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SECOND-ORDER HAMILTON–JACOBI EQUATIONS IN INFINITE DIMENSIONS*

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Abstract. Some second-order Hamilton–Jacobi equations connected to stochastic optimal control problems for infinite-dimensional systems driven by a white noise are studied. A direct method to prove existence and uniqueness of mild solutions is developed. Then this solution is identified as the value function of the related stochastic control problem, and a feedback formula for optimal controls is derived.

Key words. Hamilton–Jacobi equations, stochastic optimal control, dynamic programming, viscosity solutions, white noise, infinite dimensions

AMS(MOS) subject classifications. 49C10, 49A60, 93E20

1. Introduction. Second-order Hamilton–Jacobi equations in infinite dimensions have been studied by several authors in connection with the stochastic optimal control of distributed parameter systems; see [Lecture Notes in Mathematics, Vol. 1390, Springer-Verlag, Berlin, 1989] and the references quoted therein. Most of the works on this subject concern systems governed by stochastic partial differential equations driven by a Hilbert space-valued Wiener process. In this paper, we focus our attention on the case of stochastic systems that are driven by a white noise. For such problems fewer results are available in the literature.

In order to explain the context we have in mind, let X be a separable Hilbert space, and consider the problem of minimizing:

$$J_\varepsilon(t, x; z) = \mathscr{E}\left\{\int_t^T [g(y_\varepsilon(s)) + \frac{1}{2}|z(s)|^2]\, ds + \phi(y(T))\right\} \tag{1.1}$$

over all controls $z \in M^2_W(t, T; X)$ satisfying $|z(s)| \leqq R$ almost surely for all $s \in [t, T]$. Here ε, R, and T are given positive numbers and $g, \phi : X \to \mathbf{R}$ are bounded uniformly continuous functions. In (1.1), y_ε is the mild solution of the stochastic differential equation

$$\begin{aligned} &dy_\varepsilon(s) = (Ay_\varepsilon(s) + F(y_\varepsilon(s)) + z(s))\, dt + \sqrt{\varepsilon}\, dW(s), \qquad t \leqq s \leqq T, \\ &y_\varepsilon(t) = x, \end{aligned} \tag{1.2}$$

where W is a cylindrical Wiener process (or *white noise*) on a probability space $(\Omega, \mathscr{F}, P)$. Moreover, $M^2_W(t, T; X)$ denotes the space of the X-valued processes $x(s)$ that are adapted to W and satisfy

$$\mathscr{E}\left(\int_t^T |x(s)|^2\, ds\right) < \infty.$$

As is well known, the dynamic programming approach to problem (1.1), (1.2) consists of studying the *value function* V_ε, defined as

$$V_\varepsilon(t, x) = \inf\{J_\varepsilon(t, x; z) \colon z \in M^2_W(t, T; X), |z(s)| \leqq R \text{ a.s. } \forall s \in [t, T]\}. \tag{1.3}$$

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The function $u_\varepsilon(t, x) = V_\varepsilon(T-t, x)$ is related to the Hamilton-Jacobi-Bellman equation

$$\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\varepsilon}{2}\operatorname{Tr}(u_{xx}) + \langle Ax + F(x), u_x\rangle - H(u_x) + g(x) \quad \text{in }]0, T[\times X, \\ &u(0, x) = \phi(x), \end{aligned} \tag{1.4}$$

where

$$H(p) = \begin{cases} \frac{1}{2}|p|^2 & \text{if } |p| \leqq R, \\ R|p| - \frac{R^2}{2} & \text{if } |p| \geqq R. \end{cases} \tag{1.5}$$

The main goal of this paper is to develop a direct method of solution for equation (1.4). By "direct method" we mean a method that makes no use of the control theoretic interpretation of problem (1.4). More precisely, we will solve the above problem as an initial value problem for a semilinear parabolic equation. Then, after having identified the direct solution of (1.4) as the value function (1.3), we can transfer information from a partial differential equation context into a variational setting, by deriving a feedback formula for optimal controls.

We now explain the main ideas of our method. As in finite dimensions, we first consider the linear problem

$$\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\varepsilon}{2}\operatorname{Tr}(u_{xx}) + \langle Ax, u_x\rangle \quad \text{in }]0, T[\times X, \\ &u(0, x) = \phi(x) \end{aligned} \tag{1.6}$$

whose solution can be represented by the probabilistic formula

$$u(t, x) = \mathscr{E}\left[\phi\left(e^{tA}x + \sqrt{\varepsilon}\int_0^t e^{(t-s)A}\, dW(s)\right)\right] =: (T_t\phi)(x). \tag{1.7}$$

Indeed, when A is self-adjoint, strictly negative and A^{-1} is nuclear, it is shown in [7] that (1.7) is the unique *classical* solution of problem (1.6). This result is recalled and improved in § 3 of this paper, by proving the uniform convergence of some finite-dimensional approximations of (1.7).

Then, we define a *mild* solution of (1.4) as a solution of the integral equation

$$u(t, \cdot) = T_t\phi + \int_0^t T_{t-s}(\langle F, u_x(s, \cdot)\rangle - H(u_x(s, \cdot)) + g)\, ds. \tag{1.8}$$

We solve (1.8) by fixed-point arguments in a space of functions which are C^1 in x for $t > 0$ and satisfy a suitable blowup condition in zero, (see § 4).

In order to apply the above result to problem (1.1), (1.2) we have to identify the function $u(T-t, x)$ as the value function $V_\varepsilon(t, x)$. This could be done by standard verification techniques, computing the Itô differential $du(T-t, y_\varepsilon(t))$, if u were sufficiently smooth and the covariance of W had finite trace. To overcome this difficulty, we study a suitable finite-dimensional approximation of (1.4), for which the smoothness of the solution u_n is well known. We then apply the Itô formula to u_n and pass to the limit as $n \to \infty$. To make this procedure rigorous, it is essential to show that u_n converges to u, uniformly on the bounded sets of $[\tau, T]\times X$ for all $0 < \tau < T$ (see Theorem 4.5).

The techniques of this paper could be easily arranged to study the equation

$$\begin{aligned} &\frac{\partial u}{\partial t}=\frac{\varepsilon}{2}\operatorname{Tr}(Qu_{xx})+\langle Ax+F(x), u_x\rangle-H(u_x)+g(x) \quad \text{in }]0, T[\times X, \\ &u(0,x)=\phi(x), \end{aligned} \tag{1.9}$$

where Q is a self-adjoint positive nuclear operator in X. This problem is related to the optimal control of a system driven by a "genuine" Wiener process. Unlike (1.4), for which no other result seems available in the literature, equation (1.9) has been considered by several authors. In [1], problem (1.9) is studied with $F=0$ and assuming g and ϕ to be convex. In [9], the case of $A=0$ is treated by the theory of abstract Wiener spaces. Note that, even though the equation considered in [9] looks like (1.4), it is indeed equivalent to (1.9). The general equation (1.9) is solved in [5] by using a probabilistic formula like (1.7). In this case, however, we do not get C^1 regularity, but only differentiability in some special directions related to Q.

We conclude this Introduction by recalling that several results on viscosity solutions are available today for Hamilton–Jacobi equations in infinite dimensions (see [4] for first-order equations). Second-order equations have been treated by Lions in [10]–[12]. In this theory, the existence of solutions to (1.9), for weakly continuous data, is usually obtained by variational methods based on the representation formula (1.3).

2. Preliminaries. Let X be a separable Hilbert space, with norm $|\cdot|$. For any $R>0$, we set

$$B_R=\{x\in X;\ |x|\leqq R\}.$$

For any $x, y\in X$ we denote by $x\otimes y$ the operator defined by

$$x\otimes y\cdot z=\langle y, z\rangle x.$$

Let Y be another Hilbert space. We denote by $C_b(X, Y)$ the Banach space of all bounded uniformly continuous mappings $\phi: X\to Y$ endowed with the sup norm $\|\cdot\|_0$. Likewise, $C_b^h(X, Y)$, $h=0,1,2,\cdots$, endowed with the natural norm $\|\cdot\|_h$, is the set of all the mappings $\phi: X\to Y$ which are h times Fréchet differentiable and such that the kth derivative $\phi^{(k)}$ is uniformly continuous and bounded for all $k\leqq h$. Moreover, we set $C_b^h(X,\mathbf{R})=C_b^h(X)$.

