SIAM J. CONTROL AND OPTIMIZATION
Vol. 25, No. 3, May 1987

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SYNTHESIS OF OPTIMAL CONTROL FOR AN INFINITE DIMENSIONAL PERIODIC PROBLEM*

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Abstract. We prove an existence and uniqueness result on periodic solutions of an infinite dimensional Riccati equation.

Key words. optimal control, periodic control, dynamic programming

AMS(MOS) subject classifications. 93C, 49B

1. Introduction. Consider the following optimal control problem: minimize

$$J(u)=\frac{1}{2}\int_0^\tau [\langle M(t)y(t), y(t)\rangle+\langle N(t)u(t), u(t)\rangle]\, dt \tag{1.1}$$

over all $u\in L^2(0, \tau; U)$ subject to

$$y'(t)=A(t)y(t)+B(t)u(t)+f(t), \qquad y(0)=y(\tau). \tag{1.2}$$

Here $A(t)$ is a linear operator in a Hilbert space H, U is the Hilbert space of the controls, $M(t)$ is a linear operator in H, $N(t)$ is a linear operator in U, $B(t)$ is a linear operator from U into H and $f\in L^2(0, \tau, H)$. We give precise notations and assumptions in § 2. In § 3 we study existence and uniqueness of periodic solutions of the infinite dimensional Riccati equation

$$Q'+A^*Q+QA-QBB^*Q+M=0 \tag{1.3}$$

and in §4 we prove that the optimal control for problem (1.1), (1.2) is a feedback control. We shall use an argument of dynamic programming, which follows closely [2] where a similar problem was studied in a finite dimensional space.

2. Notation and hypotheses. Let U and H be Hilbert spaces (scalar product $\langle\ ,\ \rangle$). We shall denote by $L(H)$ the Banach algebra of all linear bounded operators in H. We set

$$\Sigma(H)=\{T\in L(H);\ T=T^*\}, \qquad \Sigma^+(H)=\{T\in\Sigma(H);\ T\geqq 0\} \tag{2.1}$$

where T^* represents the adjoint of T.

Given any interval $[a, b]$ we shall denote by $C_s([a, b]; L(H))$ the set of all the mappings $[a, b]\to L(H)$, $t\to T(t)$ such that $T(\cdot)x$ is continuous for any $x\in H$. If a and b are finite, then $C_s([a, b]; L(H))$, endowed with the norm

$$\|T\|=\operatorname{Sup}\{\|T(t)\|;\ t\in[a, b]\}, \tag{2.2}$$

is a Banach space (by the uniform boundedness theorem). We set moreover

$$C_s([a, b]; \Sigma(H))=\{T\in C_s([a, b]; L(H));\ T(t)\in\Sigma(H)\}, \tag{2.3}$$

$$C_s([a, b]; \Sigma^+(H))=\{T\in C_s([a, b]; L(H));\ T(t)\in\Sigma^+(H)\}. \tag{2.4}$$

* Received by the editors November 12, 1984; accepted for publication (in revised form) April 14, 1986.
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$C_s([a, b]; L(U))$ and $C_s([a, b]; L(U, H))$ are defined analogously. Concerning the operators $A(t)$, $t \in \mathbb{R}$, we shall assume:

(2.5) (i) $A(t) = A(t+\tau)$, $t \in \mathbb{R}$.

(ii) There exists an evolution operator $U(t, s)$, $0 \leqq s \leqq t$ such that the initial value problem

$$z'(t) = A(t)z(t) + g(t), \qquad z(0) = x$$

with $g \in L^2(0, \tau; H)$ and $x \in H$ has a unique mild solution z given by

$$z(t) = U(t, 0)x + \int_0^t U(t, s)g(s)\, ds.$$

(iii) $A_n(t) = n^2(n - A(t))^{-1} - nI$ is defined for n sufficiently large. Moreover we have $z_n \to z$ in $C([0, \tau]; H)$, where z_n is the strict solution of the approximating problem

$$z_n'(t) = A_n(t)z_n(t) + g(t), \qquad z_n(0) = x.$$

We shall denote by $U_n(t, s)$ the evolution operator relative to $A_n(t)$. We remark that (2.5) are fulfilled under the usual hypotheses of Tanabe and Kato–Tanabe (see for instance [3], [6], [8]).

