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UNBOUNDED SOLUTIONS TO THE LINEAR QUADRATIC CONTROL PROBLEM*

G. DA PRATO† AND M. C. DELFOUR‡

Abstract. Examples are presented to show that the solution of the operational algebraic Riccati equation can be an unbounded operator for infinite dimensional systems in a Hilbert space even with bounded control and observation operators. This phenomenon is connected to the presence of a continuous spectrum in one of the operators. The object of this paper is to fill up the gap in the classical linear quadratic theory. The key step is the introduction of the set of stabilizable initial conditions. Then a new simple approach to the linear-quadratic problem is presented that provides the connection with the notion of approximate stabilizability for the triplet (A, B, C).

Key words. linear quadratic, stabilizability, Riccati equation

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1. Introduction. The infinite time, linear quadratic, optimal control theory for infinite-dimensional systems in Hilbert spaces with bounded control and observation operators has been extensively studied (see, for instance, the book by Curtain and Pritchard [1]). Typically, let H (*state space*), U (*control space*), and Y (*observation space*) be three Hilbert spaces. Let $A: D(A) \subset H \to H$ be the infinitesimal generator of a strongly continuous semigroup e^{tA} and let $B: U \to H$ and $C: H \to Y$ be continuous linear operators. The state $x(t)$ at time $t \geqq 0$ is given by

$$x(t) = e^{tA}h + \int_0^\infty e^{(t-s)A}Bu(s)\, ds, \qquad t \geqq 0, \tag{1.1}$$

and the cost function by

$$J(u, h) = \int_0^\infty \{|Cx(s)|^2 + |u(s)|^2\}\, ds. \tag{1.2}$$

Under the standard (A, B, C) stabilizability hypothesis for the triplet (A, B, C),

$$\forall h \in H, \quad \exists u \in L^2(0, \infty; U) \quad \text{such that } J(u, h) < \infty, \tag{1.3}$$

it is well known that the corresponding algebraic operator Riccati equation

$$A^*P + PA - PBB^*P + C^*C = 0 \tag{1.4}$$

has a minimum positive symmetrical bounded solution $\underline{P}$; that is,

$$\underline{P}: H \to H \text{ is linear and continuous (bounded),} \tag{1.5}$$

$$\underline{P}^* = \underline{P} \text{ (symmetry)}, \quad \forall h \in H, (\underline{P}h, h) \geqq 0 \text{ (positivity)}, \tag{1.6}$$

and for any other solution of (1.4) verifying (1.5) and (1.6)

$$\forall h \in H, \quad (Qh, h) \geqq (\underline{P}h, h) \text{ (minimality)}. \tag{1.7}$$

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† Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa, Italy. The author has been partially supported by the Italian National Project M.P.I. Equazioni di Evoluzione e Applicazioni Fisico-Matematiche.

‡ Centre de Recherches Mathématiques et Département de Mathématiques et de Statistique, Université de Montréal, C.P. 6128 A, Montréal, Québec, Canada H3C 3J7. The author has been supported by a Killam fellowship from Canada Council, Natural Sciences and Engineering Research Council of Canada operating grant A-8730, and a Fonds pour la Formation de Chercheurs et l'Aide à la Recherche (le Fonds FCAR, Québec.) grant from the Ministère de l'Éducation du Québec.

The authors have recently constructed examples where the system is not stabilizable, and yet the algebraic Riccati equation has a positive selfadjoint unbounded solution (cf. [2]). This phenomenon is intimately related to the fact that only a dense subset of initial conditions are (A, B, C) stabilizable. This has many interesting implications for infinite-dimensional control systems. For instance, it points out that definitions of stabilizability (here, (A, B, I) stabilizability) that assume the existence of a bounded feedback operator really contain two hypotheses in one: the existence of a feedback operator that stabilizes all initial conditions in H, and the boundedness of this operator. Example 6.1 in § 6 describes a control system that can only be stabilized by an unbounded feedback operator for all initial conditions in H.

The object of this paper is to fill the gap in the theory. Under no stabilizability hypothesis, we a priori define the set Σ of initial states that can be (A, B, C) stabilized and show that it can be given a natural Hilbert space structure. When Σ is dense in the space of initial conditions, we construct the smallest or minimum positive selfadjoint unbounded solution to the algebraic Riccati equation. A new technique is introduced to directly obtain the semigroup associated with the closed loop system and the properties of the feedback operator. If the usual detectability hypothesis is added, we recover that the closed loop system is exponentially stable. Examples are also included to illustrate the theoretical considerations. Extensions to systems with unbounded control and observation operators are possible and will be reported in a forthcoming paper. We felt that it was more instructive to first illustrate the phenomenon and the main features of the theory for the bounded case.

Notation. The space of continuous linear operators from a Hilbert space X to another Hilbert space Y will be denoted by $\mathscr{L}(X; Y)$. When $X = Y$, the cone of continuous linear operators in $\mathscr{L}(X; X)$ verifying conditions (1.5) and (1.6) will be denoted $\Sigma^+(X)$. $\mathbf{R}$ will be the field of all real numbers and $\mathbf{N}$ the set of integers greater than or equal to 1.

2. Problem formulation. Let H, U, Y, A, B, and C be as defined in § 1. Consider the mild solution of the system

$$x'(s) = Ax(s) + Bu(s), \quad s \geqq 0, \qquad x(0) = h, \tag{2.1}$$

and the associated cost function

$$J(u, h) = \int_0^\infty \{|Cx(s)|^2 + |u(s)|^2\}\, ds. \tag{2.2}$$

A *mild solution* of (2.1) is a continuous function $x:[0, \infty[\to H$ verifying (1.1). Denote by V the value function

$$V(h) = \inf\{J(u, h): u \in L^2(0, \infty; U)\} \tag{2.3}$$

with domain

$$\operatorname{dom} V = \{h \in H: V(h) < \infty\}, \tag{2.4}$$

which will be referred to as the *domain of stabilizability* for the triple (A, B, C). Observe that under the (A, B, C) stabilizability condition (1.3) dom $V = H$.

3. An example of unbounded solution to the Riccati equation. Let $H = \ell^2$ be the Hilbert space of all sequences $x = \{x_n\}_{n\in\mathbf{N}}$, with norm

$$|x|^2 = \sum_{k=1}^{\infty} x_k^2. \tag{3.1}$$

Let $\{e_k\}$ be the orhonormal basis in ℓ^2

$$(e_k)_n = \delta_{kn}, \quad k \in \mathbf{N}. \tag{3.2}$$

Define the bounded operators

$$Ae_k = \frac{k}{k+1} e_k, \qquad Be_k = \frac{\sqrt{2k+1}}{k+1} e_k, \quad k \in \mathbf{N}. \tag{3.3}$$

Note that their spectra are made up of a point and a continuous part

$$\sigma_p(A) = \left\{ \frac{k}{k+1} : k \in \mathbf{N} \right\}, \qquad \sigma_c(A) = \{1\}, \tag{3.4}$$

$$\sigma_p(B) = \left\{ \frac{\sqrt{2k+1}}{k+1} : k \in \mathbf{N} \right\}, \qquad \sigma_c(B) = \{0\}. \tag{3.5}$$

Associate with A and B the control system

$$\begin{aligned} &x'(s) = Ax(s) + Bu(s), \quad s \geqq 0, \\ &x(0) = h, \end{aligned} \tag{3.6}$$

and the cost function

$$J(u, h) = \int_0^\infty \{|x(s)|^2 + |u(s)|^2\} \, ds. \tag{3.7}$$

Here the observation operator C is the identity operator on H. If the pair (A, B) was stabilizable, there would exist a symmetric positive bounded linear operator P_∞ on H that would be the minimum solution of the algebraic operator Riccati equation

$$P_\infty A + A^* P_\infty - P_\infty BB^* P_\infty + I = 0, \tag{3.8}$$

in the sense of conditions (1.5) and (1.6). Here A, B, and I are diagonal operators, and it is easy to check that the only positive selfadjoint solution to (3.8) is the diagonal unbounded operator

$$P_\infty e_k = (k+1) e_k, \qquad k \in \mathbf{N}. \tag{3.9}$$

This means that only initial conditions h in the domain $D(P_\infty^{1/2})$ of $P_\infty^{1/2}$

$$\begin{aligned} &D(P_\infty^{1/2}) = \left\{ x \in \ell^2 : \sum_{k=1}^\infty (k+1) x_k^2 < \infty \right\}, \\ &P_\infty^{1/2} e_k = \sqrt{k+1} \, e_k \end{aligned} \tag{3.10}$$

can be stabilized, and that for all others

$$J(u, h) = \infty, \qquad h \notin D(P_\infty^{1/2}). \tag{3.11}$$

Hence dom $V = D(P_\infty^{1/2})$ in this example.

