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On a Long-standing Conjecture of E. De Giorgi: Symmetry in 3D for General Nonlinearities and a Local Minimality Property

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Abstract. This paper studies a conjecture made by E. De Giorgi in 1978 concerning the one-dimensional character (or symmetry) of bounded, monotone in one direction, solutions of semilinear elliptic equations $\Delta u = F'(u)$ in all of $\mathbf{R}^n$. We extend to all nonlinearities $F \in C^2$ the symmetry result in dimension $n = 3$ previously established by the second and the third authors for a class of nonlinearities F which included the model case $F'(u) = u^3 - u$. The extension of the present paper is based on a new energy estimates which follow from a local minimality property of u.
In addition, we prove a symmetry result for semilinear equations in the halfspace $\mathbf{R}^4_+$. Finally, we establish that an asymptotic version of the conjecture of De Giorgi is true when $n \leq 8$, namely that the level sets of u are flat at infinity.

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1 Introduction

In 1978, E. De Giorgi [16] stated the following conjecture:

Conjecture (DG). *Let $u : \mathbf{R}^n \to (-1,1)$ be a smooth entire solution of the semilinear equation $\Delta u = u^3 - u$ satisfying the monotonicity condition*

$$\partial_{x_n} u > 0 \quad \textit{in } \mathbf{R}^n. \tag{1.1}$$

Then all level sets $\{u = s\}$ of u are hyperplanes, at least if $n \leq 8$.

The flatness of the level sets of u can be rephrased by saying that u depends only on one variable. For the model equation chosen by De Giorgi, this is equivalent to the existence of a unit vector $a \in \mathbf{R}^n$ and a constant $b \in \mathbf{R}$ such that

$$u(x) = \tanh\left(\frac{1}{\sqrt{2}}\langle a, x\rangle + b\right) \qquad \forall x \in \mathbf{R}^n.$$

When $n = 2$, this conjecture was proved in 1997 by N. Ghoussoub and C. Gui [23] (see also [21] for further extensions of this result). More recently, the second and third authors [2] have established the conjecture in the case $n = 3$. The higher dimensional cases are still open. The proofs for $n = 2$ and 3 use some techniques developed by H. Berestycki, L. Caffarelli and L. Nirenberg in [8] for the study of symmetry properties of positive solutions of semilinear elliptic equations in halfspaces.

More generally, the same symmetry question can be raised for bounded entire solutions of semilinear equations of the form

$$\Delta u - F'(u) = 0 \quad \text{in } \mathbf{R}^n, \tag{1.2}$$

under the monotonicity assumption (1.1). By "u is an entire solution", we simply mean that u is a solution in all space $\mathbf{R}^n$. The results of [23] for $n = 2$ and the results of the present paper for $n = 3$ establish the following:

Theorem 1.1 *Assume that $F \in C^2(\mathbf{R})$. Let u be a bounded solution of* (1.2) *satisfying* (1.1)*. If $n = 2$ or $n = 3$, then all level sets of u are hyperplanes, i.e., there exist $a \in \mathbf{R}^n$ and $g \in C^2(\mathbf{R})$ such that $u(x) = g(\langle a, x\rangle)$ for all $x \in \mathbf{R}^n$.*

In this paper we make a short survey on this problem and, at the same time, we prove Theorem 1.1 in dimension three. This extends to all nonlinearities $F \in C^2$ the results of the second and third authors [2], where Theorem 1.1 was proved when $n = 3$ for a class of nonlinearities F which included the model case $F'(u) = u^3 - u$. The extension of the present paper is based on new energy estimates which follow from a local minimality property of u, discussed in Section 4 below. In addition, we prove in Section 6 a symmetry result for semilinear equations in the halfspace $\mathbf{R}^4_+$. Finally, in Section 7 we establish that an asymptotic version of the conjecture of De Giorgi (already considered by L. Modica in [30]) is true when $n \le 8$, namely that the level sets of u are flat at infinity. As we will see below, this result is related to the Bernstein problem about the flatness of entire minimal graphs.

In some cases it is helpful to make the additional hypothesis (consistent with the original conjecture of De Giorgi, but not present in it) that

$$\lim_{x_n \to -\infty} u(x', x_n) = \inf u \quad \text{and} \quad \lim_{x_n \to +\infty} u(x', x_n) = \sup u \quad \forall x' \in \mathbf{R}^{n-1}, \tag{1.3}$$

where $x = (x', x_n)$, $x' \in \mathbf{R}^{n-1}$ and $x_n \in \mathbf{R}$. Here, the limits are *not* assumed to be uniform in $x' \in \mathbf{R}^{n-1}$. Even in this simpler form, conjecture (DG) was first proved in [23] for $n = 2$, in [2] for $n = 3$, and it remains open for $n \ge 4$.

In Theorem 1.1 the direction a of the variable on which u depends is not known a priori. Indeed, if u is a one-dimensional solution satisfying (1.1), we can slightly rotate coordinates to obtain a new one-dimensional solution still satisfying (1.1). The same remark holds in the case when the additional assumption (1.3) is made. Instead, if one further assumes that the limits in (1.3) are uniform in $x' \in \mathbf{R}^{n-1}$ then an a priori choice of the direction a is imposed, namely $a \cdot x = x_n$, and furthermore one knows a priori that every level set of u

is contained between two parallel hyperplanes. With the additional assumption that the limits in (1.3) are uniform in $x' \in \mathbf{R}^{n-1}$, the question of De Giorgi is known as "Gibbons conjecture", and it is by now completely settled. The conclusion $u = u(x_n)$ has been recently proved by N. Ghoussoub and C. Gui [23] for the case $n \leq 3$ and, independently and using different techniques, for general n by M.T. Barlow, R.F. Bass and C. Gui [4], H. Berestycki, F. Hamel and R. Monneau [9], and A. Farina [20]. These results apply to equation (1.2) for various classes of nonlinearities F which always include the Ginzburg-Landau model $\Delta u = u^3 - u$.

The first positive partial result on conjecture (DG) was established in 1980 by L. Modica and S. Mortola [34]. They proved the flatness of the level sets in the case $n = 2$, under the additional assumption that the level sets $\{u = s\}$ are the graphs of an equi-Lipschitz family of functions of x'. Note that, since $\partial_{x_n} u > 0$, each level set of u is the graph of a function of x'. Their proof was based on a Liouville-type theorem for non-uniformly elliptic equations in divergence form, due to J. Serrin, and on the observation that the bounded ratio $\sigma := \partial_{x_1} u / \partial_{x_2} u$ solves, after an appropriate change of independent variables, an equation of this type.

The idea of considering σ occurs also in [8, 23, 2]. But this time a different Liouville-type theorem, due to H. Berestycki, L. Caffarelli and L. Nirenberg [8], is used (see Theorem 3.1 below). This theorem does not require the assumption that σ is bounded, but instead a suitable compatibility condition between the growth of σ and the degeneracy of the coefficients of the equation.

L. Modica proved in [31] that if u is a bounded solution of (1.2) and $F \geq 0$ on the range of u, then the pointwise gradient bound $|\nabla u|^2 \leq 2F(u)$ holds in $\mathbf{R}^n$. This bound was extended in [12] by L. Caffarelli, N. Garofalo and F. Segala to more general equations. They also proved that if equality holds at some point, then the level sets of u are hyperplanes (regarding this fact, see also the survey article [27] by B. Kawohl).

2 On the relation between (DG) and the Bernstein problem

In this section we describe the heuristic argument (that we make rigorous in Section 7) establishing a relation between the conjecture of De Giorgi and the Bernstein problem about the flatness of entire minimal graphs. This problem, after a long series of partial results starting with S. Bernstein, was completely settled in 1969 with the famous work [10] of E. Bombieri, E. De Giorgi and E. Giusti (see also the nice presentations in [25, 29] on this and related problems). It is now known that the following two statements hold:

(a) *Every smooth entire solution $\psi : \mathbf{R}^m \to \mathbf{R}$ of the minimal surface equation*

$$\operatorname{div}\left(\frac{\nabla\psi}{\sqrt{1+|\nabla\psi|^2}}\right) = 0 \tag{2.1}$$

is an affine function if $m \leq 7$.

(b) *If $m \geq 8$ there exist non affine entire and smooth solutions of* (2.1).

A natural method in the analysis of entire solutions of PDE's, first used by W.H. Fleming in [18] (precisely in connection with the Bernstein problem), is the analysis of the *blow-down* family of functions associated to the solution. This leads in some cases to an understanding of the behaviour at infinity of the solution.

