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NONLINEAR OPTIMAL CONTROL WITH INFINITE HORIZON FOR DISTRIBUTED PARAMETER SYSTEMS AND STATIONARY HAMILTON-JACOBI EQUATIONS*

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Abstract. Optimal control problems, with no discount, are studied for systems governed by nonlinear "parabolic" state equations, using a dynamic programming approach.

If the dynamics are stabilizable with respect to cost, then the fact that the value function is a generalized viscosity solution of the associated Hamilton-Jacobi equation is proved. This yields the feedback formula. Moreover, uniqueness is obtained under suitable stability assumptions.

Key words. optimal control, Hamilton-Jacobi equations, viscosity solutions, evolution equations, unbounded operators

AMS(MOS) subject classifications. 49C20, 34G20

1. Introduction and setting of the problem. Let us consider two separable reflexive Banach spaces, X (the *state space*) and U (the *control space*). We denote by $|\ |$ the norm of X, which we assume to be continuously differentiable in $X\backslash\{0\}$, by X^* the dual space of X and by $\langle\ ,\ \rangle$ the pairing between X and X^*. We denote by $\partial|x|$ the subgradient of $|x|$, which is obviously single-valued on $X\backslash\{0\}$. The same symbols will also be used in the Banach space U. Moreover, we will use the following notation:

(i) For any Banach space K and any nonnegative integer k we denote by $C^k(X; K)$ the set of all the mappings $f: X \to K$ that are continuous and bounded on all bounded sets of X, together with their derivatives of order less than or equal to k.

(ii) We denote by $C^{k,1}(X; K)$ (respectively, $C^{k,1}(X; K)_{\text{loc}}$), the set of all the mappings f in $C^k(X; K)$ whose derivative of order k is Lipschitz continuous in X (in every bounded set of X).

We are interested in the following optimal control problem.

Minimize

$$J_\infty(u, x) = \int_0^\infty \{g(y(s)) + h(u(s))\}\, ds \tag{1.1}$$

over all $u \in L^1(0, \infty; U)_{\text{loc}}$, subject to state equation

$$y' = Ay + F(y) + Bu, \qquad y(0) = x. \tag{1.2}$$

Following the dynamic programming approach, we will study the Hamilton-Jacobi equation

$$H(B^*DV(x)) - \langle Ax + F(x), DV(x)\rangle - g(x) = 0 \tag{1.3}$$

where H denotes the Legendre transform of h, that is,

$$H(v) = \sup_{v \in U} \{-\langle u, v\rangle - h(u)\}. \tag{1.4}$$

The connections between (1.3) and problem (1.1)-(1.2) are well known.

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We assume the following hypotheses.

(SL) (i) $A: D(A) \subset X \to X$ generates an analytic semigroup e^{tA} in X and there exists $\omega \in \mathbf{R}$ such that $\|e^{tA}\| \leqq e^{\omega t}$.

(ii) The embedding $D(A) \to X$ is compact.

(iii) $B \in \mathscr{L}(U; X)$.

(iv) $F \in C^{1,1}(X, X)_{\text{loc}}$ and there exists $a \in \mathbf{R}$ such that $\langle F(x), x^* \rangle \leqq a|x|$, for all $x^* \in \partial |x|$, for all $x \in X$.

(v) $g \in C^{1,1}(X, \mathbf{R})_{\text{loc}}$ and $g(x) \geqq 0$ for all $x \in X$.

(vi) $h \in C^{1,1}(U, \mathbf{R})_{\text{loc}}$ is strictly convex and there exists $p > 1$ such that $h(u) \geqq \gamma |u|^p$ for all $u \in U$ and some $\gamma > 0$.

We remark that, if (SL) are fulfilled, then, by classical arguments (see, for instance, [21]), problem (1.2) has a unique global mild solution $y \in C([0, \infty[; X)$.

In the analysis of (1.3), we meet with two immediate difficulties: the nonsmoothness of solutions and the unboundedness of A. In fact, first-order partial differential equations have, in general, no global classical solutions even in finite dimensions. Therefore, a suitable notion of weak solution is required. Moreover, such a generalized solution will have to take care of the fact that Ax is defined only on a dense subspace of X.

The first problem can be successfully treated by the notion of viscosity solution, introduced by Crandall and Lions [12]–[15]. In [16] they have also extended their definition of solutions to problems involving unbounded operators. Further results in these directions have been obtained in [4] and [10] by an approximation procedure.

Stationary Hamilton–Jacobi equations have been extensively studied (see [20] for general references and results; see also [13]–[16]) mainly in the case when the equation contains an additional term of the form λV with $\lambda > 0$. This corresponds to the introduction of a discount factor $e^{-\lambda t}$ in the cost.

However, in many applications we are required not to have such a discount, as in linear quadratic optimal control problems. A large amount of work has been devoted to the analysis of this case (see, for instance, the review paper [22]). For linear quadratic optimal control problems, the Hamilton–Jacobi equation is replaced by the algebraic Riccati equation, as it is well known. In general, uniqueness is false for this equation. Therefore, we do not expect to have uniqueness for (1.3).

Optimal control problems with a linear state equation and a convex cost functional are also studied in [3] and [6]. Some generalizations to the nonconvex case, by using variational methods, are contained in [4], [6], and [7].

The main idea of our approach is to obtain a viscosity solution V of (1.3) as

$$V(x) = \lim_{t \to \infty} \phi(t, x) \tag{1.5}$$

where ϕ solves the forward equation (in the generalized sense of [10]):

$$\phi_t(t, x) + H(B^* \nabla \phi(t, x)) - \langle Ax + F(x), \nabla \phi(t, x) \rangle - g(x) = 0, \qquad \phi(0, x) = 0. \tag{1.6}$$

For the value function of the control problem (1.1)–(1.2) to be finite, we introduce the notion of stabilizability that generalizes a well-known concept in linear quadratic control (see, e.g., [22]).

DEFINITION 1.1. We say that $(A + F, B, h)$ is *stabilizable with respect to the observation* g (or, for brevity, that problem (1.1)–(1.2) is stable) if for any $x \in X$ there exists $u_x \in L^1(0, \infty; U)_{\text{loc}}$ such that $J_\infty(u_x, x) < \infty$. Such a control u_x will be called an *admissible control* at x.