The interpretation of this phenomenon is that, for initial conditions $h \notin D(P_\infty^{1/2})$, the corresponding state x cannot be stabilized with a finite energy control u in $L^2(0, \infty; H)$. Yet the closed loop system is given by the operator

$$A - BB^* P_\infty = -I, \tag{3.12}$$

which is exponentially stable in H, and for all h in H the solution x^* of the closed loop system

$$\begin{aligned} &x'(s) = [A - BB^* P_\infty] x(s), \quad s \geqq 0, \\ &x(0) = h, \end{aligned} \tag{3.13}$$

is given by $x^*(s)=e^{-s}h$, which belongs to $L^2(0,\infty;H)$, whereas the optimal control u^* is given by

$$u^*(s)=-B^*P_\infty x^*(s)=-B^*P_\infty e^{-s}h. \tag{3.14}$$

So u belongs to $L^2(0,\infty;H)$ if and only if $h\in D(P_\infty^{1/2})$.

Finally, it is useful to note that

$$B^*P_\infty e_k=\sqrt{2k+1}\,e_k,\qquad D(B^*P_\infty)=D(P_\infty^{1/2}) \tag{3.15}$$

and that

$$BB^*P_\infty e_k=\frac{2k+1}{k+1}\,e_k,\qquad D(BB^*P_\infty)=H. \tag{3.16}$$

4. Asymptotic behaviour of the solution $P(t)$ to the associated differential operator Riccati equation. It is well known that for any fixed $T>0$, we can associate with the control problem (2.1)-(2.2) the mild solution $P\in C_s([0,\infty[;\Sigma^+(H))$ of the differential operator Riccati equation

$$\begin{aligned}&P'(t)=A^*P(t)+P(t)A-P(t)BB^*P(t)+C^*C\quad\text{in }[0,T],\\&P(0)=0.\end{aligned} \tag{4.1}$$

We say that P in $C_s([0,T];\Sigma^+(H))$ is a *mild solution* of the Riccati differential equation (4.1) if P verifies the integral equation

$$P(t)x=\int_0^t\{e^{(t-s)A^*}[C^*C-P(s)BB^*P(s)]\,e^{(t-s)A}x\}\,ds$$

for all x in H (for example, see Curtain and Pritchard [1] for basic results on existence and uniqueness). We have denoted by $C_s([0,\infty[;\Sigma^+(H))$ the set of all mappings $T:[0,\infty[\to\Sigma^+(H)$, such that $T(\cdot)x$ is continuous for all $x\in H$.

For each $h\in H$ the function $(P(\cdot)h,h)$ is nondecreasing. Moreover, the following identity holds:

$$(P(t)h,h)+\int_0^t|u(s)+B^*P(t-s)x(s)|^2\,ds=\int_0^t\{|Cx(s)|^2+|u(s)|^2\}\,ds,\qquad\forall u\in L^2_{\mathrm{loc}}(0,\infty;U). \tag{4.2}$$

To obtain identity (4.2) fix $t>0$, multiply both sides of (4.1) evaluated at $t-s$, $t\geqq s\geqq 0$, by $x(s)$, use (2.1) to eliminate $x'(s)$, and integrate with respect to s from 0 to t.

Define the function

$$h\to\phi(h)=\lim_{t\to\infty}(P(t)h,h):H\to[0,\infty]. \tag{4.3}$$

The function ϕ is convex, proper,[1] and lower semicontinuous with domain

$$\Sigma=\{h\in H:\ \phi(h)<\infty\}. \tag{4.4}$$

LEMMA 4.1. *The following properties are verified*:

(i) *For all h and k in Σ, $(P(\cdot)h,k)$ is bounded*;

(ii) *Σ is a vector subspace of H*;

(iii) *For all h and k in Σ, the following limit exists*

$$\psi(h,k)=\lim_{t\to\infty}(P(t)h,k). \tag{4.5}$$

[1] A convex function $f:H\to[0,\infty]$ is said to be proper if $f(x)<\infty$ for at least one x and $f(x)>-\infty$ for every x (cf. R. T. Rockafellar [7, p. 24]).

Moreover, ψ is a bilinear form on $\Sigma\times\Sigma$ and

$$\psi(h, h)=\phi(h), \qquad \forall h\in H. \tag{4.6}$$

Proof. (i) For all h and k in Σ and $t\geqq 0$, we have

$$|(P(t)h, k)|^2\leqq(P(t)h, h)(P(t)k, k)\leqq\phi(h)\phi(k), \tag{4.7}$$

and the conclusion follows.

(ii) For all h in Σ and λ in $\mathbf{R}$, $\phi(\lambda h)=\lambda^2\phi(h)$, and hence $\lambda h\in\Sigma$. For all h and k in Σ

$$(P(t)(h+k), h+k)=(P(t)h, h)+(P(t)k, k)+2(P(t)h, k)$$

and from (i), $\phi(h+k)\leqq[\phi(h)^{1/2}+\phi(k)^{1/2}]^2$. Thus Σ is a linear subspace of H. Part (iii) is an immediate consequence of parts (i) and (ii). □

Define the following inner product on Σ

$$(h, k)_\Sigma=(h, k)+\psi(h, k), \tag{4.8}$$

which makes it a pre-Hilbert space.

LEMMA 4.2. *The space Σ endowed with the inner product* (4.8) *is a Hilbert space.*

Proof. It is sufficient to show that Σ is complete with respect to the norm

$$|h|_\Sigma=[|h|^2+\phi(h)]^{1/2}. \tag{4.9}$$

Let $\{h_n\}$ be a Cauchy sequence in Σ. Then there exists $h\in H$ such that $h_n\to h$. Moreover, there exists $\lambda\geqq 0$ such that

$$|h_n|^2+\phi(h_n)\to\lambda \tag{4.10}$$

and

$$\phi(h_n)\to\lambda-|h|^2. \tag{4.11}$$

By lower semicontinuity of ϕ, we have

$$\lambda-|h|^2=\lim_{n\to\infty}\phi(h_n)\geqq\phi(h), \tag{4.12}$$

and, by definition of Σ, h belongs to Σ. Finally, for each $\varepsilon>0$, there exists a positive integer $N(\varepsilon)$ such that

$$|h_n-h_m|_\Sigma^2=|h_n-h_m|^2+\phi(h_n-h_m)\leqq\varepsilon, \qquad \forall m, n\geqq N(\varepsilon).$$

As n goes to infinity, we get

$$|h-h_m|^2+\phi(h-h_m)\leqq\varepsilon, \qquad \forall m\geqq N(\varepsilon),$$

by continuity of the norm in H and lower semicontinuity of ϕ. This shows that $h_n\to h$ in Σ and completes the proof. □

We have constructed the space Σ of initial conditions for which the expression $(P(t)h, h)$ has a limit. In general, its closure in H will not be dense, and it will be natural to decompose H as a direct sum

$$H=\bar{\Sigma}\oplus\Sigma^\perp, \tag{4.13}$$

where $\bar{\Sigma}$ is the closure of Σ in H, and $\Sigma^\perp$ is the orthogonal complement to Σ in H. In the following, we identify the elements of the dual H' of H with those of H. We denote by Σ' the dual of Σ.