In our case, we assume that u is a bounded solution of (1.2) satisfying (1.1) and (1.3), and that $F \in C^2(\mathbf{R})$ satisfies

$$F > F(m) = F(M) \qquad \text{in } (m, M)$$

where $m = \inf u$ and $M = \sup u$. Note that for the model case $\Delta u = u^3 - u$ in conjecture (DG), we have $F(u) = (1-u^2)^2/4$, $m = -1$ and $M = 1$, so that the previous condition is satisfied. We then define $u_R(x) = u(Rx)$ and study the behaviour of u_R as $R \to \infty$. The functions u_R are bounded entire solutions of the rescaled PDE's

$$\frac{1}{R}\Delta u_R = RF'(u_R)$$

corresponding to the first variation of the functionals $\mathcal{E}_R(\cdot, \Omega)$, defined by

$$\mathcal{E}_R(v, \Omega) := \int_\Omega \left\{ \frac{1}{2R}|\nabla v|^2 + RF(v) \right\} dx \tag{2.2}$$

(where we emphasize also the dependence on the domain of integration). A classical result of L. Modica and S. Mortola [33] states that the functionals $\mathcal{E}_R(\cdot, \Omega)$ Γ-converge to a constant multiple of the area functional. Specifically, setting

$$c_F = \int_m^M \sqrt{2F}(s)\, ds,$$

the following three properties hold:

(i) (Lower semicontinuity) If E has locally finite perimeter in $\mathbf{R}^n$, then

$$\liminf_{i\to\infty} \mathcal{E}_{R_i}(u_i, \Omega) \geq c_F P(E, \Omega)$$

whenever $\Omega \subset \mathbf{R}^n$ is an open set, $R_i \to \infty$ and u_i converge to 1_E in $L^1_{\mathrm{loc}}(\mathbf{R}^n)$. Here 1_E denotes the function equal to M on E and equal to m on $\mathbf{R}^n \setminus E$, and $P(E, \Omega)$ denotes the *perimeter* of E in Ω, which coincides with the surface measure of $\Omega \cap \partial E$ if ∂E is sufficiently regular (see for instance [25]).

(ii) (Approximation) If E has locally finite perimeter in $\mathbf{R}^n$, then there exists a family $(v_R) \subset H^1_{\mathrm{loc}}(\mathbf{R}^n)$ converging to 1_E in $L^1_{\mathrm{loc}}(\mathbf{R}^n)$ and such that

$$\limsup_{R\to\infty} \mathcal{E}_R(v_R, \Omega) \leq c_F P(E, \Omega)$$

whenever $\Omega \subset \mathbf{R}^n$ is a bounded open set with $P(E, \partial\Omega) = 0$.

(iii) (Coercivity) If $R_i \to \infty$ and $(u_i) \subset H^1_{\rm loc}(\mathbf{R}^n)$ satisfies

$$\sup_i \mathcal{E}_{R_i}(u_i, \Omega) < \infty \qquad\qquad \forall \Omega \subset\subset \mathbf{R}^n,$$

then there exists a subsequence $u_{i(k)}$ and a set of locally finite perimeter E in $\mathbf{R}^n$ such that $u_{i(k)}$ converge to 1_E in $L^1_{\rm loc}(\mathbf{R}^n)$ as $k \to \infty$.

The Modica-Mortola theorem states that the functionals $\mathcal{E}_R$ converge (in an appropriate variational sense) to the perimeter functional as $R \to \infty$. Consider now a level set $\{u = s\}$ of an entire solution of (1.2) satisfying (1.1) and (1.3), and an arbitrary radius $r > 0$. Then $\{u_R = s\} \cap B_r$, a rescaled copy of $\{u = s\} \cap B_{Rr}$, is expected (heuristically) to be closer and closer in B_r to a stationary surface of the area functional, as $R \to \infty$. We notice also that, due to (1.1), $\{u = s\}$ is a graph along the x_n direction, so we may expect the limiting stationary surface to be a graph as well. For the purpose of this heuristic discussion we have identified stationary solutions and local minimizers, but this issue is far from being trivial (see Section 4).

Since r is arbitrary and we know that every entire minimal graph defined on $\mathbf{R}^m = \mathbf{R}^{n-1}$ is a hyperplane for $m = n - 1 \le 7$, we may conclude that the level sets $\{u = s\}$ are expected to be "flat" at infinity whenever $n \le 8$. This provides a strong indication of why De Giorgi's conjecture should be true, at least asymptotically, for $n \le 8$.

This argument will be made rigorous in Section 7, where we establish that the rescaled level sets $R^{-1}(\{u = s\} \cap B_{Rr})$ are closer and closer to a minimal graph in B_r as R tends to ∞ through subsequences —see also the nice results in [26] on convergence of stationary solutions of (1.2).

3 Proof of Theorem 1.1 for $n = 2$

In this section we present the method leading to Theorem 1.1 in dimensions two and three. In dimension two, the proof coincides with the one given by Ghoussoub and Gui in [23], and it will be completed in this section. When $n = 3$ the method described here is the first step towards the theorem, but in this dimension the proof needs some additional work and will be completed in Section 5. Note that in dimension three, Theorem 1.1 extends the results of [2], in which only a particular class of nonlinearities (including the model case $F'(u) = u^3 - u$) was considered.

The idea is the following. Consider the functions

$$\varphi := \partial_{x_n} u > 0 \qquad \text{and} \qquad \sigma_i := \frac{\partial_{x_i} u}{\partial_{x_n} u} = \frac{\partial_{x_i} u}{\varphi}$$

for each $i = 1, \dots, n-1$. The goal is to prove that every σ_i is constant in $\mathbf{R}^n$, since this clearly implies that $\nabla u = a|\nabla u|$ for some constant unit vector a, and hence that the level sets of u are hyperplanes orthogonal to a.

Notice that since

$$\varphi^2 \, \nabla \sigma_i = \varphi \, \nabla \partial_{x_i} u - \partial_{x_i} u \, \nabla \varphi$$

and since both $\partial_{x_i} u$ and φ solve the same linear equation $\Delta v = F''(u)v$, we have

$$\operatorname{div}(\varphi^2 \nabla \sigma_i) = \varphi \Delta \partial_{x_i} u - \partial_{x_i} u \Delta \varphi = 0.$$

The conclusion that σ_i is necessarily constant in dimensions two and three uses the following Liouville-type theorem due to H. Berestycki, L. Caffarelli and L. Nirenberg [8].

Theorem 3.1 *Let $\varphi \in L^\infty_{\rm loc}(\mathbf{R}^n)$ be a positive function. Assume that $\sigma \in H^1_{\rm loc}(\mathbf{R}^n)$ satisfies*

$$\sigma \operatorname{div}(\varphi^2 \nabla \sigma) \geq 0 \qquad \text{in } \mathbf{R}^n \tag{3.1}$$

in the distributional sense. For every $R > 1$, let $B_R = \{|x| < R\}$ and assume that there exists a constant C independent of R such that

$$\int_{B_R} (\varphi\sigma)^2 \, dx \leq CR^2 \qquad \forall R > 1. \tag{3.2}$$

Then σ is constant.

The proof of this result is based on a simple Caccioppoli type estimate for the function σ (see [8] or [2]).

To apply this theorem to the conjecture of De Giorgi, note that $\varphi\sigma_i = \partial_{x_i} u$. Therefore, in this case, condition (3.2) will hold if

$$\int_{B_R} |\nabla u|^2 \, dx \leq CR^2 \qquad \forall R > 1, \tag{3.3}$$

for some constant C independent of R.

Next, we point out that since $u \in L^\infty(\mathbf{R}^n)$ is a solution of $\Delta u - F'(u) = 0$, then $|\nabla u|$ also belongs to $L^\infty(\mathbf{R}^n)$. This is easily proved using standard interior $W^{2,p}$ estimates for the Laplacian in every ball of radius 1 in $\mathbf{R}^n$. Therefore, estimate (3.3) is obviously true when $n = 2$. This finishes the proof of Theorem 1.1 for $n = 2$.

We conclude this section with some comments on the sharpness of the previous argument. In Section 5 we will prove the energy upper bound $\int_{B_R} |\nabla u|^2 \, dx \leq CR^{n-1}$ in every dimension n, and wee will see that this bound is sharp. However, the optimal (maximal) exponent γ_n such that

$$\int_{B_R} (\varphi\sigma)^2 \, dx \leq CR^{\gamma_n} \quad \forall R > 1 \quad \Longrightarrow \quad \sigma \text{ constant} \tag{3.4}$$

in the Liouville-type theorem above (assuming that equality holds in (3.1)) is not presently known; this is an interesting *open problem.* In [3] it is proved that γ_n is strictly less than n for $n \geq 3$. Also, a sharp choice of the exponents in the counterexamples of [23] shows that $\gamma_n < 2 + 2\sqrt{n-1}$ when $n \geq 7$. Finally, note that if we had $\gamma_n \geq n - 1$ for some n, then the argument above would establish the conjecture of De Giorgi in dimension n.

4 The monotonicity assumption and local minimality

In this section we investigate in detail the consequences of the monotonicity assumption (1.1). We begin by introducing the notion of stability.

