag{A9}$$

Symmetric results hold exchanging asset 1 *with asset* 2.

Proof. We write $\bar{Z}_{j,n} = \bar{B}^{(1)}_{j+1,n}\, \bar{B}^{(2)}_{j+1,n} \left(W^{(1)}_{j,n} - W^{(1)}_{j-K^{(1)}_{j,n},n} \right)^2$ and $\bar{L}_{j,n} = \bar{B}^{(1)}_{j+1,n}\, \bar{B}^{(2)}_{j+1,n} \left(W_j - W_{j-K^{(1)}_{j,n}} \right)^2 \mathbb{1}_{\{K^{(1)}_{j,n} \leqslant K^{(2)}_{j,n}\}}$ and we notice that they form a triangular array of *dependent* random variables. To prove the convergence in Eq.(A7) and Eq.(A8) we need to show that both $\left(\bar{Z}_{j,n}\right)$ and $\left(\bar{L}_{j,n}\right)$ satisfy conditions i) and ii). We exploit results in Lemma 2. We start from $\bar{Z}_{j,n}$'s. Condition i) is easily verified by noticing that:

$$\mathbb{E}\left[\bar{Z}_{j,n}\right] = \mathbb{E}\left[\mathbb{E}\left[\bar{Z}_{j,n} | K^{(1)}_{j,n}\right]\right] = (1-x_n)(1-y_n)\, \Delta_n\, \mu_{j,n}$$

and

$$\mathbb{E}\left[\bar{Z}^2_{j,n}\right] = 3\,(1-x_n)\,(1-y_n)\,\Delta^2_n \mathbb{E}\left[\left(K^{(1)}_{j,n}\right)^2\right],$$

so that $\mathbb{V}\mathrm{ar}\left[\bar{Z}_{j,n}\right] = 2\,\Delta^2_n\, \mathbb{V}\mathrm{ar}\left[Z_{j,n}\right]$. We now prove that:

$$\mathbb{C}\mathrm{ov}\left[\bar{Z}_{j,n}, \bar{Z}_{j+k,n}\right] = \Delta^2_n \mathbb{C}\mathrm{ov}\left[Z_{j,n}, Z_{j+k,n}\right] + \Delta^2_n\, R_{j,k},$$

which implies that:

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{j=1}^{n} \sum_{k=1}^{n-j} \mathbb{C}\mathrm{ov}\left(\bar{Z}_{j,n}, \bar{Z}_{j+k,n}\right) = \underbrace{\lim_{n\to\infty} \frac{1}{n^4} \sum_{j=1}^{n} \sum_{k=1}^{n-j} \mathbb{C}\mathrm{ov}\left(Z_{j,n}, Z_{j+k,n}\right)}_{=0 \text{ from Lemma 2}} + \lim_{n\to\infty} \frac{1}{n^4} \sum_{j=1}^{n} \sum_{k=1}^{n-j} R_{j,k}.$$

Therefore, it is sufficient to show that the remainder $R_{j,k}$ is such that:

$$\lim_{n\to\infty} \frac{1}{n^4} \sum_{j=1}^{n} \sum_{k=1}^{n-j} R_{j,k} = 0.$$

To do so, let us set for sake of notation $\bar{C}^{(1,2)}_{j,n} = \bar{B}^{(1)}_{j,n} \bar{B}^{(2)}_{j,n}$ and compute $\mathbb{E}\left[\bar{Z}_{j,n}\, \bar{Z}_{j+k,n}\right]$. Notice that it is necessary to distinguish the case in which the Brownian increment $W^{(1)}_{j+k,n} - W^{(1)}_{j+k-K^{(1)}_{j+k,n},n}$ has no overlap, partial overlap or a total overlap with $W^{(1)}_{j,n} - W^{(1)}_{j-K^{(1)}_{j,n}}$. Figure A1 clarifies the situation.

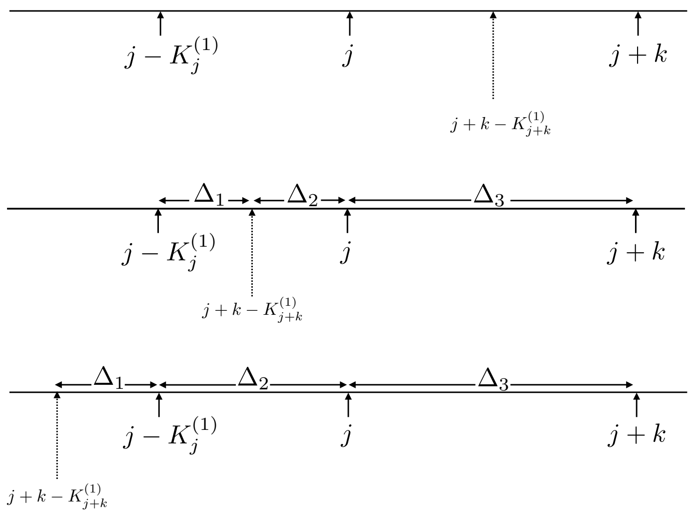

Figure A1.: Schematic representation of the three possible cases regarding the entity of the overlap of the Brownian increment $W^{(1)}_{j+k,n} - W^{(1)}_{j+k-K^{(1)}_{j+k,n},n}$ with $W^{(1)}_{j,n} - W^{(1)}_{j-K^{(1)}_{j,n},n}$: i) No overlap (top figure), ii) partial overlap (middle figure) and iii) total overlap (bottom figure).

Let us calculate

$$
\begin{aligned}
&\mathbb{E}\left[\bar{Z}_{j+k,n}\,\bar{Z}_{j,n}\right] = \\
&= \mathbb{E}\left[\bar{C}^{(1,2)}_{j+k+1,n}\left(W^{(1)}_{j+k,n} - W^{(1)}_{j+k-K^{(1)}_{j+k,n},n}\right)^2 \bar{C}^{(1,2)}_{j+1,n}\left(W^{(1)}_{j,n} - W^{(1)}_{j-K^{(1)}_{j,n},n}\right)^2\right] \\
&= \mathbb{E}\left[\bar{C}^{(1,2)}_{j+k+1,n}\left(W^{(1)}_{j+k,n} - W^{(1)}_{j+k-K^{(1)}_{j+k,n},n}\right)^2 \bar{C}^{(1,2)}_{j+1,n}\left(W^{(1)}_{j,n} - W^{(1)}_{j-K^{(1)}_{j,n},n}\right)^2 \mathbb{1}_{\left\{K^{(1)}_{j+k,n}\leqslant k\right\}}\right] + \\
&\quad \mathbb{E}\left[\bar{C}^{(1,2)}_{j+k+1,n}\left(W^{(1)}_{j+k,n} - W^{(1)}_{j+k-K^{(1)}_{j+k,n},n}\right)^2 \bar{C}^{(1,2)}_{j+1,n}\left(W^{(1)}_{j,n} - W^{(1)}_{j-K^{(1)}_{j,n},n}\right)^2 \mathbb{1}_{\left\{k<K^{(1)}_{j+k,n}\leqslant k+K^{(1)}_{j,n}\right\}}\right] + \\
&\quad \mathbb{E}\left[\bar{C}^{(1,2)}_{j+k+1,n}\left(W^{(1)}_{j+k,n} - W^{(1)}_{j+k-K^{(1)}_{j+k,n},n}\right)^2 \bar{C}^{(1,2)}_{j+1,n}\left(W^{(1)}_{j,n} - W^{(1)}_{j-K^{(1)}_{j,n},n}\right)^2 \mathbb{1}_{\left\{k+K^{(1)}_{j,n}<K^{(1)}_{j+k,n}\right\}}\right] \\
&\doteq \mathrm{E}_1 + \mathrm{E}_2 + \mathrm{E}_3.
\end{aligned}
$$

Term E_1 corresponds to the case depicted in the top of Figure A1, E_2 to the case depicted in the middle, and E_3 to the case depicted in the bottom. Let $\mathcal{F}^{(1,2)}_{j+k+1,n}$ be $\mathcal{F}^{(1,2)}_{j+k+1,n} = \sigma\left(K^{(1)}_{j,n}, K^{(1)}_{j+k+1,n}, K^{(2)}_{j+k+1,n}\right)$. We consider each term separately.

