A stochastic volatility framework with analytical filtering

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Abstract Motivated by the fact that realized measures of volatility are affected by measurement errors, we introduce a new family of discrete-time stochastic volatility models having two measurement equations relating both the observed returns and realized measures to the latent conditional variance.

Key words: Bayesian Inference, Monte Carlo Markov Chain, High-frequency, Realized volatility, ARG, Stochastic volatility

1 Introduction

In this paper we introduce a new family of discrete-time Stochastic Volatility (SV) models, for the joint modelling of returns and realized measures of volatility. The proposed model is characterized by having two *measurement equations* for the latent volatility: (i) a Normal density for the daily returns and (ii) a Gamma density for the RV measure. We then term the general version of the proposed model as SV-ARG. A salient feature of the SV-ARG is that it allows for analytical filtering and smoothing recursions for the latent factor that guides the dynamics of daily returns. This permits us to develop an effective Bayesian inference procedure for both parameters and latent factor.

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2 The model

Consider a financial log-return process r_t, a realized variance process y_t and a latent volatility process h_t. Let $\mathscr{F}_t \doteq \sigma(r_t, y_t)$ be the σ-algebra containing the information about observable quantities (log-return and realized variance y_t) available at time t, and $\widetilde{\mathscr{F}}_t^H \doteq \sigma(\mathscr{F}_{t-1}, h_t)$. We assume the following model for the dynamics of the log-returns:

$$r_t = \mu + \gamma h_t + \sqrt{h_t}\varepsilon_t, \quad \varepsilon_t \overset{i.i.d.}{\sim} \mathscr{N}(0,1), \tag{1}$$

$t = 1, \ldots, T$, where μ is the risk-free rate and γ is the market price of risk. $\mathscr{N}(m, \sigma^2)$ indicates the univariate normal distribution with mean m and variance σ^2. The dynamics in Equation (1) differs from that employed in Corsi et al. (2013); Majewski et al. (2015) for daily log-returns inasmuch in these works authors consider as driving process for returns a realized measure of volatility. Specifically, they employ the continuous part of the realized variance, hereafter RV, defined as the sum of squared returns over non-overlapping intervals within a sampling period. We refer to Equation (1) as *return equation*.

Since the RV contains information on the latent volatility process, we follow authors in Hansen and Lunde (2006); Engle and Gallo (2006); Shephard and Sheppard (2010); Takahashi et al. (2009) and introduce another measurement equation, termed *realized variance equation*, which relates the observed RV to the latent process h_t. Specifically, we assume that the realized variance y_t is sampled from a Gamma distribution

$$y_t | \widetilde{\mathscr{F}}_t^H \overset{i.d.}{\sim} \mathscr{G}(\alpha, h_t), \tag{2}$$

where $\alpha \in \mathbb{R}_+$ is constant. In the previous equation, $\mathscr{G}(k, \vartheta)$ denotes a Gamma distribution with positive shape, k, and scale parameter, ϑ.

We assume that h_t follows an autoregressive gamma process with transition distribution (see Gouriéroux and Jasiak, 2006):

$$h_t | \widetilde{\mathscr{F}}_{t-1}^H, r_{t-1}, y_{t-1} \overset{d}{\sim} \bar{\mathscr{G}}(\nu, \frac{\phi}{c} h_{t-1}, c). \tag{3}$$

In the previous equation, $\bar{\mathscr{G}}(\nu, \frac{\phi}{c} h_{t-1}, c)$ denotes the non-central gamma distribution with shape $\nu > 0$, scale $c > 0$ and non-centrality $\frac{\phi}{c} h_{t-1}$. Using the Poisson mixture representation for the non-central gamma distribution (see Gouriéroux and Jasiak, 2006, for more details), we rewrite Equation (3) as

$$\begin{aligned} h_t | z_t &\overset{i.d}{\sim} \mathscr{G}(\nu + z_t, c), \\ z_t | h_{t-1} &\overset{i.d}{\sim} \mathscr{P}(\varphi h_{t-1}), \end{aligned}$$

where, in general, $\mathscr{P}o(v)$ indicates the Poisson distribution with intensity parameter v. The latter representation is useful for both the characterization of h_t and the inference procedure. The stationarity conditions, the conditional moment generating function of this process and its risk neutral dynamics are given in (Bormetti et al., 2016).

