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OPTIMAL CONTROL FOR INTEGRODIFFERENTIAL EQUATIONS OF PARABOLIC TYPE*

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Abstract. Quadratic control problems for integrodifferential equations of parabolic type are considered. A state-space representation of the system is obtained by choosing an appropriate product space. By using the standard method based on Riccati equation, a unique optimal control over a finite horizon and under a stabilizability condition is obtained and the quadratic problem over an infinite horizon is solved. It is shown that the approach is also valid for some integrodifferential equations of different types. Two examples covered by the model are given.

Key words. optimal control, stabilizability, integrodifferential equations

AMS subject classifications. 93D15, 93C22, 93C25

1. Introduction. Let H and U be Hilbert spaces. Consider the control system

$$(1)\qquad \begin{cases} y'(t) = Ay(t) + \int_0^t K(t-r)y(r)\,dr + Bu(t), \\ y(0) = y_0, \end{cases}$$

where A is the infinitesimal generator of an analytic semigroup e^{tA} in H. We denote by $D(A)$ the domain of A and by $|\cdot|_{D(A)}$ the graph norm of A. $K(\cdot)$ is an $L(D(A);H)$-valued operator, and $B \in L(U;H)$. Under suitable conditions (see Hypothesis 1 below) there exists a resolvent operator (see [3], [13], [16]) associated with (1) and a unique classical solution to (1). For each $u \in L^2(0,T;U)$ we can define a mild solution to (1) in $C([0,T];H)$. We then wish to minimize the functional

$$(2)\qquad J(u) = \int_0^T \left\{|My(t)|^2 + |u(t)|^2\right\} dt + \langle Gy(T), y(T)\rangle$$

over all $u \in L^2(0,T;U)$. Here $M \in L(H;H_0)$, H_0 is a Hilbert space, and $G \in L^+(H)$ is the space of selfadjoint nonnegative operators on H. Under a stabilizability condition (see Hypothesis 4 below) we also wish to minimize the functional

$$(3)\qquad J(u) = \int_0^\infty \left\{|My(t)|^2 + |u(t)|^2\right\} dt$$

over all $u \in L^2(0,\infty;U)$. To our knowledge there is no direct method to solve these problems. In this paper we give a state-space representation of (1) similar to those in [15] and [19]. As in [9]–[11] we then reduce our problems to linear quadratic problems of standard type [1].

We recall some fundamental results concerning the resolvent operator associated with (1). It is convenient to introduce equations

$$(4)\qquad \begin{cases} y'(t) = Ay(t) + \int_0^t K(t-r)y(r)\,dr, \\ y(0) = y_0, \end{cases}$$

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$$\begin{cases} y'(t) = Ay(t) + \int_0^t K(t-r)y(r)\,dr + f(t), \\ y(0) = y_0, \end{cases} \tag{5}$$

In [6] and [16] the existence of a resolvent operator for (4) is shown under the following conditions.

Hypothesis 1.

(i) $K(\cdot) \in L^1(0,\infty; L(D(A); H))$.

(ii) For all $h \in D(A)$, the Laplace transform $\widetilde{K}(\cdot)h$ can be extended to a sector $S = \{\lambda \in C : \lambda \neq \omega, |\arg(\lambda - \omega)| < \varphi\}$, where $\omega \in R, \varphi \in]\pi/2, \pi[$.

(iii) There exist $\beta \in]0,1]$ and $c > 0$ such that $|\lambda^\beta \widetilde{K}(\cdot)h| \leq c|h|_{D(A)}$, $\lambda \in S$, $h \in D(A)$.

The following result is proved in [3] and [16].

THEOREM 1.1. *There exists an analytic resolvent operator $R(t) \in L(H; D(A))$, $t \geq 0$, such that*

(i) *$R(t)y_0$ is continuous for any $y_0 \in H$ and $R(0) = I$.*

(ii) *For each $y_0 \in D(A)$ and $T > 0$,*

$$R(t)y_0 \in C([0,T]; D(A)) \cap C^1([0,T]; H)$$

and it satisfies (4).

(iii) *For each $y_0 \in D(A)$ and $f \in C^\alpha([0,T]; H)$ (α-Hölder continuous), $y(t)$, given by*

$$y(t) = R(t)y_0 + \int_0^t R(t-r)f(r)\,dr,$$

is a unique classical solution (*see* [3]), *in*

$$C([0,T]; H) \cap C(]0,T]; D(A)) \cap C^1(]0,T]; H).$$

(iv) *There exist $r_0 > 0$ and $\varphi_0 \in]\frac{\pi}{2}, \varphi]$ such that for any $\lambda \in S$ with $|\lambda| \geq r_0$, $|\arg\lambda| \leq \varphi_0$, the linear operator $\lambda - A - \widetilde{K}(\lambda) : D(A) \to H$ is invertible and $(\lambda - A - \widetilde{K}(\lambda))^{-1} \in L(H; D(A))$ coincides with the Laplace transform of $R(t)$.*

For each $y_0 \in H$ and $u \in L^2(0,T;U)$

$$y(t) = R(t)y_0 + \int_0^t R(t-r)Bu(r)\,dr \tag{6}$$

is well defined and is in $C([0,T]; H)$. It is a mild solution of (1) in the sense

$$y(t) = e^{tA}y_0 + \int_0^t e^{(t-r)A} \int_0^r K(r-s)y(s)\,ds\,dr + \int_0^t e^{(t-r)A}Bu(r)\,dr. \tag{7}$$

Note that the cost function (2) makes sense for the mild solution.

For later use we establish additional properties of $R(t)$ that are not given in [3]. Let $D_A(\alpha, 2)$, $\alpha \in]0,1[$ be the real interpolation space between $D(A)$ and H. Consider the problem

$$y'(t) = Ay(t) + f(t), \qquad y(0) = y_0. \tag{8}$$

THEOREM 1.2. (i) *Let* $y_0 \in D_A(\frac{1}{2},2)$, *and let* $f \in L^2(0,T;H)$. *Then the mild solution of* (8) *lies in*

$$L^2(0,T;D(A)) \cap W^{1,2}(0,T;H) \subset C([0,T];D_A(\tfrac{1}{2},2)).$$

There exists a unique solution y *to* (4) *in* $L^2(0,T;D(A)) \cap W^{1,2}(0,T;H)$. *Hence* $R(t)y_0, R\star f \in L^2(0,T;D(A)) \cap W^{1,2}(0,T;H)$.