E_1*:*

$$\begin{aligned}
\mathrm{E}_1 &= \mathbb{E}\left[\bar{C}^{(1,2)}_{j+k+1,n}\left(W^{(1)}_{j+k,n} - W^{(1)}_{j+k-K^{(1)}_{j+k,n},n}\right)^2 \bar{C}^{(1,2)}_{j+1,n}\left(W^{(1)}_{j,n} - W^{(1)}_{j-K^{(1)}_{j,n},n}\right)^2 \mathbb{1}_{\left\{K^{(1)}_{j+k,n}\leqslant k\right\}}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\left(W^{(1)}_{j+k,n} - W^{(1)}_{j+k-K^{(1)}_{j+k,n},n}\right)^2 \left(W^{(1)}_{j,n} - W^{(1)}_{j-K^{(1)}_{j,n},n}\right)^2 \mathbb{1}_{\left\{K^{(1)}_{j+k,n}\leqslant k\right\}} \,\middle|\, \mathcal{F}^{(1,2)}_{j+k+1,n}\right]\right] \\
&= \Delta_n^2\, \mathbb{E}\left[\bar{C}^{(1,2)}_{j+k+1,n}\, \bar{C}^{(1,2)}_{j+1,n} K^{(1)}_{j+k,n}\, K^{(1)}_{j,n}\, \mathbb{1}_{\left\{K^{(1)}_{j+k,n}\leqslant k\right\}}\right]
\end{aligned}$$

E_2*:*

With reference to Figure A1 we define, for the sake of conciseness and clarity, the quantities

$$\begin{aligned}
\Delta_1 &\doteq \left(W^{(1)}_{j+k-K^{(1)}_{j+k,n},n} - W^{(1)}_{j-K^{(1)}_{j,n},n}\right) && \Rightarrow \mathbb{E}_{j-K^{(1)}_{j,n},n}\left[\Delta_1^2\right] = \left(K^{(1)}_{j,n} - K^{(1)}_{j+k,n} + k\right)\Delta_n \\
\Delta_2 &\doteq \left(W^{(1)}_{j,n} - W^{(1)}_{j+k-K^{(1)}_{j+k,n}}\right) && \Rightarrow \mathbb{E}_{j+k-K^{(1)}_{j+k,n}}\left[\Delta_2^2\right] = \left(K^{(1)}_{j+k,n} - k\right)\Delta_n \qquad . \\
\Delta_3 &\doteq \left(W^{(1)}_{j+k,n} - W^{(1)}_{j,n}\right) && \Rightarrow \mathbb{E}_{j,n}\left[\Delta_3^2\right] = k\,\Delta_n
\end{aligned} \tag{A10}$$

Hence

$$\begin{aligned}
\mathrm{E}_2 &= \mathbb{E}\left[\bar{C}^{(1,2)}_{j+k+1,n}\left(W_{j+k} - W_{j+k-K^{(1)}_{j+k,n}}\right)^2 \bar{C}^{(1,2)}_{j+1,n}\left(W_j - W_{j-K^{(1)}_{j,n}}\right)^2 \mathbb{1}_{\left\{k<K^{(1)}_{j+k,n}\leqslant k+K^{(1)}_{j,n}\right\}}\right] \\
&= \mathbb{E}\left[\bar{C}^{(1,2)}_{j+k+1,n}\bar{C}^{(1,2)}_{j+1,n}\mathbb{E}\left[\left(\Delta_3+\Delta_2\right)^2\left(\Delta_2+\Delta_1\right)^2 \mathbb{1}_{\left\{k<K^{(1)}_{j+k,n}\leqslant k+K^{(1)}_{j,n}\right\}} \,\middle|\, \mathcal{F}^{(1,2)}_{j+k+1,n}\right]\right] \\
&= \mathbb{E}\left[\bar{C}^{(1,2)}_{j+k+1,n}\bar{C}^{(1,2)}_{j+1,n}\mathbb{E}\left[\left(\Delta_3^2+\Delta_2^2+2\,\Delta_2\,\Delta_3\right)\left(\Delta_2^2+\Delta_1^2+2\,\Delta_1\,\Delta_2\right) \mathbb{1}_{\left\{k<K^{(1)}_{j+k,n}\leqslant k+K^{(1)}_{j,n}\right\}} \,\middle|\, \mathcal{F}^{(1,2)}_{j+k+1,n}\right]\right]
\end{aligned}$$

(Using Eq.s(A10) and after some algebraic computations)

$$\begin{aligned}
&= \Delta_n^2\mathbb{E}\left[\bar{C}^{(1,2)}_{j+k+1,n}\bar{C}^{(1,2)}_{j+1,n}\left(K^{(1)}_{j,n}K^{(1)}_{j+k,n}\right)\mathbb{1}_{\left\{k<K^{(1)}_{j+k,n}\leqslant k+K^{(1)}_{j,n}\right\}}\right] \\
&+ 2\Delta_n^2\mathbb{E}\left[\bar{C}^{(1,2)}_{j+k+1,n}\bar{C}^{(1,2)}_{j+1,n}\left(K^{(1)}_{j+k,n} - k\right)^2\mathbb{1}_{\left\{k<K^{(1)}_{j+k,n}\leqslant k+K^{(1)}_{j,n}\right\}}\right]
\end{aligned} \tag{A11}$$

E_3*:*

Mimicking the procedure in the previous point, we define

$$\begin{aligned}
\Delta_1 &\doteq \left(W^{(1)}_{j-K^{(1)}_{j,n},n} - W^{(1)}_{j+k-K^{(1)}_{j+k,n}}\right) && \Rightarrow \mathbb{E}_{j+k-K^{(1)}_{j+k,n}}\left[\Delta_1^2\right] = \left(K^{(1)}_{j+k,n} - K^{(1)}_{j,n} - k\right)\Delta_n, \\
\Delta_2 &\doteq \left(W^{(1)}_{j,n} - W^{(1)}_{j-K^{(1)}_{j,n},n}\right) && \Rightarrow \mathbb{E}_{j-K^{(1)}_{j,n},n}\left[\Delta_2^2\right] = K^{(1)}_{j,n}\,\Delta_n, \\
\Delta_3 &\doteq \left(W^{(1)}_{j+k,n} - W^{(1)}_{j,n}\right) && \Rightarrow \mathbb{E}_{j,n}\left[\Delta_3^2\right] = k\,\Delta_n.
\end{aligned}$$

Then

$$\begin{aligned}
\mathrm{E}_3 &= \mathbb{E}\left[\bar{C}^{(1,2)}_{j+k+1,n}\bar{C}^{(1,2)}_{j+1,n}\left(W^{(1)}_{j+k,n}-W_{j+k-K^{(1)}_{j+k,n},n}\right)^2\left(W^{(1)}_{j,n}-W^{(1)}_{j-K^{(1)}_{j,n},n}\right)^2 \mathbb{1}_{\left\{k+K^{(1)}_{j,n}<K^{(1)}_{j+k,n}\right\}}\right] \\
&= \mathbb{E}\left[\bar{C}^{(1,2)}_{j+k+1,n}\bar{C}^{(1,2)}_{j+1,n}\mathbb{E}\left[(\Delta_3+\Delta_2+\Delta_1)^2\,\Delta_2^2\,\mathbb{1}_{\left\{k+K^{(1)}_{j,n}<K^{(1)}_{j+k,n}\right\}}\,\middle|\,\mathcal{F}^{(1,2)}_{j+k+1,n}\right]\right] \\
&= \mathbb{E}\left[\bar{C}^{(1,2)}_{j+k+1,n}\bar{C}^{(1,2)}_{j+1,n}\mathbb{E}\left[\left(\Delta_3^2\,\Delta_2^2+\Delta_2^4+\Delta_1^2\,\Delta_2^2\right)\mathbb{1}_{\left\{k+K^{(1)}_{j,n}<K^{(1)}_{j+k,n}\right\}}\,\middle|\,\mathcal{F}^{(1,2)}_{j+k+1,n}\right]\right] \\
&= \Delta_n^2\mathbb{E}\left[\bar{C}^{(1,2)}_{j+k+1,n}\bar{C}^{(1,2)}_{j+1,n}\left(k\,K^{(1)}_{j,n}+3\left(K^{(1)}_{j,n}\right)^2+(K_{j+k}-K_j-k)\,K_j\right)\mathbb{1}_{\left\{k+K^{(1)}_{j,n}<K^{(1)}_{j+k,n}\right\}}\right] \\
&= \Delta_n^2\mathbb{E}\left[\bar{C}^{(1,2)}_{j+k+1,n}\bar{C}^{(1,2)}_{j+1,n}\left(K_jK^{(1)}_{j+k,n}+2\left(K^{(1)}_{j,n}\right)^2\right)\mathbb{1}_{\left\{k+K^{(1)}_{j,n}<K^{(1)}_{j+k,n}\right\}}\right].
\end{aligned}$$