Definition 4.1 (Stability) *We say that a solution u of* (1.2) *is stable if the second variation of energy $\delta^2\mathcal{E}_1/\delta^2_\xi$ with respect to compactly supported perturbations ξ is nonnegative, that is, if*

$$\int_{\mathbf{R}^n} \left\{ |\nabla\xi|^2 + F''(u)\xi^2 \right\} dx \geq 0 \qquad \forall \xi \in C_c^\infty(\mathbf{R}^n). \tag{4.1}$$

We have used the notation $\mathcal{E}_1$ for the energy, as in (2.2). It is a well known fact in the theory of maximum principles that the stability condition (4.1) is equivalent to the existence of a strictly positive solution φ of $\Delta\varphi = F''(u)\varphi$. That is, we have the following:

Proposition 4.2 *Let $H : \mathbf{R}^n \to \mathbf{R}$ be a bounded continuous function. Then*

$$\int_{\mathbf{R}^n} \left\{ |\nabla\xi|^2 + H(x)\xi^2 \right\} dx \geq 0 \qquad \forall \xi \in C_c^\infty(\mathbf{R}^n) \tag{4.2}$$

if and only if there exists a continuous function $\varphi : \mathbf{R}^n \to (0,\infty)$ such that $\Delta\varphi = H(x)\varphi$ in the sense of distributions.

Proof. Condition (4.2) implies that the first eigenvalue of the Schrödinger operator $-\Delta + H(x)$ in each ball B_R is nonnegative. Since the first eigenvalue in B_R is a decreasing function of R, it follows that all these first eigenvalues are positive. This implies that, for every constant $c_R > 0$, there exists a unique solution φ_R of

$$\begin{cases} \Delta\varphi_R = H(x)\varphi_R & \text{in } B_R \\ \varphi_R = c_R & \text{on } \partial B_R, \end{cases}$$

and, moreover, $\varphi_R > 0$ in B_R. We choose the constant c_R such that $\varphi_R(0) = 1$. Then, by the Harnack inequality, a subsequence of (φ_R) converges locally to a solution $\varphi > 0$ of $\Delta\varphi = H(x)\varphi$.

Conversely, multiplying the equation $\Delta\varphi = H(x)\varphi$ by ξ^2/φ, integrating by parts, and using the Cauchy-Schwarz inequality, we obtain (4.2). □

As a corollary, we can prove that every monotone solution is stable.

Corollary 4.3 (Monotonicity implies stability) *Every bounded entire solution u of* (1.2) *satisfying the monotonicity assumption* (1.1) *is stable.*

Proof. We simply have to notice that $\varphi = \partial_{x_n} u$ is strictly positive and solves the linearized equation $\Delta\varphi = F''(u)\varphi$. Then, the stability of u follows from Proposition 4.2 with $H = F''(u)$. □

We say that u is a *local minimizer* of $\mathcal{E}_1$ if the energy does not decrease under compactly supported perturbations, i.e.,

$$\mathcal{E}_1(u,\Omega) \le \mathcal{E}_1(v,\Omega) \qquad \text{whenever } \{u \neq v\} \subset \Omega \subset\subset \mathbf{R}^n.$$

Corollary 4.3 indicates a connection between the monotonicity assumption and the local minimality of u with respect to the energy $\mathcal{E}_1$, since the stability property is a necessary condition for local minimality. In fact, we will now see that the monotonicity assumption (1.1) implies the local minimality of u in a certain class of compactly supported perturbations —a stronger property than stability.

For this purpose, let us introduce the functions $\underline{u}$ and $\overline{u} : \mathbf{R}^{n-1} \to \mathbf{R}$ defined by

$$\underline{u}(x') := \lim_{x_n \to -\infty} u(x', x_n) \qquad \text{and} \qquad \overline{u}(x') := \lim_{x_n \to +\infty} u(x', x_n). \tag{4.3}$$

Notice that $\underline{u}$ and $\overline{u}$ are well defined if (1.1) holds. Moreover, if u is bounded and satisfies (1.2), then a simple limiting argument (see [2] for details) shows that $\underline{u}$ and $\overline{u}$ are also bounded entire solutions of the same equation (1.2), now on $\mathbf{R}^{n-1}$. In particular, $\underline{u}$ and $\overline{u}$ belong to $C^{2,\alpha}_{\rm loc}(\mathbf{R}^{n-1})$ for each $\alpha \in (0,1)$ (this follows from local $W^{2,p}$ estimates applied to the equation and to the linearized equations satisfied by $\underline{u}$, $\overline{u}$, and their first derivatives).

The following is the main result of this section.

Theorem 4.4 (Monotonicity implies local minimality) *Let u be a bounded entire solution of* (1.2) *satisfying* (1.1)*, and let $\Omega \subset \mathbf{R}^n$ be a smooth bounded domain. Then*

$$\int_\Omega \left\{ \frac{1}{2}|\nabla u|^2 + F(u) \right\} dx \le \int_\Omega \left\{ \frac{1}{2}|\nabla v|^2 + F(v) \right\} dx$$

for every function $v \in C^1(\overline{\Omega})$ such that $v \equiv u$ on $\partial\Omega$ and

$$\underline{u}(x') \le v(x', x_n) \le \overline{u}(x') \qquad \textit{for all } x = (x', x_n) \in \Omega. \tag{4.4}$$

It seems that L. Modica was already aware (see [30]) of the connection between the monotonicity assumption (1.1) and the local minimality of u, although [30] does not contain an explicit proof of this fact.

We now give the proof of Theorem 4.4, which is based on some more or less known results about calibrations for scalar functionals of the Calculus of Variations. After giving the proof of the theorem, we will explain in more detail its geometric motivation and, at the same time, we will prove an analogous result fore more general functionals.

Proof of Theorem 4.4. We denote the energy in Ω of a function $w \in C^1(\overline{\Omega})$ by

$$\mathcal{E}(w) = \int_\Omega \left\{ \frac{1}{2}|\nabla w|^2 + F(w) \right\} dx,$$

and we consider the set

$$U = \{(x,s) \in \overline{\Omega} \times \mathbf{R} \,:\, \underline{u}(x') < s < \overline{u}(x')\} \subset \overline{\Omega} \times \mathbf{R}$$

and the class of functions

$$\begin{aligned}\mathcal{A} &= \{w \in C^1(\overline{\Omega}) \,:\, \underline{u}(x') < w(x) < \overline{u}(x') \quad \forall x \in \overline{\Omega}\} \\ &= \{w \in C^1(\overline{\Omega}) \,:\, (x, w(x)) \in U \quad \forall x \in \overline{\Omega}\}.\end{aligned}$$

Note that the function v in the statement of the theorem may not belong to $\mathcal{A}$, since the inequalities in (4.4) are not strict. However, since $\partial_{x_n} u > 0$, we have that $u \in \mathcal{A}$. In particular, we have that $u + \tau(v-u) \in \mathcal{A}$ for every $\tau \in [0,1)$, and that $u + \tau(v-u) \equiv u$ on $\partial\Omega$. Hence, by letting $\tau \to 1$, we see that the theorem will be proved if we show that

$$\mathcal{E}(u) \le \mathcal{E}(w) \qquad \text{for every } w \in \mathcal{A} \text{ such that } w \equiv u \text{ on } \partial\Omega. \tag{4.5}$$

We are going to prove the last inequality using the theory of calibrations and extremal fields of the Calculus of Variations. We construct a *calibration* $\mathcal{F}$ for the functional $\mathcal{E}$ and the solution u, that is, a functional $\mathcal{F} = \mathcal{F}(w)$ defined for $w \in \mathcal{A}$ satisfying the three following properties:

(a) $\mathcal{F}(u) = \mathcal{E}(u)$.
(b) $\mathcal{F}(w) \le \mathcal{E}(w)$ for all $w \in \mathcal{A}$.
(c) $\mathcal{F}$ is a *null-lagrangian*, i.e., $\mathcal{F}(w) = \mathcal{F}(\widetilde{w})$ for every pair of functions $w \in \mathcal{A}$ and $\widetilde{w} \in \mathcal{A}$ such that $w \equiv \widetilde{w}$ on $\partial\Omega$.

The existence of such functional $\mathcal{F}$ immediately implies (4.5) and hence the theorem. Indeed, for each $w \in \mathcal{A}$ such that $w \equiv u$ on $\partial\Omega$, we have $\mathcal{F}(u) = \mathcal{F}(w)$ by (c), and therefore

$$\mathcal{E}(u) = \mathcal{F}(u) = \mathcal{F}(w) \le \mathcal{E}(w)$$

by (a) and (b).