(ii) *Let* $y_0 \in D(A)$, $f \in W^{1,2}(0,T;H)$, *and* $Ay_0 + f(0) \in D_A(\frac{1}{2},2)$. *Then the mild solution of* (8) *lies in*

$$W^{1,2}(0,T;D(A)) \cap W^{2,2}(0,T;H) \subset C^1([0,T];D_A(\tfrac{1}{2},2)).$$

Moreover, there exists a unique solution y *to* (4) *in* $W^{1,2}(0,T;D(A)) \cap W^{2,2}(0,T;H)$. *Hence* $R(t)y_0, R\star f \in W^{1,2}(0,T;D(A)) \cap W^{2,2}(0,T;H)$.

Proof. The first assertion in (i) is well known [17]. To show the second assertion of (i) we consider the corresponding integral equation of the type (7). For a small T we apply a contraction-mapping theorem on $L^2(0,T;D(A))$. The general case then follows by splitting the interval into small subintervals. The first assertion in (ii) is proved as in [15]. The second part of (ii) follows by raising regularity and considering a contraction mapping on $W^{1,2}(0,T;D(A))$. See [15] for details. □

To give a state-space representation [8], [9] of (1) we consider

$$(9) \qquad \begin{cases} y'(t) = Ay(t) + \displaystyle\int_{-\infty}^{t} K(t-r)y(r)\,dr, \\ y(0) = y_0, \\ y(\theta) = y_1(\theta), \ \theta \in]-\infty,0[, \ y_1 \in L^2(-\infty,0;D(A)). \end{cases}$$

We now rewrite this as

$$(10) \qquad \begin{cases} y'(t) = Ay(t) + \displaystyle\int_{0}^{t} K(t-r)y(r)\,dr + f(t), \\ y(0) = y_0, \end{cases}$$

where $f(t) = \int_{-\infty}^{0} K(t-\theta)y_1(\theta)\,d\theta \in L^2(0,\infty;H)$.

Hypothesis 2. $K(\cdot) \in L^2(0,\infty;L(D(A);H))$.

If we assume Hypothesis 2 is true, then the operator $\mathcal{K}$ defined by

$$(11) \qquad \mathcal{K}y_1 = \int_{-\infty}^{0} K(-\theta)y_1(\theta)d\theta, \qquad y_1 \in L^2(-\infty,0;D(A))$$

lies in $L(L^2(-\infty,0;D(A)),H)$. Moreover, $f \in W^{1,2}(0,T;H)$ for any $y_1 \in W^{1,2}(-\infty,0;D(A))$ since

$$f'(t) = K(t)y_1(0) + \int_{-\infty}^{0} K(t-\theta)y_1'(\theta)\,d\theta \in L^2(0,T;H).$$

Note also $f(0) = \mathcal{K}y_1$. Using these observations and Theorem 1.2, we have the following corollary.

COROLLARY 1.3. (i) *For each* $y_0 \in D_A(\frac{1}{2};2)$ *and* $y_1 \in L^2(-\infty,0;D(A))$ *there exists a unique solution to* (10) (*and hence to* (9)) *in the space* $L^2(0,T;D(A)) \cap W^{1,2}(0,T;H)$.

(ii) *Assume Hypothesis* 2, *and let* $y_1 \in W^{1,2}(-\infty,0;D(A))$, $y_1(0) = y_0$, *and* $Ay_0 + \mathcal{K}y_1 \in D_A(\frac{1}{2};2)$. *Then there exists a unique solution to* (9) *in*

$$W^{1,2}(-\infty,0;D(A)) \cap W^{2,2}(0,T;H) \subset C^1([0,T];D_A(\tfrac{1}{2};2)).$$

Proof. Part (i) follows directly from Theorem 1.2(i). Under Hypothesis 2 $f \in W^{1,2}(0,T;H)$ and the assumptions in (ii) of Theorem 1.2 are satisfied. Hence (10) has a unique solution in $W^{1,2}(0,T;D(A)) \cap W^{2,2}(0,T;H)$. Since $y_1(0) = y_0 \in D(A)$, there exists a unique solution to (5) in $W^{1,2}(-\infty,T;D(A))$. □

Now we write (9) in the form

$$(12)\qquad \begin{aligned} &y'(t) = Ay(t) + \int_{-\infty}^{t} K(-\theta)y(t+\theta)\,d\theta, \\ &y(0) = y_0, \\ &y(\theta) = y_1(\theta), \qquad \theta \in]-\infty,0[. \end{aligned}$$

This is a delay equation with infinite delay. A more general delay equation, but with finite delay, was considered in [15], and a semigroup was constructed on the product space $D_A(\frac{1}{2};2) \times L^2(-r,0;D(A))$. To obtain a similar result we assume Hypothesis 2 and rewrite (12) as

$$(13)\qquad \begin{aligned} &y'(t) = Ay(t) + \mathcal{K}y_t, \\ &y(0) = y_0, \\ &y(\theta) = y_1(\theta), \qquad \theta \in]-\infty,0[. \end{aligned}$$

where $y_t(\cdot) = y(t+\cdot)$. The following result is a modification of [15, Thms. 4.1 and 4.2] for the case with infinite delay.