Summing up and using the fact that $\mathbb{E}\left[\bar{Z}_{j,n}\bar{Z}_{j+k,n}\right]=\Delta_n\,\mathbb{E}\left[K^{(1)}_{j,n}\right]$ we obtain

$$\begin{aligned}
\mathbb{C}\mathrm{ov}\left[\bar{Z}_{j,n},\bar{Z}_{j+k,n}\right] &= \Delta_n^2\,\mathbb{C}\mathrm{ov}\left[Z_{j,n},Z_{j+k,n}\right]+2\,\Delta_n^2\mathbb{E}\left[\bar{C}^{(1,2)}_{j+k+1,n}\bar{C}^{(1,2)}_{j+1,n}\left(K^{(1)}_{j,n}\right)^2\mathbb{1}_{\left\{k+K^{(1)}_{j,n}<K^{(1)}_{j+k,n}\right\}}\right] \\
&\quad + 2\Delta_n^2\mathbb{E}\left[\bar{C}^{(1,2)}_{j+k+1,n}\bar{C}^{(1,2)}_{j+1,n}\left(K^{(1)}_{j+k,n}-k\right)^2\mathbb{1}_{\left\{k<K^{(1)}_{j+k,n}\leqslant k+K^{(1)}_{j,n}\right\}}\right].
\end{aligned}$$

The last two terms correspond to the reminder $R_{j,k}$ in Eq.(A). The product $\bar{C}^{(1,2)}_{j+k+1,n}\bar{C}^{(1,2)}_{j+1,n}$ is in $(0,1)$ so, to prove (A10) it is sufficient to notice that

$$\begin{aligned}
\mathbb{E}\left[\left(K^{(1)}_{j,n}\right)^2\mathbb{1}_{\left\{k+K^{(1)}_{j,n}<K^{(1)}_{j+k,n}\right\}}\right] &= \mathbb{E}\left[\left(K^{(1)}_{j,n}\right)^2\mathbb{1}_{\left\{k+K^{(1)}_{j,n}<K^{(1)}_{j+k,n}\right\}}\,\middle|\,K^{(1)}_{j,n}\leqslant j-1\right]\left(1-x_n^j\right) \\
&= \mathbb{E}\left[\left(K^{(1)}_{j,n}\right)^2\left((1-x_n)\sum_{q=K^{(1)}_{j,n}+k+1}^{j+k-1}x_n^q+x_n^{j+k}\right)\middle|\,K^{(1)}_{j,n}\leqslant j-1\right]\left(1-x_n^j\right) \\
&= x_n^{k+1}\mathbb{E}\left[\left(K^{(1)}_{j,n}\right)^2\,x_n^{K^{(1)}_{j,n}}\,\middle|\,K^{(1)}_{j,n}\leqslant j-1\right]\left(1-x_n^j\right) \\
&= \frac{\left(1-x^j\right)\left(\left(j^2\left(x^2-1\right)^2+2jx\left(x^2-1\right)-x\left(x^2+1\right)\right)x^{2j}+x^3+x\right)x^{k+2}}{(x-1)^2(x+1)^3}
\end{aligned}$$

and thus

$$\lim_{n\to\infty}\frac{1}{n^4}\sum_{j=1}^{n}\sum_{k=1}^{n-1}\frac{\left(1-x^j\right)\left(\left(j^2\left(x^2-1\right)^2+2jx\left(x^2-1\right)-x\left(x^2+1\right)\right)x^{2j}+x^3+x\right)x^{k+2}}{(x-1)^2(x+1)^3}=0,$$

together with

$$
\begin{aligned}
&\mathbb{E}\left[\left(K^{(1)}_{j+k,n}-k\right)^2 \mathbb{1}_{\left\{k<K^{(1)}_{j+k,n}\leqslant k+K^{(1)}_{j,n}\right\}}\right] \\
&=\mathbb{E}\left[\mathbb{E}\left[\left(K^{(1)}_{j+k,n}-k\right)^2 \mathbb{1}_{\left\{k<K^{(1)}_{j+k,n}\leqslant k+K^{(1)}_{j,n}\right\}}\,\middle|\, K^{(1)}_{j,n}\right]\right] \\
&=\mathbb{E}\left[\sum_{q=k+1}^{k+K^{(1)}_{j,n}} (q-k)^2\, \mathbb{1}_{\left\{k<q\leqslant k+K^{(1)}_{j,n}\right\}}\mathbb{P}\left[K^{(1)}_{j+k,n}=q\right]\,\middle|\, K^{(1)}_{j,n}\right] \\
&=\sum_{l=0}^{j}\sum_{q=k+1}^{k+l} (q-k)^2\, \mathbb{1}_{\{k<q\leqslant k+l\}}\mathbb{P}\left[K^{(1)}_{j+k,n}=q\right]\mathbb{P}\left[K^{(1)}_{j,n}=l\right] \\
&=\sum_{l=0}^{j-1}\sum_{q=k+1}^{k+l} (q-k)^2\, \mathbb{1}_{\{k<q\leqslant k+l\}}\mathbb{P}\left[K^{(1)}_{j+k,n}=q\right](1-x_n)\, x^l+\sum_{q=k+1}^{k+j} (q-k)^2\, \mathbb{1}_{\{k<q\leqslant k+j\}}\mathbb{P}\left[K^{(1)}_{j+k,n}=q\right]x_n^j \\
&=(1-x_n)^2\sum_{l=0}^{j-1}\sum_{q=k+1}^{k+l} (q-k)^2\, x_n^q\, x^l+(1-x_n)\sum_{q=k+1}^{k+j-1}(q-k)^2\, x_n^q x_n^j+j^2 x_n^{k+j}x_n^j \\
&=\frac{\left(\left(j^2\left(x^2-1\right)^2+2jx\left(x^2-1\right)-x\left(x^2+1\right)\right)x^{2j}+x^3+x\right)x^{k+1}}{(x-1)^2(x+1)^3},
\end{aligned}
$$

and thus

$$
\lim_{n\to\infty}\frac{1}{n^4}\sum_{j=1}^{n}\sum_{k=1}^{n-1}\frac{\left(\left(j^2\left(x^2-1\right)^2+2jx\left(x^2-1\right)-x\left(x^2+1\right)\right)x^{2j}+x^3+x\right)x^{k+1}}{(x-1)^2(x+1)^3}=0.
$$

Therefore condition ii) holds. So does Eq.(A7) since Lemma 3 readily applies to $s_{j,n}=\mathbb{E}\left[\bar{Z}_{j,n}\right]$.

