Scuola Normale Superiore

Classe di Scienze Matematiche, Fisiche e Naturali

Recent developments about Geometric Analysis on $\mathrm{RCD}(K, N)$ spaces

Tesi di Perfezionamento in Matematica

Candidato:

Daniele Semola

Relatori:

Prof. Luigi Ambrosio
Prof. Andrea Mondino

Anno Accademico 2019 – 2020

Abstract. This thesis is about some recent developments on Geometric Analysis and Geometric Measure Theory on $\mathrm{RCD}(K,N)$ metric measure spaces that have been obtained in [**8**, **48**, **49**, **51**, **52**, **171**].
After the preliminary Chapter 1, where we collect the basic notions of the theory relevant for our purposes, Chapter 2 is dedicated to the presentation of a simplified approach to the structure theory of $\mathrm{RCD}(K,N)$ spaces via δ-splitting maps developed in collaboration with Brué and Pasqualetto. The strategy is similar to the one adopted by Cheeger-Colding in the theory of Ricci limit spaces and it is suitable for adaptations to codimension one.
Chapter 3 is devoted to the proof of the constancy of the dimension conjecture for $\mathrm{RCD}(K,N)$ spaces. This has been obtained in a joint work with Brué, where we proved that dimension of the tangent space is the same almost everywhere with respect to the reference measure, generalizing a previous result obtained by Colding-Naber for Ricci limits. The strategy is based on the study of regularity of flows of Sobolev vector fields on spaces with Ricci curvature bounded from below, which we find of independent interest.
In Chapters 4 and 5 we present the structure theory for boundaries of sets of finite perimeter in this framework, as developed in collaboration with Ambrosio, Brué and Pasqualetto. An almost complete generalization of De Giorgi's celebrated theorem is given, opening to further developments for Geometric Measure Theory in the setting of synthetic lower bounds on Ricci curvature.
In Chapter 6 we eventually collect some results about sharp lower bounds on the first Dirichlet eigenvalue of the p-Laplacian based on a joint work with Mondino. We also address the problems of rigidity and almost rigidity, heavily relying on the compactness and stability properties of RCD spaces.

Contents

Introduction vii
Ricci curvature vii
The synthetic theory of Ricci curvature bounds x
Main contributions xiii
Other contributions xv

Chapter 1. Preliminaries 1
1. Analysis on metric measure spaces 1
2. Curvature dimension conditions 20
3. $\mathrm{RCD}(\mathrm{K}, \infty)$ metric measure spaces 22
4. $\mathrm{RCD}(\mathrm{K}, \mathrm{N})$ metric measure spaces 31

Chapter 2. Metric measure rectifiability of $\mathrm{RCD}(K, N)$ spaces 39
1. Strongly $\mathfrak{m}$-rectifiable metric measure spaces 40
2. Splitting maps on RCD spaces 41
3. Uniqueness of tangents to RCD spaces 46
4. Metric rectifiability of RCD spaces 49
5. Behaviour of the reference measure under charts 51

Chapter 3. Constancy of the dimension for $\mathrm{RCD}(K, N)$ spaces via regularity of Lagrangian flows 55
1. Green functions 58
2. G-regularity of Lagrangian Flows 70
3. Constancy of the dimension 77

Chapter 4. Sets of finite perimeter over $\mathrm{RCD}(K, N)$ spaces: existence of Euclidean tangents 87
1. Rigidity of the 1-Bakry-Émery inequality and splitting theorem 89
2. Convergence and stability results for sets of finite perimeter 97
3. Tangents to sets of finite perimeter in $\mathrm{RCD}(K, N)$ spaces 104
4. An iterated tangent theorem 110

Chapter 5. Sets of finite perimeter on $\mathrm{RCD}(K, N)$ spaces: rectifiability of the reduced boundary and Gauss-Green formula 115
1. A Gauss-Green formula on RCD spaces 116
2. Uniqueness of tangents for sets of finite perimeter 124
3. Rectifiability of the reduced boundary 129
4. Representation of the perimeter 135

Chapter 6. Polya-Szego inequality and Dirichlet p-spectral gap on $\mathrm{RCD}(N-1, N)$ spaces 139
1. Polya-Szego inequality 141
2. Spectral gap with Dirichlet boundary conditions 149

3. Rigidity 151
4. Almost rigidity in the Dirichlet p-spectral gap 154

Bibliography 157

Introduction

This thesis is about some recent developments on the Geometric Analysis and Geometric Measure Theory of metric measure spaces satisfying the Riemannian Curvature Dimension condition $\mathrm{RCD}(K, N)$. The results presented have been obtained in collaboration with Luigi Ambrosio, Elia Brué, Andrea Mondino and Enrico Pasqualetto in [**8**, **48**, **49**, **51**, **52**, **171**]. Other papers written during the PhD studies and not completely related to this topic are summarised in the last part of the introduction.

Our aim here is to review the motivations which led to the birth of the theory of Riemannian Curvature Dimension bounds and to explain how the contents of present thesis relate with this research field.

The presentation is strongly inspired by the survey papers [**7**, **66**, **126**, **211**] and the introduction of [**200**]. Written by some of the founding fathers of the theory, they deeply reflect the quick and remarkable developments it has seen in the last years and the various perspectives it combines, ranging from Riemannian Geometry to Probability and Geometric Measure Theory.

Ricci curvature

Curvature is one of the cornerstones of non Euclidean geometry. It is an infinitesimal measurement of how much the space deviates from the Euclidean model that comes in several different ways.

On a Riemannian manifold (M, g) the *sectional curvature* is a real valued function K on the Grassmannian of 2-planes in the tangent space at each point. Given $p \in M$ and orthonormal vectors $u, v \in T_pM$ spanning a 2-plane $\pi \subset T_pM$, the sectional curvature $K(\pi) = K(u, v)$ provides the dominant correction to the distance between geodesics starting at p with velocity u and v:

$$\mathsf{d}(\exp_p(tu), \exp_p(tv)) = \sqrt{2}t\left(1 - \frac{K(u,v)}{12}t^2 + O(t^3)\right) \quad \text{as } t \to 0. \tag{1}$$

Above we denoted by exp the Riemannian exponential map and we notice that positive curvature corresponds to contraction of distances.

Another way to measure the deviation from the Euclidean geometry is to look at distortion of volumes rather than distortion of distances. The *Ricci curvature* captures the behaviour of a Riemannian manifold from this point of view, even though this might be not transparent at a first glance.
The Ricci tensor is a symmetric bilinear form on the tangent space obtained via an averaging procedure from the sectional curvature. If $u \in T_pM$ is a unit vector and $(u, e_2, \dots, e_n)$ is an orthonormal basis of T_pM then

$$\mathrm{Ric}(u, u) := \sum_{i=2}^{n} K(u, e_i).$$

A very common way to think of Ricci curvature is as a negative *Laplacian of the metric*. There exists a suitable system of coordinates (x_i) called *harmonic coordinates* such that, expressing the metric and Ricci tensors in these coordinates as g_{ij} and Ric_{ij} respectively, it holds

$$\mathrm{Ric}_{ij} = -\frac{1}{2}\Delta g_{ij} + \text{lower order terms.}$$

This is usually seen as a first hint towards the regularizing effect of the evolution via Ricci flow

$$\frac{\partial}{\partial t} g_t = -2\mathrm{Ric}_{g_t},$$

in comparison with the well known heat equation $\frac{\mathrm{d}}{\mathrm{d}t}u = \Delta u$.

Here we wish to provide some different interpretations of Ricci curvature borrowed from [**210**]. On the one hand they will let its connection with distortions of volumes be more transparent, on the other one they have been crucial for the development of the synthetic treatment of Ricci curvature bounds.

Let us consider a smooth map $\psi : \mathbb{R}^n \to \mathbb{R}$ and let us introduce the family of deformations $T_t : \mathbb{R}^n \to \mathbb{R}^n$ as $T_t(x) := x + t\nabla\psi(x)$. Setting $\mathcal{J}(t) := \det(D_x T_t(x))$, it is possible to infer that

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \left(\mathcal{J}(t)\right)^{1/n} \leq 0. \tag{2}$$

The interpretation of the determinant of the Jacobian map as volume element justifies the Lagrangian interpretation of (2) as a control over the infinitesimal rate of change of the volume, when we move following the family of deformations induced by $\nabla\psi$.

There is a dual Eulerian perspective on this phenomenon motivated by fluid mechanics, where we focus on the velocity vector field instead of the trajectories, and it leads to the inequality

$$\Delta\frac{|\nabla\psi|^2}{2} - \nabla\psi\cdot\nabla\Delta\psi \geq \frac{(\Delta\psi)^2}{n}. \tag{3}$$

If we move from the flat Euclidean realm to a Riemannian manifold and we try to find the analogues of (2) and (3), we have to recognize that the right counterpart of the family of deformations above is given by

$$T_t(x) := \exp(t\nabla\psi(x)). \tag{4}$$

After facing the regularity issues due to the possible presence of a cut locus we end up with

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \left(\mathcal{J}(t)\right)^{1/n} + \frac{\mathrm{Ric}(\dot\gamma,\dot\gamma)}{n}\mathcal{J}^{1/n} \leq 0, \tag{5}$$

where $\dot\gamma := \frac{\mathrm{d}}{\mathrm{d}t}T_t(x)$, and

$$\Delta\frac{|\nabla\psi|^2}{2} - \nabla\psi\cdot\nabla\Delta\psi \geq \frac{(\Delta\psi)^2}{n} + \mathrm{Ric}(\nabla\psi,\nabla\psi). \tag{6}$$

As we claimed at the beginning of this section, (Ricci) curvature controls the distortion from the Euclidean behaviour both in (5) and (6). The second inequality in particular is known as Bochner inequality and can also be seen as a consequence of the so-called Bochner identity. A more geometric perspective on Ricci curvature, looking at the consequences of (5) can be found in [**126**]. In the Euclidean space, given a smooth and mean convex hypersurface, all its interior equidistant hypersurfaces are mean convex too (in a weak sense if they are non smooth). On a Riemannian manifold this is still the case under the assumption that the Ricci curvature is nonnegative.

Some remarks are in order, also to compare the role of Ricci curvature with that of sectional curvature. The first one is that, as it is more transparent in (5), Ricci curvature affects the behaviour of volumes rather than of distances. The second one is that the dimension plays a role too, while this was not the case in (1).

Lower bounds on Ricci curvature, coupled with upper bounds on the dimension are at the heart of Geometric Analysis and of several related fields. Among their consequences we can mention the Bishop-Gromov inequality on monotonicity of volume ratios [**127**], the splitting theorem due to Cheeger-Gromoll [**72**], the heat kernel bounds obtained by Li-Yau [**160**], several spectral gap and diameter estimates and the Lévy-Gromov isoperimetric inequality [**128**].

In [**127**] Gromov noticed that the volume monotonicity was a sufficient condition to guarantee the precompactness with respect to the Gromov-Hausdorff topology of the class $\mathcal{M}_{K,n,D}$ of Riemannian manifolds with Ricci curvature bounded from below by $K \in \mathbb{R}$ and dimension and diameter uniformly bounded from above by n and D, respectively. This remarkable observation was at the origin of a research programme stemming from the question

(*) *how does a Riemannian manifold with Ricci curvature bounded from below look like?*

Aimed at understanding the structure of *Ricci limit spaces*, i.e., those spaces arising from the compactification of $\mathcal{M}_{K,n,D}$, this theory was initiated by Cheeger and Colding in a series of papers in the mid Nineties [**68**–**71**] and it is still ongoing, with contributions from several other authors.

The role of the (pre-)compactness theorem with respect to (*) can be explained in analogy with other theories of weak solutions such as Sobolev spaces, sets of finite perimeter or currents. If we have reached a good understanding of Ricci limit spaces (as in the partial regularity theory of Sobolev functions) then:

- on the positive side we can prove that Riemannian manifolds in $\mathcal{M}_{K,n,D}$ (or in some subclass of it) cannot exhibit an arbitrarily bad behaviour of a certain sort arguing by compactness. If this is not the case, a sequence of smooth spaces with increasingly bad behaviour would origin a limit with a particular type of singularity. In case we have been a priori able to exclude the presence of this kind of singularity, then we would reach a contradiction;
- on the negative side we can show that certain quantitative statements fail on $\mathcal{M}_{K,n,D}$ just by proving that they are stable under the notion of convergence in force and exhibiting a limit space where they fail.

We refer to [**65**, **139**] for an instance of the positive argument, where uniform L^2 bounds on the Riemann curvature are obtained for manifolds with bounded Ricci curvature and volume bounded from below. Arguments of the same spirit can also lead to almost rigidity results for geometric and functional inequalities, see for instance [**58**, **171**].
The recent [**87**] is an instance of the negative one, where it is proved that it is not possible to obtain uniform C^1 estimates for harmonic functions just depending on the lower Ricci curvature bounds.

A very natural question concerning Ricci limit spaces stood open from the birth of the theory:

(**) *do Ricci limit spaces have Ricci curvature bounded from below? In which sense?*

In the investigation of (**) one should take into account the analogy with the case of sectional curvature bounds, where the successful theory of Alexandrov metric spaces [**1**, **55**, **194**] had been developed based on Toponogov's triangle comparison, mainly with contributions from the Russian and the Japanese schools.

The synthetic theory of Ricci curvature bounds

In 1991 Gromov wrote that no theory of singular spaces with Ricci curvature bounded from below existed yet [**126**]. He also claimed that any such theory should be dealing with *metric measure spaces* rather than with metric spaces (motivated also by the work of Fukaya [**104**]) and that a prominent role could be played by the heat equation and the Laplace operator.
In [**69**, Appendix 2], a few years later and despite the several results obtained about Ricci limit spaces, Cheeger and Colding pointed out that a synthetic theory of lower Ricci bounds was still missing. With the word *synthetic* they meant a characterization not depending on the existence of an underlying smooth structure nor making any reference to the notion of smoothness. A notable remark in their paper (somehow anticipating the developments of the theory) was the following: in the smooth setting Ricci curvature bounds are equivalent to mean curvature comparisons or to Bishop-Gromov inequalities. To capture more completely the implications of this condition in a synthetic framework instead, there is the need to localize the Bishop-Gromov inequality with respect to single directions.

It took the contribution of several authors to formulate a satisfactory answer to (**). The key insight which allowed to *localize* with respect to single directions the consequences of lower Ricci curvature bounds came from Optimal Transport. Later on, the bridge with the heat equation was found thanks to the theory of Gradient Flows, somehow closing the circle and confirming Gromov's prediction.

The theory of Optimal Transport takes its roots in a memoir by Monge, dating back to 1781 and inspired by economical problems, and in a paper by Kantorovich in 1942, where the same problems were investigated from a probabilistic perspective.
In the modern formulation of the problem, given Polish spaces X, Y, a lower semicontinuous *cost* function $c : X \times Y \to [0, \infty]$ and probability measures μ, ν over X and Y respectively, one looks for optimizers of the problem

$$\min \left\{ \int_{X \times Y} c(x, y) \, \mathrm{d}\pi(x, y) \right\} \tag{7}$$

where π is a probability measure on $X \times Y$ whose first and second marginals on X and Y equal μ and ν respectively. Basically Optimal Transport is about finding the cheapest way to transfer a certain prescribed distribution of sources into another one.

A revival of this theory began at the end of the Eighties. Different contributions made clear that Optimal Transport could bring new information in several apparently unrelated fields. An increasing interest was attracted by the case when $X = Y$ is either the Euclidean space or a Riemannian manifold with distance d, and $c(x, y) = \mathsf{d}^2(x, y)$. On the one hand it was realized that the ambient geometry could be lifted to a *Riemannian* geometry on the space of probability measures with finite second order moment $\mathcal{P}_2(X)$ equipped with the so-called Wasserstein distance W_2, naturally induced by Optimal Transport. This was put forward by Benamou and Brenier [**41**], Jordan, Kinderlehrer and Otto [**141**] and Otto [**179**]. On the other hand, McCann pointed out in [**167**] the convex behaviour of certain entropy type functionals over $(\mathcal{P}_2(\mathbb{R}^n), W_2)$, introducing the notions of *displacement interpolation*, as a way to interpolate between *distributions of particles* moving along straight lines from the initial position to their destination, and *displacement convexity*.

A prominent role was played by Shannon's logarithmic entropy, a functional very relevant in Information Theory and Statistical Mechanics. Also known as relative entropy, it can be defined on any metric measure space $(X, \mathsf{d}, \mathfrak{m})$ as

$$\mathrm{Ent}_{\mathfrak{m}}(\nu) := \int_X \varrho \ln \varrho \, \mathrm{d}\mathfrak{m},$$

whenever $\nu \ll \mathfrak{m}$ has density ϱ such that $\varrho \ln \varrho$ is $\mathfrak{m}$-integrable, and set to be $+\infty$ otherwise.

The connection between lower bounds on the Ricci curvature of a Riemannian manifold (M, g, vol) and the convexity of $\mathrm{Ent}_{\mathrm{vol}}$ on $(\mathcal{P}_2(M), W_2)$ was conjectured by Otto and Villani in [**178**]. This interplay was confirmed by Cordero-Erasquin, McCann and Schmuckenschläger [**78**] and Sturm and Von Renesse [**212**]. In the first paper the authors proved that the condition $\mathrm{Ric} \geq K$ on a Riemannian manifold is sufficient for the K-convexity of $\mathrm{Ent}_{\mathrm{vol}}$ on $(\mathcal{P}_2(M), W_2)$. In the second one the circle was closed confirming the equivalence of the two conditions.
Optimal transportation was giving a way of choosing a family of directions and the convexity of Ent could be obtained, very roughly, averaging (5) along these directions.

A breakthrough towards a synthetic treatment of lower Ricci curvature bounds came then with the independent works of Sturm [**200, 201**] and of Lott and Villani [**162**]. They both recognized that the condition *K-convexity of the relative entropy on the Wasserstein space* could be taken as a definition of *metric measure space with Ricci curvature bounded from below by K* and introduced the Curvature-Dimension condition $\mathrm{CD}(K, \infty)$ on the top of that. The $\mathrm{CD}(K, N)$ Curvature-Dimension condition for $1 \leq N < \infty$ was introduced as a finite dimensional refinement, coupling dimension upper bounds with lower Ricci curvature bounds, by looking at the behaviour of different power-like entropy functionals such as the Rényi-entropy.[1]

The theory attracted a lot of interest soon. It was proved that the Curvature-Dimension condition is stable with respect to suitable notions of convergence for metric measure spaces and compatible with the smooth case of (weighted) Riemannian manifolds. Ohta proved in [**175**] that Finsler manifolds verifying lower bounds on Ricci curvature are $\mathrm{CD}(K, N)$ spaces. Later Petrunin [**181**] showed that Alexandrov spaces with curvature bounded below by K and dimension n verify the $\mathrm{CD}(K(n-1), n)$ condition when endowed with the natural Hausdorff measure[2]. In this way sectional curvature lower bounds were proved to be stronger than Ricci curvature lower bounds, as expected from the smooth case.
At the same time, several geometric and analytic properties valid for Riemannian manifolds with Ricci curvature bounded from below were proved for $\mathrm{CD}(K, N)$ metric measure spaces, often with elegant arguments involving optimal transportation.

While in the smooth framework lower bounds on the Ricci curvature tensor are a local property, the analogous problem remained unsettled for the $\mathrm{CD}(K, N)$ condition for a few years. The quest for better globalization properties led Bacher and Sturm to the introduction of the *reduced* Curvature-Dimension condition $\mathrm{CD}^*(K, N)$ in [**35**]. Later T. Rajala proved in [**186**] that the globalization property for the $\mathrm{CD}(K, N)$ condition fails without further regularity conditions. In this regard, T. Rajala and Sturm introduced in [**187**] the *essentially non branching* assumption as a weak version of the non branching assumption for metric spaces, taking into account the reference measure. With this assumption, which has the negative drawback of being not stable under the usual notions of convergence, the geometries that behave too badly from the perspective of branching geodesics (such as the Euclidean space endowed with the Lebesgue measure and the ∞-norm) are ruled out, leading to more refined conclusions.

[1] The definitions in the two approaches were a bit different and [**162**] was dealing only with the $\mathrm{CD}(0, N)$ condition in the case of finite N.

[2] Actually the proof given in [**181**] covers only the case $K = 0$ but it is commonly recognized that the implication holds in the general case, see also [**216**].

In [**58**] Cavalletti and Mondino, inspired by a previous work by Klartag [**153**], found a connection between the *localization* technique of convex geometry and the synthetic theory of Curvature-Dimension bounds giving a striking proof of the Lévy-Gromov isoperimetric inequality for essentially non branching $\mathrm{CD}(K,N)$ metric measure spaces. Relying on the same technique in [**59**] they proved that essentially non branching $\mathrm{CD}^*(K,N)$ spaces satisfy basically all the known geometric and functional inequalities with the sharp constants known for the $\mathrm{CD}(K,N)$ condition. In particular they obtained the sharp Brunn-Minkowski inequality, that was considered a geometric counterpart of the $\mathrm{CD}(K,N)$ condition. Eventually, Cavalletti and E. Milman obtained in [**57**] the globalization property for the $\mathrm{CD}(K,N)$ condition and its equivalence with the $\mathrm{CD}^*(K,N)$ condition under the essential non branching assumption and the (most probably technical) assumption of finite total measure.

Despite the several properties verified by $\mathrm{CD}(K,N)$ metric measure spaces, this class was still not considered a completely appropriate answer to (**). On the one hand the splitting theorem, proved for limits of Riemannian manifolds with lower bound on the Ricci curvature going to zero in [**68**], had revealed to be a crucial tool to develop a structure theory of Ricci limits. On the other one the presence of normed Euclidean spaces $(\mathbb{R}^n, \|\cdot\|, \mathcal{L}^n)$, for any norm, in the $\mathrm{CD}(0,N)$ class, although being a sign of its generality, implies the failure of the splitting theorem in this context without additional assumptions and motivated the quest for a more restrictive class to deal with.[3]

A major role in the search for a refinement of the Curvature-Dimension condition ruling out Finsler-like geometries was played by Sobolev calculus and by a careful analysis of the heat flow.

Starting from [**141**] the identification between the heat flow and the gradient flow of the logarithmic entropy over $(\mathcal{P}_2, W_2)$ was considered at increasing levels of generality. Erbar [**95**] and Villani [**210**] obtained it for Riemannian manifolds, Gigli, Kuwada and Ohta proved it for Alexandrov spaces in [**115**], Sturm and Ohta dealt with Finsler manifolds in [**176**]. In the Alexandrov case a metric notion of gradient flow was considered, based on De Giorgi's idea of *energy dissipation inequalities* and deeply studied in the monograph [**14**] by Ambrosio, Gigli and Savaré.
Meanwhile a Sobolev calculus was developed on general metric measure spaces with different approaches introduced by Cheeger [**62**], Hajłasz [**130**] and Shanmugalingam [**193**]. In particular it was understood that there is the possibility of talking about Sobolev functions and modulus of gradient in the great generality of metric measure spaces.

In [**15**], Ambrosio, Gigli and Savaré proposed an alternative approach to Sobolev calculus based on the notion of test plan and they proved its equivalence with those proposed in [**62**] and [**193**]. They also proved the identification of the heat flow, defined as the L^2 gradient flow of the Cheeger energy, with the metric Wasserstein gradient flow of the relative entropy under the $\mathrm{CD}(K,\infty)$ assumption. Ohta and Sturm [**177**] suggested that some estimates for the heat flow, either gradient contractivity or W_2-contractivity, whose interplay was then studied by Kuwada [**154**], could give a way to distinguish between Riemannian and more general Finsler geometries.

The turning point came with the introduction of the *Riemannian* Curvature-Dimension condition $\mathrm{RCD}(K,\infty)$ by Ambrosio, Gigli and Savaré in [**16**] (dealing with finite reference measure and later extended to the case of σ-finite measure in [**13**] by Ambrosio, Gigli, Mondino and T. Rajala). The definitions were based on the coupling of the Curvature-Dimension condition with the *infinitesimally Hilbertian* assumption corresponding to the Sobolev space $H^{1,2}$ being Hilbert or, equivalently, to the linearity of the heat flow.

[3]To the best of our knowledge no structure theory for general $\mathrm{CD}(K,N)$ metric measure spaces has been developed yet.

Several properties were soon established for RCD spaces. Besides their stability and compatibility with the smooth case, it was shown that the RCD condition was equivalent to the heat flow being an EVI gradient flow of the relative entropy in the Wasserstein geometry. This led to several useful contractivity estimates. It was also a key step to establish a connection with the Eulerian approach to the curvature-dimension condition based on (6) and developed in the setting of Dirichlet forms and Γ-calculus by Bakry and Émery in [**39**], Bakry in [**37**] and Bakry, Gentil and Ledoux in [**40**]. The bridge between the $\mathrm{RCD}(K,\infty)$ condition and the Bakry-Émery $\mathrm{BE}(K,\infty)$ condition was found in [**17**].

The natural finite dimensional refinements subsequently led to the notions of $\mathrm{RCD}(K,N)$ and $\mathrm{RCD}^*(K,N)$ spaces, corresponding to $\mathrm{CD}(K,N)$ (resp. $\mathrm{CD}^*(K,N)$) coupled with linear heat flow. The class $\mathrm{RCD}(K,N)$ was proposed by Gigli [**110**], motivated by the validity of sharp Laplacian comparison and of Cheeger-Gromoll splitting theorem proved by the same author in [**108**]. The (a priori more general) $\mathrm{RCD}^*(K,N)$ condition was thoroughly analysed by Erbar-Kuwada-Sturm [**96**] and (subsequently and independently) by Ambrosio-Mondino-Savaré [**27**]. One of the main results of both [**96**] and [**27**] was the identification of $\mathrm{RCD}^*(K,N)$ and $\mathrm{BE}(K,N)$ (the natural finite dimensional counterpart of $\mathrm{BE}(K,\infty)$). In both papers, gradient flows played a key role: the approach of [**96**] was via a *dimensional* analysis of the heat flow, the one of [**27**] was instead via the non-linear porous media flow.

Thanks to [**57**] and to the fact that $\mathrm{RCD}(K,N)$ spaces are essentially non branching we eventually know that, at least in the case of finite reference measures, the $\mathrm{RCD}(K,N)$ and the $\mathrm{RCD}^*(K,N)$ conditions are equivalent. As a consequence, the Eulerian and the Lagrangian approach to lower Ricci curvature bounds are equivalent under the Riemannian assumption.

Our understanding of $\mathrm{RCD}(K,N)$ metric measure spaces has been rapidly improving in the last years.
The debate about whether they are a completely appropriate answer to (**) or not is still ongoing and the question is probably ill-posed. It has been pointed out by De Philippis, Mondino and Topping (cf. [**206**, Remark 4]) that, as a consequence of the topological manifold regularity of three dimensional non collapsed Ricci limits obtained by M. Simon [**195**] (see also the local refinement [**196**]), there are examples of three dimensional RCD spaces that are not limits of smooth Riemannian manifolds with non-negative Ricci curvature and dimension less than three, such as the cone over the projective plane $\mathbb{RP}^2$.
On the one hand this is suggesting that the RCD condition does not give a synthetic characterization of Ricci limit spaces and this opens to the search for further conditions to add to the theory as it has been the case for infinitesimal Hilbertianity. On the other hand the recent developments confirm that the interest towards RCD spaces goes beyond the theory of Ricci limits, as it is shown by the examples of Alexandrov spaces, stratified spaces [**44**] and spaces with conical singularities [**205**], without mentioning the several developments of the theory in the adimensional case.

Main contributions

Here we wish to explain how the results of the present thesis relate with the development of a structure theory for $\mathrm{RCD}(K,N)$ metric measure spaces and which questions and perspectives they open. A more detailed discussion about each contribution can be found at the beginning of the various chapters.

A *structure theory* for $\mathrm{RCD}(K,N)$ spaces should investigate how do they look like when compared to the models, the Euclidean space and smooth Riemannian manifolds. As in the regularity theory for minimal surfaces and PDEs, a very powerful tool in this perspective are blow-ups, that in this geometric setting correspond to tangents. We know that singularities can be present: a two dimensional cone with non Euclidean tangent at the tip can be obtained as a limit of smooth manifolds with Ricci curvature bounded from below by rounding off the tip. The example of the cone over $\mathbb{RP}^2$ shows that there might be wilder singularities, even at the topological level. The aim of the theory is to

- estimate the size of the *bad points* whose behaviour is singular;
- understand to which extent the spaces are regular, globally and on the complement of the set of singular points.

The structure theory moreover, besides its intrinsic theoretical interest, can be seen as a first step towards the much more challenging goal of classification up to isomorphism, at least in dimension three.

As we have seen, one of the main motivations towards a sharpening of the CD condition was the failure of the splitting theorem, later proved for $\mathrm{RCD}(0, N)$ spaces in [**108**]. Building on the top of this in [**117**] Gigli, Mondino and T. Rajala proved existence of Euclidean tangents almost everywhere with respect to the reference measure. This was the starting point for the analysis pursued by Mondino and Naber in [**170**], where the statement was improved getting uniqueness of tangents up to negligible sets and rectifiability of the regular part of the space. The result was sharpened considering also the behaviour of the reference measure with the independent contributions of Kell and Mondino, De Philippis, Marchese and Rindler and Gigli and Pasqualetto in [**85**, **120**, **144**].

In Chapter 2 we review this theory through an alternative approach developed in [**49**] and based on the use of δ-splitting functions. With respect to the existing literature there is no improvement on the regularity statements for spaces, but we improve the regularity of the charts yielding rectifiability. The use of harmonic δ-splitting maps has been crucial both in the recent developments of the structure theory of Ricci limits and in the theory of sets of finite perimeter considered in [**8**, **48**].

The state of the art of the structure theory for $\mathrm{RCD}(K, N)$ spaces after [**85**, **120**, **144**, **170**] was comparable to that of the theory of Ricci limits after [**71**]. The spaces under consideration could be stratified, up to negligible sets, into a family of regular sets of different dimensions according to the dimension of the tangent space. Cheeger and Colding conjectured that the regular set of non vanishing measure in this stratification should be exactly one and this conjecture was settled by Colding and Naber in [**77**]. Constancy of the dimension for $\mathrm{RCD}(K, N)$ spaces was conjectured, but the techniques of [**77**] seemed not suitable for this more general framework and the problem was listed among the main open questions of the theory in [**211**] and [**7**].
In [**52**] we proved the conjecture relying on a fine study of the regularity of Lagrangian flows of Sobolev vector fields under lower Ricci curvature bounds. The proof is presented in Chapter 3 with some simplifications with respect to the original approach. Besides their applications to the conjecture, the regularity estimates we obtained are expected to improve the understanding of the geometry of RCD spaces, for instance in the construction of parallel transportation [**122**].

Many questions remain open about the shape of RCD spaces. Without the non collapsing assumption several conjectures are still open even in the case of Ricci limits. For instance it is unknown whether the *essential* almost everywhere well defined dimension coincides with the Hausdorff dimension and whether there exists an open neighbourhood of the regular set which is a topological manifold. Still we have reached a good understanding of the structure of these metric measure spaces *up to negligible sets* and the attempt to push the study further, up to codimension one, sounds natural.

In Chapters 4 and 5 we present the structure theory for boundaries of sets of finite perimeter as developed in [**8**, **48**], where we obtained an almost complete extension of the

Euclidean theory originally due to De Giorgi. The classical theory of sets of finite perimeter has revealed to be an extremely powerful tool in the study of variational problems dealing with hypersurfaces when the ambient space is Euclidean or a Riemannian manifold. In the last twenty years it has attracted a lot of interest also in more general frameworks where, in most cases, a notion of smooth hypersurface might be not available at all.

No theory of codimension one surfaces had been previously developed in the case of Ricci limits, to the best of our knowledge. In this regard it seems hard to obtain regularity results for boundaries of sets of finite perimeter on Ricci limits arguing by approximation with smooth manifolds: this requires already a big deal of efforts in *codimension zero* and the structure of a general set of finite perimeter might be very poor even in smooth frameworks. Moreover, we heavily rely in our approach on the following observation: if a product metric measure space $X \times \mathbb{R}$ is $\mathrm{RCD}(0, N)$ then X is $\mathrm{RCD}(0, N-1)$. The analogous property with *Ricci limit* in place of RCD seems to be unknown.

The development of a theory of sets of finite perimeter on $\mathrm{RCD}(K, N)$ spaces might be useful to various extents. On the one hand, the fact that perimeter measures only charge the regular set, together with the versatile nature of sets of finite perimeter from the point of view of calculus of variation, might improve our knowledge of singular sets in the collapsed setting. On the other hand this could be seen as a first step towards a theory of minimal hypersurfaces in the framework of synthetic lower Ricci curvature bounds. The use of minimal surfaces in the study of manifolds with nonnegative Ricci curvature indeed has led to remarkable results in recent and less recent times [**32**, **161**, **192**].

The last chapter of the thesis is dedicated to the presentation of some results taken from [**171**], joint work with Mondino. We present some sharp lower bounds for the first eigenvalue of the p-Laplacian with Dirichlet boundary conditions on domains over $\mathrm{RCD}(N-1, N)$ metric measure spaces. The topic is apparently more related to functional analysis than with geometric measure theory, but the techniques we rely on are quite similar to those used in the development of the structure theory. In particular, thanks to the characterization of the equality cases in the Dirichlet spectral gap and to a compactness argument, we obtain an almost rigidity result which seems to be new even for smooth Riemannian manifolds.

Let us also point out that in [**64**] some connections between the almost rigidity in the spectral gap inequality and the structure theory of non collapsed Ricci limits have been established, enlightening a deep link between these two topics.

Other contributions

Here we briefly review the other papers written during the PhD studies whose content is not treated in the thesis.

Quantitative volume bounds for the singular strata of non collapsed RCD **spaces.** In [**33**], joint work with Gioacchino Antonelli and Elia Brué, we obtained some volume bounds for the tubular neighbourhoods of the quantitative singular sets of non collapsed $\mathrm{RCD}(K, N)$ metric measure spaces. In this way we generalized a result obtained by Cheeger and Naber in [**73**] for non collapsed Ricci limit spaces.

In the non quantitative stratification theory for singular sets one classifies points according to the number of independent symmetries of their tangents, that is to say according to the behaviour of the space at an infinitesimal scale.
Letting then $\mathcal{R} \subset X$ be the set of those points where the tangent cone is the N-dimensional Euclidean space, it is possible to introduce a stratification

$$\mathcal{S}^0 \subset \cdots \subset \mathcal{S}^{N-1} = \mathcal{S} = X \setminus \mathcal{R}$$

of the singular set $\mathcal{S}$ where, for any $k = 0, \ldots, N-1$, $\mathcal{S}^k$ is the set of those points where no tangent cone splits a factor $\mathbb{R}^{k+1}$. Adapting the arguments of [**70**], in [**84**] the authors obtained the Hausdorff dimension estimate $\dim_H \mathcal{S}^k \leq k$.

While in the classical stratification points are separated according to the number of symmetries of tangent cones, in the quantitative one they are classified according to the number of symmetries of balls at a fixed scale. In particular, the effective singular strata might be non empty even on smooth Riemannian manifolds while in that case there is no singular point. For any $k = 0, \ldots, N-1$ and for any $r, \eta > 0$, $\mathcal{S}^k_{\eta,r}$ is the set of those points $x \in X$ where the scale invariant Gromov-Hausdorff distance between the ball $B_s(x)$ and any ball of the same radius centred at the tip of a metric cone splitting a factor $\mathbb{R}^{k+1}$ is bigger than η for any $r < s < 1$.

Since [**73**] quantitative stratification techniques have been used in a variety of different settings, we just mention here [**65, 74, 172, 173**]. Usually the key tool to give effective estimates of certain singular sets is *quantitative differentiation*. We refer to [**67**] for an account about this tool which is extremely powerful when we are in presence of a monotone energy, whose behaviour also characterizes rigidity/regularity.

In [**33**] and [**73**], setting $v_{K,N}(r)$ the volume of the ball of radius r in the model space of sectional curvature $K/(N-1)$, the volume ratio

$$r \mapsto \frac{\mathcal{H}^N(B_r(x))}{v_{K,N}(r)} \tag{8}$$

is the right energy to look at. Its monotonicity comes from the Bishop-Gromov inequality, while the study of the rigidity case (when the map in (8) is constant) comes from the *volume cone implies metric cone* theorem [**68, 83**]. The main result we obtain is the following.

Theorem 0.1. *Given $K \in \mathbb{R}$, $N \in [2, +\infty)$, an integer $k \in [0, N)$ and $v, \eta > 0$, there exists a constant $c(K, N, v, \eta) > 0$ such that if $(X, \mathsf{d}, \mathcal{H}^N)$ is an* $\mathrm{RCD}(K,N)$ *m.m.s. satisfying*

$$\frac{\mathcal{H}^N(B_1(x))}{v_{K,N}(1)} \geq v \qquad \forall x \in X, \tag{9}$$

then, for all $x \in X$ and $0 < r < 1/2$, it holds

$$\mathcal{H}^N(\mathcal{S}^k_{\eta,r} \cap B_{1/2}(x)) \leq c(K, N, v, \eta) r^{N-k-\eta}. \tag{10}$$

In [**73**] the authors argue at the level of smooth Riemannian manifolds and the effective bounds are then passed to the possibly singular limits. In [**33**] instead we argue directly at the level of RCD metric measure spaces, putting forward some consequences of the volume bounds that we obtain at the level of the singular set of codimension one, that is not present in the theory of non collapsed Ricci limits.

The main result of [**73**] has been sharpened in the recent [**64**] where it is shown that, to some extent, singular sets of codimension k of a non collapsed Ricci limit do behave like $(n-k)$ dimensional manifolds, i.e., (10) holds even with $N-k$ in place of $N-k-\eta$ at the exponent at the right hand side. The analysis in [**33**] opens to the analogous problem in the RCD theory, since many of the estimates used in the case of non collapsed Ricci limits are not easily extended to the non smooth framework.

Spectral gap inequalities on $\mathrm{CD}(N-1,N)$ metric measure spaces. One of the most striking results of the last years in the theory of metric measure spaces verifying the $\mathrm{CD}(K,N)$ condition has been the proof of the sharp Lévy-Gromov isoperimetric inequality obtained by Cavalletti and Mondino in [**58**] under the essentially non branching assumption. In their paper (inspired by a previous work by Klartag [**153**] dealing with weighted Riemannian manifolds) the theory of synthetic curvature dimension bounds is combined with the localization technique of convex geometry yielding a proof of the isoperimetric inequality. Although quite recent, [**58**] has already generated a series of developments. Let us just mention [**59**], where a number of other geometric and functional inequalities have been proven with the tools developed in [**58**], and [**56**], where a quantitative isoperimetric inequality has been obtained still relying on localization.

In [**61**], joint work with Fabio Cavalletti and Andrea Mondino, we established a quantitative version of the Obata inequality over essentially non branching $\mathrm{CD}(N-1,N)$ metric measure spaces. More in detail, we proved the following.

Theorem 0.2. *For every real number $N>1$ there exists a real constant $C(N)>0$ with the following properties: if $(X,\mathsf{d},\mathfrak{m})$ is an essentially non branching metric measure space satisfying the* $\mathrm{CD}(N-1,N)$ *condition and* $\mathfrak{m}(X)=1$ *with* $\operatorname{supp}(\mathfrak{m})=X$*, then*

$$\pi-\operatorname{diam}(X)\le C(N)\big(\lambda^{1,2}_{(X,\mathsf{d},\mathfrak{m})}-N\big)^{1/N},$$

where we denoted by $\lambda^{1,2}_{(X,\mathsf{d},\mathfrak{m})}$ the first Neumann eigenvalue of the Laplacian on $(X,\mathsf{d},\mathfrak{m})$.

Moreover, for any Lipschitz function $u:X\to\mathbb{R}$ with $\int_X u\,\mathrm{d}\mathfrak{m}=0$ and $\int_X u^2\,\mathrm{d}\mathfrak{m}=1$, there exists a distinguished point $P\in X$ such that

$$\left\|u-\sqrt{N+1}\cos\mathsf{d}_P\right\|_{L^2(X,\mathfrak{m})}\le C(N)\left(\int_X|\nabla u|^2\,\mathrm{d}\mathfrak{m}-N\right)^{\eta}, \tag{11}$$

where d_P denotes the distance function from P and

$$\eta:=\frac{1}{8N+4}.$$

The main improvement over the existing literature on the topic is that our result covers possibly non Riemannian and non smooth ambient spaces. The main novelty in the approach is that it does not rely on PDE tools nor on smoothness of the ambient space. Still, the exponent we obtain has the same order of the one that can be obtained on smooth Riemannian manifolds relying on [**43, 180**]. With respect to [**56**], the main new difficulties are due to the necessity of handling one more constraint in Obata's inequality. Indeed, while for the isoperimetric inequality one minimizes the perimeter with volume fixed, in the Obata inequality one has to minimize the Cheeger energy subject to the constraints of zero integral and fixed L^2 norm.

The analysis pursued in the paper opens to the investigation about the optimal exponent η in (11).

CHAPTER 1

Preliminaries

This first chapter is dedicated to the introduction of the basic setting and material for the rest of the thesis. The main object of our investigation are metric measure spaces verifying lower Ricci curvature bounds and dimension upper bounds, that we are going to study from the perspectives of geometric analysis and geometric measure theory.

The first section is devoted to a review of the basic background about analysis on metric measure spaces. After introducing in Section 1.1 the notation that will be in force in the rest of the thesis, we dedicate Section 1.2 to the basics about Optimal Transport. Then in Section 1.3 we introduce some standard material about differentiation of measures and covering theorems in metric measure spaces. Section 1.4 is devoted to the theory of convergence of metric measure spaces. Eventually Section 1.5 and Section 1.6 are dedicated to the introduction of the relevant notions about Sobolev calculus and to the theory of normed modules, respectively.

In Section 2 we review the *Curvature-Dimension condition* CD as introduced by Lott-Sturm-Villani. We list some of the variants and properties that we shall need in the rest of the work.

For the sake of this thesis we will be concerned only with the theory of spaces verifying the *Riemannian* Curvature Dimension condition, that we first introduce in Section 3 in the adimensional case. Under this assumption we review the second order differential calculus in Section 3.1 and the existence and uniqueness theory for Regular Lagrangian Flows in Section 3.2.

The last section of this preliminary chapter is devoted to the dimensional side of the RCD theory. After the introduction of the notion of $\mathrm{RCD}(K,N)$ metric measure space and the statements of the basic properties, we review some geometric and analytic properties of these spaces in Section 4.1 and the theory of convergence and stability of Sobolev spaces in Section 4.2.

1. Analysis on metric measure spaces

1.1. Notation and basic tools. For our purposes, a *metric measure space* is a triple $(X,\mathsf{d},\mathfrak{m})$, where (X,d) is a complete and separable metric space, while $\mathfrak{m}\geq 0$ is a Borel measure on X with $\mathfrak{m}(X)\neq 0$ that is finite on balls. The measure $\mathfrak{m}$ will be usually referred to as *reference measure*.

In the applications presented in this thesis all the spaces will be proper, i.e. such that all bounded and closed sets are compact, and therefore locally compact.

We shall denote by $B_r(x):=\{y\in X:\mathsf{d}(x,y)<r\}$ the open ball with centre $x\in X$ and radius $r>0$ and by $\bar{B}_r(x):=\{y\in X:\mathsf{d}(x,y)\leq r\}$ the closed ball.

We will indicate by $\operatorname{supp}\mu\subset X$ the support of a nonnegtive Borel measure μ and by $\mathrm{spt} f$ the support of any continuous function $f:X\to\mathbb{R}$. The notation $\mathcal{P}(X)$ will indicate the space of probability measures over X.

We shall adopt the standard notation for the spaces of p-integrable functions $L^p(X,\mathfrak{m})$ for $0\leq p\leq\infty$, sometimes switching to the shorter notation $L^p(X)$ or $L^p(\mathfrak{m})$ whenever the ambient space or the reference measure are implicit from the context and there is no risk of confusion. Their local counterparts will be indicated by $L^p_{\mathrm{loc}}(X,\mathfrak{m})$ or by $L^p(U,\mathfrak{m})$ when we focus on restrictions to a given measurable set $U \subset X$. For $1 \le p \le \infty$ we will indicate by $\|f\|_{L^p(X,\mathfrak{m})} = \|f\|_{L^p(\mathfrak{m})} = \|f\|_p$ the L^p norm of any $f \in L^p(X,\mathfrak{m})$.

Whenever $f \in L^1_{\mathrm{loc}}(X,\mathfrak{m})$ is non-negative and $U \subset X$ is a bounded Borel set we will denote by

$$\fint_U f \, \mathrm{d}\mathfrak{m} \coloneqq \frac{1}{\mathfrak{m}(U)} \int_U f \, \mathrm{d}\mathfrak{m}$$

the average value of f on U with respect to $\mathfrak{m}$. Whenever $U = B_r(x)$ for some $x \in X$ and $r > 0$ such that $\mathfrak{m}(B_r(x)) > 0$, we will sometimes adopt the shortened notation

$$(f)_{x,r} \coloneqq \fint_{B_r(x)} f \, \mathrm{d}\mathfrak{m}.$$

Given metric spaces (X, d_X) and (Y, d_Y), a Borel map $\phi : X \to Y$ and a finite measure μ on X we will indicate by $\phi_\sharp\mu$ the finite measure on Y defined through the formula $\phi_\sharp\mu(E) \coloneqq \mu(\phi^{-1}(E))$ for any Borel subset $E \subset Y$.

Given a measure μ on X and a measurable set $E \subset X$ we will indicate by $\mu \llcorner E$ the restriction of the measure μ to E that is defined by

$$\mu \llcorner E(A) \coloneqq \mu(E \cap A),$$

for any Borel $A \subset X$.

The spaces of continuous, bounded and continuous, continuous and boundedly supported functions on X will be denoted by $\mathrm{C}(X), \mathrm{C_b}(X)$ and $\mathrm{C_{bs}}(X)$, respectively.

We shall indicate by $\mathrm{Lip}(X)$ or $\mathrm{Lip}(X, \mathsf{d})$ the space of those $f : X \to \mathbb{R}$ that are Lipschitz continuous and with $\mathrm{Lip_b}(X)$ and $\mathrm{Lip_{bs}}(X)$ the subspaces of Lipschitz and bounded functions and of Lipschitz functions with bounded support. For an open subset $U \subset X$ we shall denote by $\mathrm{Lip_{loc}}(U)$ the set of those functions that are locally Lipschitz on U.

Given a Lipschitz function $f \colon X \to \mathbb{R}$, we will denote by $\mathrm{lip} f \colon X \to [0, +\infty)$ its *slope*, which is defined as

$$\mathrm{lip} f(x) \coloneqq \limsup_{y \to x} \frac{|f(x) - f(y)|}{\mathsf{d}(x,y)} \quad \text{for every accumulation point } x \in X$$

and $\mathrm{lip} f(x) \coloneqq 0$ elsewhere.
We introduce also the notion of *asymptotic Lipschitz constant* of a Lipschitz function f defined as

$$\mathrm{lip}_a f(x) \coloneqq \lim_{r \to 0} \sup_{y,z \in B_r(x)} \frac{|f(z) - f(y)|}{\mathsf{d}(z,y)},$$

for any accumulation point $x \in X$ and $\mathrm{lip}_a f(x) = 0$ otherwise.

Given any open set $\Omega \subseteq X$, we denote by $\mathrm{Lip_c}(\Omega)$ the family of all Lipschitz functions $f \colon \Omega \to \mathbb{R}$ whose support is bounded and satisfies $\mathrm{dist}\big(\mathrm{spt}(f), X \setminus \Omega\big) > 0$ (i.e. compactly supported functions).

The space of geodesics of (X, d) is denoted by

$$\mathrm{Geo}(X) \coloneqq \{\gamma \in C([0,1], X) : \mathsf{d}(\gamma_s, \gamma_t) = |s - t| \mathsf{d}(\gamma_0, \gamma_1), \text{ for every } s, t \in [0,1]\}.$$

A metric space (X, d) is said to be a *geodesic space* if and only if for each $x, y \in X$ there exists $\gamma \in \mathrm{Geo}(X)$ such that $\gamma_0 = x, \gamma_1 = y$.

Below we list two useful lemmas. The proof of the first one, based on Cavalieri's formula, can be found for instance in [**17**, Lemma 3.3] (notice that since we are assuming that μ and all μ_n are probability measures, weak convergence in duality w.r.t. $\mathrm{C_{bs}}(Z)$ and w.r.t. $\mathrm{C_b}(Z)$ are equivalent).

Lemma 1.1. *Let (Z, d_Z) be a complete and separable metric space. Let $(\mu_n) \subset \mathcal{P}(Z)$ be weakly converging in duality with $\mathrm{C_{bs}}(Z)$ to $\mu \in \mathcal{P}(Z)$ and let f_n be Borel functions uniformly bounded from above and such that*

$$\limsup_{n\to\infty} f_n(z_n) \le f(z) \quad \textit{whenever } \operatorname{supp}\mu_n \ni z_n \to z \in \operatorname{supp}\mu, \tag{1.1}$$

for some Borel function f. Then

$$\limsup_{n\to\infty} \int_Z f_n \, \mathrm{d}\mu_n \le \int_Z f \, \mathrm{d}\mu.$$

Remark 1.2. If (Z, d_Z) is proper, f_n and f are continuous, and μ_n have uniformly bounded supports, then the uniform bound from above for f_n over the support of μ_n is a direct consequence of (1.1).

The proof of Lemma 1.1 can be easily adapted to the case when we need to estimate the liminf of $\int_Z f_n \, \mathrm{d}\mu_n$.

Lemma 1.3. *Let (Z, d_Z) be a complete and separable metric space. Let (μ_n) be a sequence of non-negative Borel measures on Z finite on bounded sets and assume that μ_n weakly converge to μ in duality w.r.t. $\mathrm{C_{bs}}(Z)$. Let (f_n) and f be non-negative Borel functions on Z such that*

$$f(z) \le \liminf_{n\to\infty} f_n(z_n) \quad \textit{whenever } \operatorname{supp}\mu_n \ni z_n \to z \in \operatorname{supp}\mu. \tag{1.2}$$

Then

$$\int f \, \mathrm{d}\mu \le \liminf_{n\to\infty} \int f_n \, \mathrm{d}\mu_n.$$

1.2. Optimal Transport tools. Here we introduce the basic notions and statements about Optimal Transport that will be needed in the thesis. We refer to [**14, 210**] for a more detailed account about this subject.

Given a metric space (X, d), the subspace of probability measures with finite second order moment, i.e. those $\mu \in \mathcal{P}(X)$ such that $\int \mathsf{d}^2(x, \bar{x}) \, \mathrm{d}\mu(x) < \infty$ for some (and thus for all) $\bar{x} \in X$ will be denoted by $\mathcal{P}_2(X)$.

We define the L^2-Kantorovich-Wasserstein distance W_2 between two measures $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ as

$$W_2(\mu_0, \mu_1)^2 := \inf_\pi \int_{X\times X} \mathsf{d}^2(x, y) \, \mathrm{d}\pi, \tag{1.3}$$

where the infimum is taken over all $\pi \in \mathcal{P}(X \times X)$ with μ_0 and μ_1 as the first and the second marginal, i.e. $(P_1)_\sharp \pi = \mu_0, (P_2)_\sharp \pi = \mu_1$. Here $P_i, i = 1, 2$ denotes the projection on the first (respectively second) factor. As (X, d) is complete, one can prove that also $(\mathcal{P}_2(X), W_2)$ is complete.