For any $\phi\in C_b(X)$ we denote by ω_ϕ a continuity modulus of ϕ, i.e., continuous function $\omega_\phi:[0,\infty[\to[0,\infty[$ satisfying $\omega_\phi(0)=0$ and such that $|\phi(x)-\phi(y)|\leqq \omega_\phi(|x-y|)$ for all $x, y\in X$. It is well known that any function $\phi\in C_b(X)$ possesses a concave continuity modulus.[1]

Lip (X, Y) is the space of all Lipschitz continuous and bounded functions from X to Y, endowed with the norm

$$\|\phi\|_1=\sup\left\{\frac{|\phi(x)-\phi(y)|}{|x-y|};\ x, y\in X;\ x\neq y\right\}+\|\phi\|_0.$$

Throughout the whole paper we fix a complete orthonormal system in X, denoted by $\{e_k\}_{k\in N}$. We define the projection Π_n of X onto the span of $\{e_1, e_2, \cdots, e_n\}$ as follows:

$$\Pi_n=\sum_{k=1}^{n} e_k\otimes e_k \quad \forall n\in\mathbf{N}. \tag{2.1}$$

[1] Indeed, set $\omega(t)=\sup\{|\phi(x)-\phi(y)|;\ |x-y|\leqq t\}$. Then ω is a nondecreasing subadditive continuity modulus for ϕ. So we can check that the concave envelope of ω has the required properties.

Now, let $\{\alpha_k\}$ be a sequence of positive real numbers. Then there exists a unique self-adjoint operator A in X such that $Ae_k = -\alpha_k e_k$. As is well known, A is densely defined and closed with domain

$$D(A) = \left\{ x \in X: \sum_{k=1}^{\infty} \alpha_k^2 \langle x, e_k \rangle^2 < \infty \right\}.$$

Moreover, since A is negative, A generates an analytic semigroup e^{tA} in X and

$$e^{tA}x = \sum_{k=1}^{\infty} e^{-t\alpha_k} \langle x, e_k \rangle \tag{2.2}$$

for all $x \in X$.

Consider now a complete probability space $\{\Omega, \mathscr{F}, \mathbf{P}\}$ and a sequence $\{\beta_k\}$ of standard one-dimensional Brownian motions, mutually independent. We denote by $W^n(t)$ the n-dimensional Brownian motion given by

$$W^n(t) = \sum_{k=1}^{n} \beta_k(t) e_k \tag{2.3}$$

for all $t \geqq 0$. We set

$$W_A^n(t) = \sum_{k=1}^{n} e_k \int_0^t e^{-\alpha_k(t-s)}\, d\beta_k(s), \tag{2.4}$$

$$W_A(t) = \sum_{k=1}^{\infty} e_k \int_0^t e^{-\alpha_k(t-s)}\, d\beta_k(s). \tag{2.5}$$

We note that $W_A^n(t)$ is the stochastic convolution

$$W_A^n(t) = \int_0^t e^{(t-s)A}\, dW^n(s).$$

In general, (2.5) is meaningless since the series in the right-hand side may not converge. The following proposition shows that it becomes meaningful, under some restrictions on the sequence $\{\alpha_k\}$.

PROPOSITION 2.1. *Assume that*

$$\sum_{k=1}^{\infty} \frac{1}{\alpha_k} < \infty. \tag{2.6}$$

Then, the series in (2.5) *converges in* $L^2(\Omega, \mathscr{F}, \mathbf{P}; H)$ *for all* $t \geqq 0$. *Moreover,* $W_A(t)$ *is a Gaussian process with mean zero and covariance operator* Q_t *given by*

$$Q_t x = \sum_{k=1}^{\infty} \frac{1 - e^{-2\alpha_k t}}{2\alpha_k} \langle x, e_k \rangle e_k, \qquad x \in X. \tag{2.7}$$

Proof. For all $t \geqq 0$, we have

$$\sum_{k=1}^{\infty} \mathscr{E}\left[\int_0^t e^{-\alpha_k(t-s)}\, d\beta_k(s) \right]^2 = \sum_{k=1}^{\infty} \int_0^t e^{-2\alpha_k s}\, ds = \sum_{k=1}^{\infty} \frac{1 - e^{-2\alpha_k t}}{2\alpha_k},$$

which is finite in view of (2.6). Therefore, the series in (2.5) converges in X for all $t > 0$ almost surely to a Gaussian process $W_A(t)$. In order to prove (2.7) it suffices to remark that, for all $x, y \in X$, we have

$$\mathscr{E}\langle W_A(t), x \rangle \langle W_A(t), y \rangle = \sum_{k=1}^{\infty} \langle x, e_k \rangle \langle y, e_k \rangle \int_0^t e^{-2\alpha_k(t-s)}\, ds. \qquad \square$$

Remark 2.2. If (2.6) is fulfilled, we write the stochastic convolution (2.5) as follows:

$$W_A(t)=\int_0^t e^{(t-s)A}\,dW(s),$$

where $W(t)=\sum_{k=1}^{\infty} e_k\beta_k(t)$ is usually interpreted as a *white noise.*

In order to show that $W_A(t)$ has continuous trajectories, we will strengthen (2.6), assuming

$$\sum_{k=1}^{\infty} \alpha_k^{2\sigma-1}<\infty \tag{2.8}$$

for some $\sigma\in\,]0,\frac{1}{2}[$. We set

$$q(t)=\sum_{k=1}^{\infty}\frac{1-e^{-2\alpha_k t}}{2\alpha_k}.$$

We note that (2.8) yields

$$q(t)\leqq Mt^{2\sigma} \tag{2.9}$$

for all $t\geqq 0$ and some constant $M>0$.

PROPOSITION 2.3. *Assume* (2.8). *Then* $W_A(t)$ *has* α*-Hölder continuous trajectories for all* $\alpha\in\,]0,\sigma[$.

Proof. Since $\{\beta_k\}$ are independent, we have

$$\begin{aligned}\mathscr{E}|W_A(t)-W_A(s)|^2&=\sum_{k=1}^{\infty}\mathscr{E}\left[\int_0^t e^{-\alpha_k(t-\rho)}\,d\beta_k(\rho)\right]^2+\sum_{k=1}^{\infty}\mathscr{E}\left[\int_0^s e^{-\alpha_k(s-\rho)}\,d\beta_k(\rho)\right]^2\\&\quad-2\sum_{k=1}^{\infty}\mathscr{E}\left[\int_0^t e^{-\alpha_k(t-\rho)}\,d\beta_k(\rho)\int_0^s e^{-\alpha_k(s-\rho)}\,d\beta_k(\rho)\right]\\&=\sum_{k=1}^{\infty}\int_0^t e^{-2\alpha_k\rho}\,d\rho+\sum_{k=1}^{\infty}\int_0^s e^{-2\alpha_k\rho}\,d\rho-2\sum_{k=1}^{\infty}\int_0^s e^{-\alpha_k(t+s-2\rho)}\,d\rho\end{aligned}$$

for all $t\geqq s\geqq 0$. Now, by changing the variable ρ with $t+s-2\rho$, we obtain

$$\mathscr{E}|W_A(t)-W_A(s)|^2=q(t)+q(s)+2\left[q\left(\frac{t-s}{2}\right)-q\left(\frac{t+s}{2}\right)\right] \tag{2.10}$$

for all $t\geqq s\geqq 0$.

Next, note that, since $q(t)-q(s)\leqq q(t-s)$, (2.9) yields $q\in C^{2\sigma}([0,\infty[)$. Therefore, there exists $C>0$ such that

$$\left|q(t)+q(s)-2q\left(\frac{t+s}{2}\right)\right|\leqq C|t-s|^{2\sigma}.$$

From (2.10) it follows that

$$\mathscr{E}|W_A(t)-W_A(s)|^2\leqq C(1+2^{1-2\sigma})|t-s|^{2\sigma}.$$

Since $W_A(t)-W_A(s)$ is a Gaussian process, $\mathscr{E}|W_A(t)-W_A(s)|^{2m}\leqq C'|t-s|^{2m\sigma}$ for all t, $s\geqq 0$ and a suitable constant C'. The Kolmogorov test yields the conclusion. □

PROPOSITION 2.4. *Assume* (2.8). *Then, for all* $T>0$,

$$\mathscr{E}\left(\sup_{0\leqq t\leqq T}|W_A(t)|\right)<\infty, \tag{2.11}$$

$$\lim_{n\to\infty}\mathscr{E}\left(\sup_{0\leqq t\leqq T}|W_A^n(t)-W_A(t)|\right)=0. \tag{2.12}$$

Proof. We use the factorization method as in [6]. Set

$$Y(s)=\sum_{k=1}^{\infty} e_k \int_0^s (s-r)^{-\sigma} e^{-\alpha_k(s-r)}\, d\beta_k(r),$$

$$Y_n(s)=\Pi_n Y(s)$$

for all $s \geqq 0$. Then, by a straightforward computation,

$$W_A(t)=\frac{\sin \pi\sigma}{\sigma}\int_0^t (t-s)^{\sigma-1} e^{(t-s)A} Y(s)\, ds, \tag{2.13}$$

$$W_A^n(t)=\frac{\sin \pi\sigma}{\sigma}\int_0^t (t-s)^{\sigma-1} e^{(t-s)A_n} Y_n(s)\, ds, \tag{2.14}$$

where $A_n = A\Pi_n$. Therefore,

$$W_A(t)-W_A^n(t)=B_n(t)+C_n(t), \tag{2.15}$$

where

$$B_n(t)=\frac{\sin \pi\sigma}{\sigma}\int_0^t (t-s)^{\sigma-1}[e^{(t-s)A}-e^{(t-s)A_n}]Y(s)\, ds,$$

$$C_n(t)=\frac{\sin \pi\sigma}{\sigma}\int_0^t (t-s)^{\sigma-1} e^{(t-s)A_n}[Y(s)-Y_n(s)]\, ds.$$