PROPOSITION 4.3. *Assume that Σ is dense in H. Then there exists a unique linear operator $P_\infty \in \mathscr{L}(\Sigma; \Sigma')$ such that*

$$\langle P_\infty h, k\rangle_\Sigma = \psi(h, k), \qquad \forall h, k \in \Sigma. \tag{4.14}$$

P_∞ can also be viewed as a closed selfadjoint positive operator on H with dense domain

$$D(P_\infty) = \{h \in \Sigma: \psi(h, \cdot) \text{ is continuous in } H\}. \tag{4.15}$$

We have

$$\phi(h) = (P_\infty h, h), \quad \forall h \in D(P_\infty), \tag{4.16}$$

$$\psi(h, k) = (P_\infty h, k), \qquad \forall h \in D(P_\infty), \forall k \in H, \tag{4.17}$$

and the subdifferential of ϕ is given by

$$\tfrac{1}{2}\partial\phi(h) = \begin{cases} P_\infty h, & \text{if } h \in D(P_\infty),\\ \varnothing, & \text{if } h \notin D(P_\infty);\end{cases} \tag{4.18}$$

that is,

$$\partial\phi(h) = \{p \in H \,|\, \forall v \in D(P_\infty), \langle p, v\rangle \leqq d\phi(h; v)\},$$

where $d\phi(h; v)$ is the Gâteaux semiderivative at h in the direction v.

Moreover, $P_\infty^{1/2}$ is well defined and

$$D(P_\infty^{1/2}) = \Sigma = [D(P_\infty), H]_{1/2}, \tag{4.19}$$

where $[X, Y]_{1/2}$ denotes the interpolation space between Y and its dense subspace X (see Lions and Peetre [6] or Lions and Magenes [5] for the theory of interpolation spaces).

Proof. By definition of the inner product on Σ, the symmetrical bilinear form ψ on $\Sigma \times \Sigma$ is continuous, and there exists a unique $P_\infty \in \mathscr{L}(\Sigma; \Sigma')$ such that (4.14) is verified. Moreover, ψ is Σ-H coercive and P_∞ is a self-adjoint operator in H with domain $D(P_\infty)$. Expression (4.18) follows from the fact that ϕ is lower semicontinuous. Hence $\partial\phi(\cdot)$ is maximal monotone on H as a set-valued function. Finally, the positive self-adjoint operator P_∞ has a positive square root $P_\infty^{1/2}$, which is a closed linear operator on H with dense domain $D(P_\infty^{1/2})$, which coincides with Σ. $\square$

Assume now that Σ is not dense in H, and denote by $\bar{\Sigma}$ the closure of Σ in H. Then we have the following similar result.

COROLLARY. *There exists a unique linear operator $P_\infty \in \mathscr{L}(\Sigma; \Sigma')$ such that*

$$\langle P_\infty h, k\rangle_\Sigma = \psi(h, k), \quad \forall h, k \in \Sigma. \tag{4.20}$$

P_∞ can also be viewed as a closed selfadjoint positive operator on $\bar{\Sigma}$ with dense domain

$$D(P_\infty) = \{h \in \Sigma: \psi(h, \cdot) \text{ is continuous in } \bar{\Sigma}\}. \tag{4.21}$$

We have

$$\phi(h) = (P_\infty h, h), \quad \forall h \in D(P_\infty), \tag{4.22}$$

$$\psi(h, k) = (P_\infty h, k), \quad \forall h \in D(P_\infty), \quad \forall k \in \bar{\Sigma}, \tag{4.23}$$

and the subdifferential of ϕ is given by

$$\tfrac{1}{2}\partial\phi(h) = \begin{cases} P_\infty h, & \text{if } h \in D(P_\infty),\\ \varnothing, & \text{if } h \notin D(P_\infty).\end{cases} \tag{4.24}$$

Moreover, $P_\infty^{1/2}$ is well defined and

$$D(P_\infty^{1/2}) = \Sigma = [D(P_\infty), \bar{\Sigma}]_{1/2}. \quad \square \tag{4.25}$$

5. Existence of the optimal control and optimal closed loop system. In this section we use the asymptotic properties obtained in § 4 to solve the optimal control problem (2.1)–(2.2). In addition, we study the mapping between the initial conditions and the optimal state and control.

THEOREM 5.1. *The following statements hold:*

(i) *Given any h in H, either $h \notin \Sigma$ and*

$$J(u, h) = +\infty, \quad \forall u \in L^2(0, \infty; U) \quad \text{and} \quad V(h) = \phi(h) = +\infty, \tag{5.1}$$

or $h \in \Sigma$ and there exists a unique optimal control $\hat{u}(\cdot, h)$ in $L^2(0, \infty; U)$ such that

$$J(\hat{u}(\cdot, h), h) = V(h) = \phi(h). \tag{5.2}$$

(ii) *The mapping*

$$\Sigma \to L^2(0, \infty; U), h \to \hat{u}(\cdot, h) \tag{5.3}$$

is linear and continuous.

(iii) *Denote by $\hat{x}(\cdot, h)$ the optimal state corresponding to the optimal control $\hat{u}(\cdot, h)$ and set*

$$S_\Sigma(t)h = \hat{x}(t, h), \qquad t \geqq 0, \quad h \in \Sigma. \tag{5.4}$$

Then $S_\Sigma(\cdot)$ is a strongly continuous semigroup in Σ.

(iv) *Let A_Σ be the infinitesimal generator of $S_\Sigma(\cdot)$. For all $h \in D(A_\Sigma)$, we have that*

$$\hat{u}(\cdot, h) \in H^1(0, \infty, U) \quad \text{and} \quad \hat{u}'(\cdot, h) = \hat{u}(\cdot, A_\Sigma h), \tag{5.5}$$

$$C\hat{x}(\cdot, h) \in H^1(0, \infty; Y), \quad \hat{x}'(\cdot, h) = \hat{x}(\cdot, A_\Sigma h), \quad \text{and} \quad D(A_\Sigma) \subset D(A). \tag{5.6}$$

(v) *For all h in $D(A_\Sigma)$ the map*

$$h \to \hat{u}(0, h) : D(A_\Sigma) \to U \tag{5.7}$$

is linear and continuous. Its closure in Σ generates an unbounded linear operator

$$K : D(K) \subset \Sigma \to U \quad \text{such that } D(A_\Sigma) \subset D(K) \tag{5.8}$$

and

$$D(A_\Sigma) = D(A) \cap D(K), \qquad A_\Sigma h = Ah + BKh. \tag{5.9}$$

Moreover, for all h in $D(A_\Sigma)$ and $t \in [0, \infty[$, $\hat{x}(t, h) \in D(A_\Sigma)$,

$$A_\Sigma \hat{x}(t, h) = A\hat{x}(t, h) + B\hat{u}(t, h) = [A + BK]\hat{x}(t, h), \tag{5.10}$$

$$\hat{u}(t, h) = K\hat{x}(t, h). \tag{5.11}$$

(vi) *For all h in $D(A_\Sigma)$,*

$$Kh = \lim_{t \to \infty} -B^* P(t)h, \tag{5.12}$$

and for all h in Σ and almost all t in $[0, \infty[$

$$\hat{u}(t, h) = K\hat{x}(t, h), \qquad \hat{x}(t, h) \in D(K).$$

When $\bar{\Sigma} = H$ the closure K_∞ of the operator $-B^* P_\infty$ in Σ coincides with K on $D(A_\Sigma)$.

