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UNIFORM APPROXIMATION OF 2 DIMENSIONAL NAVIER–STOKES EQUATION BY STOCHASTIC INTERACTING PARTICLE SYSTEMS*

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Abstract. We consider an interacting particle system modeled as a system of N stochastic differential equations driven by Brownian motions. We prove that the (mollified) empirical process converges, uniformly in time and space variables, to the solution of the two-dimensional Navier–Stokes equation written in vorticity form. The proofs follow a semigroup approach.

Key words. moderately interacting particle system, stochastic differential equations, 2d Navier–Stokes equation, vorticity equation, analytic semigroup

AMS subject classifications. 60H20, 60H10, 60F99

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1. Introduction.

The main goal of this paper is to provide a stochastic particle approximation of the two-dimensional Navier–Stokes equation. Precisely, we consider the following classical Cauchy problem which describes the evolution of the velocity field $u : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}^2$ of an incompressible fluid with kinematic viscosity coefficient $\nu > 0$: for any $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^2$,

$$\begin{cases} \partial_t u(t,x) = \nu\Delta u(t,x) - \big[u(t,x)\bullet\nabla\big]u(t,x) - \nabla p(t,x), \\ \text{div } u(t,x) = 0, \\ u(0,x) = u^{\text{ini}}(x), \end{cases} \tag{1}$$

where $\bullet$ denotes the standard Euclidean product in $\mathbb{R}^2$; the unknown quantities are the velocity $u(t,x) = (u_1(t,x), u_2(t,x)) \in \mathbb{R}^2$ of the fluid element at time t and position x and the pressure $p(t,x) \in \mathbb{R}$. Such equations are attracting the attention of a large scientific community, with a large number of publications in the literature. Since this system is very famous, we do not comment here on its derivation and rather refer to the monographs [30] and [31]. For recent developments, see also [20].

The associated (scalar) *vorticity field* $\xi = \partial_1 u_2 - \partial_2 u_1 : \mathbb{R}^2 \to \mathbb{R}$ satisfies a remarkably simple equation of *convection-diffusion propagation*, namely,

$$\partial_t \xi + u\bullet\nabla\xi = \nu\Delta\xi, \qquad x \in \mathbb{R}^2, t > 0. \tag{2}$$

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The velocity field $u(t,x)$ can be reconstructed from the vorticity distribution $\xi(t,x)$ by the convolution with the *Biot–Savart kernel* K as

$$u(t,x) = \big(K * \xi(t,\cdot)\big)(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2}\,\xi(t,y)dy, \tag{3}$$

where $(x_1,x_2)^\perp := (-x_2,x_1)$. It is well known (see [23, Lemma 1.1]) that there is a constant $c_K > 0$ such that for any $\xi \in \mathbb{L}^1(\mathbb{R}^2)\cap\mathbb{L}^\infty(\mathbb{R}^2)$

$$\|K*\xi\|_{\mathbb{L}^\infty} \le c_K\big(\|\xi\|_{\mathbb{L}^1} + \|\xi\|_{\mathbb{L}^\infty}\big), \tag{4}$$

where $\|\cdot\|_{\mathbb{L}^p}$ denotes the usual $\mathbb{L}^p(\mathbb{R}^2)$ norm. The proof of (4) simply follows from expanding the convolution and dividing $\mathbb{R}^2$ into two parts, the first one containing the points (x,y), where $|y-x|\leqslant 1$, the second one being its complement.

There is a vast literature on that model: for instance, the Cauchy problem (2) for an initial datum in $\mathbb{L}^1(\mathbb{R}^2)$ (also $\mathbb{L}^1\cap\mathbb{L}^p$) was studied in [4]. The existence of solutions of (2) for the case of an initial finite measure was proved in [13] and [19]. Uniqueness in that case is a much more difficult problem; it is shown in [13] that the solution is unique if the atomic part of the initial vorticity is sufficiently small. This last restriction has been removed recently in [12]; there, the authors obtain uniqueness when the initial datum belongs to the space of finite measures.

The question of a particle approximation to the two-dimensional (2d) Navier–Stokes equation has been already considered in the literature, as recalled in more detail in section 1.1 below. The aim of this paper is to provide a new rigorous approximation of the vorticity field ξ by stochastic particle systems, stronger than others: contrary to the previous works where only the empirical measure of the density of particles is shown to converge, here we also prove that a *mollified empirical measure* converges *uniformly*. More precisely, we consider the N-particle dynamics described, for each $N\in\mathbb{N}$, by the following system of coupled stochastic differential equations in $\mathbb{R}^2$: for any $i=1,\dots,N$,

$$dX_t^{i,N} = F\Big(\frac{1}{N}\sum_{k=1}^N (K*V^N)(X_t^{i,N}-X_t^{k,N})\Big)\,dt + \sqrt{2\nu}\,dW_t^i, \tag{5}$$

where:

- for a given $M>0$ chosen ahead (see Theorem 1.3 below), the function F is given by

$$F : \begin{pmatrix} x_1 \\ x_2\end{pmatrix} \mapsto \begin{pmatrix} (x_1\wedge M)\vee(-M) \\ (x_2\wedge M)\vee(-M)\end{pmatrix}; \tag{6}$$

- $\{W_t^i,\ i\in\mathbb{N}\}$ is a family of independent standard Brownian motions on $\mathbb{R}^2$ defined on a filtered probability space $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbb{P})$;
- the *interaction potential* $V^N:\mathbb{R}^2\to\mathbb{R}_+$ is continuous and will be specified later on.

Finally, $*$ stands for the standard convolution product, and $\wedge$ (resp., $\vee$) is the usual notation for the minimum (resp., maximum) of two real numbers.

Now let us define the *empirical process* of this particle system as

$$S_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}},$$

which is a (scalar) measure-valued process associated with the $\mathbb{R}^2$-valued processes $\{t \mapsto X_t^{i,N}\}_{i=1,\dots,N}$. Above, δ_a is the delta Dirac measure concentrated at $a \in \mathbb{R}^2$. For any test function $\phi : \mathbb{R}^2 \to \mathbb{R}$, we use the standard notation

$$\langle S_t^N, \phi \rangle := \frac{1}{N} \sum_{i=1}^{N} \phi(X_t^{i,N}).$$

Our interest lies in the investigation of the dynamical process $t \mapsto S_t^N$ in the large particle limit $N \to \infty$.

The dynamics of the empirical measure is determined by the Itô formula, which reads as follows: for any test function $\phi : \mathbb{R}^2 \to \mathbb{R}$ of class C^2, the empirical measure S_t^N satisfies

$$\begin{aligned} \langle S_t^N, \phi \rangle = \langle S_0^N, \phi \rangle &+ \int_0^t \langle S_s^N, F\big(K * V^N * S_s^N\big) \bullet \nabla \phi \rangle \, ds \\ &+ \nu \int_0^t \langle S_s^N, \Delta \phi \rangle \, ds + \frac{\sqrt{2\nu}}{N} \sum_{i=1}^{N} \int_0^t \nabla \phi(X_s^{i,N}) \bullet dW_s^i. \end{aligned} \tag{7}$$

Our main result is the uniform convergence (in the space and time variables) of the mollified empirical measure

$$g_t^N := V^N * S_t^N : x \in \mathbb{R}^2 \mapsto \int_{\mathbb{R}^2} V^N(x-y) dS_t^N(y)$$

to the solution of the Navier–Stokes equation written in vorticity form, given below in Theorem 1.3. Note that this probability measure is more regular than S_t^N, and its nicer properties allow us to obtain better convergence results. To prove the latter, we follow the new approach presented in [7] and then in [9, 8, 29], based on semigroup theory. Our source of inspiration has been the works of Oelschläger [25] and Jourdain and Méléard [18], where stochastic approximations of PDEs are investigated. We assume that the initial vorticity satisfies $\xi^{\text{ini}} \in \mathbb{L}^1(\mathbb{R}^2) \cap \mathbb{L}^\infty(\mathbb{R}^2)$, but we believe that our approach can be adapted for more irregular initial data, for instance when ξ^{ini} belongs to $\mathbb{L}^1(\mathbb{R}^2) \cap \mathbb{L}^p(\mathbb{R}^2)$ with $p \in (2, \infty)$.

Let us also note that a similar strategy based on a mild formulation for the empirical measure (*not* the mollified one) has recently been worked out in [3], where the authors prove a law of large numbers for weakly interacting particles driven by independent Brownian motions, under weak assumptions on the initial condition.

1.1. Related works. Rigorous derivations of particle approximations to the 2d Navier–Stokes equation have already been investigated in the literature. Chorin in [6] (see also [5]) proposed a heuristic probabilistic algorithm to numerically simulate the solution of the Navier–Stokes equation in two dimensions, by approximating the (scalar) vorticity function, involving cutoff kernels, by random interacting "point vortices." The convergence of Chorin's vortex method was mathematically proved in 1982, for instance, by Marchioro and Pulvirenti [22], who interpreted the vortex equation in two dimensions with bounded and integrable initial condition as a generalized McKean–Vlasov equation. Simultaneously, several authors obtained convergence proofs for Chorin's algorithm; see, for instance, Beale and Majda [2, 1] and Goodman [14]. Finally, a rate of convergence result was obtained by Long in [21]. Later Méléard [23, 24] improved the results and showed the convergence in the path space of the empirical measures of the interacting particle system. Fontbona [10] then generalized that result in dimension $d = 3$.

In addition, following the probabilistic interpretation of [22], a series of papers investigates in detail the *propagation of chaos* property. In 1987, Osada [26] proved such a result for an interacting particle system which approximates the solution of the McKean–Vlasov equation, without cutoff, by an analytical method based on generators of generalized divergence form, but only for large viscosities and bounded density initial data. The convergence of empirical measure and propagation of chaos have then been considered under more general assumptions and with innovative techniques of entropy and Fisher information by Fournier, Hauray and Mischler [11]. Finally we mention that recently, Jabin and Wang [17] showed that a mean field approximation converges to the solution of the Navier–Stokes equation written in vorticity form, and they are able to obtain quantitative optimal convergence rates for all finite marginal distributions of particles.

Besides, let us note that Marchioro and Pulvirenti [22] wished to describe a unified approach for both Navier–Stokes and Euler equations, and for that reason they did not fully exploit the stochastic nature of the Navier–Stokes equation, which is exactly what we are doing here. In fact, we strongly exploit the Brownian perturbation of the system and, therefore, we cannot cover the results obtained for the Euler equation as in [22].