To construct $\mathcal{F}$, we consider the one-parameter family of functions $\{u^t\}_{t\in\mathbf{R}}$ defined by

$$u^t(x) := u(x', x_n + t) \qquad \text{for } x \in \overline{\Omega} \text{ and } t \in \mathbf{R}.$$

The functions u^t are all solutions of the same semilinear equation (the Euler-Lagrange equation of $\mathcal{E}$) and, moreover, their graphs are pairwise disjoint (due to the assumption $\partial_{x_n} u > 0$). Because of these two properties, the family $\{u^t\}_{t\in\mathbf{R}}$ is called an *extremal field* with respect to $\mathcal{E}$. Next, we define the vector field $\phi = (\phi^x, \phi^s) : U \subset \overline{\Omega} \times \mathbf{R} \longrightarrow \mathbf{R}^{n+1}$ by

$$\phi^x(x,s) := \nabla u^t(x), \qquad \phi^s(x,s) = \frac{1}{2}|\nabla u^t(x)|^2 - F(s),$$

where $t = t(x,s)$ is the unique real number such that

$$u^t(x) = u^{t(x,s)}(x) = s. \tag{4.6}$$

Note that t exists and is unique due to the hypothesis $\partial_t u^t > 0$ and the fact that $(x,s) \in U$, i.e.,

$$\lim_{t\to-\infty} u^t(x) = \underline{u}(x') < s < \overline{u}(x') = \lim_{t\to+\infty} u^t(x).$$

Finally, we define the calibration $\mathcal{F}$ by

$$\begin{aligned}\mathcal{F}(w) &= \int_\Omega \{\langle \phi^x(x,w(x)), \nabla w(x)\rangle - \phi^s(x,w(x))\}\,dx \\ &= \int_\Omega \left\{\langle \nabla u^t, \nabla w\rangle - \frac{1}{2}|\nabla u^t|^2 + F(w)\right\} dx,\end{aligned}$$

for $w \in \mathcal{A}$, where $t = t(x,w(x))$ is defined by (4.6), i.e., by $u^t(x) = w(x)$, and ∇ denotes always the gradient with respect to $x \in \mathbf{R}^n$.

We need to show that $\mathcal{F}$ satisfies properties (a), (b) and (c). Note that (b) is obvious, by the Cauchy-Schwarz inequality. Property (a) is also immediate, since $t(x,u(x)) \equiv 0$. Property (c) will follow from the fact that ϕ is a divergence-free vector field.

To verify that $\operatorname{div}\phi = 0$, note first that u is a C^2 function in $\mathbf{R}^n$, since the linearized equation $\Delta\partial_{x_i}u = F''(u)\partial_{x_i}u$ implies (by local $W^{2,p}$ estimates) that $\partial_{x_i}u \in W^{2,p}$ and hence $u \in W^{3,p} \subset C^2$ for $p > n$. Now, the implicit function theorem applied to (4.6) gives that $t = t(x,s)$ is a $C^2(U)$ function. Moreover, differentiating (4.6) we obtain

$$\partial_t u^t\,\partial_s t = 1, \qquad \nabla u^t + \partial_t u^t\,\nabla_x t = 0. \tag{4.7}$$

In particular, ϕ is a C^1 vector field on U. We have

$$\operatorname{div}_x \phi^x = \Delta u^t + \langle \nabla\partial_t u^t, \nabla_x t\rangle = \Delta u^t - \langle \nabla\partial_t u^t, \partial_s t\,\nabla u^t\rangle$$

by (4.7), and

$$\partial_s\phi^s = \langle \nabla u^t, \partial_s t\,\nabla\partial_t u^t\rangle - F'(s),$$

and therefore $\operatorname{div}\phi = \Delta u^t - F'(s) = \Delta u^t - F'(u^t) = 0$.

Finally, we can verify property (c). Let $w \in \mathcal{A}$ and $\widetilde{w} \in \mathcal{A}$ satisfy $w \equiv \widetilde{w}$ on $\partial\Omega$. Define $\zeta = \widetilde{w} - w$ and $w_\tau = w + \tau(\widetilde{w} - w) = w + \tau\zeta$ for $0 \le \tau \le 1$. We have that $\zeta \equiv 0$ on $\partial\Omega$ and $w_\tau \in \mathcal{A}$ for all $\tau \in [0,1]$. For these values of τ, we have

$$\begin{aligned}\frac{d}{d\tau}\mathcal{F}(w_\tau) &= \frac{d}{d\tau}\int_\Omega \{\langle \phi^x(x,w_\tau), \nabla w_\tau\rangle - \phi^s(x,w_\tau)\}\,dx \\ &= \int_\Omega \{\langle \partial_s\phi^x(x,w_\tau), \nabla w_\tau\rangle\zeta + \langle \phi^x(x,w_\tau), \nabla\zeta\rangle - \partial_s\phi^s(x,w_\tau)\zeta\}\,dx.\end{aligned}$$

Integrating by parts the second term in the last expression, using $\zeta \equiv 0$ on $\partial\Omega$ and $\operatorname{div}\phi \equiv 0$ in U, we finally obtain

$$\begin{aligned}\frac{d}{d\tau}\mathcal{F}(w_\tau) &= \int_\Omega \{\langle \partial_s\phi^x(x,w_\tau), \nabla w_\tau\rangle\zeta - \operatorname{div}_x \phi^x(x,w_\tau)\zeta - \\ &\quad -\langle \partial_s\phi^x(x,w_\tau), \nabla w_\tau\rangle\zeta - \partial_s\phi^s(x,w_\tau)\zeta\}\,dx \\ &= \int_\Omega -\operatorname{div}\phi(x,w_\tau)\zeta\,dx = 0,\end{aligned}$$

and hence $\mathcal{F}(w) = \mathcal{F}(\widetilde{w})$. $\square$

Let us explain the construction of the previous proof in a more geometric way and, at the same time, include more general functionals $\mathcal{E}$ of the form

$$\mathcal{E}(w, \Omega) := \int_\Omega f(x, w, \nabla w)\, dx \qquad \text{for } w \in C^1(\Omega),$$

where the integrand $f(x, s, p)$ is bounded from below, of class C^1 in all arguments, and convex with respect to p. Here, and in the following, we use the standard notation $f(x, w, \nabla w)$ for $f(x, w(x), \nabla w(x))$.

Given a vector field $\phi = (\phi^x, \phi^s)$ defined on an open subset W of $\mathbf{R}^n \times \mathbf{R}$ and satisfying

$$\phi^s(x, s) \geq f^*(x, s, \phi^x(x, s)) \qquad \forall (x, s) \in W,$$

where $f^*(x, s, p^*)$ is the conjugate function of $f(x, s, p)$ with respect to p, there holds (by definition)

$$f(x, s, p) \geq \langle \phi^x(x, s), p\rangle - \phi^s(x, s) \qquad \forall (x, s, p) \in W \times \mathbf{R}^n. \tag{4.8}$$

Then, by integration, we obtain

$$\mathcal{E}(w, \Omega) \geq \int_\Omega \{\langle \phi^x(x, w), \nabla w\rangle - \phi^s(x, w)\}\, dx \tag{4.9}$$

for every $w \in C^1(\Omega)$ whose graph is contained in W. On the other hand, we have equality in (4.9) *for a given function* u if, in addition,

$$\begin{cases} \phi^x(x, u) = \partial_p f(x, u, \nabla u) \\ \phi^s(x, u) = -f(x, u, \nabla u) + \langle \partial_p f(x, u, \nabla u), \nabla u\rangle \end{cases} \qquad \forall x \in \Omega. \tag{4.10}$$

We are assuming, in particular, that W contains the graph $\{(x, u(x)) : x \in \Omega\}$ of u. Note that (4.8) and (4.10) imply

$$\phi^s(x, u) = f^*(x, u, \phi^x(x, u)) \qquad \forall x \in \Omega.$$

Assume now that ϕ is divergence free in W, and take any other function $v \in C^1(\Omega)$ whose graph is contained in W and such that $\{u \neq v\} \subset\subset \Omega$ (by this we mean that $\{u \neq v\}$ has compact closure contained in Ω). Then, denoting by Γu and Γv the graphs of u and v on Ω respectively (both oriented so that the s-component of the normal is negative), we get

$$\begin{aligned} \mathcal{E}(v, \Omega) &\geq \int_\Omega \{\langle \phi^x(x, v), \nabla v\rangle - \phi^s(x, v)\}\, dx \\ &= \text{flux of } \phi \text{ through } \Gamma v = \text{flux of } \phi \text{ through } \Gamma u \\ &= \int_\Omega \{\langle \phi^x(x, u), \nabla u\rangle - \phi^s(x, u)\}\, dx = \mathcal{E}(u, \Omega). \end{aligned}$$

Notice that the second equality follows by the divergence theorem using the fact that ϕ is divergence-free in W, and that Γu and Γv have the same boundary in W. Here, we have assumed that the x-slice of W is an interval for each x. We have thus proved the following result.

Theorem 4.5 *Let $\Omega \subset \mathbf{R}^n$ and $W \subset \Omega \times \mathbf{R}$ be open sets and let $u \in C^1(\Omega)$ be such that its graph Γu is contained in W. Let us assume that $\{s \in \mathbf{R} : (x,s) \in W\}$ is an interval for each $x \in \Omega$, and that there exists a divergence-free vector field $\phi = (\phi^x, \phi^s)$ in W satisfying* (4.8) *and* (4.10)*. Then u is a local minimizer of $\mathcal{E}(\cdot, \Omega)$ with respect to compact perturbations in W, i.e.,*

$$\mathcal{E}(u,\Omega) \le \mathcal{E}(v,\Omega) \qquad \textit{whenever } \{u \neq v\} \subset\subset \Omega \textit{ and } \Gamma v \subset W.$$

This method applies to minimizers with Dirichlet boundary conditions. By analogy with the theory of minimal surfaces, we call the vector field ϕ a *calibration* for u in W relative to the integrand f; see [35, 36]. The existence of a calibration is not only sufficient for minimality but, to a certain extent, also necessary (a statement in this directions in the context of Geometric Measure Theory can be derived from Sections 4 and 5 of [22]). The previous proof is based on the fact that, for divergence-free ϕ, integrals of the type

$$\mathcal{F}_\phi(w,\Omega) := \int_\Omega \{\langle \phi^x(x,w), \nabla w\rangle - \phi^s(x,w)\}\, dx \tag{4.11}$$

are invariant integrals or *null-lagrangians*, that is, as we have seen before they depend only on the value of w on the boundary of Ω (see also Section 3.3.2.2 of [14], or Sections 1.4.1, 1.4.2 and 6.1.3 of [24]). Both null-lagrangians and calibrations are classical tools to prove minimality and, in fact, are essentially the same.