Proof. (i). By definition of Σ, for all $h \notin \Sigma$ $\lim_{t \to \infty} (P(t)h, h) = \infty$ and, in view of identity (4.2),

$$(P(t)h, h) \leqq J(u, h), \quad \forall u \in L^2(0, \infty; U), \quad \forall t \geqq 0.$$

By letting t go to infinity, we obtain (5.1). When $h \in \Sigma$, identity (4.2) yields

$$\phi(h) \leqq J(u, h), \quad \forall u \in L^2(0, \infty; U).$$

For each $t > 0$, let (x_t, u_t) be defined by

$$\begin{aligned} &x_t'(s) = Ax_t(s) - BB^*P(t-s)x_t(s), \quad \text{in } [0, t], \qquad x_t(0) = h, \\ &u_t(s) = -B^*P(t-s)x_t(s), \quad \text{in } [0, t]. \end{aligned} \tag{5.13}$$

The pair (x_t, u_t) is the optimal solution on the interval $[0, t]$. Consider the extension $\hat{u}_t$ of u_t from $[0, t]$ to $[0, \infty[$

$$\hat{u}_t(s) = \begin{cases} u_t(s), & \text{if } 0 \leqq s \leqq t, \\ 0, & \text{if } s > t, \end{cases} \tag{5.14}$$

and let $\hat{x}_t$ be the corresponding extension of the solution x_t of the state equation on $[0, t]$

$$\hat{x}_t(s) = \begin{cases} x_t(s), & \text{if } 0 \leqq s \leqq t, \\ 0, & \text{if } s > t. \end{cases}$$

Again by (4.2) and (5.14)

$$(P(t)h, h) = \int_0^t \{|Cx_t(s)|^2 + |u_t(s)|^2\}\, ds \geqq \int_0^\infty |\hat{u}_t(s)|^2\, ds. \tag{5.15}$$

Hence for any sequence $\{t_n\}$, $t_n \to \infty$, the sequence $\{\hat{u}_{t_n}\}$ is bounded in $L^2(0, \infty; U)$. So there exists $\hat{u}$ in $L^2(0, \infty; U)$ and a subsequence of $\{t_n\}$ (still denoted $\{t_n\}$) such that

$$\hat{u}_{t_n} \to \hat{u} \quad \text{in } L^2(0, \infty; U)\text{-weak}. \tag{5.16}$$

Denote by $\hat{x}$ the solution of

$$\hat{x}'(s) = A\hat{x}(s) + B\hat{u}(s), \quad \text{for } s \geqq 0, \quad \hat{x}(0) = h. \tag{5.17}$$

Then for any fixed $T > 0$ and $t_n > T$

$$\hat{u}_{t_n} \to \hat{u} \quad \text{in } L^2(0, T; U)\text{-weak}, \qquad \hat{x}_{t_n} \to \hat{x} \quad \text{in } L^2(0, T; H)\text{-weak}.$$

For $t_n > T$, however,

$$(P(t_n)h, h) \geqq \int_0^T \{|Cx_{t_n}(s)|^2 + |u_{t_n}(s)|^2\}\, ds$$

and by weak lower semicontinuity

$$\phi(h) \geqq \int_0^T \{|C\hat{x}(s)|^2 + |\hat{u}(s)|^2\}\, ds.$$

As T goes to infinity

$$\phi(h) \geqq \int_0^\infty \{|C\hat{x}(s)|^2 + |\hat{u}(s)|^2\}\, ds = J(\hat{u}, h). \tag{5.18}$$

Combining (5.18) and (5.13) it follows that there exists $\hat{u} = \hat{u}(\cdot, h) \in L^2(0, \infty; U)$ such that

$$J(\hat{u}, h) \leqq \phi(h) \leqq J(u, h), \quad \forall u \in L^2(0, \infty; U).$$

It follows that $V(h) \leqq J(\hat{u}, h) \leqq \phi(h) \leqq V(h)$. This establishes (5.2). As for the uniqueness of $\hat{u}$, assume that $\hat{u}_1$ and $\hat{u}_2$ are two optimal controls in $L^2(0, \infty; U)$. Then $J(\hat{u}_1, h) = J(\hat{u}_2, h) = V(h)$. So for $\hat{u}_1 \neq \hat{u}_2$

$$\begin{aligned} J((\hat{u}_1 + \hat{u}_2)/2, h) &= \tfrac{1}{2}[J(\hat{u}_1, h) + J((\hat{u}_2, h)] - J((\hat{u}_1 - \hat{u}_2)/2, h) \\ &= V(h) - J((\hat{u}_1 - \hat{u}_2)/2, h) \leqq V(h) - \tfrac{1}{4}\|\hat{u}_1 - \hat{u}_2\|^2 < V(h), \end{aligned}$$

which contradicts the optimality of $\hat{u}_1$ and $\hat{u}_2$.

(ii) Let $\hat{u}_t$ be defined by (5.13), then

$$\|\hat{u}_t\|^2_{L^2(0,\infty; U)} \leqq (P(t)h, h) \leqq |h|^2_\Sigma. \tag{5.19}$$

Moreover, since the optimal control is unique, we have proved in part (i) that

$$\lim_{t\to\infty} \hat{u}_t = \hat{u} \quad \text{in } L^2(0, \infty; U)\text{-weak}, \quad \text{for any } h \in \Sigma.$$

We now prove that $\hat{u}_t \to \hat{u}$ in $L^2(0, \infty; U)$-strong. By optimality of the pair (x_t, u_t) on $[0, t]$

$$J^t(u_t, h) = \inf\{J^t(v, h) \colon v \in L^2(0, \infty; U)\},$$

where

$$J^t(v, h) = \int_0^t \{|Cx(s; v)|^2 + |v(s)|^2\}\, ds.$$

We want to prove that $\lim_{t\to\infty} J^t(u_t, h) = J(\hat{u}, h)$. By definition of the minimizing element u_t on $[0, t]$

$$J^t(u_t, h) \leqq J^t(\hat{u}(\cdot, h), h) = \int_0^t \{|C\hat{x}(s, \hat{u}(\cdot, h))|^2 + |\hat{u}(s, h)|^2\}\, ds$$

and necessarily

$$\limsup_{t\to\infty} J^t(u_t, h) \leqq \int_0^\infty \{|C\hat{x}(s, \hat{u}(\cdot, h))|^2 + |\hat{u}(s, h)|^2\}\, ds = J(\hat{u}(\cdot, h), h).$$

We have shown in (i) that $\hat{u}_t \to \hat{u}$, in $L^2(0, \infty; U)$-weak, and we can show by the same technique that $\{C\hat{x}_t\}$ is bounded in $L^2(0, \infty; Y)$, and that weak subsequences $\{C\hat{x}_{t_n}\}$ converging to some y in $L^2(0, \infty; Y)$ can be extracted as follows:

$$C\hat{x}_{t_n} \to y, \quad \text{in } L^2(0, \infty; Y)\text{-weak}.$$

By continuity of the state $x(\cdot; u)$ with respect to the control u on a finite time interval $[0, T]$, $T > 0$, the map $u \to x(\cdot; u) : L^2(0, T; U) \to L^2(0, T; H)$ is weakly continuous and, finally,

$$u \to Cx(\cdot; u) : L^2(0, T; U) \to L^2(0, T; Y)$$

is also weakly continuous. This implies that for all $T > 0$, $y = C\hat{x}(\hat{u}, h)$ in $L^2(0, T; Y)$ and hence in $L^2(0, \infty; Y)$. As a result,

$$\hat{u}_t \to \hat{u}, \quad \text{in } L^2(0, \infty; U)\text{-weak} \quad \text{and} \quad C\hat{x}_t \to C\hat{x}, \quad \text{in } L^2(0, \infty; Y)\text{-weak}.$$

The functional

$$(v, y) \to \int_0^\infty \{|y(s)|^2 + |v(s)|^2\}\, ds) : L^2(0, \infty; U) \times L^2(0, \infty; Y) \to \mathbf{R}$$

is, however, weakly lower semicontinuous and necessarily

$$\liminf_{t\to\infty}\int_0^\infty [|C\hat{x}_t|^2+|\hat{u}_t|^2]\,ds \geqq \int_0^\infty [|C\hat{x}|^2+|\hat{u}|^2]\,ds;$$

that is,

$$\liminf_{t\to\infty} J^t(u_t, h) \geqq J(\hat{u}, h).$$

Finally,

$$J(\hat{u}, h) \leqq \liminf_{t\to\infty} J^t(u_t, h) \leqq \limsup_{t\to\infty} J^t(u_t, h) \leqq J(\hat{u}, h),$$

and this proves that $\lim_{t\to\infty} J^t(u_t, h) = J(\hat{u}, h)$.