1.2. Notations and results. Before concluding the introduction, let us state the main results of this work. We first need to introduce some of our notations, which are listed below:

- For any measure space (S, Σ, μ), the standard $\mathbb{L}^p(S)$-spaces of real-valued functions with $p \in [1, \infty]$, are provided with their usual norm denoted by $\|\cdot\|_{\mathbb{L}^p(S)}$ or $\|\cdot\|_{\mathbb{L}^p}$ whenever the space S will be clear to the reader. With a little abuse of notation, and as soon as no confusion regarding the space S arises, we denote by $\langle f, g\rangle$ the inner product on $\mathbb{L}^2(S)$ between two functions f and g. In more general cases, if the functions take values in some space X, the notation will become $\mathbb{L}^p(S; X)$. Finally, the norm $\|\cdot\|_{\mathbb{L}^p(S)\to\mathbb{L}^p(S)}$ is the usual operator norm.
- For any $\varepsilon \in \mathbb{R}, p \geqslant 1$, and $d \in \mathbb{N}$, we denote by $\mathbb{H}^{\varepsilon}_p(\mathbb{R}^d)$ the *Bessel potential space*
$$\mathbb{H}^{\varepsilon}_p(\mathbb{R}^d) := \Big\{u \in \mathcal{S}'(\mathbb{R}^d)\ ;\ \mathcal{F}^{-1}\Big(\big(1+|\cdot|^2\big)^{\frac{\varepsilon}{2}}\ \mathcal{F}u(\cdot)\Big) \in \mathbb{L}^p(\mathbb{R}^d)\Big\},$$
where $\mathcal{F}u$ denotes the *Fourier transform* of u. These spaces are endowed with their norm
$$\|u\|_{\varepsilon,p} := \Big\|\mathcal{F}^{-1}\Big((1+|\cdot|^2)^{\frac{\varepsilon}{2}}\ \mathcal{F}u(\cdot)\Big)\Big\|^2_{\mathbb{L}^p(\mathbb{R}^d)} < \infty.$$
Note that
$$\|u\|_{0,2} = \|u\|_{\mathbb{L}^2(\mathbb{R}^d)}$$
and, moreover, for any $\varepsilon \leqslant 0$, we have (using Plancherel's identity and the fact that $(1+|\cdot|)^{\frac{\varepsilon}{2}} \leqslant 1$)
$$\|u\|_{\varepsilon,2} = \big\|(1+|\cdot|^2)^{\frac{\varepsilon}{2}}\ \mathcal{F}u(\cdot)\big\|_{\mathbb{L}^2(\mathbb{R}^d)} \le \|\mathcal{F}u\|_{\mathbb{L}^2(\mathbb{R}^d)} = \|u\|_{0,2}\,.$$
- Let us now recall the definition of *Sobolev–Slobodeckij spaces*. Let U be a general, possibly nonsmooth, open set in $\mathbb{R}^d$. Let $p \geqslant 1$. For any positive integer m we define
$$\mathbb{W}^{m,p}(U) := \Big\{f \in \mathbb{L}^p(U)\ ;\ \|f\|_{m,p} := \sum_{|s|\geqslant m} \|D^s f\|_{\mathbb{L}^p(U)} < \infty\Big\}.$$

For any $\varepsilon > 0$ not an integer, we define

$$\mathbb{W}^{\varepsilon,p}(U) := \Big\{ f \in \mathbb{W}^{[\varepsilon],p}(U) \ ; \ \|f\|_{\varepsilon,p} := \sum_{|s|=[\varepsilon]} \mathcal{I}_s(f) < \infty \Big\},$$

where

$$\mathcal{I}_s(f) := \bigg(\int_U \int_U \frac{|D^s f(x) - D^s f(y)|^p}{|x-y|^{d+(\varepsilon-[\varepsilon])p}} dxdy \bigg)^{1/p}.$$

We observe that: when $U = \mathbb{R}^d$ and $p = 2$, the Sobolev space $\mathbb{W}^{\varepsilon,2}(\mathbb{R}^d)$ and the Bessel space $\mathbb{H}_2^\varepsilon(\mathbb{R}^d)$ coincide: $\mathbb{W}^{\varepsilon,2}(\mathbb{R}^d) = \mathbb{H}_2^\varepsilon(\mathbb{R}^d)$. Moreover, note that for any open set U, $\mathbb{W}^{\varepsilon,2}(U)$ roughly corresponds to distributions f on U which are restrictions of some $f \in \mathbb{H}_2^\varepsilon(\mathbb{R}^d)$; see [32], for instance. Also we recall that $\mathbb{H}_p^\varepsilon \subset \mathbb{W}^{\varepsilon,p}$ for any $p > 1$ and $\varepsilon \geqslant 0$.

- The space of smooth real-valued functions with compact support in $\mathbb{R}^d$ is denoted by $C_0^\infty(\mathbb{R}^d)$. The space of functions of class C^k with $k \in \mathbb{N}$ is denoted by $C^k(\mathbb{R}^d)$. Finally, the space of bounded functions is denoted by $C_b(\mathbb{R}^d)$.

Now, let us give our main assumptions: first, we need to be more precise about the interaction potential V^N; second, recall that we are interested in the large N limit of the process $t \mapsto S_t^N$, and we therefore need to specify its initial condition, which is random, and supposed to be "almost chaotic"; see point 4 in Assumption 1.1 below. The expectation with respect to $\mathbb{P}$ is denoted by $\mathbb{E}$. We say that a function $f : \mathbb{R}^2 \to \mathbb{R}_+$ is a probability density if $\int_{\mathbb{R}^2} f(x)dx = 1$.

Assumption 1.1. We assume that there exists a probability density $V : \mathbb{R}^2 \to \mathbb{R}_+$, and a parameter $\beta \in [0,1]$, such that

1. for any $x \in \mathbb{R}^2$, $V^N(x) = N^{2\beta} V(N^\beta x)$;
2. $V \in C_0^\infty(\mathbb{R}^2)$;
3. there exists $p > 2$ and $\frac{2}{p} < \alpha < 1$ such that, for any $q > 0$,

$$\sup_{N\in\mathbb{N}} \mathbb{E}\Big[\big\| V^N * S_0^N \big\|_{\alpha,p}^q \Big] < \infty \ ; \tag{8}$$

4. there exists $\xi^{\text{ini}} \in \mathbb{L}^1(\mathbb{R}^2) \cap \mathbb{L}^\infty(\mathbb{R}^2)$ such that the sequence of measures $\{S_0^N\}_N$ weakly converges to the initial measure $\xi^{\text{ini}}(\cdot)dx$, as $N \to \infty$, in probability.
5. finally, the parameters (β, α, p) satisfy

$$0 < \beta < \frac{1}{4 + 2\alpha - \frac{4}{p}} < \frac{1}{4}. \tag{9}$$

Remark 1.2. In [8] the authors provide sufficient conditions for the validity of (8). To understand, very roughly, condition (8), think of dimension 1 and $\beta = \frac{1}{2}$: if we have N points on the real line, distributed very regularly, and we convolve (i.e., observe) them by a smooth kernel V^N such that it averages $\sqrt{N}$ of them, the result of the convolution is a function which does not oscillate too much; this is opposite to the case in which the concentration of the kernel V^N is such that it averages only a very few points, so that the convolution is exposed to the granularity of the sample, its minor irregularities, and concentrations. Condition (8) quantifies this control on oscillations.

Some further intuition comes from kernel smoothing algorithms, those which replace a histogram by a smooth curve; the histogram is based on a partition of the real line and simply counts the relative frequency of a sample in the intervals of the partition, kernel smoothing convolves the sample with a smooth kernel, e.g., a Gaussian kernel with standard deviation h. If, compared to the cardinality N and distribution of the sample, the partition is made of too small intervals or h is too small, we see an histogram or a kernel smoothing function which oscillates very much. This happens in particular when h is of the order of the distance between nearest neighbor points in the sample. But if we take h much larger, although very small compared to the full sample, for instance $h \sim N^{-1/2}$ (if the points are concentrated in a set of size of order one), the graph of the curve given by kernel smoothing algorithms is not oscillating anymore.

Let us emphasize, however, that condition (8) is a joint condition on the size of the smoothing kernel compared to the cardinality of the sample (the issue stressed in the previous sentences), but also on the regularity of the sample. If it has extreme concentrations around some points, the pictures above change, oscillations may reappear.

In all what follows we fix a time horizon $T \geq 0$.

THEOREM 1.3. *We assume Assumption* 1.1 *and we consider the particle system* (5) *with the parameter* M *which satisfies*

$$M \geqslant c_K\big(1 + \|\xi^{\rm ini}\|_{\mathbb{L}^\infty}\big), \tag{10}$$

where c_K *has been defined in* (4).

Then, for every $\eta \in (\frac{2}{p}, \alpha)$*, the sequence of processes* $\{t \mapsto g_t^N = V^N * S_t^N\}_{N\in\mathbb{N}}$ *converges in probability with respect to the*

- weak topology of $\mathbb{L}^2([0,T]\,;\,\mathbb{H}_p^\alpha(\mathbb{R}^2))$,
- strong topology of $C([0,T]\,;\,\mathbb{W}_{\rm loc}^{\eta,p}(\mathbb{R}^2))$,

as $N \to \infty$*, to the unique weak solution of the partial differential equation (PDE)*

$$\begin{cases} \partial_t \xi + {\rm div}\big(\xi(K * \xi)\big) = \nu \Delta \xi\,, \\ \xi(0,x) = \xi^{\rm ini}(x), \qquad\qquad x \in \mathbb{R}^2, t > 0. \end{cases} \tag{11}$$

Namely, for any real-valued test function $\phi \in C_0^\infty(\mathbb{R}^2)$ *and any* $t \geqslant 0$*, it holds that*

$$\langle \xi(t,\cdot), \phi\rangle = \big\langle \xi^{\rm ini}, \phi\big\rangle + \int_0^t \big\langle \xi(s,\cdot), (K*\xi)(s,\cdot)\bullet\nabla\phi\big\rangle\, ds + \nu\int_0^t \big\langle \xi(s,\cdot), \Delta\phi\big\rangle\, ds. \tag{12}$$

Remark 1.4. Note that the limiting PDE (11) does not depend on the value of the parameter $\beta \in (0, \frac{1}{4+2\alpha-2p})$.

Remark 1.5. The previous result implies, by the Sobolev embedding theorem (see [32, section 2] for instance), the strong convergence in $C([0,T]\times K)$ for every compact set $K \subset \mathbb{R}^2$.

Here is an outline of the paper: we start in section 2 with an exposition of the strategy to prove Theorem 1.3. We will prove the technical estimates in section 3. We chose to investigate in detail the case where $\xi^{\rm ini}$ is a probability density, in particular, is nonnegative and then, in Appendix A, we show that the same result holds in the general case $\xi^{\rm ini} \in \mathbb{L}^1(\mathbb{R}^2) \cap \mathbb{L}^\infty(\mathbb{R}^2)$, without assumptions on the sign and the value of $\int \xi^{\rm ini}(x)\, dx$, by a simple decoupling argument. Finally, covering the importance and geometrical interpretation of the uniform convergence of the mollified empirical measure, we have included a short discussion in Appendix B. In particular, the uniform convergence does not follow from the weak convergence of the empirical measure.

2. Strategy of the proof of Theorem 1.3. There are three main steps to deriving the convergence result stated in Theorem 1.3:

1. First, we write the *mild formulation* of the identity satisfied by

$$g_t^N(x) = (V^N * S_t^N)(x) = \int_{\mathbb{R}^2} V^N(x-y)\, dS_t^N(y)\,; \tag{13}$$

   see section 2.2. We will obtain a "closed" inequality (note that g^N already appears in the right-hand side of (7)), and then prove two uniform bounds; see Propositions 2.1 and 2.2. To that aim we will use two main properties of the function F: first F is Lipschitz continuous, and second it is bounded; precisely $|F(x)-F(y)| \leq |x-y|$ and $\|F\|_{\mathbb{L}^\infty(\mathbb{R}^2)} \leq M$.
2. Then we apply compacteness arguments and Sobolev embeddings to have subsequences which converge so as to pass to the limit; see sections 2.3, 2.4, and 2.6.
3. Finally, we are able to conclude the proof since the solution to the limiting PDE (11) is unique, as is proved in section 2.5.

The support of a function f is denoted by Supp$f$. When a constant C will depend on some parameter, say α, this will be highlighted in its index by C_α, but the constant may change from line to line.