Finally, we define the *value function* of problem (1.1) (1.2) as

$$V_\infty(x)=\inf\{J_\infty(u,x);\ u\in L^1(0,\infty;\ U)_{\mathrm{loc}}\}. \tag{1.7}$$

We say that $u^*\in L^1(0,\infty;\ U)_{\mathrm{loc}}$ is an *optimal control* if $J(u^*)=V_\infty(x)$; in this case, we call the corresponding solution y^* of (1.2) an *optimal state* and (u^*, y^*) an *optimal pair* at x.

In this paper we show that, if $(A+F, B, h)$ is g-stabilizable, then V_∞ is a generalized viscosity solution of (1.3). Moreover, we obtain the existence of optimal pairs as well as the feedback formula (see Theorem 4.4).

In § 3 we study the "stability" of the closed-loop system. When this system is stable and B is invertible, we prove the uniqueness of the nonnegative generalized viscosity solution of (1.3) vanishing at zero (Theorem 5.4).

An application to a nonlinear control problem for a distributed parameter system is illustrated in § 6.

We now explain our definition of generalized solutions. We define solutions of (1.3) as stationary solutions of the following evolution equation:

$$-W_t(t,x)+H(B^*\nabla W(t,x))-\langle Ax+F(x),\nabla W(t,x)\rangle=g(x). \tag{1.8}$$

More precisely, we have the following definition.

DEFINITION 1.2. Assume (SL). We say that $V\in C^{0,1}(X;\mathbf{R})_{\mathrm{loc}}$ is a *generalized viscosity solution* of (1.3) if $W(t,x):=V(x)$ is the generalized viscosity solution of (1.8) in $[0,T]\times X$ with terminal data $W(T,x)=V(x)$, for all $T>0$.

We recall below the definition of generalized viscosity solutions of the Cauchy problem (see [10])

$$\begin{aligned}&-W_t(t,x)+H(B^*\nabla W(t,x))-\langle Ax+F(x),\nabla W(t,x)\rangle=g(x)\\ &W(T,x)=\phi_0(x),\qquad x\in X,\quad t\in[0,T]\end{aligned} \tag{1.9}$$

where

$$\phi_0\in C^{0,1}(X;\mathbf{R})_{\mathrm{loc}}. \tag{1.10}$$

DEFINITION 1.3. Assume (SL) and (1.10). We say that $W\in C([0,T]\times X;\mathbf{R})$ is a *generalized viscosity solution* of (1.9) if we have

$$\lim_{n\to\infty} W_n(t,x)=W(t,x),\quad \forall x\in D(A),\quad \forall t\in[t,T] \tag{1.11}$$

where W_n is the viscosity solution (in the sense of Crandall and Lions [13]) of the problem

$$\begin{aligned}&-W_{nt}(t,x)+H(B^*\nabla W_n(t,x))-\langle A_nx+F(x),\nabla W_n(t,x)\rangle-g(x)=0,\\ &W_n(T,x)=\phi_0(x)\end{aligned} \tag{1.12}$$

where

$$A_n=nA(n-A)^{-1}. \tag{1.13}$$

We note that problem (1.12) has a unique viscosity solution (see [13] and also [10]).

A property of generalized viscosity solutions that turns out to be essential to our approach is semiconcavity (see [9]).

In applications it is also useful to consider the following more general assumptions:

(SL′) (i) Hypotheses (SL) (i), (ii), (iii), (v) and (vi) hold.

(ii) there exists a Banach space Z (with pairing denoted $\langle\ ,\ \rangle_z$), continuously embedded in X, such that the part of A in Z, A_Z, generates an analytic semigroup in Z with domain $D(A_Z)$ (not necessarily dense in Z)

$$D(A_Z)=\{x\in D(A)\cap Z;\ Ax\in Z\}.$$

Moreover, $\|e^{tA_Z}\|\leqq e^{\mu t}$ for all $t\geqq 0$ and some $\mu\in\mathbf{R}$.

(iii) There exists $\alpha\in]0,1-1/p[$, $a\in\mathbf{R}$, and two continuous functions β, $\rho:[0,\infty[\to[0,\infty[$, such that $D_A(\alpha,p)$ is embedded in Z and

$$F\in C^{1,1}(D_A(\alpha,p);X)_{\mathrm{loc}}, \tag{1.14}$$

$$\langle F(z),z^*\rangle_Z\leqq a|z|_Z\quad \forall z\in Z,\quad \forall z^*\in\partial|z|_Z, \tag{1.15}$$

$$|F(x)|\leqq\beta(|x|_Z)+\rho(|x|_Z)|x|_{\alpha,p}\quad \forall x\in D_A(\alpha,p). \tag{1.16}$$

We recall that $D_A(\alpha,p)$ is the real interpolation space between $D(A)$ and X, introduced by Lions and Peetre [19], with norm

$$|x|_{\alpha,p}=\left[\int_0^\infty \tau^{p-p\alpha-1}|Ae^{\tau A}x|^p\,d\tau\right]^{1/p}.$$

Definition 1.2 remains unchanged under assumptions (SL′), except for the fact that we assume $V\in C^{0,1}(D_A(\alpha,p);\mathbf{R})_{\mathrm{loc}}$. Moreover, in Definition 1.3 we assume $W\in C([0,T]\times D_A(\alpha,p);\mathbf{R})$ and replace (1.12) by

$$-W_{nt}(t,x)+H(B^*\nabla W_n(t,x))-\langle A_nx+F(n(n-A)^{-1}x),\nabla W_n(t,x)\rangle-g(x)=0,$$
$$W_n(T,x)=\phi_0(x).$$