THEOREM 1.4. *Assume Hypotheses* 1 *and* 2 *are true, and let* y *be the unique solution of* (12) *for* $y_0 \in D_A(\frac{1}{2};2)$ *and* $y_1 \in L^2(-\infty,0;D(A))$, *which lies in* $L^2(0,T;D(A)) \cap W^{1,2}(0,T;H)$ *for any* $T > 0$. *Then the map*

$$(14)\qquad \underline{S}(t) : \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} \to \begin{pmatrix} y(t) \\ y_t(\cdot) \end{pmatrix}$$

on $\underline{Z} = D_A(\frac{1}{2};2) \times L^2(-\infty,0;D(A))$ *is a strongly continuous semigroup. Its infinitesimal generator is given by*

$$(15)\qquad \mathcal{A}\begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \begin{pmatrix} Ay_0 + \mathcal{K}y_1 \\ \frac{dy_1}{d\theta} \end{pmatrix},$$

$$(16)\qquad \begin{aligned} D(\mathcal{A}) = \{(y_0,y_1) \in \underline{Z} : y_1 \in W^{1,2}(-\infty,0;D(A)), \\ y_1(0) = y_0, Ay_0 + \mathcal{K}y_1 \in D_A(\tfrac{1}{2};2)\} \end{aligned}$$

Proof. The only difference between (13) and the equation in [15] is the length of the memory involved. Hence one could repeat the proofs in [15]. However, we shall give a different proof for the characterization of the generator. Note first that the strong continuity and the semigroup property of $\underline{S}(t)$ follow from Corollary 1.3(i). We now show that $\mathcal{A}$, given by (15) and (16) is the infinitesimal generator of the semigroup $\underline{S}(t)$. Choose $[y_0, y_1]' \in D(\mathcal{A})$; then $\underline{S}(t)[y_0, y_1]' = [y(t), y_t(\cdot)]$, where $y(t)$ is the solution of (13) and hence of (9). Then by Corollary 1.3 (ii) we have

$$\lim_{t\to 0} \frac{y(t) - y_0}{t} = y'(0) = Ay_0 + \mathcal{K}y_1 \quad \text{in } D_A\left(\frac{1}{2}; 2\right),$$

$$\lim_{t\to 0} \frac{y_t(\cdot) - y_1(\cdot)}{t} = \frac{dy_1}{d\theta} \quad \text{in } L^2(-\infty, 0; D(A)).$$

This implies that the infinitesimal generator of the semigroup $\underline{S}(t)$ coincides with $\mathcal{A}$ on $D(\mathcal{A})$ and is an extension of $\mathcal{A}$. To see that $\mathcal{A}$ is in fact the generator we need to show only that the resolvent set of $\mathcal{A}$ is nonempty. Now choose $\lambda > 0$ and consider

$$(\lambda - \mathcal{A})[y_0, y_1]' = [z_0, z_1]' \in \underline{Z}.$$

This is equivalent to

$$\lambda y_0 - Ah - \mathcal{K}y_1 = z_0 \in D_A(\tfrac{1}{2}; 2),$$

$$\lambda y_1 - \frac{dy_1}{d\theta} = z_1 \in L^2(-\infty, 0; D(A)).$$

The second equation yields

$$y_1(\theta) = e^{\lambda\theta} y_1(0) + \int_\theta^0 e^{\lambda(\theta-\eta)} z_1(\eta)\, d\eta.$$

Setting $y_1(0) = y_0$ and substituting y_1 into the first equation, we obtain

$$(\lambda - \mathcal{A} - \mathcal{K}e^{\lambda\cdot})y_0 = z_0 + \mathcal{K}\int_\theta^0 e^{\lambda(\theta-\eta)} z_1(\eta)\, d\eta =: \overline{z}_0.$$

Noting that $\mathcal{K}e^{\lambda\cdot}y_0 = \widetilde{K}(\lambda)y_0$, we have

$$(\lambda - \mathcal{A} - \widetilde{K}(\lambda))y_0 = \overline{z}_0.$$

By virtue of Theorem 1.1(iv) this is solvable for any $\lambda > r_0$ and $y_0 = (\lambda - A - \widetilde{K}(\lambda))^{-1}\overline{z}_0 \in D(A)$. Then

$$y_1(\theta) = e^{\lambda\theta}(\lambda - \mathcal{A} - \widetilde{K}(\lambda))^{-1}\overline{z}_0 + \int_\theta^0 e^{\lambda(\theta-\eta)} z_1(\eta)\, d\eta$$

lies in $W^{1,2}(-\infty, 0; D(A))$. We also have

$$Ay_0 + \mathcal{K}y_1 = \lambda y_0 - z_0 \in D_A(\tfrac{1}{2}; 2).$$

Hence $\lambda \in \rho(\mathcal{A})$ and $\mathcal{A}$ is the infinitesimal generator of the semigroup $\underline{S}(t)$. $\square$

Remark 1.5. Let $A = A_0 + A_1$, where A_0 is selfadjoint and negative. If $A_1 \in L(D(-A)^{1/2}; H)$, then we can replace $D_A(\frac{1}{2}; 2)$ by $D(-A)^{1/2}$ and $D_A(-\frac{1}{2}; 2)$ by $D(-A^\star)^{1/2}$ (see §2).

In [14] a special case of the delay equation in [15] was considered and a quadratic control problem on $D_A(\frac{1}{2}; 2) \times L^2(-r, 0; D(A))$ was solved.

If we take $B \in L(U; D_A(\frac{1}{2}; 2))$, $M \in L(D_A(\frac{1}{2}; 2), H_0)$, and $G \in L^+(D_A(\frac{1}{2}; 2))$, then by using the semigroup $\underline{S}(t)$ in Theorem 1.4 we can solve our control problem as in [14]; however, the state space $\underline{Z}$ is not convenient in applications, and we wish to take the initial value y_0 in H rather than in $D_A(\frac{1}{2}; 2)$. Moreover, our cost functionals (2) or (3) are more natural, as we can see from examples (see Example 5.1). Thus we need a representation of our system (1) in a larger space.

2. The semigroup model. Let $D_A(-\alpha, 2)$, $\alpha \in]0, 1[$, be the extrapolation space of A (see [2]). To take y_0 in H rather than in $D_A(\frac{1}{2}; 2)$ we replace H (respectively, $D(A)$) by $D_A(-\frac{1}{2}; 2)$ (respectively, $D_A(\frac{1}{2}; 2)$) and assume, in addition to Hypothesis 1, the following hypothesis.

Hypothesis 3. $K(\cdot) \in L^2(0, \infty); L(D_A(-\frac{1}{2}; 2); D_A(\frac{1}{2}; 2))$.