2.1. Analytic semigroup. Let us first introduce the operator

$$A : \mathcal{D}(A) \subset \mathbb{L}^p(\mathbb{R}^d) \to \mathbb{L}^p(\mathbb{R}^d)$$

defined as $Af = \nu\Delta f$. It is the infinitesimal generator of an analytic semigroup (the heat semigroup) in $\mathbb{L}^p(\mathbb{R}^d)$ (see, for instance, [27]). We denote this semigroup by $\{e^{tA},\ t \geq 0\}$, which is simply given by

$$\left(e^{tA} f\right)(x) = \int_{\mathbb{R}^d} \frac{1}{(4\nu\pi t)^{d/2}} e^{-|x-y|^2/(4\nu t)} f(y)\, dy, \qquad f \in \mathbb{L}^p(\mathbb{R}^d).$$

Moreover, denoting by I the identity operator, we know that, for any $\varepsilon \in \mathbb{R}$, the domain of the operator $(\mathrm{I}-A)^{\varepsilon/2}$ is given by

$$\mathcal{D}\big((\mathrm{I}-A)^{\varepsilon/2}\big) = \mathbb{H}_p^\varepsilon(\mathbb{R}^d)$$

with equivalent norms, where $(\mathrm{I}-A)^{\varepsilon/2}$ is the Bessel potential operator given by $(\mathrm{I}-A)^{\varepsilon/2} f = \mathcal{F}^{-1}((1+|\cdot|^2)^{\frac{\varepsilon}{2}}\ \mathcal{F}f(\cdot))$. Recall also from [27] that, for every $\varepsilon > 0$ and $T > 0$, and $p > 1$, there is a constant $C_{\varepsilon,T,\nu,p} > 0$ such that, for any $t \in (0,T]$,

$$\left\| (\mathrm{I}-A)^\varepsilon e^{tA} \right\|_{\mathbb{L}^p \to \mathbb{L}^p} \leq \frac{C_{\varepsilon,T,\nu,p}}{t^\varepsilon}. \tag{14}$$

We are now ready to prove Theorem 1.3; therefore from now on the dimension is $d = 2$.

2.2. The equation for $V^N * S_t^N$ in mild form. We want to deduce an identity for $g_t^N(x)$ from (7). For every $x \in \mathbb{R}^2$ take, in identity (7), the test function $\phi_x(y) = V^N(x-y)$. We get (recall the definition (13) of g_t^N)

$$\begin{aligned} g_t^N(x) = g_0^N(x) &+ \int_0^t \left\langle S_s^N, F(K * g_s^N) \bullet \nabla V^N(x-\cdot)\right\rangle ds \\ &+ \nu \int_0^t \Delta g_s^N(x) ds + \frac{\sqrt{2\nu}}{N} \sum_{i=1}^N \int_0^t \nabla V^N\left(x - X_s^{i,N}\right) \bullet dW_s^i. \end{aligned} \tag{15}$$

In the following, let us write for the sake of clarity,

$$\left\langle S_s^N, F(K*g_s^N)\bullet\nabla V^N(x-\cdot)\right\rangle =: \Big(\nabla V^N * \big(F(K*g_s^N)S_s^N\big)\Big)(x)$$

and similarly for analogous expressions. Hence we can write (using the same idea as in [7])

$$\begin{aligned} g_t^N = e^{tA}g_0^N &+ \int_0^t e^{(t-s)A}\left(\nabla V^N * \left(F(K*g_s^N)S_s^N\right)\right)ds \\ &+ \frac{\sqrt{2\nu}}{N}\sum_{i=1}^N \int_0^t e^{(t-s)A}\left(\nabla V^N\left(\cdot - X_s^{i,N}\right)\right)\bullet dW_s^i. \end{aligned}$$

By inspection of the convolution explicit formula for $e^{(t-s)A}$, one can see that

$$e^{(t-s)A}\nabla f = \nabla e^{(t-s)A} f,$$

and then one can use the semigroup property, so as to write

$$\begin{aligned} g_t^N = e^{tA}g_0^N &+ \int_0^t \nabla e^{(t-s)A}\left(V^N * \left(F(K*g_s^N)S_s^N\right)\right)ds \\ &+ \frac{\sqrt{2\nu}}{N}\sum_{i=1}^N \int_0^t e^{(t-s)A}\left(\nabla V^N\left(\cdot - X_s^{i,N}\right)\right)\bullet dW_s^i. \end{aligned} \tag{16}$$

Recall that three parameters $p \geqslant 2$, $\alpha \in (\frac{2}{p}, 1)$ and β are fixed from Assumption 1.1 for the rest of the paper. From now on every constant C_λ which depends on some parameter λ may also depend on the three parameters α, p, β: we decide to withdraw them from the notation in order not to burden the paper. In the following we will prove two important bounds:

Proposition 2.1. *We assume Assumption* 1.1. *Let* $q \geqslant 2$. *Then there exists a positive constant* $C_{T,M,\nu,q}$ *such that, for all* $t \in (0,T]$ *and* $N \in \mathbb{N}$, *it holds:*

$$\mathbb{E}\left[\left\|(\mathrm{I}-A)^{\alpha/2} g_t^N\right\|_{\mathbb{L}^p(\mathbb{R}^2)}^q\right] \le C_{T,M,\nu,q}. \tag{17}$$

Proposition 2.2. *We assume Assumption* 1.1. *Let* $\gamma \in (0, \frac{1}{2})$ *and* $q' \geqslant 2$. *There exists a positive constant* $C_{T,M,\nu,q'}$ *such that, for any* $N \in \mathbb{N}$, *it holds that*

$$\mathbb{E}\left[\int_0^T\int_0^T \frac{\left\|g_t^N - g_s^N\right\|_{-2,2}^{q'}}{|t-s|^{1+q'\gamma}}\, ds\, dt\right] \le C_{T,M,\nu,q'}. \tag{18}$$

The proofs of Propositions 2.1 and 2.2 are postponed to sections 3.1 and 3.2, respectively.

2.3. Criterion of compactness. In this subsection we follow the arguments of [8, Section 3.1]. We start by constructing a space on which the sequence of the probability laws of $g_\cdot^N$ is tight.

We exploit Corollary 9 of Simon [28], using as far as possible the notations of that paper. Given a ball $\mathcal{B}_R \subset \mathbb{R}^2$, taken $\alpha > \frac{2}{p}$ (as in Theorem 1.3), and $\frac{2}{p} < \eta < \alpha$, we consider the space

$$X := \mathbb{W}^{\alpha,p}(\mathcal{B}_R), \qquad B := \mathbb{W}^{\eta,p}(\mathcal{B}_R), \qquad Y := \mathbb{W}^{-2,2}(\mathcal{B}_R).$$

One can check that $X \subset B \subset Y$, with compact dense embedding. Moreover, we have the interpolation inequality: for any $f \in X$

$$\|f\|_B \le C_R \, \|f\|_X^{1-\theta} \, \|f\|_Y^{\theta}$$

with

$$\theta := \frac{\alpha - \eta}{2 + \alpha}. \tag{19}$$

Now taking, in the notations of [28, Corollary 9], $s_0 := 0$, $s_1 := \gamma \in (0, \frac{1}{2})$, and choosing $q, q' \geqslant 2$ such that

$$s_1 q' = \gamma q' > 1 \qquad \text{and} \qquad s_\theta := \theta s_1 = \theta\gamma > \frac{1-\theta}{q} + \frac{\theta}{q'}, \tag{20}$$

then the corollary tells us that the space $\mathbb{L}^q([0,T]\,;\,X) \cap \mathbb{W}^{\gamma,q'}([0,T]\,;\,Y)$ is relatively compact in $C([0,T]\,;\,B)$.

Therefore, for any $\gamma \in (0, \frac{1}{2})$, for the parameter $\alpha > \frac{2}{p}$ given by point 3 of Assumption 1.1, and for $q, q' \geqslant 2$ which satisfy (20), we now consider the space

$$\mathfrak{Y}_0 := \mathbb{L}^q\big([0,T]\,;\,\mathbb{H}^\alpha_p\big) \cap \mathbb{W}^{\gamma,q'}\big([0,T]\,;\,\mathbb{H}^{-2}_2\big).$$

We use the Fréchet topology on $C\big([0,T]\,;\,\mathbb{W}^{\eta,p}_{\mathrm{loc}}(\mathbb{R}^2)\big)$ defined as

$$d\,(f,g) = \sum_{n=1}^{\infty} 2^{-n} \Big(1 \wedge \sup_{t\in[0,T]} \big\|(f-g)(t,\cdot)\big\|^2_{\mathbb{W}^{\eta,p}(\mathcal{B}_n)} \Big),$$

where $\wedge$ denotes the infimum. From the above, we conclude that $\mathfrak{Y}_0$ is compactly embedded into $C\big([0,T]\,;\,\mathbb{W}^{\eta,p}_{\mathrm{loc}}\big)$ for any $\frac{2}{p} < \eta < \alpha$. Finally, let us denote by $\mathbb{L}^2_w$ the spaces $\mathbb{L}^2$ endowed with the weak topology. We obtain that $\mathfrak{Y}_0$ is compactly embedded into

$$\mathfrak{Y} := \mathbb{L}^2_w\big([0,T]\,;\,\mathbb{H}^\alpha_p\big) \cap C\big([0,T]\,;\,\mathbb{W}^{\eta,p}_{\mathrm{loc}}\big). \tag{21}$$

Note that

$$C\big([0,T]\,;\,\mathbb{W}^{\eta,p}_{\mathrm{loc}}\big) \subset C\big([0,T]\,;\,C(D)\big)$$

for every regular bounded domain $D \subset \mathbb{R}^2$.

Let us now go back to the sequence of processes $\{g^N_\cdot\}_N$ for which we have proved several estimates. The Chebyshev inequality ensures that

$$\mathbb{P}\big(\|g^N_\cdot\|^2_{\mathfrak{Y}_0} > R\big) \le \frac{\mathbb{E}\big[\|g^N_\cdot\|^2_{\mathfrak{Y}_0}\big]}{R} \qquad \text{for any } R > 0.$$

Thus by Propositions 2.1 and 2.2 (since $q, q' \geqslant 2$), we obtain

$$\mathbb{P}\big(\|g^N_\cdot\|^2_{\mathfrak{Y}_0} > R\big) \le \frac{C}{R} \qquad \text{for any } R > 0, N \in \mathbb{N}.$$

The process $t \in [0,T] \mapsto g^N_t$ defines a probability $\mathbf{P}_N$ on $\mathfrak{Y}$. Fix $\varepsilon > 0$. The last inequality implies that there exists a bounded set $B_\epsilon \in \mathfrak{Y}_0$ such that $\mathbf{P}_N(B_\varepsilon) < 1 - \varepsilon$ for all N and, therefore, from the previous argument, there exists a compact set $K_\varepsilon \subset \mathfrak{Y}$ such that $\mathbf{P}_N(K_\varepsilon) < 1 - \varepsilon$.