2. Preliminaries. In this section we recall the basic results on the time-dependent Hamilton–Jacobi equation (1.9).

PROPOSITION 2.1. *Assume* (1.10) *and either* (SL) *or* (SL′). *Then, there exists a unique generalized viscosity solution W of problem* (1.9) *given by*

$$W(t,x)=\inf\left\{\int_t^T[g(y(s))+h(u(s))]\,ds+\phi_0(y(T));\quad u\in L^1(t,T;U)_{\mathrm{loc}}\right\} \tag{2.1}$$

where y is the solution of

$$y'(s)=Ay(s)+F(y(s))+Bu(s),\qquad t\leqq s\leqq T,\quad y(t)=x. \tag{2.2}$$

Moreover, W satisfies (1.9) *in the sense that for every* $(t,x)\in[0,T]\times D(A)$ *we have*

$$\text{(i)}\quad \forall(p_t,p_x)\in D^+W(t,x),\qquad -p_t+H(B^*p_x)-\langle Ax+F(x),p_x\rangle\leqq g(x), \tag{2.3}$$
$$\text{(ii)}\quad \forall(p_t,p_x)\in D^-W(t,x),\qquad -p_t+H(B^*p_x)-\langle Ax+F(x),p_x\rangle\geqq g(x).$$

We recall the definition of the semidifferentials D^+ and D^-:

$$D^+W(t,x)=\left\{(p_t,p_x)\in\mathbf{R}\times X^*;\ \limsup_{(s,y)\to(t,x)}\frac{W(s,y)-W(t,x)-(s-t)p_t-\langle y-x,p_x\rangle}{|s-t|+|y-x|}\leqq 0\right\},$$
$$D^-W(t,x)=\left\{(p_t,p_x)\in\mathbf{R}\times X^*;\ \liminf_{(s,y)\to(t,x)}\frac{W(s,y)-W(t,x)-(s-t)p_t-\langle y-x,p_x\rangle}{|s-t|+|y-x|}\geqq 0\right\}. \tag{2.4}$$

Remark 2.2. The results of Proposition 2.1 are proved in Theorems 3.3 and 3.7 of [10] in a slightly different form that is equivalent to the one above in view of the coercivity assumption on h.

We now recall the Maximum Principle [8], the feedback formula [4], [9], and some regularity properties of optimal pairs [9].

PROPOSITION 2.3. *Assume* (SL) (*respectively,* (SL′)) *and* (1.10). *Let* W *be given by* (2.1) *and* $(t,x)\in[0,T]\times X$ (*respectively,* $(t,x)\in[0,T]\times D_A(\alpha,p)$). *Let* (u^*,y^*) *be an optimal pair for* W *at* (t,x). *Then, there exists* $p^*\in C([t,T];X^*)$ *such that*

$$(2.5)\quad p^{*\prime}(s)+A^*p^*(s)+(DF(y^*(s))^*p^*(s)+Dg(y^*(s))=0,\ p^*(T)=D\phi(y^*(T)),$$

$$(2.6)\quad u^*(s)=-DH(B^*p^*(s)),\qquad t\leqq s\leqq T.$$

We call p^* *a* *dual arc*. *Moreover,*

$$(2.7)\qquad u^*(s)\in -DH(B^*\nabla^+W(s,y^*(s))),\qquad t\leqq s\leqq T$$

where

$$(2.8)\qquad \nabla^+W(s,x)=\left\{q\in X^*;\ \limsup_{y\to x}\frac{W(s,y)-W(s,x)-\langle y-x,q\rangle}{|y-x|}\leqq 0\right\}.$$

Furthermore, there exists $\delta\in\,]0,1[$ *such that*

$$(2.9)\qquad y^*\in C^{1,\delta}(]t,T[;X),$$

$$(2.10)\qquad p^*\in C^{1,\delta}(]t,T[;X),\qquad u^*\in C^{0,\delta}(]t,T[;X).$$

Above we have denoted by $C^{1,\delta}(I;X)$, for any real interval I, the space of functions that are Hölder continuous with exponent δ, together with their first derivative, on each subinterval $[a,b]$ contained in I.

Finally, the following results are proved in [4] and [9].

PROPOSITION 2.4. *Assume* (SL) (*respectively,* (SL′)) *and* (1.10) *and let* W *be given by* (2.1). *Then we have the following*:

(2.11) (i) $W(t,.)$ *is locally Lipschitz in* X *for all* $t\in[0,T]$;
(ii) $W(.,x)$ *is Lipschitz continuous in* $[0,T]$ *for all* $x\in D(A)$.

Furthermore, if $B^{-1}\in\mathscr{L}(H;U)$, *then* $W(t,.)$ *is semiconcave in* X *for all* $t\in[0,T[$; *that is, for all* $r>1/T$ *there exists* $C_r>0$ *such that*

$$\lambda W(t,x+(1-\lambda)x')+(1-\lambda)W(t,x-\lambda x')-W(t,x)\leqq C_r\lambda(1-\lambda)|x'|^2$$

for all $t\in[0,T-1/r]$, $|x|,|x'|\leqq r$, $\lambda\in[0,1]$.

Along with the backward Cauchy problem (1.9), we will consider the forward problem:

$$(2.12)\qquad \begin{gathered}\phi_t(t,x)+H(B^*\nabla\phi(t,x))-\langle Ax+F(x),\nabla\phi(t,x)\rangle-g(x)=0;\\ \phi(0,x)=\phi_0(x).\end{gathered}$$

We say that $\phi\in C([0,T]\times X;\mathbf{R})$ (respectively, $\phi\in([0,T]\times D_A(\alpha,p);\mathbf{R})$) is the *generalized viscosity solution* of (2.12) if $W(t,x)=\phi(T-t,x)$ is the generalized viscosity solution of (1.9).

We prove now the analogue of representation formula (2.1).

PROPOSITION 2.5. *Assume* (SL) (*respectively,* (SL′)) *and* (1.10). *Let* ϕ *be the generalized viscosity solution of* (2.12). *Then we have*

$$(2.13)\quad \phi(t,x)=\inf\left\{\int_0^t[g(y(s))+h(u(s))]\,ds+\phi_0(y(t));\ u\in L^1(0,\infty;U)_{\mathrm{loc}}\right\}$$

where y *is the solution of* (1.2).