We consider now the $\bar{L}_{j,n}$'s and we proceed as for $\bar{Z}_{j,n}$'s. Again, condition i) is easily verified by noticing that $\mathbb{V}\mathrm{ar}\left[\bar{L}_{j,n}\right]=3\Delta_n^2\mathbb{V}\mathrm{ar}\left[L_{j,n}\right]$. As regards as condition ii), we show that

$$
\mathbb{C}\mathrm{ov}\left[\bar{L}_{j,n},\bar{L}_{j+k,n}\right]=\Delta_n^2\mathbb{C}\mathrm{ov}\left[L_{j,n},L_{j+k,n}\right]+\Delta_n^2\, R_{j,k}
$$

with

$$
\lim_{n\to\infty}\frac{1}{n^4}\sum_{j=1}^{n}\sum_{k=1}^{n-j}R_{j,k}=0.
$$

To simplify notations call $I_j=\mathbb{1}_{\left\{K^{(1)}_{j,n}\leqslant K^{(2)}_{j,n}\right\}}$. Computations that led to decomposition in Eq.(A10) can be easily replicated also in this case by just extending the filtration used in the law of iterated expectation. We obtain

$$
\begin{aligned}
& \mathbb{E}\left[\bar{L}_{j+k,n}\,\bar{L}_{j,n}\right] \\
&= \mathbb{E}\left[\bar{C}^{(1,2)}_{j+k+1}\bar{C}^{(1,2)}_{j+1}\left(W^{(1)}_{j+k,n}-W^{(1)}_{j+k-K^{(1)}_{j+k,n}}\right)^2\left(W^{(1)}_{j,n}-W^{(1)}_{j-K^{(1)}_{j,n},n}\right)^2 I_{j+k}\,I_j\right] \\
&= \mathbb{E}\left[\bar{C}^{(1,2)}_{j+k+1}\bar{C}^{(1,2)}_{j+1}\left(W^{(1)}_{j+k,n}-W^{(1)}_{j+k-K^{(1)}_{j+k,n},n}\right)^2\left(W_{j}-W_{j-K^{(1)}_{j,n}}\right)^2 \mathbb{1}_{\left\{K^{(1)}_{j+k,n}\leqslant k\right\}}\,I_{j+k}\,I_j\right]+ \\
&\quad \mathbb{E}\left[\bar{C}^{(1,2)}_{j+k+1}\bar{C}^{(1,2)}_{j+1}\left(W^{(1)}_{j+k,n}-W^{(1)}_{j+k-K^{(1)}_{j+k,n},n}\right)^2\left(W^{(1)}_{j,n}-W^{(1)}_{j-K^{(1)}_{j,n},n}\right)^2 \mathbb{1}_{\left\{k<K^{(1)}_{j+k,n}\leqslant k+K^{(1)}_{j,n}\right\}}\,I_{j+k}\,I_j\right]+ \\
&\quad \mathbb{E}\left[\bar{C}^{(1,2)}_{j+k+1}\bar{C}^{(1,2)}_{j+1}\left(W^{(1)}_{j+k,n}-W^{(1)}_{j+k-K^{(1)}_{j+k,n},n}\right)^2\left(W^{(1)}_{j,n}-W^{(1)}_{j-K^{(1)}_{j,n},n}\right)^2 \mathbb{1}_{\left\{k+K^{(1)}_{j,n}<K^{(1)}_{j+k,n}\right\}}\,I_{j+k}\,I_j\right] \\
&= \mathrm{E}_1+\mathrm{E}_2+\mathrm{E}_3.
\end{aligned}
$$

with

$$
\begin{aligned}
E_1 &= \Delta_n^2\,\mathbb{E}\left[\bar{C}^{(1,2)}_{j+k+1}\bar{C}^{(1,2)}_{j+1}K^{(1)}_{j+k,n}\,K^{(1)}_{j,n}\,\mathbb{1}_{\left\{K^{(1)}_{j+k,n}\leqslant k\right\}}\,I_{j+k}\,I_j\right] \\
E_2 &= \Delta_n^2\mathbb{E}\left[\bar{C}^{(1,2)}_{j+k+1}\bar{C}^{(1,2)}_{j+1}\left(K^{(1)}_{j,n}K^{(1)}_{j+k,n}\right)\mathbb{1}_{\left\{k<K^{(1)}_{j+k,n}\leqslant k+K^{(1)}_{j,n}\right\}}\,I_{j+k}\,I_j\right] \\
&\quad + 2\Delta_n^2\mathbb{E}\left[\bar{C}^{(1,2)}_{j+k+1}\bar{C}^{(1,2)}_{j+1}\left(K^{(1)}_{j+k,n}-k\right)^2\mathbb{1}_{\left\{k<K^{(1)}_{j+k,n}\leqslant k+K^{(1)}_{j,n}\right\}}\,I_{j+k}\,I_j\right] \\
E_3 &= \Delta_n^2\mathbb{E}\left[\bar{C}^{(1,2)}_{j+k+1}\bar{C}^{(1,2)}_{j+1}\left(K_jK^{(1)}_{j+k,n}\,I_{j+k}\,I_j+2\left(K^{(1)}_{j,n}\right)^2 I_{j+k}\,I_j\right)\mathbb{1}_{\left\{k+K^{(1)}_{j,n}<K^{(1)}_{j+k,n}\right\}}\right].
\end{aligned}
$$

However since

$$
0\leqslant\mathbb{E}\left[\bar{C}^{(1,2)}_{j+k+1}\bar{C}^{(1,2)}_{j+1}\left(K^{(1)}_{j+k,n}-k\right)^2\mathbb{1}_{\left\{k<K^{(1)}_{j+k,n}\leqslant k+K^{(1)}_{j,n}\right\}}\,I_{j+k}\,I_j\right]\leqslant\mathbb{E}\left[\left(K^{(1)}_{j+k,n}-k\right)^2\mathbb{1}_{\left\{k<K^{(1)}_{j+k,n}\leqslant k+K^{(1)}_{j,n}\right\}}\right]
$$

and

$$
0\leqslant\mathbb{E}\left[2\,\bar{C}^{(1,2)}_{j+k+1}\bar{C}^{(1,2)}_{j+1}\left(K^{(1)}_{j,n}\right)^2\mathbb{1}_{\left\{k+K^{(1)}_{j,n}<K^{(1)}_{j+k,n}\right\}}\,I_{j+k}\,I_j\right]\leqslant\mathbb{E}\left[2\left(K^{(1)}_{j,n}\right)^2\mathbb{1}_{\left\{k+K^{(1)}_{j,n}<K^{(1)}_{j+k,n}\right\}}\right],
$$

by virtue of the limits computed in (A) and (A) we can conclude the $\bar{L}_{j,n}$'s satisfy condition ii) . Since Lemma 3 is readily satisfied from $s_{j,n}=\mathbb{E}\left[\bar{L}_{j,n}\right]$, Eq.(A8) hold.

Finally, to show the convergence in Eq.(A9) we prove that it holds in the $\mathbb{L}^2$-norm. This follows from the fact that, if $K^{(1)}_{j,n}\leqslant K^{(2)}_{j,n}$, then $W^{(1)}_{j,n}-W^{(1)}_{j-K^{(1)}_{j,n},n}$ is independent from $W^{(1)}_{j-K^{(1)}_{j,n},n}-W^{(1)}_{j-K^{(2)}_{j,n},n}$, which implies