The strong continuity will now be obtained by the following simple computation:

$$\begin{aligned}\|C\hat{x}_t - C\hat{x}\|^2 + \|\hat{u}_t - u\|^2 &= \|C\hat{x}_t\|^2 + \|\hat{u}_t\|^2 + \|C\hat{x}\|^2 + \|\hat{u}\|^2 - 2(C\hat{x}_t, C\hat{x}) - 2(\hat{u}_t, \hat{u})\\ &= J^t(u_t, h) + J(\hat{u}, h) - 2(C\hat{x}_t, C\hat{x}) - 2(\hat{u}_t, \hat{u}).\end{aligned}$$

As t goes to ∞, $J^t(u_t, h) \to J(u, h)$ and, by weak convergence,

$$(C\hat{x}_t, C\hat{x}) \to (C\hat{x}, C\hat{x}) = \|C\hat{x}\|^2 \quad \text{and} \quad (\hat{u}_t, \hat{u}) \to (\hat{u}, \hat{u}) = \|\hat{u}\|^2.$$

So we conclude that

$$\lim_{t\to\infty} \{\|C\hat{x}_t - C\hat{x}\|^2 + \|\hat{u}_t - u\|^2\} = 2J(\hat{u}, h) - 2[\|C\hat{x}\|^2 + \|\hat{u}\|^2] = 0$$

and that

$$\hat{u}_t \to \hat{u}, \quad \text{in } L^2(0, \infty; U)\text{-strong} \quad \text{and} \quad C\hat{x}_t \to C\hat{x}, \quad \text{in } L^2(0, \infty; Y)\text{-strong}.$$

By (5.18) and by the uniform boundedness theorem, it follows that the mapping $h \to \hat{u}(\cdot, h): \Sigma \to L^2(0, \infty; U)$ is linear and continuous.

(iii) First, note that, by Bellman's optimality principle, we have $\hat{x}(t, h) \in \Sigma$ for all $h \in \Sigma$ and

$$\hat{x}(t+s, h) = \hat{x}(t; \hat{x}(s, h)), \quad \forall t \geqq 0, \quad \forall s \geqq 0, \tag{5.20}$$

$$V(\hat{x}(t, h)) = \int_t^\infty \{|C\hat{x}(s, h)|^2 + |\hat{u}(s, h)|^2\}\,ds. \tag{5.21}$$

Thus $S_\Sigma(t)$ is a linear operator in Σ for all $t \geqq 0$. We prove now that $S_\Sigma(t)$ is bounded in Σ. By (5.17) we have

$$\hat{x}(t, h) = e^{tA}h + \int_0^t e^{(t-s)A}B\hat{u}(s, h)\,ds. \tag{5.22}$$

It follows that for any $T > 0$ there exists $C_T > 0$ such that

$$|\hat{x}(t, h)|_H^2 \leqq C_T |h|_H^2, \qquad 0 \leqq t \leqq T. \tag{5.23}$$

Moreover, from (5.21), $\phi(\hat{x}(t, h)) \leqq \phi(h)$ and the continuity of $S_\Sigma(t)$ follows. We now prove that $\lim_{t\to 0} \hat{x}(t, h) = h$, $\forall h \in \Sigma$. By (5.22) we have $\lim_{t\to 0} \hat{x}(t, h) = h$ in H. It remains to show that $\hat{x}(t)$ is continuous at $t = 0$ with respect to the seminorm $\phi(h)^{1/2}$. By the linearity of $\hat{x}(\cdot, h)$ and $\hat{u}(\cdot, h)$ in h, we have

$$\phi(\hat{x}(t, h) - h) = \int_0^\infty \{|C\hat{x}(t+s, h) - C\hat{x}(s, h)|^2 + |\hat{u}(t+s, h) - \hat{u}(s, h)|^2\}\,ds.$$

Since $C\hat{x}(\cdot, h) \in L^2(0, \infty; Y)$ and $\hat{u}(\cdot, h) \in L^2(0, \infty; U)$, we have $\lim_{t\to 0} \phi(\hat{x}(t, h) - h) = 0$. This proves (iii).

(iv) For any $h \in D(A_\Sigma)$, $\hat{x}(\cdot, h) \in C^1([0, \infty[; \Sigma)$ and $\hat{x}'(0, h) = A_\Sigma h$. Denote by $\hat{w}(\cdot) = \hat{u}(\cdot, A_\Sigma h)$ the optimal control corresponding to $A_\Sigma h$. So for all $t > 0$

$$\phi\left[\frac{\hat{x}(t, h) - h}{t} - A_\Sigma h\right] = \int_0^\infty \left\{\left|\frac{C\hat{x}(t+s, h) - C\hat{x}(s, h)}{t} - C\hat{x}'(s, h)\right|^2 + \left|\frac{\hat{u}(t+s, h) - \hat{u}(s, h)}{t} - \hat{w}(s)\right|^s\right\} ds.$$

As t goes to zero, the first two terms go to zero and necessarily

$$\lim_{t\to 0} \int_0^\infty \left|\frac{\hat{u}(t+s) - \hat{u}(s)}{t} - \hat{w}(s)\right|^2 ds = 0,$$

which implies $\hat{w} = \hat{u}'$ and $\hat{u} \in H^1(0, \infty; U)$, $\forall h \in D(A_\Sigma)$. By (5.22) we conclude that $h \in D(A)$, and (5.6) follows.

(v) We have shown in (ii) that the map $h \to \hat{u}(\cdot, h): \Sigma \to L^2(0, \infty; U)$ is linear and continuous. In particular,

$$h \to \hat{u}'(\cdot, h) = \hat{u}(\cdot, A_\Sigma h): D(A_\Sigma) \to L^2(0, \infty; U)$$

is also continuous. Hence

$$h \to \hat{u}(\cdot, h): D(A_\Sigma) \to H^1(0, \infty; U)$$

is linear and continuous when $D(A_\Sigma)$ is endowed with the following graph norm topology:

$$\|h\|^2_{D(A_\Sigma)} = \|h\|^2_\Sigma + \|A_\Sigma h\|^2.$$

In particular, $\hat{u}(\infty) = 0$, $\hat{u} \in C([0, \infty]; U)$ and the map $h \to \hat{u}(0, h): D(A_\Sigma) \to U$ is linear and continuous. We denote it by K. Equivalently, K is a closed linear unbounded operator from Σ to U with domain

$$D(K) = \{h \in \Sigma \colon Kh \in U\} \supset D(A_\Sigma).$$

In view of this and identity (5.6)

$$\forall h \in D(A_\Sigma), \quad A_\Sigma h = Ah + B\hat{u}(0, h) = [A + BK]h.$$

Conversely, if $h \in D(A) \cap D(K)$, then

$$A_\Sigma h = Ah + BKh \Rightarrow h \in D(A_\Sigma),$$

and $D(A_\Sigma) = D(A) \cap D(K)$.

(vi) To relate K and the limit of $P(t)$, we go back to formula (4.2) with $h \in \Sigma$, $u = \hat{u}(\cdot, h)$ and $x = \hat{x}(\cdot, h)$:

$$\langle P(t)h, h\rangle_\Sigma + \int_0^t |\hat{u}(s, h) + B^*P(t-s)\hat{x}(s, h)|^2\, ds = \int_0^t \{|C\hat{x}(s, h)|^2 + |\hat{u}(s, h)|^2\}\, ds. \tag{5.24}$$

As t goes to infinity, we obtain

$$\lim_{t\to\infty}\int_0^t |\hat{u}(s,h)+B^*P(t-s)\hat{x}(s,h)|^2\,ds=0.$$

Setting $P(r)=0$ for $r\leqq 0$, then

$$\lim_{t\to\infty}\int_0^\infty |\hat{u}(s,h)+B^*P(t-s)\hat{x}(s,h)|^2\,ds=0, \tag{5.25}$$

since

$$\lim_{t\to\infty}\int_t^\infty |\hat{u}(s,h)|^2\,ds=0.$$

Now repeat the same estimate with $A_\Sigma h$ instead of h and $\hat{x}(\cdot,A_\Sigma h)=\hat{x}'(\cdot,h)$, $\hat{u}(\cdot,A_\Sigma h)=\hat{u}'(\cdot,h)$. Then by the same argument

$$\lim_{t\to\infty}\int_0^\infty |\hat{u}'(s,h)+B^*P(t-s)\hat{x}'(s,h)|^2\,ds=0. \tag{5.26}$$