A more analytic proof of the null-lagrangian property of $\mathcal{F}_\phi$, which hides the geometric significance of this property, can be obtained as in the proof of Theorem 4.4. That is, first one notices that, since ϕ is divergence-free in W, the Euler-Lagrange equation of $\mathcal{F}_\phi$

$$\operatorname{div}(\phi^x(x,w)) - \langle \partial_s \phi^x(x,w), \nabla w\rangle + \partial_s \phi^s(x,w) = (\operatorname{div}_x \phi^x + \partial_s \phi^s)(x,w) = 0 \tag{4.12}$$

is satisfied by *every* $w \in C^1(\Omega)$ whose graph is contained in W (this property is also sometimes taken as definition of null-lagrangian, see [17]). Then, if w and $\widetilde{w}$ belong to $C^1(\Omega)$ are such that their graphs are contained in W and $\{w \neq \widetilde{w}\} \subset\subset \Omega$, we can define $w_\tau = w + \tau(\widetilde{w} - w)$ for $\tau \in [0,1]$ and use the fact that all functions w_τ solve (4.12), to obtain

$$\frac{d}{d\tau}\mathcal{F}_\phi(w_\tau,\Omega) = 0 \quad \forall \tau \in [0,1].$$

Hence $\mathcal{F}_\phi(w,\Omega) = \mathcal{F}_\phi(\widetilde{w},\Omega)$. As in the previous proof, the assumption that the x-slices of W are intervals plays a role (here it guarantees that the graph of each function w_τ is contained in W).

When trying to apply Theorem 4.5, the delicate part is obviously the construction of ϕ. There is however a simple way to accomplish it whenever the solution u can be embedded in a one-parameter family of solutions u^t of the Euler-Lagrange equation of $\mathcal{E}$,

$$\operatorname{div}\left(\partial_p f(x,u^t(x),\nabla u^t(x))\right) = \partial_s f(x,u^t(x),\nabla u^t(x)) \qquad \text{for all } (x,t), \tag{4.13}$$

whose graphs foliate the open region W. Such a family of solutions is called an *extremal field* with respect to $\mathcal{E}$ (see [24], Section 6.3).

More precisely, we assume that W is covered by a *regular* family of pairwise disjoint graphs Γ_t of solutions u^t of (4.13), where t belongs to an open interval $I \subset \mathbf{R}$. Then, for every $(x,s) \in W$ we take the unique $t = t(x,s) \in I$ such that $u^t(x) = s$, and set

$$\begin{cases} \phi^x(x,s) = \partial_p f(x, u^t(x), \nabla u^t(x)) \\ \phi^s(x,s) = -f(x, u^t(x), \nabla u^t(x)) + \langle \partial_p f(x, u^t(x), \nabla u^t(x)), \nabla u^t(x) \rangle. \end{cases} \tag{4.14}$$

By regular family we mean the following: locally the graphs Γ_t can be represented as level sets $\{\widetilde{u} = t\}$ of a $C^2(W)$ function $\widetilde{u}$, which obviously satisfies the condition $\partial_s \widetilde{u} \neq 0$. This property is fulfilled if, for instance, $\partial_t u^t(x) \neq 0$ for every $t \in I$ and every $x \in \Omega$. The integral $\mathcal{F}_\phi$ associated to such a vector field ϕ through (4.11) is the Hilbert invariant integral relative to the extremal field $\{u^t\}$, and in fact the following theorem holds (see [24], Section 6.3).

Theorem 4.6 *The vector field ϕ defined by* (4.14) *is a calibration of each u^t in W, that is, ϕ is a divergence-free vector field satisfying* (4.8) *and* (4.10). *Consequently, all functions u^t are local minimizers of $\mathcal{E}(\cdot, \Omega)$ in W.*

Proof. The vector field ϕ satisfies (4.10) for each u^t by construction. It also satisfies (4.8) on W, by the definition (4.14) and the assumption that f is convex with respect to p. It remains to prove that ϕ is divergence-free. This can be done in two different ways.

The first way consists of simply computing the divergence. Since $u^{t(x,s)}(x) = s$, we deduce

$$\begin{cases} \partial_t u^t \, \partial_s t = 1 \\ \nabla u^t + \partial_t u^t \, \nabla_x t = 0. \end{cases} \tag{4.15}$$

Now, using definition (4.14), we compute the divergence at a point (x,s). All expressions are evaluated at x, $s = u^t(x)$ and $p = \nabla u^t(x)$, where $t = t(x,s)$. We have

$$\begin{aligned} \mathrm{div}_x \phi^x &= \mathrm{div}\left(\partial_p f(x, u^t(x), \nabla u^t(x))\right) + \langle \partial_{ps} f, \partial_t u^t \, \nabla_x t \rangle + \\ &\quad + \langle \partial_{pp} f \cdot \nabla \partial_t u^t, \nabla_x t \rangle \end{aligned}$$

and, using (4.15),

$$\begin{aligned} \partial_s \phi^s &= -\partial_s f \, \partial_t u^t \, \partial_s t - \langle \partial_p f, \nabla \partial_t u^t \, \partial_s t \rangle + \langle \partial_{ps} f \, \partial_t u^t \, \partial_s t, \nabla u^t \rangle \\ &\quad + \langle \partial_{pp} f \cdot \nabla \partial_t u^t \, \partial_s t, \nabla u^t \rangle + \langle \partial_p f, \nabla \partial_t u^t \, \partial_s t \rangle \\ &= -\partial_s f + \langle \partial_{ps} f, \nabla u^t \rangle + \langle \partial_{pp} f \cdot \nabla \partial_t u^t \, \partial_s t, \nabla u^t \rangle. \end{aligned}$$

Using (4.13) and (4.15), we see that each one of the three terms in the last expression is the opposite of the corresponding term in the expression for $\mathrm{div}_x \phi^x$. Hence, we conclude $\mathrm{div}\, \phi = 0$ in W.

Let us present a second way, more geometric, to prove that ϕ is divergence-free. Since this property is local, we can assume in the following that, in a sufficiently small ball

$B \subset W$, there exists $\widetilde{u} \in C^2(\overline{B})$ such that $\Gamma_t \cap B = \{\widetilde{u} = t\}$ for each $t \in I$. We denote by J the interval $\widetilde{u}(B)$.

Next, we consider the following auxiliary functional: for every $\widetilde{w} \in C^1(B)$ taking values in J and such that $\partial_s \widetilde{w} > 0$, the level set $\{\widetilde{w} = t\}$ is the graph of a C^1 function w^t defined on an open subset of Ω^t of $\mathbf{R}^n$, and therefore we can set

$$\mathcal{G}(w) = \int_J \mathcal{E}(w^t, \Omega^t)\, dt.$$

Using this definition, we first prove that $\widetilde{u}$ is a stationary point for $\mathcal{G}$. The proof is based on the following lemma, whose elementary proof is left to the reader.

Lemma 4.7 *Let $B \subset \mathbf{R}^{n+1}$ be a ball and let $\widetilde{u} \in C^2(\overline{B})$ be satisfying $\partial_s \widetilde{u} > 0$ on $\overline{B}$. Let also $B' \subset\subset B$ be a concentric ball. Then, denoting by $\pi : B \to \mathbf{R}^n$ the orthogonal projection on $\{s = 0\}$ and setting $\Omega^t = \pi(\{\widetilde{u} = t\})$, there exist constants $\epsilon > 0$ and $M > 0$ such that*

$$\pi(\{\widetilde{w} = t\}) = \Omega^t \quad \text{and} \quad \|w^t - u^t\|_{C^1(\Omega^t)} \leq M \|\widetilde{w} - \widetilde{u}\|_{C^1(B)}$$

for each $t \in v(B)$, provided $\|\widetilde{w} - \widetilde{u}\|_{C^1(B)} \leq \epsilon$ and the support of $\widetilde{w} - \widetilde{u}$ is contained in B'.

Proof of Theorem 4.6 continued. Let $\zeta \in C^1_c(B)$ and consider $\widetilde{w} = \widetilde{u} + \delta\zeta$ for δ small enough. Lemma 4.7 gives

$$\frac{d}{d\delta}\, \mathcal{G}(\widetilde{u} + \delta\zeta)\bigg|_{\delta=0} = \int_J \frac{d}{d\delta} \mathcal{E}((\widetilde{u} + \delta\zeta)^t, \Omega^t)\bigg|_{\delta=0} dt = 0.$$

Hence $\widetilde{u}$ is a stationary point for $\mathcal{G}$.

