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SOME RESULTS ON BELLMAN EQUATION IN HILBERT SPACES*

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Abstract. We give an existence result on the Bellman equation related to an infinite dimensional control problem.

Key words. Bellman equation, dynamic programming, nonlinear semigroup

1. Introduction. This paper deals with the evolution equation

$$\begin{aligned}\phi_t &= \tfrac{1}{2}\operatorname{Tr}(S\phi_{xx}) + \langle Ax, \phi_x\rangle - F(x, \phi_x),\\ \phi(0, x) &= \phi_0(x),\end{aligned} \tag{1.1}$$

as well as with the stationary equation

$$\lambda\phi - \tfrac{1}{2}\operatorname{Tr}(S\phi_{xx}) - \langle Ax, \phi_x\rangle + F(x, \phi_x) = 0, \qquad \lambda > 0. \tag{1.2}$$

Here A is the infinitesimal generator of a strongly continuous semi-group in H, F a mapping from $H\times H$ into $\mathbb{R}$, ϕ a mapping from $[0, T]\times H$ into $\mathbb{R}$ (ϕ_t and ϕ_x denote derivatives with respect to t and x).

Equations (1.1) and (1.2) are relevant in the study of dynamic programming in the control of stochastic differential equations (see for instance [3], [7]). In [1] (1.1) is studied in the particular case

$$F(x, \phi_x) = \tfrac{1}{2}|\phi_x|^2 - g(x). \tag{1.3}$$

In this case it is possible to prove the existence and uniqueness of ϕ if ϕ_0 and g are convex (with polynomial growth to infinity). In applications to control theory, the hypothesis of convexity is fulfilled if the state equation is linear and the cost functional is convex. In this paper we give an approach to (1.1) and (1.2) without convexity hypotheses.

We remark that, using abstract Gauss measure, some results have been proved in [9] in the particular case when $A = 0$.

Our method consists first in solving the linear problem

$$\begin{aligned}\phi_t &= \tfrac{1}{2}\operatorname{Tr}(S\phi_{xx}) + \langle Ax, \phi_x\rangle,\\ \phi(0, x) &= \phi_0(x),\end{aligned} \tag{1.4}$$

then in considering the nonlinear term as a perturbation of the linear one. Section 2 is devoted to problem (2.4) and § 3 to (1.1), (1.2) (using the theory of nonlinear semigroups). Finally in § 4 we present an application of our results to a problem of stochastic control.

2. The linear problem. We are here concerned with the problem

$$\begin{aligned}\phi_t &= \tfrac{1}{2}\operatorname{Tr}(S\phi_{xx}) + \langle Ax, \phi_x\rangle,\\ \phi(0, x) &= \phi_0(x).\end{aligned} \tag{2.1}$$

Let us list the following hypotheses:

H1) S is a self-adjoint, positive nuclear operator in a separable Hilbert space H.

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S is given by

$$Sx = \sum_{i=1}^{\infty} \lambda_i \langle x, e_i \rangle e_i \tag{2.2}$$

where $\{e_i\}$ is a complete orthonormal system in H and $\lambda_i > 0$, $i = 1, 2, \cdots$ ($\langle \cdot \rangle$ denotes the inner product and $|\cdot|$ the norm in H).

H2) $A: D_A \subset H \to H$ is the infinitesimal generator of a strongly continuous, linear semi-group e^{tA} in H. Moreover $|e^{tA}| \leqq 1$ and $\{e_i\} \subset D_A$.

We shall denote by $C_b(H)$ the set of all mappings $\psi: H \to \mathbb{R}$ uniformly continuous and bounded. $C_b(H)$, endowed with the norm

$$\|\psi\|_\infty = \sup_{x \in H} |\psi(x)|, \tag{2.3}$$

is a Banach space. By $C_b^h(H)$, $h = 1, 2, \cdots$, we mean the set of all mappings $\psi: H \to \mathbb{R}$ uniformly continuous and bounded, with all derivatives of order less than or equal to h.

Let $\{\beta_i\}$ be a sequence of mutually independent real Brownian motions in a probability space (Ω, ε, P). Set

$$W_t = \sum_{i=1}^{\infty} \sqrt{\lambda_i} \beta_i(t) e_i; \tag{2.4}$$

then it is well known (see for instance [5]) that W_t is a H-valued Brownian motion with covariance operator S.