Concerning M, N, B and f we shall assume:

(2.6) (i) $f: \mathbb{R} \to H$ is τ-periodic and $f \in L^2(0, \tau; H)$,

(ii) $B \in C_s(\mathbb{R}, L(U, H))$ and it is τ-periodic,

(iii) $M \in C_s(\mathbb{R}; \Sigma^+(H))$ and it is τ-periodic,

(iv) $N \in C_s(\mathbb{R}, \Sigma^+(U))$, it is τ-periodic and there exists $\varepsilon > 0$ such that $N(t) \geqq \varepsilon I$, $t \leqq 0$.

Finally, in order to solve uniquely problem (1.2), we need the following assumption:

(2.7) 1 belongs to the resolvent set $\rho(U(\tau, 0))$ of $U(\tau, 0)$.

Under hypotheses (2.5)–(2.7) it is easy to prove that problem (1.2) has a unique mild solution y given by

$$y(t) = U(t, 0)(I - U(\tau, 0))^{-1} \int_0^\tau U(\tau, s)(f(s) + B(s)u(s))\, ds$$
$$+ \int_0^t U(t, s)(f(s) + B(s)u(s))\, ds. \tag{2.8}$$

Returning now to the control problem (1.1), (1.2), we remark that the functional $J: L^2(0, \tau; U) \to \mathbb{R}$ has a unique minimum u^* (since it is a coercive quadratic form); u^* is called the *optimal control* and the corresponding solution of (1.2) the *optimal state*. Finally $J(u^*)$ is the *optimal cost*.

The optimality conditions are also easily derived. Namely if u is the optimal control and y the optimal state, we have:

$$\begin{aligned} y' &= Ay + Bu + f, \qquad y(0) = y(\tau), \\ p' &= -A^*p - My, \qquad p(0) = p(\tau), \\ u &= -N^{-1}B^*p. \end{aligned} \tag{2.9}$$

Concerning the synthesis problem we shall look for a linear operator Q such that

$$p = Qy + r. \tag{2.10}$$

As easily seen, Q and r must satisfy the equations

$$Q' + A^*Q + QA - QBN^{-1}B^*Q + M = 0, \tag{2.11}$$

$$r' + (A^* - QBN^{-1}B^*)r + Qf = 0 \tag{2.12}$$

with the periodic conditions

$$Q(0) = Q(\tau), \qquad r(0) = r(\tau). \tag{2.13}$$

The differential equations in (2.9), (2.12) are intended in the mild sense, whereas the precise meaning of a solution of (2.11) will be stated in the next section.

In § 4 we will prove that the optimal control u is given by the formula

$$u = -N^{-1}B^*(Qy + r) \tag{2.14}$$

where y (the optimal state) is the solution of the closed loop equation

$$y' = Ay - BN^{-1}B^*Qy - BN^{-1}B^*r + f \tag{2.15}$$

with the condition

$$y(0) = y(\tau). \tag{2.16}$$

We remark that if the following hypothesis holds:

(2.17) 1 belongs to the resolvent sets of the evolution operators relative to $A - BN^{-1}B^*Q$ and $A^* - QBN^{-1}B^*$,

then (2.12) and (2.15) have a unique τ-periodic solution.

3. Periodic solutions of the Riccati equation. We are here concerned with periodic solutions of the Riccati equation

$$Q' + A^*Q + QA - QBN^{-1}B^*Q + M = 0. \tag{3.1}$$

We first recall some result on the final value problem

$$Q' + A^*Q + QA - QBN^{-1}B^*Q + M = 0, \qquad Q(\tau) = L \in \Sigma^+(H), \tag{3.2}$$

which we write in the following integral form:

$$\begin{aligned} Q(t)x &= U^*(\tau, t)LU(\tau, t)x \\ &\quad - \int_t^T U^*(s, t)(Q(s)B(s)N^{-1}(s)B^*(s)Q(s) - M(s))U(s, t)x\, ds, \qquad x \in H. \end{aligned} \tag{3.3}$$

Under suitable hypotheses (see Proposition 3.1 below) (3.3) has a unique solution $Q(t) = \Lambda(t, L)$.