A basic fact of W_2 geometry, is that if (X, d) is geodesic then $(\mathcal{P}_2(X), W_2)$ is geodesic as well. For any $t \in [0, 1]$, let e_t denote the evaluation map:

$$\mathrm{e}_t : \mathrm{Geo}(X) \to X, \qquad \mathrm{e}_t(\gamma) := \gamma_t.$$

Any geodesic $(\mu_t)_{t\in[0,1]}$ in $(\mathcal{P}_2(X), W_2)$ can be lifted to a measure $\nu \in \mathcal{P}(\mathrm{Geo}(X))$, called *dynamical optimal plan*, such that $(\mathrm{e}_t)_\sharp \nu = \mu_t$ for all $t \in [0, 1]$. Given $\mu_0, \mu_1 \in \mathcal{P}_2(X)$, we denote by $\mathrm{Opt}(\mu_0, \mu_1)$ the space of all $\nu \in \mathcal{P}(\mathrm{Geo}(X))$ for which $(\mathrm{e}_0, \mathrm{e}_1)_\sharp \nu$ realizes the minimum in (1.3). If (X, d) is geodesic, then the set $\mathrm{Opt}(\mu_0, \mu_1)$ is non-empty for any $\mu_0, \mu_1 \in \mathcal{P}_2(X)$.

We will denote by $\mathcal{Q}_t$ the Hopf-Lax semigroup defined by

$$\mathcal{Q}_t f(x) := \inf_{y\in X} \left\{ f(y) + \frac{\mathsf{d}^2(x, y)}{2t} \right\} \quad \text{for any } (x, t) \in X \times (0, +\infty),$$

for any $f : X \to \mathbb{R} \cup \{\pm\infty\}$, referring to [**10**, **15**] for a detailed discussion about its properties. Recall that the c-transform f^c of c is defined as $f^c := \mathcal{Q}_1(-f)$ and that $g : X \to \mathbb{R} \cup \{-\infty\}$ is said to be c-concave provided it is not identically $-\infty$ and $g = f^c$ for some $f : X \to \mathbb{R} \cup \{-\infty\}$.

Let us recall that, given $\mu, \nu \in \mathcal{P}_2(X)$, a function $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ is said *Kantorovich potential* from μ to ν provided it is c-concave and a maximizer for the dual problem of optimal transport. We quote a general result about the evolution of Kantorovich potentials along a W_2-geodesic in a metric space, referring to [**13**, Theorem 2.18] or [**210**, Theorem 7.35] for a proof.

Proposition 1.4 (Evolution of Kantorovich potentials)**.** *Let (X, d) be a metric space, (μ_t) be a W_2-geodesic and $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ be a Kantorovich potential between μ_0 and μ_1. Then for every $t \in [0, 1]$:*

i) *the function $t\mathcal{Q}_t(-\varphi) = \mathcal{Q}_1(-t\varphi)$ is a Kantorovich potential between μ_t and μ_0;*
ii) *the function $(1-t)\mathcal{Q}_{1-t}(-\varphi^c) = \mathcal{Q}_1(-(1-t)\varphi^c)$ is a Kantorovich potential from μ_t to μ_1.*

Furthermore, for every $t \in [0, 1]$ it holds

$$\mathcal{Q}_t(-\varphi) = \mathcal{Q}_{1-t}(-\varphi^c) \geq 0, \quad \textit{everywhere},$$
$$\mathcal{Q}_t(-\varphi) = \mathcal{Q}_{1-t}(-\varphi^c) = 0, \quad \textit{on } \operatorname{supp}(\mu_t).$$

Next we review the notions of non branching metric space and essentially non branching metric measure space.

A set $F \subset \mathrm{Geo}(X)$ is a *set of non-branching geodesics* if and only if for any $\gamma^1, \gamma^2 \in F$, it holds:

$$\exists\, \bar{t} \in (0,1) \text{ such that } \ \forall t \in [0, \bar{t}] \quad \gamma^1_t = \gamma^2_t \quad \Longrightarrow \quad \gamma^1_s = \gamma^2_s, \quad \forall s \in [0,1].$$

A measure μ on a measurable space $(\Omega, \mathcal{F})$ is said to be *concentrated* on $F \subset \Omega$ if there exists $E \subset F$ with $E \in \mathcal{F}$ so that $\mu(\Omega \setminus E) = 0$. With this terminology, we next recall the definition of *essentially non-branching space* from [**188**].

Definition 1.5 (Essentially non-branching space)**.** A metric measure space $(X, \mathsf{d}, \mathfrak{m})$ is *essentially non-branching* if and only if for any $\mu_0, \mu_1 \in \mathcal{P}_2(X)$, with μ_0, μ_1 absolutely continuous with respect to $\mathfrak{m}$, any element of $\mathrm{Opt}(\mu_0, \mu_1)$ is concentrated on a set of non-branching geodesics.

1.3. Hausdorff measures, covering theorems and differentiation. Let us introduce the so-called Hausdorff (type) measures on a metric measure space and the basic covering and differentiation theorems that we shall need in the thesis. The discussion is mainly borrowed from [**2**]. We refer also to [**100**, **134**] for an exhaustive treatment of the topic.

Definition 1.6 (Hausdorff measure)**.** Let (X, d) be a metric space. For any $k \in [0, +\infty)$ we let

$$\omega_k := \frac{\pi^{k/2}}{\Gamma(1 + k/2)}, \quad \text{where} \quad \Gamma(k) := \int_0^{+\infty} t^{k-1} e^{-t} \, \mathrm{d}x$$

is the Euler function. If $\delta \in (0, +\infty]$ and $A \subset X$ we let

$$\mathscr{H}^k_\delta(A) := \frac{\omega_k}{2^k} \inf \left\{ \sum_{i \in I} (\mathrm{diam}(A_i))^k : A \subset \bigcup_{i \in I} A_i, \quad \mathrm{diam} A_i < \delta \right\}. \tag{1.4}$$

Finally we define

$$\mathscr{H}^k(A) := \sup_{\delta > 0} \mathscr{H}^k_\delta(A). \tag{1.5}$$

Observe that since $\delta \mapsto \mathscr{H}^k_\delta(A)$ is nonincreasing we can replace the supremum with a limit as $\delta \downarrow 0$ in (1.5) above. The quantity $\mathscr{H}^k(A)$ will be referred throughout as the k-dimensional Hausdorff measure of A. Moreover, let us point out that when k is a natural number ω_k coincides with the Lebesgue measure of the unit ball in $\mathbb{R}^k$.

Let us point out that, for any $k \geq 0$ and for any $\delta \in (0, +\infty]$, $\mathscr{H}^k_\delta$ and $\mathscr{H}^k$ are outer measures. Moreover Borel sets are $\mathcal{H}^k$-measurable and

$$\mathscr{H}^k(A) > 0 \Rightarrow \mathscr{H}^{k'}(A) = +\infty \quad \text{for any } k > k' \geq 0.$$

Definition 1.7 (Hausdorff dimension)**.** Given any set $A \subset X$ we define the Hausdorff dimension of A as

$$\dim_{\mathscr{H}}(A) := \inf \left\{ k \geq 0 : \mathscr{H}^k(A) = 0 \right\}. \tag{1.6}$$

The Hausdorff dimension of the ambient metric space is $\dim_{\mathscr{H}}(X)$ by definition.

The construction of the Hausdorff measures is an instance of the *Carathéodory construction* which can be made with functions (usually referred as *gauge functions* in the literature) other than the k-th power of the diameter or with more specific families of covering sets. In particular, when we require that the sets A_i are balls then the outcome of the construction is the so-called *spherical* Hausdorff measure $\mathcal{S}^k$.

Remark 1.8. Given a metric measure space $(X, \mathsf{d}, \mathfrak{m})$, applying the Carathédory construction with coverings made by balls and gauge function $B_r(x) \mapsto \mathfrak{m}(B_r(x))/r^\alpha$ one obtains the so-called codimension-α Hausdorff measure (pre-measures respectively), that we shall denote by $\mathscr{H}^{h_\alpha}$ ($\mathscr{H}^{h_\alpha}_\delta$ respectively) in the following. A prominent role will be played by the case $\alpha = 1$. For this reason we shall adopt the shortened notation $\mathscr{H}^h$ in that case.

We will make appeal several times to the following covering theorem, valid on any metric space. We refer to [**2**, Theorem 2.2.3] for its proof.

Theorem 1.9 (Vitali covering lemma)**.** *Let (X, d) be a metric space and $\mathcal{F}$ be a family of balls such that*

$$\sup \{r(B) : B \in \mathcal{F}\} < +\infty,$$

where $r(B)$ denotes the radius of a ball B. Then there exists a disjoint subfamily $\mathcal{F}' \subset \mathcal{F}$ such that

$$\bigcup_{\mathcal{F}} B_r(x) \subset \bigcup_{\mathcal{F}'} B_{5r}(x).$$

Definition 1.10 (Doubling metric measure spaces)**.** A metric measure space is said to be *locally doubling* if there exists a nondecreasing function $C : (0, \infty) \to (0, \infty)$ such that, for any $x \in X$ and for any $0 < r < R$, it holds

$$\mathfrak{m}(B_{2r}(x)) \leq C(R)\mathfrak{m}(B_r(x)). \tag{1.7}$$

We will say that the m.m.s. is *doubling* if the function C can be chosen to be constant and, in that case, we shall call *doubling constant* its value.
Eventually, we shall say that the m.m.s. is *asymptotically doubling* if

$$\limsup_{r \to 0} \frac{\mathfrak{m}(B_{2r}(x))}{\mathfrak{m}(B_r(x))} < \infty \quad \text{for } \mathfrak{m}\text{-a.e. } x \in X. \tag{1.8}$$

Remark 1.11. On any doubling m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ spheres, i.e. sets of the form $\{y \in X : \mathsf{d}(x, y) = r\}$ for some $x \in X$ and $r > 0$, are $\mathfrak{m}$-negligible.

Theorem 1.12 ([**2**, Theorem 5.2.2])**.** *A metric measure space $(X, \mathsf{d}, \mathfrak{m})$ is doubling if and only if there exist constants $C', s > 0$ such that*

$$\frac{\mathfrak{m}(B_r(x))}{\mathfrak{m}(B_R(y))} \geq C' \left(\frac{r}{R}\right)^s, \tag{1.9}$$

for any $x, y \in X$ *and for any* $R \geq r > 0$ *such that* $x \in B_R(y)$.

A stronger condition, called *Ahlfors regularity* asks for a double-sided control on the measure of balls.

Definition 1.13 (Ahlfors regularity)**.** Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space. We say that it is n-Ahlfors regular provided there exist constants $A, a > 0$ such that

$$ar^n \leq \mathfrak{m}(B_r(x)) \leq Ar^n, \quad \text{for any } x \in X \text{ and any } 0 < r < \operatorname{diam}(X).$$

Definition 1.14 (Hardy-Littlewood maximal operator)**.** Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space and μ be a non-negative Borel measure on (X, d). Then we define the *maximal operator* of the measure μ with respect to $\mathfrak{m}$ as

$$M\mu(x) := \sup_{r>0} \frac{\mu(B_r(x))}{\mathfrak{m}(B_r(x))}, \quad \text{for any } x \in X.$$

If $f : X \to [0, +\infty]$ is a non-negative Borel function we shall denote by $Mf := M(f\mathfrak{m})$.

The statement below is usually referred to as *weak* $1-1$ *estimate*. We refer to [**2**, Theorem 5.2.4] for a proof based on Theorem 1.9.

Theorem 1.15 (Weak 1-1 estimate)**.** *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be a doubling metric measure space. Then there exists a constant* $C > 0$ *depending only on the doubling constant of* $\mathfrak{m}$ *such that*

$$\mathfrak{m}\left(\{x \in X : M\mu(x) > \lambda\}\right) \leq \frac{C}{\lambda}\mu(X), \quad \text{for any } \lambda > 0.$$

Relying on Theorem 1.15 one can obtain a version of the Lebesgue differentiation theorem valid on any doubling metric measure space (cf. [**2**, Theorem 5.2.6]).

Theorem 1.16. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be a doubling m.m.s. and* $f \in L^1_{\mathrm{loc}}(X, \mathfrak{m})$. *Then*

$$\lim_{r \to 0} \fint_{B_r(x)} |f(y) - f(x)| \, \mathrm{d}\mathfrak{m}(y) = 0, \quad \text{for } \mathfrak{m}\text{-a.e. } x \in X. \tag{1.10}$$

Corollary 1.17. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be a doubling m.m.s. and let* $f \in L^1_{\mathrm{loc}}(X, \mathfrak{m})$. *Then*

$$\lim_{r \to 0} \fint_{B_r(x)} f(y) \, \mathrm{d}\mathfrak{m}(y) = f(x) \quad \text{for } \mathfrak{m}\text{-a.e. } x \in X \tag{1.11}$$

Remark 1.18. We remark that Theorem 1.16 holds true even under the weaker asymptotically doubling assumption on $(X, \mathsf{d}, \mathfrak{m})$, see [**189**, Remark 3.3] and [**100**, Theorem 2.9.8].

The following fundamental result, originally due to Hardy and Littlewood, shows that the maximal operator, though nonlinear, continuously maps L^p into L^p for any $p > 1$. We refer to [**2**, Theorem 5.2.10] for a proof in the present context.

Theorem 1.19. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be a doubling m.m.s.. Then, for any* $p > 1$ *there exists a constant* C_p *such that, for any* $f \in L^p(X, \mathfrak{m})$ *it holds*

$$\|Mf\|_{L^p(X,\mathfrak{m})} \leq C_p \|f\|_{L^p(X,\mathfrak{m})}. \tag{1.12}$$

If $(X, \mathsf{d}, \mathfrak{m})$ satisfies only the local doubling condition, one can prove a local version of Theorem 1.19 with minor modifications to the arguments used in the global case.
Let us fix $1 < p \leq \infty$ and a compact set $P \subset X$. Then, there exists a constant $C > 0$, depending only on the diameter of P and the local doubling constant of $(X, \mathsf{d}, \mathfrak{m})$, such that for every $f \in L^p(X, \mathfrak{m})$ with $\mathrm{spt} f \subset P$, it holds

$$\|Mf\|_{L^p(P,\mathfrak{m})} \leq C \|f\|_{L^p(X,\mathfrak{m})}. \tag{1.13}$$

The proof of the following technical lemma is strongly inspired by the proof of the analogous statement in the Euclidean setting given in [**97**]. We refer to [**148**, Lemma 4.3] for a similar result in the present context, formulated in terms of Sobolev capacities instead of Hausdorff measures.

Lemma 1.20. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a locally doubling m.m.s.. Let $f \in L^1(X, \mathfrak{m})$, $f \geq 0$ be given. Then for any exponent $\alpha > 0$ it holds that*

$$\mathscr{H}^{h_\alpha}(\Lambda_\alpha) = 0, \qquad \textit{where we set } \Lambda_\alpha := \Big\{ x \in X \,\Big|\, \limsup_{r \downarrow 0} r^\alpha (f)_{x,r} > 0 \Big\},$$

where we recall that $\mathscr{H}^{h_\alpha}$ is the Hausdorff type measure of codimension α that we introduced in Remark 1.8.

Proof. By Corollary 1.17 we know that the limit $\lim_{r \downarrow 0} (f)_{x,r}$ exists and is finite for $\mathfrak{m}$-a.e. $x \in X$, thus for any $\alpha > 0$ we have that $\limsup_{r \downarrow 0} r^\alpha (f)_{x,r} = 0$ holds for $\mathfrak{m}$-a.e. $x \in X$. This means that $\mathfrak{m}(\Lambda_\alpha) = 0$. Calling

$$\Lambda_\alpha^k := \Big\{ x \in X \,\Big|\, \limsup_{r \downarrow 0} r^\alpha (f)_{x,r} \geq 1/k \Big\} \qquad \text{for every } k \in \mathbb{N},$$

we see that $\Lambda_\alpha = \bigcup_k \Lambda_\alpha^k$, thus in particular $\mathfrak{m}(\Lambda_\alpha^k) = 0$ for every $k \in \mathbb{N}$. Given that $f \in L^1(X, \mathfrak{m})$, for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\int_A f \, \mathrm{d}\mathfrak{m} \leq \varepsilon$ for any Borel set $A \subset X$ satisfying $\mathfrak{m}(A) < \delta$. Fix $k \in \mathbb{N}$ and pick an open set $U \subset X$ such that $\Lambda_\alpha^k \subset U$ and $\mathfrak{m}(U) < \delta$. Let us define

$$\mathcal{F} := \left\{ B_r(x) \,\middle|\, x \in \Lambda_\alpha^k, \ r \in (0, \varepsilon), \ B_r(x) \subset U, \ \int_{B_r(x)} f \, \mathrm{d}\mathfrak{m} \geq \mathfrak{m}\big(B_r(x)\big)/(r^\alpha k) \right\}.$$

Therefore by the Vitali covering Theorem 1.9 we can find a sequence $(B_i)_{i \in \mathbb{N}} \subset \mathcal{F}$ of pairwise disjoint balls $B_i = B_{r_i}(x_i)$ such that $\Lambda_\alpha^k \subset \bigcup_i B_{5r_i}(x_i)$. Since $\mathfrak{m}$ is locally doubling, there exists a constant $C_D \geq 1$ such that $\mathfrak{m}\big(B_{5r}(x)\big) \leq C_D \, \mathfrak{m}\big(B_r(x)\big)$ for every $x \in X$ and $r < \varepsilon$. Consequently

$$\begin{aligned} \mathscr{H}^{h_\alpha}_{10\varepsilon}(\Lambda_\alpha^k) &\leq \frac{1}{5^\alpha} \sum_{i=1}^\infty \frac{\mathfrak{m}\big(B_{5r_i}(x_i)\big)}{r_i^\alpha} \leq \frac{C_D}{5^\alpha} \sum_{i=1}^\infty \frac{\mathfrak{m}(B_i)}{r_i^\alpha} \leq \frac{C_D k}{5^\alpha} \sum_{i=1}^\infty \int_{B_i} f \, \mathrm{d}\mathfrak{m} \leq \frac{C_D k}{5^\alpha} \int_U f \, \mathrm{d}\mathfrak{m} \\ &\leq \frac{C_D k}{5^\alpha} \varepsilon. \end{aligned}$$

By letting $\varepsilon \downarrow 0$ we conclude that $\mathscr{H}^{h_\alpha}(\Lambda_\alpha^k) = 0$, whence $\mathscr{H}^{h_\alpha}(\Lambda_\alpha) = \lim_k \mathscr{H}^{h_\alpha}(\Lambda_\alpha^k) = 0$. □

1.4. Convergence of metric measure spaces. We wish to introduce the basic notions and results about convergence of (pointed) metric measure spaces. Basic references about this topic are the monographs [**54**, **210**] and the paper [**118**]. Here we closely follow the presentation of [**170**].

A pointed metric measure space (abbreviated to p.m.m.s. in the following) is a quadruple $(X, \mathsf{d}, \mathfrak{m}, x)$ where $(X, \mathsf{d}, \mathfrak{m})$ is a metric measure space and $x \in \operatorname{supp} \mathfrak{m}$ is a given reference point. Two p.m.m. spaces $(X, \mathsf{d}, \mathfrak{m}, x), (X', \mathsf{d}', \mathfrak{m}', x')$ are said to be isomorphic if there exists an isometry $T : (\operatorname{supp} \mathfrak{m}, \mathsf{d}) \to (\operatorname{supp} \mathfrak{m}', \mathsf{d}')$ such that $T_\sharp \mathfrak{m} = \mathfrak{m}'$ (measure preserving condition) and $T(x) = x'$.

We say that a p.m.m.s. is normalised provided $\int_{B_1(x)} (1 - \mathsf{d}(\cdot, x)) \, \mathrm{d}\mathfrak{m} = 1$. Observe that for any p.m.m.s. $(X, \mathsf{d}, \mathfrak{m}, x)$ there exists a unique constant $c > 0$ such that $(X, \mathsf{d}, c\mathfrak{m}, x)$ is normalised.

We shall denote by $\mathcal{M}_C$ the class of normalised p.m.m.s. verifying (1.7) for a given nondecreasing function $C : (0, +\infty) \to (0, +\infty)$.

Definition 1.21 (Pointed measured Gromov-Hausdorff convergence)**.** A sequence of p.m.m. spaces $(X_n, \mathsf{d}_n, \mathfrak{m}_n, x_n)$ is said to converge in the pointed measured Gromov-Hausdorff topology to $(X, \mathsf{d}, \mathfrak{m}, x)$ if there exist a separable metric space (Z, d_Z) and isometric embeddings

$\iota_n : (\operatorname{supp} \mathfrak{m}_n, \mathsf{d}_n) \to (Z, \mathsf{d}_Z)$ and $\iota : (\operatorname{supp} \mathfrak{m}, \mathsf{d}) \to (Z, \mathsf{d}_Z)$ such that, for any $\varepsilon > 0$ and $R > 0$ there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ it holds

$$\iota\left(B_R^X(x)\right) \subset B_\varepsilon^Z\left(\iota_n\left(B_{R+\varepsilon}^{X_n}(x_n)\right)\right), \quad \iota_n\left(B_R^{X_n}(x_n)\right) \subset B_\varepsilon^Z\left(\iota\left(B_{R+\varepsilon}^X(x)\right)\right) \tag{1.14}$$

and

$$\lim_{n\to\infty} \int_Z \varphi \,\mathrm{d}\left((\iota_n)_\sharp \mathfrak{m}_n\right) = \int_Z \varphi \,\mathrm{d}\left(\iota_\sharp \mathfrak{m}\right), \qquad \text{for any } \varphi \in \mathrm{C_b}(Z). \tag{1.15}$$

It is straightforward to check that this is a notion of convergence for isomorphism classes of pointed metric measure spaces.

The application of an argument originally due to Gromov [**127**] allows to prove compactness with respect to the pmGH topology for the class of uniformly doubling pointed metric measure spaces.

Proposition 1.22. *Let $C : (0, \infty) \to (0, \infty)$ be a nondecreasing function. Then there exists a distance $\mathcal{D}_C$ over $\mathcal{M}_C$ for which converging sequences are exactly those converging with respect to the pmGH topology. Moreover, the space $(\mathcal{M}_C, \mathcal{D}_C)$ is compact.*

Let us recall that there is an equivalent way to define pmGH convergence via ε-isometries as follows.

Theorem 1.23 (pmGH convergence via ε-isometries)**.** *Let $(X_n, \mathsf{d}_n, \mathfrak{m}_n, x_n)$ and $(X, \mathsf{d}, \mathfrak{m}, x)$ be as above. Then $(X_n, \mathsf{d}_n, \mathfrak{m}_n, x_n)$ converge to $(X, \mathsf{d}, \mathfrak{m}, x)$ in the pmGH sense if and only if for any $\varepsilon, R > 0$ there exists $N(\varepsilon, R) \in \mathbb{N}$ such that, for any $n \geq N(\varepsilon, R)$ there exists a Borel map $f_n^{\varepsilon,R} : B_R^{X_n}(x_n) \to X$ for which the following hold:*

i) $f_n^{\varepsilon,R}(x_n) = x$*;*
ii) $\sup_{x,y\in B_R^{X_n}(x_n)} \left|\mathsf{d}_n(x, y) - \mathsf{d}(f_n^{\varepsilon,R}(x), f_n^{\varepsilon,R}(y))\right| \leq \varepsilon$*;*
iii) *the ε-neighbourhood of $f_n^{\varepsilon,R}(B_R(x_n))$ contains $B_{R-\varepsilon}(x)$;*
iv) $\left(f_n^{\varepsilon,R}\right)_\sharp \left(\mathfrak{m}_n|_{B_R(x_n)}\right) \to \mathfrak{m}|_{B_R(x)}$ *weakly in duality with $\mathrm{C_{bs}}(X)$ as $n \to \infty$ for a.e. $R > 0$.*

Metric measured tangents will play a crucial role in the development of the note. Given a m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ and a point $x \in \operatorname{supp} \mathfrak{m}$, for any $r \in (0, 1)$ we shall consider the rescaled and normalised p.m.m.s. $(X, \mathsf{d}/r, \mathfrak{m}_r^x, x)$, where

$$\mathfrak{m}_r^x := \left(\int_{B_r(x)} \left(1 - \frac{1}{r}\mathsf{d}(x, \cdot)\right) \mathrm{d}\mathfrak{m}\right)^{-1} \mathfrak{m} = C(x, r)\mathfrak{m}. \tag{1.16}$$

Definition 1.24 (Tangent cones)**.** Let $(X, \mathsf{d}, \mathfrak{m})$ be a m.m.s. and $x \in \operatorname{supp} \mathfrak{m}$. We define the *space of tangent cones* $\mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m})$ at the point $x \in \operatorname{supp} \mathfrak{m}$ as the family of all those spaces $(Y, \varrho, \mathfrak{n}, y)$ such that

$$\lim_{n\to\infty} \mathsf{d}_{\mathrm{pmGH}}\big((X, \mathsf{d}/r_n, \mathfrak{m}_{r_n}^x, x), (Y, \varrho, \mathfrak{n}, y)\big) = 0$$

for some sequence $(r_n)_n \subseteq (0, 1)$ of radii with $r_n \downarrow 0$.

Notice that, if $(X, \mathsf{d}, \mathfrak{m}) \in \mathcal{M}_C$ for some nondecreasing function $C : (0, \infty) \to (0, \infty)$, then so is $(X, \mathsf{d}/r, \mathfrak{m}_r^x, x)$ for any $x \in \operatorname{supp} \mathfrak{m}$ and any $r \in (0, 1)$. It follows from Proposition 1.22 that $\mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m})$ is not empty for any $x \in \operatorname{supp} \mathfrak{m}$.

Next we recall the definition of *pointed measured Gromov convergence* and compare it with the one of pmGH convergence.

Definition 1.25 (Pointed measured Gromov convergence)**.** Assume that $(X_n, \mathsf{d}_n, \mathfrak{m}_n, x_n)$ and $(X, \mathsf{d}, \mathfrak{m}, x)$ are pointed metric measure spaces. Then we say that $(X_n, \mathsf{d}_n, \mathfrak{m}_n, x_n)$ converge to $(X, \mathsf{d}, \mathfrak{m}, x)$ in the pointed measured Gromov topology (abbreviated pmG topology) if there exist a complete and separable metric space (W, d_W) and isometric embeddings $(\iota_n)_n$, ι with $\iota_n : X_n \to W$ and $\iota : X \to W$ such that

$$\iota_n(x_n) \to \iota(x) \in \operatorname{supp} \mathfrak{m},$$
$$(\iota_n)_\sharp \mathfrak{m}_n \to (\iota)_\sharp \mathfrak{m} \quad \text{weakly in duality with } \mathrm{C_{bs}}(W).$$

Remark 1.26. Whenever we are dealing with families of metric measure spaces with diameter uniformly bounded from above we will employ the simpler notion of measured Gromov-Hausdorff convergence. This notion was introduced in [**104**] and it can be equivalently characterized by Definition 1.25 neglecting the condition about the convergence of the base points. We will indicate by $\mathsf{d}_{\mathrm{mGH}}$ the distance inducing convergence in the mGH topology.

The implication from pmGH convergence to pmG convergence is always true. The converse one becomes true under the additional doubling assumption (cf. [**118**], treating the case of uniformly doubling spaces. The general case can be handled reducing to balls of increasing radii where the uniform doubling assumption is in force.).

Proposition 1.27. *Let $(X_n, \mathsf{d}_n, \mathfrak{m}_n, x_n)$ be a sequence of pointed metric measure spaces that converges to $(X, \mathsf{d}, \mathfrak{m}, x)$ in the pmGH sense. Then the convergence holds also in the pmG sense. Moreover, if the spaces are C-doubling for some nondecreasing function $C : (0, \infty) \to (0, \infty)$ then also the converse implication holds true.*

Definition 1.28. Let $(X_i, \mathsf{d}_i, \mathfrak{m}_i, x_i)$, (Y, ϱ, μ, y), (Z, d_Z) be as above and $f_i : X_i \to \mathbb{R}$, $f : Y \to \mathbb{R}$. We say that $f_i \to f$ pointwise if $f_i(z_i) \to f(z)$ for every sequence of points $z_i \in X_i$ such that $z_i \to z$ in Z. If for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f_i(z_i) - f(z)| \le \varepsilon$ for every $i \ge \delta^{-1}$ and $z_i \in X_i$, $z \in Y$ with $\mathsf{d}_Z(z_i, z) \le \delta$, then we say that $f_i \to f$ uniformly.

The next proposition is a version of the Ascoli–Arzelà compactness theorem for sequences of functions defined on varying spaces. Its proof can be obtained arguing as in the case of a fixed space, see [**210**, Proposition 27.20].

Proposition 1.29. *Let $(X_i, \mathsf{d}_i, \mathfrak{m}_i, x_i)$ and (Y, ρ, μ, y) be as above and $R > 0$, $L > 0$ fixed. Then for any sequence of L-Lipschitz functions $f_i : B_R(x_i) \to \mathbb{R}$ such that $\sup_i |f_i(x_i)| < +\infty$ there exists a subsequence that converges uniformly to some L-Lipschitz function $f : B_R(y) \to \mathbb{R}$.*

Remark 1.30. Let us point out that, if all the spaces coincide and they are compact, then $f_i \to f$ pointwise according to Definition 1.28 if and only if f is continuous and $f_i \to f$ uniformly. Therefore the terminology “pointwise convergence” might be a bit misleading. Nevertheless we prefer to keep using it since it is adopted in several other works [**18, 21, 170**].

1.5. Sobolev calculus and Heat flow. In this section we recall the basic facts about Sobolev calculus and heat flow on metric measure spaces. We refer to [**15, 110**] for a more systematic treatment of this topic.

1.5.1. *Cheeger energy and minimal relaxed gradients.* For $p \in (1, \infty)$ we recall that the *p-Cheeger energy* $\mathrm{Ch}_p \colon L^p(\mathfrak{m}) \to [0, +\infty]$ is the convex and lower semicontinuous functional

$$\mathrm{Ch}_p(f) := \inf \left\{ \liminf_{n\to\infty} \int \mathrm{lip}^p(f_n) \, \mathrm{d}\mathfrak{m} \,\middle|\, (f_n)_n \subseteq L^p(\mathfrak{m}) \cap \mathrm{Lip_b}(X), \lim_{n\to\infty} \|f_n - f\|_{L^p(\mathfrak{m})} = 0 \right\}.$$

The original definition given in [**62**] involves generalized upper gradients in place of the slopes of the functions f_n and many other *pseudo-gradients* can be used, leading to the same functional. This is a consequence of a powerful identification result proved in [**10**].

The Sobolev spaces $H^{1,p}(X, \mathsf{d}, \mathfrak{m})$ are defined as the finiteness domains of the energies Ch_p. When endowed with the norm

$$\|f\|_{H^{1,p}} := \left(\|f\|^p_{L^p(X,\mathfrak{m})} + \mathrm{Ch}_p(f) \right)^{\frac{1}{p}} \tag{1.17}$$

they become Banach spaces. Furthermore, under the assumption that $(X, \mathsf{d}, \mathfrak{m})$ is doubling, the Sobolev spaces are reflexive (see [**10**, Corollary 7.5]). In that case standard functional analytic arguments yield separability and density of bounded Lipschitz functions (cf. [**10**, Proposition 7.6]).

For any $f \in H^{1,p}(X, \mathsf{d}, \mathfrak{m})$ one can define, through a minimizing procedure, an object $|\nabla f|_p$ called *p-minimal relaxed gradient* providing the integral representation

$$\mathrm{Ch}_p(f) = \int_X |\nabla f|_p^p \, \mathrm{d}\mathfrak{m}. \tag{1.18}$$

Remark 1.31. We chose to adopt the notation $|\nabla f|_p$ even though the most appropriate choice would be $|Df|_p$. Our choice is motivated by the fact that the case $p = 2$ will play a prominent role in the thesis in presence of an infinitesimally Hilbertian assumption (cf. Definition 1.35 below). We refer to [**110**] for a thorough discussion about the differential structure of metric measure spaces.

Remark 1.32. Let us point out that the p-minimal relaxed gradient can depend on p. We refer to [**91**] for the construction of an explicit example of metric measure space where this occurs. Nevertheless, in all the relevant applications for the sake of this thesis it can be proved that there is independence of the integrability exponent, as we shall see below.

Let us recall that $(X, \mathsf{d}, \mathfrak{m})$ satisfies a weak local $(1,p)$-Poincaré inequality with constants $C_P > 0$ and $\lambda \geq 1$, for some $1 \leq p < \infty$, if it holds

$$\fint_{B_r(x)} |f - (f)_{x,r}| \, \mathrm{d}\mathfrak{m} \leq C_P \, r \left(\fint_{B_{\lambda r}(x)} |Df|^p \, \mathrm{d}\mathfrak{m} \right)^{1/p} \quad \text{for all } f \in H^{1,p}(X),\ x \in X,\ r > 0. \tag{1.19}$$

In the context of metric measure spaces verifying a doubling assumption and a weak local $(1,2)$-Poincaré inequality (usually referred to as *PI spaces*) a deep identification result due to Cheeger [**62**] allows to identify (in the almost everywhere sense) the minimal relaxed gradient with the slope for Lipschitz functions, removing also the dependence of the minimal relaxed gradient on the integrability exponent. We refer to [**10**, Theorem 8.4] for the present formulation and for a different proof.

Theorem 1.33. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a m.m.s. with $\mathfrak{m}$ doubling and supporting a weak $(1,p)$-Poincaré inequality for some $1 < p < +\infty$. Then, for any $f \in H^{1,p}(X, \mathsf{d}, \mathfrak{m}) \cap \mathrm{Lip}_{\mathrm{loc}}(X)$, it holds $\mathrm{lip} f = |\nabla f|_p$ $\mathfrak{m}$-a.e. on X.*

We turn to the introduction of the notion of 2-*capacity*, referring to [**88**, **149**] for a detailed discussion on the topic.

The capacity of a given set $E \subset X$ is defined as

$$\mathrm{Cap}(E) := \inf \left\{ \|f\|^2_{H^{1,2}(X)} \;\middle|\; f \in H^{1,2}(X, \mathsf{d}, \mathfrak{m}),\ f \geq 1 \ \mathfrak{m}\text{-a.e. on some neighbourhood of } E \right\}.$$

We remark that a notion of p-capacity can be defined in an analogous way. Since we will only be concerned with the case $p = 2$ we will omit the dependence on the exponent. We refer moreover to [**88**, Remark 2.7] for a comparison with other notions of Capacity based on Newtonian functions rather than Sobolev functions. As it is pointed out therein, the notions are equivalent on PI spaces.

It turns out that Cap is a submodular outer measure on X, finite on all bounded sets, such that the inequality $\mathfrak{m}(E) \leq \mathrm{Cap}(E)$ holds for any Borel set $E \subset X$. Any function $f : X \to [0, +\infty]$ can be integrated with respect to the capacity via Cavalieri's formula

$$\int f \, \mathrm{dCap} := \int_0^{+\infty} \mathrm{Cap}(\{f > t\}) \, \mathrm{d}t,$$

since the function $t \mapsto \mathrm{Cap}(\{f > t\})$ is non-increasing, thus it is Lebesgue measurable. The integral operator $f \mapsto \int f \, \mathrm{dCap}$ is subadditive as a consequence of the submodularity of Cap. Given any set $E \subset X$, we shall use the shorthand notation $\int_E f \, \mathrm{dCap} := \int \chi_E f \, \mathrm{dCap}$.

On PI spaces the 2-capacity controls $\mathscr{H}^{h_\alpha}$ for any $\alpha < 2$. The proof of this result is inspired by the one given in [**97**] in the Euclidean context.

Theorem 1.34. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a PI space. Then it holds that $\mathscr{H}^{h_\alpha} \ll \mathrm{Cap}$ for every $\alpha \in (0, 2)$.*

Proof. Fix $\alpha \in (0, 2)$ and a set $A \subset X$ with $\mathrm{Cap}(A) = 0$. We aim to prove that $\mathscr{H}^{h_\alpha}(A) = 0$. By definition of capacity, we can find a sequence $(f_i)_i \subset H^{1,2}(X)$ such that $f_i \geq 1$ on some neighbourhood of A and $\|f_i\|_{H^{1,2}(X)} \leq 1/2^i$ for every $i \in \mathbb{N}$. Since $\sum_{i=1}^\infty \|f_i\|_{H^{1,2}(X)} < +\infty$, one has that $g := \sum_{i=1}^\infty f_i$ is a well-defined element of the Banach space $H^{1,2}(X)$. For any $k \in \mathbb{N}$ it clearly holds that $g \geq k$ on some neighbourhood of A, whence for any $x \in A$ we have $(g)_{x,r} \geq k$ for every $r < \mathrm{dist}(x, \{g < k\})$ and accordingly

$$\lim_{r \downarrow 0} (g)_{x,r} = +\infty \quad \text{for every } x \in A. \tag{1.20}$$

Furthermore, we claim that

$$\limsup_{r \downarrow 0} r^\alpha \fint_{B_r(x)} |Dg|^2 \, \mathrm{d}\mathfrak{m} = +\infty \quad \text{for every } x \in A. \tag{1.21}$$

In order to prove it, we argue by contradiction: suppose that $\limsup_{r \downarrow 0} r^\alpha \fint_{B_r(x)} |Dg|^2 \, \mathrm{d}\mathfrak{m} < +\infty$ for some $x \in A$, so that there exists a constant $M > 0$ such that

$$r^\alpha \fint_{B_r(x)} |Dg|^2 \, \mathrm{d}\mathfrak{m} \leq M \quad \text{for every } r \in (0, 1). \tag{1.22}$$

Call C_D and C_P the doubling and the Poincaré constant of $(X, \mathsf{d}, \mathfrak{m})$ (for $r < 1/2$), respectively. Therefore, for every $r < 1/(2\lambda)$ we have that

$$\begin{aligned}
\big|(g)_{x,r} - (g)_{x,2r}\big| &= \frac{1}{\mathfrak{m}(B_r(x))} \left| \int_{B_r(x)} g - (g)_{x,2r} \, \mathrm{d}\mathfrak{m} \right| \\
&\leq C_D \fint_{B_{2r}(x)} |g - (g)_{x,2r}| \, \mathrm{d}\mathfrak{m} \\
&\overset{(1.19)}{\leq} 2\, C_D\, C_P\, r \left(\fint_{B_{2\lambda r}(x)} |Dg|^2 \, \mathrm{d}\mathfrak{m} \right)^{1/2} \\
&\overset{(1.22)}{\leq} (2^{1-\alpha/2}\, C_D\, C_P\, \lambda^{-\alpha/2}\, M^{1/2})\, r^{1-\alpha/2}.
\end{aligned}$$

Let us set $C := 2^{1-\alpha/2}\, C_D\, C_P\, \lambda^{-\alpha/2}\, M^{1/2}$ and $\theta := 1 - \alpha/2 \in (0, 1)$. Then the previous computation gives $\sum_{i=2}^\infty \big|(g)_{x,2^{-i}} - (g)_{x,2^{-i+1}}\big| \leq C \sum_{i=2}^\infty (2^\theta)^{-i} < +\infty$, contradicting (1.20). This proves (1.21).

Finally, it immediately follows from (1.21) that A is contained in the set of all points $x \in X$ that satisfy $\limsup_{r \downarrow 0} r^\alpha \fint_{B_r(x)} |Dg|^2 \, \mathrm{d}\mathfrak{m} > 0$, which is $\mathscr{H}^{h_\alpha}$-negligible by Lemma 1.20. Therefore, we conclude that $\mathscr{H}^{h_\alpha}(A) = 0$. □

As we anticipated, the case $p = 2$ will play a central role in the thesis. Therefore we shall adopt the shortened notation $\mathrm{Ch} := \mathrm{Ch}_2$ when there is no risk of confusion.

We wish to emphasize that in general Ch is not a quadratic form and $H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ is not a Hilbert space. In particular, if $\|\cdot\|$ is a norm over $\mathbb{R}^n$, then the Cheeger energy associated to $(\mathbb{R}^n, \mathsf{d}_{\|\cdot\|}, \mathcal{L}^n)$ is quadratic if and only if the norm is induced by a scalar product. More in

general, if $(X, \mathsf{d}, \mathfrak{m})$ is the metric measure structure associated to a smooth Finsler manifold, then the Cheeger energy is quadratic if and only if the Finsler manifold is indeed Riemannian (cf. [**110**]).
Also motivated by the remarks above, Gigli introduced in [**110**] the notion of *infinitesimally Hilbertian* metric measure space (see also [**16**] where the condition was present and studied when coupled with lower Ricci curvature bounds).

Definition 1.35 (Infinitesimally Hilbertian space)**.** A metric measure space $(X, \mathsf{d}, \mathfrak{m})$ is said to be infinitesimally Hilbertian if the associated Cheeger energy is a quadratic form on $L^2(X, \mathfrak{m})$.

Remark 1.36. Since Ch is convex and 2-homogeneous the quadraticity is equivalent to ask for the validity of the parallelogram rule

$$\mathrm{Ch}(f+g) + \mathrm{Ch}(f-g) = 2\mathrm{Ch}(f) + 2\mathrm{Ch}(g), \qquad \text{for any } f, g \in L^2(X, \mathfrak{m}). \tag{1.23}$$

Furthermore, another equivalent condition amounts to ask that the (a priori only) Banach space $H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ is a Hilbert space.

Next we recall that the global assumption about $H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ being a Hilbert space has a series of nontrivial outcomes. We refer to [**45, 105**] for the basic terminology about Dirichlet forms, that will not play an explicit role in the development of the thesis.

Theorem 1.37. *If* Ch *is quadratic then we can introduce a symmetric bilinear operator* $\Gamma : H^{1,2}(X, \mathsf{d}, \mathfrak{m}) \times H^{1,2}(X, \mathsf{d}, \mathfrak{m}) \to L^1(X, \mathfrak{m})$ *by*

$$\Gamma(f, g) \coloneqq \lim_{\varepsilon \to 0} \frac{|\nabla(f + \varepsilon g)|^2 - |\nabla f|^2}{2\varepsilon},$$

where the limit is understood in $L^1(X, \mathfrak{m})$*. Moreover,* Γ *is a symmetric bilinear form and*

$$\mathcal{E}(f_1, f_2) \coloneqq \int_X \Gamma(f_1, f_2) \, \mathrm{d}\mathfrak{m}, \qquad \textit{for all } f_1, f_2 \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$$

defines a strongly local Dirichlet form.

In the rest of the thesis we shall adopt the notation $\nabla f \cdot \nabla g$ to indicate $\Gamma(f, g)$.
Most of the standard calculus rules can be proved when dealing with minimal relaxed gradients, let us list here the most relevant ones.

Locality on Borel sets. For any $f, g \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ it holds that $|\nabla f| = |\nabla g|$ $\mathfrak{m}$-a.e. on $\{f = g\}$.

Chain rule. For any $f \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ and for any $\phi \in \mathrm{Lip}(\mathbb{R})$ with $\phi(0) = 0$ it holds $|\nabla(\phi \circ f)| = |\phi'(f)| \, |\nabla f|$.

Leibniz rule. If $f, g \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ and $h \in \mathrm{Lip_b}(X, \mathsf{d})$, then

$$\nabla f \cdot \nabla(gh) = h \nabla f \cdot \nabla g + g \nabla f \cdot \nabla h \quad \mathfrak{m}\text{-a.e. in } X. \tag{1.24}$$

Thanks to the locality of the minimal relaxed gradient we introduce in the standard way the local Sobolev spaces on open domains of X.
Given an open set $\Omega \subseteq X$, we define $H^{1,2}_{\mathrm{loc}}(\Omega, \mathsf{d}, \mathfrak{m})$ as the space of all those $f \in L^2_{\mathrm{loc}}(\Omega, \mathfrak{m})$ such that $\eta f \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ holds for every $\eta \in \mathrm{Lip_c}(\Omega)$. Thanks to the locality property of the minimal relaxed slope, it makes sense to define, through an exhaustion procedure, $|\nabla f| \in L^2_{\mathrm{loc}}(\Omega, \mathfrak{m})$ as

$$|\nabla f| \coloneqq |\nabla(\eta f)| \;\; \mathfrak{m}\text{-a.e. on } \{\eta = 1\}, \qquad \text{for any } \eta \in \mathrm{Lip_c}(\Omega).$$

Moreover, we can argue that $H^{1,2}(\Omega, \mathsf{d}, \mathfrak{m})$ coincides with the space of all $f \in H^{1,2}_{\mathrm{loc}}(\Omega, \mathsf{d}, \mathfrak{m})$ such that $f, |\nabla f| \in L^2(\Omega, \mathfrak{m})$.
Eventually, given an open domain $\Omega \subset X$ we define the space $H^{1,2}_0(\Omega, \mathsf{d}, \mathfrak{m})$ as the closure of $\mathrm{Lip_c}(\Omega, \mathsf{d})$ in $H^{1,2}(X, \mathsf{d}, \mathfrak{m})$.

To conclude this overview about Sobolev calculus we introduce the notion of function of bounded variation on a metric measure space $(X, \mathsf{d}, \mathfrak{m})$, following [**11**].

Definition 1.38 (Function of bounded variation)**.** A function $f \in L^1(X, \mathfrak{m})$ is said to belong to the space $\mathrm{BV}(X, \mathsf{d}, \mathfrak{m})$ if there exist locally Lipschitz functions f_i converging to f in $L^1(X, \mathfrak{m})$ such that

$$\limsup_{i\to\infty} \int_X \mathrm{lip}(f_i)\,\mathrm{d}\mathfrak{m} < \infty.$$

By localizing this construction one can define

$$|Df|(A) \coloneqq \inf\left\{\liminf_{i\to\infty}\int_A \mathrm{lip}(f_i)\,\mathrm{d}\mathfrak{m} : f_i \in \mathrm{Lip}_{\mathrm{loc}}(A), \quad f_i \to f \text{ in } L^1(A, \mathfrak{m})\right\}$$

for any open $A \subset X$. In [**11**] (see also [**169**] for the case of locally compact spaces) it is proven that this set function is the restriction to open sets of a finite Borel measure that we call *total variation of* f and still denote $|Df|$.

Dropping the global integrability condition on $f = \chi_E$, let us recall now the analogous definition of set of finite perimeter in a metric measure space (see again [**5, 11, 169**]).

Definition 1.39 (Perimeter and sets of finite perimeter)**.** Given a Borel set $E \subset X$ and an open set A the perimeter $\mathrm{Per}(E, A)$ is defined in the following way:

$$\mathrm{Per}(E, A) \coloneqq \inf\left\{\liminf_{n\to\infty}\int_A \mathrm{lip}(u_n)\,\mathrm{d}\mathfrak{m} : u_n \in \mathrm{Lip}_{\mathrm{loc}}(A), \quad u_n \to \chi_E \quad \text{in } L^1_{\mathrm{loc}}(A, \mathfrak{m})\right\}.$$

We say that E has finite perimeter if $\mathrm{Per}(E, X) < +\infty$. In that case it can be proved that the set function $A \mapsto \mathrm{Per}(E, A)$ is the restriction to open sets of a finite Borel measure $\mathrm{Per}(E, \cdot)$ defined by

$$\mathrm{Per}(E, B) \coloneqq \inf\{\mathrm{Per}(E, A) : B \subset A,\ A \text{ open}\}.$$

Let us remark for the sake of clarity that $E \subset X$ with finite $\mathfrak{m}$-measure is a set of finite perimeter if and only if $\chi_E \in \mathrm{BV}(X, \mathsf{d}, \mathfrak{m})$ and that $\mathrm{Per}(E, \cdot) = |D\chi_E|(\cdot)$. We will use both the notations $\mathrm{Per}(E, \cdot)$ and $|D\chi_E|(\cdot)$ in the rest of the thesis.

In the following we will say that $E \subset X$ is a set of locally finite perimeter if χ_E is a function of locally bounded variation, that is to say $\eta\chi_E \in \mathrm{BV}(X, \mathsf{d}, \mathfrak{m})$ for any $\eta \in \mathrm{Lip}_{\mathrm{bs}}(X, \mathsf{d})$.

The following coarea formula for functions of bounded variation on metric measure spaces is taken from [**169**, Proposition 4.2], dealing with locally compact spaces and its proof works in the more general setting of metric measure spaces.

Theorem 1.40 (Coarea formula)**.** *Let $v \in \mathrm{BV}(X, \mathsf{d}, \mathfrak{m})$. Then, $\{v > r\}$ has finite perimeter for $\mathcal{L}^1$-a.e. $r \in \mathbb{R}$ and, for any Borel function $f : X \to [0, \infty]$, it holds*

$$\int_X f\,\mathrm{d}\,|Dv| = \int_{-\infty}^{+\infty}\left(\int_X f\,\mathrm{d}\mathrm{Per}(\{v > r\}, \cdot)\right)\mathrm{d}r. \tag{1.25}$$

By applying the coarea formula to the distance function we obtain immediately that, given $x \in X$, the ball $B_r(x)$ has finite perimeter for $\mathcal{L}^1$-a.e. $r > 0$, and in the sequel this fact will also be used in the quantitative form provided by (1.25). We also recall (see for instance [**4, 5**]) that sets of locally finite perimeter are an algebra, more precisely $\mathrm{Per}(E, B) = \mathrm{Per}(X \setminus E, B)$ and

$$\mathrm{Per}(E \cap F, B) + \mathrm{Per}(E \cup F, B) = \mathrm{Per}(E, B) + \mathrm{Per}(F, B).$$

We will need also the following localized version of the coarea formula, which is an easy consequence of [**169**, Remark 4.3].

Corollary 1.41. *Let $v \in \mathrm{BV}(X, \mathsf{d}, \mathfrak{m})$ be continuous and non-negative. Then, for any Borel function $f : X \to [0, \infty]$, it holds that $|Dv|(\{v = t\}) = 0$ for every $t \in [0, \infty)$ and*

$$\int_{\{s<v<t\}} f \,\mathrm{d}\,|Dv| = \int_s^t \left(\int_X f \,\mathrm{dPer}(\{v > r\}, \cdot) \right) \mathrm{d}r, \qquad 0 \le s < t < \infty. \tag{1.26}$$

After the introduction of the space of functions of bounded variation, it is natural to let $W^{1,1}(X, \mathsf{d}, \mathfrak{m})$ be the space of those functions $f \in \mathrm{BV}(X, \mathsf{d}, \mathfrak{m})$ with the following property: there exists $|\nabla f|_* \in L^1(X, \mathfrak{m})$ such that $|Df| = |\nabla f|_* \mathfrak{m}$. The space $W^{1,1}(X, \mathsf{d}, \mathfrak{m})$ endowed with the norm $\|f\|_{W^{1,1}} \coloneqq \|f\|_{L^1} + \||\nabla f|_*\|_{L^1}$ is a Banach space.

In an analogous way one can define the space $W^{1,1}_{\mathrm{loc}}(X, \mathsf{d}, \mathfrak{m})$, exploiting the strong locality of the relaxed gradient $|\nabla f|_*$ for $f \in W^{1,1}(X, \mathsf{d}, \mathfrak{m})$.

1.5.2. *Laplacian and heat flow.* Next we review the notion of *heat flow* on a metric measure space. Let us recall that, thanks to the Komura-Brezis theory (cf. [**46**]), any lowersemicontinuous convex functional on a Hilbert space admits a unique gradient flow.

Definition 1.42 (Heat flow)**.** The heat flow P_t is defined as the $L^2(X, \mathfrak{m})$-gradient flow of $\frac{1}{2}\mathrm{Ch}$.

We point out that the Brezis-Komura theory provides a continuous semigroup in $L^2(X, \mathfrak{m})$ which, under the volume growth assumption

$$\mathfrak{m}(B_r(\bar{x})) \le a e^{br^2}, \quad \text{for any } r > 0,$$

can be extended to a continuous mass preserving semigroup (still denoted by P_t) in all $L^p(X, \mathfrak{m})$ spaces, $1 \le p < \infty$. In addition, P_t preserves upper and lower bounds with constants, namely $f \le C$ $\mathfrak{m}$-a.e. (respectively $f \ge C$ $\mathfrak{m}$-a.e.) implies $P_t f \le C$ $\mathfrak{m}$-a.e. (resp. $P_t f \le C$ $\mathfrak{m}$-a.e.) for all $t \ge 0$.

For the rest of this section, unless otherwise stated, we work under the additional assumption that $(X, \mathsf{d}, \mathfrak{m})$ is infinitesimally Hilbertian.