To solve (2.1) we consider the following approximating problem:

$$\begin{aligned} &\phi_t^n = \tfrac{1}{2} \operatorname{Tr}(S_n \phi_{xx}^n) + \langle A_n x, \phi_x^n \rangle, \\ &\phi^n(0, x) = \phi_0(x), \qquad x \in H_n \end{aligned} \tag{2.5}$$

where $H_n = P_n(H)$, $P_n x = \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i$, $S_n = SP_n$, $A_n = P_n A P_n$. Note that A_n is bounded by virtue of hypothesis H2b.

The following lemma is standard (since problem (2.5) is finite dimensional).

LEMMA 2.1. *Assume that* $\phi_0 \in C_b^2(H)$. *Then problem* (2.5) *has a unique solution* ϕ^n *given by*

$$\phi^n(t, x) = E\phi_0(e^{tA_n} x + X_t^n) \quad \forall x \in H_n, \tag{2.6}$$

where

$$X_t^n = \int_0^t e^{(t-s)A_n}\, dW_s^n, \qquad W_s^n = P_n W_s \tag{2.7}$$

(*E means expectation*).

In the sequel we set

$$(T_t^n \psi)(x) = E\psi(e^{tA_n} x + X_t^n) \quad \forall x \in H_n \tag{2.8}$$

for any $\psi \in C_b(H)$. It is easy to check that T_t^n is a strongly continuous semi-group of contractions in $C_b(H_n)$ whose infinitesimal generator $\mathscr{A}_n$ is given by

$$\mathscr{A}^n \psi = \tfrac{1}{2} \operatorname{Tr}(S_n \psi_{xx}) + \langle A_n x, \psi_x \rangle \quad \forall \psi \in C_b^2(H_n). \tag{2.9}$$

Note now that X_t^n is a Gaussian random variable in H_n whose covariance Σ_t^n is given by

$$\Sigma_t^n x = \int_0^t e^{sA_n^*} S_n \, e^{sA_n} x \, ds \quad \forall x \in H_n. \tag{2.10}$$

It follows:

$$(T_t^n \psi)(x) = (2\pi)^{-n/2} \det (\Sigma_t^n)^{-1/2} \int_{H_n} \exp \left(-\tfrac{1}{2}\langle (\Sigma_t^n)^{-1} y_n, y_n\rangle\right) \psi(e^{tA_n} x + y) \, dy \qquad \forall \psi \in C_b(H_n). \tag{2.11}$$

Observe that, due to the hypothesis that $\lambda_i > 0$, we have $\det(\Sigma_t^n) \neq 0$.

We will compute now the derivative of $T_t^n \psi$.

LEMMA 2.2. *For any $\psi \in C_b(H_n)$, $t > 0$ and $x \in H_n$ the derivative of T_t^n with respect to x exists and is given by*

$$\frac{d}{dx}(T_t^n \psi)(x) = E(e^{tA_n^*} (\Sigma_t^n)^{-1} X_t^n \psi(e^{tA_n} x + X_t^n)). \tag{2.12}$$

Proof. Setting in (2.11) $z = e^{tA_n} x + y$, we get

$$(T_t^n \psi)(x) = (2\pi)^{-n/2} \det (\Sigma_t^n)^{-1/2} \int_{H_n} \exp \left(-\tfrac{1}{2}\langle (\Sigma_t^n)^{-1} (z - e^{tA_n} x), z - e^{tA_n} x\rangle\right) \psi(z) \, dz \tag{2.13}$$

from which

$$\begin{aligned} \left(\frac{dT_t^n \psi}{dx}\right)(x) &= (2\pi)^{-n/2} \det (\Sigma_t^n)^{-1/2} \int_{H_n} \exp \left(-\tfrac{1}{2}\langle (\Sigma_t^n)^{-1} (z - e^{tA_n} x), z - e^{tA_n} x\rangle\right) \\ &\qquad \cdot e^{tA_n^*} (\Sigma_t^n)^{-1} (z - e^{tA_n} x) \psi(z) \, dz \\ &= \int_{H_n} e^{tA_n^*} (\Sigma_t^n)^{-1} y \psi(e^{tA_n} x + y) f_n(y) \, dy \end{aligned} \tag{2.14}$$