Then the operator $\mathcal{K}$ in (11) belongs to $L(L^2(-\infty, 0; D_A(\frac{1}{2}; 2)); D_A(-\frac{1}{2}; 2))$. By translation we obtain all results similar to those in §1. In particular, we state results corresponding to Corollary 1.3 and Theorem 1.4, respectively.

THEOREM 2.1. (i) *For each $y_0 \in H$ and $y_1 \in L^2(-\infty, 0; D_A(\frac{1}{2}; 2))$ there exists a unique solution to* (10) *in*

$$L^2(0, T; D_A(\tfrac{1}{2}; 2)) \cap W^{1,2}(0, T; D_A(-\tfrac{1}{2}; 2)) \subset C([0, T]; H).$$

(ii) *Assume Hypothesis* 3, *and let $y_1 \in W^{1,2}(-\infty, 0; D_A(\frac{1}{2}; 2))$, $y_1(0) = y_0$, and $Ay_0 + \mathcal{K}y_1 \in H$. Then there exists a unique solution to* (9) *in*

$$W^{1,2}(-\infty, T; D_A(\tfrac{1}{2}; 2)) \cap W^{2,2}(0, T; D_A(-\tfrac{1}{2}; 2)) \subset C^1([0, T]; H).$$

THEOREM 2.2. *Assume Hypothesis* 1 *and* 3 *with H (respectively $D(A)$) replaced by $D_A(-\frac{1}{2}; 2)$ (respectively $D_A(\frac{1}{2}; 2)$). Let $y_0 \in H$, let $y_1 \in L^2(-\infty, 0; D_A(\frac{1}{2}; 2))$, and let $y(t)$ be the solution of* (9) (*and hence of* (13)) *given in Theorem* 2.1(i). *Define the map on $Z = H \times L^2(-\infty, 0; D_A(\frac{1}{2}; 2))$*

$$S(t) : \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} \to \begin{pmatrix} y(t) \\ y_t(\cdot) \end{pmatrix}. \tag{17}$$

Then $S(t)$ is a strongly continuous semigroup on Z, and its infinitesimal generator is given by

$$\mathcal{A}\begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \begin{pmatrix} Ay_0 + Ky_1 \\ \frac{dy_1}{d\theta} \end{pmatrix} \tag{18}$$

$$\begin{aligned} D(\mathcal{A}) = \{&(y_0, y_1) \in Z : y_1 \in W^{1,2}(-\infty, 0; D_A(\tfrac{1}{2}; 2)), y_1(0) = y_0, \\ &Ay_0 + Ky_1 \in H\}. \end{aligned} \tag{19}$$

Next we express $S(t)$ by using the resolvent operator. We write (9) as

$$y'(t) = Ay(t) + \int_0^t K(t-r)y(r)\,dr + K_1(t)y_1, \tag{20}$$

where

$$K_1(t)y_1 = \int_{-\infty}^0 K(t-\theta)y_1(\theta)\,d\theta \in L^2\left(0,\infty; D_A\left(-\frac{1}{2};2\right)\right) \cap C\left([0,T]; D_A\left(-\frac{1}{2};2\right)\right).$$

The solution of (20) can be written as

$$y(t) = R(t)y_0 + \int_0^t R(t-r)K_1(r)y_1\,dr$$

Set

$$S(t) = \begin{pmatrix} S_{11}(t) & S_{12}(t) \\ S_{21}(t) & S_{22}(t) \end{pmatrix};$$

then $y(t) = S_{11}(t)y_0 + S_{12}(t)y_1$. Thus we have

$$S_{11}(t)y_0 = R(t)y_0,$$

$$S_{12}(t)y_1 = \int_0^t R(t-r)K_1(r)y_1\,dr.$$

Similarly, we have

$$(S_{21}(t)y_0)(\cdot) = R(t+\cdot)y_0,$$

$$(S_{22}(t)y_1)(\cdot) = \int_0^{t+\cdot} R(t-r)K_1(r)y_1\,dr.$$

Hypotheses 1 and 3 come from physical examples such as Example 5.1. If we assume, instead of Hypothesis 3, the following hypothesis, we have, in fact, Corollary 2.3.

Hypothesis 3′. $K(\cdot) \in L^2(0,\infty; L(D_A(\frac{1}{2};2);H))) \cap L^2(0,\infty; L(H; D_A(-\frac{1}{2};2)))$. then we can find a semigroup on $H \times L^2(-\infty,0;H)$.

COROLLARY 2.3. *Assume Hypothesis* 3′. *Then for each* $y_0 \in H$ *and* $y_1 \in L^2(-\infty,0;H)$ *there exists a unique solution* $y(t)$ *to* (13) *in*

$$L^2(0,T; D_A(\tfrac{1}{2};2)) \cap C([0,T];H)$$

for any $T > 0$. *The map*

$$S(t): \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} \to \begin{pmatrix} y(t) \\ y_t(\cdot) \end{pmatrix} \tag{21}$$

is a strongly continuous semigroup on $Z = H \times L^2(-\infty,0;H)$. *Its infinitesimal generator is given by*

$$\mathcal{A}\begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \begin{pmatrix} Ay_0 + Ky_1 \\ \frac{dy_1}{d\theta} \end{pmatrix}, \tag{22}$$

$$D(\mathcal{A}) = \{(y_0, y_1) \in Z : y_1 \in W^{1,2}(-\infty, 0; H), y_1(0) = y_0, Ay_0 + Ky_1 \in H\}. \tag{23}$$

If $K(\cdot) \in L^2(0, \infty : H)$, then A need not be analytic.

Hypothesis 3″. $K(\cdot) \in L^2(0, \infty; L(H))$.

COROLLARY 2.4. *Let A be any infinitesimal generator of a strongly continuous semi-group on H. Assume Hypothesis* 3″. *Then for each $y_0 \in H$ and $y_1 \in L^2(-\infty, 0; H)$ there exists a unique solution $y(t)$ to* (13) *in $C([0,T]; H)$ for any $T > 0$. Define the map $S(t)$ as in* (21). *Then it is a strongly continuous semigroup on $Z = H \times L^2(-\infty, 0; H)$ with generator* (22), (23).