$$
\begin{aligned}
\mathrm{L}_2 &= \mathbb{E}\left[\left(\sum_{j=0}^{n-1} \bar{B}^{(1)}_{j+1,n}\,\bar{B}^{(2)}_{j+1,n}\,\eta_{j-1,n}\left(W^{(1)}_{j,n}-W^{(1)}_{j-K^{(1)}_{j,n},n}\right)\left(W^{(1)}_{j-K^{(1)}_{j,n},n}-W^{(1)}_{j-K^{(2)}_{j,n},n}\right)\mathbb{1}_{\left\{K^{(1)}_{j,n}\leqslant K^{(2)}_{j,n}\right\}}\right)^2\right] \\
&\doteq \Delta_n^2\sum_{j=0}^{n-1}\mathbb{E}\left[\bar{B}^{(1)}_{j+1,n}\,\bar{B}^{(2)}_{j+1,n}\,\eta_{j-1,n}\right]\mathbb{E}\left[K^{(1)}_{j,n}\left(K^{(2)}_{j,n}-K^{(1)}_{j,n}\right)\mathbb{1}_{\left\{K^{(1)}_{j,n}\leqslant K^{(2)}_{j,n}\right\}}\right]+ \\
&\quad +2\sum_{i<j}\mathbb{E}\Bigg[\bar{B}^{(1)}_{j+1,n}\,\bar{B}^{(2)}_{j+1,n}\,\eta_{j-1,n}\,\bar{B}^{(1)}_{i+1,n}\,\bar{B}^{(2)}_{i+1,n}\,\eta_{i-1,n}\times \\
&\quad \times\left(W^{(1)}_{j,n}-W^{(1)}_{j-K^{(1)}_{j,n},n}\right)\left(W^{(1)}_{j-K^{(1)}_{j,n},n}-W^{(1)}_{j-K^{(2)}_{j,n},n}\right)\mathbb{1}_{\left\{K^{(1)}_{j,n}\leqslant K^{(2)}_{j,n}\right\}}\times \\
&\quad \times\left(W^{(1)}_{i,n}-W^{(1)}_{i-K^{(1)}_{i,n},n}\right)\left(W^{(1)}_{i-K^{(1)}_{i,n},n}-W^{(1)}_{i-K^{(2)}_{i,n},n}\right)\mathbb{1}_{\left\{K^{(1)}_{j,n}\leqslant K^{(2)}_{j,n}\right\}}\Bigg].
\end{aligned}
\tag{A12}
$$

Nevertheless, for $i<j$ if $K^{(1)}_{j,n}\leqslant K^{(2)}_{j,n}$ and $K^{(1)}_{i,n}\leqslant K^{(2)}_{i,n}$ we get that the random variables

$$
\begin{cases}
W^{(1)}_{j,n}-W^{(1)}_{j-K^{(1)}_{j,n},n}, \\
W^{(1)}_{j-K^{(1)}_{j,n},n}-W^{(1)}_{j-K^{(2)}_{j,n},n}, \\
W^{(1)}_{i,n}-W^{(1)}_{i-K^{(1)}_{i,n},n}, \\
W^{(1)}_{i-K^{(1)}_{i,n},n}-W^{(1)}_{i-K^{(2)}_{i,n},n}
\end{cases}
$$

are mutually independent. Hence, for $i<j$

$$
\begin{aligned}
&\mathbb{E}\Bigg[\left(W^{(1)}_{j,n}-W^{(1)}_{j-K^{(1)}_{j,n},n}\right)\left(W^{(1)}_{j-K^{(1)}_{j,n},n}-W^{(1)}_{j-K^{(2)}_{j,n},n}\right)\mathbb{1}_{\left\{K^{(1)}_{j,n}\leqslant K^{(2)}_{j,n}\right\}}\times \\
&\times\left(W^{(1)}_{i,n}-W^{(1)}_{i-K^{(1)}_{i,n},n}\right)\left(W^{(1)}_{i-K^{(1)}_{i,n},n}-W^{(1)}_{i-K^{(2)}_{i,n},n}\right)\mathbb{1}_{\left\{K^{(1)}_{i,n}\leqslant K^{(2)}_{i,n}\right\}}\Bigg|K^{(1)}_{j,n},K^{(2)}_{j,n}\Bigg]=0,
\end{aligned}
$$

whence

$$
\begin{aligned}
\mathrm{L}_2 &= \Delta_n^2\sum_{j=0}^{n-1}\mathbb{E}\left[\bar{B}^{(1)}_{j+1,n}\,\bar{B}^{(2)}_{j+1,n}\,\eta_{j-1,n}\right]\mathbb{E}\left[K^{(1)}_{j,n}\left(K^{(2)}_{j,n}-K^{(1)}_{j,n}\right)\mathbb{1}_{\left\{K^{(1)}_{j,n}\leqslant K^{(2)}_{j,n}\right\}}\right] \\
&\leqslant \Delta_n^2\sum_{j=0}^{n-1}\mathbb{E}\left[K^{(1)}_{j,n}\left(K^{(2)}_{j,n}-K^{(1)}_{j,n}\right)\right]\to 0.
\end{aligned}
$$

$\square$

Appendix B: Proof of the main Theorem

Proof. Without loss of generality, we assume that the drift terms are zero and we write:

$$
\begin{aligned}
\mathsf{RC}_n &= \sum_{j=0}^{n-1} \left(X^{(1)}_{j+1,n} - X^{(1)}_{j,n}\right) \left(X^{(2)}_{j+1,n} - X^{(2)}_{j,n}\right) \\
&= \sum_{j=0}^{n-1} \left(1 - B^{(1)}_{j+1,n}\right) \left(1 - B^{(2)}_{j+1,n}\right) \left(Y^{(1)}_{j+1,n} - X^{(1)}_{j,n}\right) \left(Y^{(2)}_{j+1,n} - X^{(2)}_{j,n}\right) \\
&= \sum_{j=0}^{n-1} \left(1 - B^{(1)}_{j+1,n}\right) \left(1 - B^{(2)}_{j+1,n}\right) \left(Y^{(1)}_{j+1,n} - Y^{(1)}_{j,n} + Y^{(1)}_{j,n} - X^{(1)}_{j,n}\right) \left(Y^{(2)}_{j+1,n} - Y^{(2)}_{j,n} + Y^{(2)}_{j,n} - X^{(2)}_{j,n}\right) \\
&= \sum_{j=0}^{n-1} \left(1 - B^{(1)}_{j+1,n}\right) \left(1 - B^{(2)}_{j+1,n}\right) \left(Y^{(1)}_{j+1,n} - Y^{(1)}_{j,n}\right) \left(Y^{(2)}_{j+1,n} - Y^{(2)}_{j,n}\right) + \\
&\quad \sum_{j=0}^{n-1} \left(1 - B^{(1)}_{j+1,n}\right) \left(1 - B^{(2)}_{j+1,n}\right) \left(Y^{(1)}_{j+1,n} - Y^{(1)}_{j,n}\right) \left(Y^{(2)}_{j,n} - X^{(2)}_{j,n}\right) + \\
&\quad \sum_{j=0}^{n-1} \left(1 - B^{(1)}_{j+1,n}\right) \left(1 - B^{(2)}_{j+1,n}\right) \left(Y^{(1)}_{j,n} - X^{(1)}_{j,n}\right) \left(Y^{(2)}_{j+1,n} - Y^{(2)}_{j,n}\right) + \\
&\quad \sum_{j=0}^{n-1} \left(1 - B^{(1)}_{j+1,n}\right) \left(1 - B^{(2)}_{j+1,n}\right) \left(Y^{(1)}_{j,n} - X^{(1)}_{j,n}\right) \left(Y^{(2)}_{j,n} - X^{(2)}_{j,n}\right) = \mathrm{A}_n + \mathrm{B}_n + \mathrm{C}_n + \mathrm{D}_n.
\end{aligned} \tag{B1}
$$

Put $w_{j+1,n} \doteq \left(1 - B^{(1)}_{j+1,n}\right) \left(1 - B^{(2)}_{j+1,n}\right)$ and consider the first term