We will estimate $B_n(t)$ and $C_n(t)$ separately. After some computations we obtain

$$\mathscr{E}|Y(s)|^2=\sum_{k=1}^{\infty}\int_0^s e^{-2\alpha_k(s-r)}(s-r)^{-2\sigma}\, dr \leqq k_0 \sum_{k=1}^{\infty}\frac{1}{\alpha_k^{1-2\sigma}} =: k_1 \tag{2.16}$$

for some constant $k_0>0$. Since $Y(s)$ is a Gaussian process, (2.16) implies that, for all $m \in \mathbf{N}$,

$$\mathscr{E}|Y(s)|^{2m} \leqq k_m \tag{2.17}$$

for some constant $k_m>0$. Now, by Hölder's inequality and (2.13) it follows that

$$\mathscr{E}\left(\sup_{0\leqq t\leqq T}|W_A(t)|^{2m}\right) \leqq \left(\frac{\sin \pi\sigma}{\sigma}\right)^{2m}\left(\int_0^T s^{2m(\sigma-1)/(2m-1)}\, ds\right)^{2m-1}\int_0^T \mathscr{E}|Y(s)|^{2m}\, ds.$$

Moreover, we conclude that

$$\mathscr{E}\left(\sup_{0\leqq t\leqq T}|B_n(t)|^{2m}\right) \leqq \left(\frac{\sin \pi\sigma}{\sigma}\right)^{2m}\left(\int_0^T s^{2m(\sigma-1)/(2m-1)}\|e^{sA}-e^{sA_n}\|^{2m/(2m-1)}\, ds\right)^{2m-1} \cdot \int_0^T \mathscr{E}|Y(s)|^{2m}\, ds.$$

Now, since the semigroup e^{sA} is analytic, $\|e^{sA}-e^{sA_n}\| \to 0$ for all $s>0$ as $n\to\infty$. Thus, in view of (2.17), the dominated convergence theorem yields

$$\lim_{n\to\infty}\mathscr{E}\left(\sup_{0\leqq t\leqq T}|B_n(t)|^{2m}\right)=0 \tag{2.18}$$

provided that $(1-\sigma)2m/(2m-1)<1$. We will now estimate $C_n(t)$: we have

$$\begin{aligned}\mathscr{E}|Y(s)-Y_n(s)|^2 &= \sum_{k=n+1}^{\infty}\int_0^s e^{-2\alpha_k(s-r)}(s-r)^{-2\sigma}\, dr \\ &\leqq k_0 \sum_{k=n+1}^{\infty}\frac{1}{\alpha_k^{1-2\sigma}} := k_{1n}\end{aligned} \tag{2.19}$$

for some constant $k_{1n}>0$ such that $\lim_{n\to\infty} k_{1n}=0$. Since $Y(s)-Y_n(s)$ is a Gaussian process, (2.19) implies that, for all $m\in\mathbf{N}$,

$$\mathscr{E}|Y(s)-Y_n(s)|^{2m}\leqq c_m(k_{1n})^m \tag{2.20}$$

for some constant $c_m>0$. Now, by Hölder's inequality we conclude that

$$\mathscr{E}\left(\sup_{0\leqq t\leqq T}|C_n(t)|^{2m}\right)\leqq\left(\frac{\sin\pi\sigma}{\sigma}\right)^{2m}\left(\int_0^T s^{2m(\sigma-1)/(2m-1)}\,ds\right)^{2m-1}\cdot\int_0^T\mathscr{E}|Y(s)-Y_n(s)|^{2m}\,ds.$$

Hence

$$\lim_{n\to\infty}\mathscr{E}\left(\sup_{0\leqq t\leqq T}|C_n(t)|^{2m}\right)=0, \tag{2.21}$$

which, along with (2.18), gives the conclusion. □

3. Linear parabolic equations. In this section we study the linear problem:

$$\begin{aligned}&\frac{\partial u}{\partial t}=\frac{\varepsilon}{2}\operatorname{Tr}(u_{xx})+\langle Ax,u_x\rangle\quad\text{in }[0,T]\times X,\\&u(0,x)=\phi(x),\end{aligned} \tag{3.1}$$

where ε is a given positive number and $\phi\in C_b(X)$. Here $A:D(A)\subset X\to X$ is a self-adjoint negative operator in X satisfying, for all $k\in\mathbf{N}$,

$$Ae_k=-\alpha_k e_k, \tag{3.2}$$

with $\alpha_k>0$. In (3.1) the subscript x represents Fréchet partial derivative with respect to x and Tr denotes the trace; i.e.,

$$\operatorname{Tr}(u_{xx})=\sum_{k=1}^{\infty}\langle u_{xx}e_k,e_k\rangle.$$

The following result, proved in [7], states that, for any $\phi\in C_b(X)$, problem (3.1) has a unique classical solution given by

$$u(t,x)=\mathscr{E}(\phi(e^{tA}x+\sqrt{\varepsilon}\,W_A(t)))=:(T_t\phi)(x), \tag{3.3}$$

where $W_A(t)$ is the process defined in (2.5). The method of [7], based on a procedure of Galerkin type, uses the following sequence of semigroups which is proved to converge pointwise to the semigroup defined in (3.3):

$$\mathscr{E}(\phi(e^{tA}\Pi_n x+\sqrt{\varepsilon}\,W_A^n(t)))=:(T_t^n\phi)(x) \tag{3.4}$$

for all $\phi\in C_b(X)$. In this paper we improve the result of [7], by showing that $(T_t^n\phi)(x)$ converges to $(T_t\phi)(x)$ uniformly. We will use this convergence result in § 5. In the following we denote by $e^{tA}Q_t^{-1}$ the bounded operator defined by

$$\overline{e^{tA}Q_t^{-1}}\,e_k=\frac{2\alpha_k e^{-t\alpha_k}}{1-e^{-2t\alpha_k}}\,e_k,\qquad k\in\mathbf{N}. \tag{3.5}$$

PROPOSITION 3.1. *Assume* (3.2) *and* (2.6). *Then, for any* $\tau\in\,]0,T[$,

$$\begin{aligned}&\text{(i)}\quad\lim_{n\to\infty}(T_t^n\phi)(x)=(T_t\phi)(x),\\&\text{(ii)}\quad\lim_{n\to\infty}(T_t^n\phi)_x(x)=(T_t\phi)_x(x)\end{aligned} \tag{3.6}$$

uniformly on the bounded sets of $[\tau, T]\times X$. *Moreover, the function*

$$u:[0,\infty[\times X\to \mathbf{R},\ u(t,x)=(T_t\phi)(x)$$

is continuous. Furthermore, $u(t,\cdot)\in C_b^\infty(X)$ *for all* $t>0$, $u(\cdot,x)\in C^1(]0,\infty[)$ *for all* $x\in D(A)$ *and*

$$u_x(t,x)=\sqrt{\varepsilon}\ \mathscr{E}(\overline{e^{tA}Q_t^{-1}}\,W_A(t)\phi(e^{tA}x+\sqrt{\varepsilon}\ W_A(t))), \tag{3.7}$$

$$u_{xx}(t,x)=\varepsilon\mathscr{E}(\overline{e^{tA}Q_t^{-1}}\ W_A(t)\otimes\overline{e^{tA}Q_t^{-1}}\ W_A(t)\phi(e^{tA}x+\sqrt{\varepsilon}\ W_A(t))), \tag{3.8}$$

$$|u_x(t,x)|\leqq\sqrt{\varepsilon}\ \rho(t)\|\phi\|_0, \tag{3.9}$$

$$|\mathrm{Tr}\,[u_{xx}(t,x)]|\leqq\varepsilon\rho^2(t)\|\phi\|_0, \tag{3.10}$$

where

$$\rho^2(t)=\sum_{k=1}^{\infty}\frac{2\alpha_k\,e^{-2t\alpha_k}}{1-e^{-2t\alpha_k}}. \tag{3.11}$$

Finally, (3.1) *is fulfilled for all* $x\in D(A)$ *and* $t>0$.

Proof. First, we note that formulas (3.7) and (3.8) are derived in [7]. We will give a detailed proof of the uniform convergence in (3.6). The proofs of the remaining statements will be only sketched for the reader's convenience.