See [12] for more general cases of integrodifferential operators where A is not analytic.

3. Quadratic control on finite horizon. Now we consider (13) with control

$$\begin{cases} y'(t) = Ay(t) + Ky_t + Bu(t) \\ y(0) = y_0 \\ y(\theta) = y_1(\theta), \quad \theta \in]-\infty, 0], \end{cases} \tag{24}$$

where $y_1 \in L^2(-\infty, 0; D_A(\frac{1}{2}, 2))$. Then by setting $z(t) = [y(t), y_t(\cdot)]'$ we obtain

$$\begin{cases} z'(t) = \mathcal{A}z(t) + \widetilde{B}u(t), \\ z(0) = [y_0, y_1]', \end{cases} \tag{25}$$

where

$$\widetilde{B} = \begin{pmatrix} B \\ 0 \end{pmatrix}.$$

For each $u \in L^2(0, T; U)$ we define the mild solution of (25) by

$$z(t) = S(t)[y_0, y_1]' + \int_0^t S(t-r)\widetilde{B}u(r)\, dr. \tag{26}$$

The mild solution (6) of (1) corresponds to the first component of $z(t)$ of the special case $y_1 = 0$, i.e.,

$$z(t) = S(t)[y_0, 0] + \int_0^t S(t-r)\widetilde{B}u(r)\, dr. \tag{27}$$

The cost functional (2) can be rewritten

$$J(u) = \int_0^T \left[|\widetilde{M}z(t)|^2 + |u(t)|^2\right] dt + \langle \widetilde{G}z(T), z(T)\rangle, \tag{28}$$

where

$$\widetilde{M} = \begin{pmatrix} M \\ 0 \end{pmatrix} \in L(Z; H_0) \quad \text{and} \quad \widetilde{G} = \begin{pmatrix} G & 0 \\ 0 & 0 \end{pmatrix} \in L^+(Z).$$

The control problem (26), (28) is a standard quadratic problem [1] in the state-space form [8], [9]. As is well known, the optimal control is given by the feedback law

$$\underline{u} = -\widetilde{B}^\star Q(t)z(t), \tag{29}$$

where Q is the unique selfadjoint nonnegative solution of the Riccati equation

$$\begin{cases} Q' + \mathcal{A}^\star Q + Q\mathcal{A} + \widetilde{M}^\star\widetilde{M} - Q\widetilde{B}\widetilde{B}^\star Q = 0, \\ Q(T) = \widetilde{G}. \end{cases} \tag{30}$$

Setting

$$Q(t) = \begin{pmatrix} Q_{11}(t) & Q_{12}(t) \\ Q_{21}(t) & Q_{22}(t) \end{pmatrix},$$

we can write (29) in the form

$$\underline{u}(t) = -B^\star[Q_{11}(t)y(t) + Q_{12}(t)y_t]. \tag{31}$$

The minimal cost corresponding to $\underline{u}$ is

$$J(\underline{u}) = \left\langle Q(0)\begin{pmatrix} y_0 \\ y_1 \end{pmatrix}, \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}\right\rangle. \tag{32}$$

Hence the minimal cost for the problem (1), (2) is given by

$$J(\underline{u}) = \langle Q_{11}(0)y_0, y_0\rangle. \tag{33}$$

Summing up, we have the following theorem.

THEOREM 3.1. *Assume Hypothesis* 1 *and* 3 *are true. Then there exists a unique optimal control for the problem* (1), (2). *It is given by the feedback law* (31), *and the minimal cost is given by* (33).

For the control problem (1), (2), where $K(\cdot)$ satisfies either Hypothesis 3′ or Hypothesis 3″, the feedback law (31) is still optimal and the optimal cost is given by (33).

4. Quadratic control on infinite horizon. Here we consider the control problem (1), (3). To avoid the trivial case we make the following assumption for (27).

Hypothesis 4. For each $y_0 \in H$ and $y_1 \in L^2(-\infty, 0; D_A(\frac{1}{2}, 2)$ there exists a control $u \in L^2(0, \infty; U)$ such that

$$J(u) = \int_0^\infty \left[|\widetilde{M}z(t)|^2 + |u(t)|^2\right] dt < \infty. \tag{34}$$

Later we give sufficient conditions for Hypothesis 4. Let $Q_T(t)$ be the solution of the Riccati equation (30) with $Q_T(T) = 0$. Then the following is known.

PROPOSITION 4.1. *Assume Hypotheses* 3 *and* 4 *are true. Then there exists a strong limit* Q_∞ *of* Q_T. Q_∞ *is the minimal nonnegative solution of the algebraic Riccati equation*

$$\mathcal{A}^\star Q + Q\mathcal{A} + \widetilde{M}^\star\widetilde{M} - Q\widetilde{B}\widetilde{B}^\star Q = 0. \tag{35}$$

If $\mathcal{A}$, $\widetilde{M}$ is detectable, then Q_∞ is the unique nonnegative solution of (35). *Moreover, $A - \widetilde{B}\widetilde{B}^\star Q_\infty$ generates an exponentially stable semigroup on Z.*

THEOREM 4.2. *Assume Hypotheses* 3 *and* 4 *are true. Then there exists a unique optimal control for* (26), (34). *It is given by the feedback law*

$$\underline{u} = -\widetilde{B}^\star Q_\infty z(t), \tag{36}$$

and the minimal cost is

$$J(\underline{u}) = \left\langle Q_\infty \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}, \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} \right\rangle. \tag{37}$$

In particular, the optimal control for the problem (1), (2) *is given by*

$$\underline{u}(t) = -B^\star[Q_{11}y(t) + Q_{12}y_t] \tag{38}$$

and

$$J(\underline{u}) = \langle Q_{11}y_0, y_0\rangle, \tag{39}$$

where

$$Q_\infty = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}.$$

If $Q_{12}(y_1) = \int_{-\infty}^0 Q_{12}(-\theta)y_1(\theta)\,d\theta$ for some $Q_{12} \in L^2(0,\infty; L(D_A(\frac{1}{2},2);H))$, then the optimal closed-loop system corresponding to (38) is

$$y'(t) = (A - BB^\star Q_{11})y(t) + \int_0^t [K(t-r) - BB^\star Q_{12}(t-r)]y(r)\,dr \tag{40}$$

and of the same form as (1). If $(\mathcal{A}, \widetilde{M})$ is detectable, then the resolvent operator for (40) is exponentially stable.