where f_n is the n-dimensional density of X_t^n. Thus (2.12) follows. □

For any $\psi \in C_b(H)$ we now set

$$(T_t \psi)(x) = E\psi(e^{tA} x + X_t), \qquad t > 0, \quad x \in H, \tag{2.15}$$

where

$$X_t = \int_0^t e^{(t-s)A} \, dW_s. \tag{2.16}$$

LEMMA 2.3. *Let $\psi \in C_b(H)$, $\psi^n(x) = \psi(P_n x)$; then the following statements hold*:

a) $(T_t^n \psi^n)(x) \to T_t \psi(x) \; \forall x \in H$;

b) $T_t \psi \in C_b(H)$;

c) *T_t is a semi-group of contractions in $C_b(H)$.*

Proof. We have

$$|T_t \psi(x) - T_t^n \psi^n(x)| \leq E|\psi(e^{tA} x + X_t) - \psi(e^{tA_n} P_n x + X_t^n)|.$$

Now $e^{tA_n} P_n x \to e^{tA} x$ by the Trotter–Kato theorem; moreover $X_t^n \to X_t$ in probability since

$$
\begin{aligned}
E|X_t - X_t^n|^2 &= \sum_{i=n+1}^{\infty} \lambda_i \int_0^t |e^{(t-s)A} e_i|^2 \, ds \\
&\quad + \sum_{i=1}^{n} \lambda_i \int_0^t |e^{(t-s)A} e_i - e^{(t-s)A_n} e_i|^2 \, ds \\
&\leqq t \sum_{i=n+1}^{\infty} \lambda_i + \sum_{i=1}^{\infty} \lambda_i \int_0^t |e^{(t-s)A_n} e_i - e^{(t-s)A_n} e_i|^2 \, ds \to 0
\end{aligned}
$$

as $n \to \infty$. (Recall that $\sum_{i=1}^{\infty} \lambda_i = \operatorname{Tr}(S) < +\infty$.) Conclusion a) follows from the Lebesgue theorem. The statements b) and c) are straightforward. □

We will study now the differentiability of T_t. From (2.12) it appears (for $n \to \infty$) that we have no chance to define $(d/dx)(T_t\psi)$ for every $\psi \in C_b(H)$. To this end we need some additional hypotheses and a new definition of differentiality. The situation is similar to the Gross theory for the heat equation in Hilbert spaces (when $A = 0$, see [8]).

We set

$$
\Lambda_t^n = S_n \, e^{tA_n^*} (\Sigma_t^n)^{-1} \tag{2.17}
$$

and assume:

H3) a) There exists the limit

$$
\lim_{n \to \infty} \Lambda_t^n P_n x = \Lambda_t x \quad \forall x \in H.
$$

b) There exists a constant $\gamma > 0$ such that

$$
|\Lambda_t^n| \leqq \frac{\gamma}{t} \quad \forall t > 0.
$$

Let us give an example in which H3 is fulfilled.

Example 2.4. Assume that

$$
A e_i = -\mu_i e_i, \qquad \mu_i \geqq 0, \quad i = 1, 2, \cdots. \tag{2.18}
$$

Then

$$
\Sigma_t^n e_i = \int_0^t e^{-2\mu_i t} \lambda_i \, dt \, e_i, \qquad i = 1, 2, \cdots, \tag{2.19}
$$

so that

$$
\Lambda_t^n e_i = \frac{2 \, e^{-t\mu_i} \mu_i}{1 - e^{-2t\mu_i}} e_i, \qquad i = 1, 2, \cdots. \tag{2.20}
$$

Now the limit in H3a exists; in fact

$$
\begin{aligned}
|\Lambda_t^{n+p} P_{n+p} x - \Lambda_t^n P_n x|^2 &= \sum_{i=n+1}^{n+p} \left| \frac{2\mu_i \, e^{-t\mu_i}}{1 - e^{-2\mu_i t}} \right|^2 |\langle x, e_i \rangle|^2 \\
&\leqq \frac{\gamma}{t} \sum_{i=n+1}^{n+p} |\langle x, e_i \rangle|^2
\end{aligned} \tag{2.21}
$$

where

$$
\gamma = \sup_{\alpha > 0} \frac{\alpha \, e^{-\alpha/2}}{1 - e^{-\alpha}}. \tag{2.22}
$$

H3a, b follow easily from (2.21). □

Let us now define differentiability.