Definition 1.43 (Laplacian)**.** The Laplacian $\Delta : D(\Delta) \to L^2(X, \mathfrak{m})$ is a densely defined linear operator whose domain consists of all functions $f \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ satisfying

$$\int hg \,\mathrm{d}\mathfrak{m} = -\int \nabla h \cdot \nabla f \,\mathrm{d}\mathfrak{m} \quad \text{for any } h \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$$

for some $g \in L^2(X, \mathfrak{m})$. The unique g with this property is denoted by Δf.[1]

More generally, we say that $f \in H^{1,2}_{\mathrm{loc}}(X, \mathsf{d}, \mathfrak{m})$ is in the domain of the measure valued Laplacian, and we write $f \in D(\boldsymbol{\Delta})$, if there exists a Radon measure μ on X such that, for every $\psi \in \mathrm{Lip_c}(X)$, it holds

$$\int \psi \,\mathrm{d}\mu = -\int \nabla f \cdot \nabla \psi \,\mathrm{d}\mathfrak{m}.$$

In this case we write $\boldsymbol{\Delta} f \coloneqq \mu$. If moreover $\boldsymbol{\Delta} f \ll \mathfrak{m}$ with density in L^2_{loc} we denote by Δf the unique function in $L^2_{\mathrm{loc}}(X, \mathfrak{m})$ such that $\boldsymbol{\Delta} f = \Delta f \mathfrak{m}$ and we write $f \in D_{\mathrm{loc}}(\Delta)$.

We will also be dealing with the local counterpart of the notion above.

Definition 1.44. A function $f \in H^{1,2}(\Omega, \mathsf{d}, \mathfrak{m})$ belongs to $D(\Delta, \Omega)$ if there exists $g \in L^2(\Omega, \mathfrak{m})$ satisfying

$$\int_\Omega \nabla f \cdot \nabla h \,\mathrm{d}\mathfrak{m} = -\int_\Omega f\, g \,\mathrm{d}\mathfrak{m} \quad \text{for any } h \in H^{1,2}_0(\Omega, \mathsf{d}, \mathfrak{m}).$$

With a slight abuse of notation we write $\Delta f = g$ in Ω.

[1]The linearity of Δ follows from the quadraticity of Ch.

It is easily seen that, if $f \in D(\Delta, \Omega)$ and $\eta \in \mathrm{Lip_c}(\Omega, \mathsf{d}) \cap D(\Delta)$, $\Delta\eta \in L^\infty(X, \mathfrak{m})$ then $\eta f \in D(\Delta)$.

Having introduced the notion of Laplacian, let us point out that the heat flow can equivalently be characterized by saying that for any $u \in L^2(X, \mathfrak{m})$ the curve $t \mapsto P_t u \in L^2(X, \mathfrak{m})$ is locally absolutely continuous in $(0, +\infty)$ and satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} P_t u = \Delta P_t u \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in (0, +\infty).$$

Under our assumptions the heat flow provides a linear, continuous and self-adjoint contraction semigroup in $L^2(X, \mathfrak{m})$. Moreover, it is easily seen that

$$\lim_{t \to 0} P_t f = f \quad \text{strongly in } H^{1,2} \text{ for all } f \in H^{1,2}(X, \mathsf{d}, \mathfrak{m}).$$

We shall also extensively use the typical regularizing properties

$$P_t f \in H^{1,2}(X, \mathsf{d}, \mathfrak{m}) \ \text{ for all } f \in L^2(X, \mathfrak{m}),\ t > 0 \text{ and } \mathrm{Ch}(P_t f) \le \frac{\|f\|^2_{L^2(X,\mathfrak{m})}}{2t}, \tag{1.27}$$

$$P_t f \in D(\Delta) \ \text{ for all } f \in L^2(X, \mathfrak{m}),\ t > 0 \text{ and } \|\Delta P_t f\|^2_{L^2(X,\mathfrak{m})} \le \frac{\|f\|^2_{L^2(X,\mathfrak{m})}}{t^2}, \tag{1.28}$$

as well as the commutation rule $P_t \circ \Delta = \Delta \circ P_t$, valid for any $t > 0$.

1.5.3. *Derivations and their regularity.* The concept of derivation on a metric measure space has been introduced and deeply studied in [**213**] (see also [**214**]). In more recent papers [**89**, **112**] derivations have proved to be a natural tool for the development of a differential calculus in metric measure spaces.

Definition 1.45 (Derivation)**.** Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space. Then a *derivation* on X is a linear map $\boldsymbol{b} : \mathrm{Lip_{bs}}(X) \to L^0(\mathfrak{m})$ such that the following properties are satisfied:

i) Leibniz rule. $\boldsymbol{b}(fg) = \boldsymbol{b}(f)g + f\boldsymbol{b}(g)$ for every $f, g \in \mathrm{Lip_{bs}}(X)$.

ii) Weak locality. There exists $G \in L^0(\mathfrak{m})$ such that

$$|\boldsymbol{b}(f)| \le G \,\mathrm{lip}_a(f) \quad \mathfrak{m}\text{-a.e.} \qquad \text{for every } f \in \mathrm{Lip_{bs}}(X).$$

The least function G (in the $\mathfrak{m}$-a.e. sense) with this property is denoted by $|\boldsymbol{b}|$.

The space of all derivations on X is denoted by $\mathrm{Der}(X)$. Given any derivation $\boldsymbol{b} \in \mathrm{Der}(X)$, we define its *support* $\mathrm{supp}(\boldsymbol{b}) \subset X$ as the essential closure of $\{|\boldsymbol{b}| \neq 0\}$. For any open set $U \subset X$, we write $\mathrm{supp}(\boldsymbol{b}) \Subset U$ if $\mathrm{supp}(\boldsymbol{b})$ is bounded and $\mathrm{dist}(\mathrm{supp}(\boldsymbol{b}), X \setminus U) > 0$. Given any $\boldsymbol{b} \in \mathrm{Der}(X)$ with $|\boldsymbol{b}| \in L^1_{\mathrm{loc}}(X)$, we say that $\mathrm{div}(\boldsymbol{b}) \in L^p(\mathfrak{m})$ – for some exponent $p \in [1, \infty]$ – provided there exists a function $h \in L^p(\mathfrak{m})$ such that

$$-\int \boldsymbol{b}(f)\, \mathrm{d}\mathfrak{m} = \int f h \,\mathrm{d}\mathfrak{m} \qquad \text{for every } f \in \mathrm{Lip_{bs}}(X). \tag{1.29}$$

The function h is uniquely determined, thus it can be unambiguously denoted by $\mathrm{div}(\boldsymbol{b})$. We set

$$\mathrm{Der}^p(X) := \{\boldsymbol{b} \in \mathrm{Der}(X) \mid |\boldsymbol{b}| \in L^p(\mathfrak{m})\},$$
$$\mathrm{Der}^{p,p}(X) := \{\boldsymbol{b} \in \mathrm{Der}^p(X) \mid \mathrm{div}(\boldsymbol{b}) \in L^p(\mathfrak{m})\}$$

for any $p \in [1, \infty]$. The set $\mathrm{Der}^p(X)$ is a Banach space if endowed with the norm $\|\boldsymbol{b}\|_p := \||\boldsymbol{b}|\|_{L^p(\mathfrak{m})}$.

Remark 1.46. We claim that for every $\boldsymbol{b} \in \mathrm{Der}^{p,p}(X)$ – where $p \in [1, \infty]$ – it holds that

$$\mathrm{supp}\big(\mathrm{div}(\boldsymbol{b})\big) \subset \mathrm{supp}(\boldsymbol{b}). \tag{1.30}$$

In order to prove it, fix any open bounded subset U of $X \setminus \mathrm{supp}(\boldsymbol{b})$. Then formula (1.29) guarantees that $\int f \,\mathrm{div}(\boldsymbol{b})\, \mathrm{d}\mathfrak{m} = -\int \boldsymbol{b}(f)\, \mathrm{d}\mathfrak{m} = 0$ for every $f \in \mathrm{Lip}_{\mathrm{bs}}(U)$, whence accordingly $\mathrm{div}(\boldsymbol{b}) = 0$ holds $\mathfrak{m}$-a.e. on U. By arbitrariness of U, we conclude that (1.30) is verified.

In [**89**] it has been proven that the action of any derivation in $\mathrm{Der}^{2,2}(\mathrm{X})$ can be extended in a unique way from $\mathrm{Lip}_{\mathrm{bs}}$ to $H^{1,2}(X)$ and even to $H^{1,2}_{\mathrm{loc}}(X)$ getting a linear functional

$$\boldsymbol{b} : H^{1,2}_{\mathrm{loc}}(X, \mathsf{d}, \mathfrak{m}) \to L^1_{\mathrm{loc}}(X, \mathfrak{m}),$$

such that $|\boldsymbol{b}(f)| \leq |\boldsymbol{b}|\, |\nabla f|$ holds true $\mathfrak{m}$-a.e. on X for any $f \in H^{1,2}_{\mathrm{loc}}(X)$.

Next we introduce a class of more regular derivations for which, on $\mathrm{RCD}(K,\infty)$ metric measure spaces, Ambrosio-Trevisan have been able to prove existence and uniqueness for the associated ODE problem. We postpone a detailed discussion about this issue to Section 3.2 introducing only here the relevant class of derivations with *deformation* in L^2.

The next definition, taken from [**30**], is the natural extension of Bakry's weak definition of Hessian [**37**].
We shall denote by $\mathbb{V}_4$ the set of those $f \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ for which $|\nabla f| \in L^4(X, \mathfrak{m})$.

Definition 1.47 (Derivation with deformation in L^2)**.** Let $\boldsymbol{b} \in \mathrm{Der}^{2,2}(\mathrm{X})$ and assume that $\mathrm{Lip}_{\mathrm{bs}}$ is dense in $\mathbb{V}_4$. We write $D^{\mathrm{sym}}\boldsymbol{b} \in L^2(X, \mathfrak{m})$ if there exists $c \geq 0$ such that

$$\left| \int D^{\mathrm{sym}}\boldsymbol{b}(f,g)\,\mathrm{d}\mathfrak{m} \right| \leq c\, \||\nabla f|\|_{L^4}\, \||\nabla g|\|_{L^4}\,, \tag{1.31}$$

for all $f, g \in \mathbb{V}_4$ with $\Delta f, \Delta g \in L^4(X, \mathfrak{m})$, where

$$\int D^{\mathrm{sym}}\boldsymbol{b}(f,g)\,\mathrm{d}\mathfrak{m} := -\frac{1}{2}\int \left[\boldsymbol{b}(f)\Delta g + \boldsymbol{b}(g)\Delta f - \mathrm{div}\,\boldsymbol{b}\,(\nabla f \cdot \nabla g)\right]\,\mathrm{d}\mathfrak{m}. \tag{1.32}$$

Moreover, we let $\|D^{\mathrm{sym}}\boldsymbol{b}\|_2$ be the smallest c in (1.31).

1.6. The theory of normed modules. We briefly review the theory of tangent modules (and normed modules, more in general) in the case of infinitesimally Hilbertian metric measure spaces. This theory has been developed by Gigli in [**112**] inspired by previous works by Weaver [**213**]. Here we follow the simplified presentation of [**20**], where there are minor simplifications with respect to the general case, thanks to the Hilbertian assumption. We remark that the original approach in [**112**] starts from the construction of L^2 sections of the cotangent bundle to recover via duality the tangent module.

Let R be either $L^\infty(\mathfrak{m})$ or $L^0(\mathfrak{m})$. Let $\mathscr{M}$ be a module over the commutative ring R. Then an L^p*-pointwise norm* on $\mathscr{M}$, for some $p \in \{0\} \cup [1,\infty)$, is any mapping $|\cdot| : \mathscr{M} \to L^p(\mathfrak{m})$ such that

$$\begin{aligned} |v| &\geq 0 && \text{for every } v \in \mathscr{M}, \text{ with equality if and only if } v = 0,\\ |v+w| &\leq |v| + |w| && \text{for every } v, w \in \mathscr{M},\\ |fv| &= |f||v| && \text{for every } f \in R \text{ and } v \in \mathscr{M}, \end{aligned} \tag{1.33}$$

where all (in)equalities are in the $\mathfrak{m}$-a.e. sense. We shall consider two classes of normed modules:

- $L^p(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-modules, with $p \in [1,\infty)$. A module $\mathscr{M}^p$ over $L^\infty(\mathfrak{m})$ endowed with an L^p-pointwise norm $|\cdot|$ such that $\|v\|_{\mathscr{M}^p} := \||v|\|_{L^p(\mathfrak{m})}$ is a complete norm on $\mathscr{M}^p$.
- $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-modules. A module $\mathscr{M}^0$ over $L^0(\mathfrak{m})$ endowed with an L^0-pointwise norm $|\cdot|$ such that $\mathsf{d}_{\mathscr{M}^0}(v,w) := \int \min\{|v-w|, 1\}\,\mathrm{d}\mathfrak{m}'$ (where $\mathfrak{m}'$ is any probability measure that is mutually absolutely continuous with $\mathfrak{m}$) is a complete distance on $\mathscr{M}^0$.

We also recall a variant of the notion of L^0-normed L^0-module – where the Borel measure $\mathfrak{m}$ is replaced by the capacity – which has been proposed in [**88**] and will be relevant for the development of the theory of sets of finite perimeter in Chapters 4 and 5.

Fix a metric measure space $(X, \mathsf{d}, \mathfrak{m})$. The space of all Borel functions on X – considered up to Cap-a.e. equality – is denoted by $L^0(\mathrm{Cap})$. If continuous functions are strongly dense in $H^{1,2}(X)$ (this condition is met, for instance, if the space is infinitesimally Hilbertian), then there exists a unique "quasi-continuous representative" map $\mathsf{QCR} : H^{1,2}(X) \to L^0(\mathrm{Cap})$ that is characterized as follows: QCR is a continuous map, and for any $f \in H^{1,2}(X)$ it holds that $\mathsf{QCR}(f)$ is (the equivalence class of) a quasi-continuous function that is $\mathfrak{m}$-a.e. coinciding with f itself. Let us recall that a function $f : X \to \mathbb{R}$ is said to be quasi-continuous if for any $\varepsilon > 0$ there exists a set $E \subset X$ with $\mathrm{Cap}(E) < \varepsilon$ such that $f : X \setminus E \to \mathbb{R}$ is continuous. We refer to [**88**, Theorem 1.20] for a proof of this result and to [**149**] for a previous approach.

Given a module $\mathscr{M}_{\mathrm{Cap}}$ over the ring $L^0(\mathrm{Cap})$, we say that a mapping $|\cdot| : \mathscr{M}_{\mathrm{Cap}} \to L^0(\mathrm{Cap})$ is a *pointwise norm* provided it satisfies the (in)equalities in (1.33) in the Cap-a.e. sense for any choice of $v, w \in \mathscr{M}_{\mathrm{Cap}}$ and $f \in L^0(\mathrm{Cap})$. Then the space $\mathscr{M}_{\mathrm{Cap}}$ is said to be an $L^0(\mathrm{Cap})$*-normed* $L^0(\mathrm{Cap})$*-module* if it is complete when endowed with the distance

$$\mathsf{d}_{\mathscr{M}_{\mathrm{Cap}}}(v, w) := \sum_{k\in\mathbb{N}} \frac{1}{2^k \max\{\mathrm{Cap}(A_k), 1\}} \int_{A_k} \min\{|v - w|, 1\} \,\mathrm{dCap},$$

where $(A_k)_k$ is any increasing sequence of open subsets of X having finite capacity that is chosen in such a way that any bounded set $B \subset X$ is contained in A_k for some $k \in \mathbb{N}$ sufficiently big.

The starting point of Gigli's construction of the tangent module is provided by the formal expressions $\{(A_i, \nabla f_i)\}_{i\in I}$, where I is a finite index set, $\{A_i\}_{i\in I}$ is a $\mathfrak{m}$-measurable partition of X and $f_i \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$.

The sum of two families $\{(A_i, \nabla f_i)\}_{i\in I}, \{(B_j, \nabla g_i)\}_{j\in J}$ is $\{A_i \cap B_j, \nabla(f_i + g_j)\}_{(i,j)\in I\times J}$ and multiplication by $\mathfrak{m}$-measurable functions χ taking finitely many values is defined by

$$\chi\{(E_i, \nabla f_i)\}_{i\in I} = \{(E_i \cap F_j, \nabla(z_j f_i))\}_{(i,j)\in I\times J}, \tag{1.34}$$

where $\chi = \sum z_j \chi_{F_j}$.

Two families $\{(A_i, \nabla f_i)\}_{i\in I}, \{(B_j, \nabla g_i)\}_{j\in J}$ are said to be equivalent if $f_i = g_j$ $\mathfrak{m}$-a.e. on $A_i \cap B_j$ for all $(i, j) \in I \times J$ and one works with the vector space M of these equivalence classes, since the above defined operations are compatible with the equivalence relation.

We let the pointwise norm $|\{(A_i, \nabla f_i)\}| \in L^2(X, \mathfrak{m})$ of $\{(A_i, \nabla f_i)\}$ be $|\{(A_i, \nabla f_i)\}|(x) := |\nabla f_i|(x)$ $\mathfrak{m}$-a.e. on A_i. Thanks to the locality properties of the minimal relaxed slope, this definition does not depend on the choice of the representative and satisfies $|\chi\{(A_i, \nabla f_i)\}| = |\chi|\,|\{(A_i, \nabla f_i)\}|$ whenever χ takes finitely many values. This way, all the properties of $L^2(X, \mathfrak{m})$-normed modules are satisfied, with the only difference that multiplication is defined only for functions $\chi \in L^\infty(X, \mathfrak{m})$ having finitely many values. By completion of M with respect to the norm $\left(\int_X |\{A_i, f_i\}|^2 \,\mathrm{d}\mathfrak{m}\right)^{1/2}$ we obtain the normed module $L^2(T(X, \mathsf{d}, \mathfrak{m}))$ that we shall also denote by $L^2(TX)$ when there is no risk of confusion.

In the sequel we shall denote by V, W , etc. the typical elements of $L^2(T(X, \mathsf{d}, \mathfrak{m}))$ and by $|V|$ the pointwise norm. We start using a more intuitive notation ∇f for (the equivalence class of) $\{(X, \nabla f)\}$ and expressions like finite sums $\sum_i \chi_i \nabla f_i$

As a consequence of the very construction of the tangent module one can check that the family of finite sums

$$\sum_i \chi_i \nabla f_i, \tag{1.35}$$

where $\chi_i \in L^\infty(X,\mathfrak{m})$ and $f_i \in H^{1,2}(X,\mathsf{d},\mathfrak{m})$ is dense in $L^2(TX)$. More in general this is true whenever the functions χ are allowed to vary in a subset of $L^2(X,\mathfrak{m})\cap L^\infty(X,\mathfrak{m})$ stable with respect to truncations and dense in $L^2(X,\mathfrak{m})$.

It is easy to check that, under our assumptions, $L^2(TX)$ equipped with the norm $\|\cdot\|$ is a Hilbert space. Moreover its pointwise norm satisfies a pointwise parallelogram rule. By polarization one can then introduce a pointwise scalar product

$$L^2(TX) \ni V, W \mapsto V\cdot W \in L^1(X,\mathfrak{m}) \tag{1.36}$$

which we might think of as the *metric tensor* of the space. It can also be verified that, whenever $f,g \in H^{1,2}(X,\mathsf{d},\mathfrak{m})$ the scalar product of the tangent module $L^2(TX)$ coincides with the Carré du champ $\Gamma(f,g)$ introduced via Sobolev calculus, therefore justifying the use of the same notation.

Elements of the tangent module shall be referred to as *vector fields.* A notion of divergence can be introduced via integration by parts also in this context.

Definition 1.48 (Divergence of a vector field)**.** Let $(X,\mathsf{d},\mathfrak{m})$ be an infinitesimally Hilbertian m.m.s. and let $V \in L^2(TX)$. Then we say that V has divergence in L^2 (and write $V \in D(\mathrm{div}\,)$) if there exists $g \in L^2(X,\mathfrak{m})$ such that

$$\int_X V\cdot\nabla f\,\mathrm{d}\mathfrak{m} = -\int_X fg\,\mathrm{d}\mathfrak{m},$$

for any $f \in H^{1,2}(X,\mathsf{d},\mathfrak{m})$. It is easily seen that g is uniquely determined, if it exists and we shall denote it by $\mathrm{div}\,V$.

We shall indicate by $L^2(T^*(X,\mathsf{d},\mathfrak{m}))$ the cotangent module of $(X,\mathsf{d},\mathfrak{m})$ that we define as the dual of $L^2(T(X,\mathsf{d},\mathfrak{m}))$. The shortened notation $L^2(T^*X)$ will be adopted whenever there is no risk of confusion.

Remark 1.49. Under the additional infinitesimal Hilbertianity assumption one can introduce also an L^0*-version* of the tangent module $L^0(TX)$ which is characterized as follows: there is a unique couple $(L^0(TX),\nabla)$, where $L^0(TX)$ is an $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module and $\nabla : H^{1,2}(X)\to L^0(TX)$ is a linear *gradient* map, such that the following hold:

$$|\nabla f| \text{ coincides with the minimal relaxed slope of } f \text{ for every } f\in H^{1,2}(X),$$

$$\left\{ \sum_{i=1}^n \chi_{E_i}\nabla f_i \;\middle|\; (E_i)_{i=1}^n \text{ Borel partition of } X,\ (f_i)_{i=1}^n\subset H^{1,2}(X)\right\} \text{ is dense in } L^0(TX).$$

It can be readily checked that $L^2(TX) \coloneqq \{v\in L^0(TX)\,:\,|v|\in L^2(\mathfrak{m})\}$.

Remark 1.50. In the general situation (without infinitesimal Hilbertianity) there is a canonical *differential* operator $\mathrm{d}: H^{1,2}(X,\mathsf{d},\mathfrak{m})\to L^2(T^*X)$ such that $|\,\mathrm{d}f| = |\nabla f|$ $\mathfrak{m}$-a.e. for any $f\in H^{1,2}(X,\mathsf{d},\mathfrak{m})$, where the modulus at the left handside stands for the pointwise norm of $L^2(T^*X)$ while the modulus at the right handside stands for the minimal relaxed gradient. Recall, moreover, that the tangent bundle $L^2(TX)$ is defined to be the dual of the cotangent bundle in the general case.

The following useful result has been established as an intermediate step in the proof of [**90**, Proposition 6.5]. It allows to identify vector fields in $L^2(TX)$ with a suitable subclass of the space of derivations. The identification can be pushed up to the notion of divergence.

Proposition 1.51. *Let* $(X,\mathsf{d},\mathfrak{m})$ *be an infinitesimally Hilbertian metric measure space. Let us denote by* $\overline{\mathbb{D}}$ *the closure in* $\mathrm{Der}^2(X)$ *of the pre-Hilbert space* $\mathbb{D}\coloneqq(\mathrm{Der}^{2,2}(X),\|\cdot\|_2)$*. Then* $\overline{\mathbb{D}}$ *has a natural structure of Hilbert* $L^2(\mathfrak{m})$*-normed* $L^\infty(\mathfrak{m})$*-module and the map* $A\colon L^2(TX)\to\overline{\mathbb{D}}$*, defined as*

$$A(v)(f) \coloneqq v\cdot\nabla f \quad \textit{for every } v\in L^2(TX) \textit{ and } f\in\mathrm{Lip}_{\mathrm{bs}}(X),$$

is a normed module isomorphism between $L^2(TX)$ *and* $\overline{\mathbb{D}}$*. Moreover, it holds* $A(D(\mathrm{div})) = \mathbb{D}$ *and*

$$\mathrm{div}(A(v)) = \mathrm{div}(v) \quad \textit{for every } v \in D(\mathrm{div}).$$

Remark 1.52. In view of Proposition 1.51 in the rest of the thesis we will be dealing mainly with vector fields, even though some of the results we are going to present have originally been formulated with the language of derivations.

We proceed reviewing the basic terminology about the so-called *dimension* of the tangent module $L^2(TX)$. The present discussion is taken from [**112**, **133**].

Given a Borel set $A \subset X$ we denote the subset of $L^2(TX)$ consisting of those v such that $\chi_{A^c} v = 0$ by $L^2(TX)|_A$.

Definition 1.53 (Local independence)**.** Let $A \subset X$ be a Borel set with positive measure. We say that $\{v_i\}_{i \in I} \subset L^2(TX)$ is independent on A if

$$\sum_i f_i v_i = 0, \quad \mathfrak{m}\text{-a.e. on } A$$

holds if and only if $f_i = 0$ $\mathfrak{m}$-a.e. on A for each $i \in I$.

Definition 1.54 (Local span and generators)**.** Let $A \subset X$ be a Borel set and $V \coloneqq \{v_i\}_{i \in I} \subset L^2(TX)$. The span of V in A, denoted by $\mathrm{span}_A(V)$, is the subset of those elements of $L^2(TX)|_A$ with the following property: there exist a Borel decomposition $\{A_n\}_{n \in \mathbb{N}}$ of A and families of vectors $\{v_{i,n}\}_{i=1}^{m_n} \subset V$ and functions $\{f_{i,n}\}_{i=1}^{m_n} \subset L^\infty(X, \mathfrak{m})$, for $n \in \mathbb{N}$ such that

$$\chi_{A_n} v = \sum_{i=1}^{m_n} f_{i,n} v_{i.n}$$

for any n. We call the closure of $\mathrm{span}_A(V)$ the space generated by V on A.

We say that $L^2(TX)$ is finitely generated if there is a finite family $v_1, \dots, v_n$ spanning $L^2(TX)$ and locally finitely generated if there is a Borel partition$(E_i)_{i \in \mathbb{N}}$ of X such that $L^2(TX)|_{E_i}$ is finitely generated for every $i \in \mathbb{N}$.

Definition 1.55 (Local basis and dimension)**.** We say that a finite set $v_1, \dots, v_n$ is a basis on a Borel set A if it is independent on A and $\mathrm{span}_A\{v_1, \dots, v_n\} = L^2(TX)|_A$. If $L^2(TX)$ has a basis of cardinality n on A then we say that it has dimension n on A or that its local dimension is n on A.

It can be proved (cf. [**112**, Proposition 1.4.4]) that the definition of basis and local dimension are well-posed.

Remark 1.56. Among the powerful consequences of [**62**] there is the fact that for any metric measure space $(X, \mathsf{d}, \mathfrak{m})$ doubling and verifying a local Poincaré inequality the tangent module $L^2(TX)$ is finitely generated.

We refer to [**112**, Section 1.5] for the notion of tensor product of Hilbert modules. The only applications which will be relevant for the sake of this thesis are to the cases of the tensor product of the cotangent module $L^2(T^*X)$ with itself, that we shall denote by $L^2\left((T^*)^{\otimes^2} X\right)$, and of the tensor product of the tangent module with itself, that we shall indicate as $L^2\left((T)^{\otimes^2} X\right)$. We will indicate by $|\cdot|_{\mathrm{HS}}$ the associated pointwise norm, with the subscript HS standing for *Hilbert-Schmidt*.

Remark 1.57. The adoption of the terminology Hilbert-Schmidt for the pointwise norm of the tensor product module of the cotangent module with itself is motivated by the following observation: when the base space is the Euclidean space, the abstract construction gives rise to the space of matrix valued fields, the pointwise norm being the Hilbert-Schmidt norm on matrixes.

2. Curvature dimension conditions

Let us introduce the relevant curvature-dimension conditions for the development of the thesis, starting from the infinite dimensional case, the basic references being the seminal papers by Sturm [**200**, **201**]and Lott-Villani [**163**].

In the sequel we shall always assume that the metric measure space $(X, \mathsf{d}, \mathfrak{m})$ verifies the volume growth assumption

$$\mathfrak{m}(B_r(\bar{x})) \le a e^{br^2}, \quad \text{for some } x \in X,\ a, b \ge 0 \text{ and for all } r > 0. \tag{1.37}$$

Definition 1.58 (Logarithmic entropy)**.** We introduce the *relative entropy* functional $\mathrm{Ent}_{\mathfrak{m}} : \mathcal{P}_2(X) \to (-\infty, +\infty]$ by

$$\mathrm{Ent}_{\mathfrak{m}}(\mu) := \begin{cases} \int_X \rho \log \rho, & \textit{if } \mu = \rho\mathfrak{m}, \\ +\infty, & \text{otherwise.} \end{cases} \tag{1.38}$$

Observe that the good definition of the relative entropy follows combining the assumption $\mu \in \mathcal{P}_2(X)$ with (1.37).

Definition 1.59 ($\mathrm{CD}(K,\infty)$ condition)**.** We say that a metric measure space $(X, \mathsf{d}, \mathfrak{m})$ verifies the $\mathrm{CD}(K,\infty)$ condition if $\mathrm{Ent}_{\mathfrak{m}}$ is geodesically K-convex on $(\mathcal{P}_2(X), W_2)$, that is to say for any $\mu_0, \mu_1 \in D(\mathrm{Ent})$ there exists $(\mu_s)_{s\in[0,1]} \in \mathrm{Geo}(\mathcal{P}_2(X))$ joining μ_0 with μ_1 and such that

$$\mathrm{Ent}_{\mathfrak{m}}(\mu_s) \le (1-s)\mathrm{Ent}_{\mathfrak{m}}(\mu_0) + s\mathrm{Ent}_{\mathfrak{m}}(\mu_1) - \frac{s(1-s)}{2} K W_2^2(\mu_0, \mu_1), \quad \text{for any } s \in [0,1].$$

Remark 1.60 (Strong curvature-dimension condition)**.** Let us remark that in the definition of m.m.s. verifying the CD condition above one asks for the convexity inequality to be satisfied along one geodesic. The condition obtained forcing the convexity inequality along *any* geodesic is referred to as *strong* curvature-dimension condition.

Let us mention a useful analytic consequence of the curvature-dimension condition. It has been proved in [**185**] that $\mathrm{CD}(K,\infty)$ metric measure spaces verify a local $(1,1)$-Poincaré inequality. That is to say

$$\int_{B_r(x)} \left| f - \fint_{B_r(x)} f \, \mathrm{d}\mathfrak{m} \right| \mathrm{d}\mathfrak{m} \le 4 r e^{|K| r^2} \int_{B_{2r}(x)} |\nabla f| \, \mathrm{d}\mathfrak{m}, \tag{1.39}$$

for any $f \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$, $x \in X$ and $r > 0$.

Next we move to the finite dimensional case with the introduction of the $\mathrm{CD}(K,N)$ condition for $K \in \mathbb{R}$ and $1 < N < +\infty$. Here we follow the presentation of [**201**], remarking that in [**163**] only the case $K = 0$ was considered.

Given $k \in \mathbb{R}$, $\theta \in [0,\infty)$ and $s \in [0,1]$, let us define $\sigma_k^{(s)} : [0,\infty) \to [-\infty, \infty]$ as follows:

$$\sigma_k^{(s)}(\theta) := \begin{cases} \frac{s_k(s\theta)}{s_k(\theta)}, & \text{if } k\theta^2 \neq 0,\ k\theta^2 < \pi^2, \\ s, & \text{if } k\theta^2 = 0, \\ +\infty, & \text{if } k\theta^2 > \pi^2, \end{cases} \tag{1.40}$$

with

$$s_k(r) := \begin{cases} \frac{\sin(\sqrt{k}r)}{\sqrt{k}}, & \text{if } k > 0, \\ r, & \text{if } k = 0, \\ \frac{\sinh(\sqrt{-k}r)}{\sqrt{-k}}, & \text{if } k < 0. \end{cases}$$

Set also

$$\tau^{(s)}_{K,N}(\theta) := s^{\frac{1}{N}} \left(\sigma^{(s)}_{K/(N-1)}(\theta)\right)^{1-\frac{1}{N}} . \tag{1.41}$$

Definition 1.61 (Rényi entropy)**.** For $N \in (1,\infty)$, the *N-Rényi relative-entropy functional* $\mathcal{E}_N : \mathcal{P}_2(X) \to [-\infty, 0]$ is defined as

$$\mathcal{E}_N(\mu) := -\int_X \rho^{1-\frac{1}{N}} \,\mathrm{d}\mathfrak{m}\,,$$

where $\mu = \rho\mathfrak{m} + \mu^\perp$ is the Lebesgue decomposition of μ with $\mu^\perp$ singular with respect to $\mathfrak{m}$.

Definition 1.62 ($\mathrm{CD}(K,N)$ spaces)**.** We say that $(X,\mathsf{d},\mathfrak{m})$ verifies the curvature dimension condition $\mathrm{CD}(K,N)$ for some $K \in \mathbb{R}$ and $1 < N < \infty$ if for all $\mu_0 = \rho_0\mathfrak{m}$, $\mu_1 = \rho_1\mathfrak{m} \in \mathcal{P}_2(X)$ with bounded support and absolutely continuous with respect to $\mathfrak{m}$ there exists $\eta \in \mathrm{Opt}(\mu_0,\mu_1)$ such that $(e_t)_\sharp\eta \ll \mathfrak{m}$ for any $t \in [0,1]$ and

$$\mathcal{E}_{N'}(\mu_s) \le -\int \left[\tau^{(1-s)}_{K,N'}(\mathsf{d}(\gamma_0,\gamma_1))\,\rho_0^{-1/N'}(\gamma(0)) + \tau^{(s)}_{K,N'}(\mathsf{d}(\gamma_0,\gamma_1))\,\rho_1^{-1/N'}(\gamma_1)\right] \mathrm{d}\eta(\gamma), \tag{1.42}$$

for any $N' \ge N$ and $s \in [0,1]$, where we set $\mu_s := (e_s)_\sharp\eta$.

Remark 1.63 (Scaling properties)**.** It can be easily argued that, if $(X,\mathsf{d},\mathfrak{m})$ is a $\mathrm{CD}(K,N)$ m.m.s. then $(X,\alpha\mathsf{d},\beta\mathfrak{m})$ is a $\mathrm{CD}(K/\alpha^2,N)$ m.m.s. and the analogous property holds in the infinite dimensional case.

Remark 1.64 (The smooth case)**.** Any smooth and possibly weighted Riemannian manifold $(M,\mathsf{d}_g,e^{-V}\mathcal{H}^n)$, where $V : M \to \mathbb{R}$ is smooth, verifies the $\mathrm{CD}(K,N)$ condition if and only if the modified Ricci tensor

$$\mathrm{Ric} + \mathrm{Hess}\, V - \frac{\nabla V \otimes \nabla V}{N-n} \tag{1.43}$$

is bounded from below by K, as a symmetric bilinear form. Here we denote by n the topological dimension of the base Riemannian manifold and we point out that in the case $N = n$ only constant weights are admitted, in which case the last term in (1.43) is intended to be 0 by definition.

Let us point out that while on smooth weighted Riemannian manifolds lower bounds on the (modified) Ricci curvature tensor are local properties (indeed they are *infinitesimal*) the analogous property is false without additional assumptions for the CD condition, as it has been pointed out in [**186**].

Motivated by the quest for better *globalization properties*, Bacher-Sturm introduced in [**35**] a variant of the curvature dimension condition, called *reduced curvature-dimension* condition $\mathrm{CD}^*(K,N)$.

Definition 1.65 (Reduced curvature-dimension condition)**.** Let $K \in \mathbb{R}$ and $N > 1$. We say that a metric measure space $(X,\mathsf{d},\mathfrak{m})$ verifies the reduced curvature dimension condition $\mathrm{CD}^*(K,N)$ if for all $\mu_0 = \rho_0\mathfrak{m}$, $\mu_1 = \rho_1\mathfrak{m} \in \mathcal{P}_2(X)$ with bounded support and absolutely continuous with respect to $\mathfrak{m}$ there exists $\eta \in \mathrm{Opt}(\mu_0,\mu_1)$ such that $(e_t)_\sharp\eta \ll \mathfrak{m}$ for any $t \in [0,1]$ and

(1.44)

$$\mathcal{E}_{N'}(\mu_s) \le -\int \left[\sigma^{(1-s)}_{K/(N'-1)}(\mathsf{d}(\gamma_0,\gamma_1))\,\rho_0^{-1/N'}(\gamma(0)) + \sigma^{(s)}_{K/(N'-1)}(\mathsf{d}(\gamma_0,\gamma_1))\,\rho_1^{-1/N'}(\gamma_1)\right] \mathrm{d}\eta(\gamma),$$

for any $N' \ge N$ and $s \in [0,1]$, where we set $\mu_s := (e_s)_\sharp\eta$.

Remark 1.66. We point out that the $\mathrm{CD}^*(K,N)$ condition is weaker than the $\mathrm{CD}(K,N)$ condition in general. If $K = 0$, then the very definition of the distortion coefficients implies that the two definitions agree. Furthermore it is possible to check that, whenever $K > 0$, any

$\mathrm{CD}^*(K,N)$ m.m.s. verifies the CD condition with the same dimension upper bound and the worsened lower Ricci curvature bound $K(N-1)/N$, see [**35**].

It was known since its introduction in [**35**], that the reduced curvature dimension condition verifies a *local to global* property and that the local version of the $\mathrm{CD}^*(K,N)$ condition is equivalent to the local version of the $\mathrm{CD}(K,N)$ condition in the non branching case. Moreover, in [**96**, Corollary 3.13, Theorem 3.14, Remark 3.26] this identification was extended to the case of essentially non branching spaces. Observe that some non branching assumption is indeed necessary for the validity of the local to global property as shown in [**186**], where Rajala provided an example of complete and geodesic metric measure space verifying the local $\mathrm{CD}(0,4)$ but failing to satisfy the $\mathrm{CD}(K,N)$ condition globally for any $K\in\mathbb{R}$ and for any $N>1$.

Later on in [**57**], Cavalletti and E. Milman proved that the two conditions are equivalent under the additional essentially non branching assumption (and assuming finiteness of the measure). Due to the local nature of their arguments it is believed that the equivalence should extend to the case of a σ-finite reference measure.

As a non trivial geometric property of $\mathrm{CD}(K,N)$ metric measure spaces, we recall that they satisfy the Bishop-Gromov inequality (see [**200, 210**]), that is to say

$$\frac{\mathfrak{m}(B_R(x))}{\mathfrak{m}(B_r(x))}\le\frac{V_{K,N}(R)}{V_{K,N}(r)}, \tag{1.45}$$

for any $0<r<R$ and for any $x\in X$, where $V_{K,N}(s)$ stands for the volume of the ball of radius s in the model space for the curvature-dimension condition $\mathrm{CD}(K,N)$. In particular, when $K\ge 0$, (1.45) implies that $(X,\mathsf{d},\mathfrak{m})$ is doubling with doubling constant 2^N, i.e.

$$\mathfrak{m}(B_{2r}(x))\le 2^N\mathfrak{m}(B_r(x))\quad\text{for any } x\in X \text{ and for any } r>0. \tag{1.46}$$

In the case of a possibly negative lower Ricci curvature bound we can achieve the weaker conclusion that $(X,\mathsf{d},\mathfrak{m})$ is locally uniformly doubling, i.e. it satisfies (1.7) for a function C depending only on K and N.
In any case, as a consequence of the local doubling property, any $\mathrm{CD}(K,N)$ m.m.s. $(X,\mathsf{d},\mathfrak{m})$ is proper.

Remark 1.67. As a consequence of the Bishop Gromov inequality, it can be proved that, on any $\mathrm{CD}(K,N)$ m.m.s. $(X,\mathsf{d},\mathfrak{m})$, spheres have vanishing measure $\mathfrak{m}$, as we already remarked in Remark 1.11.

It is worth pointing out that, among the properties satisfied by $\mathrm{CD}(K,N)$ spaces, there is the $(2,2)$-Poincaré inequality. This statement can be obtained as a direct consequence of [**129**, Theorem 5.1], relying on the local doubling property of $\mathrm{CD}(K,N)$ spaces and on the $(1,1)$-Poincaré inequality (1.39).

Remark 1.68. As a consequence of the discussion above, $\mathrm{CD}(K,N)$ spaces are PI spaces, with the terminology introduced in the discussion before Theorem 1.33.

3. $\mathrm{RCD}(\mathrm{K},\infty)$ metric measure spaces

Aim of this section is to review the basic notions about spaces verifying the *Riemannian curvature dimension* condition $\mathrm{RCD}(K,\infty)$. Their introduction dates back to [**16**], that was dealing only with the case of finite reference measure, while the theory was extended to σ-finite reference measures in [**13**] to which we refer for this presentation. Let us recall that the volume growth assumption $\mathfrak{m}(B_r(x))\le a\exp(br^2)$ will be in force throughout.

Definition 1.69 ($\mathrm{RCD}(K,\infty)$ space)**.** We say that $(X,\mathsf{d},\mathfrak{m})$ is an $\mathrm{RCD}(K,\infty)$ metric measure space for some $K\in\mathbb{R}$ if it verifies the $\mathrm{CD}(K,\infty)$ curvature dimension condition and it is infinitesimally Hilbertian according to Definition 1.35.

Remark 1.70. Let us point out that $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(K, \infty)$ space if and only if (X, d) is a length space and any $\mu \in \mathcal{P}_2(X)$ is the starting point of an EVI_K gradient flow $(\mathcal{P}_t \mu)_t$ of $\mathrm{Ent}_{\mathfrak{m}}$. We refer to [**13**, Section 6] for the proof of this result, which has played a key role in the developments of the theory and to [**14**] for a general account about the theory of gradient flows on metric spaces.

Remark 1.71. Let us remark a powerful consequence of the EVI formulation of the RCD condition. Thanks to [**80**], the existence of an EVI_K gradient flow starting from any $\mu \in \mathcal{P}_2(X)$ implies K-convexity of the relative entropy along any W_2-geodesic. It follows that $\mathrm{RCD}(K, \infty)$ spaces are strong $\mathrm{CD}(K, \infty)$ spaces.

Remark 1.72. In [**187**] it has been proved that strong $\mathrm{CD}(K, \infty)$ metric measure spaces are essentially non branching. It follows from Remark 1.71 above that $\mathrm{RCD}(K, \infty)$ spaces are essentially non branching.

The approach adopted in Definition 1.69 is Lagrangian, being based on the curvature-dimension condition. Its equivalence with a dual Eulerian approach based on Bochner inequality was one of the main accomplishments of [**17**] (see also [**16**] for one of the implications).

Theorem 1.73 (Equivalence between $\mathrm{RCD}(K, \infty)$ and $\mathrm{BE}(K, \infty)$)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space verifying the volume growth assumption $\mathfrak{m}(B_r(\bar{x})) \le a \exp(br^2)$ for some $\bar{x} \in X, a, b \in \mathbb{R}$ and for any $r > 0$. Then $(X, \mathsf{d}, \mathfrak{m})$ verifies the* $\mathrm{RCD}(K, \infty)$ *condition if and only if the following are satisfied:*

i) *$(X, \mathsf{d}, \mathfrak{m})$ is infinitesimally Hilbertian;*
ii) *$(X, \mathsf{d}, \mathfrak{m})$ verifies the* Sobolev to Lipschitz property*: any $f \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ such that $|\nabla f| \in L^\infty(X, \mathfrak{m})$ admits a Lipschitz representative $\tilde{f}$ such that* $\mathrm{Lip}(\tilde{f}) = \| |\nabla f| \|_{L^\infty}$*;*
iii) *a weak Bochner inequality is satisfied: for any $f \in D(\Delta)$ and for any $g \in D(\Delta) \cap L^\infty(X, \mathfrak{m})^+$ such that $\Delta f \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ and $\Delta g \in L^\infty(X, \mathfrak{m})$ it holds*

$$\frac{1}{2} \int_X |\nabla f|^2 \, \Delta g \, \mathrm{d}\mathfrak{m} \ge \int_X \left[\nabla f \cdot \nabla \Delta f + K |\nabla f|^2 \right] g \, \mathrm{d}\mathfrak{m}. \tag{1.47}$$

Remark 1.74. We will need also the following local version of the *Sobolev to Lipschitz* property: any $f \in H^{1,2}_{\mathrm{loc}}(X, \mathsf{d}, \mathfrak{m})$ with $|\nabla f| \in L^\infty(B_{2r}(x), \mathfrak{m})$ for some $x \in X$ and $r > 0$, admits a Lipschitz representative $\bar{f}$ in $B_r(x)$ such that $\mathrm{Lip} \bar{f}_{|B_r(x)} \le \|\nabla f\|_{L^\infty(B_{2r}(x), \mathfrak{m})}$.

Remark 1.75. One of the main contributions of [**114**] has been to prove, roughly speaking, that, under the $\mathrm{RCD}(K, \infty)$ assumption, the minimal relaxed slope $|\nabla f|_p$ is independent of p, for any $1 < p < \infty$. This as a further motivation for our choice to omit the dependence on p in the notation for the minimal relaxed gradient.

We go on by stating a few regularizing properties of the heat flow on $\mathrm{RCD}(K, \infty)$ spaces, referring again to [**13, 16**] for a more detailed discussion and the proofs of these results.

Let us first explicitly point out that the semigroup P_t^* defined on $\mathcal{P}_2(X)$ (that we introduced in Remark 1.70) is the dual semigroup of P_t, that is to say

$$\int_X f \, \mathrm{d}P_t^* \mu = \int_X P_t f \, \mathrm{d}\mu \qquad \forall \mu \in \mathcal{P}_2(X), \quad \forall f \in \mathrm{Lip_b}(X).$$

Moreover, with an argument introduced in the literature by Kuwada [**154**], it can be proved that, on any $\mathrm{RCD}(K, \infty)$ metric measure space, P_t^* is K-contractive (w.r.t. the W_2-distance) and, for $t > 0$, maps probability measures into probability measures absolutely continuous w.r.t. $\mathfrak{m}$. Then, for any $t > 0$, we can introduce the so called *heat kernel* $p_t : X \times X \to [0, +\infty)$ by

$$p_t(x, \cdot)\mathfrak{m} := P_t^* \delta_x.$$

From the fact that $(P_t)_{t\geq 0}$ is self adjoint one argues that, for any $p \in [1,+\infty)$, for any $f \in L^p(X,\mathfrak{m})$ and for any $t \geq 0$

$$P_t f(x) = \int p_t(x,y) f(y)\, \mathrm{d}\mathfrak{m}(y), \quad \text{for } \mathfrak{m}\text{-a.e. } x \in X. \tag{1.48}$$

Moreover, for any $f \in L^\infty(X,\mathfrak{m})$, the formula

$$P_t f(x) = \int_X f(y) p_t(x,y)\, \mathrm{d}\mathfrak{m}(y)$$

is well defined and provides a pointwise version of the heat flow for which the so-called $L^\infty - \mathrm{Lip}$ *regularization* property is satisfied, that is to say, for any $f \in L^\infty(X,\mathfrak{m})$, we have $P_t f \in \mathrm{Lip}(X)$ with

$$\sqrt{2I_{2K}(t)}\mathrm{Lip}(P_t f) \leq \|f\|_{L^\infty}, \quad \text{for any } t > 0, \tag{1.49}$$

where $I_L(t) := \int_0^t e^{Lr}\, \mathrm{d}r$.

Then, as a consequence of the contractivity of the heat flow, we have the crucial estimate, proved in [**191**]

$$\mathrm{lip}(P_t f) \leq e^{-Kt} P_t(\mathrm{lip} f), \quad \text{pointwise on } X, \tag{1.50}$$

valid for any $t \geq 0$ and for all $f \in \mathrm{Lip}_{\mathrm{bs}}(X,\mathsf{d})$. This allows to generalize the *Bakry-Émery* contraction estimate obtained for $p = 2$ in [**16**] to the whole range of exponents $p \in (1,\infty)$:

$$|\nabla P_t f|^p \leq e^{-pKt} P_t\left(|\nabla f|^p\right), \quad \mathfrak{m}\text{-a.e. on } X, \tag{1.51}$$

for any $t \geq 0$ and for any $f \in H^{1,p}(X,\mathsf{d},\mathfrak{m})$.

Under the assumption that (X,d) is a proper metric space, in [**114**] the authors proved that the contractivity estimate holds also in the degenerate case of BV functions:

$$|DP_t f| \leq e^{-Kt} P_t^* |Df|, \tag{1.52}$$

for any $f \in \mathrm{BV}(X,\mathsf{d},\mathfrak{m})$ and for any $t \geq 0$.

To conclude this introduction to $\mathrm{RCD}(K,\infty)$ spaces we also quote from [**112**] a useful result about a dimensional decomposition of the tangent module $L^2(TX)$.

Proposition 1.76. *Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,\infty)$ metric measure space. Then there exists a unique decomposition $\{E_n\}_{n\in\mathbb{N}\cup\{\infty\}}$ of X such that:*

i) *for any $n \in \mathbb{N}$ and any $B \subset E_n$ of finite positive measure, $L^2(TX)$ has a unit orthogonal basis $\{e_i^n\}_{i=1}^n$ on B;*
ii) *for every subset $B \subset E_\infty$ with finite positive measure, there exists a unit orthogonal set $\{e_{i,B}\}_{i\in\mathbb{N}} \subset L^2(TX)|_B$ which generates $L^2(TX)|_B$.*

Definition 1.77 (Analytic dimension)**.** Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,\infty)$ metric measure space. We say that the dimension of $L^2(TX)$ is k if $k = \sup\{n \in \mathbb{N} : \mathfrak{m}(E_n) > 0\}$, where $\{E_n\}_{n\in\mathbb{N}\cup\{\infty\}}$ is the dimensional decomposition of the tangent module provided by Proposition 1.76.
We define the *analytic dimension* of $(X,\mathsf{d},\mathfrak{m})$ as the dimension of $L^2(TX)$.

3.1. Second order differential calculus. One of the main accomplishments of [**112**] has been the construction of a second order differential calculus over $\mathrm{RCD}(K,\infty)$ metric measure spaces. We refer also to the previous [**203**] for analogous constructions under the existence of a *core* of *good functions*.

Following [**112**] we introduce the space of "test" functions $\mathrm{Test}(X,\mathsf{d},\mathfrak{m})$ by

$$\mathrm{Test}(X,\mathsf{d},\mathfrak{m}) := \{f \in D(\Delta) \cap L^\infty(X,\mathfrak{m}) : |\nabla f| \in L^\infty(X) \quad \text{and} \quad \Delta f \in H^{1,2}(X,\mathsf{d},\mathfrak{m})\}. \tag{1.53}$$

We remark that, for any $g \in L^2 \cap L^\infty(X, \mathsf{m})$, it holds that $P_t g \in \mathrm{Test}(X, \mathsf{d}, \mathsf{m})$ for any $t > 0$, thanks to the regularizing properties of the heat flow that we recalled above. In particular $\mathrm{Test}(X, \mathsf{d}, \mathsf{m})$ is dense in $H^{1,2}(X, \mathsf{d}, \mathsf{m})$. Moreover, any $f \in \mathrm{Test}(X, \mathsf{d}, \mathsf{m})$ admits a Lipschitz representative thanks to the Sobolev to Lipschitz property.

Generalizing and adapting the arguments proposed by Bakry in [**37, 38**], Savaré proved in [**191**] the following crucial regularity result, which constitutes the first step towards the construction of the second order differential calculus of RCD spaces.

Theorem 1.78. *Let* $f \in \mathrm{Test}(X, \mathsf{d}, \mathsf{m})$*, then* $|\nabla f|^2 \in D(\mathbf{\Delta}) \subset H^{1,2}(X, \mathsf{d}, \mathsf{m})$ *and*

$$\mathrm{Ch}(|\nabla f|^2) \le -\int \left[K |\nabla f|^4 + |\nabla f|^2 \nabla f \cdot \nabla \Delta f\right] \mathrm{d}\mathsf{m}, \tag{1.54}$$

$$\frac{1}{2}\mathbf{\Delta} |\nabla f|^2 \ge \left(K |\nabla f|^2 + \nabla f \cdot \nabla \Delta f\right) \mathsf{m}. \tag{1.55}$$

By polarization we can deduce from (1.54) that $\nabla f \cdot \nabla g \in H^{1,2}(X, \mathsf{d}, \mathsf{m})$ for any $f, g \in \mathrm{Test}(X, \mathsf{d}, \mathsf{m})$. With this information it is readily checked, thanks to the Leibniz rule for the Laplacian, that $\mathrm{Test}(X)$ is an algebra.