From (3.3) and (3.4) we obtain

$$\begin{aligned}|(T_t\phi)(x)-(T_t^n\phi)(x)|&\leqq\mathscr{E}|\phi(e^{tA}x+\sqrt{\varepsilon}\ W_A(t))-\phi(e^{tA}\Pi_nx+\sqrt{\varepsilon}\ W_A^n(t))|\\&\leqq\mathscr{E}\omega_\phi(|e^{tA}x-e^{tA}\Pi_nx|+\sqrt{\varepsilon}\ |W_A(t)-W_A^n(t)|)\\&\leqq\omega_\phi(|e^{tA}x-e^{tA}\Pi_nx|+\sqrt{\varepsilon}\ \mathscr{E}|W_A(t)-W_A^n(t)|),\end{aligned} \tag{3.12}$$

where ω_ϕ is a concave continuity modulus for ϕ. Now, since e^{tA} is compact for $t>0$,

$$\lim_{n\to\infty}|e^{tA}x-e^{tA}\Pi_nx|=0 \tag{3.13}$$

uniformly for $(t,x)\in[\tau,T]\times B_R$, $R>0$. Moreover,

$$\mathscr{E}|W_A(t)-W_A^n(t)|\leqq\sqrt{T}\sqrt{\mathscr{E}|W_A(t)-W_A^n(t)|^2}. \tag{3.14}$$

Thus, (3.6)(i) follows from (3.13) and (3.14), in light of Proposition 2.1. Equation (3.6)(ii) follows by a similar argument, using formula (3.7) for $u_x(t,x)$. Next, we have

$$\begin{aligned}|\langle u_x(t,x),e_k\rangle|^2&=\varepsilon\,e^{-2t\alpha_k}\left(\frac{2\alpha_k}{1-e^{-2t\alpha_k}}\right)^2\left|\mathscr{E}\int_0^t e^{-2(t-s)\alpha_k}\,d\beta_k(s)\phi(e^{tA}x+W_A(t))\right|^2\\&\leqq\varepsilon\,e^{-2t\alpha_k}\left(\frac{2\alpha_k}{1-e^{-2t\alpha_k}}\right)^2\mathscr{E}\left|\int_0^t e^{-2(t-s)\alpha_k}\,d\beta_k(s)\right|^2\|\phi\|_0^2\end{aligned} \tag{3.15}$$

and (3.9) follows taking the sum over k. Similarly, (3.8) yields

$$\begin{aligned}|\langle u_{xx}(t,x)e_k,e_k\rangle|&=\varepsilon\,e^{-2t\alpha_k}\left(\frac{2\alpha_k}{1-e^{-2t\alpha_k}}\right)^2\Bigg|\mathscr{E}\\&\qquad\cdot\left\{\left[\int_0^t e^{-2(t-s)\alpha_k}\,d\beta_k(s)\right]^2\phi(e^{tA}x+W(t))\right\}\Bigg|\\&\leqq\varepsilon\,e^{-2t\alpha_k}\frac{2\alpha_k}{1-e^{-2t\alpha_k}}\|\phi\|_0,\end{aligned} \tag{3.16}$$

which in turn implies (3.10). □

Remark 3.2. Under the stronger condition (2.8), we have

$$\rho^2(t) \leqq \sup_k \left(\frac{2\alpha_k e^{-t\alpha_k}}{1-e^{-2t\alpha_k}}\right)^2 \cdot q(t) \leqq \frac{L}{t^2} q(t), \tag{3.17}$$

where

$$L = 4 \max_{\alpha \geqq 0} \frac{\alpha e^{-\alpha}}{1-e^{-2\alpha}}. \tag{3.18}$$

Hence, estimate (3.10) and (2.9) yield

$$|\mathrm{Tr}\,[u_{xx}(t, x)]| \leqq \frac{C\varepsilon}{t^{2-2\sigma}}. \tag{3.19}$$

4. Semilinear parabolic equations. Let $T > 0$ and consider the Cauchy problem

$$\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\varepsilon}{2} \mathrm{Tr}\,(u_{xx}) + \langle Ax + F(x), u_x\rangle - H(u_x) + g(x) \quad \text{in } [0, T] \times X, \\ &u(0, x) = \phi(x), \end{aligned} \tag{4.1}$$

where we assume the following, in addition to (3.2) and (2.8),

$$\begin{aligned} &\text{(i)} \quad \phi, g \in C_b(X), \\ &\text{(ii)} \quad H \in \mathrm{Lip}\,(X), H(0) = 0, \\ &\text{(iii)} \quad F \in C_b(X; X). \end{aligned} \tag{4.2}$$

Obviously, the requirement $H(0) = 0$ implies no loss of generality, as we can replace g by $g - H(0)$. We will solve problem (4.1) in the Banach space

$$\Sigma = \{v \in C_b([0, T] \times X) \colon v_x \in C(]0, T] \times X; X),\ t^{1-\sigma} v_x \in B(]0, T] \times X; X)\},$$

$$\|v\|_\Sigma = \sup\{|v(t, x)| + |t^{1-\sigma} v_x(t, x)| \colon (t, x) \in]0, T] \times X\},$$

where σ is defined in (2.8) and $B(]0, T] \times X; X)$ denotes the space of all bounded X-valued functions defined in $]0, T] \times X$.

DEFINITION 4.1. A function $u \in \Sigma$ is called a mild solution of problem (4.1) if u is a solution of the integral equation

$$u(t, \cdot) = T_t \phi + \int_0^t T_{t-s}(\langle F, u_x(s, \cdot)\rangle - H(u_x(s, \cdot)) + g)\, ds \quad \forall t \in [0, T], \tag{4.3}$$

where T_t is the semigroup defined in (3.3).

LEMMA 4.2. *Assume* (2.8) *and let* $\psi \colon]0, T] \times X \to \mathbf{R}$ *be such that*

(i) $\psi \in C_b([\tau, T] \times X)$ *for all* $\tau \in]0, T]$,

(ii) $|t^{1-\sigma}\psi(t, x)| \leqq K$ *for all* $(t, x) \in]0, T] \times X$ *and some constant* $K > 0$.

Set

$$f(t, \cdot) = \int_0^t T_{t-s}(\psi(s, \cdot))\, ds \quad \forall t \in]0, T]. \tag{4.4}$$

Then $f \in \Sigma$.

Proof. Step 1. $f \in C_b([0, T] \times X)$. Fix $\varepsilon > 0$ and let $\tau \in]0, T]$ such that $K\tau^\sigma \leqq \sigma\varepsilon$. Let $(t, x), (t', x') \in [0, T] \times X$.

We shall consider two cases separately.

Case 1. $t, t' \in [0, \tau]$. Then we have, obviously,

$$|f(t,x)-f(t',x')| \leqq \int_0^t |T_{t-s}(s^{1-\sigma}\psi(s,\cdot))(x)|s^{\sigma-1}\,ds + \int_0^{t'} |T_{t-s}(s^{1-\sigma}\psi(s,\cdot))(x')|s^{\sigma-1}\,ds$$
$$\leqq 2K\int_0^\tau s^{\sigma-1}\,ds \leqq 2\varepsilon.$$

Case 2. $t \in]\tau/2, T[$, $t' \in [\tau, T]$. Then we have

$$|f(t,x)-f(t',x')| \leqq 2\varepsilon + \left|\int_{\tau/2}^t T_{t-s}(\psi(s,\cdot))(x)\,ds - \int_{\tau/2}^{t'} T_{t'-s}(\psi(s,\cdot))(x')\,ds\right|.$$

In view of (ii), standard continuity properties of the integral in the right-hand side imply that there exists $\delta > 0$ such that, if $|t-t'|+|x-x'| < \delta$, then

$$\left|\int_\tau^t T_{t-s}(\psi(s,\cdot))(x)\,ds - \int_\tau^{t'} T_{t'-s}(\psi(s,\cdot))(x')\,ds\right| < \varepsilon.$$

To conclude the reasoning, set $\delta' = \min\{\delta, \tau/2\}$. Then, from the above analysis it follows that $|f(t,x)-f(t',x')| \leqq 3\varepsilon$ provided $|t-t'|+|x-x'| < \delta$. This proves that $f \in C_b([0,T]\times X)$.

Step 2. $t^{1-\sigma}f_x(t,x)$ is bounded on $]0,T]\times X$. First, note that, by (3.9), (2.8), and Remark 3.2, we obtain

$$|(T_{t-s}\psi(s,\cdot))_x| \leqq \frac{C_0K}{(t-s)^{1-\sigma}s^{1-\sigma}}, \qquad 0<s<t, \tag{4.5}$$

where $C_0 = \sqrt{\varepsilon LM}$, L and M being defined in (3.17) and (2.9), respectively. Hence, $f_x(t,x)$ exists for all $t>0$ and $x \in X$. Moreover, by (ii),

$$|t^{1-\sigma}f_x(t,x)| \leqq t^{1-\sigma}C_0K\int_0^t (t-s)^{\sigma-1}s^{\sigma-1}\,ds. \tag{4.6}$$

On the other hand, (4.6) yields the conclusion of Step 2 since

$$\int_0^t (t-s)^{\sigma-1}s^{\sigma-1}\,ds = t^{2\sigma-1}\beta(\sigma,\sigma) \leqq \frac{2^{2(1-\sigma)}}{\sigma}t^{2\sigma-1}, \tag{4.7}$$

where β is the Euler beta function.