Proof. By definition we have

$$\phi(t,x)=\inf\left\{\int_{T-t}^{T}\{g(y(s))+h(u(s))\}\,ds+\phi_0(y(T));\right.\tag{2.14}$$
$$\left.u\in L^1(T-t,T;U),\ y'(s)=Ay(s)+F(y(s))+Bu(s),\ y(T-t)=x\right\}.$$

Set $\sigma=s-T+t$ to obtain

$$\phi(t,x)=\inf\left\{\int_0^t g(y(\sigma+T-t))+h(u(\sigma+T-t))\,d\sigma+\phi_0(y(T));\right.$$
$$\left.u\in L^1(T-t,T\colon U),\ y'(s)=Ay(s)+F(y(s))+Bu(s),\ y(T-t)=x\right\}.$$

Now, let $\underline{y}(s)=y(s+T-t)$, $\underline{u}(s)=u(s+T-t)$; then

$$\phi(t,x)=\inf\left\{\int_0^t\{g(\underline{y}(s))+h(\underline{u}(s))\}\,ds+\phi_0(\underline{y}(t));\right.$$
$$\left.\underline{u}\in L^1(0,t;U),\ \underline{y}'(s)=A\underline{y}(s)+F(\underline{y}(s))+B\underline{u}(s),\ \underline{y}(0)=x\right\}$$

and the assertion is proved. □

3. Sufficient conditions for stabilizability. To our knowledge there are no general conditions that yield global stabilizability in the sense of Definition 1.1 (for local results see [2] and [18]). In the following we give some sufficient conditions that may be applied to various situations. For instance, the problem we analyze in § 6 fits into the framework of Proposition 3.3 below.

The simplest case for which there is stabilizability is when the dynamical system $\eta(t,x)$ generated by $A+F$, that is the solution of

$$\eta'=A\eta+F(\eta),\qquad \eta(0)=x,\tag{3.1}$$

is "exponentially stable." Indeed, in this case it suffices to take $u=0$ in (1.2). More precisely, we can easily prove the following proposition.

PROPOSITION 3.1. *Assume* (SL) (*respectively,* (SL′)). *Let* $h(0)=0$ *and suppose that there exist positive constants* C, R, σ, τ, *and* δ *such that*

$$|\eta(t,x)|\leqq C\,e^{-\delta t}|x|^{\tau}\quad \text{for all } x\in X,\tag{3.2}$$

$$|g(x)|\leqq C|x|^{\sigma}\qquad \text{for } |x|\leqq R.\tag{3.3}$$

Then, (1.1)–(1.2) *is stable.*

Remark 3.2. A typical assumption that implies (3.2) is that $A+F+\varepsilon$ be dissipative for some $\varepsilon>0$, i.e.,

$$\langle Ax+F(x)+\varepsilon x,x^*\rangle\leqq 0\quad \text{for all } x\in D(A) \text{ and } x^*\in\partial|x|.\tag{3.4}$$

Next, when B is invertible, we can prove a quite general result.

PROPOSITION 3.3. *Assume* (SL) (*respectively,* (SL′)) *and let* $B^{-1}\in\mathscr{L}(X;U)$. *Suppose further that there exist positive constants* C, R, *and* σ *such that*

$$|F(x)|\leqq C|x|^{\sigma}\ \text{ for } |x|\leqq R\ (\text{respectively, } |F(x)|\leqq C(|x|_{\alpha,p})^{\sigma} \text{ for } |x|_{\alpha,p}\leqq R),\tag{3.5}$$

$$|g(x)|\leqq C|x|^{\sigma}\quad \text{for } |x|\leqq R,\tag{3.6}$$

$$|h(u)|\leqq C|u|^{\sigma}\quad \text{for } |u|\leqq R.\tag{3.7}$$

Then, (1.1)–(1.2) *is stable.*

Proof. We set

$$u(t) = -B^{-1}\{(\omega+1)\, e^{t(A-\omega-1)}x + F(e^{t(A-\omega-1)}x)\}. \tag{3.8}$$

Then

$$|u(t)| \leqq \|B^{-1}\|_{\mathscr{L}(X;U)}\{|\omega+1||x|\, e^{-t} + C|x|\, e^{-\gamma t}\} \quad \text{for } t > \log(|x|/R)/\gamma.$$

So, the corresponding solution of the state equation (1.2) is given by $y(t) = e^{t(A-\omega-1)}x$. In view of (3.5), (3.6), and (3.7), u is an admissible control at x and the proof is complete. □

Now we consider the case when F is "small."

PROPOSITION 3.4. *Assume* (SL) (*respectively* (SL′)) *and that there exist positive constants C, R, and σ such that* (3.5), (3.6), *and* (3.7) *hold. Assume in addition that there exists $K \in \mathscr{L}(X;U)$ such that $A - BK$ is exponentially stable, i.e., that (A, B) is stabilizable by a feedback K. There exists $\varepsilon_0 > 0$ such that if*

$$\begin{gathered} |F(x) - F(y)| \leqq \varepsilon_0|x-y| \quad \textit{for all } x, y \in X \\ (\textit{respectively, } |F(x) - F(y)| \leqq \varepsilon_0|x-y|_{\alpha,p} \textit{ for all } x, y \in D_A(\alpha, p)); \end{gathered} \tag{3.9}$$

then (1.1)-(1.2) *is stable.*

Proof. Assume that (3.9) hold for some ε, and let $x \in X$. We will show that, if ε is sufficiently small, then the following control

$$u_x(t) = e^{t(A-BK)}x \tag{3.10}$$

is admissible. Let $N > 0$ and $c > 0$ be such that

$$\|e^{t(A-BK)}\| \leqq N e^{-2ct}, \qquad t \geqq 0 \tag{3.11}$$

and set

$$\begin{gathered} \|v\|_c = \sup\{e^{ct}|v(t)|;\, t \geqq 0\}, \qquad v \in C([0,\infty[;\, X), \\ (\text{respectively, } \|v\|_c = \sup\{e^{ct}|v(t)|_{\alpha,p};\, t \geqq 0\}, \qquad v \in C([0,\infty[;\, D_A(\alpha,p)), \end{gathered} \tag{3.12}$$

$$\begin{gathered} \Sigma = \{v \in C([0,\infty[;\, X);\, \|v\|_c < \infty\} \\ (\text{respectively, } \Sigma = \{v \in C([0,\infty[;\, D_A(\alpha,p));\, \|v\|_c < \infty\}). \end{gathered} \tag{3.13}$$