We proceed introducing the notion of Hessian and the space $W^{2,2}(X, \mathsf{d}, \mathsf{m})$. The idea behind the definition in [**112**] is the observation that the identity

$$2\,\mathrm{Hess}\, f(\nabla g_1, \nabla g_2) = \nabla g_1 \cdot \nabla(\nabla f \cdot \nabla g_2) + \nabla g_2 \cdot \nabla(\nabla f \cdot \nabla g_1) - \nabla f \cdot \nabla(\nabla g_1 \cdot \nabla g_2) \tag{1.56}$$

valid for a sufficiently large class of test functions g_1, g_2 characterizes the hessian $\mathrm{Hess}\, f$ of f.

Definition 1.79 (The space $W^{2,2}$ and the Hessian)**.** The space $W^{2,2}(X, \mathsf{d}, \mathsf{m}) \subset H^{1,2}(X, \mathsf{d}, \mathsf{m})$ is the space of all $f \in H^{1,2}(X, \mathsf{d}, \mathsf{m})$ for which there exists $A \in L^2\left((T^*)^{\otimes 2} X\right)$ such that for any $h, g_1, g_2 \in \mathrm{Test}(X)$ it holds

$$\begin{aligned} &2\int hA(\nabla g_1, \nabla g_2)\, \mathrm{d}\mathsf{m} \\ &= -\int \left[\nabla f \cdot \nabla g_1 \,\mathrm{div}\,(h\nabla g_2) + \nabla f \cdot \nabla g_2 \,\mathrm{div}\,(h \nabla g_1) + h\nabla f \cdot \nabla(\nabla g_1 \cdot \nabla g_2)\right] \mathrm{d}\mathsf{m}. \end{aligned} \tag{1.57}$$

In this case the operator A will be called Hessian of f and denoted as $\mathrm{Hess}\, f$.

We endow $W^{2,2}(X, \mathsf{d}, \mathsf{m})$ with the norm $\|\cdot\|_{W^{2,2}(X)}$ defined by

$$\|f\|^2_{W^{2,2}} := \|f\|^2_{L^2} + \|\nabla f\|^2_{L^2(TX)} + \|\mathrm{Hess}\, f\|^2_{L^2((T^*)^{\otimes 2} X)}\,. \tag{1.58}$$

Existence of many functions in $W^{2,2}(X, \mathsf{d}, \mathsf{m})$ comes from [**112**, Theorem 3.3.8] that we state below.

Theorem 1.80. *Any function* $f \in \mathrm{Test}(X, \mathsf{d}, \mathsf{m})$ *belongs to* $W^{2,2}(X, \mathsf{d}, \mathsf{m})$*. Moreover, for any* $f, g_1, g_2 \in \mathrm{Test}(X, \mathsf{d}, \mathsf{m})$*, the identity* (1.56) *is verified.*

Corollary 1.81. *It holds that* $D(\Delta) \subset W^{2,2}(X, \mathsf{d}, \mathsf{m})$*. Moreover, the quantitative estimate*

$$\int_X |\mathrm{Hess}\, f|^2 \,\mathrm{d}\mathsf{m} \le \int_X \left\{(\Delta f)^2 - K |\nabla f|^2\right\} \mathrm{d}\mathsf{m} \tag{1.59}$$

is verified for any $f \in D(\Delta)$*.*

Given the inclusion above, following [**112**, Definition 3.3.17] we let $H^{2,2}(X, \mathsf{d}, \mathsf{m})$ be defined as the $W^{2,2}$-closure of $\mathrm{Test}(X)$ in $W^{2,2}(X, \mathsf{d}, \mathsf{m})$. In [**112**, Proposition 3.3.18] it is then established that $H^{2,2}$ is the closure of $D(\Delta)$ in $W^{2,2}$.

Let us recall that the Hessian enjoys the following locality property that has been proved in [**112**, Proposition 3.3.24].

Proposition 1.82. *Given $f_1, f_2 \in H^{2,2}(X, \mathsf{d}, \mathfrak{m})$ it holds*

$$|\operatorname{Hess} f_1| = |\operatorname{Hess} f_2| \quad \mathfrak{m}\text{-}a.e.\ in\ \{f_1 = f_2\}.$$

Moreover, for $f_1, f_2 \in H^{2,2}(X, \mathsf{d}, \mathfrak{m})$ the expected identity

$$\mathrm{d}\,(\nabla f_1 \cdot \nabla f_2) = \operatorname{Hess} f_1(\nabla f_2, \cdot) + \operatorname{Hess} f_2(\nabla f_1, \cdot), \quad \mathfrak{m}\text{-a.e.}, \tag{1.60}$$

is satisfied, see [**112**, Proposition 3.2.22]. As a useful consequence of (1.60) we have that

$$|\nabla\,(\nabla f \cdot \nabla g)| \le |\operatorname{Hess} f|\,|\nabla g| + |\operatorname{Hess} g|\,|\nabla f|\,, \tag{1.61}$$

for any $f, g \in H^{2,2}(X, \mathsf{d}, \mathfrak{m})$.

Next we move from functions to vector fields. Also in this case it will be important to have a class of regular test vector fields to work with.

Definition 1.83 (Test vector fields)**.** We introduce the class of test vector fields $\mathrm{TestV}(X) \subset L^2(TX)$ as

$$\mathrm{TestV}(X) \coloneqq \left\{ \sum_{i=1}^{n} g_i \nabla f_i \;:\; n \in \mathbb{N},\ f_i, g_i \in \mathrm{Test}(X) \right\}. \tag{1.62}$$

Below we introduce our working definition of Sobolev vector field with symmetric covariant derivative in L^2.

Definition 1.84. The Sobolev space $H^{1,2}_{C,s}(TX) \subset L^2(TX)$ is the space of all $b \in L^2(TX)$ with $\operatorname{div} b \in L^2(X, \mathfrak{m})$ for which there exists a tensor $S \in L^2(T^{\otimes 2}X)$ such that, for any choice of $h, g_1, g_2 \in \mathrm{Test}(X, \mathsf{d}, \mathfrak{m})$, it holds

$$\begin{aligned} &\int hS : (\nabla g_1 \otimes \nabla g_2)\,\mathrm{d}\mathfrak{m} \\ &= \frac{1}{2} \int \left\{ -b(g_2) \operatorname{div}(h \nabla g_1) - b(g_1) \operatorname{div}(h \nabla g_2) + \operatorname{div}(hb) \nabla g_1 \cdot \nabla g_2 \right\}\,\mathrm{d}\mathfrak{m}. \end{aligned} \tag{1.63}$$

In this case we shall call S the symmetric covariant derivative of b and we will denote it by $\nabla_{\mathrm{sym}} b$. We endow the space $H^{1,2}_{C,s}(TX)$ with the norm $\|\cdot\|_{H^{1,2}_{C,s}(TX)}$ defined by

$$\|b\|^2_{H^{1,2}_{C,s}(TX)} \coloneqq \|b\|^2_{L^2(TX)} + \|\nabla_{\mathrm{sym}} b\|^2_{L^2(T^{\otimes 2}X)}\,.$$

Remark 1.85. It is not difficult to check that when the ambient space is a smooth Riemannian manifold and b is a smooth vector field, the above defined symmetric covariant derivative coincides with the symmetric part of the covariant derivative ∇b. We refer to [**165**] for this verification in the case of gradient vector fields.

Remark 1.86. With an abuse of notation, from now on we shall indicate by $S(\nabla g_1, \nabla g_2) = S : (\nabla g_1 \otimes \nabla g_2)$. Let us remark that we denoted by : the canonical pointwise scalar product induced by the Hilbert-Schmidt norm on $L^2(T^{\otimes^2} X)$.

Remark 1.87. It easily follows from the definition that the symmetric covariant derivative of any vector field in $H^{1,2}_{C,s}(TX)$ is a symmetric tensor.
Moreover, any $b \in H^{1,2}_{C}(TX)$ such that $\operatorname{div} b \in L^2(X, \mathfrak{m})$ belongs to $H^{1,2}_{C,s}(TX)$ and $\nabla_{\mathrm{sym}} b$ is the symmetric part of ∇b (we refer to [**112**, Section 3.4] for the definition of space $H^{1,2}_{C}(TX)$ and of the associated notion of covariant derivative).
In particular, it holds that $\mathrm{TestV}(X)$ is included in the space of vector fields with symmetric covariant derivative in L^2 and that, for any $f \in W^{2,2}(X, \mathsf{d}, \mathfrak{m}) \cap D(\Delta)$, it holds $\nabla f \in H^{1,2}_{C}(TX)$ and, a fortiori, $\nabla f \in H^{1,2}_{C,s}(TX)$.

Remark 1.88. Let us point out, since this observation will be relevant for the applications to the theory of Regular Lagrangian Flows we are going to present in the next subsection, that vector fields in $H^{1,2}_{C,s}(TX)$ have deformation in L^2 according to Definition 1.47. This can be readily checked passing to the modules in (1.63).

Thanks to the existence of a second order differential calculus on $\mathrm{RCD}(K,\infty)$ spaces there is the possibility to consider vector fields which are defined capacity almost everywhere, while on a general metric measure space without further regularity assumptions, roughly speaking, they are only defined $\mathfrak{m}$-almost everywhere.

Let us recall, since this fact plays a crucial role in the discussion below, that $|\nabla f| \in H^{1,2}(X)$ for any $f \in \mathrm{Test}(X)$ (see [**88**]). In particular, for any $f \in \mathrm{Test}(X)$, $|\nabla f|$ admits a quasi-continuous representative.

Theorem 1.89 (Tangent $L^0(\mathrm{Cap})$-module [**88**])**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K,\infty)$ space, then the following holds: there exists a unique couple $(L^0_{\mathrm{Cap}}(TX), \tilde\nabla)$, where $L^0_{\mathrm{Cap}}(TX)$ is an $L^0(\mathrm{Cap})$-normed $L^0(\mathrm{Cap})$-module and $\tilde\nabla : \mathrm{Test}(X) \to L^0_{\mathrm{Cap}}(TX)$ is a linear operator, such that*

$$|\tilde\nabla f| = \mathsf{QCR}(|\nabla f|) \quad \textit{in the } \mathrm{Cap}\textit{-a.e. sense} \quad \textit{for every } f \in \mathrm{Test}(X),$$

$$\left\{ \sum_{n\in\mathbb{N}} \chi_{E_n} \tilde\nabla f_n \;\middle|\; (E_n)_n \textit{ Borel partition of } X,\ (f_n)_n \subset \mathrm{Test}(X) \right\} \quad \textit{is dense in } L^0_{\mathrm{Cap}}(TX).$$

The space $L^0_{\mathrm{Cap}}(TX)$ is called capacitary tangent module *on X, while $\tilde\nabla$ is the* capacitary gradient*.*

To conclude this subsection we review the theory of the Holdge Laplacian studied in [**112**]. The discussion in this part is less introductory and contains some results borrowed from [**48**], where these tools were developed and used in the proof of the Gauss-Green formula for sets of finite perimeter (cf. Chapter 5).

Let us consider the space $H^{1,2}_{\mathrm{H}}(TX) \subset H^{1,2}_C(TX)$ and the Hodge Laplacian $\Delta_{\mathrm{H}} : D(\Delta_{\mathrm{H}}) \subset H^{1,2}_{\mathrm{H}}(TX) \to L^2(TX)$, which have been defined in [**112**, Definition 3.5.13] and [**112**, Definition 3.5.15], respectively (cf. the first paragraph of [**111**, Section 2.6] for the identification between vector and covector fields).

It follows from its definition that the Hodge Laplacian is self-adjoint, namely that

$$\int \langle \Delta_{\mathrm{H}} v, w\rangle \, \mathrm{d}\mathfrak{m} = \int \langle v, \Delta_{\mathrm{H}} w\rangle \, \mathrm{d}\mathfrak{m} \quad \text{for every } v, w \in D(\Delta_{\mathrm{H}}). \tag{1.64}$$

Let us consider the *augmented Hodge energy functional* $\tilde{\mathcal{E}}_{\mathrm{H}} : L^2(TX) \to [0,+\infty]$, which is defined in [**112**, eq. (3.5.16)] (up to identifying $L^2(T^*X)$ with $L^2(TX)$ via the musical isomorphism). Then we denote by $(\mathsf{h}_{\mathrm{H},t})_{t\geq 0}$ the gradient flow in $L^2(TX)$ of the functional $\tilde{\mathcal{E}}_{\mathrm{H}}$. This means that for any vector field $v \in L^2(TX)$ it holds that $t \mapsto \mathsf{h}_{\mathrm{H},t}(v) \in L^2(TX)$ is the unique continuous curve on $[0,+\infty)$ with $\mathsf{h}_{\mathrm{H},0}(v) = v$, which is locally absolutely continuous on $(0,+\infty)$ and satisfies

$$\mathsf{h}_{\mathrm{H},t}(v) \in D(\Delta_{\mathrm{H}}) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\mathsf{h}_{\mathrm{H},t}(v) = -\Delta_{\mathrm{H}} \mathsf{h}_{\mathrm{H},t}(v) \quad \text{for every } t > 0.$$

It also holds that

$$\mathsf{h}_{\mathrm{H},t}(\nabla f) = \nabla P_t f \quad \text{for every } f \in H^{1,2}(X) \text{ and } t \geq 0. \tag{1.65}$$

Finally, we recall that vector fields satisfy the following Bakry-Émery contraction estimate (see [**112**, Proposition 3.6.10]):

$$|\mathsf{h}_{\mathrm{H},t}(v)|^2 \le e^{-2Kt} P_t(|v|^2) \quad \mathfrak{m}\text{-a.e.} \quad \text{for every } v \in L^2(TX) \text{ and } t \ge 0. \tag{1.66}$$

Lemma 1.90 ($\mathsf{h}_{\mathrm{H},t}$ is self-adjoint)**.** *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K,\infty)$ *space. Then it holds that*

$$\int \langle \mathsf{h}_{\mathrm{H},t}(v), w\rangle \, \mathrm{d}\mathfrak{m} = \int \langle v, \mathsf{h}_{\mathrm{H},t}(w)\rangle \, \mathrm{d}\mathfrak{m} \quad \textit{for every } v, w \in L^2(TX) \textit{ and } t \ge 0. \tag{1.67}$$

Proof. Fix $v, w \in L^2(TX)$ and $t > 0$. We define the function $\varphi\colon [0,t] \to \mathbb{R}$ as

$$\varphi(s) := \int \langle \mathsf{h}_{\mathrm{H},s}(v), \mathsf{h}_{\mathrm{H},t-s}(w)\rangle \, \mathrm{d}\mathfrak{m} \quad \text{for every } s \in [0,t].$$

Therefore, the function φ is absolutely continuous and satisfies

$$\varphi'(s) = -\int \langle \Delta_{\mathrm{H}} \mathsf{h}_{\mathrm{H},s}(v), \mathsf{h}_{\mathrm{H},t-s}(w)\rangle \, \mathrm{d}\mathfrak{m} + \int \langle \mathsf{h}_{\mathrm{H},s}(v), \Delta_{\mathrm{H}} \mathsf{h}_{\mathrm{H},t-s}(w)\rangle \, \mathrm{d}\mathfrak{m} \overset{(1.64)}{=} 0 \text{ for a.e. } t > 0.$$

Then φ is constant, thus in particular $\int \langle \mathsf{h}_{\mathrm{H},t}(v), w\rangle \, \mathrm{d}\mathfrak{m} = \varphi(t) = \varphi(0) = \int \langle v, \mathsf{h}_{\mathrm{H},t}(w)\rangle \, \mathrm{d}\mathfrak{m}$. □

Proposition 1.91. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K,\infty)$ *space. Then for any* $v \in D(\mathrm{div})$ *it holds that*

$$\mathsf{h}_{\mathrm{H},t}(v) \in H^{1,2}_C(TX) \cap D(\mathrm{div}) \quad \textit{and} \quad \mathrm{div}(\mathsf{h}_{\mathrm{H},t}(v)) = P_t(\mathrm{div}(v)) \quad \textit{for every } t > 0.$$

Proof. First of all, observe that $\mathsf{h}_{\mathrm{H},t}(v) \in H^{1,2}_{\mathrm{H}}(TX) \subset H^{1,2}_C(TX)$ by [**112**, Corollary 3.6.4]. Moreover, let $f \in H^{1,2}(X)$ be given. Then it holds that

$$\begin{aligned}\int \langle \nabla f, \mathsf{h}_{\mathrm{H},t}(v)\rangle \, \mathrm{d}\mathfrak{m} &\overset{(1.67)}{=} \int \langle \mathsf{h}_{\mathrm{H},t}(\nabla f), v\rangle \, \mathrm{d}\mathfrak{m} \overset{(1.65)}{=} \int \langle \nabla P_t f, v\rangle \, \mathrm{d}\mathfrak{m} = -\int P_t f \, \mathrm{div}(v) \, \mathrm{d}\mathfrak{m} \\ &= -\int f P_t(\mathrm{div}(v)) \, \mathrm{d}\mathfrak{m}.\end{aligned}$$

By arbitrariness of f, we conclude that $\mathsf{h}_{\mathrm{H},t}(v) \in D(\mathrm{div})$ and $\mathrm{div}(\mathsf{h}_{\mathrm{H},t}(v)) = P_t(\mathrm{div}(v))$. □

3.2. Regular Lagrangian Flows of Sobolev vector fields. In this subsection we review the theory of regular Lagrangian Flows (RLF for short) of Sobolev vector fields in $\mathrm{RCD}(K,\infty)$ metric measure spaces. This theory has been firstly introduced in the Euclidean setting by Ambrosio in [**6**], inspired by the earlier work by Di Perna-Lions [**93**]. In more recent times the theory has been extended to the setting of metric measure spaces verifying suitable regularity assumptions by Ambrosio-Trevisan in [**30**]. In this discussion we follow the presentation of the lecture notes [**31**], assuming throughout that $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(K,\infty)$ m.m.s..

Let us first point out that a notion of time dependent vector field over $(X, \mathsf{d}, \mathfrak{m})$ can be introduced in the natural way.

Definition 1.92. Let us fix $T > 0$. We say that $b : [0,T] \to L^2(TX)$ is a time dependent vector field if, for every $f \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$, the map

$$(t,x) \mapsto b_t \cdot \nabla f(x)$$

is measurable with respect to the completion of the product sigma-algebra $\mathcal{L}^1(\mathbb{R}) \otimes \mathscr{B}(X)$. We say that b is bounded if

$$\|b\|_{L^\infty} := \||b|\|_{L^\infty([0,T]\times X)} < \infty,$$

and that $b \in L^1((0,T); L^2(TX))$ if

$$\int_0^T \|b_s\|_{L^2(TX)} \, \mathrm{d}s < \infty.$$

In the sequel we shall stress the dependence of a vector field b on the time variable only in case it is relevant for the sake of clarity.

In the context of $\mathrm{RCD}(K,\infty)$ spaces the definition of Regular Lagrangian flow reads as follows.

Definition 1.93. Let us fix a possibly time dependent vector field b. We say that $\mathbf{X} : [0,T] \times X \to X$ is a Regular Lagrangian flow associated to b if the following conditions hold true:

1) $\mathbf{X}(0,x) = x$ and $\mathbf{X}(\cdot,x) \in C([0,T]; X)$ for every $x \in X$;
2) there exists $L \geq 0$, called compressibility constant, such that
$$\mathbf{X}(t,\cdot)_\sharp \mathfrak{m} \leq L\mathfrak{m}, \qquad \text{for every } t \in [0,T];$$
3) for every $f \in \mathrm{Test}(X,\mathsf{d},\mathfrak{m})$ the map $t \mapsto f(\mathbf{X}(t,x))$ belongs to $\mathrm{AC}([0,T])$ for $\mathfrak{m}$-a.e. $x \in X$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{X}(t,x)) = b_t \cdot \nabla f(\mathbf{X}(t,x)) \qquad \text{for a.e. } t \in (0,T). \tag{1.68}$$

We remark that the notion of RLF is stable under modification in a negligible set of initial conditions, but we prefer to work with a pointwise defined map in order to avoid technical issues.

Remark 1.94. Under the additional assumption $b \in L^1((0,T); L^2(TX))$, (1.68) holds true for every $g \in H^{1,2}(X,\mathsf{d},\mathfrak{m})$ (where it is understood that in this case the map $t \mapsto g(\mathbf{X}_t(x))$ belongs to $W^{1,1}((0,T))$ for $\mathfrak{m}$-a.e. $x \in X$) if and only if it holds for every $h \in D$ with $D \subset H^{1,2}(X,\mathsf{d},\mathfrak{m})$ dense with respect to the strong topology.
Indeed, if this is the case, for any $g \in H^{1,2}(X,\mathsf{d},\mathfrak{m})$ and every $\varepsilon > 0$ we can find $h \in D$ such that $\|g-h\|_{H^{1,2}(X,\mathsf{d},\mathfrak{m})} < \varepsilon$. Hence, since (1.68) holds true for h, we can estimate

$$\begin{aligned}
\int_X \Big| g(\mathbf{X}(t,x)) - g(x) - & \int_0^t b_s \cdot \nabla g(\mathbf{X}(s,x)) \, \mathrm{d}s \Big|^2 \mathrm{d}\mathfrak{m}(x) \\
&\leq 2 \int_X |g(\mathbf{X}(t,x)) - h(\mathbf{X}(t,x))|^2 \, \mathrm{d}\mathfrak{m}(x) + 2 \int_X |g(x) - h(x)|^2 \, \mathrm{d}\mathfrak{m}(x) \\
&\quad + 2 \int_X \left| \int_0^t b_s \cdot \nabla (g-h)(\mathbf{X}(s,x)) \, \mathrm{d}s \right|^2 \mathrm{d}\mathfrak{m}(x) \\
&\leq 2(L+1) \|g-h\|^2_{L^2(X,\mathfrak{m})} + 2L \|g-h\|^2_{H^{1,2}(X,\mathsf{d},\mathfrak{m})} \sqrt{t} \int_0^t \|b_s\|^2_{L^2} \, \mathrm{d}s \\
&\leq \varepsilon^2 C(L, t, \|b\|_{L^1((0,T);L^2(TX))}),
\end{aligned}$$

that, together with an application of Fubini's theorem, implies the validity of (1.68) for g. Moreover, one can easily prove via a localization procedure that also functions in the class $H^{1,2}_{\mathrm{loc}}(X,\mathsf{d},\mathfrak{m})$ are admissible tests in (1.68).

Next we can review the well-posedness theorem for the existence and uniqueness problem for regular Lagrangian flows of Sobolev vector fields over RCD spaces, as proved in [**30**].

Theorem 1.95. *Let* $(X,\mathsf{d},\mathfrak{m})$ *be an* $\mathrm{RCD}(K,\infty)$ *m.m.s. and* $(b_t) \in L^1((0,T); L^2(TX))$ *be a time dependent vector field. If*

$$\|D^{\mathrm{sym}} b_t\|_2 \in L^1(0,T), \quad \textit{and } |\mathrm{div}\, b_t| \in L^1((0,T); L^\infty(X,\mathfrak{m})), \tag{1.69}$$

then there exists a unique regular Lagrangian flow $\mathbf{X}_t$ *of* b_t *on* $(0,T)$*, where we stress that uniqueness is understood in the pathwise sense: for any* $\mathbf{X}$ *and* $\mathbf{X}'$ *regular Lagrangian flows it holds that* $\mathbf{X}_t(x) = \mathbf{X}'_t(x)$ *for any* $t \in (0,T)$ *for* $\mathfrak{m}$*-a.e.* $x \in X$*.*

Remark 1.96. The bounded compressibility assumption in the definition of regular Lagrangian flow can be improved to a quantitative control as follows:

$$(\mathbf{X}_t)_\sharp \mathfrak{m} \le e^{\int_0^t \|\operatorname{div} b_s\|_\infty \, \mathrm{d}s} \mathfrak{m} \quad \text{for any } t \in [0,T]. \tag{1.70}$$

Remark 1.97. Let us point out that in order to obtain existence and uniqueness of regular Lagrangian flows in [**30**] the duality with the Eulerian perspective based on the continuity equation was crucially exploited.
In particular, under the assumptions of Theorem 1.95, for any $\bar{u} \in L^1(X,\mathfrak{m}) \cap L^\infty(X,\mathfrak{m})$ there exists $u \in L^\infty_{\mathrm{loc}}((0,T); L^1(X,\mathfrak{m}) \cap L^\infty(X,\mathfrak{m}))$ such that $(\mathbf{X}_t)_\sharp (u\mathfrak{m}) = u_t \mathfrak{m}$ and it solves the *continuity equation*: for any $\phi \in \mathrm{Test}(X,\mathsf{d},\mathfrak{m})$ the map $t \mapsto \int_X \phi u_t \, \mathrm{d}\mathfrak{m}$ is locally absolutely continuous with distributional derivative

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_X \phi u_t \, \mathrm{d}\mathfrak{m} = \int_X (b \cdot \nabla \phi) u_t \, \mathrm{d}\mathfrak{m}.$$

Remark 1.98. As a consequence of the uniqueness part of Theorem 1.95, the following semigroup law is verified: for any $t \in (0,T)$ it holds that for $\mathfrak{m}$-a.e. $x \in X$

$$\mathbf{X}_t(\mathbf{X}_s(x)) = \mathbf{X}_{t+s}(x) \qquad \text{for all } s \in (0,T) \quad \text{such that } t+s \in (0,T). \tag{1.71}$$

In Chapter 3 we will need a specialization of the general existence and uniqueness theorem to the case when the vector field is time independent, it has vanishing divergence and vanishing deformation.

Theorem 1.99. *Let* $(X,\mathsf{d},\mathfrak{m})$ *be an* $\mathrm{RCD}(K,\infty)$ *space for some* $K \in \mathbb{R}$*. Let* b *be a bounded vector field with* $\operatorname{div} b = 0$ *and* $D^{\mathrm{sym}} b = 0$ *(i.e.* $\|D^{\mathrm{sym}} b\|_2 = 0$*). Then*

- (i) *there exists a unique regular Lagrangian flow* $\mathbf{X} : \mathbb{R} \times X \to X$ *associated to* b[2]
- (ii) $\mathbf{X}$ *admits a representative satisfying a pointwise semigroup property: for any* $s,t \in \mathbb{R}$ *and for any* $x \in X$ *it holds that*

$$\mathbf{X}_t(\mathbf{X}_s(x)) = \mathbf{X}_{t+s}(x) \tag{1.72}$$

and $\mathbf{X}_t$ *is a measure-preserving isometry for any* $t \in \mathbb{R}$*.*

Proof. Part (i) is a consequence of Theorem 1.95. Let us prove (ii). From (1.70) we conclude that $(\mathbf{X}_t)_\sharp \mathfrak{m} = \mathfrak{m}$ for any $t \in \mathbb{R}$. Let us now take $\bar{u} \in L^\infty(X,\mathfrak{m}) \cap H^{1,2}(X,\mathsf{d},\mathfrak{m})$ and u a solution of the continuity equation as in Remark 1.97. Thanks to [**30**, Lemma 5.8] we get that $P_\alpha u_t \in \mathrm{Test}(X,\mathsf{d},\mathfrak{m})$ is still a solution of the continuity equation for any $\alpha \in (0,1)$. Then we can compute

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \int_X |\nabla P_\alpha u_t|^2 \, \mathrm{d}\mathfrak{m} &= -\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \int_X P_\alpha u_t \Delta P_\alpha u_t \, \mathrm{d}\mathfrak{m} \\ &= -\int_X b \cdot \nabla \Delta P_\alpha u_t \; P_\alpha u_t \, \mathrm{d}\mathfrak{m}. \end{aligned}$$

Since $\operatorname{div} b = 0$ and $D^{\mathrm{sym}} b = 0$, we deduce

$$-\int_X b \cdot \nabla \Delta P_\alpha u_t \; P_\alpha u_t \, \mathrm{d}\mathfrak{m} = \int_X b \cdot \nabla P_\alpha u_t \; P_\alpha \Delta u_t \, \mathrm{d}\mathfrak{m} = 0,$$

[2]To be more precise, there exist unique Regular Lagrangian flows $\mathbf{X}^+, \mathbf{X}^- : [0,+\infty) \times X \to X$ associated to b and $-b$ respectively and we let $\mathbf{X}_t = \mathbf{X}^+_t$ for $t \ge 0$ and $\mathbf{X}_t = \mathbf{X}^-_{-t}$ for $t \le 0$.

therefore

$$\int_X |\nabla P_\alpha u_t|^2 \, d\mathfrak{m} = \int_X |\nabla P_\alpha \bar{u}|^2 \, d\mathfrak{m} \qquad \forall t \in \mathbb{R}, \quad \forall \alpha \in (0,1). \tag{1.73}$$

Taking the limit in (1.73) as $\alpha \to 0$ it easily follows that $u_t \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ for any $t \in \mathbb{R}$ and that $\int_X |\nabla u_t|^2 \, d\mathfrak{m}$ does not depend on $t \in \mathbb{R}$. Using the identity $u_t(x) = \bar{u}(\mathbf{X}(-t, x))$ (which can be checked using the semigroup property (1.72) and $(\mathbf{X}_t)_\sharp \mathfrak{m} = \mathfrak{m}$) we deduce that, for any $t \in \mathbb{R}$,

$$\mathrm{Ch}(\bar{u} \circ \mathbf{X}_t) = \mathrm{Ch}(\bar{u}) \qquad \forall \bar{u} \in L^\infty(X, \mathfrak{m}) \cap H^{1,2}(X, \mathsf{d}, \mathfrak{m}).$$

The conclusion follows from arguments that have been used several times in the literature, as in [**108**, Proposition 4.20]. ☐

4. RCD(K, N) metric measure spaces

The notion of $\mathrm{RCD}(K, N)$ metric measure space was proposed in [**110**] as a finite dimensional counterpart of $\mathrm{RCD}(K, \infty)$ metric measure spaces, coupling the curvature-dimension condition $\mathrm{CD}(K, N)$ with the infinitesimal Hilbertianity assumption.

Definition 1.100 ($\mathrm{RCD}(K, N)$ space)**.** A metric measure space $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(K, N)$ space for some $K \in \mathbb{R}$ and $1 < N < \infty$ if it verifies the $\mathrm{CD}(K, N)$ condition and it is infinitesimally Hilbertian.

Later on, after the introduction of the $\mathrm{CD}^*(K, N)$ condition in [**36**], also the following variant was naturally introduced.

Definition 1.101 ($\mathrm{RCD}^*(K, N)$ space)**.** A metric measure space $(X, \mathsf{d}, \mathfrak{m})$ is said to be an $\mathrm{RCD}^*(K, N)$ m.m.s. if it verifies the $\mathrm{CD}^*(K, N)$ condition and it is infinitesimally Hilbertian.

Inspired by the infinite dimensional case, in the two independent papers [**27, 96**] an Eulerian counterpart for the Riemannian curvature dimension condition was proposed and its equivalence with the previous Lagrangian approach was studied.

Recall that we keep assuming that all the metric measure spaces verify the volume growth assumption $\mathfrak{m}(B_r(x)) \le a e^{br^2}$ for some $x \in X$, any $r > 0$ and some $a, b \ge 0$.

Definition 1.102 ($\mathrm{BE}(K, N)$ condition)**.** We say that a metric measure space $(X, \mathsf{d}, \mathfrak{m})$ verifies the $\mathrm{BE}(K, N)$ condition for some $K \in \mathbb{R}$ and $1 \le N < +\infty$ if:

i) the Cheeger energy is quadratic (i.e. $(X, \mathsf{d}, \mathfrak{m})$ is infinitesimally Hilbertian);
ii) a weak dimensional Bochner inequality is satisfied: for any $f \in D(\Delta)$ and for any $g \in D(\Delta) \cap L^\infty(X, \mathfrak{m})^+$ such that $\Delta f \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ and $\Delta g \in L^\infty(X, \mathfrak{m})$ it holds

$$\frac{1}{2} \int_X |\nabla f|^2 \, \Delta g \, d\mathfrak{m} \ge \int_X \left[\nabla f \cdot \nabla \Delta f + K |\nabla f|^2 + \frac{1}{N} (\Delta f)^2 \right] g \, d\mathfrak{m}. \tag{1.74}$$

iii) any $f \in H^{1,2}(X)$ such that $|\nabla f| \le 1$ $\mathfrak{m}$-a.e. on X admits a 1-Lipschitz representative.

With two different approaches, in [**27, 96**], the following equivalence was established.

Theorem 1.103. *A metric measure space $(X, \mathsf{d}, \mathfrak{m})$ verifies the $\mathrm{BE}(K, N)$ condition for some $K \in \mathbb{R}$ and $1 < N < +\infty$ if and only if it verifies the $\mathrm{RCD}^*(K, N)$ condition.*

Remark 1.104. Let us point out that with arguments analogous to those leading to the conclusion of Remark 1.72, it is possible to prove that $\mathrm{RCD}(K, N)$ metric measure spaces are essentially non branching. Therefore we are in force to apply [**57**] to obtain that, at least in the case of finite reference measure, the $\mathrm{RCD}^*(K, N)$ and the $\mathrm{RCD}(K, N)$ condition are equivalent, yielding also equivalence of the $\mathrm{RCD}(K, N)$ and the Eulerian $\mathrm{BE}(K, N)$ condition. As we already pointed out, it is thought that the arguments leading to the identification between CD^* and CD in the essentially non branching case should extend to the case of a σ-finite reference measure due to their local nature.

A class of $\mathrm{RCD}(K,N)$ metric measure spaces enjoying further regularity properties is that of *non collapsed* spaces, that has been introduced by De Philippis-Gigli in [**84**] (see also the previous work by Kitabeppu [**151**] where a similar condition was proposed) as a natural synthetic counterpart of Cheeger-Colding's non collapsed Ricci limits [**69**].

Definition 1.105 (Non collapsed spaces)**.** Let $K \in \mathbb{R}$ and $N \geq 1$. We say that a metric measure space $(X,\mathsf{d},\mathfrak{m})$ is a *non collapsed* $\mathrm{RCD}(K,N)$ ($\mathrm{ncRCD}(K,N)$ for short) space provided it is an $\mathrm{RCD}(K,N)$ m.m.s. and $\mathfrak{m} = \mathcal{H}^N$.

In [**84**] also the more general class of *weakly non collapsed* spaces has been introduced. In order to do recall this notion let us introduce the notation

$$\theta_N(x) := \lim_{r\to 0} \frac{\mathfrak{m}(B_r(x))}{V_{K,N}(r)} = \sup_{r>0} \frac{\mathfrak{m}(B_r(x))}{V_{K,N}(r)}. \tag{1.75}$$

Observe that the second equality above is a consequence of the Bishop-Gromov inequality (1.45).

Definition 1.106 (Weakly non collapsed spaces)**.** Let $K \in \mathbb{R}$ and $N \geq 1$. We say that $(X,\mathsf{d},\mathfrak{m})$ is a weakly non collapsed $\mathrm{RCD}(K,N)$ space (wncRCD(K, N) for short) provided

$$\theta_N(x) < +\infty, \quad \text{for } \mathfrak{m}\text{-a.e. } x \in X.$$

4.1. Geometric analysis on $\mathrm{RCD}(K,N)$ **spaces.** We collect here some basic properties of $\mathrm{RCD}(K,N)$ spaces that will be relevant in the thesis.

Let us first point out that since, as we already remarked, $\mathrm{CD}(K,N)$ spaces are PI spaces, the same is true a fortiori for $\mathrm{RCD}(K,N)$ spaces. In particular Cheeger's Theorem 1.33 about the identification of minimal relaxed gradients with the slope applies.

Moreover (see [**114**, Remark 3.5]), since $\mathrm{RCD}(K,N)$ spaces are proper, if it holds that $f \in W^{1,1}(X,\mathsf{d},\mathfrak{m}) \cap L^p(X,\mathfrak{m})$ and $|\nabla f|_* \in L^p(X,\mathfrak{m})$ for some $p > 1$, then $f \in H^{1,p}(X,\mathsf{d},\mathfrak{m})$ with $|\nabla f|_p = |\nabla f|_*$ $\mathfrak{m}$-a.e.. Vice versa, if $f \in H^{1,p}(X,\mathsf{d},\mathfrak{m}) \cap L^1(X,\mathfrak{m})$ and $|\nabla f|_p \in L^1(X,\mathfrak{m})$, then $f \in W^{1,1}(X,\mathsf{d},\mathfrak{m})$ and $|\nabla f|_* = |\nabla f|_p$ $\mathfrak{m}$-a.e.. Therefore the identification result for minimal relaxed gradients extends to the whole range of exponents $p \in [1,+\infty)$.

The following result establishes existence of regular cut-off functions in this context.

Lemma 1.107 (Good cut-off functions [**26**])**.** *Let* $(X,\mathsf{d},\mathfrak{m})$ *be an* $\mathrm{RCD}(K,N)$ *space. Let* $\Omega \subseteq X$ *be an open set and* $K \subseteq \Omega$ *a compact set. Then there exists* $\eta \in \mathrm{Test}(X)$ *such that* $0 \leq \eta \leq 1$ *on* X*, the support of* η *is compactly contained in* Ω*, and* $\eta = 1$ *on some open neighbourhood of* K*.*

Moreover, in the particular case of balls, for every $x \in X$*,* $R > 0$*,* $0 < r < R$ *there exists* $\eta \in \mathrm{Test}(X,\mathsf{d},\mathfrak{m})$ *as above with respect to the domains* $B_r(x) \subset B_R(x)$*, with support contained in* $B_{2r}(x)$ *and such that*

$$r^2\,|\Delta\eta| + r\,|\nabla\eta| \leq C(K,N,R). \tag{1.76}$$

We shall denote in the sequel by $\mathrm{Test}_c(X,\mathsf{d},\mathfrak{m})$ the space of test functions with compact support.

The following remark plays a role in the definition of δ-splitting maps in Section 2.

Remark 1.108. Given an open set $\Omega \subseteq X$ and a function $f \in D(\Omega,\Delta)$, we say that f is *harmonic* if $\Delta f = 0$. If in addition f is Lipschitz, then one can define (the modulus of) its Hessian as follows:

$$\left|\mathrm{Hess}(f)\right| := \left|\mathrm{Hess}(\eta f)\right| \ \ \mathfrak{m}\text{-a.e. on } \{\eta = 1\}, \qquad \text{for } \eta \in \mathrm{Test}(X) \text{ with } \mathrm{spt}(\eta) \subseteq \Omega. \tag{1.77}$$

This way we obtain a well-defined function $\left|\mathrm{Hess}(f)\right| : \Omega \to [0,+\infty)$, thanks to the locality property of the Hessian and the fact that $\eta f \in D(\Delta) \subset H^{2,2}(X,\mathsf{d},\mathfrak{m})$, for every good cut-off function η.

Corollary 1.109. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *m.m.s..Then there exists a constant* $C := C_{K,N} \geq 0$ *such that for any* $f \in \mathrm{Test}(X)$ *and for any* $x \in X$ *it holds*

$$\begin{aligned}(1.78)\qquad \int_{B_1(x)} |\operatorname{Hess} f|^2 \, \mathrm{d}\mathfrak{m} \leq & C_{N,K} \left(\int_{B_2(x)} |\Delta f|^2 \, \mathrm{d}\mathfrak{m} + \inf_{m \in \mathbb{R}} \int_{B_2(x)} \left| |\nabla f|^2 - m \right| \, \mathrm{d}\mathfrak{m} \right) \\ & - K \int_{B_2(x)} |\nabla f|^2 \, \mathrm{d}\mathfrak{m}.\end{aligned}$$

Proof. The sought estimate can be obtained integrating with respect to η, where η is a good cut off function as in Lemma 1.107 with respect to $B_1(x) \subset B_2(x)$, the improved Bochner inequality with Hessian term obtained in [**112**, Theorem 3.3.8]. □

Since $\mathrm{RCD}(K, N)$ spaces are locally doubling and satisfy a local Poincaré inequality, the general theory of Dirichlet forms as developed in [**197–199**] guarantees that we can find a locally Hölder continuous representative of the heat kernel p on $X \times X \times (0, +\infty)$.

Moreover in [**138**] the following finer properties of the heat kernel have been proved relying on the previous [**106, 137**]: there exist constants $C_1 > 1$ and $c \geq 0$ such that

$$(1.79)\qquad \frac{1}{C_1 \mathfrak{m}(B_{\sqrt{t}}(x))} \exp\left\{ -\frac{\mathsf{d}^2(x,y)}{3t} - ct \right\} \leq p_t(x,y) \leq \frac{C_1}{\mathfrak{m}(B_{\sqrt{t}}(x))} \exp\left\{ -\frac{\mathsf{d}^2(x,y)}{5t} + ct \right\}$$

for any $x, y \in X$ and for any $t > 0$. Moreover it holds

$$(1.80)\qquad |\nabla p_t(x, \cdot)|(y) \leq \frac{C_1}{\sqrt{t}\, \mathfrak{m}(B_{\sqrt{t}}(x))} \exp\left\{ -\frac{\mathsf{d}^2(x,y)}{5t} + ct \right\} \quad \text{for } \mathfrak{m}\text{-a.e. } y \in X,$$

for any $t > 0$ and for any $x \in X$. We remark that in (1.79) and (1.80) above one can take $c = 0$ whenever $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(0, N)$ m.m.s..

In [**108**] (see also [**109**]) Gigli generalized the splitting theorem, originally due to Cheeger-Gromoll [**72**] in the setting of smooth Riemannian manifolds with non-negative Ricci curvature and to Cheeger-Colding [**68**] for Ricci limit spaces (with lower Ricci bounds converging to 0), to the framework of $\mathrm{RCD}(0, N)$ spaces.

Recall that, for a metric space (X, d), a line $\gamma : (-\infty, +\infty) \to X$ is a curve such that

$$(1.81)\qquad \mathsf{d}(\gamma(s), \gamma(t)) = |t - s|, \quad \text{for any } s, t \in \mathbb{R}.$$

Moreover, when dealing with the product $Z := X \times Y$ of two metric measure spaces $(X, \mathsf{d}_X, \mathfrak{m}_X)$ and $(Y, \mathsf{d}_Y, \mathfrak{m}_Y)$ we will always consider it endowed with the product distance

$$(1.82)\qquad \mathsf{d}_Z^2\left((x,y),(x',y')\right) := \mathsf{d}_X^2(x,x') + \mathsf{d}_Y^2(y,y')$$

and the product measure $\mathfrak{m}_Z := \mathfrak{m}_X \otimes \mathfrak{m}_Y$. In that case we will write write $Z = (X, \mathsf{d}_X, \mathfrak{m}_X) \times (Y, \mathsf{d}_Y, \mathfrak{m}_Y)$.

Remark 1.110 (Tensorization of the Cheeger energy)**.** In [**13, 16, 17**] (see also [**28**] where the problem is treated in absence of curvature assumptions) it has been proved that on product spaces the Cheeger energy has the tensorization property. That is to say, for any given $f \in L^2(Z, \mathfrak{m}_Z)$, it holds that $f \in H^{1,2}(Z, \mathsf{d}_Z, \mathfrak{m}_Z)$ if and only if the following holds. Denoting by $f^x(y) := f(x,y)$ and $f^y(x) := f(x,y)$, $f^x \in H^{1,2}(Y, \mathsf{d}_Y, \mathfrak{m}_Y)$ for $\mathfrak{m}_X$-a.e. $x \in X$ and $\int \mathrm{Ch}_Y(f^x) \, \mathrm{d}\mathfrak{m}_X < +\infty$, if and only if $f^y \in H^{1,2}(X, \mathsf{d}_X, \mathfrak{m}_X)$ for $\mathfrak{m}_Y$-a.e. $y \in Y$ and $\int \mathrm{Ch}_X(f^y) \, \mathrm{d}\mathfrak{m}_Y < +\infty$. In that case it holds that

$$(1.83)\qquad |\nabla f|_Z^2(x,y) = |\nabla f^y|_X^2(x,y) + |\nabla f^x|_Y^2(x,y), \quad \text{for } \mathfrak{m}_Z\text{-a.e. } (x,y) \in X \times Y.$$

Theorem 1.111 (Splitting theorem for $\mathrm{RCD}(0,N)$ spaces)**.** *Let* $(X,\mathsf{d},\mathfrak{m})$ *be an* $\mathrm{RCD}(0,N)$ *metric measure space. Assume that there exists a line* $\gamma:(0,+\infty)\to X$*. Then there exists an* $\mathrm{RCD}(0,N-1)$ *m.m.s.* $(Y,\mathsf{d}_Y,\mathfrak{m}_Y)$ *such that*

$$(X,\mathsf{d},\mathfrak{m})=(\mathbb{R},\mathsf{d}_{\mathrm{eucl}},\mathcal{L}^1)\times(Y,\mathsf{d}_Y,\mathfrak{m}_Y), \tag{1.84}$$

where the product is intended in the sense of metric measure spaces.

We next recall the notion of warped product between metric measure spaces, generalizing the well known Riemannian construction. This is going to play a role in Chapter 6.
Given two geodesic metric measure spaces $(B,\mathsf{d}_B,\mathfrak{m}_B)$ and $(F,\mathsf{d}_F,\mathfrak{m}_F)$ and a Lipschitz function $f:B\to[0,+\infty)$ one can define a length structure on the product $B\times F$ as follows: for any absolutely continuous curve $\gamma:[0,1]\to B\times F$ with components (α,β), define

$$L(\gamma):=\int_0^1\left(\left|\alpha'\right|^2(t)+(f\circ\alpha(t))^2\left|\beta'\right|^2(t)\right)^{\frac{1}{2}}\,\mathrm{d}t$$

and consider the associated pseudo-distance

$$\mathsf{d}((p,x),(q,y)):=\inf\left\{L(\gamma):\gamma(0)=(p,x),\ \gamma(1)=(q,y)\right\}.$$

The f-warped product of B with F is the metric space defined by

$$B\times_f F:=(B\times F/_\sim,\mathsf{d}),$$

where $(p,x)\sim(q,y)$ if and only if $\mathsf{d}((p,x),(q,y))=0$. One can also associate a natural measure and obtain

$$B\times_f^N F:=(B\times_f F,\mathfrak{m}_C),\quad \mathfrak{m}_C:=f^N\mathfrak{m}_B\otimes\mathfrak{m}_F,$$

that we will call warped product metric measure space of $(B,\mathsf{d}_B,\mathfrak{m}_B)$ and $(F,\mathsf{d}_F,\mathfrak{m}_F)$.

In Proposition 1.112 below we collect some results concerning the improved regularity of W_2-geodesics on $\mathrm{RCD}(K,N)$ metric measure spaces. The results are mainly taken from [**107**, **184**].

Proposition 1.112. *Let* $(X,\mathsf{d},\mathfrak{m})$ *be an* $\mathrm{RCD}(K,N)$ *m.m.s. for some* $K\in\mathbb{R}$ *and* $1<N<+\infty$*. Let* $\mu_0,\mu_1\in\mathcal{P}_2(X)$ *be absolutely continuous w.r.t.* $\mathfrak{m}$*, with bounded densities and bounded supports. Then:*

(i) *there exists a unique* W_2*-geodesic* $(\mu_t)_{t\in[0,1]}$ *joining* μ_0 *and* μ_1*. Moreover, it holds* $\mu_t\le C\mathfrak{m}$ *for any* $t\in[0,1]$ *for some* $C>0$*;*

(ii) *letting* ρ_t *be the density of* μ_t *w.r.t.* $\mathfrak{m}$*, it holds that, for any* $t\in[0,1]$ *and for any sequence* $(t_k)_{k\in\mathbb{N}}$ *converging to* t*, there exists a subsequence* $(t_{n_k})_{k\in\mathbb{N}}$ *such that*

$$\rho_{t_{n_k}}\to\rho_t\quad \mathfrak{m}\text{-a.e. as } k\to\infty.$$

Equivalently, the map $t\mapsto\rho_t$ *is continuous in* $L^1(\mathfrak{m})$*.*

4.2. Convergence and stability results. A crucial role in the development of the thesis will be played by the stability of relevant geometric and analytic properties along sequences of $\mathrm{RCD}(K,N)$ metric measure spaces converging in the pmGH sense (see Section 1.4 for the relevant definitions in the general framework of pointed metric measure spaces).

We start recalling the basic stability result for the $\mathrm{RCD}(K,N)$ condition. Stability of the $\mathrm{RCD}(K,\infty)$ condition was among the outcomes of [**118**]. Its refinement to the finite dimensional case can be obtained relying on the stability of the $\mathrm{CD}(K,N)$ condition and exploiting the various equivalences between different notions of convergence that hold true in the framework of locally compact spaces. Observe that $\mathrm{RCD}(K,N)$ spaces for finite N satisfy indeed this last assumption.

Theorem 1.113 (Stability of the $\mathrm{RCD}(K,N)$ condition)**.** *Let* $(X_i, \mathsf{d}_i, \mathfrak{m}_i, x_i)$ *be* $\mathrm{RCD}(K_i, N_i)$ *pointed metric measure spaces. Suppose that* $K_i \to K$ *and* $N_i \to N$ *as* $i \to \infty$ *and that* $(X_i, \mathsf{d}_i, \mathfrak{m}_i, x_i)$ *converge in the pmGH topology to* $(Y, \mathsf{d}_Y, \mathfrak{m}_Y, y)$*. Then* $(Y, \mathsf{d}_Y, \mathfrak{m}_Y)$ *is an* $\mathrm{RCD}(K,N)$ *metric measure space.*

Remark 1.114 (Tangents to $\mathrm{RCD}(K,N)$ spaces)**.** As a consequence of the scaling and stability properties of the $\mathrm{RCD}(K,N)$ condition, combined with Gromov's compactness argument, it can be easily argued that for any $\mathrm{RCD}(K,N)$ m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ and for any $x \in X$, $\mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m})$ is not empty and all its elements are $\mathrm{RCD}(0,N)$ spaces.

More in general, any sequence of normalized $\mathrm{RCD}(K,N)$ pointed metric measure spaces admits a subsequence converging in the pmGH sense to a normalized $\mathrm{RCD}(K,N)$ pointed m.m.s..

From now on in this subsection we are concerned with the stability properties of functions and spaces of functions along a fixed sequence of $\mathrm{RCD}(K_i, N_i)$ (pointed) metric measure spaces $(X_i, \mathsf{d}_i, \mathfrak{m}_i, x_i) \to (Y, \varrho, \mu, y)$ converging in the (pointed) measured Gromov-Hausdorff topology to the $\mathrm{RCD}(K,N)$ p.m.m.s. $(Y, \mathsf{d}_Y, \mathfrak{m}_Y, y)$ and such that $K_i \to K$ and $N_i \to N$ as $i \to \infty$. The basic references for this part are [**18**, **19**, **118**].

Recall that, since the dimension upper bounds and the lower Ricci curvature bounds are converging to finite limits, the spaces in the sequence are uniformly doubling. Therefore, as we have seen in Section 1.4, pmGH convergence is equivalent to pmG convergence. From now on we assume that the convergence is realized by means of isometric embeddings of the spaces into a common separable metric space (Z, d_Z).

We recall below the notions of convergence in L^p and Sobolev spaces for functions defined over converging sequences of metric measure spaces. We will be concerned only with the cases $p = 2$ and $p = 1$ in the rest of the thesis.

Definition 1.115. We say that $f_i \in L^2(X_i, \mathfrak{m}_i)$ converge in L^2-weak to $f \in L^2(Y, \mu)$ if $f_i \mathfrak{m}_i \rightharpoonup f\mu$ in duality with $\mathrm{C_{bs}}(Z)$ and $\sup_i \|f_i\|_{L^2(X_i, \mathfrak{m}_i)} < +\infty$.

We say that $f_i \in L^2(X_i, \mathfrak{m}_i)$ converge in L^2-strong to $f \in L^2(Y, \mu)$ if $f_i \mathfrak{m}_i \rightharpoonup f\mu$ in duality with $\mathrm{C_{bs}}(Z)$ and $\lim_i \|f_i\|_{L^2(X_i, \mathfrak{m}_i)} = \|f\|_{L^2(Y, \mu)}$.