We say that $Q \in C_s([0, \tau]; \Sigma^+(H))$ is a *τ-periodic solution* of (3.1) if it is a solution of (3.3) with $Q(\tau) = Q(0)$; this is equivalent to

$$Q(\tau) = \Lambda(0, Q(\tau)). \tag{3.4}$$

We shall consider also the approximating problem

$$Q_n' + A_n^*Q_n + Q_nA_n - Q_nBN^{-1}B^*Q_n + M = 0, \qquad Q_n(\tau) = L \tag{3.5}$$

where $A_n(t) = n^2(n - A(t))^{-1} - nI$. Problem (3.5) has clearly a unique solution that we denote by $Q_n(t) = \Lambda_n(t, L)$.

PROPOSITION 3.1. *Assume* (2.5), (2.6) *and let L belong to* $\Sigma^+(H)$. *Then*

(3.6) (i) *There exists a unique solution* Q (*resp.* Q_n) *of* (3.3) (*resp.* (3.5)). *Moreover* $Q_n \to Q$ *in* $C_s([0, \tau]; \Sigma^+(H))$.

(ii) *If* $L \leqq \bar{L}$ *we have*:

$$\Lambda(t, L) \leqq \Lambda(t; \bar{L}).$$

(iii) *If* $\{L_K\}$ *is an increasing sequence in* $\Sigma^+(H)$ *that converges strongly to* L, *then* $\Lambda(\cdot, L_k)$ *converges to* $\Lambda(\cdot, \bar{L})$ *in* $C_s([0, \tau]; \Sigma^+(H))$.

Proof. Statement (i) is essentially proved in [4] (see also [1, Thm. 1, p. 64]). The proof of (ii) is completely similar to that of [1, Lemma 16, p. 83]. Let us prove (iii). Setting $Q(t) = \Lambda(t, L)$, $Q_k(t) = \Lambda(t, L_k)$ we have

$$Q_k(t)x = U^*(\tau, t)L_k U(\tau, t)x$$

$$(3.7) \qquad -\int_t^\tau U^*(s, t)(Q_k(s)B(s)N^{-1}(s)B^*(s)Q_k(s) - M(s))U(s, t)x\, ds, \qquad x \in H.$$

By (3.6), $\{Q_k(t)\}$ is increasing for any t and $Q_k(t) \leqq Q(t)$. It follows that there exists $\bar{Q}(t) \leqq Q(t)$ such that $Q_k(t) \to \bar{Q}(t)x$ for any $x \in H$. By the dominated convergence theorem, taking the limit, as $k \to \infty$ in (3.7), we obtain

$$\bar{Q}(r)x = U^*(\tau, t)LU(\tau, t)x$$

$$(3.8) \qquad -\int_t^\tau U^*(s, t)(\bar{Q}(s)B(s)N^{-1}(s)B^*(s)\bar{Q}(s) - M(s))U(s, t)x\, ds, \qquad x \in H.$$

From (3.8) it follows that $\bar{Q} \in C_s([0, \tau]; \Sigma^+(H))$ so that, by uniqueness, we have $\bar{Q} = Q$. □

In order to prove the existence of a periodic solution of (3.1), we need a stabilizability assumption:

(3.9) There exists a τ-periodic function $K \in C_s(\mathbb{R}; L(H, U))$ and two numbers, $\omega > 0$, $\mu > 0$ such that $\|U_{A-BK}(t, s)\| \leqq \mu\, e^{-\omega(t-s)}$, $t > s$, where U_{A-BK} is the evolution operator relative to $A(t) - B(t)K(t)$, $t \in [0, \tau]$.

This hypothesis reduces to the usual one for the algebraic Riccati equation when A, B and M are time-independent (see [7]).