Σ, equipped with the norm $\|\ \|_c$, is a Banach space. Now consider the problem

$$z' = (A - BK)z + F(z), \qquad z(0) = x. \tag{3.14}$$

By a fixed point argument we can easily show that if ε is small, then (3.14) has a unique solution in Σ. Since z coincides with the solution of the state equation (1.2) when $u = u_x$, we have obtained the conclusion. □

4. Existence. In this section we prove that the existence of solutions to the Hamilton-Jacobi equation

$$H(B^*DV_\infty(x)) - \langle Ax + F(x), DV_\infty(x)\rangle - g(x) = 0 \tag{4.1}$$

is equivalent to the fact that $(A+F, B, h)$ is g-stabilizable. We will obtain V_∞ as the limit of the generalized viscosity solution to the problem

$$\phi_t(t,x) + H(B^*\nabla\phi(t,x)) - \langle Ax + F(x), \nabla\phi(t,x)\rangle - g(x) = 0, \qquad \phi(0,x) = 0 \tag{4.2}$$

when $t \to +\infty$.

PROPOSITION 4.1. *Assume* (SL) *and suppose that problem* (1.1)–(1.2) *is stable. Let* ϕ *be the generalized viscosity solution to* (4.2) *and let* V_∞ *be given by* (1.4). *Then, for all* $x \in X$ *we have*

$$V_\infty(x) = \lim_{t \uparrow \infty} \phi(t, x). \tag{4.3}$$

Proof. By Proposition 2.5 it follows that $\phi(t, x)$ is increasing in t for any $x \in X$ and $\phi(t, x) \leqq V_\infty(x)$. Thus

$$\phi_\infty(x) = \lim_{t \uparrow \infty} \phi(t, x) \leqq V_\infty(x). \tag{4.4}$$

Now let (u_t, y_t) be such that

$$\phi(t, x) = \int_0^t \{g(y_t(s)) + h(u_t(s))\}\, ds$$

where $u \in L^1(0, t; U)$ and $y_t'(s) = Ay_t(s) + f(y_t(s)) + Bu_t(s)$; $y_t(0) = x$. Then we have

$$V_\infty(x) \geqq \int_0^t h(u_t(s))\, ds \geqq \gamma \|u_t\|^p_{L^p(0,t;H)}. \tag{4.5}$$

Set $\underline{u}_t(s) = u_t(s)$ if $s \in [0, t]$ and $\underline{u}_t(s) = 0$ is $s > t$; since by (4.5) $\{\underline{u}_t\}$ is bounded in $L^p(0, \infty; U)$, there exists

$$t_n \uparrow +\infty \text{ such that } v_n := \underline{u}_{t_n} \to u^* \text{ weakly in } L^p(0, \infty; U); \text{ set } z_n = \underline{y}_{t_n}.$$

Now fix $T > 0$; since e^{tA} is compact for all $t > 0$ (by hypothesis (SL)(ii)) we have that $z_n \to y^*$ in $C([0, T]; X)$, where y^* is the solution of (1.2) with $u = u^*$. Since h is convex it follows that

$$\phi_\infty(x) \geqq \int_0^T \{g(y^*(s)) + h(u^*(s))\}\, ds.$$

But T is arbitrary, so $g(y^*)$ and $h(u^*)$ belong to $L^1(0, \infty; \mathbf{R})$ and

$$\phi_\infty(x) \geqq \int_0^\infty \{g(y^*(s)) + h(u^*(s))\}\, ds \geqq V_\infty(x). \qquad \square$$

Under assumptions (SL′) a similar result can be proved.

PROPOSITION 4.2. *Assume* (SL′) *and suppose that problem* (1.1)–(1.2) *is stable. Let* ϕ *be the generalized viscosity solution to* (4.1) *and* V_∞ *the value function given by* (1.4). *Then, for all* $x \in D_A(\alpha, p)$ *we have*

$$V_\infty(x) = \lim_{t \uparrow \infty} \phi(t, x). \tag{4.6}$$

Proof. The reasoning is similar to the one above. Since F is only defined in $D_A(\alpha, p)$, now we must prove that

$$z_n \to y^* \quad \text{in } C([0, t]; D_A(\alpha, p)). \tag{4.7}$$

From (SL′)(ii) and (1.15) it follows that

$$\frac{d^+}{dt} |z_n(t)|_Z \leqq (a + \omega)|z_n(t)|_Z + |Bv_n(t)| \tag{4.8}$$

where d^+/dt denotes the right derivative. Thus, there exists $C(T)>0$ such that $|z_n(t)|_Z \leqq C(T)$ for every $t \in [0, T]$. We set $\zeta_n = F(z_n) + Bv_n$. Then, from the representation formula

$$z_n(t) = e^{tA}x + \int_0^t e^{(t-s)A}\zeta_n(s)\, ds \tag{4.9}$$

and the fact that v_n is bounded in $L^p(0, \infty; U)$, we conclude that there exists $C_1(T)>0$ such that $|z_n(t)|_{\alpha,p} \leqq C_1(T)$ for every $t \in [0, T]$. Therefore, $\{\zeta_n\}$ is bounded in $L^p(0, T; X)$ and we can find a subsequence, still denoted by $\{\zeta_n\}$, such that $\zeta_n \to \zeta^*$ weakly in $L^p(0, T; X)$. Moreover, $z_n \to y^*$ in $C([0, t]; X)$.

To show (4.7) note that, for all $t, \varepsilon \in]0, T[$,

$$\begin{aligned} |y^*(t) - z_n(t)|_{\alpha,p} \leqq & \int_\varepsilon^T |e^{(t-s)A}\zeta_n(s)|_{\alpha,p}\, ds \\ & + \left(\int_0^\varepsilon \|e^{(t-s)A}\|^{p/(p-1)}_{\mathscr{L}(X, D_A(\alpha,p))}\, ds\right)^{(p-1)/p} \left(\int_0^\varepsilon |\zeta_n(s)|^p\, ds\right)^{1/p}. \end{aligned} \tag{4.10}$$

Also,

$$\|e^{tA}\|_{\mathscr{L}(X, D_A(\alpha,p))} \leqq \frac{\text{const}}{t^\alpha} \quad \forall t>0. \tag{4.11}$$

So, using the fact that e^{tA}, $t>0$, is a compact operator from X into $D_A(\alpha, p)$ and recalling that $\alpha \in]0, 1-1/p[$, we can easily derive (4.7) from (4.10) and (4.11). □

To prove our existence result, we need a lemma.