Definition 1.116. We say that a sequence $(f_i) \subset L^1(X_i, \mathfrak{m}_i)$ converges L^1-strongly to $f \in L^1(Y, \mu)$ if

$$\sigma \circ f_i \mathfrak{m}_i \rightharpoonup \sigma \circ f\mu \qquad \text{and} \qquad \int_{X_i} |f_i| \, \mathrm{d}\mathfrak{m}_i \to \int_Y |f| \, \mathrm{d}\mu,$$

where $\sigma(z) := \mathrm{sign}(z)\sqrt{|z|}$ and the weak convergence is understood in duality with $\mathrm{C_{bs}}(Z)$. Equivalently, if $\sigma \circ f_i$ L^2-strongly converge to $\sigma \circ f$.

We say that $f_i \in \mathrm{BV}(X_i, \mathfrak{m}_i)$ converge in energy in BV to $f \in \mathrm{BV}(Y, \mu)$ if f_i converge L^1-strongly to f and

$$\lim_{i \to \infty} |Df_i|(X_i) = |Df|(Y).$$

Remark 1.117. The presence of the function σ in the definition of L^1-strong convergence is necessary due to the lack of reflexivity of L^1. Indeed the counterpart of Definition 1.115 in the case $p = 1$ is easily seen to be not equivalent to convergence in L^1 norm when all the spaces coincide.

Next we deal with the stability of L^p convergence with respect to the basic operations.

Proposition 1.118. *Let us fix* $p = 1, 2$.

(i) *For any* $f_i, g_i \in L^p(X_i, \mathfrak{m}_i)$ *such that* $f_i \to f \in L^p(Y, \mu)$ *and* $g_i \to g \in L^p(Y, \mu)$ *strongly in* L^p *one has* $f_i + g_i \to f + g$ *strongly in* L^p.

(ii) *If* $f_i \to f$ *and* $g_i \to g$ *in* L^2*-strong then* $f_i g_i \to fg$ *in* L^1*-strong.*

(iii) *If* $f_i \to f$ *in* L^1*-strong and* $\sup_{i\in\mathbb{N}} \|f_i\|_{L^\infty(X_i,\mathfrak{m}_i)} < \infty$ *then* $\|f_i\|_{L^2(X_i,\mathfrak{m}_i)} \to \|f\|_{L^2(Y,\mu)}$. *In particular,* $f_i \to f$ *in* L^2*-strong.*

(iv) *More in general, if* $f_i \in L^p(X_i,\mathfrak{m}_i)$ *converge in* L^p*-strong to* $f \in L^p(Y,\mu)$ *then* $\phi\circ f_i$ *converge to* $\phi\circ f$ *in* L^p*-strong for any* $\phi \in \mathrm{Lip}(\mathbb{R})$ *such that* $\phi(0)=0$.

Let us now introduce the notion of $H^{1,2}$-convergence, along with its local counterpart.

Definition 1.119 ($H^{1,2}$ convergence)**.** We say that $f_i \in H^{1,2}(X_i,\mathsf{d}_i,\mathfrak{m}_i)$ are weakly convergent to $f \in H^{1,2}(Y,\varrho,\mu)$ if they converge in L^2-weak and $\sup_i \mathrm{Ch}^i(f_i) < \infty$. Strong $H^{1,2}$-convergence is defined asking that f_i converge to f in L^2-strong and $\lim_i \mathrm{Ch}^i(f_i) = \mathrm{Ch}(f)$.

Definition 1.120 (Local $H^{1,2}$ convergence)**.** We say that $f_i \in H^{1,2}(B_R(x_i),\mathsf{d}_i,\mathfrak{m}_i)$ are weakly convergent in $H^{1,2}$ to $f \in H^{1,2}(B_R(y),\varrho,\mu)$ on $B_R(y)$ if f_i are L^2-weakly (or L^2-strongly, equivalently) to f on $B_R(y)$ with $\sup_{i\in\mathbb{N}}\|f_i\|_{H^{1,2}} < \infty$. Strong convergence in $H^{1,2}$ on $B_R(y)$ is defined by requiring

$$\lim_{i\to\infty}\int_{B_R(x_i)} |\nabla f_i|^2 \,\mathrm{d}\mathfrak{m}_i = \int_{B_R(y)} |\nabla f|^2 \,\mathrm{d}\mu.$$

Below we list some useful results about convergence of Sobolev spaces taken from [**18**].

Proposition 1.121 ([**18**, Lemma 5.8])**.** *Let* $f_i \in H^{1,2}(X_i,\mathsf{d}_i,\mathfrak{m}_i)$ *be weakly converging in* $H^{1,2}$ *to* $f \in H^{1,2}(Y,\varrho,\mu)$*. Then*

$$\liminf_{i\to\infty}\int_Z g\,|\nabla f_i| \,\mathrm{d}\mathfrak{m}_i \geq \int_Z g\,|\nabla f| \,\mathrm{d}\mu, \qquad \textit{for any non-negative } g \in \mathrm{Lip}_{\mathrm{bs}}(Z).$$

Proposition 1.122 ([**18**, Corollary 5.5])**.** *The following stability results hold true:*

(a) *if* $f_i \in H^{1,2}(X_i,\mathsf{d}_i,\mathfrak{m}_i)$*,* $f_i \in \mathcal{D}(\Delta_i)$ *converge in* L^2*-strong to* f *and* $\Delta_i f_i$ *are uniformly bounded in* L^2*, then* $f \in \mathcal{D}(\Delta)$*,* $\Delta_i f_i$ *converge in* L^2*-weak to* Δf *and* f_i *converge in* $H^{1,2}$*-strong to* f*;*

(b) *for all* $t>0$*,* $P^i_t f_i$ *converge in* $H^{1,2}$*-strong to* $P_t f$ *whenever* f_i *converge in* L^2*-strong to* f*.*

Theorem 1.123 ([**18**, Theorem 5.7])**.** *Let* $v_i, w_i \in H^{1,2}(X_i,\mathsf{d}_i,\mathfrak{m}_i)$ *be strongly convergent in* $H^{1,2}$ *to* $v, w \in H^{1,2}(Y,\varrho,\mu)$*, respectively. Then* $\nabla v_i\cdot\nabla w_i$ *converge* L^1*-strongly to* $\nabla v\cdot\nabla w$*.*

The following useful compactness criterion is borrowed from [**118**, Theorem 6.3] (see also [**18**, Theorem 7.4]).

Theorem 1.124. *Let* $f_i \in H^{1,2}(X_i,\mathsf{d}_i,\mathfrak{m}_i)$ *be such that*

$$\sup_i\left\{\int_Z |f_i|^2 \,\mathrm{d}\mathfrak{m}_i + \mathrm{Ch}^i(f_i)\right\} < \infty$$

and

$$\lim_{R\to\infty}\limsup_{i\to\infty}\int_{Z\setminus B_R(\bar z)} |f_i|^2 \,\mathrm{d}\mathfrak{m}_i = 0,$$

for some (and thus for all) $\bar z \in Z$*. Then* (f_i) *has a* L^2*-strongly convergent subsequence to* $f \in H^{1,2}(Y,\varrho,\mu)$*.*

We end up collecting some useful results about local convergence of Sobolev spaces.

Lemma 1.125 ([**19**, Lemma 2.10])**.** *Let* $f \in \mathrm{Lip}_c(B_R(y),\varrho)$*. Then there exists a sequence of functions* $f_i \in \mathrm{Lip}_c(B_R(x_i),\mathsf{d}_i)$ *satisfying*

$$\sup_{i\in\mathbb{N}}\||\nabla f_i|\|_{L^\infty(X_i,\mathfrak{m}_i)} < \infty$$

and strongly convergent to f *in* $H^{1,2}$*.*

Theorem 1.126 ([**19**, Theorem 4.4])**.** *Let* $f_i \in D(\Delta, B_R(x_i))$ *with*

$$\sup_{i\in\mathbb{N}} \int_{B_R(x_i)} (|f_i|^2 + |\nabla f_i|^2 + (\Delta f_i)^2)\ \mathrm{d}\mathfrak{m}_i < \infty,$$

and let f *be an* L^2*-strong limit function of* f_i *on* $B_R(y)$*. Then:*

(i) $f \in D(\Delta, B_R(y))$*;*
(ii) $\Delta f_i \to \Delta f$ *on* $B_R(y)$ *weakly in* L^2*;*
(iii) $|\nabla f_i|^2 \to |\nabla f|^2$ *on* $B_R(y)$ *strongly in* L^1*.*

Proposition 1.127 ([**19**, Corollary 4.12])**.** *Let* $f \in H^{1,2}(B_R(y), \varrho, \mu)$ *be a harmonic function. Then, for any* $0 < r < R$ *there exist* $f_i \in H^{1,2}(B_r(x_i), \mathsf{d}_i, \mathfrak{m}_i)$ *harmonic such that* $f_i \to f$ *on* $B_r(y)$ *strongly in* $H^{1,2}$*.*

CHAPTER 2

Metric measure rectifiability of $\mathrm{RCD}(K,N)$ spaces

In this chapter we present the contents of [**49**], joint work with Bruè and Pasqualetto, where we obtained simplified proofs of the rectifiability of $\mathrm{RCD}(K,N)$ metric measure spaces via the theory of δ-splitting maps.

Many efforts have been recently aimed at understanding the so-called structure theory of $\mathrm{RCD}(K,N)$ spaces. As in the more classical regularity theory in Partial Differential Equations and Geometric Measure Theory, this amounts to find points where the object looks like a regular one, usually referred to as *regular* points, to distinguish them from the *singular* ones and to control both the geometry o the regular part and the size of the singular one. Part of the job in this study is of course to understand which is the right notion of regularity.

After [**170**] by Mondino-Naber, we know that $\mathrm{RCD}(K,N)$ spaces are rectifiable as metric spaces and later, in the three independent works by De Philippis-Marchese-Rindler, Kell-Mondino and Gigli-Pasqualetto [**85**, **120**, **146**], the analysis was sharpened taking into account the behaviour of the reference measure and getting rectifiability as metric measure spaces. The development of this theory was inspired in turn by the results obtained for Ricci limit spaces in the seminal papers by Cheeger-Colding [**69**–**71**]. In the proofs given in those papers, a crucial role was played by (k,δ)-splitting maps:

Definition 2.1. Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(-1,N)$ space. Let $x\in X$ and $\delta>0$ be given. Then a map $u=(u_1,\dots,u_k)\colon B_r(x)\to\mathbb{R}^k$ is said to be a (k,δ)-*splitting map* provided:

i) $u_a\colon B_r(x)\to\mathbb{R}$ is harmonic and C_N-Lipschitz for every $a=1,\dots,k$,
ii) $r^2 \fint_{B_r(x)} |\mathrm{Hess}(u_a)|^2\,\mathrm{d}\mathfrak{m}\le\delta$ for every $a=1,\dots,k$,
iii) $\fint_{B_r(x)} |\nabla u_a\cdot\nabla u_b-\delta_{ab}|\,\mathrm{d}\mathfrak{m}\le\delta$ for every $a,b=1,\dots,k$.

These maps provide approximations, in the integral L^2-sense and up to the second order, of k independent coordinate functions in the Euclidean space and they were introduced in [**68**] in the study of Riemannian manifolds with lower Ricci curvature bounds.
Item ii) in the definition of δ-splitting maps is about smallness of the L^2-norm of the Hessian, in scale invariant sense. Let us point out that in [**69**, **71**] and in more recent works about Ricci limits as [**64**], δ-splitting maps are built only at the level of the smooth approximating sequence, where there is a clear notion of Hessian available. The metric information they encode (ε-GH closeness to spaces splitting Euclidean factors, cf. Section 2 below) is then passed to the limit.
Prior than [**112**], there was no notion of Hessian available in the RCD framework. This fact, together with the absence of smooth approximating sequences, motivated the necessity to find an alternative approach to rectifiability in [**85**, **120**, **146**, **170**] with respect to the Cheeger-Colding theory. A new *almost splitting via excess* theorem was the main ingredient playing the role of the theory of δ-splitting maps in [**170**] while, studying the behaviour of the reference measure with respect to charts, a crucial role was played in both [**85**, **120**, **146**] by a recent and powerful result obtained by De Philippis-Rindler [**86**].

Nowadays we have at our disposal both a second order differential calculus on RCD spaces [**112**] and general convergence and stability results for Sobolev functions on converging sequences of RCD(K,N) spaces [**18, 19, 118**]. In a previous collaboration with Bruè and Pasqualetto [**48**] we exploited all these tools to prove rectifiability for reduced boundaries of sets of finite perimeter in this context. The study of [**48**] was devoted to the theory in codimension one, which required some additional ideas and technical efforts, but it was evident that similar arguments could provide more direct proofs of rectifiability for RCD(K,N) spaces in the spirit of those in [**69, 71**]. In the treatment of the reference measure the necessity of a new tool was also motivated by the failure in the setting of weighted Riemannian manifolds of one of the key lemmas valid in the non weighted case and used in [**70**].

Taking as a starting point existence of Euclidean tangents almost everywhere with respect to the reference measure, obtained by Gigli-Mondino-Rajala in [**117**], in [**49**] we provided the arguments to get uniqueness (almost everywhere) of tangents and rectifiability of RCD(K,N) spaces as metric measure spaces via δ-splitting maps. Moreover, we recovered via a different strategy the lower semicontinuity of the so called essential dimension proved firstly in [**152**]. A beautiful argument to obtain uniqueness of tangents from the lower semicontinuity of the dimension partially bypassing [**117**] has been pointed out to us by one of the reviewers of [**49**] and can be found in the paper. For the sake of this presentation we chose the original argument, since the strategy of the present proof can be generalized to the theory in codimension one.

This chapter is organised as follows: the preliminary Section 1 is dedicated to the introduction of the main notions of rectifiability in this context. Section 2 is then devoted to the introduction of the notion of δ-splitting map and of their main properties, in particular to the equivalence between closeness to products with Euclidean factors and existence of splitting maps. Sections 3 to 5 are then dedicated to establish uniqueness of tangents, rectifiability as metric spaces and rectifiability of the reference measure, respectively. In order to achieve the absolute continuity of the reference measure with respect to the relevant Hausdorff measure we rely on [**86**], as in the previous approaches.

Since the results contained in this chapter and in the reference paper [**49**] are not new a comparison is in order, mainly with the strategy adopted in [**170**]. The two approaches have indeed the same key ingredient, that is a good enough approximation of coordinate maps on Euclidean tangent spaces. While in [**170**] the approximating maps are built in terms of distance functions, in our approach we deal with the harmonic approximations of the coordinates, relying on [**19**]. While the improvement upon the regularity of the charts, from distance functions to harmonic functions, does not lead to new results in the theory in codimension zero, it is expected that this can lead to a better understanding of the structure in positive codimension, as it has been the case for (non collapsed) Ricci limit spaces in [**64, 73, 139**] and for the theory fo sets of finite perimeter we shall present in Chapters 4 and 5.

The role of this chapter in the context of this thesis is twofold. On the one hand we report about the new proofs for the structure theorems obtained in [**49**], on the other one all the the results we present below played a crucial preliminary role both in the development of the theory of sets of finite perimeter [**8, 49**] (cf. Chapters 4 and 5) and in the proof of constancy of the dimension in [**52**] (cf. Chapter 3).

1. Strongly $\mathfrak{m}$-rectifiable metric measure spaces

Below we report the classical definition of $\mathfrak{m}$-*rectifiable metric space* and quote from [**121**] the definition of *(strongly)* $\mathfrak{m}$-*rectifiable space*. The rest of the chapter will be dedicated to the proof of strong $\mathfrak{m}$-rectifiability for RCD(K,N) metric measure spaces.

Definition 2.2 (𝔪-rectifiable metric space)**.** Given a metric measure space $(X, \mathsf{d}, \mathfrak{m})$ we say that it is $\mathfrak{m}$-rectifiable as a metric space provided it can be covered by a countable disjoint union $\bigcup_{k\in\mathbb{N}} A_k$, where $A_k \in \mathcal{B}(X)$ for any $n, k \in \mathbb{N}$ there exists a countable union $\bigcup_{n\in\mathbb{N}} U_n^k$ with $U_k^n \in \mathcal{B}(X)$ such that

i) $\mathfrak{m}(A_k \setminus \bigcup_{n\in\mathbb{N}} U_n^k) = 0$;
ii) for any $n \in \mathbb{N}$ there exist a biLipschitz map $\varphi_n^k : U_n^k \to \varphi(U_n^k) \subset \mathbb{R}^k$ that we shall indicate by *chart*.

Whenever we want to stress the dimension of the target space of the charts and that we can take $(1+\varepsilon)$ as biLipschitz constant, we might also say that A_k is $(\mathfrak{m}, k, \varepsilon)$-rectifiable. If A_k is $(\mathfrak{m}, k, \varepsilon)$-rectifiable for any $\varepsilon > 0$ then we will say that it is a strongly $\mathfrak{m}$-rectifiable metric space.

Definition 2.3 (𝔪-rectifiable metric measure space)**.** We say that $(X, \mathsf{d}, \mathfrak{m})$ is $\mathfrak{m}$-rectifiable as metric measure space provided the charts in Definition 2.2 above can be chosen in such a way that

$$(\varphi_n^k)_\# \mathfrak{m} \ll \mathcal{L}^k, \quad \text{for any } k, n \in \mathbb{N}.$$

Furthermore, we say that $(X, \mathsf{d}, \mathfrak{m})$ is strongly $\mathfrak{m}$-rectifiable as metric measure space if, for any $\varepsilon > 0$, the charts φ_n^k can be chosen to be $(1+\varepsilon)$-biLipschitz.

2. Splitting maps on RCD spaces

This section is devoted to the study of δ-splitting maps. Let us recall that their introduction in the study of spaces with lower Ricci curvature bounds dates back to [**68**]. Since then they have been used for a wide range of applications in the setting of Ricci limit spaces (see [**63**–**65**, **69**–**71**]). The use of δ-splitting maps in the RCD framework has remained elusive in the first developments of the theory. Indeed, prior than [**112**], there was no notion of Hessian available to give a meaning to condition ii) in Definition 2.5 below. While the use of this tool was implicit in [**20**], in two collaborations with Bruè and Pasqualetto [**48**, **49**] we explicitly stated and proved some useful properties of δ-splitting maps.

Let us make a few preliminary observations in order to motivate the rest of the discussion. The first one is that on a space of the form $\mathbb{R}^k \times Z$ the k coordinate functions of the Euclidean factor are harmonic, they have vanishing Hessian and they have gradients which are orthogonal in the $\mathfrak{m}$-a.e. sense. These properties can be easily verified relying on the tensorization of the Cheeger energy on product spaces, see Remark 1.110. The second remark is that also the converse implication is true, as we argue in the next lemma. We refer to [**33**, Lemma 1.21] for a proof.

Lemma 2.4. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an* $\mathrm{RCD}(0, N)$ *m.m.s. and suppose that there exist functions $u_1, \dots, u_k$ such that $\Delta u_i = 0$, $|\nabla u_i| = 1$ and $\nabla u_i \cdot \nabla u_j = 0$ for any $i, j = 1, \dots, k$. Then there exists an* $\mathrm{RCD}(0, N-k)$ *m.m.s. $(Z, \mathsf{d}_Z, \mathfrak{m}_Z)$ such that $(X, \mathsf{d}, \mathfrak{m})$ is isomorphic to* $(\mathbb{R}^k, \mathsf{d}_{\mathrm{Eucl}}, \mathcal{L}^k) \times (Z, \mathsf{d}_Z, \mathfrak{m}_Z)$.

Maps of δ-splitting are (local) approximations in the integral L^2-sense of coordinate functions over the Euclidean factor of a split space. In the definition below $C_N > 1$ is a constant depending only on the upper dimension bound, whose explicit value can be computed following [**136**].

Definition 2.5 (Splitting map)**.** Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(-1, N)$ space. Let $x \in X$ and $\delta > 0$ be given. Then a map $u = (u_1, \dots, u_k) \colon B_r(x) \to \mathbb{R}^k$ is said to be a *δ-splitting map* provided:

i) $u_a \colon B_r(x) \to \mathbb{R}$ is harmonic and C_N-Lipschitz for every $a = 1, \dots, k$,
ii) $r^2 \fint_{B_r(x)} |\mathrm{Hess}(u_a)|^2 \,\mathrm{d}\mathfrak{m} \le \delta$ for every $a = 1, \dots, k$,

iii) $\fint_{B_r(x)} |\nabla u_a \cdot \nabla u_b - \delta_{ab}| \, \mathrm{d}\mathfrak{m} \le \delta$ for every $a, b = 1, \dots, k$.

Remark 2.6. We refer to Remark 1.108 where we clarified the meaning of $|\operatorname{Hess} u|$ when $u : B_r(x) \to \mathbb{R}$ is harmonic and not necessarily globally defined thanks to the existence of good cut-off functions.

Remark 2.7. Let us point out that if $u : B_r(x) \to \mathbb{R}^k$ is a δ-splitting map then for any orthogonal linear transformation $A : \mathbb{R}^k \to \mathbb{R}^k$ the map $v := A \circ u : B_r(x) \to \mathbb{R}^k$ is still a δ-splitting map. More in general this is the case even under the assumption that A is δ-close to an orthogonal transformation, up to turn the δ-splitting property into a $C\delta$-splitting property for some constant $C > 0$.

Remark 2.8. Let us explicitly point out, since this remark will play a role in the forthcoming arguments, that the integral quantities appearing in ii) and iii) of Definition 2.5 are invariant if the metric measure space and the function are scaled accordingly.

Remark 2.9. With respect to the definition of δ-splitting map which is nowadays adopted within the theory of Ricci limits (see for instance [**65**, Definition 1.20]) the main difference is condition (i). Therein the sharper bound $|\nabla u| \le 1 + \delta$ is imposed in the definition though, as they observe, it can be obtained as a consequence of the bound $|\nabla u| \le C_N$ and of the other defining properties (when working in the smooth framework), see in particular [**65**, (3.45)-(3.48)].

Remark 2.10. Let us point out that, if we assume that $r^2 \le \delta$, then a map $u : B_r(x) \to \mathbb{R}^k$ verifying i) and iii) in Definition 2.5 is a $C_N\delta$-splitting map. In order to prove this statement it is sufficient to apply Corollary 1.109 to each component u^i of u, taking into account the assumption that u^i is harmonic (therefore the first term at the right handside vanishes) and that condition iii), applied with choice $a = b = i$, yields

$$\fint_{B_r(x)} \left| \left|\nabla u^i\right|^2 - 1 \right| \mathrm{d}\mathfrak{m} \le \delta, \quad \text{for any } i = 1, \dots, k.$$

Even though in the applications to the theory of rectifiability we will build δ-splitting maps only on balls that, when rescaled to radius one, have lower Ricci bound $-\delta$, we chose to include also the bound on the scale invariant Hessian in the defining conditions for two reasons:

- coherence with the literature of Ricci limit spaces;
- to always have clear the properties at our disposal.

2.1. δ-splitting maps and ϵ-closeness. The power of δ-splitting maps in the theory of lower Ricci bounds is that, roughly speaking, they allow to pass from analysis to geometry and vice-versa. Namely, the existence of a δ-splitting map with k components on a Riemannian manifold with Ricci bounded below by $-\delta$ can be turned into ε-GH closeness (in the scale invariant sense) to a space which splits a factor $\mathbb{R}^k$ and vice-versa (see [**68**] and [**65**, Lemma 1.21]).

Below we wish to provide rigorous statements of the above-mentioned statements in the framework of RCD spaces. The convergence and stability results of [**18**, **19**, **118**] allow us to argue by compactness avoiding the explicit constructions of [**68**]. The price we have to pay is that the statements become less local in nature w.r.t. [**65**, Lemma 1.21], still they are sufficient for our purposes. Let us also mention that in a work in progress with Bruè and Naber [**47**] we prove the local version of the equivalence between existence of δ-splitting maps and ε-closeness to spaces splitting Euclidean factors, therefore closing the gap with the theory of Ricci limits.

Proposition 2.11 ensures that, over an RCD$(-\varepsilon, N)$ space ε-close to a product $\mathbb{R}^k \times Z$, one can build a δ-splitting map with k components. Proposition 2.12 instead corresponds to the rough statement "the existence of a δ-splitting map with k components implies that the m.m.s. is ε-close to a product $\mathbb{R}^k \times Z$".

Proposition 2.11 (From GH-isometry to δ-splitting)**.** *Let $N > 1$ be given. Then for any $\delta > 0$ there exists $\varepsilon = \varepsilon_{N,\delta} > 0$ such that the following property holds. If $(X, \mathsf{d}, \mathfrak{m})$ is an* RCD(K,N) *space, $x \in X$, $r > 0$ with $r^2|K| \le \varepsilon$, and there is an* RCD$(0, N-k)$ *space $(Z, \mathsf{d}_Z, \mathfrak{m}_Z, z)$ such that*

$$\mathsf{d}_{\mathrm{pmGH}}\Big((X, \mathsf{d}/r, \mathfrak{m}_x^r, x), (\mathbb{R}^k \times Z, \mathsf{d}_{\mathrm{Eucl}} \times \mathsf{d}_Z, \mathcal{L}^k \otimes \mathfrak{m}_Z, (0^k, z))\Big) \le \varepsilon,$$

then there exists a δ-splitting map $u\colon B_{5r}(x) \to \mathbb{R}^k$.

Proof. Let us begin pointing out that, by scaling, it is sufficient to prove the following statement: for any $\delta > 0$ there exists $\varepsilon = \varepsilon_{N,\delta} > 0$ such that, if $(X, \mathsf{d}, \mathfrak{m})$ is an RCD$(-\varepsilon, N)$ m.m.s., $x \in X$ and

$$\mathsf{d}_{pmGH}\left((X, \mathsf{d}, \mathfrak{m}, x), \left(\mathbb{R}^k \times Z, (0^k, z)\right)\right) < \varepsilon$$

for some pointed RCD$(0, N-k)$ metric measure space $(Z, \mathsf{d}_Z, \mathfrak{m}_Z, z)$, then there exists a δ-splitting map $u : B_5(x) \to \mathbb{R}^k$.

We are going to build upon the local convergence and stability results that we recalled in Section 4.2, arguing by contradiction.

Suppose the conclusion to be false, then we could find a sequence of RCD$(-1/n, N)$ pointed m.m. spaces $(X_n, \mathsf{d}_n, \mathfrak{m}_n, x_n)$ such that, for some RCD$(0, N-k)$ pointed m.m.s. $(Z, \mathsf{d}_Z, \mathfrak{m}_Z, z)$ it holds that

$$\mathsf{d}_{pmGH}\left((X_n, \mathsf{d}_n, \mathfrak{m}_n, x_n), \left(\mathbb{R}^k \times Z, (0^k, z)\right)\right) < 1/n$$

for any $n \ge 1$. Furthermore there should be $\delta_0 > 0$ such that there is no δ_0-splitting map over $B_5(x_n)$ for any $n \ge 1$.

Let $v : \mathbb{R}^k \times Z \to \mathbb{R}^k$ be defined by $v(p, x) = x$ and denote by $v^1, \dots, v^k$ its components (they are the coordinate functions of the split factor). Observe that $\Delta v^i = 0$ for any $i = 1, \dots, k$ and $\nabla v^i \cdot \nabla v^j = \delta_{ij}$ for any $i, j = 1, \dots, k$. In particular, v^i is harmonic on $B_{10}((z, 0^k))$. Hence we can apply Proposition 1.127 to get harmonic functions $v_n^i : B_9(x_n) \to \mathbb{R}$ that converge strongly in $H^{1,2}$ to v^i on $B_9((z, 0^k))$.

Observe that, thanks to [**136**, Theorem 1.1], we can assume that v_n^i is C_N-Lipschitz for any $n \in \mathbb{N}$ and for any $i = 1, \dots, k$. We wish to prove that $v_n = (v_n^1, \dots, v_n^k)$ is a δ_0-splitting map on $B_5(x_n)$ for n sufficiently big.
To this aim let us recall that Theorem 1.126 yields strong L^1-convergence of $\nabla v_n^i \cdot \nabla v_n^j$ to δ_{ij} on $B_9((z, 0^k))$ and on $B_5((z, 0^k))$ for any $i, j = 1, \dots, k$ (as a consequence of the L^1 convergence of $\nabla v_n^i \cdot \nabla v_n^i$ and of $\nabla(v_n^i + v_n^j) \cdot \nabla(v_n^i + v_n^j)$). In particular, due to the uniform boundedness of the gradients, we get

$$\lim_{n \to \infty} \fint_{B_R(x_n)} \left| \nabla v_n^i \cdot \nabla v_n^j - \delta_{ij} \right| \mathrm{d}\mathfrak{m}_n = 0, \tag{2.1}$$

for any $i, j = 1, \dots, k$ and for any $R = 5, 9$. The choice $R = 5$ gives the second defining condition of δ-splitting map for n sufficiently large and we are left with the verification of the third one.
We wish to prove that

$$\lim_{n \to \infty} \int_{B_5(x_n)} \left| \mathrm{Hess}\, v_n^i \right|^2 \mathrm{d}\mathfrak{m}_n = 0 \tag{2.2}$$

for any $i = 1, \dots, k$. As we already pointed out in Remark 2.10, (2.2) follows from the improved Bochner inequality with Hessian term integrated against a good cut-off function Corollary 1.109 taking into account (2.1).

□

Proposition 2.12 (From δ-splitting to GH-isometry). *Let $N > 1$ be given. Then for any $\varepsilon > 0$ there exists $\delta = \delta_{N,\varepsilon} > 0$ such that the following property holds. Let $(X, \mathsf{d}, \mathfrak{m})$ be an* RCD(K,N) *space, $x \in X$, and let $u\colon B_r(x) \to \mathbb{R}^k$ be such that $u\colon B_s(x) \to \mathbb{R}^k$ is a δ-splitting map for all $s < r$. Then for any $(Y, \varrho, \mathfrak{n}, y) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m})$ it holds that*

$$\mathsf{d}_{\mathrm{pmGH}}\Big((Y, \varrho, \mathfrak{n}, y), (\mathbb{R}^k \times Z, \mathsf{d}_{\mathrm{Eucl}} \times \mathsf{d}_Z, \mathcal{L}^k \otimes \mathfrak{m}_Z, (0^k, z))\Big) \le \varepsilon,$$

for some pointed RCD$(0, N-k)$ *space $(Z, \mathsf{d}_Z, \mathfrak{m}_Z, z)$.*

Proof. We claim that the following statement holds: for any $\varepsilon > 0$, there exists $\delta = \delta_{N,\varepsilon} > 0$ such that, for any RCD$(-\delta, N)$ normalized p.m.m.s. $(X, \mathsf{d}, \mathfrak{m}, x)$, if there exists a map $u : B_{\delta^{-1}}(x) \to \mathbb{R}^k$ such that u is a δ-splitting map over $B_s(x)$ for any $0 < s < \delta^{-1}$, then

$$\mathsf{d}_{pmGH}\left((X, \mathsf{d}, \mathfrak{m}, x), (\mathbb{R}^k \times Z, (0^k, z))\right) < \varepsilon$$

for some pointed RCD$(0, N-k)$ metric measure space $(Z, \mathsf{d}_Z, \mathfrak{m}_Z, z)$.

Let us see how to conclude the proof given this statement.
Choose $\delta = \delta(K, N, \varepsilon/2)$ given by the claim above. If $(Y, \varrho, \mu, y) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m})$ then there exists $t > 0$ such that $t^{-1}r > \delta^{-1}$, $t^2 |K| \le \delta$ and

$$\mathsf{d}_{pmGH}\left((X, t^{-1}\mathsf{d}, \mathfrak{m}_x^t, x), (Y, \varrho, \mu, y)\right) < \varepsilon/2. \tag{2.3}$$

Thanks to the claim applied to $(X, t^{-1}\mathsf{d}, \mathfrak{m}_x^t, x)$, there exists an RCD$(0, N-k)$ p.m.m.s. $(Z, \mathsf{d}_Z, \mathfrak{m}_Z, z)$ such that

$$\mathsf{d}_{pmGH}\left((X, t^{-1}\mathsf{d}, \mathfrak{m}_x^t, x), (Z \times \mathbb{R}^k, (z, 0^k))\right) < \varepsilon/2. \tag{2.4}$$

The conclusion follows from (2.3) and (2.4) by the triangle inequality.

Let us pass to the proof of the claim.
We wish to prove it arguing by contradiction. To this aim let us suppose that, for any $n \ge 1$, there exist an RCD$(-1/n, N)$ m.m.s. $(X_n, \mathsf{d}_n, \mathfrak{m}_n)$, a point $x_n \in X_n$ and a map $u_n : B_n(x_n) \to \mathbb{R}^k$ which is a $1/n$-splitting map when restricted to $B_s(x_n)$ for any $0 < s < n$. Up to extracting a subsequence that we do not relabel, we can assume that $(X_n, \mathsf{d}_n, \mathfrak{m}_n, x_n)$ converge in the pmGH-topology to an RCD$(0,N)$ p.m.m.s. $(X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, x_\infty)$. Here we have used the stability and compactness property of RCD(K,N) spaces, cf. Remark 1.114. We claim that X_∞ splits off a factor $\mathbb{R}^k$. Observe that, if this is the case, then we reach the sought contradiction. The rest of this proof is dedicated to establishing the claim.

We wish to prove that there exists a function $v : X_\infty \to \mathbb{R}^k$ such that, letting $v := (v^1, \dots, v^k)$, it holds that v^i is Lipschitz, harmonic and with vanishing Hessian for any $i = 1, \dots, k$ and $\nabla v^i \cdot \nabla v^j = \delta_{ij}$ $\mathfrak{m}_\infty$-a.e. for any $i, j = 1, \dots, k$. The function v will be obtained as a limit function of the $1/n$-splitting maps $u_n : B_n(x_n) \to \mathbb{R}^k$. Indeed, by Definition 2.5 (i) the u_n are C_N-Lipschitz for any $n \in \mathbb{N}$. Moreover we can assume without loss of generality that $u_n(x_n) = 0^k$ for any $n \in \mathbb{N}$. Therefore by the generalized Ascoli–Arzelà theorem (Proposition 1.29) and a diagonal argument we can infer the existence of $v : X_\infty \to \mathbb{R}^k$ such that u_n converge to v locally uniformly on $B_R(x_n)$ for any $R > 0$. As a consequence, it is easy to check that u_n converge strongly in L^2 (see Definition 1.115) to v on $B_R(x_n)$ for any $R > 0$. Since the functions u_n are harmonic on $B_{2R}(x_n)$, at least for n sufficiently large, by

Theorem 1.126 and Proposition 1.118 it follows that v is harmonic and that, for any $R > 0$ and $i, j = 1, \dots, k$,

$$\fint_{B_R(x_\infty)} \left|\nabla v^i \cdot \nabla v^j - \delta_{ij}\right| \,\mathrm{d}\mathfrak{m}_\infty = \lim_{n\to\infty} \fint_{B_R(x_n)} \left|\nabla u_n^i \cdot \nabla u_n^j - \delta_{ij}\right| \,\mathrm{d}\mathfrak{m}_n = 0.$$

Hence $\nabla v^i \cdot \nabla v^j = \delta_{ij}$ $\mathfrak{m}_\infty$-a.e. on X_∞.

Since $(X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty)$ is an $\mathrm{RCD}(0, N)$ m.m.s., from $\Delta v^i = 0$ and $\left|\nabla v^i\right|^2 = 1$ we infer by (1.78) that $\mathrm{Hess}\, v^i = 0$, for any $i = 1, \dots, k$. Thanks to Lemma 2.4 the space X_∞ splits of a factor $\mathbb{R}^k$. □

2.2. Propagation of regularity. With the aim of proving uniqueness of tangents and rectifiability, given the results of Section 2.1, it will be useful to propagate the information "there exists a δ-splitting map over a given ball" at many locations and at scales. In order to do so we rely on a maximal function argument, exploiting the integral nature of the defining conditions of δ-splitting maps.

Let us point out that analogous arguments were used also in the structure theory of Ricci limits by Cheeger-Colding and by Mondino-Naber in [**170**]. Moreover, in Chapter 5 we will provide a generalization of the statement below, suited for the theory in codimension one (cf. in particular Proposition 5.15).

Proposition 2.13 (Propagation of the δ-splitting property)**.** *Let $N > 1$ be given. Then there exists a constant $C_N > 0$ such that the following property holds. If $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(K, N)$ space and $u\colon B_{2r}(p) \to \mathbb{R}^k$ is a δ-splitting map for some $p \in X$, $r > 0$ with $r^2|K| \le 1$, and $\delta \in (0, 1)$, then there exists a Borel set $G \subseteq B_r(p)$ such that $\mathfrak{m}(B_r(p) \setminus G) \le C_N\sqrt{\delta}\,\mathfrak{m}(B_r(p))$ and*

$$u\colon B_s(x) \to \mathbb{R}^k \text{ is a } \sqrt{\delta}\text{-splitting map}, \quad \text{for every } x \in G \text{ and } s \in (0, r).$$

Proof. Thanks to a scaling argument, it is sufficient to prove the claim in the case in which $r = 1$ and $|K| \le 1$. Let us define $G \subseteq B_1(p)$ as $G := \bigcap_{a=1}^k G_a \cap \bigcap_{a,b=1}^k G_{a,b}$, where we set

$$G_a := \left\{x \in B_1(p) \;\middle|\; \sup_{s\in(0,1)} \fint_{B_s(x)} \left|\mathrm{Hess}(u_a)\right|^2 \,\mathrm{d}\mathfrak{m} \le \sqrt{\delta}\right\},$$

$$G_{a,b} := \left\{x \in B_1(p) \;\middle|\; \sup_{s\in(0,1)} \fint_{B_s(x)} \left|\nabla u_a \cdot \nabla u_b - \delta_{ab}\right|^2 \,\mathrm{d}\mathfrak{m} \le \sqrt{\delta}\right\}.$$

It holds that $u\colon B_s(x) \to \mathbb{R}^k$ is a $\sqrt{\delta}$-splitting map for all $x \in G$ and $s \in (0, 1)$. To prove the claim, it remains to show that $\mathfrak{m}(B_1(p) \setminus G_a), \mathfrak{m}(B_1(p) \setminus G_{a,b}) \le C_N\sqrt{\delta}\,\mathfrak{m}(B_1(p))$ for all $a, b = 1, \dots, k$.

Given any $x \in B_1(p)\setminus G_a$, we can choose $s_x \in (0, 1)$ such that $\fint_{B_{s_x}(x)} \left|\mathrm{Hess}(u_a)\right|^2 \,\mathrm{d}\mathfrak{m} > \sqrt{\delta}$. By using Vitali covering Theorem 1.9, we can find a sequence $(x_i)_i \subseteq B_1(p) \setminus G_a$ such that $\left\{B_{s_{x_i}}(x_i)\right\}_i$ are pairwise disjoint and $B_1(p) \setminus G_a \subseteq \bigcup_i B_{5s_{x_i}}(x_i)$. Therefore

$$\begin{aligned}\mathfrak{m}(B_1(p) \setminus G_a) &\le \sum_{i\in\mathbb{N}} \mathfrak{m}(B_{5s_{x_i}}(x_i)) \le C_N \sum_{i\in\mathbb{N}} \mathfrak{m}(B_{s_{x_i}}(x_i)) \le \frac{C_N}{\sqrt{\delta}} \sum_{i\in\mathbb{N}} \int_{B_{s_{x_i}}(x_i)} \left|\mathrm{Hess}(u_a)\right|^2 \,\mathrm{d}\mathfrak{m} \\ &\le \frac{C_N \mathfrak{m}(B_2(p))}{\sqrt{\delta}} \fint_{B_2(p)} \left|\mathrm{Hess}(u_a)\right|^2 \,\mathrm{d}\mathfrak{m} \le C_N\sqrt{\delta}\,\mathfrak{m}(B_1(p)),\end{aligned}$$

where we used the doubling property of $\mathfrak{m}$, the defining property of s_{x_i}, and the fact that u is a δ-splitting map on $B_2(p)$.

An analogous argument shows that $\mathfrak{m}(B_1(p) \setminus G_{a,b}) \le C_N\sqrt{\delta}\,\mathfrak{m}(B_1(p))$ for all $a, b = 1, \dots, k$, thus the statement is achieved. □

3. Uniqueness of tangents to RCD spaces

Aim of this section is to give a proof of uniqueness and regularity of tangents over RCD(K,N) metric measure spaces up to measure zero sets, relying on the theory of δ-splitting maps. As we already pointed out this result was proved for the first time in [**170**] in this framework, while the proof for Ricci limits dates back to [**69**].

As in [**170**], we take as a starting point [**117**] by Gigli-Mondino-Rajala, where the authors proved existence of Euclidean tangents outside of a negligible set.
The fundamental tools for the proof of Theorem 2.15 given in [**117**] were the splitting theorem for RCD$(0,N)$ spaces [**108, 109**] and an instance of the so called *iterated tangent property* that we quote below.

Theorem 2.14 (Iterated tangent property [**117**])**.** *Let* $(X,\mathsf{d},\mathfrak{m})$ *be an* RCD(K,N) *space. Then for* $\mathfrak{m}$*-a.e. point* $x\in X$ *it holds that*

$$\mathrm{Tan}_z(Y,\varrho,\mathfrak{n})\subseteq \mathrm{Tan}_x(X,\mathsf{d},\mathfrak{m}) \quad \textit{for every } (Y,\varrho,\mathfrak{n},y)\in\mathrm{Tan}_x(X,\mathsf{d},\mathfrak{m}) \textit{ and } z\in Y.$$

Let us remark that Theorem 2.14 above sharpens a previous result obtained by Le Donne in [**157**] dealing with doubling metric spaces, which, in turn, was inspired by the seminal paper by Preiss [**182**]. In the forthcoming Chapter 4 (see in particular Theorem 4.41) we will provide another instance of this phenomenon in the case of sets of finite perimeter over RCD(K,N) spaces, taken from the joint work with Ambrosio and Bruè [**8**].

Theorem 2.15 (Euclidean tangents to RCD spaces [**117**])**.** *Let* $(X,\mathsf{d},\mathfrak{m})$ *be an* RCD(K,N) *space. Then for* $\mathfrak{m}$*-a.e. point* $x\in X$ *there exists* $k(x)\in\mathbb{N}$ *with* $k(x)\le N$ *such that*

$$(\mathbb{R}^{k(x)},\mathsf{d}_{\mathrm{Eucl}},c_{k(x)}\mathcal{L}^{k(x)},0^{k(x)})\in\mathrm{Tan}_x(X,\mathsf{d},\mathfrak{m}),$$

where we set $c_k:=\mathcal{L}^k(B_1(0^k))/(k+1)$ *for every* $k\in\mathbb{N}$.

In order to prove that there exist Euclidean tangents almost everywhere, in [**117**] the author first argue that almost every point is an intermediate point of a geodesic. Therefore, any element of the tangent cone at those points contains a line and hence it splits a Euclidean factor by the splitting Theorem 1.111. Iterating this argument one can recover Theorem 2.15 by Theorem 2.14.

Motivated by the analogous definitions given in [**69**], in [**170**] the following notion of regular points and sets were introduced.

Definition 2.16. Let $(X,\mathsf{d},\mathfrak{m})$ be an RCD(K,N) space. Then we define

$$\mathcal{R}_k:=\left\{x\in X\;\middle|\;\mathrm{Tan}_x(X,\mathsf{d},\mathfrak{m})=\{(\mathbb{R}^k,\mathsf{d}_{\mathrm{Eucl}},c_k\mathcal{L}^k,0^k)\}\right\}\quad\text{for every } k\in\mathbb{N}\text{ with } k\le N.$$

The elements of $\mathcal{R}_k$ are said to be the *k-regular points* in X.

Remark 2.17. The value of the normalizing constant c_k depends on the scaling for the measure we chose in Definition 1.24.

Observe that, if $x\in\mathcal{R}_k$, then one has

$$\lim_{r\to 0}\frac{\int_{B_r(x)}\left(1-\frac{\mathsf{d}(x,y)}{r}\right)\,\mathrm{d}\mathfrak{m}(y)}{\mathfrak{m}(B_r(x))}=\frac{1}{k+1}. \tag{2.5}$$

Moreover it can be easily checked that $x\in\mathcal{R}_k$ if and only if

$$\lim_{r\to 0}\mathsf{d}_{pmGH}\left(\left(X,r^{-1}\mathsf{d},\frac{\mathfrak{m}}{\mathfrak{m}(B_r(x))},x\right),\left(\mathbb{R}^k,\mathsf{d}_{\mathrm{eucl}},\frac{1}{\omega_k}\mathcal{L}^k,0^k\right)\right)=0.$$

Let us introduce also some auxiliary terminology. Given any point $x\in X$ and any $k\in\mathbb{N}$, we say that an element $(Y,\varrho,\mathfrak{n},y)\in\mathrm{Tan}_x(X,\mathsf{d},\mathfrak{m})$ *splits off a factor* $\mathbb{R}^k$ provided

$$(Y,\varrho,\mathfrak{n},y)\cong(\mathbb{R}^k\times Z,\mathsf{d}_{\mathrm{Eucl}}\times\mathsf{d}_Z,\mathcal{L}^k\otimes\mathfrak{m}_Z,(0^k,z))$$

for some pointed $\mathrm{RCD}(0, N-k)$ space $(Z, \mathsf{d}_Z, \mathfrak{m}_Z, z)$.

Theorem 2.18 (Uniqueness of tangents)**.** *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *space. Then*

$$\mathfrak{m}\Big(X \setminus \bigcup_{k \le N} \mathcal{R}_k\Big) = 0.$$

Proof. The proof will be achieved through three intermediate steps where we cover the metric measure space by auxiliary sets.

Step 1. Fix any $k \in \mathbb{N}$ with $k \le N$. We define the auxiliary sets $A_k, A'_k \subseteq X$ as follows:

i) A_k is the family of all points $x \in X$ such that $(\mathbb{R}^k, \mathsf{d}_{\mathrm{Eucl}}, c_k \mathcal{L}^k, 0^k) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m})$, but no other element of $\mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m})$ splits off a factor $\mathbb{R}^k$.

ii) A'_k is the family of all points $x \in X$ which satisfy $(\mathbb{R}^k, \mathsf{d}_{\mathrm{Eucl}}, c_k \mathcal{L}^k, 0^k) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m})$ and $(\mathbb{R}^\ell, \mathsf{d}_{\mathrm{Eucl}}, c_\ell \mathcal{L}^\ell, 0^\ell) \notin \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m})$ for every $\ell \in \mathbb{N}$ with $\ell > k$.

Observe that $\mathcal{R}_k \subseteq A_k \subseteq A'_k$. The $\mathfrak{m}$-measurability of the sets $\mathcal{R}_k, A_k, A'_k$ can be proven by adapting the proof of [**170**, Lemma 6.1]. It also follows from the very definitions that $\mathfrak{m}(X \setminus \bigcup_{k \le N} A'_k) = 0$.

Step 2. We aim to prove that $\mathfrak{m}(A'_k \setminus A_k) = 0$ arguing by contradiction. Suppose $\mathfrak{m}(A'_k \setminus A_k) > 0$, then we can find a point $x \in A'_k \setminus A_k$ where the iterated tangent property of Theorem 2.14 holds. Since $x \notin A_k$, there exists a pointed $\mathrm{RCD}(0, N-k)$ space $(Y, \varrho, \mathfrak{n}, y)$ with $\mathrm{diam}(Y) > 0$ such that

$$(\mathbb{R}^k \times Y, \mathsf{d}_{\mathrm{Eucl}} \times \varrho, \mathcal{L}^k \otimes \mathfrak{n}, (0^k, y)) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}).$$

Theorem 2.15 yields the existence of a point $z \in Y$ such that it holds $(\mathbb{R}^\ell, \mathsf{d}_{\mathrm{Eucl}}, c_\ell \mathcal{L}^\ell, 0^\ell) \in \mathrm{Tan}_z(Y, \varrho, \mathfrak{n})$, for some $\ell \in \mathbb{N}$ with $0 < \ell \le N - k$. This implies that

$$(\mathbb{R}^{k+\ell}, \mathsf{d}_{\mathrm{Eucl}}, c_{k+\ell} \mathcal{L}^{k+\ell}, 0^{k+\ell}) \in \mathrm{Tan}_{(0^k, z)}(\mathbb{R}^k \times Y, \mathsf{d}_{\mathrm{Eucl}} \times \varrho, \mathcal{L}^k \otimes \mathfrak{n}).$$

Therefore, Theorem 2.14 guarantees that $(\mathbb{R}^{k+\ell}, \mathsf{d}_{\mathrm{Eucl}}, c_{k+\ell} \mathcal{L}^{k+\ell}, 0^{k+\ell}) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m})$, which contradicts the fact that $x \in A'_k$. Consequently, we have proven that $\mathfrak{m}(A'_k \setminus A_k) = 0$, as desired.