Step 3. $f_x \in C_b([t_0,T]\times X; X)$ for all $t_0 \in]0,T[$. Fix $t_0 \in]0,T[$ and $t_0 \leqq t \leqq t' \leqq T$, $x, x' \in X$. Then

$$\begin{aligned} &|f_x(t,x)-f_x(t',x')| \\ &\leqq \left|\int_0^t [(T_{t-s}\psi(s,\cdot))_x(x) - (T_{t'-s}\psi(s,\cdot))_x(x')]\,ds\right| \\ &\quad + \left|\int_t^{t'} (T_{t'-s}\psi(s,\cdot))_x(x)\,ds\right|. \end{aligned} \tag{4.8}$$

Moreover, recalling (4.6), we obtain

$$\left|\int_t^{t'} (T_{t'-s}\psi(s,\cdot))_x(x)\,ds\right| \leqq C_0K\int_t^{t'} (t'-s)^{\sigma-1}s^{\sigma-1}\,ds \leqq C_0\frac{K}{\sigma}t_0^{\sigma-1}(t'-t)^\sigma. \tag{4.9}$$

On the other hand, for all $\eta > 0$ there exists $\tau \in]0, t_0/2]$ such that

$$\begin{aligned} &\left| \int_0^t [(T_{t-s}\psi(s,\cdot))_x(x) - (T_{t'-s}\psi(s,\cdot))_x(x')]\, ds \right| \\ &\qquad \leqq 2\eta + \left| \int_t^{t-\tau} [(T_{t-s}\psi(s,\cdot))_x(x) - (T_{t'-s}\psi(s,\cdot))_x(x')]\, ds \right|. \end{aligned} \tag{4.10}$$

Indeed, it suffices to take τ so that

$$C_0 \frac{K}{\sigma} \left(\frac{t_0}{2}\right)^{\sigma-1} \tau^\sigma < \eta. \tag{4.11}$$

Finally, the conclusion follows from (4.8)-(4.10) and the uniform continuity of the mapping $(s, t, x) \to (T_{t-s}\psi(s,\cdot))_x(x)$ on $\{(s, t, x)\colon t_0 \leqq t \leqq T,\ \tau \leqq s \leqq t-\tau,\ x \in X\}$. The proof of the lemma is thus complete. □

THEOREM 4.3. *Assume* (3.2), (2.8), *and* (4.2). *Then problem* (4.1) *has a unique mild solution.*

Proof. Suppose first that T is sufficiently small, i.e.,

$$(1 + 2^{2(1-\sigma)} C_0) \frac{\|F\|_0 + \|H\|_1}{\sigma} T^\sigma \leqq \frac{1}{2}, \tag{4.12}$$

where $C_0 = \sqrt{\varepsilon LM}$, L and M being defined in (3.17) and (2.9), respectively. Define a map Γ on Σ as follows:

$$(\Gamma v)(t,\cdot) = T_t\phi + \int_0^t T_{t-s}(\langle F, v_x(s,\cdot)\rangle - H(v_x(s,\cdot)) + g)\, ds \tag{4.13}$$

for all $t \in [0, T]$. From Lemma 4.2, it follows that $\Gamma : \Sigma \to \Sigma$. Moreover,

$$|(\Gamma v)(t,x) - (\Gamma z)(t,x)| \leqq \frac{\|F\|_0 + \|H\|_1}{\sigma} T^\sigma \|v - z\|_\Sigma, \tag{4.14}$$

$$t^{1-\sigma} |(\Gamma v)_x(t,x) - (\Gamma z)_x(t,x)| \leqq C_0(\|F\|_0 + \|H\|_1)\beta(\sigma,\sigma) T^\sigma \|v - z\|_\Sigma, \tag{4.15}$$

where β is the Euler beta function. Since $\beta(\sigma,\sigma) \leqq 2^{2(1-\sigma)}/\sigma$, (4.13)-(4.15) imply that Γ is a contraction in Σ and the conclusion follows by the contraction mapping principle. Finally, condition (4.12) can be removed by a finite number of iterations of the previous fixed-point argument. □

In the sequel we will consider the following "finite-dimensional" approximation of (4.1):

$$\begin{aligned} &\frac{\partial u_n}{\partial t} = \frac{\varepsilon}{2} \operatorname{Tr}(u_{n,xx}) + \langle A\Pi_n x + \Pi_n F(\Pi_n x), u_{n,x}\rangle - H(u_{n,x}) + g(\Pi_n x), \\ &u_n(0,x) = \phi(\Pi_n x), \end{aligned} \tag{4.16}$$

which has the integral form

$$u_n(t,\cdot) = T_t^n \phi \circ \Pi_n + \int_0^t T_{t-s}^n(\langle F \circ \Pi_n, u_{n,x}(s,\cdot)\rangle - H(u_{n,x}(s,\cdot)) + g \circ \Pi_n)\, ds. \tag{4.17}$$

THEOREM 4.5. *Assume* (3.2), (2.8), *and* (4.2) *and let* u *and* u_n *be the solutions of* (4.1) *and* (4.16), *respectively. Then, for all* $\tau \in]0, T[$,

$$\text{(4.18)} \qquad \begin{aligned} &\text{(i)} \quad \lim_{n\to\infty} |u(t,x) - u_n(t,x)| = 0, \\ &\text{(ii)} \quad \lim_{n\to\infty} |u_x(t,x) - u_{n,x}(t,x)| = 0 \end{aligned}$$

uniformly for $t \in [\tau, T]$ *and* x *in bounded sets of* X.

We first prove the following lemma.

LEMMA 4.6. *Assume* (2.8) *and let* $\psi_n, \psi :]0, T] \times X \to \mathbf{R}$, $n \in N$, *be such that*

(i) $\psi_n, \psi \in C_b([\tau, T] \times X)$ *for all* $\tau \in]0, T]$

(ii) $|t^{1-\sigma}\psi_n(t,x)| \leqq K$, $|t^{1-\sigma}\psi(t,x)| \leqq K$ *for all* $(t,x) \in]0, T] \times X$ *and some constant* $K > 0$.

(iii) $\lim_{n\to\infty} \sup \{|t^{1-\sigma}(\psi(t,x) - \psi_n(t,x))| : t \in [0, T], |x| \leqq R\} = 0$ *for all* $R > 0$.

Set

$$f_n(t,\cdot) = \int_0^t (T_{t-s}\psi_n)(s,\cdot)\, ds, \qquad f(t,\cdot) = \int_0^t (T_{t-s}\psi)(s,\cdot)\, ds.$$

Then, for all $R > 0$

$$\text{(4.19)} \qquad \lim_{n\to\infty} |f_n(t,x) - f(t,x)| = 0,$$

$$\text{(4.20)} \qquad \lim_{n\to\infty} t^{1-\sigma}|f_{n,x}(t,x) - f_x(t,x)| = 0$$

uniformly for $t \in [0, T]$ *and* $|x| \leqq R$.

Proof. First, we note that $f_n, f \in \Sigma$ in view of Lemma 4.2. Now, fix $R > 0$ and let $t \in [0, T]$, $|x| \leqq R$. We have

$$\text{(4.21)} \qquad \begin{aligned} |f(t,x) - f_n(t,x)| \leqq &\left| \int_0^t T^n_{t-s}[\psi(s,\cdot) - \psi_n(s,\cdot)]\, ds \right| \\ &+ \left| \int_0^t [T_{t-s} - T^n_{t-s}]\psi(s,\cdot)\, ds \right|. \end{aligned}$$

We claim that

$$\text{(4.22)} \qquad \lim_{n\to\infty} \left| \int_0^t [T_{t-s} - T^n_{t-s}]\psi(s,\cdot)\, ds \right| = 0$$

uniformly for $t \in [0, T]$, $|x| \leqq R$. Indeed, fix $\eta > 0$ and let $\tau \in]0, T[$ be such that $(K/\sigma)\tau^\sigma < \eta$. Then, if $0 \leqq t \leqq \tau$,

$$\text{(4.23)} \qquad \left| \int_0^t [T_{t-s} - T^n_{t-s}]\psi(s,\cdot)\, ds \right| \leqq 2\eta.$$

On the other hand, if $\tau < t \leqq T$, then

$$\text{(4.24)} \qquad \begin{aligned} \left| \int_0^t [T_{t-s} - T^n_{t-s}]\psi(s,\cdot)\, ds \right| &\leqq 2\eta + \left| \int_{\tau/2}^{t-\tau/2} [T_{t-s} - T^n_{t-s}]\psi(s,\cdot)\, ds \right| \\ &\quad + \left| \int_{t-\tau/2}^{t} [T_{t-s} - T^n_{t-s}]\psi(s,\cdot)\, ds \right| \\ &\leqq 4\eta + \left| \int_{\tau/2}^{t-\tau/2} [T_{t-s} - T^n_{t-s}]\psi(s,\cdot)\, ds \right| \end{aligned}$$

and (4.22) follows from (3.6). Next, let us show that

$$\lim_{n\to\infty}\left|\int_0^t T^n_{t-s}[\psi(s,\cdot)-\psi_n(s,\cdot)]\,ds\right|=0 \tag{4.25}$$

uniformly for $t\in[0,T]$, $|x|\leqq R$. Recalling (3.4), we obtain

$$\begin{aligned}T^n_{t-s}[\psi(s,\cdot)-\psi_n(s,\cdot)](x)=\mathscr{E}[&\psi(s,e^{(t-s)A}\Pi_n x+\sqrt{\varepsilon}\,W^n_A(t-s))\\&-\psi_n(s,e^{(t-s)A}\Pi_n x+\sqrt{\varepsilon}\,W^n_A(t-s))].\end{aligned} \tag{4.26}$$