DEFINITION 2.5. We assume that $\psi \in C_b(H)$ is S-differentiable if:

a) For any $x, y \in H$ there exists the limit

$$\lim_{h\to 0} \frac{1}{h}(\psi(x+hSy)-\psi(x)) = L_x(y); \tag{2.23}$$

b) $L_x(y)$ is linear, continuous in y.

If ψ is S-differentiable we denote by $S\psi_x$ the element of H defined by

$$L_x(y) = \langle S\psi_x(x), y\rangle. \tag{2.24}$$

We shall denote by $C^1_S(H)$ the set of all mappings ψ in $C_b(H)$ such that

i) ψ is S-differentiable,

ii) $S\psi_x \in C_b(H)$,

and $C^1_S(H)$, endowed with the norm

$$\|\psi\|_{C^1_S(H)} = \|\psi\|_\infty + \|S\psi_x\|_\infty, \tag{2.25}$$

is a Banach space.

We are ready now to prove the main result of this section.

PROPOSITION 2.6. *Assume that* H1, H2 *and* H3 *are fulfilled. Let* $\psi \in C_b(H)$ *and* $t>0$; *then* $T_t\psi \in C^1_S(H)$ *and*

$$S(T_t\psi)_x(x) = E(\Lambda_t X_t \psi(e^{tA}x + X_t)) = \lim_{n\to\infty} S_n(T^n_t\psi)_x(P_n x). \tag{2.26}$$

Moreover

$$\begin{aligned} &\|S_n(T^n_t\psi)_x\|_\infty \leqq \frac{\gamma}{\sqrt{t}}\sqrt{\operatorname{Tr}(S)},\\ &\|S(T_t\psi)_x\|_\infty \leqq \frac{\gamma}{\sqrt{t}}\sqrt{\operatorname{Tr}(S)}, \end{aligned} \tag{2.27}$$

where γ *is the constant in* H3b.

Proof. For any $x, y \in H$ we set

$$F(h) = (T_t\psi)(x+hSy), \tag{2.28}$$

$$F_n(h) = (T^n_t\psi)(P_n x + hS_n y). \tag{2.29}$$

Clearly $F_n(h) \to F(h)$ uniformly in [0, 1]. Moreover from Lemma 2.2 we have

$$F'_n(h) = \langle E(\Lambda^n_t X^n_t \psi(e^{tA_n}(P_n x + hS_n y) + X^n_t)), y\rangle \tag{2.30}$$

so that, as $h \to 0$,

$$F'_n(h) \to \langle E(\Lambda_t X_t \psi(e^{tA}(x+hSy)+X_t)), y\rangle \quad \text{uniformly in } [0, 1]. \tag{2.31}$$

Thus $F(h)$ is differentiable in h and equality (2.26) follows. Concerning (2.27) we have

$$\begin{aligned} \|S(T_t\psi)_x\|_\infty &\leqq \frac{\gamma}{t}\|\psi\|_\infty (E(|X_t|^2))^{1/2}\\ &= \left(\sum_{i=1}^\infty \lambda_i \int_0^t |e^{(t-s)A}e_i|^2\, dt\right)^{1/2} \leqq \frac{\gamma}{\sqrt{t}}\sqrt{\operatorname{Tr} S}. \end{aligned} \tag{2.32}$$

□

We remark now that the semi-group T_t on $C_b(H)$ is not strongly continuous (when H is infinite-dimensional and A is unbounded). Since we cannot use the Hille–Yosida theorem, we use the following procedure to define the "infinitesimal generator" of T_t.

We set

$$\begin{aligned}(F_\lambda\psi)(x)&=\int_0^\infty e^{-\lambda t}(T_t\psi)(x)\,dt\\ &=\int_0^\infty e^{-\lambda t}E\psi(e^{tA}x+X_t)\,dt\quad \forall\psi\in C_b(H),\,x\in H.\end{aligned}\tag{2.33}$$

Clearly there exists a linear operator $\mathscr{A}$ in $C_b(H)$ such that

$$R(\lambda,\mathscr{A})\psi=F_\lambda\psi\quad\forall\lambda>0;\tag{2.34}$$

moreover

$$\|R(\lambda,\mathscr{A})\|_\infty\leqq\frac{1}{\lambda}\quad\forall\lambda>0\tag{2.35}$$

so that $\mathscr{A}$ is m-dissipative in $C_b(H)$. $\mathscr{A}$ can be viewed as the abstract realization of the linear operator

$$\tfrac{1}{2}\operatorname{Tr}(S\psi_{xx})+\langle Ax,\psi_x\rangle.$$

The following corollary is straightforward:

COROLLARY 2.7. *Assume that* H1, H2 *and* H3 *are fulfilled. Let* $\psi\in C_b(H)$ *and* $\lambda>0$. *Then* $R(\lambda,\mathscr{A})\psi\in C^1_S(H)$ *and*

$$S(R(\lambda,\mathscr{A})\psi)_x(x)=\lim_{n\to\infty}S_n(R(\lambda,\mathscr{A}_n)\psi_n)_x(P_nx),\tag{2.36}$$

where the operators $\mathscr{A}_n$ *and* $\mathscr{A}$ *are defined by* (2.9) *and* (2.34) *respectively. Moreover,*

$$\|S(R(\lambda,\mathscr{A})\psi)_x\|_\infty\leqq\frac{\gamma\Gamma(1/2)\sqrt{\operatorname{Tr}(S)}}{\sqrt{\lambda}}=\frac{\gamma'}{\sqrt{\lambda}}.\tag{2.37}$$

3. The nonlinear problem.

We consider here the problem.

$$\begin{aligned}&\phi_t=\tfrac{1}{2}\operatorname{Tr}(S\phi_{xx})+\langle Ax,\phi_x\rangle-F(S\phi_x),\\ &\phi(0,x)=\phi_0(x).\end{aligned}\tag{3.1}$$

Denote by Lip (H) the set of all mappings $\psi\colon H\to\mathbb{R}$ Lipschitz continuous and set

$$\|F\|_L=\sup\left\{\frac{|f(x)-f(y)|}{|x-y|},\,x,y\in H,\,x\neq y\right\}.\tag{3.2}$$

Let $F\in\operatorname{Lip}(H,H)$ and $\mathscr{B}$ be the mapping in $C_b(H)$ defined by

$$\mathscr{B}\phi=-F(S\phi_x)\quad\forall\phi\in C^1_{S(H)}.\tag{3.3}$$

We are going to prove that $\mathscr{A}+\mathscr{B}$ is m-dissipative, and then we shall invoke the Crandall–Ligget theorem [4] to solve (3.1).

Let us also introduce the approximating operator

$$\mathscr{B}_n\phi=-F(S_n\phi_x)\quad\forall\phi\in C^1_b(H_n).\tag{3.4}$$

LEMMA 3.1. *Assume that the hypotheses* H1, H2 *and* H3 *hold. Let* $F \in \mathrm{Lip}(H)$; *then* $\mathcal{A}_n + \mathcal{B}_n$ *is m-dissipative. Moreover, if*

$$\lambda > 4(\gamma' \|F\|_L)^2 \tag{3.5}$$

we have

$$\|\mathcal{B}_n(R(\lambda, \mathcal{A}_n))\|_L \leqq \tfrac{1}{2} \tag{3.6}$$

and

$$(\lambda - \mathcal{A}_n - \mathcal{B}_n)^{-1} g = R(\lambda, \mathcal{A}_n)(1 - \mathcal{B}_n(R(\lambda, \mathcal{A}_n)))^{-1} g \quad \forall g \in C_b(H_n). \tag{3.7}$$

Proof. The dissipativity of $\mathcal{A}_n + \mathcal{B}_n$ can be easily checked (it is a finite-dimensional operator). For m-dissipativity it suffices to show (see for instance [6]) that $\lambda - \mathcal{A}_n - \mathcal{B}_n$ is surjective for some $\lambda > 0$. To this purpose choose $g \in C_b(H_n)$ and consider the equation

$$\lambda\phi - \mathcal{A}_n\phi - \mathcal{B}_n\phi = g, \qquad \lambda > 0. \tag{3.8}$$

If we set $\psi = \lambda\phi - \mathcal{A}_n\phi$, (3.8) is equivalent to

$$\psi - \Sigma_n(\psi) = g \tag{3.9}$$

where

$$\Sigma_n\psi = -F(S_n(R(\lambda, \mathcal{A}_n)\psi_x)). \tag{3.10}$$

Recalling (2.27) we have

$$\|\Sigma_n\|_L \leqq \|F\|_L \frac{\gamma'}{\sqrt{\lambda}} \tag{3.11}$$

and the conclusion follows from the contraction principle. □

The proof of the following lemma is quite similar so it will be omitted.