Remark 3.2. Hypothesis (3.9) is fulfilled if either $\|U(t, s)\| \leqq a\, e^{-b(t-s)}$ with $b > 0$ or $B(t) \geqq \sigma > 0$ and $a = 1$. □

We are ready now to prove the following theorem:

THEOREM 3.3. *Assume* (2.5), (2.6) *and* (3.9). *Then there exists a* τ*-periodic solution of* (3.1).

Proof. We first recall a well-known identity (see for instance [1]). Let $u \in L^2(0, T; U)$, $T > 0$ and let y be the mild solution of the problem

$$(3.10) \qquad y' = Ay + Bu, \quad y(0) = x, \quad x \in H.$$

Let W be the solution of the final value problem

$$(3.11) \qquad W' + A^*W + WA - WBN^{-1}B^*W + M = 0, \qquad W(T) = 0;$$

then we have:

$$(3.12) \quad \langle W(0)x, x\rangle + \int_0^T \|N^{-1/2}B^*Wy + N^{1/2}u\|^2\, ds = \int_0^T [\langle My, y\rangle + \langle Nu, u\rangle]\, ds.$$

We prove now the existence of a τ-periodic solution of (3.1). Set

$$(3.13) \qquad S_0 = 0, \qquad S_{n+1}(t) = \Lambda(t, S_n(0)), \qquad n \in \mathbb{N}.$$

By (3.6) $\{S_n\}$ is increasing. For any $k\in\mathbb{N}$ we set

(3.14) $$W_k(t)=S_h(t-(k-h-1)\tau),\quad t\in[(k-h-1)\tau,(k-h)\tau],\qquad h=1,\cdots,k.$$

As easily checked, W_k is a solution of the problem

(3.15) $$W_k'+A^*W_k+W_kA-W_kBN^{-1}B^*W_k+M=0,\quad W_k(k\tau)=0,\qquad 0\leqq t\leqq k\tau.$$

We now resort to (3.9) and (3.12) with u and y given by

(3.16) $$u(t)=-K(t)U_{A-BK}(t,0)x,\quad y(t)=U_{A-BK}(t,0)x,\quad x\in H$$

and we get

(3.17) $$\langle W_k(0)x,x\rangle\leqq\frac{\mu^2}{2\eta}(\|M\|+\|N\|\,\|K\|)\|x\|^2,$$

which implies that the sequence $\{S_n(0)\}$ is bounded in $\Sigma^+(H)$. By a well-known result on the monotone sequences of linear operators it follows that there exists $\bar{S}\in\Sigma^+(H)$ such that $S_n(0)x\to\bar{S}x$ for any $x\in H$. Now, by Proposition 3.1(iii) and by (3.13) we have, as $n\to\infty$ $\bar{S}=\Lambda(0,\bar{S})$ so that $\Lambda(t,\bar{S})$ is the required periodic solution. □

Remark 3.4. Theorem 3.3 generalizes a result in [9].

We consider now uniqueness and to this purpose we introduce a detectability assumption which reduces to the usual one for the algebraic Riccati equation (see [7]). We assume:

(3.18) There exists a τ-periodic function $K_1\in C_s(\mathbb{R},L(H))$ and two numbers $\omega_1>0$, $\mu_1>0$ such that

$$\|U_{A-K_1\sqrt{M}}(t,s)\|\leqq\mu_1\,e^{-\omega_1(t-s)},\qquad t\geqq s$$

where $U_{A-K_1\sqrt{M}}$ is the evolution operator relative to

$$A(t)-K_1(t)\sqrt{M(t)},\qquad t\in[0,\tau].$$

We remark that (3.18) implies (2.17).

We first prove two lemmas as follows.