Using the coarea formula (see for instance [25]), we can give a canonical representation of $\mathcal{G}$ with the Lagrangian

$$L(y, q) = f\left(y, -\frac{p^x}{p^s}\right) p^s \quad \text{where } y = (x, s) \text{ and } q = (p^x, p^s).$$

Indeed, we have

$$\begin{aligned}
\int_J \mathcal{E}(w^t, \Omega^t)\, dt &= \int_J \int_{\Omega^t} f(x, w^t, \nabla w^t)\, dx\, dt \\
&= \int_J \int_{\Gamma_t} \frac{f(y, -\partial_x \widetilde{w}/\partial_s \widetilde{w}))}{\sqrt{1 + |\nabla w^t|^2}}\, d\mathcal{H}^n(y)\, dt \\
&= \int_J \int_{\{\widetilde{w}=t\}} f\left(y, -\frac{\partial_x \widetilde{w}}{\partial_s \widetilde{w}}\right) \frac{\partial_s \widetilde{w}}{|\nabla \widetilde{w}|}\, d\mathcal{H}^n(y) dt \\
&= \int_B f\left(y, -\frac{\partial_x \widetilde{w}}{\partial_s \widetilde{w}}\right) \partial_s \widetilde{w}\, dy.
\end{aligned}$$

Since $\widetilde{u}$ is a stationary point of $\mathcal{G}$, it follows that $\widetilde{u}$ satisfies the Euler-Lagrange equation

$$\operatorname{div}\left(\partial_q L(y, \nabla \widetilde{u})\right) = 0 \qquad \text{in } B.$$

Substituting the expression for L, we eventually obtain

$$\operatorname{div}\left(-\partial_p f(x,s,-\frac{\partial_x\widetilde{u}}{\partial_s\widetilde{u}}), f(x,s,-\frac{\partial_x\widetilde{u}}{\partial_s\widetilde{u}}) - \langle\partial_p f(x,s,-\frac{\partial_x\widetilde{u}}{\partial_s\widetilde{u}}), \frac{\partial_x\widetilde{u}}{\partial_s\widetilde{u}}\rangle\right) = 0 \tag{4.16}$$

for each $(x,s) \in B$. Finally, if we take ϕ as in (4.14) and take into account that

$$-\frac{\partial_x\widetilde{u}(x,s)}{\partial_s\widetilde{u}(x,s)} = \nabla u^t(x),$$

(4.16) reduces to $\operatorname{div}\phi = 0$ in B. □

5 Energy estimates. Completion of the proof of Theorem 1.1 for $n = 3$

In this section we establish some a priori estimates for bounded entire solutions u of (1.2). The first one, stated below, does not require the monotonicity assumption on u and was proved by L. Modica in [32]. Throughout this section, we consider the constant

$$c_u = \min\{F(s) : \inf u \le s \le \sup u\},$$

that is, the infimum of F on the range of u.

Theorem 5.1 (Monotonicity formula and lower bounds) *Let u be a bounded entire solution of* (1.2). *Then*

$$\phi(R) := R^{1-n}\int_{B_R}\left\{\frac{1}{2}|\nabla u|^2 + F(u) - c_u\right\}dx$$

is nondecreasing in $(0,\infty)$. *In particular, if u is not constant then there exists a positive constant c such that*

$$\int_{B_R}\left\{\frac{1}{2}|\nabla u|^2 + F(u) - c_u\right\}dx \ge c\,R^{n-1} \qquad \forall R > 1. \tag{5.1}$$

Modica also proved in [31] the pointwise gradient bound

$$|\nabla u|^2 \le 2(F(u) - c_u) \qquad \text{in } \mathbf{R}^n,$$

where, as before, the monotonicity assumption on u is not required.

The proof of the energy upper bound

$$\int_{B_R}\left\{\frac{1}{2}|\nabla u|^2 + F(u) - c_u\right\}dx \le CR^{n-1} \qquad \forall R > 1, \tag{5.2}$$

which plays a crucial role in our proof of the De Giorgi conjecture in dimension three, is more delicate and requires some additional work. Here, the monotonicity assumption on

u is needed. In [2] we gave a simple proof of this estimate for a special class of nonlinearities F. It was based on a "sliding" argument using the functions $u^t(x) = u(x', x_n + t)$ of the previous section. In the present paper, we use the local minimality property of u to extend the energy upper bound (5.2) to every nonlinearity $F \in C^2$. The precise result is the following:

Theorem 5.2 (Upper bounds) *Let u be an entire solution of* (1.2) *satisfying* (1.1). *If either $n \le 3$ or u satisfies* (1.3), *then* (5.2) *holds for some constant C independent of R. In particular, we have that*

$$\int_{B_R} |\nabla u|^2 \, dx \le CR^{n-1} \qquad \forall R > 1.$$

When $n = 3$, the previous theorem establishes estimate (3.3). As we saw in Section 3, this completes the proof of Theorem 1.1 and of De Giorgi's conjecture in dimension three.

Proof of Theorem 5.2. Let $m = \inf u$, $M = \sup u$, and $s \in [m, M]$ be such that $c_u = F(s)$. If u satisfies (1.3) then $\underline{u} \equiv m$ and $\overline{u} \equiv M$, and hence we can perform a simple energy comparison argument. Indeed, let $\phi_R \in C^\infty(\mathbf{R}^n)$ satisfy $0 \le \phi_R \le 1$ in $\mathbf{R}^n$, $\phi_R \equiv 1$ in B_{R-1}, $\phi_R \equiv 0$ in $\mathbf{R}^n \setminus B_R$ and $\|\nabla \phi_R\|_\infty \le 2$, and consider

$$v_R := (1 - \phi_R)u + \phi_R s.$$

This function satisfies the conditions stated for v in Theorem 4.4 when $\Omega = B_R$, and hence we can compare the energy of u with the energy of v_R in B_R. Taking into account that $F(s) = c_u$, we obtain

$$\begin{aligned}
&\int_{B_R} \left\{ \frac{1}{2}|\nabla u|^2 + F(u) - c_u \right\} dx \\
&\quad \le \int_{B_R} \left\{ \frac{1}{2}|\nabla v_R|^2 + F(v_R) - c_u \right\} dx \\
&\quad = \int_{B_R \setminus B_{R-1}} \left\{ \frac{1}{2}|\nabla v_R|^2 + F(v_R) - c_u \right\} dx \le C|B_R \setminus B_{R-1}| \le CR^{n-1}
\end{aligned}$$

for every $R > 1$, with C independent of R.

We now consider the case when condition (1.3) is dropped but $n \le 3$. When $n = 2$, Theorem 5.2 is a consequence of Theorem 1.1, which is already proved in dimension 2. Indeed, that the energy is bounded by CR (and not only by CR^2) follows easily from the one-dimensionality of u and an ODE argument (see [2]).

Assume now that $n = 3$. Notice that $m = \inf \underline{u}$ and $M = \sup \overline{u}$, and define

$$\widetilde{m} = \sup \underline{u} \qquad \text{and} \qquad \widetilde{M} = \inf \overline{u}.$$

Obviously $\widetilde{m}$ and $\widetilde{M}$ belong to $[m, M]$. By Lemma 3.1 and Lemma 3.2 of [2] we know that $\underline{u}$ and $\overline{u}$ are either constant or monotone one dimensional solutions in $\mathbf{R}^2$. The proof of this fact is based on the stability of u and on the proof of the conjecture of De Giorgi in $\mathbf{R}^{n-1} = \mathbf{R}^2$ (see [2] or next section, where we will recall the ideas involved in the proof). Moreover, by a simple ODE argument (see also [2]), we have

$$F > F(m) = F(\widetilde{m}) \qquad \text{in } (m, \widetilde{m}) \tag{5.3}$$

in case $m < \widetilde{m}$ (i.e., $\underline{u}$ is not constant) and

$$F > F(\widetilde{M}) = F(M) \qquad \text{in } (\widetilde{M}, M) \tag{5.4}$$

in case $\widetilde{M} < M$ (i.e., $\overline{u}$ is not constant).

In all four possible cases (that is, each $\underline{u}$ and $\overline{u}$ is constant or one dimensional) we deduce from (5.3) and (5.4) that $\widetilde{m} \leq \widetilde{M}$ and that there exists $s \in [\widetilde{m}, \widetilde{M}]$ such that $F(s) = c_u$ (recall that c_u is the infimum of F in the range of u). We conclude that

$$\underline{u}(x') \leq \widetilde{m} \leq s \leq \widetilde{M} \leq \overline{u}(x') \qquad \forall x' \in \mathbf{R}^2,$$

and hence we can apply Theorem 4.4 to make the comparison argument with the function $v_R = (1 - \phi_R)u + \phi_R s$ as before, and hence obtain the desired energy upper bound. $\square$

6 A symmetry result in $\mathbf{R}^4_+$

In this section we continue the study of symmetry properties for semilinear elliptic equations, that we write here in the form $\Delta u + f(u) = 0$, but now in halfspaces $\mathbf{R}^n_+ = \{x \in \mathbf{R}^n : \ x_n > 0\}$. More precisely, we study bounded solutions of the problem

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \mathbf{R}^n_+ \\ u > 0 & \text{in } \mathbf{R}^n_+ \\ u = 0 & \text{on } \partial\mathbf{R}^n_+. \end{cases} \tag{6.1}$$

We always assume that f is a Lipschitz function on $[0, \infty)$ and that $u \in C^2(\mathbf{R}^n_+)$ is a bounded solution of (6.1) continuous up to the boundary of $\mathbf{R}^n_+$. Applying standard $W^{2,p}$ estimates in every ball or half ball of radius 1 in $\mathbf{R}^n_+$, we see that $|\nabla u|$ is globally bounded in $\mathbf{R}^n_+$ as well.