Finally, denoting by $\{\mathtt{L}^N\}_{N\in\mathbb{N}}$ the laws of the processes $\{g^N\}_{N\in\mathbb{N}}$ on $\mathfrak{Y}_0$, we have proved that $\{\mathtt{L}^N\}_{N\in\mathbb{N}}$ is tight in $\mathfrak{Y}$, hence relatively compact, by Prohorov's theorem. From every subsequence of $\{\mathtt{L}^N\}_{N\in\mathbb{N}}$ it is possible to extract a further subsequence which converges to a probability measure $\mathtt{L}$ on $\mathfrak{Y}$. Moreover by a theorem of Skorokhod (see [16, Theorem 2.7]), we are allowed, eventually after choosing a suitable probability space where all our random variables can be defined, to assume

$$g^N \to \xi \quad \text{in } \mathfrak{Y}, \qquad \text{a.s.,} \tag{22}$$

where the law of ξ is $\mathtt{L}$.

2.4. Passing to the limit. Next step is to characterize the limit. First, recall formula (15), which reads

$$g_t^N(x) = g_0^N(x) + \int_0^t \left\langle S_s^N, F(K*g_s^N)\bullet \nabla V^N(x-\cdot)\right\rangle ds + \nu\int_0^t \Delta g_s^N(x)ds + \frac{\sqrt{2\nu}}{N}\sum_{i=1}^N \int_0^t \nabla V^N\left(x - X_s^{i,N}\right)\bullet dW_s^i.$$

Taking a test function $\phi : \mathbb{R}^2 \to \mathbb{R}$ we have

$$\left\langle g_t^N,\phi\right\rangle = \left\langle g_0^N,\phi\right\rangle + \int_0^t \left\langle S_s^N, F(K*g_s^N)\bullet \nabla(V^N*\phi)\right\rangle ds + \nu\int_0^t \left\langle g_s^N,\Delta\phi\right\rangle ds + \frac{\sqrt{2\nu}}{N}\sum_{i=1}^N \int_0^t \nabla(V^N*\phi)\left(X_s^{i,N}\right)\bullet dW_s^i. \tag{23}$$

It is clear from (22) that

$$\left\langle g_t^N,\phi\right\rangle \xrightarrow[N\to\infty]{} \langle \xi,\phi\rangle, \qquad \left\langle g_0^N,\phi\right\rangle \xrightarrow[N\to\infty]{} \left\langle \xi^{\mathrm{ini}},\phi\right\rangle, \tag{24}$$

and also that

$$\mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^N \int_0^t \nabla(V^N*\phi)\left(X_s^{i,N}\right)\bullet dW_s^i\right|^2\right] = \frac{1}{N^2}\sum_{i=1}^N\int_0^t \mathbb{E}\left[\left|\nabla(V^N*\phi)\left(X_s^{i,N}\right)\right|^2\right]ds \leq \frac{t}{N}\|\nabla\phi\|_{\mathbb{L}^\infty}^2 \xrightarrow[N\to\infty]{} 0. \tag{25}$$

We now claim that

$$\lim_{N\to\infty}\int_0^t \left\langle S_s^N, F(K*g_s^N)\bullet\nabla(V^N*\phi)\right\rangle ds = \int_0^t\int_{\mathbb{R}^2}\xi(s,x)F(K*\xi)\bullet\nabla\phi(x)\,dxds. \tag{26}$$

Proof of (26). First, we observe that $\left\|g_t^N\right\|_{\mathbb{L}^1} = 1$ and $g_\cdot^N$ is uniformly bounded in $\mathbb{L}^2([0,T]\,;\,\mathbb{H}_p^\eta)$ for any $\eta > \frac{2}{p}$. Then by the Sobolev embedding theorem (see [32, section 2.8.1]), we have that $g_\cdot^N$ is uniformly bounded in $\mathbb{L}^2\big([0,T]\,;\,C_b(\mathbb{R})\big)$.

By interpolation we also know that $g_\cdot^N \in \mathbb{L}^2([0,T]\,;\,\mathbb{L}^a(\mathbb{R}^2))$ for any $a\in[1,+\infty]$, and there is a constant $C_K>0$ such that

$$\left\|K*g_t^N\right\|_{\mathbb{L}^a} \leq C_K\left\|g_t^N\right\|_{\mathbb{L}^b} \tag{27}$$

with $\frac{1}{a} = \frac{1}{b} - \frac{1}{2}$ and $1<b<2$. The kernel K is singular, of Calderon–Zygmund type.

Hence, there exists a constant $C>0$ such that, for any $a\in(1,+\infty)$,

$$\big\|\nabla(K*g_t^N)\big\|_{\mathbb{L}^a}\le C\big\|g_t^N\big\|_{\mathbb{L}^a}. \tag{28}$$

Let us introduce $f_\cdot^N=K*g_\cdot^N$. By the Sobolev embedding theorem, estimates (27) and (28) imply that, for any $\tilde\eta>0$ (take $\tilde\eta=1-\frac{2}{a}$ with $a>2$),

$$\|f^N\|_{\mathbb{L}^2([0,T]\,;\,C^{\tilde\eta}(\mathbb{R}^2))}\le C.$$

Now, let $\chi:\mathbb{R}^2\to[0,1]\in C_0^\infty$ be a *cutoff function*, such that

$$0\le\chi(x)\le 1\qquad\text{and}\qquad \chi(x)=1\text{ if }|x|\le 1.$$

Let $\tilde\chi=1-\chi$. We can decompose

$$\begin{aligned}&\big(K*\big(g_t^N-\xi\big)\big)(x)\\&\quad=\int_{\mathbb{R}^2}\chi(y)K(y)(g_t^N-\xi)(x-y)dy+\int_{\mathbb{R}^2}\tilde\chi(x-y)K(x-y)(g_t^N-\xi)(y)dy.\end{aligned}$$

We observe that $\chi(\cdot)K(\cdot)\in\mathbb{L}^c(\mathbb{R}^2)$ with $c<2$, and $\tilde\chi(x-\cdot)K(x-\cdot)\in\mathbb{L}^d(\mathbb{R}^2)$ with $d>2$. Since $g_\cdot^N$ is uniformly bounded in $\mathbb{L}^2([0,T]\,;\,\mathbb{L}^a(\mathbb{R}^2))$ for all $a\in[1,+\infty]$ we obtain that $K*g_t^N$ converges to $K*\xi$.

Finally, we can bound as follows:

$$\begin{aligned}&\Big|\Big\langle S_s^N,F(K*g_s^N)\bullet\nabla(V^N*\phi)\Big\rangle-\Big\langle g_s^N,F(K*g_s^N)\bullet\nabla(V^N*\phi)\Big\rangle\Big|\\&\qquad\le\sup_{x\in\mathbb{R}^2}\Big|F(K*g_s^N)\bullet\nabla(V^N*\phi)(x)-\big(F(K*g_s^N)\bullet\nabla(V^N*\phi)\big)*V^N\big)(x)\Big|.\end{aligned}$$

Let us control the last term, using the facts that

- V is a density (denoted below by ($\int V=1$));
- F is Lipschitz and bounded (denoted below by ($F\in\mathrm{Lip}\cap L^\infty$));
- V is compactly supported (denoted below by (V is c.s.));
- and ϕ is compactly supported and smooth,

as follows (the norm $\|\cdot\|$ below is the Euclidean norm on $\mathbb{R}^2$): for any $x\in\mathbb{R}^2$,

$$\begin{aligned}&\Big|F(K*g_s^N)(x)\bullet\nabla(V^N*\phi)(x)-\big(F(K*g_s^N)\bullet\nabla(V^N*\phi)\big)*V^N\big)(x)\Big|\\
&\overset{(\int V=1)}{\le}\int_{\mathbb{R}^2}V(y)\,\Big\|\nabla(V^N*\phi)(x)\Big\|\,\Big\|F(K*g_s^N)(x)-F(K*g_s^N)\big(x-\tfrac{y}{N^\beta}\big)\Big\|dy\\
&\qquad+\int_{\mathbb{R}^2}V(y)\,\Big\|\nabla(V^N*\phi)(x)-\nabla(V^N*\phi)\big(x-\tfrac{y}{N^\beta}\big)\Big\|\,\big\|(K*g_s^N)(x)\big\|dy\\
&\overset{(F\in\mathrm{Lip}\cap L^\infty)}{\le}C\int_{\mathbb{R}^2}V(y)\,\|\nabla(V^N*\phi)(x)\|\,\Big\|f_s^N(x)-f_s^N\big(x-\tfrac{y}{N^\beta}\big)\Big\|dy\\
&\qquad+\frac{C}{N^\beta}\int_{\mathbb{R}^2}V(y)\|y\|dy\\
&\overset{(V\text{ is c.s.})}{\le}\frac{C}{N^{\tilde\eta\beta}}\sup_{x,y\in\mathbf{K}}\frac{\big\|f_s^N(x)-f_s^N(y)\big\|}{\|x-y\|^{\tilde\eta}}\int_{\mathbb{R}^2}V(y)\|y\|^{\tilde\eta}\,dy\\
&\qquad+\frac{C}{N^\beta}\int_{\mathbb{R}^2}V(y)\|y\|dy,\end{aligned}$$

where $\mathbf{K} \subset \mathbb{R}^2$ is a compact set. Therefore we have obtained

$$\Big| F(K * g_s^N)(x)\bullet \nabla(V^N * \phi)(x) - \big(F(K * g_s^N)\bullet \nabla(V^N * \phi)\big) * V^N\big)(x)\Big| \leq \frac{C}{N^{\tilde{\eta}\beta}},$$

where the constant C depends on $\|f^N\|_{C^{\tilde{\eta}}(\mathbb{R}^2)}$. Thus,

$$\begin{aligned}\lim_{N\to\infty}\int_0^t \Big\langle S_s^N, F(K * g_s^N) \bullet \nabla(V^N * \phi)\Big\rangle \, ds \\ &= \lim_{N\to\infty}\int_0^t \Big\langle g_s^N, F(K * g_s^N)\bullet \nabla(V^N * \phi)\Big\rangle \, ds \\ &= \lim_{N\to\infty}\int_0^t \int_{\mathbb{R}^2} g_s^N(x) F(K * g_s^N)(x)\bullet \nabla(V^N * \phi)(x) \; dxds \\ &= \int_0^t \int_{\mathbb{R}^2} \xi(s,x) F(K * \xi(s,\cdot))(x)\bullet \nabla\phi(x) \, dxds,\end{aligned}$$

where in the last equality we used that $g_s^N \to \xi$ strongly in $\mathbb{L}^2\left([0,T]\,;\, C(\mathbb{R}^2)\right)$. We have proved (26).

Therefore from (23), (24), (25), and (26), we conclude that the limit point ξ is then a weak solution to the PDE

$$\partial_t \xi + \operatorname{div}\big(\xi\; F(K * \xi)\big) = \nu\Delta\xi, \qquad \xi\big|_{t=0} = \xi^{\text{ini}}. \tag{29}$$

Note that there is one more step to recover (44) in Theorem 1.3, which will be achieved in section 2.7 below. Before that, we need to prove that the solution to (11) is unique.

2.5. Uniqueness of the solution.

THEOREM 2.3. *We assume that* $\xi^{\text{ini}} \in \mathbb{L}^1 \cap \mathbb{L}^\infty(\mathbb{R}^2)$. *Then there is at most one weak solution of* (11) *which belongs to* $\mathbb{L}^2\big([0,T]\,;\, \mathbb{L}^1 \cap \mathbb{L}^\infty(\mathbb{R}^2)\big)$.