3 Analytical filtering and smoothing

Applying similar argument as in Creal (2015), we are able to provide analytical expressions for the: (i) conditional likelihood, (ii) Markov transition, (iii) initial distribution of z_t, (iv) filtering and the smoothing of the latent process h_t. In particular, the following two propositions hold.

Proposition 1. *For the* SV-ARG *model described in Equation* (1)*,* (2) *and* (3) *the conditional likelihood, $p(r_t, y_t|z_t, \theta)$, the Markov transition, $p(z_t|z_{t-1}, r_{t-1}, y_{t-1}, \theta)$, and the initial distribution of z_t, $p(z_1;\theta)$, are respectively given by:*

$$
\begin{aligned}
p(r_t, y_t|z_t;\theta) &= 2\eta(z_t, y_t;\theta) K_{\lambda(z_t)}\left(\sqrt{\psi\chi^{(t)}}\right)\left(\sqrt{\frac{\chi^{(t)}}{\psi}}\right)^{\lambda(z_t)}, \\
p(z_t|z_{t-1}, r_{t-1}, y_{t-1};\theta) &\propto \mathscr{S}\left(\lambda(z_{t-1}), \chi^{(t-1)}\frac{\phi^{(d)}}{c}, \psi\frac{c}{\phi^{(d)}}\right), \\
p(z_1;\theta) &\propto \mathscr{N}\mathscr{B}\left(\nu, \phi^{(d)}\right),
\end{aligned}
$$

with

$$
\begin{aligned}
\eta(z_t, y_t;\theta) &= \frac{\exp(\gamma\mu_{1t})}{\sqrt{2\pi}}\frac{y_t^{\alpha_t-1}}{\Gamma(\alpha_t)}\frac{1}{\Gamma(\nu+z_t)c^{\nu+z_t}}, \\
\mu_{1t} &= r_t - \mu, \\
\alpha_t &= \alpha, \\
\lambda(z_t) &= \nu + z_t - \alpha_t - 1/2, \\
\chi^{(t)} &= \mu_{1t}^2 + 2\mu_{2t}, \\
\mu_{2t} &= y_t, \\
\psi &= \gamma^2 + \frac{2}{c}.
\end{aligned}
$$

Proof. See Bormetti et al. (2016).

Proposition 2. *Let $\lambda(z_t)$, $\chi^{(t)}$ and ψ be the quantities defined in Proposition 1. The marginal filtered, $p(h_t|\mathbf{r}_{1:t}, \mathbf{y}_{1:t}, \mathbf{z}_{1:t}, \mathbf{x}_{1:t};\theta)$, and smoothed, $p(h_t|\mathbf{r}_{1:T}, \mathbf{y}_{1:T}, \mathbf{z}_{1:T}, \mathbf{x}_{1:T};\theta)$ distributions are*

$$
\begin{aligned}
p(h_t|\mathbf{r}_{1:t}, \mathbf{y}_{1:t}, \mathbf{z}_{1:t}, \mathbf{x}_{1:t};\theta) &\propto \mathscr{G}ig\left(\lambda(z_t), \chi^{(t)}, \psi\right), \\
p(h_t|\mathbf{r}_{1:T}, \mathbf{y}_{1:T}, \mathbf{z}_{1:T}, \mathbf{x}_{1:T};\theta) &\propto \mathscr{G}ig\left(\lambda(z_t) + z_{t+1}, \chi^{(t)}, \psi + 2\frac{\phi^{(d)}}{c}\right),
\end{aligned}
$$

$t = 1, \cdots, T.$

Proof. See Bormetti et al. (2016).