LEMMA 4.4. *Assume* (SL) (*respectively,* (SL′)). *For any* $T>0$ *and* $x \in X$ (*respectively,* $x \in D_A(\alpha, p)$) *we have*

$$\begin{aligned} V_\infty(x) = \inf \Bigg\{ & \int_0^T [g(y(s)) + h(u(s))]\, ds + V_\infty(y(T)); \\ & u \in L^1(0, T; U), y'(s) + Ay(s) + F(y(s)) + Bu(s), y(0) = x \Bigg\}. \end{aligned} \tag{4.12}$$

Proof. Denote by V^* the right-hand side of (4.12). Let u be an admissible control and let y be the corresponding solution of (1.2). Then,

$$\begin{aligned} \int_0^\infty \{g(y(s)) + h(u(s))\}\, ds = & \int_0^T \{g(y(s)) + h(u(s))\}\, ds \\ & + \int_0^\infty \{g(y(\sigma+T)) + h(u(\sigma+T))\}\, d\sigma \end{aligned}$$

whence

$$\int_0^\infty \{g(y(s)) + h(u(s))\}\, ds \geqq \int_0^T \{g(y(s)) + h(u(s))\}\, ds + V_\infty(y(T)),$$

which implies that $V^* \geqq V_\infty(x)$. We now prove the reverse inequality. Fix $T>0$ and $u \in L^1(0, T; U)$; let $y \in C([0, T]; X)$ be the corresponding solution of (1.2) and (u_T, y_T) be an optimal pair for problem (1.1), (1.2) with $x = y(T)$. Set

$$\underline{u}(s) = u(s) \quad \text{if } 0 \leqq s \leqq T, \qquad \underline{u}(s) = u_T(s-T) \quad \text{if } s \geqq T.$$

Since $y_T(0) = y(T)$, we have

$$\underline{y}(s) = y(s) \quad \text{if } 0 \leqq s \leqq T, \qquad \underline{y}(s) = y_T(s-T) \quad \text{if } s \geqq T.$$

Then,

$$V_\infty(x) \leqq \int_0^T \{g(y(s)) + h(u(s))\}\, ds + \int_T^\infty \{g(y_T(s-T)) + h(y_T(s-T))\}\, ds$$
$$= \int_0^T \{g(y(s)) + h(u(s))\}\, ds + V_\infty(y(T)),$$

which implies $V_\infty(x) \leqq V^*$. □

The main result of this section is the following theorem.

THEOREM 4.4. *Assume* (SL) (*respectively,* (SL′)) *and suppose that problem* (1.1)-(1.2) *is stable. Then* V_∞ *is a generalized viscosity solution of equation* (4.1). *Moreover, for any* $x \in X$ (*respectively,* $x \in D_A(\alpha, p)$) *there exists an optimal pair* (u^*, y^*) *and the following feedback formula holds*:

$$u^*(t) \in -DH(B^* \nabla^+ V_\infty(y^*(t))), \qquad t \geqq 0. \tag{4.13}$$

Proof. By Lemma 4.3 and by Proposition 2.1 it follows that $V_\infty(x) = W(t, x)$ where W is the generalized viscosity solution of the problem

$$\begin{gathered} -W_t(t, x) + H(B^* DW(t, x)) - \langle Ax + F(x), DW(t, x)\rangle - g(x) = 0, \\ W(T, x) = V_\infty(x). \end{gathered} \tag{4.14}$$

Then, V_∞ is a generalized viscosity solution of (4.1). The existence of an optimal pair (u^*, y^*) was implicitly obtained in the proof of Proposition 4.1 (respectively, Proposition 4.2). Finally, Proposition 2.3 yields the feedback formula (4.13). □

Remark 4.5. From (2.3) we also obtain that, for all $x \in D(A)$,

$$H(B^*p) - \langle Ax + F(x), p\rangle - g(x) \leqq 0 \quad \forall p \in D^+ V_\infty(x), \tag{4.15}$$

$$H(B^*p) - \langle Ax + F(x), p\rangle - g(x) \geqq 0 \quad \forall p \in D^- V_\infty(x). \tag{4.16}$$

Remark 4.6—(Maximum principle). Assume (SL) (respectively, (SL′)). From Proposition 2.3 we conclude that, if $x \in X$ (respectively, $x \in D_A(\alpha, p)$) and (u^*, y^*) is an optimal pair at x, then there exists $p^* \in C([0, \infty[; X)$ such that

$$\begin{aligned} &p^{*\prime}(s) + A^* p^*(s) + (DF(y^*(s))^* + Dg(y(^*s))) = 0, \\ &p^*(s) \in D^+ V_\infty(y^*(s)), \\ &u^*(s) \in -DH(B^* D^+ V_\infty(y^*(s))) \end{aligned} \tag{4.17}$$

for any $s \in [0, T]$.

Remark 4.7—(Feedback dynamical system). Assume (SL) (respectively, (SL′)) and let (u^*, y^*) be an optimal pair at $x \in X$ (respectively, $x \in D_A(\alpha, p)$). Then, by Remark 4.6, y^* is a solution of the closed loop equation

$$y'(t) \in Ay(t) + F(y(t)) - BDH(B^* D^+ V_\infty(y(t))), \quad y(0) = x, \quad t \geqq 0. \tag{4.18}$$

Moreover, by Proposition 2.3, there exists $\delta \in]0, 1[$ such that

$$y^* \in C^{1,\delta}(]0, \infty[; X). \tag{4.19}$$

Now, we denote by S_t the dynamical system generated by (4.18), that is,

$$S_t(x) = y(t), \qquad t \geqq 0, \quad x \in X. \tag{4.20}$$

Then, from the Dynamic Programming Principle (4.12) it follows that S_t is a semigroup of nonlinear operators in X.

We remark that no theory is available to directly solve the initial value problem (4.18) except for special situations such as

$$X = U \text{ Hilbert space}, \quad B = 1, \quad h(x) = \tfrac{1}{2}\|x\|^2, \quad V_\infty \text{ convex.}$$

In this case the operator in the right-hand side of (4.18) becomes m-dissipative (see [6]).