LEMMA 3.2. *Under the same hypotheses of Lemma* 3.1, *if* (3.5) *holds then* $(\lambda - \mathcal{A} - \mathcal{B})^{-1}$ *exists and is given by*

$$(\lambda - \mathcal{A} - \mathcal{B})^{-1} g = R(\lambda, \mathcal{A})(1 - \mathcal{B}R(\lambda, \mathcal{A}))^{-1} g \quad \forall g \in C_b(H). \tag{3.12}$$

Note that at this stage we cannot assert that $\mathcal{A} + \mathcal{B}$ is m-dissipative (we did not prove that $\mathcal{A} + \mathcal{B}$ is dissipative). This will be proved by the following proposition.

PROPOSITION 3.3. *Assume that hypotheses* H1, H2, H3 *hold. Let* $F \in \mathrm{Lip}(H)$; *then* $\mathcal{A} + \mathcal{B}$ *is m-dissipative. Moreover, for any* $g \in C_b(H)$ *we have*

$$((\lambda - \mathcal{A} - \mathcal{B})^{-1} g)(x) = \lim_{n \to \infty} ((\lambda - \mathcal{A}_n - \mathcal{B}_n)^{-1} g_n)(x) \quad \forall x \in H, \tag{3.13}$$

$$S((\lambda - \mathcal{A} - \mathcal{B})^{-1} g)_x(x) = \lim_{n \to \infty} S_n((\lambda - \mathcal{A}_n - \mathcal{B}_n)^{-1} g_n)_x(x) \quad \forall x \in H, \tag{3.14}$$

where

$$g_n(x) = g(P_n x). \tag{3.15}$$

Proof. Set

$$\begin{aligned} \psi_n &= (1 - \mathcal{B}_n R(\lambda, \mathcal{A}_n))^{-1} g_n, \\ \psi &= (1 - \mathcal{B} R(\lambda, \mathcal{A}))^{-1} g. \end{aligned} \tag{3.16}$$

By virtue of Corollary 2.7, in order to prove (3.13) and (3.14) it suffices to prove that

$$\psi(x)=\lim_{n\to\infty}\psi_x(x)\quad \forall x\in H. \tag{3.17}$$

By the contraction principle we have

$$\begin{aligned}\psi_n&=\lim_{m\to\infty}\psi_n^m\quad \text{in } C_b(H_n),\\ \psi&=\lim_{m\to\infty}\psi^m\quad \text{in } C_b(H)\end{aligned} \tag{3.18}$$

where

$$\begin{aligned}&\psi_n^0=g_n,\qquad \psi^0=g,\\ &\psi_n^{m+1}=g_n+\Sigma_n(\psi_n^m),\qquad \psi^{m+1}=g+\Sigma(\psi^m).\end{aligned} \tag{3.19}$$

However, since g_n does not go to g in $C_b(H)$ (as $n\to 0$), the conclusion (3.17) does not follow immediately.

Fix now $x\in H$; then we have

$$\begin{aligned}|\psi(x)-\psi_n(P_nx)|\leqq|\psi(x)-\psi^m(x)|+|\psi^m(x)-\psi_n^m(P_nx)|\\ +|\psi_n(P_nx)-\psi_n^m(P_nx)|.\end{aligned} \tag{3.20}$$

The first and the third term of the right-hand side of (3.20) go to zero (as $m\to\infty$) uniformly in n; moreover, for any fixed m we have $|\psi^m(x)-\psi_n^m(P_nx)|\to 0$ as $n\to\infty$; thus (3.17) is proved. Now dissipativity of $\mathcal{A}+\mathcal{B}$ follows from (3.13), and m-dissipativity from Lemma 3.2.

Let now $\rho\in \operatorname{Lip}(H, H)$ and set

$$\begin{aligned}\mathcal{C}\phi&=\langle\rho(x), S\phi_x\rangle\quad \forall\phi\in C_S^1(H),\\ \mathcal{C}_n\phi&=\langle S_n\rho(x), \phi_x\rangle\quad \forall\phi\in C_S^1(H_n).\end{aligned} \tag{3.21}$$

Then by similar arguments we can prove the following.