Proof. For any function $u : [0,T]\times\mathbb{R}^2 \to \mathbb{R}$ we introduce the notation

$$\|u\|_{\mathbb{L}^1\cap\mathbb{L}^\infty} := \|u(t,\cdot)\|_{\mathbb{L}^1} + \|u(t,\cdot)\|_{\mathbb{L}^\infty}.$$

Let ξ^1, ξ^2 be two weak solutions of (11) with the same initial condition ξ^{ini} which satisfies $\xi \in \mathbb{L}^1 \cap \mathbb{L}^\infty(\mathbb{R}^2)$. By hypothesis, from [4], $\xi^1, \xi^2 \in \mathbb{L}^2([0,T]\,;\, \mathbb{L}^\infty)$.

Let $\{h_\varepsilon\}_\varepsilon$ be a family of standard symmetric mollifiers on $\mathbb{R}^2$. For any $\varepsilon > 0$ and $x \in \mathbb{R}^2$ we can use $h_\varepsilon(x-\cdot)$ as a test function in (12). Therefore we set $\xi_\varepsilon^i(t,x) = (\xi^i(t,\cdot) * h_\varepsilon)(x)$ for $i = 1,2$. Then we have, for any $(t,x) \in [0,T]\times\mathbb{R}^2$,

$$\xi_\varepsilon^i(t,x) = (\xi^{\text{ini}} * h_\varepsilon)(x) + \nu\int_0^t \Delta\xi_\varepsilon^i(s,x)\, ds + \int_0^t \big((\nabla h_\varepsilon \bullet F(K * \xi^i)) * \xi^i\big)(s,x)\, ds.$$

Writing this identity in mild form we obtain (writing with a little abuse of notation $\xi^i(t)$ for the function $\xi^i(t,\cdot)$ and $\mathfrak{S}(t)$ for e^{tA})

$$\xi_\varepsilon^i(t) = \mathfrak{S}(t)\bullet (\xi^{\text{ini}} * h_\varepsilon) + \int_0^t \mathfrak{S}(t-s)\bullet \big((\nabla h_\varepsilon \bullet F(K * \xi^i)) * \xi^i\big)(s)ds.$$

The function $X = \xi^1 - \xi^2$ satisfies

$$h_\varepsilon * X(t) = \int_0^t \nabla\mathfrak{S}(t-s)\bullet\Big(h_\varepsilon * \Big(F(K * \xi^1)\xi^1 - F(K * \xi^2)\xi^2\Big)\Big)\, ds.$$

Thus we obtain

$$\|h_\varepsilon * X(t)\|_{\mathbb{L}^\infty} \leq \int_0^t \left\| \nabla\mathfrak{G}(t-s)\bullet\Big(h_\varepsilon * \Big(F(K*\xi^1)\xi^1 - F(K*\xi^2)\xi^2\Big)\Big)\right\|_{\mathbb{L}^\infty} ds.$$

Therefore, there is a constant $C_\nu > 0$ such that

$$\|h_\varepsilon * X(t)\|_{\mathbb{L}^\infty} \leq \int_0^t \frac{C_\nu}{(t-s)^{\frac{1}{2}}} \left\| h_\varepsilon * \Big(F(K*\xi^1)\xi^1 - F(K*\xi^2)\xi^2\Big)\right\|_{\mathbb{L}^\infty} ds.$$

Taking the limit as $\varepsilon \to 0$ we arrive at

$$\|X(t)\|_{\mathbb{L}^\infty} \leq \int_0^t \frac{C_\nu}{(t-s)^{\frac{1}{2}}} \left\| F(K*\xi^1)\xi^1 - F(K*\xi^2)\xi^2\right\|_{\mathbb{L}^\infty} ds.$$

With similar arguments we have the same estimate in the $\mathbb{L}^1$-norm as follows:

$$\|X(t)\|_{\mathbb{L}^1} \leq \int_0^t \frac{C'_\nu}{(t-s)^{\frac{1}{2}}} \left\| F(K*\xi^1)\xi^1 - F(K*\xi^2)\xi^2\right\|_{\mathbb{L}^1} ds.$$

By an easy calculation we have

$$\begin{aligned}
\|X(t)\|_{\mathbb{L}^\infty} &\leq \int_0^t \frac{C_\nu}{(t-s)^{\frac{1}{2}}} \Big(\big\|XF(K*\xi^1)\big\|_{\mathbb{L}^\infty} + \big\|\xi^2\big(F(K*\xi^1) - F(K*\xi^2)\big)\big\|_{\mathbb{L}^\infty}\Big)\, ds \\
&\leq C_\nu \int_0^t \frac{\|\xi^1\|_{\mathbb{L}^1\cap\mathbb{L}^\infty}}{(t-s)^{\frac{1}{2}}} \|X\|_{\mathbb{L}^\infty}\, ds + C_\nu \int_0^t \frac{\|\xi^2\|_{\mathbb{L}^1\cap\mathbb{L}^\infty}}{(t-s)^{\frac{1}{2}}} \big(\|X\|_{\mathbb{L}^\infty} + \|X\|_{\mathbb{L}^1}\big) ds \\
&\leq C_\nu \int_0^t \frac{\|\xi^1\|_{\mathbb{L}^1\cap\mathbb{L}^\infty} + \|\xi^2\|_{\mathbb{L}^1\cap\mathbb{L}^\infty}}{(t-s)^{\frac{1}{2}}} \big(\|X\|_{\mathbb{L}^\infty} + \|X\|_{\mathbb{L}^1}\big) ds.
\end{aligned}$$

On the other hand, in a similar way we have

$$\begin{aligned}
\|X(t)\|_{\mathbb{L}^1} &\leq C'_\nu \int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} \Big(\big\|XF(K*\xi^1)\big\|_{\mathbb{L}^1} + \big\|\xi^2\big(F(K*\xi^1) - F(K*\xi^2)\big)\big\|_{\mathbb{L}^1}\Big)\, ds \\
&\leq C'_\nu \int_0^t \frac{\|\xi^1\|_{\mathbb{L}^1\cap\mathbb{L}^\infty} + \|\xi^2\|_{\mathbb{L}^1\cap\mathbb{L}^\infty}}{(t-s)^{\frac{1}{2}}} \big(\|X\|_{\mathbb{L}^\infty} + \|X\|_{\mathbb{L}^1}\big) ds.
\end{aligned}$$

Therefore we have, for a constant $C''_\nu > 0$, that

$$\|X(t)\|_{\mathbb{L}^1\cap\mathbb{L}^\infty} \leq C''_\nu \int_0^t \frac{\|\xi^1\|_{\mathbb{L}^1\cap\mathbb{L}^\infty} + \|\xi^2\|_{\mathbb{L}^1\cap\mathbb{L}^\infty}}{(t-s)^{\frac{1}{2}}} \|X(s)\|_{\mathbb{L}^1\cap\mathbb{L}^\infty}\, ds.$$

By Gronwall's lemma we conclude $X = 0$. ∎

2.6. Convergence in probability.

COROLLARY 2.4. *The sequence $\{g^N\}_{N\in\mathbb{N}}$ converges in probability to ξ.*

Proof. We denote the joint law of (g^N, g^M) by $\nu^{N,M}$. Similarly to the proof of tightness for g^N (section 2.3) we have that the family $\{\nu^{N,M}\}$ is tight in $\mathfrak{Y}\times\mathfrak{Y}$, where $\mathfrak{Y}$ has been defined in (21).

Let us take any subsequence ν^{N_k,M_k}. By Prohorov's theorem, it is relatively weakly compact hence it contains a weakly convergent subsequence. Without loss of generality we may assume that the original sequence $\{\nu^{N,M}\}$ itself converges weakly to a measure ν. According to the Skorokhod immersion theorem, we infer the existence of a probability space $(\overline{\Omega},\overline{\mathcal{F}},\overline{\mathbb{P}})$ with a sequence of random variables $(\overline{g}^N,\overline{g}^M)$ converging almost surely in $\mathfrak{Y}\times\mathfrak{Y}$ to random variable $(\bar{u},\breve{u})$ and the laws of $(\overline{g}^N,\overline{g}^M)$ and $(\bar{u},\breve{u})$ under $\overline{\mathbb{P}}$ coincide with $\nu^{N,M}$ and ν, respectively.

Analogously, it can be applied to both $\overline{g}^N$ and $\overline{g}^M$ in order to show that $\bar{u}$ and $\breve{u}$ are two solutions of the PDE (11). By Theorem 2.3, which gives the uniqueness of the solution, we have $\bar{u}=\breve{u}$. Therefore

$$\nu\big((x,y)\in\mathfrak{Y}\times\mathfrak{Y}\ ;\ x=y\big)=\overline{\mathbb{P}}(\bar{u}=\breve{u})=1.$$

Now, we have all in hand to apply Gyongy–Krylov's characterization of convergence in probability, which is written as follows.

LEMMA 2.5 (Gyongy–Krylov [15]). *Let $\{X_n\}$ be a sequence of random elements in a Polish space Ψ equipped with the Borel σ-algebra. Then X_n converges in probability to a Ψ-valued random element if and only if for each pair (X_ℓ,X_m) of subsequences, there exists a subsequence $\{v_k\}$ given by*

$$v_k=(X_{\ell(k)},X_{m(k)}),$$

converging weakly to a random element $v(x,y)$ supported on the diagonal set

$$\{(x,y)\in\Psi\times\Psi : x=y\}.$$

This lemma implies that the original sequence defined on the initial probability space converges in probability in the topology of $\mathfrak{Y}$ to a random variable μ. □

2.7. Conclusion. Let $\xi(t,x)$ be the unique solution of the vorticity equation (11) with initial condition $\xi^{\rm ini}\in\mathbb{L}^\infty\cap\mathbb{L}^1(\mathbb{R}^2)$. From [4] we have $\|\xi\|_\infty\le\|\xi^{\rm ini}\|_\infty$. Then, by definition of the Biot–Savart kernel K, there is a positive constant c_K (given by (4)) such that

$$\|K*\xi\|_\infty\le c_K(1+\|\xi^{\rm ini}\|_\infty).$$

Therefore if we take $M\ge c_K(1+\|\xi^{\rm ini}\|_\infty)$, we conclude that $\xi(t,x)$ coincides with the unique solution of (29), which is satisfied by the limit point of the sequence $\{g^N\}$.

3. Technical proofs.

In this last section we prove Propositions 2.1 and 2.2.

3.1. Proof of Proposition 2.1. Let us prove the first estimate on g^N given in Proposition 2.1, namely, (17). Let $q\geqslant 2$.