Finally we note that

$$(4.21) \qquad g(S_t x) \in L^1(0, \infty; X) \quad \forall x \in X \quad (\text{respectively, } x \in D_A(\alpha, p)).$$

5. Uniqueness. To make the context of this section clearer to the reader, we recall some known results from linear quadratic control that correspond to the following choice of data:

$$(5.1) \qquad H(x) = \tfrac{1}{2}|x|^2, \quad f(x) = 0, \quad g(x) = \tfrac{1}{2}|Cx|^2, \quad C \in \mathscr{L}(X) \quad x \in X$$

where X is a Hilbert space. In this case, setting $V(x) = \frac{1}{2}\langle Px, x\rangle$, (1.3) reduces to the algebraic Riccati equation:

$$(5.2) \qquad A^*P + PA - PBB^*P + C^*C = 0.$$

As it is well known, if (A, B) is stabilizable with respect to the observation C, then there exists a minimal positive solution P_∞ of (5.2). Moreover, if the feedback operator

$$(5.3) \qquad L = A - BB^*P_\infty$$

is exponentially stable, then P_∞ is unique among the positive solutions of (5.2).

In general, no necessary and sufficient condition for uniqueness of positive solutions is known. A sufficient condition for L to be exponentially stable (which would yield uniqueness), is that C be invertible (more generally that (A, C) be *detectable*; see [25]).

The aim of this section is to generalize the previous results to the general Hamilton–Jacobi equation

$$(5.4) \qquad H(B^*DV(x)) - \langle Ax + F(x), DV(x)\rangle - g(x) = 0.$$

Throughout this section we assume either (SL) or (SL′) and that

(5.5) (i) Problem (1.1)–(1.2) is stable;
(ii) $g(0) = 0$, $h(0) = 0$.

By Theorem 4.5 we know that (5.4) has a generalized viscosity solution given by V_∞.

First we remark that V_∞ is minimal.

LEMMA 5.1. *Assume* (SL) (*respectively,* (SL′)) *and* (5.5). *Let V be a nonnegative generalized viscosity solution of* (5.4) *such that* $V(0) = 0$. *Then* $V_\infty(x) \leqq V(x)$, *for all* $x \in X$ (*respectively,* $x \in D_A(\alpha, p)$).

Proof. By Proposition 2.5 it follows that $\phi(t, x) \leqq V(x)$ where ϕ is the solution of (4.2). Then, Propositions 4.1 and 4.2 yield the conclusion. □

Now, to prove uniqueness we must show that V_∞ is maximal. A sufficient condition for maximality is that $B^{-1} \in \mathscr{L}(H; U)$ and the semigroup of nonlinear operators $S_t(x)$, defined in (4.20), be "stable" for any x in X.

LEMMA 5.2. *Assume* (SL) (*respectively,* (SL′)) *and* (5.5). *Suppose that* $B^{-1} \in \mathscr{L}(H; U)$ *and*

(5.6) $\forall x \in X$ (*respectively,* $x \in D_A(\alpha, p)$) $\exists r \geqq 1$ *such that* $t \to S_t(x)$ *belongs to* $L^r(0, \infty; X)$.

Then V_∞ *is maximal, that is if* V *is a generalized viscosity solution of* (5.4) *such that* $V(0)=0$, *then*

$$V_\infty(x) \geqq V(x) \quad \forall x \in X. \tag{5.7}$$

Proof. Let $x \in X$ (respectively, $x \in D_A(\alpha, p)$) be fixed. Set $y(t)=S_t(x)$. Recalling Proposition 2.4, we have that $D^+V=\partial V$ (see [9]), where ∂V denotes the generalized gradient in the sense of Clarke [11]. So, by (4.19) and Theorem 2.3.10 of [11], we can differentiate the function $t \to V(y(t))$ in the following sense. There exists $q(t) \in D^+V(y(t))$ such that

$$\frac{d}{dt}V(y(t))=\langle Ay(t)+F(y(t))+Bu(t), q(t)\rangle \geqq -g(y(t))+\langle u(t), B^*q(t)\rangle + H(B^*q(t))$$
$$\geqq -g(y(t))-h(u(t)).$$

The first of the inequalities above follows from (4.15). Hence,

$$V(y(t))+\int_0^t \{g(y(s))+h(u(s))\} \geqq V(x).$$

Since $y \in L^r(0,\infty; X)$, there exists a sequence $\{t_n\} \uparrow \infty$, such that $y(t_n) \to 0$. Thus, by the above inequality, we conclude that $V_\infty(x) \geqq V(x)$ as required. □

Remark 5.3. A sufficient condition that yields (5.6) is the coercivity of g, that is,

$$g(x) \geqq C|x|^r \quad \forall x \in X \tag{5.8}$$

for some constant $C>0$.

From Lemmas 5.1 and 5.2 we deduce the following uniqueness result.

THEOREM 5.4. *Assume* (SL) (*respectively,* (SL′)), (5.5), *and* (5.6). *Then* (5.4) *has a unique generalized viscosity solution that is nonnegative and vanishes at* $x=0$.

6. Application to a semilinear parabolic state equation. Let Ω be a bounded open set of $\mathbf{R}^n$ with smooth boundary $\partial\Omega$. Consider the following optimal control problem:

Minimize

$$J(u,x)=\frac{1}{p}\int_0^\infty dt \int_\Omega \{|y(t,\xi)|^p+|u(t,\xi)|^p\}\,d\xi \tag{6.1}$$

over all controls $u \in L^p([0,\infty[\times\Omega)$, $p>1$, and states y satisfying

$$\frac{\partial y}{\partial t}(t,\xi)=\Delta_\xi y(t,\xi)+\Gamma(y(t,\xi), \nabla_\xi y(t,\xi))+u(t,\xi) \quad \text{in } [0,\infty[\times\Omega, \tag{6.2}$$

$$y(t,\xi)=0 \quad \text{on } [0,\infty[\times\partial\Omega, \tag{6.3}$$

$$y(0,\xi)=x(\xi) \quad \text{on } \Omega \tag{6.4}$$

where $\Gamma(r,s)$ is a real-valued function defined in $\mathbf{R}\times\mathbf{R}^n$ and $x \in L^p(\Omega)$.