PROPOSITION 3.4. *Assume that hypotheses* H1, H2, H3 *hold. Let* $F\in\operatorname{Lip}(H)$, $\rho\in\operatorname{Lip}(H,H)$; *then* $\mathcal{A}+\mathcal{B}+\mathcal{C}$ *is m-dissipative. Moreover, for any* $g\in C_b(H)$ *we have*

$$((\lambda-\mathcal{A}-\mathcal{B}-\mathcal{C})^{-1}g)(x)=\lim_{n\to\infty}((\lambda-\mathcal{A}_n-\mathcal{B}_n-\mathcal{C}_n)^{-1}g_n)(x)\quad\forall x\in H, \tag{3.22}$$

$$S((\lambda-\mathcal{A}-\mathcal{B}-\mathcal{C})^{-1}g)_x(x)=\lim_{n\to\infty}S((\lambda-\mathcal{A}_n-\mathcal{B}_n-\mathcal{C}_n)^{-1}g_n)_x(x)\quad\forall x\in H, \tag{3.23}$$

where g_n *is given by* (3.15).

Remark 3.5. Under the hypotheses of Proposition 3.4 we draw the following conclusions.

a) For any $\lambda>0$, $g\in C_b(H)$ the equations

$$\lambda\phi-\tfrac{1}{2}\operatorname{Tr}(S\phi_{xx})-\langle Ax,\phi_x\rangle+F(S\phi_x)-\langle\rho(x), S\phi_x\rangle=g, \tag{3.24}$$

$$\lambda\phi^n-\tfrac{1}{2}\operatorname{Tr}(S_n\phi^n_{xx})-\langle A_nx,\phi^n_x\rangle+F(S_n\phi^n_x)-\langle S_n\rho(x),\phi^n_x\rangle=g \tag{3.25}$$

have unique solutions ϕ and ϕ^n; moreover

$$\phi^n(P_nx)\to\phi(x),\quad (S_n\phi^n_x)(P_nx)\to(S\phi_x)(x)\quad\forall x\in H. \tag{3.26}$$

b) $\mathcal{A}+\mathcal{B}+\mathcal{C}$ verifies the hypotheses of the Crandall–Liggett theorem; thus we can conclude that the problem

$$\begin{aligned}\phi_t &= \tfrac{1}{2}\operatorname{Tr}(S\phi_{xx}) + \langle Ax, \phi_x\rangle - F(S\phi_x) + \langle \rho(x), S\phi_x\rangle = g,\\ \phi(0, x) &= \phi_0 \in C_b(H)\end{aligned} \tag{3.27}$$

has a unique weak solution. □

4. An application to control theory. We shall study the following control problem. Minimize

$$J(x, u) = E\int_0^\infty e^{-\lambda t}(g(y(s)+\tfrac{1}{2}|u(s)|^2))\,ds, \qquad \lambda > 0 \text{ fixed}, \tag{4.1}$$

over all $u \in U$ subject to the state equation

$$\begin{aligned}dy &= (Ay + S\rho(y) + Su)\,dt + dW_t,\\ y(0) &= x.\end{aligned} \tag{4.2}$$

U (the control space) is the set of all stochastic processes u adapted to W_t and such that $|u(t)| \leqq R$ where $R > 0$ is fixed. We shall assume in the whole of this section that hypotheses H1, H2, H3 hold and moreover that $\rho \in \operatorname{Lip}(H)$.

Let $J(x) = \inf_{u\in U} J(x, u)$ be the value function of problem (4.1). The corresponding Bellman equation is see for instance [3]:

$$\lambda\phi - \tfrac{1}{2}\operatorname{Tr}(S\phi_{xx}) - \langle Ax, \phi_x\rangle - \langle \rho(x), S\phi_x\rangle + F(S\phi_x) = g(x), \tag{4.3}$$

where

$$F(x) = \begin{cases} \tfrac{1}{2}|x|^2 & \text{if } |x| \leqq R,\\ R|x| - \dfrac{R^2}{2} & \text{if } |x| \geqq R.\end{cases} \tag{4.4}$$

Clearly $F \in \operatorname{Lip}(H)$, so that by Proposition (3.4) (see also Remark 3.5a), (4.3) has a unique solution $\phi \in C_S^1(H)$. Moreover, by (3.26) ϕ can be approximated by the solution ϕ^n to the equation

$$\lambda\phi^n - \tfrac{1}{2}\operatorname{Tr}(S_n\phi^n_{xx}) - \langle A_n x, \phi^n_x\rangle - \langle S_n\rho(x), \phi^n_x\rangle + F(S_n\phi^n_x) = g(x), \qquad x \in H_n. \tag{4.5}$$