Step 1. From (16) after a multiplication by $(\mathrm{I}-A)^{\alpha/2}$ and by triangular inequality we have

$$\Big\|(\mathrm{I}-A)^{\alpha/2}g_t^N\Big\|_{\mathbb{L}^p(\mathbb{R}^2)}\le\Big\|(\mathrm{I}-A)^{\alpha/2}\,e^{tA}g_0^N\Big\|_{\mathbb{L}^p(\mathbb{R}^2)} \tag{30}$$

$$+\int_0^t\Big\|(\mathrm{I}-A)^{\alpha/2}\,\nabla e^{(t-s)A}\left(V^N*\left(F(K*g_s^N)S_s^N\right)\right)\Big\|_{\mathbb{L}^p(\mathbb{R}^2)}ds \tag{31}$$

$$+\Big\|\frac{\sqrt{2\nu}}{N}\sum_{i=1}^N\int_0^t(\mathrm{I}-A)^{\alpha/2}\,\nabla e^{(t-s)A}\left(V^N\left(\cdot-X_s^{i,N}\right)\right)dW_s^i\Big\|_{\mathbb{L}^p(\mathbb{R}^2)}. \tag{32}$$

We denote $H := \mathbb{L}^p(\mathbb{R}^2)$. Then

$$\begin{aligned}\left\| (\mathrm{I}-A)^{\alpha/2} g_t^N \right\|_{\mathbb{L}^q(\Omega,H)} &\leq \left\| (\mathrm{I}-A)^{\alpha/2} e^{tA} g_0^N \right\|_{\mathbb{L}^q(\Omega,H)} \\ &+ \int_0^t \left\| (\mathrm{I}-A)^{\alpha/2} \nabla e^{(t-s)A} \left(V^N * \left(F(K*g_s^N) S_s^N \right) \right) \right\|_{\mathbb{L}^q(\Omega,H)} ds \\ &+ \left\| \frac{\sqrt{2\nu}}{N} \sum_{i=1}^N \int_0^t (\mathrm{I}-A)^{\alpha/2} \nabla e^{(t-s)A} \left(V^N \left(\cdot - X_s^{i,N} \right) \right) dW_s^i \right\|_{\mathbb{L}^q(\Omega,H)}.\end{aligned}$$

Step 2. The first term (30) can be estimated by

$$\left\| (\mathrm{I}-A)^{\alpha/2} e^{tA} g_0^N \right\|_{\mathbb{L}^q(\Omega,H)} \leq \left\| (\mathrm{I}-A)^{\alpha/2} g_0^N \right\|_{\mathbb{L}^q(\Omega,H)} \leq C_q.$$

The boundedness of g_0^N follows from Assumption 1.1, item 3.

Step 3. Let us come to the second term (31):

$$\begin{aligned}&\int_0^t \left\| (\mathrm{I}-A)^{\alpha/2} \nabla e^{(t-s)A} \left(V^N * \left(F(K*g_s^N) S_s^N \right) \right) \right\|_{\mathbb{L}^q(\Omega,H)} ds \\ &\quad \leq C \int_0^t \Big\{ \left\| (\mathrm{I}-A)^{(1+\alpha)/2} e^{(t-s)A} \right\|_{\mathbb{L}^p \to \mathbb{L}^p} \\ &\qquad\qquad \times \left\| V^N * \left(F(K*g_s^N) S_s^N \right) \right\|_{\mathbb{L}^q(\Omega,H)} \Big\} ds.\end{aligned}$$

We have

$$\left\| (\mathrm{I}-A)^{(1+\alpha)/2} e^{(t-s)A} \right\|_{\mathbb{L}^p \to \mathbb{L}^p} \leq \frac{C_{\nu,T}}{(t-s)^{(1+\alpha)/2}}.$$

On the other hand, for any $x \in \mathbb{R}^2$,

$$\left| \left(V^N * \left(F(K*g_s^N) S_s^N \right) \right)(x) \right| \leq \left\| F(K*g_s^N) \right\|_\infty \left| V^N * S_s^N(x) \right| \leq M \left| g_s^N(x) \right|.$$

Hence,

$$\left\| V^N * \left(F(K*g_s^N) S_s^N \right) \right\|_{\mathbb{L}^q(\Omega,H)} \leqslant M \|g_s^N\|_{\mathbb{L}^q(\Omega,H)} \leqslant C_M \left\| (\mathrm{I}-A)^{\alpha/2} g_s^N \right\|_{\mathbb{L}^q(\Omega,H)}.$$

To summarize, we have proved

$$\begin{aligned}&\int_0^t \left\| (\mathrm{I}-A)^{\alpha/2} \nabla e^{(t-s)A} \left(V^N * \left(F(K*g_s^N) S_s^N \right) \right) \right\|_{\mathbb{L}^q(\Omega,H)} ds \\ &\qquad \leq C_{\nu,M,T} \int_0^t (t-s)^{(1+\alpha)/2} \left\| (\mathrm{I}-A)^{\alpha/2} g_s^N \right\|_{\mathbb{L}^q(\Omega,H)} ds.\end{aligned}$$

This bounds the second term. Recall that $\alpha < 1$ therefore $(t-s)^{-(1+\alpha)/2}$ is integrable.

Step 4. The estimate of the third term (32) is quite tricky and we postpone it to Lemma 3.1 below; see (33). Collecting the three bounds together, we get

$$\left\| (\mathrm{I}-A)^{\alpha/2} g_t^N \right\|_{\mathbb{L}^q(\Omega,H)} \leq C_{q,T} + C_{\nu,M,T} \int_0^t (t-s)^{(1+\alpha)/2} \left\| (\mathrm{I}-A)^{\alpha/2} g_s^N \right\|_{\mathbb{L}^q(\Omega,H)} ds.$$

We may apply Gronwall's Lemma we deduce

$$\left\|(\mathrm{I}-A)^{\alpha/2}\, g_t^N\right\|_{\mathbb{L}^q(\Omega,H)} \le C_{q,T,M,\nu},$$

and Proposition 2.1 follows.

LEMMA 3.1. *We assume Assumption* 1.1. *Let* $q \geqslant 2$. *Then there exists a constant* $C_{q,T}>0$ *such that for all* $t\in[0,T]$,

$$\left\|\frac{1}{N}\sum_{i=1}^N \int_0^t (\mathrm{I}-A)^{\alpha/2}\, \nabla e^{(t-s)A}\left(V^N\left(\cdot - X_s^{i,N}\right)\right) dW_s^i\right\|_{\mathbb{L}^q(\Omega,\mathbb{L}^p(\mathbb{R}^2))}^q \le C_{q,T}. \tag{33}$$

Proof. From Sobolev embeddings we have

$$\begin{aligned}&\left\|\frac{1}{N}\sum_{i=1}^N \int_0^t (\mathrm{I}-A)^{\alpha/2}\, \nabla e^{(t-s)A}\left(V^N\left(\cdot - X_s^{i,N}\right)\right) dW_s^i\right\|_{\mathbb{L}^q(\Omega,\mathbb{L}^p(\mathbb{R}^2))}^q\\ &\qquad \leqslant C\left\|\frac{1}{N}\sum_{i=1}^N \int_0^t (\mathrm{I}-A)^{(1+\alpha-\frac{2}{p})/2}\, \nabla e^{(t-s)A}\left(V^N\left(\cdot - X_s^{i,N}\right)\right) dW_s^i\right\|_{\mathbb{L}^q(\Omega,\mathbb{L}^2(\mathbb{R}^2))}^q.\end{aligned}$$

From the Burkholder–Davis–Gundy inequality (see [33] for instance) we obtain

$$\begin{aligned}&\left\|\frac{1}{N}\sum_{i=1}^N \int_0^t (\mathrm{I}-A)^{(1+\alpha-\frac{2}{p})/2}\, \nabla e^{(t-s)A}\left(V^N\left(\cdot - X_s^{i,N}\right)\right) dW_s^i\right\|_{\mathbb{L}^q(\Omega,\mathbb{L}^2(\mathbb{R}^2))}^q\\ &\quad \le C_q\, \mathbb{E}\left[\frac{1}{N^2}\sum_{i=1}^N \int_0^t \left\|(\mathrm{I}-A)^{(1+\alpha-\frac{2}{p})/2}\, \nabla e^{(t-s)A}\left(V^N\left(\cdot - X_s^{i,N}\right)\right)\right\|_{\mathbb{L}^2(\mathbb{R}^2)}^2 ds\right]^{q/2}.\end{aligned}$$

Moreover, we can estimate

$$\begin{aligned}&\frac{1}{N^2}\int_{\mathbb{R}^2}\sum_{i=1}^N \int_0^t \left|\left((\mathrm{I}-A)^{(1+\alpha-\frac{2}{p})/2}\, \nabla e^{(t-s)A}\left(V^N\left(\cdot - X_s^{i,N}\right)\right)\right)(x)\right|^2 ds dx\\ &\qquad = \frac{1}{N}\int_0^t \left\|(\mathrm{I}-A)^{(1+\alpha-\frac{2}{p})/2}\, \nabla e^{(t-s)A} V^N\right\|_{\mathbb{L}^2(\mathbb{R}^2)}^2\, ds\\ &\qquad = \frac{1}{N}\int_0^t \left\|(\mathrm{I}-A)^{-\delta/2}\, \nabla e^{(t-s)A}\,(\mathrm{I}-A)^{(1+\alpha-\frac{2}{p}+\delta)/2}\, V^N\right\|_{\mathbb{L}^2(\mathbb{R}^2)}^2\, ds\\ &\qquad \le \frac{1}{N}\int_0^t \frac{1}{(t-s)^{1-\delta}}\left\|V^N\right\|_{1+\alpha-\frac{2}{p}+\delta,2}^2\, ds\\ &\qquad \le C_{T,\delta}\, N^{\beta(2+2\delta+2\alpha+2-\frac{4}{p})-1}. \end{aligned} \tag{34}$$

Therefore (34) is bounded by some constant $C_{q,T}$ if we take $\beta < \frac{1}{4+2\alpha-\frac{4}{p}}$, δ close enough to zero. This provides the bound of the lemma. ∎

3.2. Proof of Proposition 2.2. Let us now prove the second estimate on g^N given in Proposition 2.2, namely, (18). Let $q' \geqslant 2$. In this proof we use the fact that $\mathbb{L}^2(\mathbb{R}^2)\subset \mathbb{H}_2^{-2}$ with continuous embedding, and that the linear operator Δ is bounded from $\mathbb{L}^2(\mathbb{R}^2)$ to $\mathbb{H}_2^{-2}$.

Let us first recall that, from interpolation, from Proposition 2.1 and using the fact that $\|g_t^N\|_{\mathbb{L}^1(\mathbb{R}^2)} = 1$, we have: for any $\theta \in (0,1)$,

$$\mathbb{E}\Big[\|g_t^N\|_{0,2}^{q'}\Big] \leqslant \mathbb{E}\Big[\|g_t^N\|_{0,p}^{\theta q'}\ \ \|g_t^N\|_{\mathbb{L}^1(\mathbb{R}^2)}^{(1-\theta)q'}\Big] \leqslant \mathbb{E}\Big[\|g_t^N\|_{0,p}^{\theta q'}\Big] \leqslant C_{q',T}. \tag{35}$$

We then observe that

$$g_t^N(x) - g_s^N(x) = \int_s^t \left\langle S_r^N, (K * F(g_r^N))\ \nabla V^N\,(x-\cdot)\right\rangle dr + \nu \int_s^t \Delta g_r^N(x) dr$$
$$+ \frac{\sqrt{2\nu}}{N}\sum_{i=1}^N \int_s^t \nabla\left(V^N\right)\left(x - X_r^{i,N}\right) dW_r^i.$$

Therefore we have

$$\mathbb{E}\Big[\|g_t^N(x) - g_s^N(x)\|_{-2,2}^{q'}\Big]$$
$$\leq (t-s)^{q'-1}\int_s^t \mathbb{E}\Big[\,\|\langle S_r^N, F(K*g_r^N)\ \nabla V^N\,(x-\cdot)\rangle\|_{-2,2}^{q'}\Big] dr \tag{36}$$
$$+ (t-s)^{q'-1}\,\frac{1}{2}\int_s^t \mathbb{E}\Big[\,\|\Delta g_r^N(x)\|_{-2,2}^{q'}\Big]\ dr \tag{37}$$
$$+ \mathbb{E}\Bigg[\bigg\|\frac{\sqrt{2\nu}}{N}\sum_{i=1}^N \int_s^t \nabla\left(V^N\right)\left(x - X_r^{i,N}\right) dW_r^i\bigg\|_{-2,2}^{q'}\Bigg]. \tag{38}$$