Introduce the notation, and use (5.25) and (5.26) as follows:

$$v_t(s)=\hat{u}(s,h)+B^*P(t-s)\hat{x}(s,h),\qquad v_t\to 0\quad\text{in } L^2(0,\infty;U),$$
$$w_t(s)=\hat{u}'(s,h)+B^*P(t-s)\hat{x}'(s,h),\qquad w_t\to 0\quad\text{in } L^2(0,\infty;U).$$

For h in $D(A_\Sigma)$, differentiate (5.24) with respect to t

$$\frac{d}{dt}\langle P(t)h,h\rangle+|\hat{u}(0,h)+B^*P(t)h|^2+2\int_0^t (v_t(s),w_t(s))\,ds=|C\hat{x}(t,h)|^2+|\hat{u}(t,h)|^2.$$

For $t'\geqq t$, however,

$$\langle P(t')h,h\rangle-\langle P(t)h,h\rangle\geqq 0\Rightarrow\frac{d}{dt}\langle P(t)h,h\rangle\geqq 0,$$

and note that

$$\lim_{t\to\infty}\int_0^t (v_t(s),w_t(s))\,ds=\lim_{t\to\infty}\int_0^\infty (v_t(s),w_t(s))\,ds\to 0\quad\text{as } t\to\infty.$$

Hence

$$0\leqq\limsup_{t\to\infty}\frac{d}{dt}<P(t)h,h\rangle\leqq\limsup_{t\to\infty}\{|C\hat{x}(t,h)|^2+|\hat{u}(t,h)|^2\},$$
$$0\leqq\limsup_{t\to\infty}|\hat{u}(0,h)+B^*P(t)h|^2\leqq\limsup_{t\to\infty}\{|C\hat{x}(t,h)|^2+|\hat{u}(t,h)|^2\},$$

and the lim inf are positive. Recall, however, that $C\hat{x}(\cdot,h)\in H^1(0,\infty;Y)$ and $\hat{u}(\cdot,h)\in H^1(0,\infty;U)$, and this implies that $\lim_{t\to\infty}C\hat{x}(\cdot,h)=0$ and $\lim_{t\to\infty}\hat{u}(\cdot,h)=0$, and the limit of the two terms exists and is equal to 0. So, finally, for all h in $D(A_\Sigma)$ $Kh=\lim_{t\to\infty}[-B^*P(t)h]$. □

Remark 5.1. Theorem 5.1 shows that

$$V(h)=\phi(h)=V(\hat{x}(0,h))=\int_0^\infty\{|C\hat{x}(s,h)|^2+|\hat{u}(s,h)|^2\}\,ds. \tag{5.27}$$

Hence dom $V = \text{dom}\ \phi = \Sigma$, and Σ coincides with the domain of stabilization of the triple (A, B, C).

Moreover, by the linearity of $\hat{x}(s, h)$ and $\hat{u}(s, h)$ in h, it follows that

$$\psi(h, k) = \int_0^\infty \{(C\hat{x}(s, h), C\hat{x}(s, k)) + (\hat{u}(s, h), \hat{u}(s, k))\}\, ds, \quad \forall h, k \in \Sigma. \tag{5.28}$$

6. The algebraic Riccati equation. Recall that we have identified the elements of the dual H' of H with those of H. Our first task is to give a meaning to a solution of the operator algebraic Riccati equation. Let Q be a positive selfadjoint closed linear operator from H to H with a dense domain $D(Q)$. Define

(6.1) $\Sigma_Q = D(Q^{1/2})$ endowed with its graph norm topology,

(6.2) $A_Q = A - BB^*Q$ on $D(A) \cap D(Q)$, and

(6.3) $\bar{A}_Q$ = closure of A_Q in Σ_Q (closure of the graph of A_Q in $\Sigma_Q \times \Sigma_Q$).

DEFINITION 6.1. We say that a positive selfadjoint closed linear operator Q with dense domain in H is a solution of the operator algebraic Riccati equation if

(i) $\bar{A}_Q$ is the infinitesimal generator of a strongly continuous semigroup $\{S_Q(t)\}$ of class C_0 on Σ_Q, and

(ii) Q verifies the following equation:

$$\begin{gathered}(Qh, Ak) + (Qk, Ah) - (B^*Qh, B^*Qk)_U + (Ch, Ck)_Y = 0, \\ \forall h, k \in D(A) \cap D(Q).\end{gathered} \tag{6.4}$$

DEFINITION 6.2. We say that the triplet (A, B, C) is *approximately stabilizable* (respectively, *stabilizable*) if $\bar{\Sigma} = H$ (respectively, $\Sigma = H$).

Remark 6.1. Note that our definition of approximate stabilizability does not assume the existence of a bounded linear feedback operator. In the literature on the control of infinite dimensional systems, many papers use a definition of stabilizability that assumes the existence of a bounded feedback operator (cf., for instance, Jacobson and Nett [8]). As we will see in Example 6.1, there are simple control systems for which there exists only an unbounded feedback operator, which makes the closed loop system stable for all initial conditions in the state space H. So for infinite dimensional control systems a hypothesis using the existence of a bounded feedback really contains two hypotheses in one. To clarify this question we would have to systematically go over this literature. However, this is not the objective of this paper.

Proposition 6.1. (i) If the triplet (A, B, C) is approximately stabilizable, then the operator P_∞ on H defined by (4.15) is a positive selfadjoint closed linear solution of the operator algebraic Riccati equation (6.4). Moreover, for any other positive selfadjoint closed linear solution Q to (6.4), P_∞ is the minimum solution, that is,

$$D(Q^{1/2}) \subset D(P_\infty^{1/2}), \quad \text{and} \quad \forall h \in D(Q), (Qh, h) \geqq (P_\infty h, h). \tag{6.5}$$

(ii) The operator algebraic Riccati equation (6.4) has a positive selfadjoint solution in the sense of Definition 6.1 if and only if the triplet (A, B, C) is approximately stabilizable.

Proof. (i) Recall that from (5.28) for all h and k in $D(A_\Sigma)$

$$\langle P_\infty A_\Sigma h, k\rangle = \int_0^\infty \{(C\hat{x}'(s, h), C\hat{x}(s, k))_Y + (\hat{u}'(s, h), \hat{u}(s, k))_U\}\, ds,$$

$$\langle P_\infty h, A_\Sigma k\rangle = \int_0^\infty \{(C\hat{x}(s, h), C\hat{x}'(s, k))_Y + (\hat{u}(s, h), \hat{u}'(s, k))_U\}\, ds.$$

Now $C\hat{x}(\cdot, h)$ and $C\hat{x}(\cdot, k)$ belong to $H^1(0, \infty; Y)$; $\hat{u}(\cdot, h)$ and $\hat{u}(\cdot, k)$ belong to $H^1(0, \infty; Y)$, and their limits as t goes to infinity are 0. Therefore

$$\begin{aligned}\langle P_\infty A_\Sigma h, k\rangle + \langle P_\infty h, A_\Sigma k\rangle &= \int_0^\infty \frac{d}{dt}\{(C\hat{x}(s, h), C\hat{x}(s, k))_Y + (\hat{u}(s, h), \hat{u}(s, k))_U\}\, ds \\ &= -(Ch, Ck)_Y - (\hat{u}(0, h), \hat{u}(0, k))_U.\end{aligned}$$

In view of expression (5.10) to (5.12) in Theorem 5.1 we readily obtain (6.4) by specializing to h and k in $D(Q) \cap D(A)$.