A sufficient condition for Hypothesis 4 can be found in [7]. For the sake of completeness we quote some results from [7]. Let $\widehat{K}(\cdot)$ be a maximal analytic extension of the Laplace transform of K, and let Ω_0 be its domain of definition. We set

$$\begin{cases} \rho_0 = \{\lambda \in \Omega : \exists\, (\lambda - A - \widehat{K}(\lambda))^{-1}\}, \\ F(\lambda) = (\lambda - A - \widehat{K}(\lambda))^{-1} \quad \text{for } \lambda \in \rho_0, \end{cases}$$

and we denote by ρ_1 the set of all isolated removable singularity of $F(\cdot)$. Moreover, we set

$$\begin{cases} \rho = \rho_0 \cup \rho_1, \\ F(\lambda) = \lim_{z\to\lambda} F(z), \qquad \lambda \in \rho\backslash\rho_0. \end{cases}$$

Define the *generalized spectrum* $\sigma = C\backslash\rho$. If λ_0 is a pole of $F(\cdot)$ of order m_0, we set, for λ sufficiently close to λ_0,

$$F(\lambda) = \sum_{n=0}^{\infty} S_n(\lambda - \lambda_0)^n + \sum_{n=0}^{m_0-1} Q_n(\lambda-\lambda_0)^{-n-1},$$

where

$$\begin{cases} Q_n = \frac{1}{2\pi i} \int_{C(\lambda_0,\varepsilon)} F(\lambda)(\lambda-\lambda_0)^n \, d\lambda, \\ S_n = \frac{1}{2\pi i} \int_{C(\lambda_0,\varepsilon)} F(\lambda)(\lambda-\lambda_0)^{-n-1} \, d\lambda, \end{cases}$$

and $C(\lambda_0, \varepsilon)$ is the circle with center λ_0 having sufficiently small radius $\varepsilon > 0$.

Let $\omega > 0$ be such that $\sigma \cap \{\lambda \in C : \text{Re}\lambda = -\omega\} = \emptyset$, and let $\sigma_+(\omega) = \sigma \cap \{\lambda \in C : \text{Re}\lambda > -\omega\}$, $\sigma_-(\omega) = \sigma \cap \{\lambda \in C : \text{Re}\lambda < -\omega\}$. We can make the following assumption.

Hypothesis 5. (i) $\sigma_+(\omega) = \{\lambda_1, \ldots, \lambda_N\}$, where for each $j = 1, \ldots, N$, λ_j is a pole of $F(\cdot)$ of order $m_j < \infty$.

(ii) The residues $R_{j,k}$, $k = 0, 1, \ldots, m_j$, of $F(\cdot)$ at $\lambda = \lambda_j$ are finite-rank operators.

The condition below is called a *Hautus condition.*

Hypothesis 6. Range $Q^\star_{j,k} \cap$ Ker $B^\star = \{0\}$ for all $j = 1, 2, \ldots, N$ and $k = 0, 1, \ldots, m_j$.

Let X be a Banach space, and let $C_\omega([0, \infty[; X)$ be the space of bounded continuous functions $x(t)$ in X with property $\sup_{t>0} \|x(t)e^{\omega t}\|_X < +\infty$. Under Hypothesis 5 it is shown [7] that Hypothesis 6 holds if and only if the following is true:

For each $y_0 \in H$ there exists a control $u \in C_\omega([0, \infty[; U)$ (in fact, $u \in C^\alpha_\omega([0, \infty[; U))$ such that $y \in C_\omega([0, \infty[; H)$, where y is the solution of (1). Hence if Hypotheses 5 and 6 hold, then the control problem (1), (3) is well defined. Modifying slightly the proof of [7, Thm. 2.3], we can show that under Hypothesis 6 system (13) is stabilizable in the above sense. We have, in fact, the following result, the proof of which was suggested to us by A. Lunardi.

THEOREM 4.3. *If Hypothesis* 6 *holds, then the system* (13) *is stabilizable, i.e., for each* $y_0 \in H$ *there exists a control* $u \in C_\omega([0, \infty[; U)$ *such that* $y \in C_\omega([0, \infty[; H)$ *for some* $\omega \in]0, \omega_0[$.

Proof. Define

$$R_{\lambda_j}(t) = \frac{1}{2\pi i} \int_{C(\lambda_j,\varepsilon)} e^{\lambda t} F(\lambda \, d\lambda) = \sum_{k=0}^{m_j-1} \frac{e^{\lambda_j t} t^k}{k!} Q_{j,k}$$

and

$$R^{\omega_0}_+(t) = \sum_{j=1}^{N} R_{\lambda_j}(t).$$

Then as in [7, Prop. 1.1] we can show that problem (13) is stabilizable in the above sense if and only if for each $y_0 \in H$ and $y_1 \in L^2(-\infty, 0, D_A(\frac{1}{2}, 2))$ there exists $u \in C_\omega([0, \infty[; U)$ such that

$$(41) \quad R^\omega_+(t) y_0 + \int_0^{+\infty} R^\omega_+(t-s) K_1(s) y_1 \, ds = -\int_0^{+\infty} R^\omega_+(t-s) Bu(s) \, ds, \qquad t \geq 0,$$

where $K_1(\cdot)$ is as given in §2. First, we assume $\text{Re}\,\lambda_j > 0$, $j = 1, 2, \ldots, N$, and show (41) with $\omega = \omega_0$. If there exists λ_j with $\text{Re}\,\lambda_j = 0$, then we can set $\underline{v}(t) = e^{\varepsilon t} y(t)$ for sufficiently small $\varepsilon > 0$ and reduce the problem to the case for which $\omega = \omega_0 - \varepsilon$.