Moreover, Proposition 2.4 implies that there exists a random variable C, such that

$$|e^{(t-s)A}\Pi_n x+\sqrt{\varepsilon}\,W^n_A(t-s)|\leqq R+C \tag{4.27}$$

for $n\in\mathbf{N}$, $0\leqq s\leqq t\leqq T$ and $|x|\leqq R$. So, in light of hypothesis (iii),

$$\begin{aligned}\lim_{n\to\infty}\sup_{|x|\leqq R,\tau\leqq s\leqq t}&|\psi(s,e^{(t-s)A}\Pi_n x+\sqrt{\varepsilon}\,W^n_A(t-s))\\&-\psi_n(s,e^{(t-s)A}\Pi_n x+\sqrt{\varepsilon}\,W^n_A(t-s))|=0\end{aligned} \tag{4.28}$$

almost surely for all $\tau\in\,]0,T]$. Hence, by the dominated convergence theorem, for all $\tau\in\,]0,T]$.

$$\lim_{n\to\infty}\left|\int_\tau^t T^n_{t-s}[\psi(s,\cdot)-\psi_n(s,\cdot)](x)\,ds\right|=0 \tag{4.29}$$

uniformly for $|x|\leqq R$ and $\tau\in\,]0,T]$. Now fix $\eta>0$ and choose τ so that $(K/\sigma)\tau^\sigma<\eta$. Then

$$\left|\int_0^t T^n_{t-s}[\psi(s,\cdot)-\psi_n(s,\cdot)](x)\,ds\right|\leqq 2\eta+\left|\int_\tau^t T^n_{t-s}[\psi(s,\cdot)-\psi_n(s,\cdot)](x)\,ds\right| \tag{4.30}$$

and (4.25) follows from (4.28). Finally, (4.21), (4.22), and (4.25) imply (4.19). Next, we prove (4.20). For all $t\in\,]0,T]$, $|x|\leqq R$ we have

$$\begin{aligned}t^{1-\sigma}|f_x(t,x)-f_{n,x}(t,x)|\leqq\, &t^{1-\sigma}\left|\int_0^t(T^n_{t-s}[\psi(s,\cdot)-\psi_n(s,\cdot)])_x\,ds\right|\\&+t^{1-\sigma}\left|\int_0^t([T_{t-s}-T^n_{t-s}]\psi)_x(s,\cdot)\,ds\right|.\end{aligned} \tag{4.31}$$

Now, fix $\eta>0$ and let $\tau\in\,]0,T[$ be such that $8C_0Kt^\sigma<\sigma\eta$, where $C_0=\sqrt{\varepsilon LM}$, L and M being defined in (3.17) and (2.9), respectively. Then, by (4.5) and (4.7) we conclude that, if $0\leqq t\leqq T$,

$$t^{1-\sigma}\left|\int_0^t([T_{t-s}-T^n_{t-s}]\psi)_x(s,\cdot)\,ds\right|\leqq 2C_0Kt^{1-\sigma}\int_0^t(t-s)^{\sigma-1}s^{\sigma-1}\,ds\leqq\frac{8}{\sigma}C_0Kt^\sigma<\eta.$$

On the other hand, if $\tau<t\leqq T$, we obtain, as in (4.24),

$$\begin{aligned}&t^{1-\sigma}\left|\int_0^t([T_{t-s}-T^n_{t-s}]\psi)_x(s,\cdot)\,ds\right|\\&\quad\leqq 2\eta+t^{1-\sigma}\left|\int_{\tau/2}^{t-\tau/2}([T_{t-s}-T^n_{t-s}]\psi)_x(s,\cdot)\,ds\right|.\end{aligned} \tag{4.32}$$

So, by (3.6)(ii),

$$\lim_{n\to\infty}t^{1-\sigma}\left|\int_0^t([T_{t-s}-T^n_{t-s}])_x\psi(s,\cdot)\,ds\right|=0 \tag{4.33}$$

uniformly for $t \in]0, T]$ and $|x| \leqq R$. Next we prove that

$$\lim_{n\to\infty} t^{1-\sigma} \left| \int_0^t (T^n_{t-s}[\psi(s,\cdot)-\psi_n(s,\cdot)])_x \, ds \right| = 0 \tag{4.34}$$

uniformly for $t \in]0, T]$ and $|x| \leqq R$. From (3.7) it follows that

$$\begin{aligned} &(T^n_{t-s}[\psi(s,\cdot)-\psi_n(s,\cdot)])_x(x) \\ &\quad = \sqrt{\varepsilon}\, \mathscr{E}\{\overline{e^{(t-s)A}Q^{-1}_{t-s}}\, \Pi_n W_A(t-s)[\psi(s, e^{(t-s)A}\Pi_n x + \sqrt{\varepsilon}\, W^n_A(t-s)) \\ &\qquad - \psi_n(s, e^{(t-s)A}\Pi_n x + \sqrt{\varepsilon}\, W^n_A(t-s))]\}. \end{aligned} \tag{4.35}$$

Recalling (4.28) and (2.11), we conclude that, if $t \geqq \tau$, then

$$\begin{aligned} &\lim_{n\to\infty} \sup_{|x|\leqq R, \tau\leqq s\leqq t} |\overline{e^{(t-s)A}Q^{-1}_{t-s}}\, \Pi_n W_A(t-s)[\psi-\psi_n] \\ &\qquad \cdot (s, e^{(t-s)A}\Pi_n x + \sqrt{\varepsilon}\, W^n_A(t-s))| = 0 \end{aligned} \tag{4.36}$$

almost surely for $|x| \leqq R$, $\tau/2 \leqq s \leqq t-\tau/2$. Now, arguing as above, we can prove (4.34) and so (4.20). □

Proof of Theorem 4.5. Set

$$\begin{aligned} G_n(x,p) &= \langle F(\Pi_n x), p\rangle - H(p) + g(\Pi_n x), \\ G(x,p) &= \langle F(\Pi x), p\rangle - H(p) + g(\Pi x) \end{aligned} \tag{4.37}$$

for all $x, p \in X$ and

$$(\Gamma_n v)(t,\cdot) = T^n_t \phi + \int_0^t T^n_{t-s} G(\cdot, v_x(s,\cdot))\, ds \quad \forall v \in \Sigma. \tag{4.38}$$

From Lemma 4.2 it follows that Γ_n maps Σ into Σ. Arguing as in the proof of Theorem 4.3 it follows that

$$\|\Gamma_n v - \Gamma_n z\|_\Sigma \leqq \tfrac{1}{2}\|v-z\|_\Sigma \tag{4.39}$$

provided that T satisfies (4.12). Therefore, Γ_n has a unique fixed-point u_n, which is the unique mild solution of (4.16). Moreover,

$$\|u_n - \Gamma^\mu_n(0)\|_\Sigma \leqq 2^{1-\mu}(T\|g\|_0 + \|\phi\|_0), \tag{4.40}$$

where Γ^μ_n denotes the μ-iterate of Γ_n. We claim that for all $\mu \in \mathbf{N}$ and all $R > 0$,

$$\lim_{n\to\infty} \Gamma^\mu_n(0)(t,x) = \Gamma^\mu(0)(t,x) \tag{4.41}$$

uniformly for $t \in [\tau, T]$ and $|x| \leqq R$. In fact, (4.41) is true for $\mu = 1$, in view of (3.6)(i). Now, suppose that (4.41) holds for $\mu \in \mathbf{N}$. Then the functions

$$\psi(t,x) = G(x, \Gamma^\mu(0))_x(t,x), \qquad \psi_n(t,x) = G(x, \Gamma^\mu_n(0))_x(t,x) \tag{4.42}$$

satisfy the assumptions of Lemma 4.6. Consequently, by (4.19)

$$\lim_{n\to\infty} \Gamma^{\mu+1}_n(0)(t,x) = \Gamma^{\mu+1}(0)(t,x) \tag{4.43}$$

uniformly for $t \in [0, T]$, $|x| \leqq R$. Therefore (4.41) holds for all $\mu \in \mathbf{N}$.