To estimate the first term (36) we observe first that

$$\mathbb{E}\Big[\|\,\langle S_r^N, F(K*g_r^N)\ \nabla V^N\,(x-\cdot)\rangle\,\|_{-2,2}^{q'}\Big] = \mathbb{E}\Big[\,\|\nabla(S_r^N F(K*g_r^N) * V^N)\|_{-2,2}^{q'}\Big]$$
$$\leq \mathbb{E}\Big[\,\|(S_r^N F(K*g_r^N) * V^N\|_{-1,2}^{q'}\Big].$$
$$\leq C_M \mathbb{E}\Big[\,\|g_t^N\|_{\mathbb{L}^2(\mathbb{R}^2)}^{q'}\Big] \leq C. \tag{39}$$

Moreover, for the second term (37) we write

$$\mathbb{E}\Big[\,\|\Delta g_r^N\|_{-2,2}^{q'}\Big] \leq C\mathbb{E}\Big[\,\|g_r^N\|_{\mathbb{L}^2(\mathbb{R}^2)}^{q'}\Big] \leq C_{q',T}, \tag{40}$$

from (35). Finally we bound the last term (38):

$$\mathbb{E}\Bigg[\bigg\|\frac{1}{N}\sum_{i=1}^N \int_s^t \nabla\left(V^N\right)\left(x - X_r^{i,N}\right) dW_r^i\bigg\|_{-2,2}^{q'}\Bigg]$$
$$\leq C_{q'}\mathbb{E}\Bigg[\frac{1}{N^2}\sum_{i=1}^N \int_s^t \Big\|\nabla\left(V^N\right)\left(x - X_r^{i,N}\right)\Big\|_{-2,2}^{q'} dr\Bigg]^{q'/2}$$

and we observe that

$$\frac{1}{N^2}\int_{\mathbb{R}} \sum_{i=1}^N \int_s^t \Big\|(\mathrm{I}-A)^{-1}\,\nabla\left(V^N\right)\left(x - X_r^{i,N}\right)\Big\|^2 dr dx$$
$$= (t-s)\frac{1}{N}\big\|V^N\big\|_{-1,2}^2 \leq (t-s)\frac{1}{N}\big\|V^N\big\|_{0,2}^2 \leq CN^{2\beta-1}(t-s) \leq C(t-s).$$

In order to conclude the lemma, we need to divide (36)–(38) by $|t-s|^{1+q'\gamma}$. From the previous estimates, we always get a term of the form $|t-s|^{\varepsilon}$ with $\varepsilon < 1$ (using the assumption $\gamma < \frac{1}{2}$).

Appendix A. More general initial data. Assume that the initial vorticity ξ^{ini} has variable sign. Define

$$\xi_{+}^{\text{ini}} := \xi^{\text{ini}} \vee 0, \qquad \xi_{-}^{\text{ini}} := \left(-\xi^{\text{ini}}\right) \vee 0,$$

and

$$\Gamma_{\pm} := \int \xi_{\pm}^{\text{ini}}(x)\, dx > 0$$

(they are finite, since we assume $\xi^{\text{ini}} \in \mathbb{L}^1$). Let $\{X_0^{i,\pm}\}_{i\in\mathbb{N}}$ be a double sequence of random variables in $\mathbb{R}^2$ such that, for the empirical measures

$$S_0^{N,\pm} := \frac{\Gamma_{\pm}}{N} \sum_{i=1}^{N} \delta_{X_0^{i,\pm}}$$

one has, for some $\alpha > \frac{2}{p}$, for any $q > 0$,

$$\sup_{N\in\mathbb{N}} \mathbb{E}\left[\left\| V^N * S_0^{N,\pm} \right\|_{\alpha,p}^{q} \right] < \infty \tag{41}$$

and the two sequences of measures $\{S_0^{N,\pm}\}_{N\in\mathbb{N}}$ weakly converge to the initial measures $\xi_{\pm}^{\text{ini}}(\cdot)dx$, as $N \to \infty$, in probability, i.e.,

$$S_0^{N,\pm} \xrightarrow[N\to\infty]{} \xi_{\pm}^{\text{ini}}(\cdot)dx \qquad \text{in probability.} \tag{42}$$

Consider the system of PDEs, given for $x \in \mathbb{R}^2, t > 0$, by

$$\begin{aligned} \partial_t \xi^{+} + u \bullet \nabla \xi^{+} &= \nu \Delta \xi^{+}, \\ \partial_t \xi^{-} + u \bullet \nabla \xi^{-} &= \nu \Delta \xi^{-}, \\ u &= K * \left(\xi^{+} - \xi^{-}\right), \\ \xi^{+}|_{t=0} = \xi_0^{+}, &\qquad \xi^{-}|_{t=0} = \xi_0^{-}. \end{aligned} \tag{43}$$

It is not difficult to prove the same results of existence and uniqueness as the ones obtained for the usual Navier–Stokes equations

$$\partial_t \xi + u \bullet \nabla \xi = \nu \Delta \xi, \qquad x \in \mathbb{R}^2, t > 0. \tag{44}$$

Moreover, if (ξ^+, ξ^-) is a solution of the system, then $\xi = \xi^+ - \xi^-$ is a solution of (44); if ξ is a solution of (44) and ξ^+ is a solution of the first equation of the system, with $u = K * \xi$, then $\xi^- = \xi^+ - \xi$ is a solution of the second equation of the system, and $u = K * (\xi^+ - \xi^-)$ holds. In this sense the system and (44) are equivalent.

Let $\{W_t^{i,\pm}\}_{i\in\mathbb{N}}$ be a family of independent $\mathbb{R}^2$-valued Brownian motions, defined on the same probability space as $\{X_0^{i,\pm}\}_{i\in\mathbb{N}}$ and independent of them. Given $N \in \mathbb{N}$, consider particles with positions $\{X_t^{i,N,\pm}\}_{i\in\mathbb{N}}$ satisfying

$$\begin{aligned} dX_t^{i,N,+} &= F\bigg(\Gamma_+ \frac{1}{N}\sum_{k=1}^{N}(K*V^N)(X_t^{i,N,+}-X_t^{k,N,+}) \\ &\qquad\qquad - \Gamma_- \frac{1}{N}\sum_{k=1}^{N}(K*V^N)(X_t^{i,N,+}-X_t^{k,N,-})\bigg)\, dt \\ &\qquad + \sqrt{2\nu}\, dW_t^{i,+}, \\ dX_t^{i,N,-} &= F\bigg(\Gamma_+ \frac{1}{N}\sum_{k=1}^{N}(K*V^N)(X_t^{i,N,-}-X_t^{k,N,+}) \\ &\qquad\qquad - \Gamma_- \frac{1}{N}\sum_{k=1}^{N}(K*V^N)(X_t^{i,N,-}-X_t^{k,N,-})\bigg)\, dt \\ &\qquad + \sqrt{2\nu}\, dW_t^{i,-} \end{aligned}$$

with initial conditions $\{X_0^{i,\pm}\}_{i\in\mathbb{N}}$. Consider the associated empirical measures

$$S_t^{N,\pm} := \frac{\Gamma_\pm}{N}\sum_{i=1}^{N}\delta_{X_t^{i,\pm}}$$

and empirical densities

$$g_t^{N,\pm} := V^N * S_t^{N,\pm}.$$

THEOREM A.1. *Assume on V, β, α, p, the same conditions of Assumption* 1.1 *and in addition assume* (41) *and* (42). *Consider the particle system $\{X_t^{i,\pm}\}_{i\in\mathbb{N}}$ with the parameter M which satisfies*

$$M \geqslant c_K\left(1+\|\xi^{\text{ini}}\|_{\mathbb{L}^\infty}\right). \tag{45}$$

Then, for every $\eta\in(\frac{2}{p},\alpha)$, the sequence of processes $\{g_t^{N,+}, g_t^{N,-}\}$ converges in probability with respect to the

- weak topology of $\left(\mathbb{L}^2\left([0,T]\,;\,\mathbb{H}_p^\alpha(\mathbb{R}^2)\right)\right)^2$,
- strong topology of $\left(C\left([0,T]\,;\,\mathbb{W}_{\text{loc}}^{\eta,p}(\mathbb{R}^2)\right)\right)^2$,

as $N\to\infty$, to the unique weak solution of the PDE system (43) *and thus $g_t^{N,+}-g_t^{N,-}$ converges, in the same topologies, to the unique weak solution of the PDE* (44).

We do not repeat the full proof in this case but only sketch the main points. The empirical measures satisfy

$$\begin{aligned} \left\langle S_t^{N,+},\phi\right\rangle &= \left\langle S_0^{N,+},\phi\right\rangle + \int_0^t \left\langle S_s^{N,+}, F\left(K*V^N*S_s^{N,+} - K*V^N*S_s^{N,-}\right)\bullet\nabla\phi\right\rangle ds \\ &\qquad + \nu\int_0^t \left\langle S_s^{N,+},\Delta\phi\right\rangle ds + \frac{\sqrt{2\nu}}{N}\sum_{i=1}^{N}\int_0^t \nabla\phi(X_s^{i,N,+})\bullet dW_s^{i,+} \end{aligned}$$

and

$$\left\langle S_t^{N,-}, \phi \right\rangle = \left\langle S_0^{N,-}, \phi \right\rangle + \int_0^t \left\langle S_s^{N,-}, F\left(K * V^N * S_s^{N,+} - K * V^N * S_s^{N,-}\right) \bullet \nabla\phi \right\rangle ds$$
$$+ \nu \int_0^t \left\langle S_s^{N,-}, \Delta\phi \right\rangle ds + \frac{\sqrt{2\nu}}{N} \sum_{i=1}^N \int_0^t \nabla\phi(X_s^{i,N,-}) \bullet dW_s^{i,-}$$

and the empirical densities satisfy

$$g_t^{N,+}(x) = g_0^{N,+}(x) + \int_0^t \left\langle S_s^{N,+}, F(K * g_s^{N,+} - K * g_s^{N,-}) \bullet \nabla V^N(x - \cdot) \right\rangle ds$$
$$+ \nu \int_0^t \Delta g_s^{N,+}(x) ds + \frac{\sqrt{2\nu}}{N} \sum_{i=1}^N \int_0^t \nabla V^N\left(x - X_s^{i,N,+}\right) \bullet dW_s^{i,+}$$

and

$$g_t^{N,-}(x) = g_0^{N,-}(x) + \int_0^t \left\langle S_s^{N,-}, F(K * g_s^{N,+} - K * g_s^{N,-}) \bullet \nabla V^N(x - \cdot) \right\rangle ds$$
$$+ \nu \int_0^t \Delta g_s^{N,-}(x) ds + \frac{\sqrt{2\nu}}{N} \sum_{i=1}^N \int_0^t \nabla V^N\left(x - X_s^{i,N,-}\right) \bullet dW_s^{i,-}$$

which in mild form are

$$g_t^{N,+} = e^{tA} g_0^{N,+} + \int_0^t \nabla e^{(t-s)A} \left(V^N * \left(F(K * g_s^{N,+} - K * g_s^{N,-}) S_s^{N,+}\right)\right) ds$$
$$+ \frac{\sqrt{2\nu}}{N} \sum_{i=1}^N \int_0^t e^{(t-s)A} \left(\nabla V^N\left(\cdot - X_s^{i,N,+}\right)\right) \bullet dW_s^{i,+}$$

and

$$g_t^{N,-} = e^{tA} g_0^{N,-} + \int_0^t \nabla e^{(t-s)A} \left(V^N * \left(F(K * g_s^{N,+} - K * g_s^{N,-}) S_s^{N,-}\right)\right) ds$$
$$+ \frac{\sqrt{2\nu}}{N} \sum_{i=1}^N \int_0^t e^{(t-s)A} \left(\nabla V^N\left(\cdot - X_s^{i,N,-}\right)\right) \bullet dW_s^{i,-}.$$