As in the previous sections, the goal is to establish that the level sets of u are hyperplanes or, equivalently, that $u = u(x_n)$ is a function of x_n alone. This symmetry question is slightly easier than the question in the whole of $\mathbf{R}^n$, since we know that at least one level set of u is a hyperplane, namely the 0-level set.

In [6] and [8], H. Berestycki, L. Caffarelli and L. Nirenberg proved that every solution u of (6.1) (not necessarily bounded) satisfies $\partial_{x_n} u > 0 \quad \text{in } \mathbf{R}^n_+$ when $n = 2$ or when $n \geq 3$ and $f(0) \geq 0$. In [8] the same authors developed the technique that we have used in Section 3 (that is, the idea of applying a Liouville theorem to the equation satisfied by the quotient of partial derivatives of u) to prove the following result.

Theorem 6.1 (Symmetry in $\mathbf{R}^2_+$ and $\mathbf{R}^3_+$) *Let u be a bounded solution of* (6.1). *If $n = 2$ then u depends only on x_n, i.e., $u = u(x_n)$. If $n = 3$ the same conclusion holds if one assumes in addition that $f(0) \geq 0$ and $f \in C^1([0, \infty))$.*

S.B. Angenent [1], P. Clément and G. Sweers [13], and H. Berestycki, L. Caffarelli and L. Nirenberg [7], have also proved the same symmetry property in any number of dimensions, but under more restrictive assumptions on f. More precisely, in [7] the authors established (in all dimensions n) that every bounded solution of (6.1) is symmetric, i.e. $u = u(x_n)$, if one assumes in addition the three following conditions on f:

Condition (1). For some $\mu > 0$ we have $f > 0$ in $(0, \mu)$ and $f \leq 0$ in $[\mu, \infty)$.
Condition (2). For some $s_0 \in (0, \mu)$ and some $\delta_0 > 0$, $f(s) \geq \delta_0 s$ in $[0, s_0]$.
Condition (3). For some $s_1 \in (s_0, \mu)$, f is nonincreasing in (s_1, μ).

When $n = 4$ we can improve this symmetry result by requiring essentially only condition (1) on f. The precise statement is the following:

Theorem 6.2 (Symmetry in $\mathbf{R}^4_+$) *Let $f \in C^1([0, \infty))$ and assume that $f \geq 0$ in $[0, \infty)$ or that $f \geq 0$ in $[0, \mu]$ and $f \leq 0$ in $[\mu, \infty)$ for some $\mu > 0$. Then, when $n = 4$, every bounded solution u of* (6.1) *depends only on x_4.*

For the proof of this theorem we will need the following result established in [5], which was also used by the same authors in [8] to prove Theorem 6.1 in $\mathbf{R}^n_+$, $n = 2, 3$. This result states that if u is a bounded solution of (6.1) in $\mathbf{R}^n_+$ and if

$$f(\sup u) \leq 0 \tag{6.2}$$

then u is symmetric, i.e., $u = u(x_n)$.

Proof of Theorem 6.2. We first prove, following [7], that

$$\Delta u \leq 0 \quad \text{in } \mathbf{R}^4_+. \tag{6.3}$$

This is obvious in the case when f is nonnegative. Suppose now that $f \geq 0$ in $[0, \mu]$ and $f \leq 0$ in $[\mu, \infty)$ for some $\mu > 0$, and define $M = \sup u$. If we show that $M \leq \mu$ then (6.3) follows immediately. Arguing by contradiction, suppose that $M > \mu$. Then the open set $A = \{u > \mu\}$ is not empty, is contained in $\mathbf{R}^4_+$, and the bounded function $u - \mu$ vanishes on ∂A and is subharmonic in A. A version of the maximum principle in unbounded domains having an exterior open cone (see Section 2 of [7]) gives that $u - \mu \leq 0$ in A, a contradiction. We have therefore established (6.3).

Next, we use (6.3) to show that

$$\int_{Q_R} |\nabla u|^2 \, dx \leq C R^3 \tag{6.4}$$

for every cylinder $Q_R = B'_R \times (a, a+R) \subset \mathbf{R}^4_+$, where C is a constant independent of R and a. Indeed, since $-\Delta u \geq 0$ we have $u(-\Delta u) \leq M(-\Delta u)$ and hence

$$\begin{aligned}
\int_{Q_R} |\nabla u|^2\, dx &= \int_{\partial Q_R} u \frac{\partial u}{\partial \nu}\, d\mathcal{H}^3 + \int_{Q_R} u(-\Delta u)\, dx \\
&\leq \int_{\partial Q_R} u \frac{\partial u}{\partial \nu}\, d\mathcal{H}^3 - M \int_{Q_R} \Delta u\, dx \\
&= \int_{\partial Q_R} (u - M) \frac{\partial u}{\partial \nu}\, d\mathcal{H}^3 \leq C\mathcal{H}^3(\partial Q_R) = CR^3.
\end{aligned}$$

Since $f(0) \geq 0$, the result of [6] previously mentioned gives that $\partial_{x_4} u > 0$ in $\mathbf{R}^4_+$. Hence the function

$$\overline{u}(x') = \lim_{x_4 \to +\infty} u(x', x_4) \qquad \text{for } x' \in \mathbf{R}^3$$

is well defined and satisfies $\Delta \overline{u} + f(\overline{u}) = 0$ in $\mathbf{R}^3$. Estimate (6.4) implies, by letting $a \to \infty$, that

$$\int_{B'_R} |\nabla \overline{u}|^2\, dx' \leq CR^2 \tag{6.5}$$

for every ball $B'_R \subset \mathbf{R}^3$ of radius R.

Next we show that there exists a strictly positive function φ in $\mathbf{R}^3$ such that

$$\Delta \varphi + f'(\overline{u})\varphi = 0 \qquad \text{in } \mathbf{R}^3. \tag{6.6}$$

This is shown using the ideas on stability of Section 4 and arguing as in Section 3 of [8] or as in Section 3 of [2]. Let us recall briefly the argument; since $\partial_{x_4} u > 0$ solves the linearized equation then

$$\int_{\mathbf{R}^4_+} \left\{ |\nabla \xi|^2 - f'(u)\xi^2 \right\} dx \geq 0 \qquad \forall \xi \in C_c^\infty(\mathbf{R}^4_+)$$

(see Proposition 4.2). Then, using the continuity of f', one deduces that

$$\int_{\mathbf{R}^3} \left\{ |\nabla \eta|^2 - f'(\overline{u})\eta^2 \right\} dx \geq 0 \qquad \forall \eta \in C_c^\infty(\mathbf{R}^3)$$

(see the proof of Lemma 3.1 of [2] for details). Using Proposition 4.2 we deduce the existence of $\varphi > 0$ satisfying (6.6).

For this choice of function φ, we can now consider the functions $\sigma_i = \partial_{x_i} \overline{u} / \varphi$ and argue as in the proof of De Giorgi's conjecture given in Section 3, since we have (6.5) and (6.6). We deduce that each partial derivative of $\overline{u}$ is a constant multiple of φ (and hence has constant sign). In particular, $\overline{u}$ is either a constant or a monotone function of only one variable. Then the ODE $\overline{u}'' + f(\overline{u}) = 0$ gives that $f(\sup \overline{u}) = 0$. Since $\sup u = \sup \overline{u}$ we deduce $f(\sup u) = 0$ and hence (6.2). This implies that $u = u(x_4)$, by the result of [5] mentioned above. $\square$

7 A partial result for $n \le 8$

In this section we make rigorous the heuristic discussion of Section 2 and prove that in dimensions $n \le 8$ all entire solutions of (1.2) satisfying (1.1) and (1.3) are “flat at infinity”. Precisely, we can prove the following result.

Theorem 7.1 *Assume that $n \le 8$, that u is a bounded entire solution of* (1.2)*, that both* (1.1) *and* (1.3) *hold and*

$$F > F(\inf u) = F(\sup u) \qquad \text{in } (\inf u, \sup u).$$

Let $(R_i) \subset (0,\infty)$ be converging to ∞. Then there exist a subsequence $R_{i(k)}$ and a unit vector $a \in \mathbf{R}^n$ such that

$$\lim_{k\to\infty} R_{i(k)}^{1-n} \int_{B_{R_{i(k)}}} \left\{ |\nabla u|^2 - |\partial_a u|^2 \right\} dx = 0. \tag{7.1}$$

Moreover, $u_k(x) = u(R_{i(k)}x)$ converge in $L^1_{\rm loc}(\mathbf{R}^n)$ to the characteristic function (*with values* $\inf u$ *and* $\sup u$) *of a halfspace orthogonal to a.*

The theorem above is a slight improvement of an analogous result stated by L. Modica, under the local minimality assumption, in [30].