$$
\begin{aligned}
\mathrm{A}_n &= \sum_{j=0}^{n-1} w_{j+1,n}\, \Delta_{j+1} Y^{(1)} \Delta_{j+1} Y^{(2)} \\
&= \frac{1}{4} \sum_{j=0}^{n-1} w_{j+1,n} \left[\left(\Delta_{j+1} Y^{(1)} + \Delta_{j+1} Y^{(2)}\right)^2 - \left(\Delta_{j+1} Y^{(1)} - \Delta_{j+1} Y^{(2)}\right)^2\right] \\
&= \underbrace{\frac{1}{4} \sum_{j=0}^{n-1} w_{j+1,n} \left[\left(\int_{t_j}^{t_{j+1}} \left(\sigma^{(1)}_s + \sigma^{(2)}_s \rho_s\right) dW^{(1)}_s\right)^2 + \left(\int_{t_j}^{t_{j+1}} \sigma^{(2)}_s \xi_s \, dW^{(2)}_s\right)^2\right]}_{\mathrm{A}^{(+)}_n} \\
&\quad \underbrace{- \frac{1}{4} \sum_{j=0}^{n-1} w_{j+1,n} \left[\left(\int_{t_j}^{t_{j+1}} \left(\sigma^{(1)}_s - \sigma^{(2)}_s \rho_s\right) dW^{(1)}_s\right)^2 + \left(\int_{t_j}^{t_{j+1}} \sigma^{(2)}_s \xi_s \, dW^{(2)}_s\right)^2\right]}_{\mathrm{A}^{(-)}_n} \\
&\quad + \underbrace{\frac{1}{2} \sum_{j=0}^{n-1} w_{j+1,n} \left(\int_{t_j}^{t_{j+1}} \left(\sigma^{(1)}_s + \sigma^{(2)}_s \rho_s\right) dW^{(1)}_s\right) \left(\int_{t_j}^{t_{j+1}} \sigma^{(2)}_s \xi_s \, dW^{(2)}_s\right)}_{\mathrm{A}^{(+)}_{0,n}} \\
&\quad + \underbrace{\frac{1}{2} \sum_{j=0}^{n-1} w_{j+1,n} \left(\int_{t_j}^{t_{j+1}} \left(\sigma^{(1)}_s - \sigma^{(2)}_s \rho_s\right) dW^{(1)}_s\right) \left(\int_{t_j}^{t_{j+1}} \sigma^{(2)}_s \xi_s \, dW^{(2)}_s\right)}_{\mathrm{A}^{(-)}_{0,n}}.
\end{aligned}
$$

By standard arguments we have that:

$$\tfrac{1}{4}\sum_{j=0}^{n-1}\omega_{j+1,n}\left(\int_{t_j}^{t_{j+1}}\left(\sigma_s^{(1)}\pm\sigma_s^{(2)}\rho_s\right)dW_s^{(1)}\right)^2\xrightarrow{p}\tfrac{1}{4}\left(1-p_1\right)\left(1-p_2\right)\int_0^1\left(\sigma_s^{(1)}\pm\sigma_s^{(2)}\rho_s\right)^2ds$$

and so:

$$\begin{aligned}\mathrm{A}_n^{(+)}-\mathrm{A}_n^{(-)}&\xrightarrow{p}\frac{1}{4}\left(1-p_1\right)\left(1-p_2\right)\int_0^1\left(\left(\sigma_s^{(1)}+\sigma_s^{(2)}\rho_s\right)^2-\left(\sigma_s^{(1)}-\sigma_s^{(2)}\rho_s\right)^2\right)ds\\&=\left(1-p_1\right)\left(1-p_2\right)\int_0^1\sigma_s^{(1)}\sigma_s^{(2)}\rho_s\,ds\end{aligned}\tag{B2}$$

We show now that $\mathrm{A}_{0,n}^{(+)}\xrightarrow{p}0$. For this purpose consider the L^2-norm:

$$\begin{aligned}\mathbb{E}\left[\left|\mathrm{A}_{0,n}^{(+)}\right|^2\right]&=\sum_{j=1}^{n}\mathbb{E}\left[\omega_{j,n}\left|\int_{t_{j-1}}^{t_j}\left(\sigma_s^{(1)}+\sigma_s^{(2)}\right)dW_s^{(1)}\right|^2\left|\int_{t_{j-1}}^{t_j}\sigma_s^{(2)}\sqrt{1-\rho_s^2}\,dW_s^{(2)}\right|^2\right]\\&=\sum_{j=1}^{n}\mathbb{E}\left[\omega_{j,n}\,\mathbb{E}_j\left[\left|\int_{t_{j-1}}^{t_j}\left(\sigma_s^{(1)}+\sigma_s^{(2)}\right)dW_s^{(1)}\right|^2\left|\int_{t_{j-1}}^{t_j}\sigma_s^{(2)}\sqrt{1-\rho_s^2}\,dW_s^{(2)}\right|^2\right]\right]\\&=\sum_{j=1}^{n}\mathbb{E}\left[\omega_{j,n}\,\mathbb{E}_j\left[\left|\int_{t_{j-1}}^{t_j}\left(\sigma_s^{(1)}+\sigma_s^{(2)}\right)dW_s^{(1)}\right|^2\right]\mathbb{E}_j\left[\left|\int_{t_{j-1}}^{t_j}\sigma_s^{(2)}\sqrt{1-\rho_s^2}\,dW_s^{(2)}\right|^2\right]\right]\\&=\sum_{j=1}^{n}\mathbb{E}\left[\omega_{j,n}\,C_j^2\Delta_n^2\right]=O\left(\Delta_n\right)\to 0.\end{aligned}$$

The last identity is easily explained by defining Z_s either $\sigma_s^{(1)}+\sigma_s^{(2)}$ or $\sigma_s^{(2)}\sqrt{1-\rho_s^2}$ and dW_s either $dW_s^{(1)}$ or $dW_s^{(2)}$ and noticing that

$$\mathbb{E}_j\left[\left|\int_{t_{j-1}}^{t_j}Z_s\,dW_s\right|^2\right]=\left(\int_{t_{j-1}}^{t_j}Z_s^2\,ds\right)\frac{2^{\frac{1}{2}}\Gamma(\frac{3}{2})}{\sqrt{\pi}}\leqslant C_j\Delta_n.$$

So since $\mathrm{A}_{0,n}^{(+)}\xrightarrow{\mathrm{L}^2}0$ we get $\mathrm{A}_{0,n}^{(+)}\xrightarrow{p}0$ and, with an identical reasoning, also $\mathrm{A}_{0,n}^{(-)}\xrightarrow{p}0$. Summarizing,

$$\mathrm{A}_n=\mathrm{A}_n^{(+)}-\mathrm{A}_n^{(-)}+\mathrm{A}_{0,n}^{(+)}+\mathrm{A}_{0,n}^{(-)}\xrightarrow{p}\left(1-p_1\right)\left(1-p_2\right)\int_0^t\sigma_s^{(1)}\sigma_s^{(2)}\rho_s\,ds.$$

We consider now

$$\begin{aligned}
\mathrm{B}_n &= \sum_{j=0}^{n-1} \left(1 - B^{(1)}_{j+1,n}\right) \left(1 - B^{(2)}_{j+1,n}\right) \left(Y^{(1)}_{j+1,n} - Y^{(1)}_{j,n}\right) \left(Y^{(2)}_{j,n} - X^{(2)}_{j,n}\right) \\
&= \sum_{j=0}^{n-1} \left(1 - B^{(1)}_{j+1,n}\right) \left(1 - B^{(2)}_{j+1,n}\right) \left(Y^{(1)}_{j+1,n} - Y^{(1)}_{j,n}\right) \left(Y^{(2)}_{j,n} - Y^{(2)}_{j-K^{(2)}_{j,n},n}\right) \\
&= \sum_{j=0}^{n-1} \omega_{j+1,n} \left(Y^{(1)}_{j+1,n} - Y^{(1)}_{j,n}\right) \left(Y^{(2)}_{j,n} - Y^{(2)}_{j-K^{(2)}_{j,n},n}\right).
\end{aligned}$$