Let us also consider the approximating state equations

$$\begin{aligned}dy_n &= (A_n y_n + S_n\rho(y_n) + S_n u)\,dt + dW^n_t,\\ y_n(0) &= x \in H_n.\end{aligned} \tag{4.6}$$

LEMMA 4.1. *Let $x \in H$, $u \in U$, y be the corresponding solution of* (4.2) *and ϕ the solution of* (4.3). *Then the following identity holds,*

$$\begin{aligned}&\phi(x) + \tfrac{1}{2}E\int_0^t [|u + S\phi_x|^2 - \chi(|S\phi_x| - R)]\,ds\\ &\qquad = E\int_0^t (g(y(s)+\tfrac{1}{2}|u(s)|^2)\,ds + e^{-\lambda t}(y(t)),\end{aligned} \tag{4.7}$$

where

$$\chi(\alpha) = \begin{cases} 0 & \text{if } \alpha \leqq 0,\\ \alpha^2 & \text{if } \alpha \geqq 0.\end{cases} \tag{4.8}$$

Proof. Let y_n be the solution of (4.6) and ϕ^n the solution of (4.5). By the Itô formula we have

$$d\,e^{-\lambda t}\phi^n(y_n)=\{F(S_n\phi_x^n)+\langle S_n u, \phi_x^n\rangle-g(y_n)\}\,dt+\langle\phi_x^n, dW_t\rangle. \tag{4.9}$$

By integrating and taking expectations, we get

$$\begin{aligned}\phi^n(x_n)+\tfrac{1}{2}E\int_0^t [|u_n+S_n\phi_x^n|^2-\chi(|S_n\phi_x^n|-R)]\,ds\\ =E\int_0^t (g(y_n)+\tfrac{1}{2}|u_n(s)|^2)\,ds+e^{-\lambda t}\phi^n(y_n(t)),\end{aligned} \tag{4.10}$$

where $u_n=P_n u$, and (4.9) follows by letting n go to infinity.

PROPOSITION 4.2. *The solution* ϕ *to* (4.3) *coincides with the value function* J *of problem* (4.1). *Moreover, there exists a unique optimal control* u^* *for problem* (4.1) *which is related to the optimal state by the synthesis formula*:

$$u^*(t)=-h(S\phi_x(y^*(t))),\qquad t\geqq 0, \tag{4.11}$$

where

$$h(z)=\begin{cases}|z| & \text{if } |z|\leqq R,\\ \dfrac{z}{|z|}R & \text{if } |z|\geqq R.\end{cases} \tag{4.12}$$

Proof. First of all we remark that the following inequality holds

$$|u+S\phi_x|^2-\chi(|S\phi_x|-R)\geqq 0, \tag{4.13}$$

the equality being fulfilled if

$$u=-h(S\phi_x). \tag{4.14}$$

Thus, from (4.7) it follows that $\phi(x)\leqq J(x)$. To prove the converse let $\bar{y}$ be the solution of the closed loop equation

$$\begin{aligned}d\bar{y}&=(A\bar{y}+Sp(\bar{y})-h(S\phi_x(\bar{y})))\,dt+dW_t,\\ \bar{y}(0)&=x.\end{aligned} \tag{4.15}$$

The existence and uniqueness of Eq. (4.15) are standard because $h\in\operatorname{Lip}(H)$ and $S\phi_x\in C_b(H)$. By setting $u=\bar{u}$, $y=\bar{y}$ in (4.7), and letting λ go to infinity, we obtain

$$\phi(x)=E\int_0^\infty (g(\bar{y}(s))+\tfrac{1}{2}|\bar{u}(s)|^2)\,ds \tag{4.16}$$

so that $(\bar{u},\bar{y})$ is an optimal couple for problem (4.1). Finally let $(\tilde{u},\tilde{y})$ be another optimal couple; again by (4.7) we get

$$E\int_0^\infty [|\tilde{u}+S\phi_x(\tilde{y})|^2-\chi(|S\phi_x(\tilde{y})|-R)]\,ds=0 \tag{4.17}$$

which implies $\tilde{u}=-h(S\phi_x(\tilde{y}))$; due to the uniqueness of (4.15) we have $\tilde{u}=\bar{u}$. □

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