The distance between the property proved in Theorem 7.1 and the full De Giorgi conjecture could be significant because of the following two facts:

(a) A priori the direction a depends on the sequence (R_i), or the subsequence $(R_{i(k)})$. Notice that Theorem 7.1 implies, by a simple contradiction argument, the existence of unit vectors vectors a_R such that

$$\lim_{R\to\infty} R^{1-n} \int_{B_R} \left\{ |\nabla u|^2 - |\partial_{a_R} u|^2 \right\} dx = 0.$$

However, it is not clear how a “spiraling” behaviour of the level sets of u at infinity could be ruled out.

(b) Even if we were able to solve the problem raised in (a), and prove that

$$\lim_{R\to\infty} R^{1-n} \int_{B_R} \left\{ |\nabla u|^2 - |\partial_a u|^2 \right\} dx = 0$$

for some unit vector a, then it is not clear how the stronger conclusion that ∇u is *everywhere* parallel to a could be drawn.

Sketch of the proof of Theorem 7.1. In the proof we denote by $D_a u$ the distributional derivative along $a \in \mathbf{R}^n$ of a function u. Notice that, since

$$\begin{aligned} |D_a u|(\Omega) &= \sup \left\{ D_a u(\zeta) :\ \zeta \in C^1_c(\Omega),\ \|\zeta\|_\infty \le 1 \right\} \\ &= \sup \left\{ -\int_\Omega u\, \partial_a \zeta \, dx :\ \zeta \in C^1_c(\Omega),\ \|\zeta\|_\infty \le 1 \right\} \end{aligned}$$

the mapping $u \mapsto |D_a u|(\Omega)$ is lower semicontinuous with respect to the $L^1_{\rm loc}(\Omega)$ convergence for every $a \in \mathbf{R}^n$ and every open set $\Omega \subset \mathbf{R}^n$.

Step 1. Let $u_i(x) = u(R_i x)$. With no loss of generality, we may assume that $c_u = 0$, $m = -1$ and $M = 1$. By the energy upper bound (5.2) we infer

$$\mathcal{E}_{R_i}(u_i, B_r) = R_i^{1-n}\mathcal{E}_1(u, B_{R_i r}) \leq C r^{n-1}$$

as soon as $R_i \geq 1$ and $r \geq 1$ (we use the notation $\mathcal{E}_{R_i}$ from Section 2). Hence, the coercivity property (iii) of Γ-convergence (see Section 2) gives a subsequence $u_{i(k)}$ and a set E of locally finite perimeter in $\mathbf{R}^n$ such that

$$\lim_{k\to\infty} u_{i(k)} = 1_E \qquad \text{in } L^1_{\rm loc}(\mathbf{R}^n).$$

For notational simplicity in the following we assume convergence of the original sequence (u_i).

Step 2. It is well known that properties (i) and (ii) of Γ-convergence ensure convergence of global minimizers to global minimizers. It is perhaps less known that under some additional assumptions (satisfied by the functionals $\mathcal{E}_R$) also the *local* minimality property is preserved in the limit (see Section 5 in [15] and [30] for details). Hence, since by Theorem 4.4 the functions u_i are local minimizers of $\mathcal{E}_{R_i}$, we obtain that E is a local minimizer of the perimeter, i.e.

$$P(E,\Omega) \leq P(F,\Omega) \qquad \text{whenever } E\Delta F \subset\subset \Omega \subset\subset \mathbf{R}^n \tag{7.2}$$

(where $E\Delta F$ denotes the symmetric diference). Moreover, the same proof of this fact shows that the energies $\mathcal{E}_{R_i}(u_i,\cdot)$ are locally weakly* converging as measures to $c_F P(E,\cdot)$, i.e.,

$$\lim_{i\to\infty}\int_{\mathbf{R}^n}\left\{\frac{1}{2R_i}|\nabla u_i|^2 + R_i F(u_i)\right\}\xi\,dx = c_F\int_{\mathbf{R}^n}\xi\,dP(E,\cdot) \qquad \forall \xi \in C_c(\mathbf{R}^n).$$

By De La Vallée Poussin theorem (see for instance [25], Appendix A) we obtain

$$\lim_{i\to\infty} R_i^{1-n}\int_{B_{rR_i}}\left\{\frac{1}{2}|\nabla u|^2 + F(u)\right\}dx = \lim_{i\to\infty}\mathcal{E}_{R_i}(u_i,B_r) = c_F P(E,B_r) \tag{7.3}$$

for each $r > 0$ such that $P(E,\partial B_r) = 0$ (this condition holds with at most countably many exceptions). In particular (5.1) gives $P(E,B_r) \geq c r^{n-1}/c_F > 0$, and hence E is neither the empty set nor the whole space.

Step 3. If $n \leq 7$ it is well known that every set E satisfying (7.2) and with nonzero perimeter is a halfspace. This is not the case in $\mathbf{R}^8$, a counterexample being the Simons cone (see for instance [25]). In our case we can extend the conclusion up to $\mathbf{R}^8$ noticing that the condition (1.1) yields $D_{x_n}\chi_E \geq 0$ in the sense of distributions. This implies that χ_E is the hypograph of a function $\psi : \mathbf{R}^{n-1} \to \overline{\mathbf{R}}$ (with values in the extended real line $\overline{\mathbf{R}}$), i.e., $E = \{(x', x_n) : x_n > \psi(x')\}$.

Since E is a local minimizer, we may consider ψ to be an entire solution of the mean curvature equation (2.1) in a generalized sense. Indeed, this viewpoint was adopted by M. Miranda in [28, 29] (see also Chapter 16 in E. Giusti [25]) to define generalized solutions of least area problems. He proved that for every such generalized solution ψ the sets

$$P := \left\{x \in \mathbf{R}^{n-1} :\ \psi(x) = +\infty\right\}, \qquad N := \left\{x \in \mathbf{R}^{n-1} :\ \psi(x) = -\infty\right\}$$

are both local minimizers of the perimeter in $\mathbf{R}^{n-1}$, and this allows us to prove that the solution to the Bernstein problem is unchanged if we consider generalized solutions of (2.1) instead of classical ones. Indeed, we distinguish the following two cases:

(a) Both P and N are negligible. In this case M. Miranda proved that ψ is (equivalent to) a *classical* solution of (2.1). In particular ψ is an affine function, since $n-1 \le 7$.

(b) N has positive measure. Then N must be a halfspace of $\mathbf{R}^{n-1}$ (since $n-1 \le 7$) and therefore E contains a halfspace of $\mathbf{R}^n$. This implies (see [25], Theorem 17.4) that E itself is a halfspace. The case when P has positive measure can be handled in a similar way.

Step 4. In the previous step we proved that E is a halfspace. Let a be a unit vector perpendicular to ∂E. We will prove that

$$\liminf_{i\to\infty} R_i^{1-n} \int_{B_{R_i}} \left\{\frac{1}{2}|\partial_a u|^2 + F(u)\right\} dx \ge c_F P(E, B_1) \tag{7.4}$$

which, together with (7.3) with $r = 1$, gives (7.1).

In order to show (7.4) we follow the same path of the Modica and Mortola proof of the lower semicontinuity inequality (i) of Γ-convergence. Indeed, let

$$G(t) := \int_{-1}^{t} \sqrt{2F(s)}\, ds$$

and notice that the Young inequality gives

$$\begin{aligned} & R_i^{1-n} \int_{B_{R_i}} \left\{\frac{1}{2}|\partial_a u|^2 + F(u)\right\} dx \\ &= \int_{B_1} \left\{\frac{1}{2R_i}|\partial_a u_i|^2 + R_i F(u_i)\right\} dx \\ &\ge \int_{B_1} \sqrt{2F(u_i)}\, |\partial_a u_i|\, dx = \int_{B_1} |\partial_a (G\circ u_i)|\, dx. \end{aligned}$$

Notice that $G\circ u_i$ converge in $L^1_{\mathrm{loc}}(\mathbf{R}^n)$ to $G\circ 1_E$, i.e., $c_F\chi_E$ (χ_E is equal to 1 on E and equal to 0 on $\mathbf{R}^n \setminus E$). Hence, the lower semicontinuity of directional derivatives under $L^1_{\mathrm{loc}}(\mathbf{R}^n)$ convergence gives

$$\begin{aligned} \liminf_{i\to\infty} R_i^{1-n} \int_{B_{R_i}} \left\{\frac{1}{2}|\partial_a u|^2 + F(u)\right\} dx &\ge c_F |D_a \chi_E|(B_1) \\ &= c_F \int_{B_1\cap\partial E} |\langle \nu_E, a\rangle|\, d\mathcal{H}^{n-1}, \end{aligned}$$

where ν_E is the inner normal to E. Since a is parallel to ν_E, (7.4) follows. $\square$

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