LEMMA 3.5. *Assume* (2.5), (2.6) *and* (3.18) *and set* $L=A-BN^{-1}B^*Q$ *where* Q *is a* τ*-periodic solution of* (3.1). *Then there exists* $c>0$ *such that*

(3.19) $$\int_s^\infty\|U_L(t,s)x\|^2\,dt\leqq c\|x\|^2.$$

Proof. Let Q be a τ-periodic solution of (3.1), fix $k\in\mathbb{N}$. Then we have

(3.20) $$Q'+L^*Q+QL+QBN^{-1}B^*Q+M=0,\quad Q(k\tau)=Q(0),\quad t\in[0,k\tau].$$

Let Q_n be the solution of the approximating problem

(3.21) $$Q_n'+L_n^*Q_n+Q_nL_n+Q_nBN^{-1}B^*Q_n+M=0,\qquad Q_n(k\tau)=Q(0)$$

where $L_n=A_n-BN^{-1}B^*Q_n$. We remark that Q_n is not necessarily periodic. For any $x\in H$ we have

(3.22) $$\frac{d}{dt}<Q_n(t)U_{L_n}(t,s)x,U_{L_n}(t,s)x\rangle=-\|N^{1/2}BQ_nU_{L_n}(t,s)x\|^2-\|\sqrt{M}\,U_{L_n}(t,s)x\|^2.$$

By integrating in $[s, t]$ and letting n go to infinity we find

$$\langle Q(s)x, x\rangle = \langle Q(k\tau)U_L(k\tau, s)x, U_L(k\tau, s)x\rangle + \int_0^{k\tau} [\|N^{-1/2}Q(\sigma)U_L(\sigma, s)x\|^2 + \|\sqrt{M(\sigma)}U_L(\sigma, s)x\|^2]\, d\sigma. \tag{3.23}$$

Then functions $N^{-1/2}QU_L(\cdot, s)x$ and $\sqrt{M}U_L(\cdot, s)x$ belong to $L^2(s, \infty; H)$. Let now Π be defined by

$$L = \Pi + (K_1\sqrt{M} - BN^{-1}B^*Q) \tag{3.24}$$

and remark that, by (2.18),

$$\|U_\Pi(t, s)\| \leqq \mu_1 e^{-\omega_1(t-s)}. \tag{3.25}$$

By (3.24) it follows

$$U_L(t, s)x = U_\Pi(t, s)x + \int_s^t U_\Pi(t, \sigma)(K_1(\sigma)\sqrt{M(\sigma)} - B(\sigma)N^{-1}(\sigma)B^*(\sigma)Q(\sigma))U_L(\sigma, s)x\, d\sigma; \tag{3.26}$$

now, by the Young inequality $U_L(\cdot, s)x$ belongs to $L^2(s, \infty; H)$ as required. □

LEMMA 3.6. *Under the same hypotheses of Lemma* 3.5 *there exists a constant* $c_1 > 0$ *such that*

$$\|U_L(t, s)\| \leqq \frac{c_1}{(t-s)}, \qquad t \geqq s. \tag{3.27}$$

Proof. Since L is τ-periodic, there exist $\mu_2 > 0$ and $\xi \in \mathbb{R}$ such that

$$\|U_L(t, s)\| \leqq \mu_2 e^{\xi(t-s)}, \qquad t > s. \tag{3.28}$$

For any $x \in H$ we have

$$\begin{aligned}\frac{1}{2\xi}(e^{2\xi(t-s)} - 1)\|U_L(t, s)x\| &= \int_s^t e^{2\xi(\sigma-s)}\|U_L(t, s)x\|^2\, d\sigma \\ &\leqq \int_s^t e^{2\xi(\sigma-s)}\|U_L(\sigma, s)x\|^2\|U_L(t, \sigma)\|^2\, d\sigma \\ &\leqq c\mu_2^2 e^{2\xi(t-s)}\|x\|^2\end{aligned}$$

by (3.19); thus there exists $\gamma > 0$ such that

$$\|U_L(t, s)\| \leqq \gamma, \qquad t \geqq s. \tag{3.29}$$

We have finally

$$\begin{aligned}(t-s)\|U_L(t, s)x\|^2 &= \int_s^t \|U_L(t, s)x\|^2\, d\sigma \\ &\leqq \int_s^t \|U_L(\sigma, s)x\|^2\|U_L(t, \sigma)\|^2\, d\sigma \leqq \gamma^2 c\|x\|^2\end{aligned}$$

and the conclusion follows. □

Remark 3.7. The above proof is inspired by the proof of the Datko theorem given in [7].