As in the proof of Theorem 2.3 in [7], we can show that (41) is equivalent to

$$\sum_{j=1}^{N}\sum_{n=0}^{m_j-1} e^{\lambda_j t}\frac{t^n}{n!}Q_{j,n}y_0+\sum_{j=1}^{N}\sum_{n=0}^{m_j-1}\sum_{k=n}^{m_j-1} e^{\lambda_j t}\frac{t^n}{n!}\int_0^{+\infty} e^{-\lambda_j s}(-s)^{k-n}Q_{j,k}K_1(s)y_1\,ds \tag{42}$$
$$=-\sum_{j=1}^{N}\sum_{n=0}^{m_j-1}\sum_{k=n}^{m_j-1} e^{\lambda_j t}\frac{t^n}{n!}\int_0^{+\infty} e^{-\lambda_j s}(-s)^{k-n}Q_{j,k}Bu(s)\,ds.$$

Note that the second term is well defined since $K_1(\cdot)y_1$ is bounded and $\operatorname{Re}\lambda_j > 0$. Since the functions $t \to e^{\lambda_j t}t^n$ are linearly independent, (42) is equivalent to $\Gamma u = Q[y_0, K_1(\cdot)y_1]$, where

$$\Gamma : C_\omega([0,+\infty[;U) \to H^K,\ K=\sum_{j=1}^{N} m_j,$$

$$\Gamma u = \left\{\sum_{k=n}^{m_j-1}\int_0^{+\infty} e^{-\lambda_j s}(-s)^{k-n}Q_{j,k}Bu(s)ds\right\}_{j=1,\dots,N;n=0,\dots,m_j-1},$$

and $Q : H \to H^K$,

$$Q(y_0, K_1(\cdot))y_1)$$
$$=\left\{Q_{j,n}y+\textstyle\sum_{k=n}^{m_j-1}\int_0^{+\infty} e^{-\lambda_j s}(-s)^{k-n}Q_{j,k}K_1(s)y_1ds\right\}_{j=1,\dots,N;n=0,\dots,m_j-1}.$$

Since the range of Γ and Q are finite-dimensional, (42) holds if and only if

$$\operatorname{Ker} Q^\star \supset \operatorname{Ker}\Gamma^\star. \tag{43}$$

For each $(h_{jn}) = (h_{jn})_{j=1,\dots,N;n=0,\dots,m_j-1} \in H^K$ we have

$$\Gamma^\star(h_{jn})u = -\sum_{j=1}^{N}\sum_{k=0}^{m_j-1}\sum_{h=0}^{m_j-1-k}\int_0^{+\infty}\frac{(-s)^k}{k!}e^{-\lambda_j s} < u(s), B^\star Q^\star_{j,k+h}h_{j,k+h} > ds$$
$$+\,Q^*(h_{jn})(y_0, K_1(\cdot))y_1)$$
$$=\sum_{j=1}^{N}\sum_{h=0}^{m_j-1}\langle y_0, Q^*_{jh}h_{jh}\rangle$$
$$+\sum_{j=1}^{N}\sum_{k=0}^{m_j-1}\sum_{h=0}^{m_j-1-k}\int_0^{+\infty}\frac{(-s)^k}{k!}e^{-\lambda_j s}\langle K_1(s)y_1, Q^\star_{j,k+n}h_{j,k+n}\rangle ds$$

so that

$$\operatorname{Ker}\Gamma^\star = \left\{(h_{jn})\in H^K : B^\star\left(\sum_{h=k}^{m_j-1}Q^\star_{jh}h_{jn}\right),\ j=1,\dots,N,\ k=0,1,\dots,m_j-1\right\}$$
$$=\{(h_{jn})\in H^K : B^\star Q^\star_{jh}h_{jn}, j=1,\dots,N, k=0,1,\dots,m_j-1\},$$

$$\text{Ker } Q^\star \supset \{(h_{jn}) \in H^K : \sum_{j=1}^{N} \sum_{h=0}^{m_j-1} Q^\star_{jh} h_{jn} = 0\}.$$

Now we can show that (13) is stabilizable and is equivalent to (43). It is easy to see that if (13) is stabilizable, then (43) holds. Conversely, assume that (23) holds and let $h \in \text{Ker}\, B^\star Q^\star_{j_0 h_0}$ for some j_0, h_0. Set $h_{jn} = \delta_{j,j_0}\delta_{n,n_0} h$; then $B^\star Q^\star_{jn} h_{jn} = 0$ for each $j = 1, \ldots, N, h = 0, \ldots, m_j - 1$. By (43) we have

$$\sum_{j=1}^{N} \sum_{h=0}^{m_j-1} Q^\star_{jn} h_{jn} = Q^\star_{j_0 n_0} h = 0.$$

Therefore, for each j_0, h_0 we have

$$\text{Ker}\, Q^\star_{j_0 n_0} \supset B^\star Q^\star_{j_0 n_0},$$

and (13) is stabilizable. □

Now consider the detectability of $\mathcal{A}, \widetilde{M}$. It is useful to consider the following:

$$(44) \qquad \begin{cases} \underline{\eta}'(t) = -A^\star \underline{\eta}(t) - \int_t^T K^\star(r-t)\underline{\eta}(r)\, dr, \\ \underline{\eta}(T) = \eta_1. \end{cases}$$

Suppose we have classical solutions for (4) and (44). Then by differentiating $\langle y(t), \underline{\eta}(t)\rangle$, integrating from $t = 0$ to T, and using the Fubini theorem we obtain

$$\langle y(T), \eta_1\rangle = \langle y_0, \underline{\eta}(0)\rangle.$$

Hence (44) is the adjoint system of (4). Equations (44) can be also written

$$(45) \qquad \begin{cases} \eta'(t) = A^\star \eta(t) + \int_0^t K^\star(t-r)\eta(r)\, dr, \\ \eta(0) = \eta_1. \end{cases}$$

Thus the detectability of $(\mathcal{A}, \widetilde{M})$ is translated into the stabilizability of the following system:

$$(46) \qquad \eta'(t) = A^\star \eta(t) + \int_0^t K^\star(t-r)\eta(r)\, dr + \widetilde{M} v(t).$$

If $A^\star$ and $K^\star$ have properties similar to those of A and K we can obtain sufficient conditions for detectability. Finally, we note that we can also solve control problems for inhomogeneous systems as in [4], and [5].