Call

$$\mathrm{Z}_{j,n} = \omega_{j+1,n} \left(Y^{(1)}_{j+1,n} - Y^{(1)}_{j,n}\right) \left(Y^{(2)}_{j,n} - Y^{(2)}_{j-K^{(2)}_{j,n},n}\right),$$

so that $\mathrm{B}_n = \sum_{j=0}^{n-1} \mathrm{Z}_{j,n}$. The $\mathbb{L}^2$-norm of B_n is now computed as

$$\mathbb{E}\left[\left(\sum_{j=0}^{n-1} Z_j\right)^2\right] = \sum_{j=0}^{n-1} \mathbb{E}\left[Z_j^2\right] + 2\,\mathrm{EC}_n$$

where EC_n contains the expected value of the cross-products, that is

$$\begin{aligned}
\mathrm{EC}_n &= \sum_{j=0}^{n-1} \sum_{i=j+1}^{n-1} \mathbb{E}\left[\omega_{j+1,n} \left(Y^{(1)}_{j+1,n} - Y^{(1)}_{j,n}\right) \omega_{i+1,n} \left(Y^{(1)}_{i+1,n} - Y^{(1)}_{i,n}\right) \left(Y^{(2)}_{i,n} - Y^{(2)}_{i-K^{(2)}_{i,n},n}\right) \left(Y^{(2)}_{j,n} - Y^{(2)}_{j-K^{(2)}_{j,n},n}\right)\right] \\
&= \sum_{j=0}^{n-1} \sum_{k=1}^{n-j-1} \mathbb{E}\Bigg[\omega_{j+1,n} \left(Y^{(1)}_{j+1,n} - Y^{(1)}_{j,n}\right) \omega_{j+k+1,n} \left(Y^{(1)}_{j+k+1,n} - Y^{(1)}_{j+k,n}\right) \times \\
&\quad \times \left(Y^{(2)}_{j+k,n} - Y^{(2)}_{j+k-K^{(2)}_{j+k,n},n}\right) \left(Y^{(2)}_{j,n} - Y^{(2)}_{j-K^{(2)}_{j,n},n}\right)\Bigg] \\
&= \sum_{j=0}^{n-1} \sum_{k=1}^{n-j-1} \mathbb{E}\Bigg[\omega_{j+1,n}\, \omega_{j+k+1,n} \left(Y^{(2)}_{j,n} - Y^{(2)}_{j-K^{(2)}_{j,n},n}\right) \left(Y^{(1)}_{j+1,n} - Y^{(1)}_{j,n}\right) \times \\
&\quad \times \left(Y^{(2)}_{j+k,n} - Y^{(2)}_{j+k-K^{(2)}_{j+k,n},n}\right) \left(Y^{(1)}_{j+k+1,n} - Y^{(1)}_{j+k,n}\right)\Bigg].
\end{aligned}$$

since $\left(Y^{(1)}_{j+k+1,n} - Y^{(1)}_{j+k,n}\right)$ is independent from the other terms for all $j = 0, ..., n-1$ and for all $k = 1, ..., n-j-1$ and since, besides, $\mathbb{E}\left[\left(Y^{(1)}_{j+k+1,n} - Y^{(1)}_{j+k,n}\right)\right] = 0$ we get $\mathrm{EC}_n = 0$. Concerning $\mathbb{E}\left[Z_j^2\right]$ note that, by the boundedness of $\sigma^{(1)}$ and $\sigma^{(2)}$ we have

$$\mathbb{E}\left[Z_j^2\right] \leqslant C\,\Delta_n^2\,\mathbb{E}\left[K^{(2)}_{j+k,n}\right].$$

The expected value of $K_{j,n}^{(2)}$ for $j \geqslant 1$ (remember that $K_0^{(1)} = 0$ identically) can be computed analytically as

$$\mathbb{E}\left[K_{j,n}^{(2)}\right] = j\,(p_{n,2})^j + (1-p_{n,2})\sum_{q=1}^{j-1} q\,(p_{n,2})^q$$

so that

$$\sum_{j=1}^{n-1}\mathbb{E}\left[K_{j,n}^{(2)}\right] = \frac{p_{n,2}\left((p_{n,2})^n + n\,(1-p_{n,2}) - 1\right)}{(1-p_{n,2})^2} = O\left(\Delta_n^{-1}\right)$$

and so

$$\sum_{j=0}^{n-1}\mathbb{E}\left[Z_j^2\right] \leqslant C\,\Delta_n \to 0.$$

whence $B_n \xrightarrow{p} 0$ and, by symmetry, $C_n \xrightarrow{p} 0$. We consider the last term

$$\begin{aligned}
\mathrm{D}_n &= \sum_{j=0}^{n-1}\left(1-B_{j+1,n}^{(1)}\right)\left(1-B_{j+1}^{(2)}\right)\left(Y_{j,n}^{(1)}-X_{j,n}^{(1)}\right)\left(Y_{j,n}^{(2)}-X_{j,n}^{(2)}\right)\\
&= \sum_{j=0}^{n-1}\left(1-B_{j+1,n}^{(1)}\right)\left(1-B_{j+1}^{(2)}\right)\left(Y_{j,n}^{(1)}-Y_{j-K_j^{(1)},n}^{(1)}\right)\left(Y_{j,n}^{(2)}-Y_{j-K_j^{(2)},n}^{(2)}\right)\\
&= \sum_{j=0}^{n-1}\left(1-B_{j+1,n}^{(1)}\right)\left(1-B_{j+1}^{(2)}\right)\left(Y_{j,n}^{(2)}-Y_{j-K_j^{(2)},n}^{(2)}\right)\left(Y_{j,n}^{(1)}-Y_{j-K_j^{(1)},n}^{(1)}\right)\\
&= \underbrace{\sum_{j=0}^{n-1}\left(1-B_{j+1,n}^{(1)}\right)\left(1-B_{j+1}^{(2)}\right)\int_{t_{j-K_j^{(2)},n}}^{t_{j,n}}\sigma_s^{(2)}\rho_s\,dW_s^{(1)}\int_{t_{j-K_j^{(1)},n}}^{t_{j,n}}\sigma_s^{(1)}\,dW_s^{(1)}}_{\mathrm{D}_n^{(1)}}\\
&\quad+\underbrace{\sum_{j=0}^{n-1}\left(1-B_{j+1,n}^{(1)}\right)\left(1-B_{j+1}^{(2)}\right)\int_{t_{j-K_j^{(2)},n}}^{t_{j,n}}\sigma_s^{(2)}\sqrt{1-\rho_s^2}\,dW_s^{(2)}\int_{t_{j-K_j^{(1)},n}}^{t_{j,n}}\sigma_s^{(1)}\,dW_s^{(1)}}_{\mathrm{D}_n^{(2)}}.
\end{aligned}$$

We have now

$$\begin{aligned}
\mathrm{D}_n^{(1)} &= \sum_{j=0}^{n-1}\omega_{j+1,n}\left(\sigma_{j,n}^{(2)}\rho_j\left(W_{t_{j,n}}^{(1)}-W_{t_{j-K_j^{(2)},n}}^{(1)}\right)+O_p\left(K_j^{(2)}\Delta_n\right)\right)\left(\sigma_{j,n}^{(1)}\left(W_{t_{j,n}}^{(1)}-W_{t_{j-K_j^{(1)},n}}^{(1)}\right)+O_p\left(K_j^{(1)}\Delta_n\right)\right)\\
&= \underbrace{\sum_{j=0}^{n-1}\omega_{j+1,n}\Bigg[\sigma_{j,n}^{(1)}\sigma_{j,n}^{(2)}\rho_j\left(W_{t_{j,n}}^{(1)}-W_{t_{j-K_j^{(2)},n}}^{(1)}\right)\left(W_{t_{j,n}}^{(1)}-W_{t_{j-K_j^{(1)},n}}^{(1)}\right)}_{\mathrm{F}_n}+O_p\left(\left(K_j^{(2)}\right)^{1/2}K_j^{(1)}\Delta_n^{3/2}\right)\\
&\quad+O_p\left(\left(K_j^{(1)}\right)^{1/2}K_j^{(2)}\Delta_n^{3/2}\right)+O_p\left(K_j^{(1)}K_j^{(2)}\Delta_n^2\right)\Bigg]
\end{aligned}$$

$$\begin{aligned}
\mathrm{D}_n^{(2)} &= \sum_{j=0}^{n-1} \omega_{j+1,n} \left(\sigma_{j,n}^{(2)} \sqrt{1-\rho_j^2} \left(W_{t_{j,n}}^{(2)} - W_{t_{j-K_j^{(2)},n}}^{(2)} \right) + O_p\left(K_j^{(2)} \Delta_n \right) \right) \left(\sigma_{j,n}^{(1)} \left(W_{t_{j,n}}^{(1)} - W_{t_{j-K_j^{(1)},n}}^{(1)} \right) + O_p\left(K_j^{(1)} \Delta_n \right) \right) \\
&= \underbrace{\sum_{j=0}^{n-1} \omega_{j+1,n} \left[\sigma_{j,n}^{(1)} \sigma_{j,n}^{(2)} \sqrt{1-\rho_j^2} \left(W_{t_{j,n}}^{(2)} - W_{t_{j-K_j^{(2)},n}}^{(2)} \right) \left(W_{t_{j,n}}^{(1)} - W_{t_{j-K_j^{(1)},n}}^{(1)} \right) \right.}_{\mathrm{G}_n} + O_p\left(K_j^{(1)} \left(K_j^{(2)} \right)^{1/2} \Delta_n^{3/2} \right) \\
&\left. + O_p\left(\left(K_j^{(1)} \right)^{1/2} K_j^{(2)} \Delta_n^{3/2} \right) + O_p\left(K_j^{(1)} K_j^{(2)} \Delta_n^2 \right) \right].
\end{aligned}$$