Step 3. In order to complete the proof of the statement, it suffices to show that

$$\mathfrak{m}\big(B_R(p) \cap (A_k \setminus \mathcal{R}_k)\big) = 0 \quad \text{for every } p \in X \text{ and } R > 0. \tag{2.6}$$

Let $p \in X$ and $R, \eta > 0$ be fixed. Choose any $\delta \in (0, \eta)$ associated with η as in Proposition 2.12. Moreover, choose any $\varepsilon \in (0, 1/7)$ associated with δ^2 as in Proposition 2.11. Given a point $x \in A_k$, we can find $r_x \in (0,1)$ such that $4r_x^2 |K| \le \varepsilon$ and

$$\mathsf{d}_{\mathrm{pmGH}}\Big((X, \mathsf{d}/(2r_x), \mathfrak{m}^x_{2r_x}, x), (\mathbb{R}^k, \mathsf{d}_{\mathrm{Eucl}}, c_k \mathcal{L}^k, 0^k)\Big) \le \varepsilon.$$

By applying Vitali covering Theorem 1.9 to the family $\{B_{r_x}(x) : x \in A_k \cap B_R(p)\}$, we obtain a sequence $(x_i)_i \subseteq A_k \cap B_R(p)$ such that $\{B_{r_{x_i}}(x_i)\}_i$ are pairwise disjoint and $A_k \cap B_R(p) \subseteq \bigcup_i B_{5r_{x_i}}(x_i)$. For any $i \in \mathbb{N}$, we know from Proposition 2.11 that there exists a δ^2-splitting map $u^i \colon B_{10r_{x_i}}(x_i) \to \mathbb{R}^k$. Furthermore, Proposition 2.13 guarantees the existence of a Borel set $G^i_\eta \subseteq B_{5r_{x_i}}(x_i)$ such that $\mathfrak{m}(B_{5r_{x_i}}(x_i) \setminus G^i_\eta) \le C_N \delta\, \mathfrak{m}(B_{5r_{x_i}}(x_i))$ and $u^i \colon B_s(x) \to \mathbb{R}^k$ is a δ-splitting map for every $x \in G^i_\eta$ and $s \in (0, 5r_{x_i})$. Hence, Proposition 2.12 guarantees that for any $x \in G^i_\eta$ the following property holds:

$$\begin{aligned} &\text{Given any element } (Y, \varrho, \mathfrak{n}, y) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}), \text{ there exists} \\ &\text{a pointed } \mathrm{RCD}(0, N-k) \text{ space } (Z, \mathsf{d}_Z, \mathfrak{m}_Z, z) \text{ such that} \\ &\mathsf{d}_{\mathrm{pmGH}}\Big((Y, \varrho, \mathfrak{n}, y), (\mathbb{R}^k \times Z, \mathsf{d}_{\mathrm{Eucl}} \times \mathsf{d}_Z, \mathcal{L}^k \otimes \mathfrak{m}_Z, (0^k, z))\Big) \le \eta. \end{aligned} \tag{2.7}$$

Then let us define $G_\eta \coloneqq \bigcup_i G^i_\eta$. Clearly, each element of G_η satisfies (2.7). Moreover, it holds

$$\begin{aligned}\mathfrak{m}\big(B_R(p)\cap(A_k\setminus G_\eta)\big) &\leq \sum_{i\in\mathbb{N}}\mathfrak{m}\big(B_{5r_{x_i}}(x_i)\setminus G^i_\eta\big)\leq C_N\delta\sum_{i\in\mathbb{N}}\mathfrak{m}\big(B_{5r_{x_i}}(x_i)\big)\\ &\leq C_N\eta\sum_{i\in\mathbb{N}}\mathfrak{m}\big(B_{r_{x_i}}(x_i)\big)\leq C_N\eta\,\mathfrak{m}\big(B_{R+1}(p)\big).\end{aligned} \tag{2.8}$$

Now consider the Borel set $G \coloneqq \bigcap_i \bigcup_j G_{1/2^{i+j}}$. It follows from (2.8) that $\mathfrak{m}\big(B_R(p)\cap(A_k\setminus G)\big)=0$. Moreover, let $x\in A_k\cap G$ and $(Y,\varrho,\mathfrak{n},y)\in \mathrm{Tan}_x(X,\mathsf{d},\mathfrak{m})$ be fixed. Then by using (2.7) we can find a sequence $\big\{(Z_i,\mathsf{d}_{Z_i},\mathfrak{m}_{Z_i},z_i)\big\}_i$ of pointed RCD$(0,N-k)$ spaces such that

$$(\mathbb{R}^k\times Z_i,\mathsf{d}_{\mathrm{Eucl}}\times\mathsf{d}_{Z_i},\mathcal{L}^k\otimes\mathfrak{m}_{Z_i},(0^k,z_i))\overset{\mathrm{pmGH}}{\longrightarrow}(Y,\varrho,\mathfrak{n},y)\quad\text{ as } i\to\infty. \tag{2.9}$$

Up to a not relabeled subsequence, we can suppose that $(Z_i,\mathsf{d}_{Z_i},\mathfrak{m}_{Z_i},z_i)\to(Z,\mathsf{d}_Z,\mathfrak{m}_Z,z)$ in the pmGH-topology for some pointed RCD$(0,N-k)$ space $(Z,\mathsf{d}_Z,\mathfrak{m}_Z,z)$. Consequently, (2.9) ensures that $(Y,\varrho,\mathfrak{n},y)$ is isomorphic to $\big(\mathbb{R}^k\times Z,\mathsf{d}_{\mathrm{Eucl}}\times\mathsf{d}_Z,\mathcal{L}^k\otimes\mathfrak{m}_Z,(0^k,z)\big)$. Given that $x\in A_k$, we deduce that Z must be a singleton. In other words, we have proven that any element of $\mathrm{Tan}_x(X,\mathsf{d},\mathfrak{m})$ is isomorphic to $(\mathbb{R}^k,\mathsf{d}_{\mathrm{Eucl}},c_k\mathcal{L}^k,0^k)$, so that $x\in\mathcal{R}_k$. This shows that $A_k\cap G\subseteq\mathcal{R}_k$, whence the claim (2.6) follows. □

By combining Theorem 2.18 with the properties of δ-splitting maps discussed in Section 2, we can prove the following result, which constitutes a strengthening of [**152**, Theorem 1.2]:

Theorem 2.19. *Let $(X,\mathsf{d},\mathfrak{m})$ be an* RCD(K,N) *space. Let $k\in\mathbb{N}$, $k\leq N$ be the maximal number such that $\mathfrak{m}(\mathcal{R}_k)>0$. Then for any $x\in X$ and $\ell>k$ we have that no element of* $\mathrm{Tan}_x(X,\mathsf{d},\mathfrak{m})$ *splits off a factor $\mathbb{R}^\ell$. In particular, it holds that $\mathcal{R}_\ell=\emptyset$ for every $\ell>k$.*

Proof. First of all, we claim that for any given $\ell>k$ there exists $\varepsilon>0$ such that

$$\mathsf{d}_{\mathrm{pmGH}}\Big((\mathbb{R}^j,\mathsf{d}_{\mathrm{Eucl}},c_j\mathcal{L}^j,0^j),(\mathbb{R}^\ell\times Z,\mathsf{d}_{\mathrm{Eucl}}\times\mathsf{d}_Z,\mathcal{L}^\ell\otimes\mathfrak{m}_Z,(0^\ell,z))\Big)>\varepsilon \tag{2.10}$$

for every $j\leq k$ and for every pointed normalised RCD$(0,N-\ell)$ space $(Z,\mathsf{d}_Z,\mathfrak{m}_Z,z)$. To prove it, we argue by contradiction: suppose there exist $j\leq k$ and a sequence $(Z_n,\mathsf{d}_{Z_n},\mathfrak{m}_{Z_n},z_n)$, $n\in\mathbb{N}$ of pointed RCD$(0,N-\ell)$ spaces such that $\big(\mathbb{R}^\ell\times Z_n,\mathsf{d}_{\mathrm{Eucl}}\times\mathsf{d}_{Z_n},\mathcal{L}^\ell\otimes\mathfrak{m}_{Z_n},(0^\ell,z_n)\big)\to(\mathbb{R}^j,\mathsf{d}_{\mathrm{Eucl}},c_j\mathcal{L}^j,0^j)$ in the pmGH-topology. Up to taking a not relabelled subsequence, we have that $(Z_n,\mathsf{d}_{Z_n},\mathfrak{m}_{Z_n},z_n)$ pmGH-converge to some pointed RCD$(0,N-\ell)$ space $(Z,\mathsf{d}_Z,\mathfrak{m}_Z,z)$ by Remark 1.114. Therefore, we deduce that $(\mathbb{R}^j,\mathsf{d}_{\mathrm{Eucl}},c_j\mathcal{L}^j,0^j)$ and $\big(\mathbb{R}^\ell\times Z,\mathsf{d}_{\mathrm{Eucl}}\times\mathsf{d}_Z,\mathcal{L}^\ell\otimes\mathfrak{m}_Z,(0^\ell,z)\big)$ are isomorphic, which is not possible as $\ell>j$. This leads to a contradiction, thus proving the claim.

We prove the main statement by contradiction: suppose there exist $x\in X$ and $\ell>k$ such that

$$(\mathbb{R}^\ell\times Z,\mathsf{d}_{\mathrm{Eucl}}\times\mathsf{d}_Z,\mathcal{L}^\ell\otimes\mathfrak{m}_Z,(0^\ell,z))\in\mathrm{Tan}_x(X,\mathsf{d},\mathfrak{m}) \tag{2.11}$$

for some pointed RCD$(0,N-\ell)$ space $(Z,\mathsf{d}_Z,\mathfrak{m}_Z,z)$. Consider $\varepsilon>0$ associated with ℓ as in the first part of the proof. Choose $\delta>0$ associated with ε as in Proposition 2.12, then $\eta>0$ associated with δ^2 as in Proposition 2.11. It follows from (2.11) that there is $r>0$ such that $r^2|K|\leq\eta$ and

$$\mathsf{d}_{\mathrm{pmGH}}\Big((X,\mathsf{d}/r,\mathfrak{m}^x_r,x),(\mathbb{R}^\ell\times Z,\mathsf{d}_{\mathrm{Eucl}}\times\mathsf{d}_Z,\mathcal{L}^\ell\otimes\mathfrak{m}_Z,(0^\ell,z))\Big)\leq\eta.$$

Then Proposition 2.11 guarantees the existence of a δ^2-splitting map $u\colon B_{5r}(x)\to\mathbb{R}^\ell$. Therefore, by Proposition 2.13 and Proposition 2.12 there exists a Borel set $G\subseteq B_r(x)$ with $\mathfrak{m}(G)>0$ satisfying the following property: for any $y\in G$, each element of $\mathrm{Tan}_y(X,\mathsf{d},\mathfrak{m})$ is ε-close (with respect to the distance $\mathsf{d}_{\mathrm{pmGH}}$) to some space that splits off a factor $\mathbb{R}^\ell$. Given that $\mathfrak{m}(X \setminus (\mathcal{R}_1 \cup \cdots \cup \mathcal{R}_k)) = 0$ by Theorem 2.18, there exist $y \in G$ and $j \leq k$ for which $(\mathbb{R}^j, \mathsf{d}_{\rm Eucl}, c_j\mathcal{L}^j, 0^j)$ is the only element of $\mathrm{Tan}_y(X, \mathsf{d}, \mathfrak{m})$. Consequently, we have that

$$\mathsf{d}_{\rm pmGH}\Big((\mathbb{R}^j, \mathsf{d}_{\rm Eucl}, c_j\mathcal{L}^j, 0^j), (\mathbb{R}^\ell \times Z', \mathsf{d}_{\rm Eucl} \times \mathsf{d}_{Z'}, \mathcal{L}^\ell \otimes \mathfrak{m}_{Z'}, (0^\ell, z'))\Big) \leq \varepsilon$$

for some pointed normalised $\mathrm{RCD}(0, N-\ell)$ space $(Z', \mathsf{d}_{Z'}, \mathfrak{m}_{Z'}, z')$. This is in contradiction with (2.10). □

4. Metric rectifiability of RCD spaces

Aim of this section is to exploit the properties of δ-splitting maps discussed in Section 2 to show that finite-dimensional $\mathrm{RCD}(K,N)$ spaces are metrically rectifiable in the sense of Definition 2.2. The first proof of this result was obtained in [**170**].

A key tool in our proof will be the following lemma.

Lemma 2.20. *Let $N > 1$ be given. Then for any $\eta > 0$ there exists $\delta = \delta_{N,\eta} > 0$ such that the following property holds. If $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(K,N)$ space and $u \colon B_r(x) \to \mathbb{R}^k$ is a δ-splitting map for some radius $r > 0$ with $r^2|K| \leq 1$ and some point $x \in X$ satisfying*

$$\mathsf{d}_{\rm pmGH}\big((X, \mathsf{d}/r, \mathfrak{m}_r^x, x), (\mathbb{R}^k, \mathsf{d}_{\rm Eucl}, c_k\mathcal{L}^k, 0^k)\big) < \delta^2,$$

then

$$\Big||u(y) - u(z)| - \mathsf{d}(y,z)\Big| \leq \eta r, \qquad \textit{for every } y, z \in B_r(x). \tag{2.12}$$

Proof. Thanks to a scaling argument, it suffices to prove the statement for $r = 1$ and $|K| \leq 1$. We argue by contradiction: suppose there exist $\eta > 0$, a sequence $(X_n, \mathsf{d}_n, \mathfrak{m}_n, x_n)$, and a sequence of maps $u^n \colon B_1(x_n) \to \mathbb{R}^k$, such that the following properties are satisfied:

i) $(X_n, \mathsf{d}_n, \mathfrak{m}_n, x_n)$ is a normalised $\mathrm{RCD}(K,N)$ space;
ii) u^n is a $1/n$-splitting map with $u^n(x_n) = 0^k$;
iii) it holds that $\mathsf{d}_{\rm pmGH}((X_n, \mathsf{d}_n, \mathfrak{m}_n, x_n), (\mathbb{R}^k, \mathsf{d}_{\rm Eucl}, c_k\mathcal{L}^k, 0^k)) \leq 1/n$;
iv) there exist points $y_n, z_n \in B_1(x_n)$ such that

$$\Big||u^n(y_n) - u^n(z_n)| - \mathsf{d}_n(y_n, z_n)\Big| > \eta. \tag{2.13}$$

Observe that item iii) guarantees that $(X_n, \mathsf{d}_n, \mathfrak{m}_n, x_n) \to (\mathbb{R}^k, \mathsf{d}_{\rm Eucl}, c_k\mathcal{L}^k, 0^k)$ in the pmGH-topology. Possibly taking a not relabelled subsequence, it holds that $u^n \to u^\infty$ strongly in $H^{1,2}$ on $B_1(0^k)$, for some limit map $u^\infty \colon B_1(0^k) \to \mathbb{R}^k$ (cf. Section 4.2 for the theory of convergence of Sobolev spaces along pmGH converging sequences). Moreover, thanks to the uniform Lipschitz continuity of the maps u_n, an Ascoli-Arzelà type argument implies that the convergence is also pointwise in the following sense: whenever $X_n \ni w_n \to w_\infty \in \mathbb{R}^k$, it holds $u^n(w_n) \to u^\infty(w_\infty)$. We also deduce from item ii) above that $\mathrm{Hess}(u^\infty_a) = 0$ and $\nabla u^\infty_a \cdot \nabla u^\infty_b = \delta_{ab}$ on $B_1(0^k)$ for all $a, b = 1, \ldots, k$, whence u^∞ is the restriction to $B_1(0^k)$ of an orthogonal transformation of $\mathbb{R}^k$. Indeed the vanishing of the Hessian, together with the condition $u^\infty(0) = 0$, yields the linearity of the components, while the orthogonality of the transformation comes from the orthogonality of the gradients, since the ambient space is Euclidean. However, this contradicts the fact that by letting $n \to \infty$ in (2.13) we obtain that

$$\Big||u^\infty(y_\infty) - u^\infty(z_\infty)| - |y_\infty - z_\infty|\Big| \geq \eta,$$

where $y_\infty, z_\infty \in B_1(0^k)$ stand for the limit points of $(y_n)_n$ and $(z_n)_n$, respectively. □

Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ space and $k \in \mathbb{N}$ be such that $k \leq N$. Recall that k-regular points are defined according to the behaviour of the tangent cone. For the sake of the arguments we are going to present below it is relevant to introduce auxiliary sets collecting

points where balls (with center at the given point) behave like the Euclidean ones for any radius below a certain threshold and not only after the blow-up.
In order to do so, following [**70**], we define

$$(\mathcal{R}_k)_{r,\delta} := \Big\{ x \in \mathcal{R}_k \,\Big|\, \mathsf{d}_{\rm pmGH}\big((X, \mathsf{d}/s, \mathfrak{m}_s^x, x), (\mathbb{R}^k, \mathsf{d}_{\rm Eucl}, c_k\mathcal{L}^k, 0^k)\big) < \delta \text{ for every } s < r \Big\}$$

for every $r,\delta > 0$. Observe that for any given $\delta > 0$ it holds that $(\mathcal{R}_k)_{r,\delta} \nearrow \mathcal{R}_k$ as $r \searrow 0$, as one can easily argue relying on the definition of tangent cone.

Theorem 2.21. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an* RCD(K,N) *space. Let $k \in \mathbb{N}$ be such that $k \leq N$. Then the k-regular set $\mathcal{R}_k$ of X is $(\mathfrak{m}, k, \varepsilon)$-rectifiable for every $\varepsilon > 0$.*

Proof. The proof is divided into two intermediate steps. First we prove that δ-splitting map are biLpischitz charts when restricted to suitable subsets of the starting space. Then we get rectifiability via a covering argument.

Step 1. First of all, we claim that for any $\eta > 0$ there exists $\delta = \delta_{N,\eta} \in (0,1)$ such that the following property holds: if $(X, \mathsf{d}, \mathfrak{m})$ is an RCD(K,N) space and $u\colon B_{5r}(p) \to \mathbb{R}^k$ is a δ-splitting map for some radius $r > 0$ satisfying $r^2|K| \leq 1$ and some point $p \in (\mathcal{R}_k)_{2r,\delta}$, then there exists a Borel set $G \subseteq B_r(p)$ such that $\mathfrak{m}(B_r(p) \setminus G) \leq C_N \eta\, \mathfrak{m}(B_r(p))$ and

$$\Big| |u(x) - u(y)| - \mathsf{d}(x,y) \Big| \leq \eta\, \mathsf{d}(x,y) \quad \text{for every } x,y \in (\mathcal{R}_k)_{2r,\delta} \cap G. \tag{2.14}$$

To prove it, choose any $\delta \in (0,\eta^2)$ so that $\sqrt{\delta}$ is associated with $\sqrt{\eta}$ as in Lemma 2.20. Now let us consider an RCD(K,N) space $(X, \mathsf{d}, \mathfrak{m})$ and a δ-splitting map $u\colon B_{5r}(p) \to \mathbb{R}^k$, for some $r > 0$ with $r^2|K| \leq 1$ and $p \in (\mathcal{R}_k)_{2r,\delta}$. By using Proposition 2.13, we can find a Borel set $G \subseteq B_r(p)$ such that $\mathfrak{m}(B_r(p) \setminus G) \leq C_N \eta\, \mathfrak{m}(B_r(p))$ and $u\colon B_s(x) \to \mathbb{R}^k$ is a $\sqrt{\delta}$-splitting map for all $x \in G$ and $s \in (0,2r)$. Then Lemma 2.20 guarantees that the map $u\colon B_s(x) \to \mathbb{R}^k$ verifies the scale invariant version of (2.12) for every $x \in (\mathcal{R}_k)_{2r,\delta} \cap G$ and $s \in (0,2r)$; here we used that $x \in (\mathcal{R}_k)_{2r,\delta} \subseteq (\mathcal{R}_k)_{s,\delta}$.

Fix any $x,y \in (\mathcal{R}_k)_{2r,\delta} \cap G$. Being $\mathsf{d}(x,y) < 2r$, we know that the map $u\colon B_{\mathsf{d}(x,y)}(x) \to \mathbb{R}^k$ verifies the condition above, therefore $\big||u(x) - u(y)| - \mathsf{d}(x,y)\big| \leq \eta\, \mathsf{d}(x,y)$. This yields (2.14).

Step 2. Let $\varepsilon > 0$ be fixed. We aim to show that $\mathcal{R}_k$ is $(\mathfrak{m}, k, \varepsilon)$-rectifiable. Fix $\bar{x} \in X$ and $j \in \mathbb{N}$. Choose any sequence $\eta_n \searrow 0$ such that $1 - \eta_n \geq 1/(1+\varepsilon)$ for every $n \in \mathbb{N}$. Let $\delta_n \in (0,1)$ be associated with η_n as in **Step 1**, then let $\varepsilon_n \in (0,\delta_n)$ be associated with δ_n as in Proposition 2.11.

Now choose a sequence $(r_n)_n \subseteq (0,1)$ of radii such that $r_n^2|K| \leq 1$ and

$$\mathfrak{m}\Big(B_j(\bar{x}) \cap \big(\mathcal{R}_k \setminus (\mathcal{R}_k)_{2r_n,\varepsilon_n}\big)\Big) \leq \frac{1}{n} \quad \text{for every } n \in \mathbb{N}. \tag{2.15}$$

Let $n \in \mathbb{N}$ be fixed. By Vitali covering Theorem 1.9, we find points $x_1, \dots, x_\ell \in B_j(\bar{x}) \cap (\mathcal{R}_k)_{2r_n,\varepsilon_n}$ for which $\{B_{r_n/5}(x_i)\}_{i=1}^{\ell}$ are pairwise disjoint and $B_j(\bar{x}) \cap (\mathcal{R}_k)_{2r_n,\varepsilon_n} \subseteq B_{r_n}(x_1) \cup \dots \cup B_{r_n}(x_\ell)$. Proposition 2.11 guarantees the existence of a δ_n-splitting map $u^i\colon B_{5r_n}(x_i) \to \mathbb{R}^k$ for every $i = 1, \dots, \ell$. Therefore **Step 1** yields a Borel set $G'_i \subseteq B_{r_n}(x_i)$ such that $\mathfrak{m}(B_{r_n}(x_i) \setminus G'_i) \leq C_N \eta_n\, \mathfrak{m}(B_{r_n}(x_i))$ and $\big||u^i(x) - u^i(y)| - \mathsf{d}(x,y)\big| \leq \eta_n\, \mathsf{d}(x,y)$ for every $x,y \in G_i^{jn} := (\mathcal{R}_k)_{2r_n,\varepsilon_n} \cap G'_i$. Since it holds that $1 - \eta_n \geq 1/(1+\varepsilon)$ by assumption, we deduce that u^i is $(1+\varepsilon)$-biLipschitz with its image when restricted to G_i^{jn}, whence

$G^{jn} := B_j(\bar{x}) \cap \bigcup_{i=1}^{\ell} G_i^{jn}$ is $(\mathfrak{m}, k, \varepsilon)$-rectifiable. Observe that

$$\begin{aligned}\mathfrak{m}\Big((B_j(\bar{x}) \cap (\mathcal{R}_k)_{2r_n,\varepsilon_n}) \setminus G^{jn}\Big) &\leq \sum_{i=1}^{\ell} \mathfrak{m}(B_{r_n}(x_i) \setminus G_i^{jn}) \leq C_N \eta_n \sum_{i=1}^{\ell} \mathfrak{m}(B_{r_n}(x_i)) \\ &\leq C_N \eta_n \sum_{i=1}^{\ell} \mathfrak{m}(B_{r_n/5}(x_i)) \leq C_N \eta_n \, \mathfrak{m}(B_{j+1}(\bar{x})).\end{aligned}$$

By taking (2.15) into account, we conclude that $\mathfrak{m}(\mathcal{R}_k \setminus \bigcup_{j,n\in\mathbb{N}} G^{jn}) = 0$. Given that the set $\bigcup_{j,k\in\mathbb{N}} G^{jn}$ is $(\mathfrak{m}, k, \varepsilon)$-rectifiable by construction, the statement is achieved. □

5. Behaviour of the reference measure under charts

In this section we deal with the behaviour of the reference measure $\mathfrak{m}$ of an $\mathrm{RCD}(K,N)$ m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ with respect to charts. The only difference with respect to the original proofs in [**85**, **120**, **146**] is that the charts in our approach are components of harmonic δ-splitting maps. Still, we heavily rely on one of the corollaries of the main achievement of [**86**]. Before quoting it let us make a couple of remarks about vector fields on weighted Euclidean spaces and normal currents. The discussion and the statements are borrowed from [**120**].

Suppose that μ is a non-negative Radon measure on $\mathbb{R}^k$. Then there are at least two reasonable notions for the space of $L^2(\mu)$ vector fields: the first one is to consider the abstract construction of the L^2 tangent module of a metric measure space (cf. Section 1.6), that we shall denote by $L^2_\mu(T\mathbb{R}^k)$. The second possibility is to consider the space of Borel maps from $\mathbb{R}^k$ to $\mathbb{R}^k$ that are in $L^2(\mu)$. We will denote by $L^2(\mathbb{R}^k, \mathbb{R}^k; \mu)$ this space. It turns out that $L^2(\mathbb{R}^k, \mathbb{R}^k; \mu)$ is an $L^2(\mu)$-normed $L^\infty(\mu)$-module generated by $\{\nabla f : f \in C_c^\infty(\mathbb{R}^k)\}$, where $\nabla f \colon \mathbb{R}^k \to \mathbb{R}^k$ stands for the 'classical' gradient of f.

In [**120**, Proposition 2.10] it has been proven that there exists a map $\iota : L^2_\mu(T\mathbb{R}^k) \to L^2(\mathbb{R}^k, \mathbb{R}^k, \mu)$ which is an L^∞-module morphism preserving the pointwise norm.

With the embedding above at our disposal we can go further associating to any vector field $v \in L^2_\mu(T\mathbb{R}^k)$ a normal one dimensional current in the following way: for any smooth compactly supported one differential form ω we let

$$\mathcal{J}(v)(\omega) := \int \omega(\iota(v)) \, \mathrm{d}\mu.$$

It is easy to check that $\mathcal{J}(v)$ has finite mass and, since ι preserves the pointwise norm, it holds $\|\mathcal{J}(v)\| = |v| \, \mu$, where we denoted by $\|\cdot\|$ the mass measure.

By the definition of the boundary operator on currents, it holds

$$(\partial \mathcal{J}(v))\,(f) = (\mathcal{J}(v))\,(\,\mathrm{d}f) = \int \,\mathrm{d}f(v) \, \mathrm{d}\mu.$$

Hence we can argue that $\partial \mathcal{J}(v)$ is a Radon measure if and only if $\iota(v)\mu$ has measure valued divergence and, in that case, there is the expected identification between boundary measure and divergence.

Lemma 2.22 (Corollary 2.12 in [**120**]). *Let $v \in L^2_\mu(T\mathbb{R}^k)$ be compactly supported. Then $\mathcal{J}(v)$ is a normal one dimensional current if and only if $\iota(v)\mu$ has measure valued divergence. In that case*

$$\partial \mathcal{J}(v) = -\mathrm{div}_\mu v.$$

Below we state a crucial result from [**86**], that has been obtained as corollary of a deep structural result for measure solutions of linear PDEs.

Theorem 2.23 (Corollary 1.12 in [**86**])**.** *Let* $T_1 = \overrightarrow{T_1}\,\|T_1\|,\dots,T_k = \overrightarrow{T_k}\,\|T_k\| \in \boldsymbol{N}_1(\mathbb{R}^k)$ *be one dimensional normal currents on* $\mathbb{R}^k$ *such that there exists a positive Radon measure* $\mu \in \mathcal{M}_+(\mathbb{R}^k)$ *with the following properties:*

i) $\mu \ll \|T_i\|$ *for* $i = 1,\dots,k$*;*
ii) *for* μ*-a.e.* x *it holds* $\mathrm{span}\Big\{\overrightarrow{T_1},\dots,\overrightarrow{T_k}\Big\} = \mathbb{R}^k$.

Then $\mu \ll \mathcal{L}^k$*.*

As anticipated, our aim is to prove that, given any of the (k,δ)-splitting maps $u : X \to \mathbb{R}^k$ providing rectifiability of a region $\mathcal{K}$ of the RCD(K,N) m.m.s. $(X,\mathsf{d},\mathfrak{m})$, it holds that $u_\sharp\left(\mathfrak{m}|_{\mathcal{K}}\right) \ll \mathcal{L}^k$. In order to be in position to apply Theorem 2.23, we need a way to build normal one currents by means of the chart.

Let us introduce some auxiliary notation. Let X, Y be Polish spaces, fix a finite Borel measure $\mu \geq 0$ on X and a Borel map $\varphi\colon X \to Y$. Then we define

$$\mathsf{Pr}_\varphi(f) := \frac{\mathrm{d}\varphi_\sharp(f\mu)}{\mathrm{d}\varphi_\sharp\mu} \quad \text{for every } f \in L^1(\mu). \tag{2.16}$$

The resulting map $\mathsf{Pr}_\varphi\colon L^1(\mu) \to L^1(\varphi_\sharp\mu)$ is linear and continuous. Given any $p \in (1,\infty]$, it holds that Pr_φ maps continuously $L^p(\mu)$ to $L^p(\varphi_\sharp\mu)$. The *essential image* of a Borel set $E \subseteq X$ is defined as $\mathsf{Im}_\varphi(E) := \{\mathsf{Pr}_\varphi(\chi_E) > 0\} \subseteq Y$.

The key tool is the proposition below, that we borrow from [**120**], reporting a sketch of the proof for sake of completeness.

Proposition 2.24 (Differential of an $\mathbb{R}^k$-valued Lipschitz map)**.** *Let* (X,d,μ) *be an infinitesimally Hilbertian metric measure space such that* μ *is finite. Let* $\varphi\colon X \to \mathbb{R}^k$ *be a Lipschitz map. Then there exists a unique linear and continuous operator* $\mathsf{D}_\varphi\colon L^2(TX) \to L^2(\mathbb{R}^k,\mathbb{R}^k;\varphi_\sharp\mu)$ *such that*

$$\int_F \nabla f \cdot \mathsf{D}_\varphi(v)\,\mathrm{d}\varphi_\sharp\mu = \int_{\varphi^{-1}(F)} \nabla(f\circ\varphi)\cdot v\,\mathrm{d}\mu \tag{2.17}$$

for any $f \in C_c^\infty(\mathbb{R}^k)$*,* $v \in L^2(TX)$ *and any Borel* $F \subseteq \mathbb{R}^k$*. In particular, if* $v \in D(\mathrm{div}\,)$*, then the distributional divergence of* $\mathsf{D}_\varphi(v)$ *is* $\mathsf{Pr}_\varphi(\mathrm{div}\,(v))$*.*

Moreover, if the map φ *is biLipschitz with its image when restricted to some Borel set* $E \subseteq X$ *and* $v_1,\dots,v_k \in L^2(TX)$ *are independent on* E*, then the family of vectors* $\mathsf{D}_\varphi(\chi_E\,v_1)(y),\dots,\mathsf{D}_\varphi(\chi_E\,v_k)(y)$ *constitutes a basis of* $\mathbb{R}^k$ *for* $\varphi_\sharp\mu$*-a.e. point* $y \in \mathsf{Im}_\varphi(E)$*.*

Proof. Existence of the map D_φ is proven in [**120**]: it suffices to define $\mathsf{D}_\varphi := \iota\circ\mathsf{Pr}_\varphi\circ\mathrm{d}\varphi$. The fact that this map satisfies (2.17) follows from [**120**, Proposition 2.7] and the very definition of ι (we do not need to require properness of φ, as μ is a finite measure). Uniqueness of D_φ follows from the fact that $\{\nabla f : f \in C_c^\infty(\mathbb{R}^k)\}$ generates $L^2(\mathbb{R}^k,\mathbb{R}^k;\varphi_\sharp\mu)$.

Suppose now that $v \in D(\mathrm{div}\,)$. Then for every $f \in C_c^\infty(\mathbb{R}^k)$ it holds that $f\circ\varphi \in H^{1,2}(X,\mathsf{d},\mu)$, whence

$$\begin{aligned}\int \nabla f\cdot\mathsf{D}_\varphi(v)\,\mathrm{d}\varphi_\sharp\mu &\overset{(2.17)}{=} \int \nabla(f\circ\varphi)\cdot v\,\mathrm{d}\mu = -\int f\circ\varphi\,\mathrm{div}\,(v)\,\mathrm{d}\mu = -\int f\,\mathrm{d}\varphi_\sharp(\mathrm{div}\,(v)\mu)\\ &\overset{(2.16)}{=} -\int f\,\mathsf{Pr}_\varphi(\mathrm{div}\,(v))\,\mathrm{d}\varphi_\sharp\mu.\end{aligned}$$

This shows that the distributional divergence of $\mathsf{D}_\varphi(v)$ is represented by $\mathsf{Pr}_\varphi(\mathrm{div}\,(v))$. Finally, the last claim of the statement follows from [**120**, Proposition 2.2] and [**120**, Proposition 2.10]. □

Theorem 2.25 (Behaviour of $\mathfrak{m}$ under charts)**.** *Let* $(X,\mathsf{d},\mathfrak{m})$ *be an* $\mathrm{RCD}(K,N)$ *space. Consider a* δ*-splitting map* $u\colon B_r(p)\to\mathbb{R}^k$ *which is* $(1+\varepsilon)$*-biLipschitz with its image (for some* $\varepsilon<1/k$*) when restricted to some compact set* $\mathcal{K}\subseteq B_r(p)$*. Then it holds that*

$$u_\sharp(\mathfrak{m}\llcorner_{\mathcal{K}})\ll\mathcal{L}^k.$$

In particular, for any $k\in\mathbb{N}$*,* $k\le N$*,* $\mathfrak{m}\llcorner\mathcal{R}_k$ *is absolutely continuous with respect to the* k*-dimensional Hausdorff measure on* (X,d)*.*

Proof. First of all, fix a good cut-off function $\eta\colon X\to\mathbb{R}$ for the pair $\mathcal{K}\subseteq B_r(p)$, in the sense of Lemma 1.107. Define $\mu:=\mathfrak{m}\llcorner B_r(p)$ and $\varphi:=\eta u\colon X\to\mathbb{R}^k$. Observe that the components $\varphi_1,\dots,\varphi_k$ of φ are test functions and $\varphi|_{\mathcal{K}}$ is $(1+\varepsilon)$-biLipschitz with its image. Consider the differential $\mathsf{D}_\varphi\colon L^2(TX)\to L^2(\mathbb{R}^k,\mathbb{R}^k;\varphi_\sharp\mu)$ defined in Proposition 2.24. Fix a sequence $(\psi_i)_i$ of compactly-supported, Lipschitz functions $\psi_i\colon X\to[0,1]$ that pointwise converge to $\chi_{\mathcal{K}}$. We then set

$$v^i_a:=\mathsf{D}_\varphi(\psi_i\nabla\varphi_a)\in L^2(\mathbb{R}^k,\mathbb{R}^k;\varphi_\sharp\mu)\quad\text{ for every } i\in\mathbb{N}\text{ and } a=1,\dots,k.$$

Note that $\psi_i\nabla\varphi_a\in D(\mathrm{div}\,)$ by the Leibniz rule for divergence and the fact that $\varphi_a\in D(\Delta)$, whence Proposition 2.24 ensures that the distributional divergence of each vector field v^i_a is an $L^2(\varphi_\sharp\mu)$-function. Hence, it holds that $\mathcal{I}_{ia}:=v^i_a\,\varphi_\sharp\mu$ is a normal 1-current in $\mathbb{R}^k$ by Lemma 2.22. Note also that

$$\overrightarrow{\mathcal{I}_{ia}}=\chi_{\{|v^i_a|>0\}}\frac{v^i_a}{|v^i_a|}\ \text{ and }\ \|\mathcal{I}_{ia}\|=|v^i_a|\,\varphi_\sharp\mu\quad\text{ for every } i\in\mathbb{N}\text{ and } a=1,\dots,k.$$

Call A_i the set of $y\in\mathbb{R}^k$ such that $v^i_1(y),\dots,v^i_k(y)$ form a basis of $\mathbb{R}^k$. Since $(\varphi_\sharp\mu)|_{A_i}\ll\|\mathcal{I}_{ia}\|$ holds for all $a=1,\dots,k$, we deduce by Theorem 2.23 that

$$(\varphi_\sharp\mu)\llcorner A_i\ll\mathcal{L}^k\quad\text{ for every } i\in\mathbb{N}. \tag{2.18}$$

Now define $v_a:=\mathsf{D}_\varphi(\chi_{\mathcal{K}}\nabla_\mu\varphi_a)\in L^2(\mathbb{R}^k,\mathbb{R}^k;\varphi_\sharp\mu)$ for every $a=1,\dots,k$. It can readily be checked that $\nabla_\mu\varphi_1,\dots,\nabla_\mu\varphi_k$ are independent on $\mathcal{K}$ (here the assumption $\varepsilon<1/k$ plays a role), whence the vectors $v_1(y),\dots,v_k(y)$ are linearly independent for $\varphi_\sharp\mu$-a.e. $y\in\mathsf{Im}_\varphi(\mathcal{K})$ by Proposition 2.24.

Furthermore, for any given $a=1,\dots,k$, we can see (by using dominated convergence theorem) that $\psi_i\nabla_\mu\varphi_a\to\chi_{\mathcal{K}}\nabla_\mu\varphi_a$ in $L^2(TX)$ as $i\to\infty$, thus $v^i_a\to v_a$ in $L^2(\mathbb{R}^k,\mathbb{R}^k;\varphi_\sharp\mu)$ as $i\to\infty$ by continuity of D_φ. In particular, possibly passing to a not relabelled subsequence, we can assume that $\lim_i v^i_a(y)=v_a(y)$ for $\varphi_\sharp\mu$-a.e. $y\in\mathbb{R}^k$. This implies that $(\varphi_\sharp\mu)\big(\mathsf{Im}_\varphi(K)\setminus\bigcup_i A_i\big)=0$, thus (2.18) yields $(\varphi_\sharp\mu)\llcorner_{\mathsf{Im}_\varphi(\mathcal{K})}\ll\mathcal{L}^k$. Since $\mathsf{Im}_\varphi(\mathcal{K})=\{\mathsf{Pr}_\varphi(\chi_{\mathcal{K}})>0\}$ by definition, we conclude that

$$u_\sharp(\mathfrak{m}\llcorner_{\mathcal{K}})=\varphi_\sharp(\mu\llcorner_{\mathcal{K}})=\frac{\mathrm{d}\varphi_\sharp(\chi_{\mathcal{K}}\mu)}{\mathrm{d}\varphi_\sharp\mu}\,\varphi_\sharp\mu=\mathsf{Pr}_\varphi(\chi_{\mathcal{K}})\,\varphi_\sharp\mu\ll\mathcal{L}^k.$$

Therefore, the first part of the statement is achieved.

The last part of the statement follows from the first one, the inner regularity of $\mathfrak{m}$, and (the proof of) Theorem 2.21. □

In order to recapitulate the results of this chapter and some of their consequences let us collect them in a single statement.
Before doing that we introduce a refined subset of the k-dimensional regular set $\mathcal{R}^*_k$ by

$$\mathcal{R}^*_k:=\left\{x\in\mathcal{R}_k:\exists\lim_{r\to0}\frac{\mathfrak{m}(B_r(x))}{\omega_k r^k}\in(0,+\infty)\right\}.$$

Theorem 2.26. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *m.m.s. for some* $K \in \mathbb{R}$ *and* $1 < N < +\infty$. *Then* $(X, \mathsf{d}, \mathfrak{m})$ *is strongly* $\mathfrak{m}$*-rectifiable as a metric measure space. Moreover, for* $\mathfrak{m}$*-a.e.* $x \in \mathcal{R}_k$ *the tangent cone at* x *is the Euclidean space of dimension* k, $\mathfrak{m}(\mathcal{R}_k \setminus \mathcal{R}_k^*) = 0$ *and*

$$\lim_{r \to 0} \frac{\mathfrak{m}(B_r(x))}{\omega_k r^k} = \frac{\mathrm{d}\left(\mathfrak{m} \llcorner_{\mathcal{R}_k^*}\right)}{\mathrm{d}\left(\mathcal{H}^k \llcorner_{\mathcal{R}_k^*}\right)} \quad \text{for } \mathfrak{m}\text{-a.e. } x \in \mathcal{R}_k^*. \tag{2.19}$$

We do not prove the parts of the statement that do not follow from the discussion of this chapter, referring to [**21**] for the detailed arguments. We just point out that the conclusions $\mathfrak{m}(\mathcal{R}_k \setminus \mathcal{R}_k^*) = 0$ and (2.19) follow from the first part via standard geometric measure theory arguments (see [**2**, **100**, **150**]).

CHAPTER 3

Constancy of the dimension for $\mathrm{RCD}(K,N)$ spaces via regularity of Lagrangian flows

This chapter is dedicated to the proof of the *constancy of the dimension conjecture* for $\mathrm{RCD}(K,N)$ metric measure spaces.

Theorem 3.1 (Constancy of the dimension)**.** *Let* $(X,\mathsf{d},\mathfrak{m})$ *be an* $\mathrm{RCD}(K,N)$ *m.m.s. for some* $K\in\mathbb{R}$ *and* $1\le N<\infty$*. Then there is exactly one regular set* $\mathcal{R}_n$ *having positive* $\mathfrak{m}$*-measure in the Mondino-Naber decomposition of* $(X,\mathsf{d},\mathfrak{m})$.

This conclusion has been achieved in [**52**], written in collaboration with Brué. For the sake of this thesis we chose to slightly modify the presentation, avoiding some of the technical tools developed in [**52**] to reach the regularity result for Lagrangian Flows of Sobolev vector fields and focusing only on the minimal regularity needed for the proof of the constancy of the dimension.

Let us point out that some of the ideas which led to the proof of Theorem 3.1 had been developed in a previous joint work with Brué [**51**] and that in another ongoing collaboration [**50**], more focused on the sharpness of the regularity statements, we improve upon some of the results of [**52**].

The starting point of our analysis is the structure theory of $\mathrm{RCD}(K,N)$ spaces as developed in [**85**, **120**, **144**, **170**] and reviewed in Chapter 2.
Let us mention that an analogous structure theory for Ricci limit spaces had been already developed by Cheeger-Colding in the nineties. Moreover, in [**69**], they conjectured that there should be exactly one k-dimensional regular set $\mathcal{R}_k$ having positive measure. It took more than ten years before the work [**77**], where Colding-Naber affirmatively solved this conjecture. The analogous problem in the framework of $\mathrm{RCD}(K,N)$ metric measure spaces was open since [**170**].

In order to motivate the developments in [**52**], that we shall present below, we find it relevant to explain why it seemed hard to adapt the strategy pursued by Colding-Naber in the case of Ricci limits to the setting of $\mathrm{RCD}(K,N)$ spaces and then to present the heuristic standing behind our new approach.

The technique of [**77**] is based on fine estimates on the behaviour of balls of small radii centred along the interior of a minimizing geodesic over a smooth Riemannian manifold (with Ricci bounded from below) that are stable enough to pass through the possibly singular measured Gromov-Hausdorff limits.
When dealing with an $\mathrm{RCD}(K,N)$ space, in general, there is no smooth approximating sequence one can appeal on. Nevertheless, one could try to reproduce their main estimate (see [**77**, Theorem 1.3]) directly at the level of the given metric measure space: up to our knowledge, the calculus tools available at this stage, although being quite powerful (see for instance [**112**]), are still not sufficient to such an issue.

This being said, the study of the flow of a suitably chosen vector field, which was at the technical core of the proof for Ricci limit spaces, is the starting point in our approach too. A key idea is the following: one would expect the geometry to change continuously along a flow

and that, as a consequence, flow maps might be a useful tool to prove that the space has a certain "homogeneity" property.
Let us illustrate what we mean with a completely elementary example. Consider a smooth and connected differentiable manifold M. Given $x, y \in M$, there exists a diffeomorphism $\phi : M \to M$ such that $\phi(x) = y$. Moreover, a rather common way to build such a map is as flow map at a fixed time of a suitably chosen smooth vector field and we could rephrase this statement by saying that "flows of smooth vector fields act transitively on M". Of course this construction gives nothing more, at the level of the qualitative structure of M, than a confirmation of the fact that that different points in M have diffeomorphic neighbourhoods. Instead, a quantitative analysis based on this strategy was at the core of [**77**] and a non smooth version of it has as deep consequence the proof of Theorem 3.1.

Trying to pursue such a plan, we are left with a series of questions: given an RCD(K, N) metric measure space $(X, \mathsf{d}, \mathfrak{m})$,

i) can we find a notion of vector field, a notion of flow associated to it and a class of vector fields "rich enough" for the sake of the applications and "regular enough" to prove existence and uniqueness for such a generalized notion of flow?
ii) What do we mean by "rich enough" in the question above? And, can we gain some transitivity in the spirit of the smooth elementary example?
iii) Are the flows considered in i) regular in some sense?
iv) Eventually, are the regularity in iii) and the transitivity in ii) strong enough to be incompatible with the possibility of having regular sets of different dimensions with positive measure in Theorem 2.26?

The remainder of this introduction is dedicated to a brief overview of this plan, that we are going to pursue in the rest of the chapter. While addressing i) is a matter of collecting ingredients available in the literature the main novelty of [**52**] stands in the study of points ii)-iv).

Concerning i), we recall that a first order differential calculus that can be built on any metric measure space. In particular, it is possible to talk about vector fields in such framework (cf. Sections 1.5 and 1.5.3). Moreover, in Section 3.1 we presented the second order differential structure of RCD(K, ∞) spaces. At the level of vector fields, which are *first order differential objects*, this further regularity allows to define a notion of covariant derivative such that the class of Sobolev vector fields with covariant derivative in L^2 is a rich one.
Moreover, as we have seen in Section 3.2, in [**30**] Ambrosio-Trevisan developed an existence and uniqueness theory for Regular Lagrangian Flows of Sobolev vector fields over RCD spaces, completing the picture about the first point of our plan.

At this stage a comment is in order to motivate why different choices for the class of vector fields to deal with seem to be not well suited for the study of spaces with lower Ricci curvature bounds. In the case of smooth ambient spaces the most natural class to develop an existence and regularity theory for flows would have been that of Lipschitz vector fields. However, as it has been recently pointed out in [**87**], in the non smooth setting it is out of hope to expect continuity even for the norms of gradients of harmonic functions, which might be thought to be the smoothest functions available. Both in the smooth and in the synthetic framework one can obtain contraction and regularity estimates even for flows associated to monotone vector fields (cf. [**132**]) and gradient flows of semiconvex functions (cf. [**202**]). However, while in the case of Alexandrov spaces the existence of *many* semiconvex functions is a direct consequence of the definition, in the setting of RCD spaces even the existence of a single semiconvex function is not clear. These remarks motivate the necessity to look for vector fields with lower regularity assumptions with the aim of developing a geometrically relevant theory.

Let us deal with ii), the "transitivity" issue. Over an RCD(K,N) metric measure space $(X,\mathsf{d},\mathfrak{m})$ a pointwise notion of transitivity might be out of reach. Nevertheless, some of the known constructions and results of the smooth category can be recovered in this more general framework by looking at measures absolutely continuous with respect to the reference measure and curves of measures instead of points and curves (cf. [**10**, **15**, **108**, **113**]).
In this regard, here is a natural question towards a weak form of transitivity: is it true that, for any pair of probability measures μ_0, μ_1 absolutely continuous and with bounded densities with respect to $\mathfrak{m}$, we can find a Sobolev vector field such that, calling F its regular Lagrangian flow at a fixed time, it holds $F_\sharp\mu_0 = \mu_1$? In Section 3.2, we will see how the Lewy-Stampacchia inequality, proved in this abstract framework by Gigli-Mosconi in [**119**], allows to give an (almost) affirmative answer to this question.

Next we pass to the regularity issue. As in the case of the well-posedness problem, the study of regularity for Lagrangian flows associated to Sobolev vector fields is far from being trivial also in the Euclidean setting. The first result in this direction was obtained by Crippa-De Lellis in [**79**] building upon some ideas that have previously appeared in [**25**]. Therein it was proved that Regular Lagrangian flows of Sobolev vector fields are Lusin Lipschitz maps.

When trying to control the behaviour of the distance between two trajectories of the flow in the Euclidean setting as in [**79**], a key tool is the so-called maximal estimate for Sobolev functions. Here comes a key issue when moving from the case of a flat ambient space to Riemannian manifolds or more general metric measure spaces: while on the Euclidean space one can deal with vector fields arguing componentwise, in the more general setting of curved or non smooth spaces this is not the case. Therefore, the scalar valued maximal estimate for Sobolev functions, holding in a broad class of metric measure spaces including RCD spaces (cf. [**9**, **10**]), cannot be lifted from Sobolev functions to Sobolev vector fields.

In Section 2.1 we make a digression to explain why, in the Euclidean case, looking at the Green function of the Laplacian (in place of the distance function) along trajectories of the flow, one is led to a different perspective about the problem, suitable for generalizations to the setting of our interest.

In [**51**] we pursued this strategy obtaining an extension of the result by Crippa-De Lellis to compact non collapsed RCD(K,N) spaces. Therein the Green function played the role of an auxiliary tool in the measurement of regularity in Lusin Lipschitz terms.
The key novelty in the approach that we present in this chapter is a change of perspective: regularity is measured in terms of Green functions, not anymore in terms of distance functions. Quite surprisingly, to obtain our key estimates we do not rely on the second order differentiation formula proved on RCD(K,N) spaces by Gigli-Tamanini in [**123**, **124**].

In Section 1 we prove some useful estimates for the Green function of the Laplacian and for its counterpart in the case of a negative lower bound on the Ricci curvature. Key references about the Green function in geometric analysis are [**158**–**160**, **207**–**209**], while to the best of our knowledge, the related tools had not been previously developed in the RCD framework. Proposition 3.16 and its counterpart Proposition 3.21 in the case of an arbitrary lower Ricci curvature bound, are new even in the smooth setting and play the role of the Lusin-Lipschitz property for Sobolev functions in the case of vector fields.
These estimates are the key tool for the regularity theory that we develop in Section 2. In Theorem 3.26 we prove that, on any RCD($0,N$) metric measure space $(X,\mathsf{d},\mathfrak{m})$ satisfying suitable volume growth assumptions at infinity, a weak version of Crippa-De Lellis' result holds if we understand Lusin-Lipschitz regularity with respect to a newly defined quasi-metric $\mathsf{d}_G = 1/G$, G being the minimal positive Green function of the Laplacian over $(X,\mathsf{d},\mathfrak{m})$. In Theorem 3.27 we also adapt our arguments to cover the case of an arbitrary lower Ricci

curvature bound, using the Green function of a modified elliptic operator as measurement of regularity.

After i), ii) and iii) we address iv). Let us point out that having at our disposal a perfect extension of Crippa-De Lellis' result to the metric setting it would have been rather easy to exclude the possibility of regular sets of different dimensions with positive measure in the Mondino-Naber decomposition of an RCD(K, N) metric measure space, just building on the transitivity result in ii) and the observation that, given $k < n$, it is impossible to find a Lipschitz map $\Phi : \mathbb{R}^k \to \mathbb{R}^n$ such that $\Phi_\sharp \mathcal{L}^k \ll \mathcal{L}^n$.

Here we exploit a modification of this idea. The RCD(K, N) condition concerns neither the distance nor the reference measure by themselves but a coupling of these objects, as it happens for the Laplacian, the heat flow and the Green function. Moreover, it is possible to catch, in a quantitative way, the asymptotic behaviour of the Green function near to regular points of the metric measure space in terms of distance, measure and dimension (see Lemma 3.35). This allows to find a counterpart for the "preservation of the Hausdorff dimension via biLipschitz maps" formulated just in terms of Green functions (see Theorem 3.36) and to complete the proof of Theorem 3.1, the spirit being that a control over two among distance, reference measure and Green function gives in turn a control over the remaining one.
While Green functions have proved to be very useful in the theory of non collapsed Ricci spaces to establish deep regularity results, such as in [**76**, **139**], to the best of our knowledge [**52**] has been the first application of this tool to the collapsed setting.

As we anticipated, here we chose to state and prove weaker regularity results for Lagrangian flows with respect to [**52**]. This choice allows to simplify the presentation, avoiding the discussion of some properties of the (modified) Green functions and the implementation of the Crippa-De Lellis regularity scheme. However, some more efforts are needed in Section 3 in order to achieve Theorem 3.1. Let us also mention that the stronger regularity results obtained of [**52**], although being not necessary for the solution of the constancy of the dimension conjecture, are expected to be useful in the future development of a refined theory of RCD(K, N) spaces, e.g. in the construction of parallel transport (cf. [**122**]).

1. Green functions

The aim fo this section is to introduce and study the main properties of the Green function of the Laplacian on an RCD$(0, N)$ metric measure space verifying suitable volume growth assumptions. Then, in Section 1.2, we show how the theory can adapted to cover the case of a possibly negative lower Ricci curvature bound.

The role of Green functions in geometric analysis has been prominent in the literature (cf. [**158**–**160**, **207**–**209**]) and also, more recently, in the theory of Ricci limit spaces (see for instance [**65**, **75**, **92**]).

1.1. Non-negative Ricci curvature. A natural setting to have existence of a positive Green function is that of RCD$(0, N)$ metric measure spaces satisfying suitable volume growth conditions (see Assumption 3.2 and Assumption 3.14 below).

Up to the end of this subsection we assume that $(X, \mathsf{d}, \mathfrak{m})$ is an RCD$(0, N)$ metric measure space. Further assumptions on the space will be added in the sequel.

We set

$$G(x, y) := \int_0^\infty p_t(x, y) \, \mathrm{d}t$$

and, for every $\varepsilon > 0$,

$$G^\varepsilon(x,y) := \int_\varepsilon^\infty p_t(x,y)\,\mathrm{d}t. \tag{3.1}$$

We shall adopt in the sequel also the notation $G_x(\cdot) := G(x,\cdot)$ (and analogously for G^ε).

Before going on let us observe that, at least at a formal level, the Green function is the fundamental solution of the Laplace operator. Indeed

$$\boldsymbol{\Delta}_y G_x(\cdot) = \boldsymbol{\Delta}_y \left(\int_0^\infty p_t(x,\cdot)\,\mathrm{d}t\right) = \int_0^\infty \Delta_y p_t(x,\cdot)\,\mathrm{d}t = \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t} p_t(x,\cdot)\,\mathrm{d}t = [p_t(x,\cdot)]_0^\infty = -\delta_x.$$

In order to get the good definition of both G and G^ε, up to the end of this subsection, unless otherwise stated, we will work under the following assumption.

Assumption 3.2. There exists $x \in X$ such that

$$\int_1^\infty \frac{s}{\mathfrak{m}(B_s(x))}\,\mathrm{d}s < \infty. \tag{3.2}$$

Recall that, for a non compact Riemannian manifold with non-negative Ricci curvature, it was proved by Varopoulos (cf. [**207**–**209**]) that (3.2) is a necessary and sufficient condition for the existence of a positive Green function of the Laplacian (and this last condition is known as *non-parabolicity* in the literature).