Example 4.4. Let Ω be a bounded open domain in R^n with C^2 boundary $\partial\Omega$. Consider the heat equation in materials of the *fading-memory* type introduced by Nunziato [20]:

$$(47) \qquad \begin{cases} b_0 \dfrac{\partial y}{\partial t}(t,x) + \dfrac{\partial}{\partial t}\displaystyle\int_0^t \beta(t-r) y(r,x)\, dr \\ \qquad = c_0 \Delta y(t,x) - \displaystyle\int_0^t \gamma(t-r)\Delta y(r,x)\, dr + B_0 u(t,x), \ t > 0, x \in \overline{\Omega}, \\ y(0,x) = y_0(x), \qquad x \in \overline{\Omega}, \\ \Gamma y(t,x) = 0, \qquad t > 0, \quad x \in \partial\Omega, \end{cases}$$

where $\Gamma y = y$ or $\Gamma y = \partial y/\partial n$. $y(t,x)$ represents the temperature at $x \in \overline{\Omega}$ at time t, b_0, and c_0 are positive constants, and u is the heat supply. β and γ are completely monotone kernels with

$$\beta(t) = \int_0^\infty e^{-\omega t}\mu(d\omega), \qquad \gamma(t) = \int_0^\infty e^{-\omega t}\nu(d\omega),$$

where μ and ν are positive Borel measures with compact support $\operatorname{supp}\mu$ and $\operatorname{supp}\nu$ contained in $]a,\infty[$, $a > 0$. Then the heat equation (47) can be written as (1) in $H = L^2(\Omega)$ with

$$Ah = \frac{1}{b_0}(c_0\Delta h - \beta(0)h), \qquad D(A) = \{h \in H : \Delta h \in H,\ \Gamma h = 0\},$$

$$K(t)h = \frac{1}{b_0}(-\beta'(t)h - \gamma(t)\Delta h),$$

$$B = \frac{1}{b_0}B_0.$$

Let $\widetilde{\beta}(\cdot)$ and $\widetilde{\gamma}(\cdot)$ be analytic extensions of the Laplace transforms of β and γ to $C\backslash\operatorname{supp}\mu$ and $C\backslash\operatorname{supp}\nu$, respectively. Then

$$\rho_0 = \left\{\lambda \in C : c_0 - \widetilde{\gamma}(\lambda) \neq 0, \frac{\lambda(b_0 + \widetilde{\beta}(\lambda))}{c_0 - \widetilde{\gamma}(\lambda)} \neq -\lambda_n\right\},$$

$$F(\lambda) = \frac{b_0}{\frac{\lambda(b_0+\widetilde{\beta}(\lambda))}{c_0-\widetilde{\gamma}(\lambda)}} R\left(\frac{\lambda(b_0 + \widetilde{\beta}(\lambda))}{c_0 - \widetilde{\gamma}(\lambda)}, \Delta\right),$$

where $\{-\lambda_n\}$ is the decreasing sequence of the eigenvalues of Δ. Hence

$$\{\lambda \in: \ \operatorname{Re}\lambda \geq 0\} = \begin{cases} \emptyset & \text{if } \Gamma y = y \text{ on } \partial\Omega, \\ \{0\} & \text{if } \Gamma y = \frac{\partial y}{\partial n} \text{ on } \partial\Omega. \end{cases}$$

If $\Gamma y = y$, then (47) with $u = 0$ is stable. Hence our control problems (1), (2) and (1), (3) are well defined. If $\Gamma y = \frac{\partial y}{\partial n}$, then (47) with $u = 0$ is not stable, but satisfies Hypothesis 5. This implies that if $h(x) = h_0$ (constant different from 0) $\notin$ Ker $B^\star$, then (47) is stabilizable. If $Bu = b(x)u(t)$, $b(\cdot) \in H$, and u is a scalar, then $\int_\Omega b(x)\,dx \neq 0$ implies stabilizability. Hence the quadratic problems (1), (3) and, of course, (1), (2) are well defined. See [6] for more discussions on this example and other examples of system (1). See also [19] for heat equations with memory.

The following example was introduced to us by J. Zabczyk and is covered by our model.

Example 4.5. Consider the delay equation

$$(48) \qquad \begin{cases} x''(t) = -k(0)x(t) - \int_{-\infty}^0 k'(-r)x(t+r)\,dr + u(t), \\ x(0) = x_0, \qquad x'(0) = x_1, \end{cases}$$

which can be regarded as a model of the oscillation of a particle suspended by light linearly viscoelastic string, where the relaxation modulus $k : [0,\infty[\to R$ is differentiable and such that

$$k(t) > 0, \quad k'(t) < 0, \quad k''(t) > 0, \quad \lim_{t\to\infty} k'(t) = 0.$$

Setting

$$y = \begin{pmatrix} x \\ x' \end{pmatrix},$$

we can write (48) as

$$y'(t) = \begin{pmatrix} 0 & 1 \\ -k(0) & 0 \end{pmatrix} - \int_{\infty}^{0} k(-r) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} x(t+r)\,dr + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t),$$

$$y(0) = y_0 = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}.$$

If $k \in L^2(0,\infty)$, then this is a special case covered by Theorem 4.2. If $k(t) = e^{-at}$, $a > 0$, the spectrum σ is given by

$$\sigma = \{\lambda : \lambda^3 + a\lambda^2 + k(0)\lambda + ak(0) - 1 = 0\}.$$

Set

$$A = \begin{pmatrix} 0 & 1 \\ -k(0) & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad M = [1,0];$$

then (A,B) is controllable and (M,A) is observable. Hence the minimization problem

$$J(u) = \int_0^{\infty} \left[|x(t)|^2 + |u(t)|^2\right] dt$$

is well defined.

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