We consider now the following decomposition of F_n

$$\begin{aligned}
\mathrm{F}_n^{(<)} &= \sum_{j=0}^{n-1} \omega_{j+1,n} \sigma_{j,n}^{(1)} \sigma_{j,n}^{(2)} \rho_j \left(W_{t_{j,n}}^{(1)} - W_{t_{j-K_j^{(1)},n}}^{(1)} \right)^2 1_{\left\{ K_{j,n}^{(1)} \leqslant K_{j,n}^{(2)} \right\}} \\
&+ \sum_{j=0}^{n-1} \omega_{j+1,n} \sigma_{j,n}^{(1)} \sigma_{j,n}^{(2)} \rho_j \left(W_{t_{j,n}}^{(1)} - W_{t_{j-K_j^{(1)},n}}^{(1)} \right) \left(W_{t_{j-K_j^{(1)},n}}^{(1)} - W_{t_{j-K_j^{(2)},n}}^{(1)} \right) 1_{\left\{ K_{j,n}^{(1)} \leqslant K_{j,n}^{(2)} \right\}},
\end{aligned}$$

$$\begin{aligned}
\mathrm{F}_n^{(>)} &= \sum_{j=0}^{n-1} \omega_{j+1,n} \sigma_{j,n}^{(1)} \sigma_{j,n}^{(2)} \rho_j \left(W_{t_{j,n}}^{(1)} - W_{t_{j-K_j^{(2)},n}}^{(1)} \right)^2 1_{\left\{ K_{j,n}^{(2)} < K_{j,n}^{(1)} \right\}} \\
&+ \sum_{j=0}^{n-1} \omega_{j+1,n} \sigma_{j,n}^{(1)} \sigma_{j,n}^{(2)} \rho_j \left(W_{t_{j,n}}^{(1)} - W_{t_{j-K_j^{(2)},n}}^{(1)} \right) \left(W_{t_{j-K_j^{(2)},n}}^{(1)} - W_{t_{j-K_j^{(1)},n}}^{(1)} \right) 1_{\left\{ K_{j,n}^{(1)} > K_{j,n}^{(2)} \right\}} \\
&= \sum_{j=0}^{n-1} \omega_{j+1,n} \sigma_{j,n}^{(1)} \sigma_{j,n}^{(2)} \rho_j \left(W_{t_{j,n}}^{(1)} - W_{t_{j-K_j^{(2)},n}}^{(1)} \right)^2 1_{\left\{ K_{j,n}^{(2)} \leqslant K_{j,n}^{(1)} \right\}} \\
&- \sum_{j=0}^{n-1} \omega_{j+1,n} \sigma_{j,n}^{(1)} \sigma_{j,n}^{(2)} \rho_j \left(W_{t_{j,n}}^{(1)} - W_{t_{j-K_j^{(2)},n}}^{(1)} \right)^2 1_{\left\{ K_{j,n}^{(2)} = K_{j,n}^{(1)} \right\}} \\
&+ \sum_{j=0}^{n-1} \omega_{j+1,n} \sigma_{j,n}^{(1)} \sigma_{j,n}^{(2)} \rho_j \left(W_{t_{j,n}}^{(1)} - W_{t_{j-K_j^{(2)},n}}^{(1)} \right) \left(W_{t_{j-K_j^{(2)},n}}^{(1)} - W_{t_{j-K_j^{(1)},n}}^{(1)} \right) 1_{\left\{ K_{j,n}^{(1)} > K_{j,n}^{(2)} \right\}},
\end{aligned}$$

and of G_n

$$\begin{aligned}
\mathrm{G}_n^{(<)} &= \sum_{j=0}^{n-1} \omega_{j+1,n} \sigma_{j,n}^{(1)} \sigma_{j,n}^{(2)} \sqrt{1-\rho_j^2} \left(W_{t_{j,n}}^{(2)} - W_{t_{j-K_j^{(1)},n}}^{(2)} \right) \left(W_{t_{j,n}}^{(1)} - W_{t_{j-K_j^{(1)},n}}^{(1)} \right) 1_{\left\{ K_{j,n}^{(1)} \leqslant K_{j,n}^{(2)} \right\}} \\
&+ \sum_{j=0}^{n-1} \omega_{j+1,n} \sigma_{j,n}^{(1)} \sigma_{j,n}^{(2)} \sqrt{1-\rho_j^2} \left(W_{t_{j,n}}^{(2)} - W_{t_{j-K_j^{(1)},n}}^{(2)} \right) \left(W_{t_{j-K_j^{(1)},n}}^{(1)} - W_{t_{j-K_j^{(2)},n}}^{(1)} \right) 1_{\left\{ K_{j,n}^{(1)} \leqslant K_{j,n}^{(2)} \right\}},
\end{aligned}$$

$$\begin{aligned}
\mathrm{G}_n^{(>)} &= \sum_{j=0}^{n-1} \omega_{j+1,n} \sigma_{j,n}^{(1)} \sigma_{j,n}^{(2)} \sqrt{1-\rho_j^2} \left(W_{t_{j,n}}^{(2)} - W_{t_{j-K_j^{(2)},n}}^{(2)} \right) \left(W_{t_{j,n}}^{(1)} - W_{t_{j-K_j^{(2)},n}}^{(1)} \right) 1_{\left\{ K_{j,n}^{(1)} > K_{j,n}^{(2)} \right\}} \\
&+ \sum_{j=0}^{n-1} \omega_{j+1,n} \sigma_{j,n}^{(1)} \sigma_{j,n}^{(2)} \sqrt{1-\rho_j^2} \left(W_{t_{j,n}}^{(2)} - W_{t_{j-K_j^{(2)},n}}^{(2)} \right) \left(W_{t_{j-K_j^{(2)},n}}^{(1)} - W_{t_{j-K_j^{(1)},n}}^{(1)} \right) 1_{\left\{ K_{j,n}^{(1)} > K_{j,n}^{(2)} \right\}}.
\end{aligned}$$

By applying Lemma 4 we obtain the following convergences in probability:

$$\mathrm{F}_n^{(<)} \xrightarrow{p} \frac{(1-p_1)^2\,(1-p_2)\,p_1p_2}{(1-p_1\,p_2)^2}\int_0^1 \sigma_s^{(1)}\,\sigma_s^{(2)}\,\rho_s\,ds,$$

$$\mathrm{F}_n^{(>)} \xrightarrow{p} \left(\frac{(1-p_2)^2\,(1-p_1)\,p_1p_2}{(1-p_1\,p_2)^2} - \frac{(1-p_1)^2\,(1-p_2)^2\,p_1p_2}{(1-p_1p_2)^2}\right)\int_0^1 \sigma_s^{(1)}\,\sigma_s^{(2)}\,\rho_s\,ds,$$

$\mathrm{G}_n^{(>)} \xrightarrow{p} 0$ and $\mathrm{G}_n^{(<)} \xrightarrow{p} 0$.

Thus, by applying simple algebraic computations we obtain

$$\mathsf{RC}_n \xrightarrow{p} \frac{(1-p_1)\,(1-p_2)}{(1-p_1\,p_2)}\int_0^1 \sigma_s^{(1)}\sigma_s^{(2)}\rho_s\,ds,$$

whence the thesis. $\square$

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