We are now ready to prove uniqueness.

THEOREM 3.8. *Assume* (2.5), (2.6) *and* (3.18). *Then* (3.1) *has at most one* τ-*periodic solution.*

Proof. Let Q, Q_1 be τ-periodic solutions of (3.1); set $R = Q - Q_1$. Then R verifies the equation

$$R' + L^*R + RL + RBN^{-1}B^*R = 0, \qquad t \in [0, k\tau] \tag{3.30}$$

for any $k \in \mathbb{N}$. Let R_n be the solution of the final value problem

$$R_n' + L_n^*R_n + R_nL_n + R_nBN^{-1}B^*R_n = 0, \qquad R_n(k\tau) = R(k\tau). \tag{3.31}$$

It follows that

$$\begin{aligned}&\frac{d}{dt}\langle R_n(t)U_{L_n}(t,s)x, U_{L_n}(t,s)x\rangle \\ &\qquad = -\|N^{-1/2}B^*(t)R_n(t)U_{L_n}(t,s)x\|^2 \leqq 0, \qquad x \in H,\end{aligned} \tag{3.32}$$

which implies

$$\langle R(0)U_L(k\tau, s)x, U_L(k\tau, s)x\rangle \leqq \langle R(s)x, x\rangle. \tag{3.33}$$

Letting k go to infinity and using (3.27), we get $\langle R(s)x, x\rangle \geqq 0$, that is, $Q(s) \geqq Q_1(s)$; by interchanging Q and Q_1 we find $Q(s) \geqq Q_1(s)$ and finally that $Q = Q_1$. □

Remark 3.9. Stability. Assume the hypotheses of Theorems 3.3 and 3.8; let Q be the unique periodic solution of (3.1) and S a solution of the final value problem

$$S' + A^*S + SA - SBN^{-1}B^*S + M = 0, \qquad S(0) = S_0 \in \Sigma^+(H), \qquad -\infty < t \leqq 0.$$

Setting $Z = Q - S$, $L = A - BN^{-1}B^*Q$, we have

$$Z' + L^*Z + ZL + ZBN^{-1}B^*Z = 0.$$

Thus, by (3.27), it follows that

$$\lim_{\|S_0\| \to 0} \|Q(t) - S(t)\| = 0 \quad \text{uniformly in } t$$

and the periodic solution Q is stable.

4. Dynamic programming. The Hamilton-Jacobi-Bellman equation corresponding to the control problem (1.1)-(1.2) is

$$\begin{aligned}&\psi_t(t,x) - \tfrac{1}{2}\|N(t)^{-1/2}B^*(t)\psi_x(t,x)\|^2 \\ &\qquad + \langle Ax + f(t), \psi_x(t,x)\rangle + \tfrac{1}{2}\langle M(t)x, x\rangle = 0.\end{aligned} \tag{4.1}$$

The following result is easily proved.

PROPOSITION 4.1. *Assume* (2.5)-(2.7), (2.17) *and* (3.9). *Let Q be a τ-periodic solution of* (3.1) *and r the periodic solution of* (2.12). *Then the function*

$$\psi(t,x) = \tfrac{1}{2}\langle Q(t)x, x\rangle + \langle r(t), x\rangle + s(t) \tag{4.2}$$

is a solution of (4.1) *if and only if we have*

$$s' - \tfrac{1}{2}\|N^{-1}B^*r\|^2 + \langle f(t), r(t)\rangle = 0. \tag{4.3}$$

LEMMA 4.2. *Assume the hypotheses of Proposition* 4.1. *Let ψ be given by* (4.2), $u \in L^2(0, \tau; U)$, *y be defined by* (1.2) *and J by* (1.1). *Then the following identity holds*:

$$\begin{aligned}J(u) &= \int_0^\tau \|N^{-1/2}B^*(Qy + r) + N^{1/2}u\|^2\,dt \\ &\qquad + \int_0^\tau [\langle f, r\rangle - \tfrac{1}{2}\|B^*r\|^2]\,dt.\end{aligned} \tag{4.4}$$