Remark 3.3. Let us observe that all the metric measure spaces obtained as tensor products between an $\mathrm{RCD}(0,N)$ m.m.s. $(X,\mathsf{d},\mathfrak{m})$ and an Euclidean factor $(\mathbb{R}^k, \mathsf{d}_{\mathbb{R}^k}, \mathcal{L}^k)$ for $k \geq 3$ do satisfy Assumption 3.2.

We introduce the auxiliary functions $F, H : X \times (0,+\infty) \to (0,+\infty)$ by

$$F(x,r) := \int_r^\infty \frac{s}{\mathfrak{m}(B_s(x))}\,\mathrm{d}s \tag{3.3}$$

and

$$H(x,r) := \int_r^\infty \frac{1}{\mathfrak{m}(B_s(x))}\,\mathrm{d}s. \tag{3.4}$$

They are the objects we will use to estimate the Green function and its gradient (see [**125**] for analogous results in the smooth setting). As for the Green function, we will often write $F_x(r)$ or $H_x(r)$ in place of $F(x,r)$ and $H(x,r)$.

Remark 3.4. Let us remark that both F and H are continuous w.r.t. the first variable. This property can be verified recalling that spheres are negligible on doubling m.m. spaces (cf. Remark 1.11) and using the continuity of the function $x \mapsto \mathfrak{m}(B_r(x))$ (with $r > 0$ fixed).

The next proposition has the aim to provide estimates for the Green function and its gradient in terms of $F_x(\mathsf{d}(x,y))$ and $H_x(\mathsf{d}(x,y))$ that are simpler objects to work with. Its proof is postponed after a series of useful lemmas.

Proposition 3.5 (Main estimates for G)**.** *Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(0,N)$ m.m.s. satisfying Assumption 3.2. Then there exists a constant $C_2 \geq 1$, depending only on N, such that, for any $x \in X$,*

$$\frac{1}{C_2} F_x(\mathsf{d}(x,y)) \leq G_x(y) \leq C_2 F_x(\mathsf{d}(x,y)) \quad \text{for any } y \in X. \tag{3.5}$$

Moreover for any $x \in X$ it holds that $G_x \in W^{1,1}_{\mathrm{loc}}(X,\mathsf{d},\mathfrak{m})$ and

$$|\nabla G_x|(y) \leq \int_0^\infty |\nabla p_t(x,\cdot)|(y)\,\mathrm{d}t \leq C_2 H_x(\mathsf{d}(x,y)), \quad \text{for } \mathfrak{m}\text{-a.e. } y \in X. \tag{3.6}$$

The first lemma below deals with the integrability properties of the maps $y \mapsto F_x(\mathsf{d}(x,y))$ and $y \mapsto H_x(\mathsf{d}(x,y))$. Since its formulation and its proof do not require any regularity assumption for the metric measure space, apart from the validity of Assumption 3.2, we state it in this generality.

Lemma 3.6. *Let $(X,\mathsf{d},\mathfrak{m})$ be a m.m.s. satisfying Assumption 3.2. Then for every $x \in X$, the functions $y \mapsto F_x(\mathsf{d}(x,y))$ and $y \mapsto H_x(\mathsf{d}(x,y))$ belong to $L^1_{\mathrm{loc}}(X,\mathfrak{m})$. Moreover the map $(w,z) \mapsto H(w,\mathsf{d}(w,z))$ belongs to $L^1_{\mathrm{loc}}(X \times X, \mathfrak{m}\times\mathfrak{m})$.*

Proof. Let $g : \mathbb{R} \to [0,+\infty)$ be a Borel function, define $f(r) := \int_r^\infty g(s)\,\mathrm{d}s$. Observe that

$$\int_{B_R(x)} f(\mathsf{d}_x(w))\,\mathrm{d}\mathfrak{m}(w) = \int_0^R g(s)\mathfrak{m}(B_s(x))\,\mathrm{d}s + f(R)\mathfrak{m}(B_R(x)), \quad \text{for any } R>0, \tag{3.7}$$

as an application of Fubini's theorem shows. Fix now any $x \in X$. Applying (3.7), first with $g(s) = \frac{s}{\mathfrak{m}(B_s(x))}$ and then with $g(s) = \frac{1}{\mathfrak{m}(B_s(x))}$, we get

$$\int_{B_R(x)} F_x(\mathsf{d}_x(w))\,\mathrm{d}\mathfrak{m}(w) = \frac{R^2}{2} + F_x(R)\mathfrak{m}(B_R(x)), \tag{3.8}$$

and

$$\int_{B_R(x)} H_x(\mathsf{d}_x(w))\,\mathrm{d}\mathfrak{m}(w) = R + H_x(R)\mathfrak{m}(B_R(x)), \tag{3.9}$$

that imply in turn that $y \mapsto F_x(\mathsf{d}(x,y))$ and $y \mapsto H_x(\mathsf{d}(x,y))$ belong to $L^1_{\mathrm{loc}}(X,\mathfrak{m})$.

We now prove the local integrability of $(w,z) \mapsto H(w,\mathsf{d}(w,z))$. It suffices to show that

$$\int_{B_R(\bar{x})}\int_{B_R(\bar{x})} H(w,\mathsf{d}(w,z))\,\mathrm{d}\mathfrak{m}(z)\,\mathrm{d}\mathfrak{m}(w) < \infty, \qquad \forall R>0,\ \forall \bar{x} \in X. \tag{3.10}$$

Observe that for every $w \in B_R(\bar{x})$ it holds $B_R(\bar{x}) \subset B_{2R}(w)$. Hence

$$\begin{aligned}
&\int_{B_R(\bar{x})}\int_{B_R(\bar{x})} H(w,\mathsf{d}(w,z))\,\mathrm{d}\mathfrak{m}(z)\,\mathrm{d}\mathfrak{m}(w)\\
&\qquad\le \int_{B_R(\bar{x})}\int_{B_{2R}(w)} H(w,\mathsf{d}(w,z))\,\mathrm{d}\mathfrak{m}(z)\,\mathrm{d}\mathfrak{m}(w)\\
&\qquad\overset{(3.9)}{=} \int_{B_R(\bar{x})} \left[2R + \mathfrak{m}(B_{2R}(w))H_w(2R)\right]\,\mathrm{d}\mathfrak{m}(w)\\
&\qquad\le 2R\mathfrak{m}(B_R(\bar{x})) + \mathfrak{m}(B_{3R}(\bar{x}))\int_{B_R(\bar{x})} H_w(2R)\,\mathrm{d}\mathfrak{m}(w).
\end{aligned}$$

Since $B_{s/2}(\bar{x}) \subset B_s(w)$ for every $w \in B_R(\bar{x})$ and $s > 2R$, we obtain

$$\begin{aligned}
\int_{B_R(\bar{x})} H_w(2R)\,\mathrm{d}\mathfrak{m}(w) &= \int_{2R}^\infty \int_{B_R(\bar{x})} \frac{1}{\mathfrak{m}(B_s(w))}\,\mathrm{d}\mathfrak{m}(w)\,\mathrm{d}s\\
&\le \int_{2R}^\infty \int_{B_R(\bar{x})} \frac{1}{\mathfrak{m}(B_{s/2}(\bar{x}))}\,\mathrm{d}\mathfrak{m}(w)\,\mathrm{d}s\\
&= \mathfrak{m}(B_R(\bar{x}))\int_{2R}^\infty \frac{1}{\mathfrak{m}(B_{s/2}(\bar{x}))}\,\mathrm{d}s < \infty.
\end{aligned}$$

□

The following lemma deals with the regularity of the approximations of the Green function G_x^ε.

Lemma 3.7. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(0, N)$ *space satisfying Assumption 3.2 and fix* $x \in X$. *For every* $0 < \varepsilon < 1$ *the function* G_x^ε *belongs to* $\mathrm{Lip_b}(X) \cap D_{\mathrm{loc}}(\Delta)$ *and it holds* $\Delta G_x^\varepsilon = -p_\varepsilon(x, \cdot)$. *Moreover* $G_x \in W^{1,1}_{\mathrm{loc}}(X, \mathsf{d}, \mathfrak{m})$ *and*

$$\lim_{\varepsilon \to 0} G_x^\varepsilon = G_x \qquad \text{in } W^{1,1}_{\mathrm{loc}}(X, \mathsf{d}, \mathfrak{m}). \tag{3.11}$$

Proof. First of all let us prove that $G_x^\varepsilon \in L^\infty(X, \mathfrak{m})$. Using (1.79) and Assumption 3.2 we have

$$G_x^\varepsilon(y) = \int_\varepsilon^\infty p_t(x,y)\,\mathrm{d}y \le \int_\varepsilon^\infty \frac{C_1}{\mathfrak{m}(B_{\sqrt{t}}(x))}\,\mathrm{d}t = 2C_1 \int_{\sqrt{\varepsilon}}^\infty \frac{t}{\mathfrak{m}(B_t(x))}\,\mathrm{d}t < \infty.$$

The proof of the property $G_x^\varepsilon \in \mathrm{Lip_b}(X)$ will follow by (1.49) after proving that the identity $G_x^{\alpha+t} = P_t G_x^\alpha$ holds true for any $\alpha, t \in (0, +\infty)$, since we proved that $G^\alpha \in L^\infty$. To this aim, for any $x, y \in X$ and for any $t, \alpha > 0$, we compute

$$\begin{aligned} P_t G_x^\alpha(y) &= \int_X p_t(y,z) G_x^\alpha(z)\,\mathrm{d}\mathfrak{m}(z) = \int_\alpha^\infty \int_X p_t(y,z) p_s(x,z)\,\mathrm{d}\mathfrak{m}(z)\,\mathrm{d}s \\ &= \int_\alpha^\infty p_{t+s}(x,y)\,\mathrm{d}s = \int_{\alpha+t}^\infty p_s(x,y)\,\mathrm{d}s = G_x^{\alpha+t}(y). \end{aligned}$$

In order to prove that $G_x^\varepsilon \in D_{\mathrm{loc}}(\Delta)$ and $\Delta G_x^\varepsilon = p_\varepsilon(x, \cdot)$ we consider a function $f \in \mathrm{Test}(X, \mathsf{d}, \mathfrak{m})$ and we compute

$$\int_X G_x^\varepsilon(w) \Delta f(w)\,\mathrm{d}\mathfrak{m}(w) = \int_\varepsilon^\infty P_t \Delta f(x)\,\mathrm{d}t = -P_\varepsilon f(x),$$

where the last equality follows from the observation that $P_r f \to 0$ pointwise as $r \to \infty$ for any $f \in L^1 \cap L^2(X, \mathfrak{m})$, that is a consequence of the estimates for the heat kernel (1.79) and the fact that $\mathfrak{m}(X) = \infty$.

Let us prove (3.11). We preliminary observe that $G_x^\varepsilon \to G_x$ in $L^1_{\mathrm{loc}}(X, \mathfrak{m})$, since $G_x - G_x^\varepsilon \ge 0$ and

$$\int_X \left(G_x(y) - G_x^\varepsilon(y)\right)\,\mathrm{d}\mathfrak{m}(y) = \int_X \int_0^\varepsilon p_t(x,y)\,\mathrm{d}t\,\mathrm{d}\mathfrak{m}(y) = \int_0^\varepsilon \int_X p_t(x,y)\,\mathrm{d}\mathfrak{m}(y)\,\mathrm{d}t = \varepsilon.$$

To conclude the proof it suffices to show that G_x^ε is a Cauchy sequence in $W^{1,1}_{\mathrm{loc}}(X, \mathsf{d}, \mathfrak{m})$. We claim that, for every $0 < \varepsilon_1 < \varepsilon_2 < 1$,

$$|\nabla(G_x^{\varepsilon_1} - G_x^{\varepsilon_2})|(y) = \mathrm{lip}(G_x^{\varepsilon_1} - G_x^{\varepsilon_2})(y) \le \int_{\varepsilon_2}^{\varepsilon_1} \mathrm{lip}\left(p_t(x, \cdot)\right)(y)\,\mathrm{d}t, \quad \text{for } \mathfrak{m}\text{-a.e. } y \in X. \tag{3.12}$$

As a consequence of the Bishop-Gromov inequality (1.46) we get

$$\begin{aligned} \sup_{t>0} \int_X \frac{e^{-\frac{\mathsf{d}^2(x,y)}{5t}}}{\mathfrak{m}(B_{\sqrt{t}}(x))}\,\mathrm{d}\mathfrak{m}(y) &= \sup_{t>0} \frac{1}{\mathfrak{m}(B_{\sqrt{t}}(x))} \int_X \int_{\mathsf{d}^2(x,y)/t}^\infty \frac{e^{-s/5}}{5}\,\mathrm{d}s\,\mathrm{d}\mathfrak{m}(y) \\ &= \sup_{t>0} \int_0^\infty \frac{e^{-s/5}}{5} \frac{\mathfrak{m}(B_{\sqrt{st}}(x))}{\mathfrak{m}(B_{\sqrt{t}}(x))}\,\mathrm{d}s \\ &\le \int_0^\infty \frac{e^{-s/5}}{5} \max\{s; 1\}^{N/2}\,\mathrm{d}s < \infty, \end{aligned}$$

that, together with the estimate for the gradient of the heat kernel (1.80), implies

$$\int_0^1 \int_X |\nabla p_t(x, \cdot)|(y)\,\mathrm{d}\mathfrak{m}(y)\,\mathrm{d}t \le \int_0^1 \frac{C_2}{\sqrt{t}} \int_X \frac{e^{-\frac{\mathsf{d}^2(x,y)}{5t}}}{\mathfrak{m}(B_{\sqrt{t}}(x))}\,\mathrm{d}\mathfrak{m}(y)\,\mathrm{d}t < \infty,$$

therefore (3.12) will yield the desired conclusion.

Let us pass to the verification of (3.12). Observe that the $\mathfrak{m}$-a.e. identifications between slopes and minimal weak upper gradients follow from the local Lipschitz regularity of the heat kernel and G_x^ε for $\varepsilon > 0$ thanks to Theorem 1.33. Moreover, the very definition of G^ε guarantees that

$$\operatorname{lip}(G_x^{\varepsilon_1} - G_x^{\varepsilon_2})(y) \leq \limsup_{z \to y} \int_{\varepsilon_2}^{\varepsilon_1} \frac{|p_t(x,y) - p_t(x,z)|}{\mathsf{d}(y,z)} \, \mathrm{d}t, \quad \text{for every } y \in X. \tag{3.13}$$

Moreover, for any $r < \frac{1}{2} d(x,y)$, the gradient estimate for the heat kernel (1.80) yields

$$|\nabla p_t(x,\cdot)|(w) \leq \frac{C_1 e^{-\frac{r^2}{5t}}}{\sqrt{t}\mathfrak{m}(B_{\sqrt{t}}(x))} \quad \text{for } \mathfrak{m}\text{-a.e } w \in B_r(y). \tag{3.14}$$

Hence $p_t(x,\cdot)$ is Lipschitz in $B_{r/2}(y)$ with Lipschitz constant bounded from above by the right hand side of (3.14), thanks to Remark 1.74. Summarizing we obtained the bound

$$\frac{|p_t(x,y) - p_t(x,z)|}{\mathsf{d}(y,z)} \leq \frac{C_1 e^{-\frac{r^2}{5t}}}{\sqrt{t}\mathfrak{m}(B_{\sqrt{t}}(x))}, \tag{3.15}$$

for every $z \in B_{r/2}(y)$ and every $t \in (0, \infty)$. Hence we can apply Fatou's lemma and pass from (3.13) to (3.12). $\square$

Remark 3.8. Proceeding as in the proof of Lemma 3.7 above, one can prove that, for any $\eta \in \mathrm{Test}(X, \mathsf{d}, \mathfrak{m})$ with compact support, it holds that $\eta G_x^\varepsilon \in \mathrm{Test}(X, \mathsf{d}, \mathfrak{m})$ for any $x \in X$ and for any $\varepsilon > 0$.

The elementary proof of Lemma 3.9 below can be obtained with minor modifications to the proof of [**125**, Lemma 5.50].

Lemma 3.9. *Let $\phi : (0, +\infty) \to (0, +\infty)$ be monotone increasing and set*

$$\psi(r) := \int_0^{+\infty} \frac{1}{\phi(\sqrt{t})} \exp\left(-\frac{r^2}{t}\right) \mathrm{d}t.$$

If ϕ satisfies the local doubling property

$$\phi(2r) \leq C(R)\phi(r) \quad \textit{for any } 0 < r < R,$$

for some nondecreasing function $C : (0, +\infty) \to (0, +\infty)$, then there exists a nondecreasing function $\Lambda : (0, +\infty) \to (0, +\infty)$, whose values depend only on the function C, such that

$$\frac{1}{\Lambda(R)} \int_r^\infty \frac{s}{\phi(s)} \, \mathrm{d}s \leq \psi(r) \leq \Lambda(R) \int_r^\infty \frac{s}{\phi(s)} \, \mathrm{d}s, \tag{3.16}$$

for any $0 < r < R$ and for any $R \in (0, +\infty)$. Moreover, when C is constant, we can choose Λ to be constant.

Proof of Proposition 3.5. The proof of (3.5) follows from the estimates for the heat kernel (1.79) applying Lemma 3.9 with $\phi(r) := \mathfrak{m}(B_r(x))$.

In order to prove (3.6) we observe that, arguing exactly as in the proof of (3.11), one obtains that, for any $\varepsilon > 0$ and any $x \in X$,

$$|\nabla G_x^\varepsilon|(y) \leq \int_\varepsilon^\infty |\nabla p_t(x,\cdot)|(y) \quad \text{for } \mathfrak{m}\text{-a.e. } y \in X. \tag{3.17}$$

The sought conclusion follows from (3.11).

The proof of the inequality

$$\int_0^\infty |\nabla p_t(x,\cdot)|(y) \, \mathrm{d}t \leq C_2 H_x(\mathsf{d}(x,y)), \qquad \text{for } \mathfrak{m}\text{-a.e. } y \in X$$

follows from the gradient estimate for the heat kernel (1.80), applying Lemma 3.9 with choice $\phi(r) := r\mathfrak{m}(B_r(x))$. ☐

Remark 3.10. It is clear from the proof of Proposition 3.5 that the regularized functions G^ε satisfy

$$|\nabla G_x^\varepsilon|(y) \leq \int_\varepsilon^\infty |\nabla p_t(x,\cdot)|(y)\,\mathrm{d}t \leq C_2 H_x(\mathsf{d}(x,y)), \qquad \text{for } \mathfrak{m}\text{-a.e. } y \in X. \tag{3.18}$$

Remark 3.11. As a consequence of (3.6) and of the continuity of the map $x \mapsto H_x(r)$, exploiting the monotonicity w.r.t. r of H and Remark 1.74, one can prove that G_x is continuous in $X \setminus \{x\}$.

The maximal estimate below is a key tool to bound the rate of change of the Green function along trajectories of a Lagrangian flow. It will be crucial in the proof of the vector-valued maximal estimate Proposition 3.16.

Proposition 3.12 (Maximal estimate, scalar version)**.** *Let* $(X,\mathsf{d},\mathfrak{m})$ *be an* $\mathrm{RCD}(0,N)$ *m.m.s. satisfying Assumption 3.2. Then there exists* $C_M > 0$*, depending only on* N*, such that, for any Borel function* $f : X \to [0,+\infty)$*, it holds*

$$\int f(w)\,|\nabla G_x(w)|\,|\nabla G_y(w)|\,\mathrm{d}\mathfrak{m}(w) \leq C_M G(x,y)\left(Mf(x) + Mf(y)\right), \tag{3.19}$$

for every $x, y \in X$*.*

Proof. Fix two different points in $x, y \in X$. Thanks to (3.6) we can estimate the left hand side of (3.19) with

$$C_2^2 \int_0^\infty \int_0^\infty \int_X f(w) \frac{\mathbb{I}_{B_r(x)}(w)}{\mathfrak{m}(B_r(x))} \frac{\mathbb{I}_{B_s(y)}(w)}{\mathfrak{m}(B_s(y))}\,\mathrm{d}\mathfrak{m}(w)\,\mathrm{d}s\,\mathrm{d}r.$$

By splitting the domain $(0,+\infty)\times(0,+\infty)$ into A_1, A_2 and A_3, with $A_1 := \{(s,r)|\ \mathsf{d}(x,y)+s \leq r\}$, $A_2 := \{(s,r)|\ \mathsf{d}(x,y)+r \leq s\}$ and $A_3 := \{(s,r)|\ \mathsf{d}(x,y) > |r-s|\}$ we are left with the estimates of the following quantities:

$$I_1 := \int_{A_1} \int_X f(w) \frac{\mathbb{I}_{B_r(x)}(w)}{\mathfrak{m}(B_r(x))} \frac{\mathbb{I}_{B_s(y)}(w)}{\mathfrak{m}(B_s(y))}\,\mathrm{d}\mathfrak{m}(w)\,\mathrm{d}s\,\mathrm{d}r,$$

$$I_2 := \int_{A_2} \int_X f(w) \frac{\mathbb{I}_{B_r(x)}(w)}{\mathfrak{m}(B_r(x))} \frac{\mathbb{I}_{B_s(y)}(w)}{\mathfrak{m}(B_s(y))}\,\mathrm{d}\mathfrak{m}(w)\,\mathrm{d}s\,\mathrm{d}r$$

and

$$I_3 := \int_{A_3} \int_X f(w) \frac{\mathbb{I}_{B_r(x)}(w)}{\mathfrak{m}(B_r(x))} \frac{\mathbb{I}_{B_s(y)}(w)}{\mathfrak{m}(B_s(y))}\,\mathrm{d}\mathfrak{m}(w)\,\mathrm{d}s\,\mathrm{d}r.$$

In order to estimate I_1, we observe that $B_s(y) \subset B_r(x)$ for every $(s,r) \in A_1$, thus

$$\begin{aligned}
I_1 =& \int_{A_1} \frac{1}{\mathfrak{m}(B_r(x))} \fint_{B_s(y)} f(w)\,\mathrm{d}\mathfrak{m}(w)\,\mathrm{d}s\,\mathrm{d}r \\
\leq& Mf(y) \int_{\mathsf{d}(x,y)}^\infty \int_0^{r-\mathsf{d}(x,y)} \frac{1}{\mathfrak{m}(B_r(x))}\,\mathrm{d}s\,\mathrm{d}r \\
\leq& Mf(y) \int_{\mathsf{d}(x,y)}^\infty \frac{r}{\mathfrak{m}(B_r(x))}\,\mathrm{d}r \\
\leq& C_2 G(x,y) Mf(y).
\end{aligned}$$

By symmetry we get

$$I_2 \leq C_2 G(x,y) Mf(x).$$

To estimate I_3 let us observe that, if $r+s<\mathsf{d}(x,y)$, then $B_r(x)\cap B_s(y)=\emptyset$. Thus the integration can be restricted to the smaller domain $B:=\{(s,r)|\ \mathsf{d}(x,y)>|r-s|,\ r+s\geq \mathsf{d}(x,y)\}$ that we split once more into $B_1:=\{(s,r)|\ \mathsf{d}(x,y)>r-s,\ r+s\geq \mathsf{d}(x,y),\ r\geq s\}$ and $B_2:=\{(s,r)|\ \mathsf{d}(x,y)>s-r,\ r+s\geq \mathsf{d}(x,y),\ r<s\}$. Therefore we have

$$\begin{aligned}I_3 =&\int_B\int_X f(w)\frac{\mathbb{I}_{B_r(x)}(w)}{\mathfrak{m}(B_r(x))}\frac{\mathbb{I}_{B_s(y)}(w)}{\mathfrak{m}(B_s(y))}\,\mathrm{d}\mathfrak{m}(w)\,\mathrm{d}s\,\mathrm{d}r\\ =&\int_{B_1}\int_X f(w)\frac{\mathbb{I}_{B_r(x)}(w)}{\mathfrak{m}(B_r(x))}\frac{\mathbb{I}_{B_s(y)}(w)}{\mathfrak{m}(B_s(y))}\,\mathrm{d}\mathfrak{m}(w)\,\mathrm{d}s\,\mathrm{d}r\\ &+\int_{B_2}\int_X f(w)\frac{\mathbb{I}_{B_r(x)}(w)}{\mathfrak{m}(B_r(x))}\frac{\mathbb{I}_{B_s(y)}(w)}{\mathfrak{m}(B_s(y))}\,\mathrm{d}\mathfrak{m}(w)\,\mathrm{d}s\,\mathrm{d}r\\ =:&I_3^1+I_3^2.\end{aligned}$$

We now deal with I_3^1. Using the rough estimate $\mathbb{I}_{B_r(x)}\leq 1$ we obtain

$$\begin{aligned}I_3^1\leq&\int_{B_1}\frac{1}{\mathfrak{m}(B_r(x))}\fint_{B_s(y)}f(w)\,\mathrm{d}\mathfrak{m}(w)\,\mathrm{d}s\,\mathrm{d}r\\ \leq& Mf(y)\int_{\mathsf{d}(x,y)/2}^{\infty}\int_{|\mathsf{d}(x,y)-r|}^{r}\frac{1}{\mathfrak{m}(B_r(x))}\,\mathrm{d}s\,\mathrm{d}r\\ \leq& Mf(y)\int_{\mathsf{d}(x,y)/2}^{\infty}\frac{r}{\mathfrak{m}(B_r(x))}\,\mathrm{d}r\\ =&Mf(y)\frac{1}{4}\int_{\mathsf{d}(x,y)}^{\infty}\frac{r}{\mathfrak{m}(B_{r/2}(x))}\,\mathrm{d}r.\end{aligned}$$

With a simple application of (1.46) and (3.5) we conclude that $I_3^1\leq C(C_2,N)Mf(y)G(x,y)$. By symmetry we also have $I_3^2\leq C(C_2,N)Mf(x)G(x,y)$. Putting all these estimates together we obtain the desired result. □

Remark 3.13. It is clear from the proof of Proposition 3.12 and from Remark 3.10 that the same estimate holds true if one puts ∇G_x^ε and ∇G_y^ε in place of ∇G_x and ∇G_y at the left hand side of (3.19). More precisely it holds that

$$\int f(z)\,|\nabla G_x^\varepsilon(z)|\,|\nabla G_y^\varepsilon(z)|\,\mathrm{d}\mathfrak{m}(z)\leq C_M G(x,y)\,(Mf(x)+Mf(y))\,, \tag{3.20}$$

for every $x,y\in X$.

Next we introduce a key object, the Green quasi-metric d_G:

$$\mathsf{d}_G(x,y):=\begin{cases}\frac{1}{G(x,y)} & \text{if } x\neq y,\\ 0 & \text{otherwise.}\end{cases} \tag{3.21}$$

In [**52**] we proved that (under the additional volume growth assumption Assumption 3.14) d_G is a *quasi-metric* on X (i.e. it satisfies a triangle inequality up to a multiplicative constant) and that $\mathfrak{m}$ is still a doubling measure over (X,d_G). The terminology, quite common in the literature about analysis on metric spaces, is borrowed from [**134**, Chapter 14].

Assumption 3.14. There exists an RCD$(0,N-3)$ metric measure space $(\bar X,\bar{\mathsf{d}},\bar{\mathfrak{m}})$ such that $(X,\mathsf{d},\mathfrak{m})$ is the tensor product between $(\bar X,\bar{\mathsf{d}},\bar{\mathfrak{m}})$ and $(\mathbb{R}^3,\mathsf{d}_{\mathbb{R}^3},\mathcal{L}^3)$.

First of all observe that d_G is symmetric and positive whenever $x\neq y$. Moreover, for every $x\in X$, the map $y\mapsto \mathsf{d}_G(x,y)$ is continuous. Indeed, thanks to the continuity of G_x in $X\setminus\{x\}$ (see Remark 3.11 above), we need only to show that $\mathsf{d}_G(x,\cdot)$ is continuous at x, and this is the content of the following lemma.

Lemma 3.15. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(0, N)$ *m.m.s. satisfying Assumption 3.14. Then for any* $x \in X$ *it holds that* $\mathsf{d}_G(x, y) \to 0$ *if and only if* $\mathsf{d}(x, y) \to 0$.

Proof. Suppose that $\mathsf{d}_G(x, y) \to 0$. Then, by the very definition of d_G, it must be $G(x, y) \to +\infty$. Hence, since we have the uniform control $G(x, y) \le C_2 F(x, \mathsf{d}(x, y))$ and $F(x, \cdot)$ is bounded away from 0, we conclude $\mathsf{d}(x, y) \to 0$.

In order to prove the converse implication let us observe that, if $\mathsf{d}(x, y) \to 0$, then $F(x, \mathsf{d}(x, y)) \to \infty$. Indeed under our assumptions $s \mapsto s/\mathfrak{m}(B(x, s))$ is not integrable at 0 and to conclude we just need to exploit the bound $G(x, y) \ge 1/C_2 F(x, \mathsf{d}(x, y))$ (see Proposition 3.5 above). □

Proposition 3.16 (Maximal estimate, vector-valued version)**.** *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(0, N)$ *m.m.s. satisfying Assumption 3.2. Assume that* $b \in H^{1,2}_{C,s}(TX)$ *is a compactly supported and bounded vector field. Then, setting* $g := |\nabla_{\mathrm{sym}} b| + |\mathrm{div}\, b|$, *it holds*

$$|b \cdot \nabla G_x(y) + b \cdot \nabla G_y(x)| \le 2C_M G(x, y)(Mg(x) + Mg(y)), \tag{3.22}$$

for $\mathfrak{m} \times \mathfrak{m}$*-a.e.* $(x, y) \in X \times X$, *where* M *stands for the maximal operator.*

Proof. The heuristic standing behind the proof of this result is the following one: assuming that b is divergence free we can formally compute

$$\begin{aligned} b \cdot \nabla G_x(y) + b \cdot \nabla G_y(x) &= -\int_X b \cdot \nabla G_x(w)\, \mathrm{d}\boldsymbol{\Delta} G_y(w) - \int_X b \cdot \nabla G_y(w)\, \mathrm{d}\boldsymbol{\Delta} \boldsymbol{G}_x(w) \\ &= 2\int_X \nabla_{\mathrm{sym}} b(w)(\nabla G_x(w), \nabla G_y(w))\, \mathrm{d}\mathfrak{m}(w), \end{aligned}$$

so that, taking the moduli and applying Proposition 3.12, we would reach the desired conclusion.

The proof will be divided into two steps: in the first one we are going to prove an estimate for the regularized functions G^ε; in the second one the sought conclusion will be recovered by an approximation procedure.

Step 1 We start proving that, for every $\varepsilon \in (0, 1)$ and for every $x, y \in X$, it holds

$$\left| \int_X \left\{ b \cdot \nabla G^\varepsilon_x \Delta G^\varepsilon_y + b \cdot \nabla G^\varepsilon_y \Delta G^\varepsilon_x \right\} \mathrm{d}\mathfrak{m} \right| \le 2C_M G(x, y) \left(Mg(x) + Mg(y) \right). \tag{3.23}$$

To this aim, we choose a cut-off function with compact support $\eta \in \mathrm{Test}(X, \mathsf{d}, \mathfrak{m})$ such that $\eta \equiv 1$ on $\mathrm{supp}\, b$ (the existence of such function follows from Lemma 1.107). Applying (1.63) with $h = \eta$, $f = \eta G^\varepsilon_x$ and $g = \eta G^\varepsilon_y$ (observe that they are admissible test functions in the definition of symmetric covariant derivative thanks to Remark 3.8) we obtain:

$$\begin{aligned} &\left| \int_X \left\{ b \cdot \nabla G^\varepsilon_x \Delta G^\varepsilon_y + b \cdot \nabla G^\varepsilon_y \Delta G^\varepsilon_x \right\} \mathrm{d}\mathfrak{m} \right| \\ &\qquad \le \left| \int_X \left\{ b \cdot \nabla G^\varepsilon_x \Delta G^\varepsilon_y + b \cdot \nabla G^\varepsilon_y \Delta G^\varepsilon_x - \mathrm{div}\, b\, \nabla G^\varepsilon_x \cdot \nabla G^\varepsilon_y \right\} \mathrm{d}\mathfrak{m} \right| \\ &\qquad\quad + \left| \int_X \mathrm{div}\, b\, \nabla G^\varepsilon_x \cdot \nabla G^\varepsilon_y \, \mathrm{d}\mathfrak{m} \right| \\ &\qquad = 2\left| \int_X \nabla_{\mathrm{sym}} b\, (\nabla G^\varepsilon_x, \nabla G^\varepsilon_y)\, \mathrm{d}\mathfrak{m} \right| + \left| \int_X \mathrm{div}\, b\, \nabla G^\varepsilon_x \cdot \nabla G^\varepsilon_y \, \mathrm{d}\mathfrak{m} \right| \\ &\qquad \le 2\int_X g(w) |\nabla G^\varepsilon_x(w)| |\nabla G^\varepsilon_y(w)|\, \mathrm{d}\mathfrak{m}(w). \end{aligned} \tag{3.24}$$

The estimate in (3.23) follows from (3.24) applying (3.20).

Step 2 In the second step of the proof we prove that, as $\varepsilon \to 0$, it holds

$$\left| \int_X \left\{ b \cdot \nabla G_x^\varepsilon \Delta G_y^\varepsilon + b \cdot \nabla G_y^\varepsilon \Delta G_x^\varepsilon \right\} \mathrm{d}\mathfrak{m} \right| \to |b \cdot \nabla G_x(y) + b \cdot \nabla G_y(x)| \tag{3.25}$$

in $L^1_{\mathrm{loc}}(X \times X, \mathfrak{m} \times \mathfrak{m})$. This will allow us to get (3.22) by choosing a sequence $\varepsilon_i \downarrow 0$ such that the convergence in (3.25) holds true $\mathfrak{m} \times \mathfrak{m}$-a.e. on $X \times X$ and exploiting Step 1.

In order to prove (3.25), we start recalling that $\Delta G_y^\varepsilon(w) = -p_\varepsilon(y,w)$ for any $\varepsilon > 0$ (see Lemma 3.7). Thus

$$\int_X b \cdot \nabla G_x^\varepsilon \Delta G_y^\varepsilon \, \mathrm{d}\mathfrak{m} = -P_\varepsilon(b \cdot \nabla G_x^\varepsilon)(y) \quad \text{for any } x, y \in X. \tag{3.26}$$

For our purposes it suffices to check that $\int_K \int_K |P_\varepsilon(b \cdot \nabla G_x^\varepsilon)(y) - b \cdot \nabla G_x(y)| \, \mathrm{d}\mathfrak{m}(x) \, \mathrm{d}\mathfrak{m}(y) \to 0$ as $\varepsilon \to 0$, for every compact $K \subset X$. Adding and subtracting $P_\varepsilon(b \cdot \nabla G_x)(y)$ (that is well defined since $b \cdot \nabla G_x \in L^1(X, \mathfrak{m})$), we obtain

$$\begin{aligned}
&\int_K \int_K |P_\varepsilon(b \cdot \nabla G_x^\varepsilon)(y) - b \cdot \nabla G_x(y)| \, \mathrm{d}\mathfrak{m}(x) \, \mathrm{d}\mathfrak{m}(y) \\
&\le \int_K \|P_\varepsilon(b \cdot \nabla (G_x^\varepsilon - G_x))\|_{L^1(X,\mathfrak{m})} \, \mathrm{d}\mathfrak{m}(x) + \int_K \|P_\varepsilon(b \cdot \nabla G_x) - b \cdot \nabla G_x\|_{L^1(X,\mathfrak{m})} \, \mathrm{d}\mathfrak{m}(x).
\end{aligned}$$

Using the L^1-contractivity of the semigroup P_ε, we deduce that

$$\|P_\varepsilon(b \cdot \nabla (G_x^\varepsilon - G_x))\|_{L^1(X,\mathfrak{m})} \le \|b \cdot \nabla (G_x^\varepsilon - G_x)\|_{L^1(X,\mathfrak{m})} \quad \text{for any } x \in X.$$

Hence, for any $x \in X$,

$$\|P_\varepsilon(b \cdot \nabla (G_x^\varepsilon - G_x))\|_{L^1(X,\mathfrak{m})} \to 0 \quad \text{as } \varepsilon \downarrow 0,$$

since $G_x^\varepsilon \to G^x$ in $W^{1,1}_{\mathrm{loc}}(X, \mathsf{d}, \mathfrak{m})$ by Proposition 3.5 and b has compact support by assumption. Also the term $\|P_\varepsilon(b \cdot \nabla G_x) - b \cdot \nabla G_x\|_{L^1(X,\mathfrak{m})}$ goes to zero for every $x \in X$ since, as just remarked, $b \cdot \nabla G_x \in L^1(X, \mathfrak{m})$. Moreover both these terms are uniformly bounded by the function $x \mapsto C \|b\|_{L^\infty} \|H_x(\mathsf{d}_x(\cdot))\|_{L^1(\mathrm{supp}\,(b),\mathfrak{m})}$ that is locally integrable, since the map $(x,y) \mapsto H(x, \mathsf{d}(x,y))$ belongs to $L^1_{\mathrm{loc}}(X \times X, \mathfrak{m} \times \mathfrak{m})$ by Lemma 3.6. The conclusion of (3.25) can now be recovered applying the dominated convergence theorem. □

1.2. Extension to the case of an arbitrary lower Ricci bound.

Aim of this subsection is to study more general Green functions (that, formally, will be fundamental solutions of modified elliptic operators). These modified Green functions will play the role of the Green function of the Laplacian in the case of an arbitrary lower Ricci curvature bound when developing a regularity theory for Lagrangian flows of Sobolev vector fields as we shall do in Section 2.

This being said, the spirit of this part will be to show how to adapt the estimates we obtained in the case of non-negative Ricci curvature to this more general setting up to pay the price that they become local and less intrinsic.

Assumption 3.17. Throughout this section we assume that $(X, \mathsf{d}, \mathfrak{m})$ is the tensor product between an arbitrary RCD$(K, N-3)$ m.m.s. for some $K \in \mathbb{R}$ and $4 < N < \infty$ and a Euclidean factor $(\mathbb{R}^3, \mathsf{d}_{\mathbb{R}^3}, \mathcal{L}^3)$.

Let us stress once more that, for the purposes of the upcoming Section 3, it will be not too restrictive to have a regularity result for Lagrangian flows just over spaces satisfying Assumption 3.17.

Let $c \ge 0$ be the constant appearing in (1.79) and (1.80) and set

$$\bar{G}(x,y) := \int_0^\infty e^{-ct} p_t(x,y) \, \mathrm{d}t \quad \text{for any } x, y \in X, \tag{3.27}$$

and, in analogy with (3.1),

$$\bar{G}^{\varepsilon}(x,y) := \int_{\varepsilon}^{\infty} e^{-ct} p_t(x,y) \quad \text{for any } \varepsilon > 0 \text{ and any } x, y \in X. \tag{3.28}$$

As in the case of the Green function G, we shall adopt in the sequel also the notation $\bar{G}_x(\cdot) = \bar{G}(x,\cdot)$ (and analogously for $\bar{G}^{\varepsilon}$).
Observe that, assuming that $c > 0$, $\bar{G}_x$ is well defined and belongs to $L^1(X, \mathfrak{m})$ for every $x \in X$. Indeed an application of Fubini's theorem yields

$$\int_X \bar{G}_x(w)\, \mathrm{d}\mathfrak{m}(w) = \int_0^{\infty} e^{-ct} \int_X p_t(x,w)\, \mathrm{d}\mathfrak{m}(w)\, \mathrm{d}t = \int_0^{\infty} e^{-ct}\, \mathrm{d}t < \infty. \tag{3.29}$$

We can also remark that (3.29) holds true without any extra hypothesis on the $\mathrm{RCD}(K,N)$ m.m.s. $(X, \mathsf{d}, \mathfrak{m})$. Nevertheless, Assumption 3.17 will be crucial in order to obtain meaningful estimates for $\bar{G}$ and its gradient in terms of the functions F and H introduced in (3.3), (3.4).

At least at a formal level one can check that $\bar{G}$ solves the equation $\mathbf{\Delta} \bar{G}_x = -\delta_x + c\bar{G}_x$. Indeed

$$\begin{aligned}
\mathbf{\Delta}_y \bar{G}_x(\cdot) =& \mathbf{\Delta}_y \left(\int_0^{\infty} e^{-ct} p_t(x,\cdot)\, \mathrm{d}t \right) \\
=& \int_0^{\infty} e^{-ct} \Delta_y p_t(x,\cdot)\, \mathrm{d}t = \int_0^{\infty} e^{-ct} \frac{\mathrm{d}}{\mathrm{d}t} p_t(x,\cdot)\, \mathrm{d}t \\
=& [p_t(x,\cdot)]_0^{\infty} + c \int_0^{\infty} e^{-ct} p_t(x,\cdot)\, \mathrm{d}t = -\delta_x + c\bar{G}_x(\cdot).
\end{aligned}$$

To let this computation become rigorous, one can proceed as in the proof of Lemma 3.7 and check firstly that $\bar{G}^{\varepsilon}_x \in \mathrm{Lip}_b \cap D_{\mathrm{loc}}(\Delta)$ for any $x \in X$ and any $\varepsilon > 0$, with

$$\Delta \bar{G}^{\varepsilon}_x(y) = -e^{-c\varepsilon} p_{\varepsilon}(x,y) + c\bar{G}^{\varepsilon}_x(y), \qquad \text{for } \mathfrak{m}\text{-a.e. } y \in X, \tag{3.30}$$

and then that

$$\lim_{\varepsilon \to 0} \bar{G}^{\varepsilon}_x \to \bar{G}_x \qquad \text{in } W^{1,1}(X, \mathsf{d}, \mathfrak{m}). \tag{3.31}$$

Our primary goal is now to obtain useful local estimates for $\bar{G}$ and its gradient in terms of F and H.

Proposition 3.18 (Main estimates for $\bar{G}$)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ m.m.s. satisfying Assumption 3.17. Then, for any compact $P \subset X$, there exists $\bar{C} = \bar{C}(P) \geq 1$ such that*

$$\frac{1}{\bar{C}} F_x(\mathsf{d}(x,y)) \leq \bar{G}_x(y) \leq \bar{C} F_x(\mathsf{d}(x,y)) \quad \textit{for any } x, y \in P. \tag{3.32}$$

Moreover for any $x \in X$ it holds $\bar{G}_x \in W^{1,1}_{\mathrm{loc}}(X, \mathsf{d}, \mathfrak{m})$ and, for any $x \in P$,

$$|\nabla \bar{G}_x|(y) \leq \int_0^{\infty} e^{-ct} |\nabla p_t(x,\cdot)|(y)\, \mathrm{d}t \leq \bar{C} H_x(\mathsf{d}(x,y)) \quad \textit{for } \mathfrak{m}\textit{-a.e. } y \in P. \tag{3.33}$$

Proof. Applying the estimates for the heat kernel (1.79) we find out that

$$\frac{1}{C_1} \int_0^{\infty} \frac{e^{-2ct} e^{-\frac{\mathsf{d}(x,y)^2}{3t}}}{\mathfrak{m}(B_{\sqrt{t}}(x))}\, \mathrm{d}t \leq \bar{G}_x(y) \leq C_1 \int_0^{\infty} \frac{e^{-\frac{\mathsf{d}(x,y)^2}{5t}}}{\mathfrak{m}(B_{\sqrt{t}}(x))}\, \mathrm{d}t \quad \text{for any } x, y \in X. \tag{3.34}$$

Exploiting (1.7) (which is a consequence of the Bishop-Gromov inequality) and Lemma 3.9, we obtain from (3.34) that

$$\bar{G}_x(y) \leq C_1 \Lambda(R) F_x\left(\frac{\mathsf{d}(x,y)}{\sqrt{5}} \right) \quad \text{for any } x, y \text{ such that } \mathsf{d}(x,y) < R, \tag{3.35}$$

where Λ is the function in the statement of Lemma 3.9.
The bound from above in (3.32) follows from (3.35) together with the following observation, that will play a role also in the sequel: for any compact $P \subset X$, for any $R > 0$ and for any $\lambda < 1$, there exists $C(P, R, \lambda) \geq 0$ such that

$$F_x(\lambda r) \leq C(P, R, \lambda) F_x(r) \quad \text{for any } x \in P \text{ and any } 0 < r < R. \tag{3.36}$$

Indeed (3.36) can be checked splitting

$$F_x(\lambda r) = \int_{\lambda r}^{\lambda R} \frac{s}{\mathfrak{m}(B_s(x))} \, \mathrm{d}s + \int_{\lambda R}^{\infty} \frac{s}{\mathfrak{m}(B_s(x))} \, \mathrm{d}s, \tag{3.37}$$

$$F_x(r) = \int_{r}^{R} \frac{s}{\mathfrak{m}(B_s(x))} \, \mathrm{d}s + \int_{R}^{\infty} \frac{s}{\mathfrak{m}(B_s(x))} \, \mathrm{d}s \tag{3.38}$$

and using the local doubling property (1.7) together with a change of variables to bound the first term in (3.37) with the first one in (3.38) and the continuity of $x \mapsto F_x(R)$ to compare the second terms (here the compactness of P comes into play).

To obtain the lower bound in (3.32) we proceed as follows. Starting from the lower bound in (3.34), exploiting the elementary inequality $e^{-d^2/3t} \geq e^{-1/3} \mathbb{I}_{[d,\infty]}(\sqrt{t})$ and changing variables, we obtain

$$\int_0^{\infty} \frac{e^{-2ct} e^{-\frac{\mathsf{d}(x,y)^2}{3t}}}{\mathfrak{m}(B_{\sqrt{t}}(x))} \, \mathrm{d}t \geq e^{-1/3} \int_{\mathsf{d}(x,y)}^{\infty} e^{-2ct^2} \frac{t}{\mathfrak{m}(B_t(x))} \, \mathrm{d}t.$$

To conclude it suffices to observe that, splitting the integral in two parts and using a continuity argument, as in the verification of (3.36) above, we find a constant $C(P) > 0$ such that

$$\int_{\mathsf{d}(x,y)}^{\infty} e^{-2ct^2} \frac{t}{\mathfrak{m}(B_t(x))} \, \mathrm{d}t \geq C(P) \int_{\mathsf{d}(x,y)}^{\infty} \frac{t}{\mathfrak{m}(B_t(x))} \, \mathrm{d}t = C(P) F_x(\mathsf{d}(x,y)),$$

for any $x, y \in P$.

The proof of (3.33) can be obtained with arguments analogous to those presented above, starting from (1.80) and following the strategy we adopted to prove (3.6). □

Proposition 3.19 (Maximal estimate, scalar version). *Let $(X, \mathsf{d}, \mathfrak{m})$ be an* RCD(K,N) *metric measure space satisfying Assumption 3.17. For any compact set $P \subset X$, there exists $C_M(P) > 0$ such that, for any Borel function $f : X \to [0, +\infty)$ supported in P, it holds*

$$\int_X f(w) \left|\nabla \bar{G}_x(w)\right| \left|\nabla \bar{G}_y(w)\right| \mathrm{d}\mathfrak{m}(w) \leq \bar{C}_M(P) \bar{G}(x,y) \left(Mf(x) + Mf(y)\right), \tag{3.39}$$

for any $x, y \in P$.

Proof. We begin by recalling that, as an intermediate step in the proof of Proposition 3.12, we obtained the following inequality:

$$\begin{aligned} &\int_X f(w) H_x(\mathsf{d}(x,w)) H_y(\mathsf{d}(y,w)) \, \mathrm{d}\mathfrak{m}(w) \\ &\leq C \left(F_x\left(\frac{\mathsf{d}(x,y)}{2}\right) + F_y\left(\frac{\mathsf{d}(x,y)}{2}\right) + F_x(\mathsf{d}(x,y)) + F_y(\mathsf{d}(x,y)) \right) \left(Mf(x) + Mf(y)\right), \end{aligned} \tag{3.40}$$

for any $x, y \in X$, for some numerical constant $C > 0$ (the assumptions concerning the m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ played no role in that part of the proof).

Then let us observe that, thanks to (3.33),
(3.41)
$$\int_X f(w)\left|\nabla \bar{G}_x(w)\right|\left|\nabla \bar{G}_y(w)\right| \,\mathrm{d}\mathfrak{m}(w) \le \bar{C}(P)^2 \int_X f(w) H_x(\mathsf{d}(x,w)) H_y(\mathsf{d}(y,w))\,\mathrm{d}\mathfrak{m}(w)$$
for any $x,y \in P$. Exploiting (3.36) with $\lambda = 1/2$, (3.40) and (3.41), we obtain that, up to increasing the constant $\bar{C}(P)$, it holds
(3.42)
$$\int_X f(w)\left|\nabla \bar{G}_x(w)\right|\left|\nabla \bar{G}_y(w)\right| \,\mathrm{d}\mathfrak{m}(w) \le \bar{C}(P)\left(F_x(\mathsf{d}(x,y)) + F_y(\mathsf{d}(x,y))\right)(Mf(x) + Mf(y)),$$
for any $x,y \in P$.
The sought conclusion (3.39) follows from (3.42) and the lower bound in (3.32). □

Remark 3.20. It follows from the proof of Proposition 3.19 above that also the estimate

$$\int_X f(w)\left|\nabla \bar{G}^\varepsilon_x(w)\right|\left|\nabla \bar{G}^\varepsilon_y(w)\right| \,\mathrm{d}\mathfrak{m}(w) \le \bar{C}_M(P)\bar{G}(x,y)\left(Mf(x) + Mf(y)\right) \tag{3.43}$$

holds true, for any $\varepsilon > 0$ and for any $x, y \in P$.

By analogy with (3.21), we introduce a function $\mathsf{d}_{\bar{G}}$ that we will use to measure the regularity of RLFs, in the following way:

$$\mathsf{d}_{\bar{G}}(x,y) := \begin{cases} 1/\bar{G}(x,y) & \text{if } x \neq y, \\ 0 & \text{otherwise.} \end{cases} \tag{3.44}$$

It is immediate to check that it is symmetric, non-negative and that $\mathsf{d}_{\bar{G}}(x,y) = 0$ if and only if $x = y$. Moreover, following verbatim the proof of Lemma 3.15 and exploiting the two-sided bounds in (3.32), it is easy to prove that, for any $x \in X$, the map $y \mapsto \mathsf{d}_{\bar{G}}(x,y)$ is continuous with respect to d.

We end this section about the properties of the modified Green function $\bar{G}$ with a vector valued maximal estimate. In the proof of Theorem 3.26 it plays the same role that Proposition 3.16 plays in the proof of Theorem 3.27.

Proposition 3.21 (Maximal estimate, vector-valued version)**.** *Assume that* $(X,\mathsf{d},\mathfrak{m})$ *is an* $\mathrm{RCD}(K,N)$ *m.m.s. satisfying Assumption 3.17 and let* $P \subset X$ *be a compact set. Then, for any* $b \in H^{1,2}_{C,s}(TX)$ *bounded and with compact support in* P*, there exists a positive function* $F \in L^2(P,\mathfrak{m})$ *such that*

$$\left|b\cdot\nabla\bar{G}_x(y) + b\cdot\nabla\bar{G}_y(x)\right| \le \bar{G}(x,y)(F(x)+F(y)) \quad \textit{for } \mathfrak{m}\times\mathfrak{m}\textit{-a.e. } (x,y)\in P\times P, \tag{3.45}$$

and

$$\|F\|_{L^2(P,\mathfrak{m})} \le C_V \left\||\nabla_{\mathrm{sym}} b| + |\mathrm{div}\, b|\right\|_{L^2(X,\mathfrak{m})}, \tag{3.46}$$

where $C_V = C_V(P) > 0$.

Proof. The strategy we follow is the same proposed in the proof of Proposition 3.16.
First we are going to prove that there exists F as above such that

$$\left|\int_X \left\{b\cdot\nabla\bar{G}^\varepsilon_x(w)p_\varepsilon(y,w) + b\cdot\nabla\bar{G}^\varepsilon_y(w)p_\varepsilon(x,w)\right\}\mathrm{d}\mathfrak{m}(w)\right| \le \bar{G}(x,y)\left(F(x)+F(y)\right), \tag{3.47}$$

for any $x,y \in P$ and for any $0 < \varepsilon < 1$. The stated conclusion will then follow from (3.47), taking into account (3.31) and following verbatim the second step of the proof of Proposition 3.16.