Let Q be another posiitve selfadjoint solution of the operator algebraic Riccati equation (6.4). Then we can rearrange the terms in the following way:

$$([A - BB^*Q]h, Qk) + (Qh, [A - BB^*Q]k) + (B^*Qh, B^*Qk)_U + (Ch, Ck)_Y = 0, \qquad \forall h, k \in D(Q) \cap D(A).$$

By hypothesis

$$(6.6) \qquad (\bar{A}_Q h, Qk) + (Qh, \bar{A}_Q k) + (B^*Qh, B^*Qk)_U + (Ch, Ck)_Y = 0$$

and

$$(B^*Qh, B^*Qk)_U = -[(Ch, Ck)_Y - (Q^{1/2}\bar{A}_Q h, Q^{1/2}k) - (Q^{1/2}h\, Q^{1/2}\bar{A}_Q k)].$$

However, $D(Q) \cap D(A) \subset D(\bar{A}_Q)$ and, by linearity and density, the above equation extends to all h and k in $D(\bar{A}_Q)$. In particular, the operator $K_Q = -B^*Q$ has a continuous linear extension $\bar{K}_Q$: $D(\bar{K}_Q) \subset H \to U$ such that $D(\bar{K}_Q) \supset D(\bar{A}_Q)$.

For all h in $D(\bar{A}_Q)$,

$$2(\bar{A}_Q S_Q(t)h, Qh) + |B^*QS_Q(t)h|_U^2 + |CS_Q(t)h|_Y^2 = 0$$

and for all $t \geqq 0$

$$|Q^{1/2}S_Q(t)h|^2 + \int_0^t \{|B^*QS_Q(s)h|_U^2 + |CS_Q(s)h|_Y^2\}\, ds = |Q^{1/2}h|^2.$$

Therefore

$$(6.7) \qquad \forall t \geqq 0, \quad \forall h \in D(\bar{A}_Q), \quad \int_0^t \{|u_Q(s)|_U^2 + |Cx_Q(s)|_Y^2\}\, ds \leqq |Q^{1/2}h|^2,$$

where

$$u_Q(s) = -B^*Qx_Q(s) \quad \text{and} \quad x_Q(s) = S_Q(s)h, \quad s \geqq 0.$$

Using the monotone increasing property of the integral, inequality (6.7) holds with $t = \infty$ and extends to all h in $D(Q^{1/2})$. Recall that for all h in $\Sigma = D(P_\infty^{1/2})$

$$\int_0^\infty \{|C\hat{x}(s, h)|^2 + |\hat{u}(s, h)|^2\}\, ds) = |P_\infty^{1/2}h|^2$$

for the control, and state

$$\hat{u}(s, h) = -B^*Q\hat{x}(s, h) \quad \text{and} \quad \hat{x}(s, h) = S_\Sigma(s)h, \quad s \geqq 0.$$

Hence, by minimality of the optimal control $\hat{u}(\cdot, h)$,

$$|P_\infty^{1/2}h|^2 = J(\hat{u}(\cdot, h), h) \leqq J(u_Q, h) \leqq |Q^{1/2}h|^2$$

and, necessarily, $D(Q^{1/2}) \subset D(P_\infty^{1/2}) = \Sigma$.

(ii) From part (i) we have already established that (6.4) has a positive selfadjoint solution if (A, B, C) is stabilized. Conversely, if Q is a positive selfadjoint solution to the operator algebraic Riccati equation (6.4), then we can repeat the step in part (i) and obtain (6.7), which says that the dense subset $D(\bar{A}_Q)$ of initial conditions is (A, B, C) stabilizable. In particular, $D(\bar{A}_Q) \subset \Sigma$ and $\bar{\Sigma} = H$. □

Example 6.1. Recall the example in § 3. We have seen that

$$H = D(A) = \ell^2, \quad \Sigma = \left\{ h \in \ell^2 : \sum_{k=1}^{\infty} (k+1) h_k^2 < \infty \right\}, \quad \bar{\Sigma} = H, \tag{6.8}$$

$$D(P_\infty) = \left\{ h \in \ell^2 : \sum_{k=1}^{\infty} (k+1)^2 h_k^2 < \infty \right\}, \qquad P_\infty e_k = (k+1) e_k, \quad k \in \mathbf{N}. \tag{6.9}$$

Moreover, $D(A_\Sigma) = \Sigma$, and K is the closed operator in H

$$D(K) = \Sigma, \quad K e_k = \sqrt{2k+1}\, e_k, \qquad k \in \mathbf{N}. \tag{6.10}$$

The space Σ is the set of all initial conditions that can be stabilized with a finite energy. However, for all h in H

$$\int_0^\infty |x(s)|_H^2 \, ds < \infty, \tag{6.11}$$

and for all h in Σ

$$\int_0^\infty |x(s)|_\Sigma^2 \, ds = \int_0^\infty \{ |x(s)|_H^2 + \langle P_\infty x(s), x(s) \rangle_\Sigma \} \, ds \leqq c|h|^2. \tag{6.12}$$

We remark that, in general, the closed loop system is not exponentially stable, as the following example shows.

Example 6.2. Let $H = D(A) = \ell^2$,

$$A e_k = 0, \quad B e_k = \frac{1}{k} e_k, \quad C e_k = e_k. \tag{6.13}$$

Then

$$P_\infty e_k = k e_k, \quad k \in \mathbf{N}, \tag{6.14}$$

$$\Sigma = \left\{ h \in \ell^2 : \sum_{k=1}^{\infty} k h_k^2 < \infty \right\}, \quad \bar{\Sigma} = H, \tag{6.15}$$

$$F e_k = (A - BB^* P_\infty) e_k = -\frac{1}{k} e_k. \tag{6.16}$$

Thus F is stable but not exponentially stable both in H and in Σ.

PROPOSITION 6.2. *If the triplet (A, B, C) is approximatively stabilizable and the pair (A^*, C^*) is stabilizable, then*

$$\int_0^\infty |\hat{x}(t, h)|_H^2 \, dt < \infty, \quad \textit{for all } h \in \Sigma \tag{6.17}$$

and the triplet (A, B, I) is approximatively stabilizable.

Proof. If (A^*, C^*) is stabilizable, then there exists a minimum positive bounded solution to the Riccati equation

$$AQ + QA^* - QC^*CQ + I = 0 \tag{6.18}$$

and the closed loop system

$$y'(t) = [A^* - C^*CQ]y(t), \qquad y(0) = k \tag{6.19}$$

is L^2-stable. Denote by $T(t)$ the semigroup associated with the above system. For all h in Σ consider the optimal state $\hat{x}(\cdot, h)$ and control $\hat{u}(\cdot, h)$, then

$$\hat{x}'(t, h) = [A^* - C^*CQ]^*\hat{x}(t, h) + QC^*C\hat{x}(t, h) + B\hat{u}(t, h), \qquad \hat{x}(0, h) = h \tag{6.20}$$

and

$$\hat{x}(t, h) = T^*(t)h + \int_0^t T^*(t-s)[QC^*C\hat{x}(s, h) + B\hat{u}(s, h)]\, ds. \tag{6.21}$$

It follows that

$$\|\hat{x}(t; h)\|_{L^2(0,\infty;H)} \leqq \|T^*(\cdot)h\|_{L^2(0,\infty;H)} + \|T^*(\cdot)h\|_{L^2(0,\infty;H)} \|QC^*C\hat{x}(\cdot, h) + B\hat{u}(\cdot, h)\|_{L^2(0,\infty;H)}.$$

The right-hand side is finite since T^* is exponentially decreasing, QC^* and B are bounded, and $C\hat{x}(\cdot, h)$ and $\hat{u}(\cdot, h)$ are $L^2(0, \infty; H)$ functions. □

Remark 6.2. To show the L^2-stability with respect to the Σ norm, we would have to prove that

$$\int_0^\infty \langle P_\infty \hat{x}(t; h), \hat{x}(t, h)\rangle_\Sigma\, dt = \int_0^\infty \int_t^\infty [|C\hat{x}(s, h)|^2 + |\hat{u}(s, h)|^2]\, ds\, dt < \infty. \tag{6.22}$$

7. A condition for approximative stabilizability. In this section we examine the connection between the (A, B, I) approximative stabilizability and the Hautus condition. We present a set of conditions (Hypothesis 7.1) under which the equivalence is verified (Proposition 7.1). We complete this section with an application of Hypothesis 7.1 to the nerve axon system (Example 7.1).