To apply the results of §§ 3–5, we proceed to check the assumptions (SL′). Let $X=U=L^p(\Omega)$, A be defined by

$$D(A)=W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega) \quad Az=\Delta_\xi z \quad \forall z \in D(A) \tag{6.5}$$

and let $B=1$. Then A generates an analytic semigroup in $L^p(\Omega)$ by [1] and the embedding of $D(A)$ in $L^p(\Omega)$ is compact in view of the Rellich Theorem. Also, we set

$$g(x)=\frac{1}{p}\int_\Omega |x(\xi)|^p\,d\xi, \qquad h(u)=\frac{1}{p}\int_\Omega |u(\xi)|^p\,d\xi. \tag{6.6}$$

Then, it is well known that $g, h \in C^2(L^p(\Omega))$, provided that

$$p \geqq 2. \tag{6.7}$$

So far, we have shown that assumptions (SL')(i) are satisfied.

Now we check (SL')(ii). For this purpose we define

$$Z = C(\bar{\Omega})$$

and note that by the results of [23] the part of A in Z, A_Z, generates an analytic semigroup. This semigroup is also contracting in view of the maximum principle and so (SL')(ii) holds with $\mu = 0$. Next, to verify (SL')(iii), recall the following well-known characterization of the interpolation spaces $D_A(\alpha, p)$ (see, for instance, [24]):

$$D_A(\alpha, p) = \begin{cases} \{f \in W^{2\alpha,p}(\Omega); f_{|\partial\Omega} = 0\} & \text{if } \alpha \in \left]\dfrac{1}{2p}, 1\right[, \\ W^{2\alpha,p}(\Omega) & \text{if } \alpha \in \left]0, \dfrac{1}{2p}\right[. \end{cases}$$

By the Sobolev Embedding Theorem,

$$D_A(\alpha, p) \subset C(\bar{\Omega}) = z \quad \text{if } \alpha > \frac{n}{2p}. \tag{6.8}$$

Note that the constraint in (6.8) is compatible with the requirement $\alpha \in]0, 1 - 1/p[$ if

$$p > \frac{n+2}{2}. \tag{6.9}$$

Let $F(x) = \Gamma(x, \nabla_\xi x)$ and assume

$$\Gamma \in C^2(\mathbf{R} \times \mathbf{R}^n). \tag{6.10}$$

From the Sobolev Embedding Theorem it follows that

$$\alpha > \frac{n+p}{2p} \Rightarrow W^{2\alpha,p}(\Omega) \subset C^1(\bar{\Omega}), \tag{6.11}$$

which in turn implies that F fulfills (1.14). Note again that the constraint in (6.11) is compatible with the requirement $\alpha \in]0, 1 - 1/p[$ if

$$p > n + 2. \tag{6.12}$$

We will now show that the condition

$$r\Gamma(r, 0) \leqq ar^2 \quad \text{for all } r \in \mathbf{R} \text{ and some } a \in \mathbf{R} \tag{6.13}$$

implies (1.15). The argument is known; nevertheless, we recall it for the reader's convenience. First, let

(6.14) $z \in C^1(\bar{\Omega})$ be such that $|z|$ has a unique maximum point, say $\xi_0 \in \Omega$.

Then, we can easily show that $\partial|z| = \{z^*\}$, where

$$z^* = \begin{cases} \delta_{\xi_0} & \text{if } z(\xi_0) = |z|_Z, \\ -\delta_{\xi_0} & \text{if } z(\xi_0) = -|z|_Z, \end{cases}$$

and δ denotes the Dirac measure. Thus,

$$\langle F(z), z^* \rangle = \begin{cases} \Gamma(|z|_Z, 0) & \text{if } z(\xi_0) = |z|_Z, \\ -\Gamma(-|z|_Z, 0) & \text{if } z(\xi_0) = -|z|_Z. \end{cases} \tag{6.15}$$

From (6.13) and (6.15) we get

$$\langle F(z), z^*\rangle \leqq a|z|_Z \tag{6.16}$$

for all z satisfying (6.14). On the other hand, it is well known (see, for instance, [17, Lemma II-7-1]) that $z \in Z$ satisfies (6.16) if and only if

$$|z| \leqq |z+\lambda(F(z)-az)| \quad \forall \lambda > 0. \tag{6.17}$$

Since the set of functions z satisfying (6.14) is dense in Z, the proof of (1.15) is complete. Finally, if we assume that

$$|\Gamma(r, s)| \leqq \beta(|r|)+\rho(|r|)|s| \tag{6.18}$$

where $\beta, \rho : [0, \infty[\to [0, \infty[$ are continuous functions, then (1.16) easily follows. Therefore, assumptions (SL′) are fulfilled if

(6.19) $(n+p)/2p < \alpha < 1-1/p$, $p > n+2$, and (6.10), (6.13), and (6.18) hold.

Our next goal is to show that $(A+F, B, h)$ is g-stabilizable. This will be given by Proposition 3.3 if we assume that the function β in (6.18) satisfies

$$\beta(r) \leqq Cr^{\sigma} \quad \forall r \in [0, R] \tag{6.20}$$

for some constants $C, R \geqq 0$.

Now, Theorem 5.4 yields the following theorem.

THEOREM 6.1. *Assume* (6.19) *and* (6.20). *Then the Hamilton-Jacobi equation*

$$(p-1)|DV(x)|_{X^*}^{p'} - p\langle \Delta_\xi x + \Gamma(x, \nabla_\xi x), DV(x)\rangle - |x|_X^p = 0, \qquad p' = \frac{p}{p-1} \tag{6.21}$$

has a unique generalized viscosity solution $V_\infty \geqq 0$ *such that* $V_\infty(0)=0$. V_∞ *is the value function of the control problem* (6.1)–(6.4). *Moreover, for any* $x \in D_A(\alpha, p)$ *there exists an optimal pair* (u^*, y^*) *at* x *and the following feedback formula holds*:

$$u^*(t) \in |D^+V(y^*(t))|^{p'-2} D^+V(y^*(t)). \tag{6.22}$$

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