Proof. Let $Q_n(t)=\Lambda_n(t, Q(\tau))$, let r_n be the solution of the problem

$$r_n'+(A_n^*-Q_nBN^{-1}B^*)r_n+Q_nf=0, \qquad r_n(\tau)=r(\tau). \tag{4.5}$$

Let s_n be such that

$$s_n'-\tfrac{1}{2}\|N^{-1/2}B^*r_n\|^2+\langle f, r_n\rangle=0 \tag{4.6}$$

and, finally, let y_n be the solution of the problem

$$y_n'=A_ny_n+Bu+f, \qquad y_n(0)=y(0). \tag{4.7}$$

Setting

$$\psi_n(t, y)=\tfrac{1}{2}\langle Q_n(t)y, y\rangle+\langle r_n(t), y\rangle+s_n(t) \tag{4.8}$$

we have

$$\frac{d}{dt}\psi_n(t, y_n)=\frac{1}{2}\|N^{-1/2}B^*(Q_ny_n+r_n)\|^2-\frac{1}{2}[\langle My_n+y_n\rangle+\langle Nu, u\rangle]. \tag{4.9}$$

Now the conclusion follows by integrating (4.9) in $[0, \tau]$ and by letting n go to infinity. □

THEOREM 4.3. *Assume* (2.5)-(2.7), (2.17) *and* (3.9). *Let Q be a τ-periodic solution of* (3.1), *let r be the corresponding τ-periodic solution of* (2.12) *and y the solution of the closed loop equation* (2.15) *with $y(0)=y(\tau)$. Then the optimal control u^* is given by*

$$u^*=-N^{-1}B^*(Qy+r) \tag{4.10}$$

and the optimal cost results from

$$J(u^*)=\int_0^\tau \left[\langle f, r\rangle-\frac{1}{2}\|B^*r\|^2\right] dt. \tag{4.11}$$

Proof. By (4.4) it follows that

$$J(u)\geqq\int_0^\tau \left[\langle f, r\rangle-\frac{1}{2}\|B^*r\|^2\right] dt=\Gamma; \tag{4.12}$$

now, if u is given by (4.10) we have $J(u^*)=\Gamma$ so that u is optimal. □

Example 4.4. Let Ω be a bounded subset of R^n with smooth boundary $\partial\Omega$. Consider the following problem:

Minimize

$$J(u)=\frac{1}{2}\int_0^\tau dt\int_\Omega d\xi\,[|y(t, \xi)|^2+|u(t, \xi)|^2] \tag{4.13}$$

over all $u\in L^2([0, \tau]\times\Omega)$

subject to

$$\begin{aligned}&\frac{d}{dt}y(t, \xi)=\Delta_\xi y(t, \xi)-\phi(t)y(t, \xi)+u(t, \xi)+f(t, \xi),\\ &y(t, \xi)=0, \qquad t\in[0, \tau], \quad \xi\in\partial\Omega,\\ &y(0, \xi)=y(t, \xi)\end{aligned} \tag{4.14}$$

where f and ϕ are continuous, τ-periodic in t and ϕ is nonnegative. Δ_ξ is the Laplace operator acting in the variable ξ.

Set $H = U = L^2(\Omega)$, $M(t) = N(t) = B = I$ and

$$A(t) = \Delta_\xi - \phi(t), \qquad D(A(t)) = H^2(\Omega) \cap H_0^1(\Omega). \tag{4.15}$$

As easily seen, hypotheses (2.5) and (2.6) hold; moreover

$$U(t, s) = \exp\left(C(t-s) - \int_s^t \phi(\sigma)\, d\sigma\right) \tag{4.16}$$

where $Cy = \Delta_\xi y$ and $D(C) = D(A(t))$. By the maximum principle we have

$$\|U(t, s)\| \leqq 1 \tag{4.17}$$

so that $1 \in \rho(U(\tau, 0))$ and (2.7) is fulfilled. Moreover, (2.17) also holds because $A - BN^{-1}B^*Q \doteq A - Q$ and Q is positive. Finally (3.9) holds by virtue of Remark 3.2. □

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