Hypothesis 7.1. Let A be the infinitesimal generator of an analytic semigroup on H. Denote by $\sigma(A)$ the spectrum of A, and by $\rho(A)$ the resolvent set of A. Assume that the following properties are verified:

(i) *$\sigma(A)$ consists of a convergent sequence $\{\lambda_i\}$ of semisimple*[2] *eigenvalues plus the limit point $\lambda_\infty = \lim_{i\to\infty} \lambda_i$;*

(ii) $\sigma(A) = \sigma^-(A) \cup \sigma^+(A)$, *where* $\sigma^-(A) = \{\lambda\colon \operatorname{Re}\lambda < 0\}$ *and* $\sigma^+(A) = \{\lambda\colon \operatorname{Re}\lambda > 0\}$. *We set $P_+ = 1/(2\pi i)\int_\gamma (\lambda - A)^{-1}\, d\lambda$, where γ is a suitable curve around $\sigma^+(A)$ and define $P = I - P_+$;*

(iii) *Setting $P_i = 1/(2\pi i)\int_{C(\lambda_i, \varepsilon_i)} (\lambda - A)^{-1}\, d\lambda$, where $C(\lambda_i, \varepsilon_i)$ is a circle in $\rho(A)$, we have $e^{tA}P_+x = \sum_{i=1}^\infty e^{t\lambda_i}P_i x$.*

PROPOSITION 7.1. *Assume that Hypothesis* 7.1 *is verified and that $B \in \mathscr{L}(U; H)$. Then the following statements are equivalent*:

(i) *The triple (A, B, I) is approximatively stabilizable,*

(ii) $\operatorname{Ker}(B^*) \cap \operatorname{Ker}(A^* - \lambda_i I) = \{0\}$ *for all $\lambda_i \in \sigma^+(A)$.*

Proof. (i) $\Rightarrow$ (ii). Assume, by contradiction, that (A, B, I) is approximatively stabilizable and that there exists $\lambda \in \sigma(A)$ and h in H, $|h| = 1$, such that $A^*h = \bar{\lambda}h$, $B^*h = 0$. By (A, B, I) approximative stabilizability for any k in Σ there exists a control u in $L^2(0, \infty; U)$ such that the corresponding solution x of (2.1) belongs to $L^2(0, \infty; U)$. Define the function $g(t) = (h, x(t))$. Then g is the solution of the equation

$$g'(t) = \lambda g(t), \quad t \geqq 0, \quad g(0) = (h, k) \Rightarrow g(t) = (h, k)\, e^{\lambda t}, \quad t \geqq 0.$$

[2] An eigenvalue is said to be semisimple if it is an isolated point of the spectrum and a simple pole of the resolvent operator (cf. T Kato [4, p. 41]).

Hence

$$|(h, k)| \int_0^\infty e^{\operatorname{Re}\lambda t}\, dt = \|g\|_{L^2(0,\infty)} \leqq |h|\, \|x\|_{L^2(0,\infty;U)} < \infty.$$

However,

$$\int_0^\infty e^{\operatorname{Re}\lambda t}\, dt = \infty \Rightarrow \forall k \in \Sigma, \quad (h, k) = 0, \tag{7.1}$$

and by density of Σ in H, $h = 0$, which is in contradiction with our hypothesis.

(ii) $\Rightarrow$ (i). Let $h \in H$ and $u \in L^2(0, \infty; U)$. We can write the solution of problem (2.1) as

$$\begin{aligned} x(t) &= e^{tA}Ph + \int_0^t e^{(t-s)A}P_-Bu(s)\, ds - \int_t^\infty e^{(t-s)A}P_+Bu(s)\, ds \\ &\quad + e^{tA}\left\{ P_+h + \int_0^\infty e^{-sA}P_+Bu(s)\, ds \right\}. \end{aligned} \tag{7.2}$$

Thus the control u is admissible if and only if $P_+h + \int_0^\infty e^{-sA}P_+Bu(s)\, ds = 0$. Consider now the mapping

$$u \to \gamma(u) = \int_0^\infty e^{-sA}P_+Bu(s)\, ds : L^2(0, \infty; U) \to P_+H = H_+ \tag{7.3}$$

and its adjoint

$$h \to (\gamma^* h)(s) = B^* e^{-sA^*} h : H_+^* \to L^2(0, \infty; U). \tag{7.4}$$

Clearly the triple (A, B, I) is approximatively stabilizable if and only if $\operatorname{Ker}(\gamma^*) = \{0\}$. Now assume that (ii) holds and, by contradiction, that $\operatorname{Ker}(\gamma^*) \neq \{0\}$. In view of Hypothesis 7.1 (iii) for any $h \in \operatorname{Ker}(\gamma^*)$ we have

$$B^* e^{-sA^*} h = B^* \sum_{i=1}^\infty e^{-t\lambda_i} P_i^* h = 0, \tag{7.5}$$

which implies $P_i^* h \in \operatorname{Ker}(B^*)$. Since $P_i^* h \in \operatorname{Ker}(A^* - \lambda_i I)$ (because λ_i is semisimple) we have found a contradiction with (ii). □

Example 7.1 (The nerve axon system). Let Ω be an open bounded set in $\mathbf{R}$ and consider the system (introduced in [3])

$$\begin{aligned} &\frac{\partial x_1}{\partial t}(t, \xi) = \alpha \Delta x_1(t,) + b_{11}x_1(t, \xi) + b_{12}x_2(t, x) + \sum_{j=1}^J f_j(t)\phi_j(t, x), \\ &t > 0, \quad \xi \in \Omega, \\ &\frac{\partial x_2}{\partial t}(t, \xi) = b_{21}x_1(t, x) + b_{22}x_2(t, x) + \sum_{j=1}^J g_j(t)\psi_j(t, x), \quad t > 0, \quad \xi \in \Omega, \\ &x_1(0, x) = h_1(x), \quad \xi \in \Omega, \\ &x_2(0, x) = h_2(x), \quad \xi \in \Omega, \\ &x_1(t, \xi) = 0, \quad x_2(t, \xi) = 0, \quad t > 0, \quad \xi \in \partial\Omega, \end{aligned} \tag{7.6}$$

where we assume that α, $b_{ij} : \mathbf{R} \to \mathbf{R}$ are given real numbers, with $\alpha > 0$, $b_{12}b_{21} \neq 0$, and $\phi_1, \ldots, \phi_J$; $\psi_1, \ldots, \psi_J \in C(\bar{\Omega})$ are linearly independent functions.

Choose $H = L^2(\Omega) \times L^2(\Omega)$, $U = \mathbf{R}^J \times \mathbf{R}^J$. Setting

$$(7.7) \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad h = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}, \quad u = \begin{bmatrix} (f_1, \dots, f_J) \\ (g_1, \dots, g_J) \end{bmatrix}, \quad Bu = \begin{bmatrix} \sum_{j=1}^{J} f_j \phi_j(t, \cdot) \\ \sum_{j=1}^{J} g_j \psi_j(t, \cdot) \end{bmatrix}$$

and

$$(7.8) \qquad b = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix},$$

we can write system (7.6) in the abstract form (2.1). The spectrum $\sigma(A)$ of A consists in two sequences of semisimple eigenvalues $\{\lambda_\pm(k)\}_{k \in \mathbf{N}}$ and the accumulation point

$$(7.10) \qquad \lambda_\infty = b_{22}.$$

The eigenvalues $\lambda_\pm(k)$ are defined by

$$(7.11) \qquad \lambda_\pm(k) = \tfrac{1}{2}\{-\alpha\mu_k + \mathrm{Tr}\,(b) \pm \sqrt{[-\alpha\mu_k + \mathrm{Tr}\,(b)]^2 + 4[\alpha\mu_k b_{22} - \det(b)]}\},$$

where the μ_k's are the eigenvalues of the Laplacian with Dirichlet boundary conditions. Now it is easy to check Hypothesis 7.1, so that we can apply Proposition 7.1.

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