4 Simulation results

For the SV-ARG model we simulate 50 data-series of 1,000 observations. For each data-series we run the Gibbs sampler in Bormetti et al. (2016) for 100,000 iterations, discard the first 20,000 draws to avoid dependence from initial conditions, and finally apply a thinning procedure to reduce the dependence between consecutive draws. We test the efficiency of the algorithm in three different scenarios: LOW-PERSISTENCE ($\beta = 0.3$), MEDIUM PERSISTENCE ($\beta = 0.6$), and finally, HIGH PERSISTENCE ($\beta = 0.9$). The true values for the other parameters used in the simulations are reported in Table 1together with the grand average of the parameter posterior means along with their robust standard deviations. The results in Table 1 indicates the accuracy of the MCMC scheme is remarkable for all the scenarios (LOW PERSISTENCE, MEDIUM PERSISTENCE, HIGH PERSISTENCE). As regards the efficiency, the magnitudes of the inefficiency factor after applying a thinning procedure are below ten.

Table 1 SUMMARY OUTPUT OF THE PARAMETER ESTIMATES FOR THE SV-ARG MODEL

		LOW PERSISTENCE		MEDIUM PERSISTENCE		HIGH PERSISTENCE	
θ	TRUE	ESTIMATE	STD	ESTIMATE	STD	ESTIMATE	STD
μ	0.0	0.0018	0.0118	-0.0051	0.0177	-0.0074	0.0358
γ	1.0	1.0552	0.0738	1.0523	0.0720	1.0685	0.0784
α	0.8	0.8428	0.0572	0.8327	0.0575	0.8474	0.0647
ν	0.8	0.8033	0.0371	0.7981	0.0394	0.8182	0.0576
c	1.0	0.9654	0.0938	0.9706	0.0909	0.9395	0.0790
β		0.3 118	0.0595	0.6376	0.0746	0.9702	0.0839

5 Conclusions

Motivated by the presence of measurement errors in the empirically computed realized volatility measures we introduce a new family of discrete-time models. We derive the analytical filtering and smoothing and show that they can be used for efficient inference on the parameters and the latent volatility process.

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References

Bormetti, G., Casarin, R., Corsi, F., and Livieri, G. (2016). Smiles at errors: A discrete-time stochastic volatility framework for pricing options with realized measures. *Working Paper, University Ca' Foscari of Venice*.

Chib, S., Nardari, F., and Shephard, N. (2002). Markov chain Monte Carlo methods for stochastic volatility models. *Journal of Econometrics*, 108(2):281–316.

Corsi, F., Fusari, N., and La Vecchia, D. (2013). Realizing smiles: Options pricing with realized volatility. *Journal of Financial Economics*, 107(2):284–304.

Creal, D. D. (2015). A class of non-Gaussian state space models with exact likelihood inference. *Journal of Business & Economic Statistics*, (just-accepted).

Engle, R. F. and Gallo, G. M. (2006). A multiple indicators model for volatility using intra-daily data. *Journal of Econometrics*, 131(1):3–27.

Gouriéroux, C. and Jasiak, J. (2006). Autoregressive gamma processes. *Journal of Forecasting*, 25(2):129–152.

Hansen, P. R. and Lunde, A. (2006). Realized variance and market microstructure noise. *Journal of Business & Economic Statistics*, 24(2):127–161.

Majewski, A. A., Bormetti, G., and Corsi, F. (2015). Smile from the past: A general option pricing framework with multiple volatility and leverage components. *Journal of Econometrics*, 187(2):521–531.

Shephard, N. and Sheppard, K. (2010). Realising the future: Forecasting with high-frequency-based volatility (HEAVY) models. *Journal of Applied Econometrics*, 25(2):197–231.

Takahashi, M., Omori, Y., and Watanabe, T. (2009). Estimating stochastic volatility models using daily returns and realized volatility simultaneously. *Computational Statistics & Data Analysis*, 53(6):2404–2426.