Finally, to prove (4.18)(i) note that for all $\mu, n \in \mathbf{N}$

$$\begin{aligned} &|u(t,x) - u_n(t,x)| \\ &\quad \leqq |u(t,x) - \Gamma^\mu(0)(t,x)| + |\Gamma^\mu(0)(t,x) - \Gamma^\mu_n(0)(t,x)| + |\Gamma^\mu_n(0)(t,x) - u_n(t,x)| \\ &\quad \leqq 2^{2-\mu}(T\|g\|_0 + \|\phi\|_0) + |\Gamma^\mu(0)(t,x) - \Gamma^\mu_n(0)(t,x)|. \end{aligned} \tag{4.44}$$

Fix $\eta > 0$ and let $\mu_\eta \in \mathbf{N}$ be such that $2^{2-\mu_\eta}(T\|g\|_0 + \|\phi\|_0) < \eta$. Then, (4.44) yields

$$|u(t,x) - u_n(t,x)| \leqq 2\eta + |\Gamma^{\mu_\eta}(0)(t,x) - \Gamma_n^{\mu_\eta}(0)(t,x)|. \tag{4.45}$$

Now, (4.18)(ii) can be easily derived by minor modifications of the above argument (using (3.6)(ii) instead of (3.6)(i) and (4.20) instead of (4.19)). Therefore, the proof is complete. □

5. Application to stochastic optimal control. Let $\{\Omega, \mathscr{F}, \mathbf{P}\}$ be a complete probability space and $\{\beta_k\}$ a sequence of standard one-dimensional Brownian motions, mutually independent. For any $s \geqq 0$ let $\mathscr{F}_t$ be the σ-algebra generated by $\{\beta_k(s)\colon k = 1, 2, \cdots; 0 \leqq s \leqq t\}$. Let $M_W^2(t, T; X)$ denote the space of the X-valued processes x such that $x(s)$ in $\mathscr{F}_s$-measurable for all $t \leqq s \leqq T$ and

$$\mathscr{E}\left(\int_t^T |x(s)|^2\, ds\right) < \infty.$$

Consider a stochastic system governed by the state equation

$$y(s) = e^{(s-t)A}x + \int_t^s e^{(s-r)A}[F(y(r)) + z(r)]\, dr + \sqrt{\varepsilon}\, W_A(t,s), \qquad s \geqq t \geqq 0, \tag{5.1}$$

where $x \in X$, A is a self-adjoint operator satisfying (3.2) and (2.8), $F \in \mathrm{Lip}\,(X, X)$, $z \in M_W^2(t, T; X)$, and $W_A(t,s)$ is defined by

$$W_A(t,s) = \sum_{k=1}^{\infty} e_k \int_t^s e^{-\alpha_k(s-r)}\, d\beta_k(r), \qquad s \geqq t \geqq 0. \tag{5.2}$$

Equation (5.1) can be regarded as the "mild" form of the stochastic differential equation

$$\begin{aligned} &dy(s) = \{Ay(s) + F(y(s)) + z(s)\}\, ds + \sqrt{\varepsilon}\, dW(s), \qquad t \leqq s \leqq T, \\ &y(t) = x, \end{aligned} \tag{5.3}$$

where $W(t)$ is a cylindrical Wiener process (see Remark 2.2).

We now prove the existence of solutions to (5.3) as well as a Galerkin approximation result.

PROPOSITION 5.1. *Assume* (3.2), (2.8) *and let* $F \in \mathrm{Lip}\,(X, X)$. *Then, for all* $z \in M_W^2(t, T; X)$, *equation* (5.1) *has a unique solution* $y_\varepsilon(\cdot\,; t, x, z)$, *which is continuous with probability one.*

Proof. Let $\Lambda = \{v \in M_W^2(t, T; X)\colon \mathscr{E}(\sup_{t \leqq s \leqq T} |v(s)|^2) < +\infty\}$ and define a map λ on Λ as follows:

$$\lambda(v)(s) = e^{(s-t)A}x + \int_t^s e^{(s-r)A}[F(v(r)) + z(r)]\, dr + \sqrt{\varepsilon}\, W_A(t,s), \qquad t \leqq s \leqq T. \tag{5.4}$$

From Proposition 2.4 it follows that $W_A(t, \cdot) \in \Lambda$. Hence, $\lambda : \Lambda \to \Lambda$. Moreover, λ is a contraction provided that $T - t < 1/\|F\|_1$ and the conclusion follows by standard fixed-point arguments. □

PROPOSITION 5.2. *Assume* (3.2), (2.8) *and let* $F \in \mathrm{Lip}\,(X, X)$. *Let* $y_{\varepsilon,n}(\cdot\,; t, x, z)$ *be the solution of*

$$\begin{aligned} &dy_{\varepsilon,n}(s) = \{A\Pi_n y_{\varepsilon,n}(s) + \Pi_n F(\Pi_n y_{\varepsilon,n}(s)) + \Pi_n z(s)\}\, ds + \sqrt{\varepsilon}\, dW^n(s), \\ &y_{\varepsilon,n}(t) = \Pi_n x, \qquad\qquad t \leqq s \leqq T, \end{aligned} \tag{5.5}$$

where $z \in M^2_W(t, T; X)$ *and* $W^n(t)$ *is defined in* (2.3). *Then,*

$$\lim_{n\to\infty} \mathscr{E}\left(\sup_{t\leqq s\leqq T} |y_\varepsilon(s; t, x, z) - y_{\varepsilon,n}(s; t, x, z)|\right) = 0 \tag{5.6}$$

for all $x \in X$.

Proof. Let Λ be the space defined in the previous proof and define a map $\lambda_n n$ on Λ as follows:

$$\lambda_n(v)(s) = e^{(s-t)A}\Pi_n x + \int_t^s e^{(s-r)A}\Pi_n[F(\Pi_n v(r)) + \Pi_n z(r)]\, dr + \sqrt{\varepsilon}\, \Pi_n W_A(t, s), \qquad t \leqq s \leqq T.$$

Then, λ_n is a contraction in Λ uniformly with respect to $n \in \mathbf{N}$, provided that $T - t < 1/\|F\|_1$. Moreover, Proposition 2.4 implies that

$$\lim_{n\to\infty} \mathscr{E}\left(\sup_{t\leqq s\leqq T} |\lambda_n(v)(s) - \lambda(v)(s)|\right) = 0$$

for all $v \in M^2_W(t, T; X)$. The conclusion then follows by the contraction mapping theorem depending on a parameter. □

We will now study the following stochastic optimal control problem.

Given $R > 0$, minimize the cost functional

$$J_\varepsilon(t, x; z) = \mathscr{E}\left\{\int_t^T \left[g(y_\varepsilon(s; t, x, z)) + \frac{1}{2}|z(s)|^2\right] ds + \phi(y_\varepsilon(T; t, x, z))\right\} \tag{5.7}$$

over all controls $z \in M^2_W(t, T; X)$ satisfying $|z(s)| \leqq R$ almost surely for all $s \in [t, T]$.

The *value function* of problem (5.7) is given by

$$V_\varepsilon(t, x) = \inf\{J_\varepsilon(t, x; z) \colon z \in M^2_W(t, T; X), |z(s)| \leqq \mathbf{R}\}. \tag{5.8}$$

The corresponding Hamilton-Jacobi-Bellman equation reads as follows:

$$\begin{aligned} &\frac{\partial v}{\partial t} + \frac{\varepsilon}{2}\operatorname{Tr}(v_{xx}) + \langle Ax + F(x), v_x\rangle - H(v_x) + g(x) = 0 \quad \text{in } [0, T]\times X, \\ &v(T, x) = \phi(x), \end{aligned} \tag{5.9}$$

where H is defined by

$$H(p) = \begin{cases} \dfrac{1}{2}|p|^2 & \text{if } |p| \leqq R, \\ R|p| - \dfrac{R^2}{2} & \text{if } |p| \geqq R. \end{cases} \tag{5.10}$$

From Theorem 4.3 we obtain the result below.

THEOREM 5.3. *Assume* (2.1), (2.8), (4.2)(i) *and let* $F \in \operatorname{Lip}(X, X)$. *Then problem* (5.9) *has a unique mild solution, which coincides with the value function* V_ε. *Moreover, for any* $(t, x) \in [0, T]\times X$, *there exists an optimal control for problem* (5.7). *Furthermore, any optimal control* z^* *is related to the corresponding optimal state* y^* *by the feedback formula*

$$z^*(s) = -h\left(\frac{\partial V_\varepsilon}{\partial x}(s, y^*(s))\right), \qquad t \leqq s \leqq T, \tag{5.11}$$

where

$$h(p) = \begin{cases} p & \text{if } |p| \leqq R, \\ \dfrac{pR}{|p|} & \text{if } |p| \geqq R. \end{cases} \tag{5.12}$$

Proof. First, we note that the existence and uniqueness of a mild solution v to (5.9) follows from Theorem 4.3, since the function H defined in (5.10) fulfills (4.2)(ii).