The proof of the estimate

$$\mathbb{E}\left[\left\| (\mathrm{I} - A)^{\alpha/2} g_t^{N,+} \right\|_{\mathbb{L}^p(\mathbb{R}^2)}^q\right] \le C_{T,M,\nu,q}$$

(and similarly for $g_t^{N,-}$) is similar to the case of a single sign (Proposition 2.1): precisely, the estimates on

$$\left\| (\mathrm{I} - A)^{\alpha/2} e^{tA} g_0^{N,+} \right\|_{\mathbb{L}^p(\mathbb{R}^p)}$$

and on

$$\left\| \frac{1}{N} \sum_{i=1}^N \int_0^t (\mathrm{I} - A)^{\alpha/2} \nabla e^{(t-s)A} \left(V^N\left(\cdot - X_s^{i,N,+}\right)\right) dW_s^{i,+} \right\|_{\mathbb{L}^p(\mathbb{R}^2)}$$

are obviously the same. But the middle term can also be studied in the same way: one has

$$\begin{aligned}&\left|\left(V^N*\left(F\left(K*g_s^{N,+}-K*g_s^{N,-}\right)S_s^{N,+}\right)\right)(x)\right|\\&\qquad\le\left\|F(K*g_s^{N,+}-K*g_s^{N,-})\right\|_\infty\left|V^N*S_s^{N,+}(x)\right|\le M\;\left|g_s^{N,+}(x)\right|\end{aligned}$$

and the same conclusion follows, using Gronwall's Lemma.

Moreover, in the proof of

$$\mathbb{E}\Big[\int_0^T\int_0^T\frac{\left\|g_t^{N,+}-g_s^{N,+}\right\|_{-2,2}^{q'}}{|t-s|^{1+q'\gamma}}\,ds\,dt\Big]\le C_{T,M,\nu,q'}$$

(and similarly for $g_t^{N,-}$) the only part which a priori may change is

$$\begin{aligned}&\mathbb{E}\Big[\left\|\left\langle S_r^{N,+},F(K*g_s^{N,+}-K*g_s^{N,-})\,\nabla V^N(x-\cdot)\right\rangle\right\|_{-2,2}^{q'}\Big]\\&\quad=\mathbb{E}\left[\left\|\nabla(S_r^{N,+}F(K*g_s^{N,+}-K*g_s^{N,-})*V^N)\right\|_{-2,2}^{q'}\right]\\&\quad\le\mathbb{E}\left[\left\|S_r^{N,+}F(K*g_s^{N,+}-K*g_s^{N,-})*V^N\right\|_{-1,2}^{q'}\right]\\&\quad\le C_M\mathbb{E}\left[\left\|g_t^N\right\|_{\mathbb{L}^2}^{q'}\right]\le C,\end{aligned}$$

so in fact this part remains the same. Then one can apply the same arguments used for tightness (see section 2.3).

In the passage to the limit (section 2.4), the arguments are similar. We use the weak formulation

$$\begin{aligned}\left\langle g_t^{N,+},\phi\right\rangle&=\left\langle g_0^{N,+},\phi\right\rangle+\int_0^t\left\langle S_s^{N,+},F(K*g_s^{N,+}-K*g_s^{N,-})\bullet\nabla(V^N*\phi)\right\rangle ds\\&\quad+\nu\int_0^t\left\langle g_s^{N,+},\Delta\phi\right\rangle(x)ds+\frac{\sqrt{2\nu}}{N}\sum_{i=1}^N\int_0^t\nabla(V^N*\phi)\left(X_s^{i,N,+}\right)\;\bullet dW_s^{i,+},\end{aligned}$$

where ϕ is a smooth test function with compact support. The only difficult step is proving that

$$\begin{aligned}&\lim_{N\to\infty}\int_0^t\left\langle S_s^{N,+},F\big(K*g_s^{N,+}-K*g_s^{N,-}\big)\bullet\nabla(V^N*\phi)\right\rangle ds\\&\qquad=\int_0^t\int_{\mathbb{R}^2}\xi^+(s,x)\,F\Big(K*\big(\xi^+(s,x)-\xi^-(s,x)\big)\Big)\bullet\nabla\phi(x)\,ds.\end{aligned}$$

The proof is analogous to the case of a single sign; let us recall the main steps. After application of Skorohod's theorem, due to the a.s. convergence of $g_s^{N,\pm}$ to $\xi^\pm$ in $\mathbb{L}_w^2\left([0,T]\,;\,\mathbb{H}_p^\alpha\right)$ and called $f_s^{N,\pm}:=K*g_s^{N,\pm}$, thanks to the properties of the Biot–Savart operator we get that (a.s.) $f_s^{N,\pm}$ converge to $K*\xi^\pm$ in $\mathbb{L}_w^2\left([0,T]\,;\,\mathbb{L}^p\right)$ and, moreover, $f_s^{N,\pm}$ are (a.s.) bounded in $\mathbb{L}^2\left([0,T]\,;\,C^{\widetilde{\eta}}\right)$. The last property is used to prove that

$$\begin{aligned}&\Big|F(f_s^{N,+}-f_s^{N,-})(x)\bullet\nabla(V^N*\phi)(x)\\&\qquad-\big(F(f_s^{N,+}-f_s^{N,-})\bullet\nabla(V^N*\phi)\big)*V^N(x)\Big|\le C/N^{\widetilde{\eta}\beta}.\end{aligned}$$

From this the other steps are easier and equal to the one-sign case.

Appendix B. Uniform convergence. All past papers dealing with particle approximation of 2d Navier–Stokes equations prove weak convergence of the empirical measures S_t^N to the probability law $\xi_t dx$. The novelty here is that we prove a stronger convergence, namely, the convergence in suitable function spaces of the mollified empirical measure $g_t^N := V^N * S_t^N$. Consider for instance the property of uniform convergence in space on compact sets ($\mathbb{L}^2$ in time). It is not possible to deduce this result from the weak convergence $S_t^N \rightharpoonup \xi_t dx$ (see below). If one only knows that $S_t^N \rightharpoonup \xi_t dx$, and one considers a classical kernel smoothing algorithm $\theta_{\epsilon_N} * S_t^N$ to approximate the profile ξ_t by means of S_t^N, it is not clear how to choose θ_{ϵ_N} in such a way so as to have uniform convergence of $\theta_{\epsilon_N} * S_t^N$ to ξ_t. The method described in this paper indicates a strategy for a better particle approximation of solutions to 2d Navier–Stokes equations.

Let us understand more closely the strength of the uniform convergence. It is a strong indication that the particles are distributed quite uniformly in space, they do not have too much concentration or aggregation. Let us make this remark more quantitative.

Proposition B.1. *Assume that the probability density V has the property*

$$h\mathbf{1}_{[-r,r]^2} \leq V$$

for some constants $h, r > 0$. Assume that for some (t, ω, R) we have

$$\sup_{|x|\leq R} g_t^N(x) \leq C.$$

Then

$$\sup_{|x|\leq R} \operatorname{Card}\left\{ i = 1, \ldots, N \; ; \; X_t^{i,N} \in \left[x - \frac{r}{N^\beta}, x + \frac{r}{N^\beta} \right]^2 \right\} \leq \frac{C}{h} N^{1-2\beta}. \tag{46}$$

Proof. The result simply follows from the inequalities

$$hN^{2\beta}\mathbf{1}_{[-\frac{r}{N^\beta}, \frac{r}{N^\beta}]^2} \leq V^N$$

and thus

$$h\frac{\operatorname{Card}\left\{ i = 1, \ldots, N : X_t^{i,N} \in \left[x - \frac{r}{N^\beta}, x + \frac{r}{N^\beta} \right]^2 \right\}}{N^{1-2\beta}} \leq g_t^N(x). \qquad \square$$

Let us give a heuristic interpretation of the previous result.

First consider the case of points $X_t^{i,N}$ geometrically uniform in $\left[-\frac{1}{2}, \frac{1}{2}\right]^2$: we consider a uniform grid in $\left[-\frac{1}{2}, \frac{1}{2}\right]^2$ of side length $\frac{1}{N^{1/2}}$, hence, with (roughly) N grid points, and we put one particle per grid point.

In a box $\left[x - \frac{r}{N^\beta}, x + \frac{r}{N^\beta}\right]^2 \subset \left[-\frac{1}{2}, \frac{1}{2}\right]^2$ we have (approximatively)

$$\left(\frac{\frac{2r}{N^\beta}}{\frac{1}{N^{1/2}}} \right)^2 = 4r^2 N^{1-2\beta}$$

points. This is exactly estimate (46).

Now, let us break this uniformity. Divide $\left[-\frac{1}{2}, \frac{1}{2}\right]^2$ into two sets:

$$Q = \left[0, \frac{1}{N^{1/2}}\right]^2 \qquad \text{and} \qquad Q^c := \left[-\frac{1}{2}, \frac{1}{2}\right]^2 \backslash Q.$$

Call X_t^i the first $N - N^{1-\beta}$ particles, $\widetilde{X}_t^i$ the last $N^{1-\beta}$ ones. Put the $N - N^{1-\beta}$ particles X_t^i in the grid points of Q^c, no more than one particle per grid point. Put the $N^{1-\beta}$ particles $\widetilde{X}_t^i$ into $Q_0 = \left[0, \frac{r}{N^\beta}\right]^2$, which is contained in Q (for large N), being that $\beta < \frac{1}{4}$.

Then, all cubes $Q_x := \left[x - \frac{r}{N^\beta}, x + \frac{r}{N^\beta}\right]^2 \subset Q^c$ contain, as above, at most $4r^2 N^{1-2\beta}$ particles. But Q_0 contains $N^{1-\beta}$ particles, much more than $4r^2N^{1-2\beta}$ for large N. This is an example of a configuration which does not fulfill estimate (46). For such a configuration, however, we still have

$$\left\langle S_t^N, \phi \right\rangle \to \left\langle \xi_t, \phi \right\rangle$$

for all continuous bounded ϕ. Let us only check that this is true for $\phi = \mathbf{1}_{\mathcal{O}}$ when $\mathcal{O}$ is an open set (it is not continuous but it is a good heuristic indication). We have

$$\begin{aligned} \left\langle S_t^N, \phi \right\rangle &= \left\langle S_t^N, \mathbf{1}_{\mathcal{O}\cap Q} \right\rangle + \left\langle S_t^N, \mathbf{1}_{\mathcal{O}\cap Q^c} \right\rangle \leq \left\langle S_t^N, \mathbf{1}_Q \right\rangle + \left\langle S_t^N, \mathbf{1}_{\mathcal{O}\cap Q^c} \right\rangle \\ &\sim \frac{1}{N} N^{1-\beta} + |\mathcal{O}| \xrightarrow[N\to\infty]{} |\mathcal{O}| = \left\langle \xi_t, \phi \right\rangle, \end{aligned}$$

where $\xi_t = \mathbf{1}$. The excessive concentration in Q_0 does not prevent weak convergence but it is not allowed by the stronger convergence proved here.

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