Let us show that $v = V_\varepsilon$. We claim that v satisfies the dynamic programming principle below: for any $t \in]0, T[$, $x \in X$ and $z \in M^2_W(t, T; X)$ such that $|z(s)| \leqq R$ almost surely, we have

$$\begin{aligned} v(t,x) &+ \frac{1}{2}\mathscr{E}\int_t^T \{|z(s)+v_x(s, y_\varepsilon(s;t,x,z))|^2 - \chi(v_x(s, y_\varepsilon(s;t,z,z)) - R)\}\, ds \\ &= \mathscr{E}\left\{\int_t^T \left[g(y_\varepsilon(s;t,x,z)) + \frac{1}{2}|z(s)|^2\right] ds + \phi(y_\varepsilon(T;t,x,z))\right\}, \end{aligned} \tag{5.13}$$

where $\chi(a) = 0$ if $a \leqq 0$ and $\chi(a) = a^2$ if $a \geqq 0$.

Indeed, let u_n be the solution of problem (4.16) with H given by (5.10) and set $v_n(t,x) = u_n(T-t, x)$. We claim that v_n is regular. To show this fact let $\zeta(t, x_1, \cdots, x_n)$ be defined as $\zeta(t, x_1, \cdots, x_n) = v_n(t, x_1e_1 + \cdots + x_ne_n)$, for all $(t, x_1, \cdots, x_n) \in]0, T[\times \mathbf{R}^n$. Then ζ is a classical solution of the problem

$$\begin{aligned} &\frac{\partial \zeta}{\partial t} + \Delta\zeta - \sum_{i=1}^n [\alpha_i x_i - \langle F(x_1e_1 + \cdots + x_ne_n), e_i\rangle]\frac{\partial \zeta}{\partial x_i} \\ &\quad - H\left(\frac{\partial \zeta}{\partial x_1}e_1 + \cdots + \frac{\partial \zeta}{\partial x_n}e_n\right) + g(x_1e_1 + \cdots + x_ne_n) = 0, \\ &\zeta(T, x_1, \cdots, x_n) = \phi(x_1e_1 + \cdots + x_ne_n). \end{aligned}$$

So, we can use the Itô formula to differentiate $v_n(s, y_{\varepsilon,n}(s))$ where $y_{\varepsilon,n}(s) = y_{\varepsilon,n}(s; t, x, z)$. Thus, we obtain

$$dv_n(s, y_{\varepsilon,n}(s)) = \frac{\partial v_n}{\partial t}(s, y_{\varepsilon,n}(s))\, ds + \langle dy_{\varepsilon,n}(s), v_{n,x}(s, y_{\varepsilon,n}(s))\rangle + \frac{\varepsilon}{2}\operatorname{Tr}(v_{n,xx}(s, y_{\varepsilon,n}(s)))\, ds.$$

Now, recall (4.16) and (5.5), integrate on $[t, T]$ and take expectation to obtain

$$\begin{aligned} v_n(t,x) &+ \frac{1}{2}\mathscr{E}\int_t^T \{|\Pi_n z(s) + v_{n,x}(s, y_{\varepsilon,n}(s))|^2 - \chi(|v_{n,x}(s, y_{\varepsilon,n}(s))| - R\}\, ds \\ &= \mathscr{E}\left\{\int_t^T \left[g(y_{\varepsilon,n}(s;t,x,z)) + \frac{1}{2}|\Pi_n z(s)|^2\right] ds + \phi(y_{\varepsilon,n}(T;t,x,z))\right\}. \end{aligned}$$

By Proposition 5.2 and Theorem 4.5, we obtain (5.13) in the limit as $n \to \infty$. Next, we note that the following inequality holds:

$$|z(s) + v_x(s, y_\varepsilon(s;t,x,z))|^2 - \chi(v_x(s, y_\varepsilon(s;t,x,z)) - R) \geqq 0. \tag{5.14}$$

Thus, from (5.13) and (5.14) it follows that $v(t,x) \leqq V_\varepsilon(t,x)$.

To prove the reverse inequality, let us consider the closed-loop equation

$$\begin{aligned} y(s) = e^{(s-t)A}x + \int_t^s e^{(s-r)A}[F(y(r)) - h(v_x(r, y(r)))]\, dr + \sqrt{\varepsilon}\, W_A(t,s), \\ T > s \geqq t \geqq 0, \end{aligned} \tag{5.15}$$

which can be solved by the Schauder fixed-point theorem (see, e.g., [13, Cor. 2.3]). Indeed, from (2.8) it follows that e^{tA} is compact for $t > 0$. Let y^* be a mild solution

of (5.15). Taking

$$z(s) = -h(v_x(s, y^*(s))), \tag{5.16}$$

we have the equality in (5.14), and so $v(t, x) \geqq V_\varepsilon(t, x)$ for all $t < T$. Moreover, the choice (5.16) provides an optimal control at (t, x). Finally, the feedback formula (5.11) follows from (5.13) and the fact that $v(t, x) = V_\varepsilon(t, x)$. □

Example 5.4. Let $X = L^2(0, \pi)$ and define

$$\begin{aligned} &D(A) = H^2(0, \pi) \cap H_0^1(0, \pi), \\ &Ax = \frac{\partial^2 x}{\partial \xi^2} \quad \forall x \in D(A), \end{aligned} \tag{5.17}$$

$$F(x)(\xi) = f(x(\xi)), \quad g(x) = \int_0^\pi \alpha(x(\xi))\, d\xi, \quad \phi(x) = \int_0^\pi \beta(z(\xi))\, d\xi,$$

where $f \in C_b^1(\mathbf{R})$, $\alpha, \beta \in C_b(\mathbf{R})$. By inspection, F, g, and ϕ fulfill the hypotheses of Theorem 5.3. As for operator A, (3.2) is satisfied by $\alpha_k = k^2$, and so (2.8) holds true for any $\sigma \in]0, \frac{1}{2}[$. Therefore, the results of this section apply to the following stochastic optimal control problem.

Minimize

$$J_\varepsilon(t, x; z) = \mathscr{E}\left\{\int_t^T \int_0^\pi \left[\alpha(y(s, \xi)) + \frac{1}{2}|z(s, \xi)|^2\right] d\xi\, ds + \int_0^\pi \beta(y(T, \xi))\, d\xi\right\} \tag{5.18}$$

over all controls $z \in M_W^2([t, T]; L^2(0, \pi))$ satisfying $\int_0^\pi |z(s, \xi)|^2\, d\xi \leqq R^2$ almost surely for all $s \in [t, T]$, where the state y is subject to

$$\begin{aligned} &dy(s, \xi) = \left\{\frac{\partial^2}{\partial x^2} y(s, \xi) + f(y(s, \xi)) + z(s, \xi)\right\} ds + \sqrt{\varepsilon}\, dW(s), \\ &y(s, 0) = y(s, \pi) = 0, \qquad s \in [t, T], \\ &y(t, \xi) = x(\xi). \end{aligned} \tag{5.19}$$

Remark 5.5. Let us consider the same problem as in Example 5.4 for an N-dimensional parabolic state equation, i.e., taking $X = L^2([0, \pi]^N)$ and $Ax = \Delta x$, with Dirichlet or Neumann boundary conditions. Then, Theorem 5.3 does not apply. In fact, we can show that $q(t) = t^{1-N/2} o(t)$ (see [7]) and (2.6) is not satisfied. However, if we consider the iterated Laplace operator $A_1 x = (-1)^{m-1}(-\Delta)^m x$, (with Dirichlet boundary conditions), we have $q(t) = t^{1-N/2m} O(t)$ and Theorem 5.3 applies if $N < 2m$.

Remark 5.6. Using Theorem 5.3 and the variational technique of [8] we can characterize the value functions of deterministic optimal control problems as limits, as $\varepsilon \downarrow 0$, of the mild solutions u_ε of (4.1). For example, consider the following problem.

Given $R > 0$, minimize

$$J(t, x; z) = \int_t^T \left[g(y(s; t, x, z)) + \frac{1}{2}|z(s)|^2\right] ds + \phi(y(T; t, x, z)) \tag{5.20}$$

over all controls $z \in L^2(t, T; X)$ satisfying $|z(s)| \leqq R$.

Here $y(\cdot; t, x, z)$ is the mild solution of the state equation

$$\begin{aligned} &y'(s) = Ay(s) + F(y(s)) + z(s), \qquad t \leqq s \leqq T, \\ &y(t) = x. \end{aligned}$$

Define the value function of problem (5.20) as

$$V(t, x) = \inf\{J(t, x; z) \colon z \in L^2(t, T; H), |z(s)| \leqq R\}.$$

Then, for all $(t, x) \in [0, T] \times H$, we can show that

$$|u_\varepsilon(T-t, x) - V(t, x)| \leqq T\omega_g(C\sqrt{\varepsilon}) + \omega_\phi(C\sqrt{\varepsilon}), \tag{5.21}$$

where ω_g (respectively, ω_ϕ) denotes a concave modulus of continuity for g (respectively, ϕ) and

$$C = \sqrt{q(T) - q(t)}\, e^{T\|F\|_1}.$$

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