Recall from (3.30) that $p_\varepsilon(x,w)=e^{c\varepsilon}[-\Delta\bar{G}^\varepsilon_x(w)+c\bar{G}^\varepsilon_x(w)]$ for $\mathfrak{m}$-a.e. $w\in X$. Hence we can estimate

$$\begin{aligned}
&\Big|\int_X b\cdot\nabla\bar{G}^\varepsilon_x(w)p_\varepsilon(y,w)+b\cdot\nabla\bar{G}^\varepsilon_y(w)p_\varepsilon(x,w)\,\mathrm{d}\mathfrak{m}(w)\Big|\\
&\quad=e^{c\varepsilon}\left|\int_X\left\{b\cdot\nabla\bar{G}^\varepsilon_x(w)(-\Delta\bar{G}^\varepsilon_y(w)+c\bar{G}^\varepsilon_y(w))+b\cdot\nabla\bar{G}^\varepsilon_y(w)(-\Delta\bar{G}^\varepsilon_x(w)+c\bar{G}^\varepsilon_x(w))\right\}\mathrm{d}\mathfrak{m}(w)\right|\\
&\quad\le e^{c\varepsilon}\left|\int_X\left\{b\cdot\nabla\bar{G}^\varepsilon_x\Delta\bar{G}^\varepsilon_y+b\cdot\nabla\bar{G}^\varepsilon_y\Delta\bar{G}^\varepsilon_x\right\}\mathrm{d}\mathfrak{m}\right|+ce^{c\varepsilon}\left|\int_X\left\{b\cdot\nabla\bar{G}^\varepsilon_x\ \bar{G}^\varepsilon_y+b\cdot\nabla\bar{G}^\varepsilon_y\ \bar{G}^\varepsilon_x\right\}\mathrm{d}\mathfrak{m}\right|\\
&\quad=:I^\varepsilon_1(x,y)+I^\varepsilon_2(x,y).
\end{aligned}$$

Arguing as in the first step of the proof of Proposition 3.16 and applying Remark 3.20, we obtain that

$$I^\varepsilon_1(x,y)\le e^{c\varepsilon}\bar{C}_M(P)\bar{G}(x,y)(Mg(x)+Mg(y)),\tag{3.48}$$

for any $x,y\in P$ and for any $0<\varepsilon<1$, where $g:=|\nabla_{\mathrm{sym}}b|+|\mathrm{div}\,b|$. Dealing with I^ε_2, integrating by parts and using the Leibniz rule we obtain that

$$I^\varepsilon_2(x,y)=ce^{c\varepsilon}\left|\int_X\mathrm{div}\,b\ \bar{G}^\varepsilon_x\bar{G}^\varepsilon_y\,\mathrm{d}\mathfrak{m}\right|\tag{3.49}$$

for any $x,y\in P$. Arguing as we did in the proofs of the previous results it is possible to find a constant $\bar{C}=\bar{C}(P)>0$ depending only on the compact set P and such that $\bar{G}^\varepsilon_x(y)\le\bar{C}H_x(\mathsf{d}(x,y))$ for any $x,y\in P$ and for any $\varepsilon>0$. Then we can estimate $I^\varepsilon_2(x,y)$ with

$$\begin{aligned}
ce^{c\varepsilon}\int_X|\mathrm{div}\,b|\,H_x(\mathsf{d}(x,w))H_y(\mathsf{d}(y,w))\,\mathrm{d}\mathfrak{m}(w)&\le ce^{c\varepsilon}\bar{C}(P)\bar{G}(x,y)\left(M\,|\mathrm{div}\,b|\,(x)+M\,|\mathrm{div}\,b|\,(y)\right)\\
&\le ce^{c\varepsilon}\bar{C}(P)\bar{G}(x,y)\left(Mg(x)+Mg(y)\right).
\end{aligned}$$

To conclude it is sufficient to observe that (3.46) follows from the local version of the Hardy Littlewood theorem (cf. with the discussion after Theorem 1.19).

□

2. G-regularity of Lagrangian Flows

In this section we achieve a regularity result for Lagrangian Flows of Sobolev vector fields which will be the key tool to establish the constancy of the dimension for RCD(K,N) spaces.

As we anticipated, for the sake of this thesis we chose to weaken the regularity achieved to the minimal one needed for proving the constancy of the dimension, in order to shorten and clarify the presentation. What we are going to prove is a kind of Lusin-Lipschitz regularity result for Lagrangian flows, understood in terms of the quasi-metrics d_G or $\mathsf{d}_{\bar{G}}$. We point out that Lusin-Lipschitz regularity, although being a quite mild notion, has revealed to be crucial in a broad range of applications, for instance in [**22**, **79**].

This section is divided in three subsections. In Section 2.1 we review in a rather informal way the classical Cauchy-Lipschitz theory and the regularity theory for Lagrangian flows of Sobolev vector fields in the Euclidean case as developed in [**79**]. These considerations motivate the necessity of an alternative approach in our setting. Then we illustrate the heuristic standing behind the theory of regularity in terms of Green functions that we developed in [**51**, **52**]. In Section 2.2 we present some preliminary technical tools that we exploit in Section 2.3 to achieve the regularity statement.

2.1. A motivating digression. Let $b : \mathbb{R}^d \to \mathbb{R}^d$ be a vector field and denote by $\mathbf{X} : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ its flow map, that we assume to be well-defined for every $t \in [0,T]$ and for every $x \in \mathbb{R}^d$. A natural way to measure the regularity of $\mathbf{X}$ is in terms of Lipschitz continuity. Moreover, it is a rather elementary fact that, whenever b is Lipschitz, the flow map $\mathbf{X}_t$ is Lipschitz as well. Indeed, willing to control the distance between trajectories starting from different points $x, y \in \mathbb{R}^d$, it is sufficient to compute

$$\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{X}_t(x) - \mathbf{X}_t(y)| \le |b(\mathbf{X}_t(x)) - b(\mathbf{X}_t(y))| \le \mathrm{Lip}(b)|\mathbf{X}_t(x) - \mathbf{X}_t(y)|, \tag{3.50}$$

to obtain that

$$|\mathbf{X}_t(x) - \mathbf{X}_t(y)| \le e^{t\mathrm{Lip}(b)}|x - y|, \qquad \text{for any } t \in [0,T].$$

Lowering the regularity assumption on the vector field from Lipschitz to Sobolev, the second inequality in (3.50) fails and we cannot expect Lipschitz continuity for the Lagrangian flow $\mathbf{X}_t$ that, in general, might even be discontinuous. However, in [**79**], Crippa-De Lellis obtained a Lusin-Lipschitz regularity result for Lagrangian flows associated to vector fields $b \in W^{1,p}(\mathbb{R}^d;\mathbb{R}^d)$ for $p > 1$. That is to say, for every bounded $K \subset \mathbb{R}^d$ and for every $\varepsilon > 0$, there exist $C = C(\varepsilon, \|b\|_{W^{1,p}}, K) > 0$ and $E \subset K$ with $\mathcal{L}^d(K \setminus E) < \varepsilon$ such that $\mathbf{X}_t$ is C-Lipschitz over E, for any $t \in [0,T]$.

The key tool exploited by Crippa-De Lellis seeking for an analogue of (3.50) is the so-called maximal estimate for Sobolev functions: there exists $C_d > 0$, such that any $f \in W^{1,p}(\mathbb{R}^d;\mathbb{R})$ admits a representative, still denoted by f, for which

$$|f(x) - f(y)| \le C_d(M|\nabla f|(x) + M|\nabla f|(y))\,|x - y|\,, \qquad \text{for any } x, y \in X, \tag{3.51}$$

where $M|\nabla f|$ is the maximal operator applied to $|\nabla f|$. Observe that, if $p > 1$, then $\|M|\nabla f|\|_{L^p} \le C_{p,d}\,\||\nabla f|\|_{L^p}$ for some constant $C_{p,d} > 0$, thanks to Theorem 1.19. Moreover, since on $\mathbb{R}^d$ a vector field is Sobolev if and only if its components are so, (3.51) holds true also for any $b \in W^{1,p}(\mathbb{R}^d;\mathbb{R}^d)$.

This being said, the replacement of (3.50) in the Sobolev case is

$$\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{X}_t(x) - \mathbf{X}_t(y)| \le C\left\{M|\nabla b|(\mathbf{X}_t(x)) + M|\nabla b|(\mathbf{X}_t(y))\right\}|\mathbf{X}_t(x) - \mathbf{X}_t(y)|. \tag{3.52}$$

The sought regularity for $\mathbf{X}_t$ does not follow any more applying Gronwall lemma to (3.52). However, one might think of (3.52) as a quantitative infinitesimal version of the regularity result for the Lagrangian flow.

Having such a perspective in mind, the situation changes significantly passing from the Euclidean space to an $\mathrm{RCD}(K,N)$ metric measure space or, more simply, to a smooth Riemannian manifold. Indeed, while the maximal estimate for real valued Sobolev functions (3.51) is a very robust result (cf. [**9**, **10**]), we are not aware of any intrinsic way to lift it to the level of vector fields.

Let us introduce the more appealing notation d for the distance function but still think for sake of simplicity to the Euclidean case. Trying to turn the Sobolev regularity of the vector field into some bound for the right hand side in the expression

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{d}(\mathbf{X}_t(x), \mathbf{X}_t(y)) = b \cdot \nabla \mathsf{d}_{\mathbf{X}_t(x)}(\mathbf{X}_t(y)) + b \cdot \nabla \mathsf{d}_{\mathbf{X}_t(y)}(\mathbf{X}_t(x)), \tag{3.53}$$

a natural attempt could be to appeal to the interpolation

$$b \cdot \nabla \mathsf{d}_x(y) + b \cdot \nabla \mathsf{d}_y(x) = \int_0^1 \nabla_{\mathrm{sym}} b(\gamma'(s), \gamma'(s))\,\mathrm{d}s, \tag{3.54}$$

where $\gamma : [0,1] \to \mathbb{R}^d$ is the geodesic joining x to y and $\nabla_{\mathrm{sym}} b$ is the symmetric part of the covariant derivative of b. However, when the bounds on $\nabla_{\mathrm{sym}} b$ are only of integral type, it is not clear how to obtain useful estimates from (3.54) without deeply involving the Euclidean structure, that is something to be avoided in view of the extensions to the metric setting.

The starting point of the study in [**51**,**52**] was, instead, the following observation: suppose that $d \geq 3$, then, calling G the Green function of the Laplacian on $\mathbb{R}^d$, it holds $G(x,y) = c_d \mathsf{d}(x,y)^{2-d}$. This implies in turn that controlling the distance between two trajectories of the flow is the same as controlling the Green function along them. Moreover, computing the rate of change of the Green function along the flow, we end up with the necessity to bound the quantity

$$b \cdot \nabla G_x(y) + b \cdot \nabla G_y(x),$$

that, assuming $\operatorname{div} b = 0$ for sake of simplicity, we can formally rewrite as

$$\begin{aligned} b \cdot \nabla G_x(y) + b \cdot \nabla G_y(x) &= -\int_{\mathbb{R}^d} b(w) \cdot \nabla G_x(w)\, \mathrm{d}\boldsymbol{\Delta} G_y(w) - \int_{\mathbb{R}^d} b(w) \cdot \nabla G_y(w)\, \mathrm{d}\boldsymbol{\Delta} G_x(w) \\ &= 2\int_{\mathbb{R}^d} \nabla_{\mathrm{sym}} b(\nabla G_x, \nabla G_y)\, \mathrm{d}\mathfrak{m}. \end{aligned} \tag{3.55}$$

Observe that, being (3.55) in integral form, we can expect it to fit better than (3.54) with the assumption $\nabla_{\mathrm{sym}} b \in L^2$ and this expectation is confirmed by the validity, for some $C > 0$, of the key estimate

$$\int_{\mathbb{R}^d} f |\nabla G_x| |\nabla G_y|\, \mathrm{d}\mathfrak{m} \leq C G(x,y)(Mf(x) + Mf(y)), \quad \text{for any } x, y \in X \tag{3.56}$$

and for any Borel function $f : \mathbb{R}^d \to [0, +\infty)$. In Proposition 3.12 and Proposition 3.19 we obtained generalizations of the estimate above to the setting of our interest and we are going to apply them in the forthcoming subsections to develop a regularity theory for Lagrangian flows.

2.2. Flows and vector fields on product spaces. This subsection is dedicated to the proof of a general result about the structure of regular Lagrangian flows associated to vector fields with product structure over product spaces. As a corollary we will obtain that, for $\mathfrak{m} \times \mathfrak{m}$-a.e. $(x,y) \in X \times X$, the map $t \mapsto G(\mathbf{X}_t(x), \mathbf{X}_t(y))$ is differentiable $\mathcal{L}^1$-a.e., with the explicit and expected formula for the derivative, together with the counterpart of this result for the modified Green function $\bar{G}$.

Let $(X, \mathsf{d}_X, \mathfrak{m}_X)$ and $(Y, \mathsf{d}_Y, \mathfrak{m}_Y)$ be RCD(K,∞) m.m. spaces. Let $Z := X \times Y$ be endowed with the product m.m.s. structure (see (1.82) and the discussion thereafter) and recall from [**13**,**16**] that $(Z, \mathsf{d}_Z, \mathfrak{m}_Z)$ is an RCD(K,∞) m.m.s itself.
We will denote by π_X and π_Y the canonical projections from Z onto X and Y respectively. This being said we introduce the so-called algebra of tensor products by

$$\mathcal{A} := \left\{ \sum_{j=1}^{n} g_j \circ \pi_X \cdot h_j \circ \pi_Y : \; g_j \in H^{1,2}_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}(X) \text{ and } h_j \in H^{1,2}_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}(Y)\ \forall j = 1, \dots, n \right\}.$$

Theorem 3.22. *Let X, Y and Z be as above. Then, for any $f \in H^{1,2}_{\mathrm{loc}}(Z, \mathsf{d}_Z, \mathfrak{m}_Z) \cap L^\infty_{\mathrm{loc}}(Z, \mathfrak{m})$ and for any compact $P \subset Z$, there exists a sequence $(f)_{n\in\mathbb{N}}$ in $\mathcal{A}$ with $\|f_n\|_{L^\infty(P)}$ uniformly bounded and such that $\|f_n - f\|_{L^2(P,\mathfrak{m}_Z)} + \||\nabla(f_n - f)|\|_{L^2(P,\mathfrak{m}_Z)} \to 0$ as $n \to \infty$.*

Proof. Let us denote by $\bar{\mathcal{A}}$ the set of functions $f \in H^{1,2}_{\mathrm{loc}}(Z, \mathsf{d}_Z, \mathfrak{m}_Z) \cap L^\infty_{\mathrm{loc}}(Z, \mathfrak{m})$ for which the statement of the theorem holds true. Let $\mathcal{A}_{\mathsf{d}}$ be the smallest subset of $\mathrm{Lip}_b(X)$ containing truncated distances from points of Z and closed with respect to sum, product and lattice operations, let $\mathcal{A}_{\mathsf{dbs}} \subset H^{1,2}(Z, \mathsf{d}_Z, \mathfrak{m}_Z) \cap L^\infty(Z, \mathfrak{m}_Z)$ be the subalgebra of $\mathcal{A}_{\mathsf{d}}$ made by functions with bounded support. In [**29**, Theorem B.1] it is proved that $\mathcal{A}_{\mathsf{dbs}}$ is dense in $H^{1,2}(Z, \mathsf{d}_Z, \mathfrak{m}_Z)$ and it is straightforward to check that one can approximate any bounded function in $H^{1,2}(Z, \mathsf{d}_Z, \mathfrak{m}_Z)$ with a sequence of uniformly bounded functions in $\mathcal{A}_{\mathsf{dbs}}$. Hence, to get the stated conclusion, it is sufficient to prove that $\mathsf{d}_Z(z,\cdot) \wedge k \in \bar{\mathcal{A}}$ for any $z \in Z$, for any $k \geq 0$, and the implication $f, g \in \bar{\mathcal{A}} \implies f \wedge g \in \bar{\mathcal{A}}$.

Let us first prove that $\mathsf{d}_Z(z,\cdot) \in \bar{\mathcal{A}}$ for any $z \in Z$. For any natural $n \geq 1$ let $(h_n^k)_{k\in\mathbb{N}}$ be a sequence of polynomials converging to $t \mapsto \sqrt{1/n + t}$ in $C^1_{\mathrm{loc}}([0,+\infty))$ as $k \to \infty$. Let us fix $z \in Z$. It is simple to see that $h_n^k(\mathsf{d}_Z(z,\cdot)^2)$ converges in $H^{1,2}_{\mathrm{loc}}(Z, \mathsf{d}_Z, \mathfrak{m}_Z) \cap L^\infty_{\mathrm{loc}}(Z, \mathfrak{m}_Z)$ to $\sqrt{1/n + \mathsf{d}_Z^2(z,\cdot)}$ when $k \to \infty$ and that $\sqrt{1/n + \mathsf{d}_Z^2(z,\cdot)} \to \mathsf{d}_Z(z,\cdot)$, in the same topology, when $n \to \infty$. Observe that the very definition of d_Z yields $\mathsf{d}_Z(z,w)^2 = \mathsf{d}_X(\pi_X(z), \pi_X(w))^2 + \mathsf{d}_Y(\pi_Y(z), \pi_Y(w))^2$ for any $w \in Z$, therefore $h_n^k(\mathsf{d}_Z(z,\cdot)^2) \in \mathcal{A}$.

Let us now prove the implication $g \in \bar{\mathcal{A}} \implies |g| \in \bar{\mathcal{A}}$. With this aim, let us fix $g \in \bar{\mathcal{A}}$ and a sequence $g_m \in \mathcal{A}$ converging to g in $H^{1,2}_{\mathrm{loc}}(Z, \mathsf{d}_Z, \mathfrak{m}_Z) \cap L^\infty_{\mathrm{loc}}(Z, \mathfrak{m}_Z)$ when $m \to \infty$. Setting $g^k_{n,m} := h^k_n \circ g^2_m$, we have $g^k_{n,m} \in \mathcal{A}$ and it is easy to check that it converges to $\sqrt{1/n + g_m^2}$ in $H^{1,2}_{\mathrm{loc}}(Z, \mathsf{d}_Z, \mathfrak{m}_Z) \cap L^\infty_{\mathrm{loc}}(Z, \mathfrak{m}_Z)$ as $k \to \infty$. Moreover $\sqrt{1/n + g_m^2} \to |g_m|$ in $H^{1,2}_{\mathrm{loc}}(Z, \mathsf{d}_Z, \mathfrak{m}_Z) \cap L^\infty_{\mathrm{loc}}(Z, \mathfrak{m}_Z)$ when $n \to \infty$ and eventually $|g_m| \to |g|$, in the same topology, when $m \to \infty$. By a diagonal argument, we recover the sought approximating sequence.

Finally we exploit the identity

$$a \wedge b = \frac{|a+b| - |a-b|}{2}, \qquad \forall a, b \in [0, \infty),$$

to deduce that $\mathsf{d}_Z(z,\cdot) \wedge k \in \bar{\mathcal{A}}$ for any $z \in Z$, for any $k \geq 0$ and the implication $f, g \in \bar{\mathcal{A}} \implies f \wedge g \in \bar{\mathcal{A}}$. □

Let us consider now $b_t^X \in L^1((0,T); L^2(TX))$ and $b_t^Y \in L^1((0,T); L^2(TY))$. We introduce the "product" vector field b_t^Z by saying that, for every $f \in H^{1,2}(Z, \mathsf{d}_Z, \mathfrak{m}_Z)$,

$$b_t^Z \cdot \nabla f(x,y) := b_t^X \cdot \nabla f_y(x) + b_t^Y \cdot \nabla f_x(y), \tag{3.57}$$

for $\mathfrak{m}_Z$-a.e. $(x,y) \in Z$, where $f_x(y) := f(x,y)$, $f_y(x) := f(x,y)$ and we are implicitly exploiting the tensorization of the Cheeger energy (see Remark 1.110). It is simple to check that $b_t^Z \in L^1((0,T); L^2_{\mathrm{loc}}(TZ))$ and

$$|b_t^Z|^2(x,y) \leq |b_t^X|^2(x) + |b_t^Y|^2(y), \qquad \text{for } \mathfrak{m} \times \mathfrak{m}\text{-a.e. } (x,y) \in X \times Y.$$

Proposition 3.23. *Let b_t^X and b_t^Y be as above and let $\mathbf{X}_t^X$ and $\mathbf{X}_t^Y$ be regular Lagrangian flows associated to b_t^X and b_t^Y, respectively. Then*

$$\mathbf{X}_t^Z(x,y) := (\mathbf{X}_t^X(x), \mathbf{X}_t^Y(y))$$

is a regular Lagrangian flow associated to b_t^Z.

Proof. We need to check the validity of the three conditions in Definition 1.93.
The first one is trivial and the bounded compressibility property of $\mathbf{X}_t^Z$ is a direct consequence of the bounded compressibility property of $\mathbf{X}_t^X$ and $\mathbf{X}_t^Y$.
Dealing with the third one, we observe that, thanks to Theorem 3.22 and Remark 1.94, it is sufficient to check its validity testing it for any $f \in \mathcal{A}$. Moreover, by the linearity of (1.68) w.r.t. the test function, we can assume without loss of generality that $f = g \circ \pi_X \cdot h \circ \pi_Y$, with $g \in H^{1,2}_{\mathrm{loc}}(X, \mathsf{d}_X, \mathfrak{m}_X) \cap L^\infty_{\mathrm{loc}}(X, \mathfrak{m}_X)$ and $h \in H^{1,2}_{\mathrm{loc}}(Y, \mathsf{d}_Y, \mathfrak{m}_Y) \cap L^\infty_{\mathrm{loc}}(Y, \mathfrak{m}_Y)$. We need to prove that for $\mathfrak{m}_Z$-a.e. $(x,y) \in X \times Y$ the map $z \mapsto f(\mathbf{X}_t^Z(z))$ belongs to $W^{1,1}((0,T))$ and has derivative given by

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{X}_t^Z(z)) = b_t^Z \cdot \nabla^Z f(\mathbf{X}_t(z)) \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in (0,T). \tag{3.58}$$

To this aim we observe that, since $\mathbf{X}_t^X$ and $\mathbf{X}_t^Y$ are regular Lagrangian flows of b_t^X and b_t^Y respectively, it holds that the maps $t \mapsto g(\mathbf{X}_t^X(x))$ and $t \mapsto h(\mathbf{X}_t^Y(y))$ are bounded and belong to $W^{1,1}((0,T))$ for $\mathfrak{m}_X$-a.e. $x \in X$ and $\mathfrak{m}_Y$-a.e. $y \in Y$ respectively. Moreover

$$\frac{\mathrm{d}}{\mathrm{d}t} g(\mathbf{X}_t^X(x)) = b_t^X \cdot \nabla g(\mathbf{X}_t^X(x)) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t} h(\mathbf{X}_t^Y(y)) = b_t^Y \cdot \nabla h(\mathbf{X}_t^Y(y)) \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in (0,T),$$

for $\mathfrak{m}_X$-a.e. $x \in X$ and $\mathfrak{m}_Y$-a.e. $y \in Y$, respectively. Applying Fubini's theorem and the Leibniz rule we obtain that, for $\mathfrak{m}_X \times \mathfrak{m}_Y$-a.e. $(x,y) \in X \times Y$, the map $t \mapsto g(\mathbf{X}_t^X(x))h(\mathbf{X}_t^Y(y))$ belongs to $W^{1,1}((0,T))$, moreover

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(g(\mathbf{X}_t^X(x))h(\mathbf{X}_t^Y(y))\right) &= \left(\frac{\mathrm{d}}{\mathrm{d}t} g(\mathbf{X}_t^X(x))\right) h(\mathbf{X}_t^Y(y)) + g(\mathbf{X}_t^X(x)) \left(\frac{\mathrm{d}}{\mathrm{d}t} h(\mathbf{X}_t^Y(y))\right) \\ &= h(\mathbf{X}_t^Y(y)) b_t^X \cdot \nabla g(\mathbf{X}_t^X(x)) + g(\mathbf{X}_t^X(x)) b_t^Y \cdot \nabla h(\mathbf{X}_t^Y(y)) \\ &= b_t^Z \cdot \nabla f(\mathbf{X}_t^Z(x,y)), \end{aligned}$$

for $\mathcal{L}^1$-a.e. $t \in (0,T)$, which implies (3.58). ☐

Corollary 3.24. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an* RCD$(0,N)$ *m.m.s. satisfying Assumption 3.2. Let moreover $b \in L^1((0,T); L^2(TX))$ and $\mathbf{X}_t$ be a regular Lagrangian flow associated to b. Then, the map*

$$t \mapsto G(\mathbf{X}_t(x), \mathbf{X}_t(y))$$

belongs to $W^{1,1}((0,T))$ for $\mathfrak{m} \times \mathfrak{m}$-a.e. $(x,y) \in X \times X$ and its derivative is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} G(\mathbf{X}_t(x), \mathbf{X}_t(y)) = b_t \cdot \nabla G_{\mathbf{X}_t(x)}(\mathbf{X}_t(y)) + b_t \cdot \nabla G_{\mathbf{X}_t(y)}(\mathbf{X}_t(x)),$$

for $\mathcal{L}^1$-a.e. $t \in (0,T)$.

Proof. Let us start observing that $G^\varepsilon \in H^{1,2}_{\mathrm{loc}}(X \times X)$ for any $\varepsilon > 0$ (actually it has locally bounded weak upper gradient as one can prove with the same techniques introduced in the proof of Proposition 3.5, taking into account Remark 3.4).
It follows from Proposition 3.23, applied with $X = Y$ and $b^X = b^Y =: b$, that

$$G^\varepsilon(\mathbf{X}_t(x), \mathbf{X}_t(y)) - G^\varepsilon(x,y) = \int_0^t \left\{ b_s \cdot \nabla G^\varepsilon_{\mathbf{X}_s(x)}(\mathbf{X}_s)(y) + b_s \cdot \nabla G^\varepsilon_{\mathbf{X}_s(y)}(\mathbf{X}_s(x)) \right\} \mathrm{d}s, \tag{3.59}$$

for $\mathfrak{m} \times \mathfrak{m}$-a.e. $(x,y) \in X \times X$ and for every $t \in [0,T]$.
We wish to pass to the limit as $\varepsilon \downarrow 0$ in (3.59) to obtain that for any $t \in [0,T]$ it holds

$$G(\mathbf{X}_t(x), \mathbf{X}_t(y)) - G(x,y) = \int_0^t \left\{ b_s \cdot \nabla G_{\mathbf{X}_s(x)}(\mathbf{X}_s(y)) + b_s \cdot \nabla G_{\mathbf{X}_s(y)}(\mathbf{X}_s(x)) \right\} \mathrm{d}s, \tag{3.60}$$

for $\mathfrak{m} \times \mathfrak{m}$-a.e. $(x,y) \in X \times X$. The sought conclusion would easily follow. To this aim let us observe that the left hand side in (3.59) converges to $G(\mathbf{X}_t(y), \mathbf{X}_t(x)) - G(x,y)$ in $L^1_{\mathrm{loc}}(X \times X, \mathfrak{m} \times \mathfrak{m})$. Thus, it suffices to prove that the right hand side in (3.59) converges to

$$\int_0^t \left\{ b_s \cdot \nabla G_{\mathbf{X}_s(x)}(\mathbf{X}_s(y)) + b_s \cdot \nabla G_{\mathbf{X}_s(y)}(\mathbf{X}_s(x)) \right\} \mathrm{d}s \qquad \text{in } L^1_{\mathrm{loc}}(X \times X, \mathfrak{m} \times \mathfrak{m}).$$

To this aim we fix $z \in X$ such that $\mathsf{d}(\mathbf{X}_s(z), z) \leq \|b\|_{L^\infty} t$ for every $s \in [0,t]$ (observe that this property holds true for $\mathfrak{m}$-a.e. point). The triangle inequality yields

$$\mathsf{d}(\mathbf{X}_s(z), \mathbf{X}_s(y)) \leq 2t \|b\|_{L^\infty} + \mathsf{d}(z,y), \qquad \text{for } \mathfrak{m}\text{-a.e. } y \in X. \tag{3.61}$$

Thus, setting $B := B(z,R)$, for some $R > 0$, and $\bar{B} := B(z, R + 2t\|b\|_{L^\infty})$, we have

$$(\mathbf{X}_s)_\#(\mathbb{I}_B \mathfrak{m}) \leq L \mathbb{I}_{\bar{B}} \mathfrak{m}. \tag{3.62}$$

The bounded compressibility property of the RLF allows us to estimate

$$\begin{aligned}\int_{B\times B}&\left|\int_0^t b_s\cdot\nabla G^{\varepsilon}_{\mathbf{X}_s(x)}(\mathbf{X}_s(y))\,\mathrm{d}s-\int_0^t b_s\cdot\nabla G_{\mathbf{X}_s(x)}(\mathbf{X}_s(y))\,\mathrm{d}s\right|\,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}\mathfrak{m}(y)\\ &\le\int_0^t\int_B\int_B|b_s|(\mathbf{X}_s(y))\cdot|\nabla(G^{\varepsilon}_{\mathbf{X}_s(x)}-G_{\mathbf{X}_s(x)})|(\mathbf{X}_s(y))\,\mathrm{d}\mathfrak{m}(y)\,\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}s\\ &\le L^2t\,\|b\|_{L^\infty}\int_{\bar B}\|\nabla(G^{\varepsilon}_x-G_x)\|_{L^1(\bar B)}\,\mathrm{d}\mathfrak{m}(x).\end{aligned}$$

Observe that the last term above goes to zero, as a simple application of the dominated convergence theorem shows (for more details about this step we refer to the proof of Proposition 3.16, where we dealt with a similar term). Arguing similarly for the term $\int_0^t b_s\cdot\nabla G^{\varepsilon}_{\mathbf{X}_s(y)}(\mathbf{X}_s(x))\,\mathrm{d}s$ we obtain the sought conclusion. □

Remark 3.25. A conclusion analogous to the one stated in Corollary 3.24 holds true with $\bar G$ in place of G assuming that $(X,\mathsf{d},\mathfrak{m})$ is an $\mathrm{RCD}(K,N)$ m.m.s. satisfying Assumption 3.17. To get this result it suffices to argue as in the proof of Corollary 3.24 using Proposition 3.18 instead of Proposition 3.5.

2.3. A Lusin-Type regularity result. In this subsection we achieve a regularity result for Lagrangian Flows of Sobolev vector fields on $\mathrm{RCD}(K,N)$ spaces that will provide the key tool to establish constancy of the dimension.

In [**51**, **52**] the starting point for our analysis has been the Euclidean regularity theory developed in [**79**] (inspired by the previous [**25**]).
In the case of compact Ahlfors regular $\mathrm{RCD}(K,N)$ spaces (including in particular ncRCD spaces) covered in [**51**] we obtained an analogue of the Crippa-De Lellis result, using the Green function of the Laplacian as an intermediate tool and relying on the global comparison between the Green function and a suitable power of the distance. Then, in [**52**], we exploited a careful analysis of the properties of Green and generalized Green functions to achieve a regularity theory in the spirit of [**79**] but formulated in terms of the quasi-metrics d_G and $\mathsf{d}_{\bar G}$. Let us also mention that in the work in progress [**50**] we sharpen the regularity statements of [**52**] relying on more careful analysis of the asymptotics of the Green functions near to the poles, obtaining the expected behaviour of the estimates with respect to time.

For the sake of the present thesis, we chose to state and prove the regularity results for Lagrangian flows in a weaker form with respect to the one of [**52**]. In this way we avoid the technicalities necessary to implement the Crippa-De Lellis scheme, bypassing also the treatment of some estimates involving the Green quasi-metrics (i.e. the quasi triangle inequality and the doubling property of [**52**]). The price we have to pay is that the results are only proved to hold $\mathfrak{m}\times\mathfrak{m}$-a.e. in the product space $X\times X$. Still, they are sufficient for the purpose of the proof of the constancy of the dimension Theorem 3.1.

Let us first deal with the case of a non-negative lower Ricci curvature bound.

Theorem 3.26. *Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(0,N)$ m.m.s. verifying Assumption 3.2. Let us fix $T>0$ and let $b_t\in L^\infty((0,T)\times X)$ be a time dependent vector field with compact support, uniformly w.r.t. time. We further assume that $b_t\in H^{1,2}_{C,s}(TX)$ for a.e. $t\in(0,T)$ and that $|\nabla_{\mathrm{sym}}b_t|\in L^1((0,T);L^2(X,\mathfrak{m}))$ and $\operatorname{div}b_t\in L^1((0,T);L^\infty(X,\mathfrak{m}))$. Let $(\mathbf{X}_t)_{t\in[0,T]}$ be the unique regular Lagrangian flow of b_t and let $L\ge 0$ be its compressibility constant. Then the following holds: for every $T>0$*

$$\mathsf{d}_G(\mathbf{X}_t(x),\mathbf{X}_t(y))\le\mathsf{d}_G(x,y)\exp\left\{\int_0^T Mg(\mathbf{X}_s(x))\,\mathrm{d}s+\int_0^T Mg(\mathbf{X}_s(y))\,\mathrm{d}s\right\},\tag{3.63}$$

for $\mathfrak{m}\times\mathfrak{m}$*-a.e.* $(x,y)\in X\times X$*, where we set* $g_t:=|\nabla_{\mathrm{sym}}b_t|+|\mathrm{div}\, b_t|$.

Furthermore, there exists a constant $C=C(N)\geq 0$ *such that, setting*

$$g^*(x):=\int_0^T Mg_t(\mathbf{X}_t(x))\,\mathrm{d}t, \tag{3.64}$$

it holds

$$\|g^*\|_{L^2}\leq CL\int_0^T \||\nabla_{\mathrm{sym}}b_s|+|\mathrm{div}\, b_s|\|_{L^2}\,\mathrm{d}s. \tag{3.65}$$

Proof. We wish to estimate the derivative with respect to time of the Green quasi metric between trajectories of the RLF $t\mapsto \mathsf{d}_G(\mathbf{X}_t(x),\mathbf{X}_t(y))$. In order to do so we use Corollary 3.24 and Proposition 3.16 obtaining

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}G(\mathbf{X}_t(x),\mathbf{X}_t(y))\right|\leq 2C_M G(\mathbf{X}_t(x),\mathbf{X}_t(y))\left(Mg_t(\mathbf{X}_t(x))+Mg_t(\mathbf{X}_t(y))\right), \tag{3.66}$$

for $\mathcal{L}^1$-a.e. $t\in(0,T)$ and for $\mathfrak{m}\times\mathfrak{m}$-a.e. $(x,y)\in X\times X$.

Integrating with respect to the time variable and recalling that $\mathsf{d}_G:=1/G$ we get, for any $t\in[0,T]$,

$$\mathsf{d}_G(\mathbf{X}_t(x),\mathbf{X}_t(y))\leq \mathsf{d}_G(x,y)\exp\left\{\int_0^T Mg_s(\mathbf{X}_s(x))\,\mathrm{d}s+\int_0^T Mg_s(\mathbf{X}_s(y))\,\mathrm{d}s\right\} \tag{3.67}$$

for $\mathfrak{m}\times\mathfrak{m}$-a.e. $(x,y)\in X\times X$. Note that the function $g^*(x):=\int_0^T Mg_s(\mathbf{X}_s(x))\,\mathrm{d}s$ belongs to L^2 with

$$\|g^*\|_{L^2}\leq CL\int_0^T \||\nabla_{\mathrm{sym}}b_s|+|\mathrm{div}\, b_s|\|_{L^2}\,\mathrm{d}s, \tag{3.68}$$

where C is a universal constant and L is as in Definition 1.93. This bound can be obtained relying on the bounded compressibility of the RLF, taking into account Theorem 1.19.

□

Passing to the case of a possibly negative lower Ricci curvature bound, with Proposition 3.21 at our disposal we can develop a regularity theory for Regular Lagrangian flows of Sobolev vector fields in terms of the quasi-metric $\mathsf{d}_{\bar{G}}$.

Theorem 3.27. *Let* $(X,\mathsf{d},\mathfrak{m})$ *be an* RCD(K,N) *m.m.s. verifying Assumption 3.17. Let* $P\subset X$ *be compact and* $T>0$ *be fixed. Let* $b_t\in L^\infty((0,T)\times X)$ *be a time dependent vector field with support contained in* P*. Let us further assume that* $b_t\in H^{1,2}_{C,s}(TX)$ *for a.e.* $t\in(0,T)$*, that* $|\nabla_{\mathrm{sym}}b_t|\in L^1((0,T);L^2(X,\mathfrak{m}))$ *and that* $\mathrm{div}\, b_t\in L^1((0,T);L^\infty(X,\mathfrak{m}))$*. Let* $(\mathbf{X}_t)_{t\in[0,T]}$ *be the Regular Lagrangian flow of* b*. Then, for every* $t\in[0,T]$ *and for* $\mathfrak{m}\times\mathfrak{m}$*-almost every* $(x,y)\in X\times X$ *it holds that*

$$\mathsf{d}_{\bar{G}}\left(\mathbf{X}_t(x),\mathbf{X}_t(y)\right)\leq \mathsf{d}_{\bar{G}}(x,y)\exp\left\{\int_0^T F_s\left(\mathbf{X}_s(x)\right)\,\mathrm{d}s+\int_0^T F_s\left(\mathbf{X}_s(y)\right)\,\mathrm{d}s\right\}, \tag{3.69}$$

for some non-negative function $F_t:(0,T)\times X\to[0,+\infty]$ *verifying*

$$\int_0^T \|F_t\|_{L^2(X,\mathfrak{m})}\,\mathrm{d}t<\infty. \tag{3.70}$$

Proof. We just sketch the proof, highlighting the main differences with respect to the case of non-negative lower Ricci curvature bound we treated in Theorem 3.26

Taking into account Remark 3.25 we can argue that the map

$$t\mapsto \bar{G}(\mathbf{X}_t(x),\mathbf{X}_t(y))$$

belongs to $W^{1,1}((0,T))$ for $\mathfrak{m}\times\mathfrak{m}$-a.e. $(x,y)\in X\times X$ and its derivative is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{G}(\mathbf{X}_t(x),\mathbf{X}_t(y)) = b_t\cdot\nabla\bar{G}_{\mathbf{X}_t(x)}(\mathbf{X}_t(y)) + b_t\cdot\nabla\bar{G}_{\mathbf{X}_t(y)}(\mathbf{X}_t(x)), \tag{3.71}$$

for $\mathcal{L}^1$-a.e. $t\in(0,T)$.

With (3.71) at our disposal we can argue as in the previously treated case, relying on Proposition 3.21 in place of Proposition 3.16 and then integrating with respect to time.

The bound in (3.70) can be achieved taking into account the bounded compressibility of the RLF as in the previous case too. □

In analogy with the case of real valued functions (where the Lipschitz regularity is understood w.r.t. the distance d), in [**52**] we introduced the notions of d_G and $\mathsf{d}_{\bar{G}}$ Lusin Lipschitz maps.

Definition 3.28 (Green Lusin Lipschitz maps)**.** Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(0,N)$ m.m.s. satisfying Assumption 3.2. We say that a map $\Phi: X\to X$ is d_G-Lusin Lipschitz if there exists a family $\{E_n : n\in\mathbb{N}\}$ of Borel subsets of X such that $\mathfrak{m}(X\setminus\cup_{n\in\mathbb{N}}E_n)=0$ and

$$\mathsf{d}_G(\Phi(x),\Phi(y))\le n\mathsf{d}_G(x,y),$$

for any $x,y\in E_n$ and for any $n\in\mathbb{N}$.
By analogy, if $(X,\mathsf{d},\mathfrak{m})$ is an $\mathrm{RCD}(K,N)$ m.m.s. satisfying Assumption 3.17, we say that $\Psi: X\to X$ is $\mathsf{d}_{\bar{G}}$-Lusin Lipschitz if it satisfies the above conditions with $\mathsf{d}_{\bar{G}}$ in place of d_G.

Remark 3.29. Let us remark that, with the above introduced terminology, we could combine [**52**, Proposition 2.19] and [**52**, Theorem 2.20] to say that the Regular Lagrangian flow of a sufficiently regular vector field over an $\mathrm{RCD}(0,N)$ m.m.s. satisfying Assumption 3.14 is a d_G-Lusin Lipschitz map (the RLF of a sufficiently regular vector field over an $\mathrm{RCD}(K,N)$ m.m.s. satisfying Assumption 3.17 is a $\mathsf{d}_{\bar{G}}$-Lusin Lipschitz map, respectively).

Given the weaker results that we chose to present in this thesis, we introduce an alternative terminology.

Definition 3.30 (Weak Green Lusin Lipschitz maps)**.** Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(0,N)$ m.m.s. satisfying Assumption 3.2. We say that a map $\Phi: X\to X$ is *weakly* d_G*-Lusin Lipschitz* (or a *weak* d_G*-Lusin Lipschiz map*) if there exists a function $g\in L^2(X,\mathfrak{m})$ such that

$$\mathsf{d}_G(\Phi(x),\Phi(y))\le\mathsf{d}_G(x,y)\exp(g(x)+g(y)),\quad\text{for }\mathfrak{m}\times\mathfrak{m}\text{-a.e. }(x,y)\in X\times X. \tag{3.72}$$

By analogy, if $(X,\mathsf{d},\mathfrak{m})$ is an $\mathrm{RCD}(K,N)$ m.m.s. satisfying Assumption 3.17, we say that $\Psi: X\to X$ is *weakly* $\mathsf{d}_{\bar{G}}$*-Lusin Lipschitz* (alternatively a *weak* $\mathsf{d}_{\bar{G}}$*-Lusin Lipschitz map*) if it satisfies (3.72) with $\mathsf{d}_{\bar{G}}$ in place of d_G, for some $g\in L^2(X,\mathfrak{m})$.

Remark 3.31. With the above introduced terminology, Theorem 3.26 can be rephrased by saying that the Regular Lagrangian Flow of a Sobolev vector field with uniformly compact support and uniformly bounded divergence on an $\mathrm{RCD}(0,N)$ m.m.s. satisfying Assumption 3.14 is a weak d_G-Lusin Lipschitz map. An analogous conclusion holds with $\mathsf{d}_{\bar{G}}$ in place of d_G when the ambient space is an $\mathrm{RCD}(K,N)$ m.m.s. verifying Assumption 3.17, as a consequence of Theorem 3.27.

3. Constancy of the dimension

The aim of this section is to prove Theorem 3.1 (see also Theorem 3.40 below), that could be restated by saying that, if $(X,\mathsf{d},\mathfrak{m})$ is an $\mathrm{RCD}(K,N)$ m.m.s. for some $K\in\mathbb{R}$ and $1\le N<\infty$, then there exists a natural number $1\le n\le N$ such that the tangent cone of $(X,\mathsf{d},\mathfrak{m})$ is the n-dimensional Euclidean space at $\mathfrak{m}$-almost every point in X. In this way we extend to this abstract framework a relatively recent result obtained by Colding-Naber in [**77**] for Ricci-limit spaces.

Let us briefly describe the strategy we are going to implement, which is different with respect to the one adopted in [**77**], since we cannot rely on the existence of a smooth approximating sequence for $(X,\mathsf{d},\mathfrak{m})$.

We begin remarking that the statement of Theorem 3.1 is not affected by taking the tensor product with Euclidean factors. By means of this simple observation, we will put ourselves in position to apply the results of Section 2.

In Section 3.1 we start proving that weak $\mathsf{d}_G/\mathsf{d}_{\bar{G}}$-Lusin Lipschitz maps having bounded compressiblity from an RCD(K,N) m.m.s. into itself are regular enough to carry an information about the dimension from their domain to their image. This rigidity result has to be compared with the standard fact that biLipschitz maps preserve the Hausdorff dimension.

Then we are going to prove that the class of RLFs of Sobolev vector fields, that we know to be weakly $\mathsf{d}_G/\mathsf{d}_{\bar{G}}$-Lusin Lipschitz from Section 2, is rich enough to gain "transitivity" at the level of probability measures with bounded support and bounded density w.r.t. $\mathfrak{m}$. Better said, the primary goal of Section 3.2 is to show that any pair of probability measures which are intermediate points of a W_2-geodesic joining probabilities with bounded support and bounded density w.r.t. $\mathfrak{m}$ can be obtained one from the other via push-forward through the RLF of a vector field satisfying the assumptions of Theorem 3.26 (or Theorem 3.27).

Eventually in Section 3.3 we will prove that the above mentioned "transitivity" is not compatible with the "rigidity" we obtain in Section 3.1 and the possibility of having non negligible regular sets of different dimensions in the Mondino-Naber decomposition of $(X,\mathsf{d},\mathfrak{m})$ (cf. [**170**] and Chapter 1).

3.1. A rigidity result for Regular Lagrangian Flows. The aim of this subsection is to prove a rigidity result for weak d_G and $\mathsf{d}_{\bar{G}}$-Lusin Lipschitz maps (see Definition 3.30) that we are going to apply later on to Regular Lagrangian Flows of Sobolev vector fields.

Roughly speaking, given an RCD(K,N) m.m.s. satisfying Assumption 3.17, we are going to prove that a weak $\mathsf{d}_G/\mathsf{d}_{\bar{G}}$-Lusin Lipschitz map with bounded compressibility cannot move a part of dimension n of $(X,\mathsf{d},\mathfrak{m})$ into a part of dimension $k<n$ (see Theorem 3.36 below for a precise statement).

Remark 3.32. Just at a speculative level, let us point out that, in the case of d-Lusin Lipschitz maps, this conclusion would have been a direct consequence of standard geometric measure theory arguments. However, a priori, it is not clear how to build directly non trivial maps from the space into itself with d-Lusin Lipschitz regularity, while in Section 2 above we were able to obtain weak d_G-Lusin Lipschitz regularity for a very rich family of maps[1].

We begin with a result about preservation of the dimension that can be considered to some extent a much simplified version of Sard's lemma (see Remark 3.34 below).
With respect to the corresponding statement in [**52**] the present one is more general, allowing to weaken the regularity assumptions on the map and for more general domain and target spaces

Proposition 3.33. *Fix $k,n\in\mathbb{N}$ such that $1\le k<n$. Let $(X,\mathsf{d},\mathfrak{m})$ be a doubling metric measure space and (Y,d_Y) be a metric space.*

Let $A\subset X$ and $\Phi:A\to Y$ be a measurable function such that

i)

$$\lim_{r\to 0^+}\operatorname{ess\,sup}_{y\in A\cap B_r(x)}\frac{\mathsf{d}_Y\left(\Phi(y),\Phi(x)\right)}{\mathsf{d}(x,y)^{\frac{n}{k}}}=0,\quad \textit{for any } x\in A; \tag{3.73}$$

ii) *any $x\in A$ is a density point of A with respect to $\mathfrak{m}$.*

[1]A posteriori, one of the consequences of Theorem 3.40 is that Regular Lagrangian Flows are also d-Lusin Lipschitz, as we proved on [**52**]

Then, if $\mathcal{H}^n(A) < \infty$, $\mathcal{H}^k(\Phi(A)) = 0$.

Proof. As a first step we reduce to the case in which the essential supremum is replaced by a supremum in (3.73).
In order to do so let us point out that, since $n > k$, (3.73) yields that

$$\lim_{r \to 0^+} \operatorname{ess\,sup}_{y \in A \cap B_r(x)} \frac{\mathsf{d}_Y(\Phi(y), \Phi(x))}{\mathsf{d}(x, y)} = 0, \quad \text{for any } x \in A. \tag{3.74}$$

Observe that the combination of i) and ii) implies the more classical condition *approximate slope of* Φ *at* x *equal to* 0. By [**94**] (see also [**100**, Theorem 3.1.8]), (3.74) combined with the density assumption implies that A is a countable union of $\mathfrak{m}$-measurable sets A_n such that $\Phi|_{A_n}$ is Lipschitz for any $n \in \mathbb{N}$. The result in [**94**] is stated for real valued functions but the proof carries over also for metric space valued functions.
The Lipschitz continuity after restriction guarantees that the essential supremum can be replaced by the supremum at any fixed $r > 0$. Therefore, for any $n \in \mathbb{N}$, it holds

$$\lim_{r \to 0^+} \sup_{y \in A_n \cap B_r(x)} \frac{\mathsf{d}_Y(\Phi(x), \Phi(y))}{\mathsf{d}(x, y)^{\frac{n}{k}}} = 0, \quad \text{for any } x \in A_n.$$

Given what we observed above, thanks to the fact that $\Phi(A) = \cup_n \Phi(A_n)$ and that the union of a countable family of $\mathcal{H}^k$-negligible sets is $\mathcal{H}^k$-negligible, we can assume without loss of generality that

$$\lim_{r \to 0^+} \sup_{y \in A \cap B_r(x)} \frac{\mathsf{d}_Y(\Phi(y), \Phi(x))}{\mathsf{d}(x, y)^{\frac{n}{k}}} = 0, \quad \text{for any } x \in A. \tag{3.75}$$

We wish to prove that $\mathcal{H}^k_\delta(\Phi(A)) = 0$ for any $\delta > 0$. Fix now $\varepsilon > 0$. It follows from (3.75) that, for any $x \in A$, we can find $r_x < \delta/10$ such that, for any $y \in B_{5r_x}(x) \cap A$, it holds

$$\mathsf{d}_Y(\Phi(y), \Phi(x)) \le \varepsilon \mathsf{d}(x, y)^{\frac{n}{k}}. \tag{3.76}$$

Next we split A into a countable disjoint union $A = \cup_l A_l$ in such a way that, for any $x \in A_l$ it holds $r_x > 1/l$. Then, for any $l \in N$ we choose a covering $(A^i_l)_{i \in \mathbb{N}}$ of A_l such that $\operatorname{diam} A^i_l \le \min\{\delta, 1/l\}$ for any $i \in \mathbb{N}$, any element of the covering intersects A_l and

$$\sum_i \left(\operatorname{diam} A^i_l\right)^n \le \mathcal{H}^n(A_l) + 2^{-l}. \tag{3.77}$$

Observe that, since we can find $x \in A_l$ for which $x \in A^i_l$ for any $i \in \mathbb{N}$, (3.76) guarantees that

$$\operatorname{diam}\left(\Phi(A^i_l)\right) \le \varepsilon \left(\operatorname{diam}(A^i_l)\right)^{n/k}. \tag{3.78}$$

Moreover,

$$\Phi(A) \subset \bigcup_{l \in \mathbb{N}} \bigcup_{i \in \mathbb{N}} \Phi\left(A^i_l\right). \tag{3.79}$$

Therefore

$$\mathcal{H}^k_{\delta^{n/k}}(A) \le \sum_{l \in \mathbb{N}} \sum_{i \in \mathbb{N}} \left(\operatorname{diam}\left(\Phi(A^i_l)\right)\right)^k \tag{3.80}$$

$$\le \varepsilon^k \sum_{l \in \mathbb{N}} \left(\operatorname{diam}(A^i_l)\right)^n \tag{3.81}$$

$$\le \varepsilon^k \sum_{l \in \mathbb{N}} \left(\mathcal{H}^n(A_l) + 2^{-l}\right) \le \varepsilon^k \left(\mathcal{H}^n(A) + 2\right), \tag{3.82}$$

where the first inequality follows from (3.79), the second one from (3.78), the third one from(3.77) and the last one from te fact that $(A_l)_l$ is a partition of A.

Letting $\varepsilon \to 0$ we infer that $\mathcal{H}^k_{\delta^{k/n}}(\Phi(A)) = 0$ for any $\delta > 0$. Eventually, letting $\delta \to 0$ we get the sought conclusion.

□

Remark 3.34. The proof of Proposition 3.33 resembles the part of the proof of Sard's lemma where it is shown that the image of the set of points where all the derivatives vanish up to a certain order is negligible (see for instance [**102**] for a proof of Sard's lemma which has been inspiring for this case and [**190**] for the original paper by Sard). Recall that the classical Sard lemma requires some regularity of the map and that the highest is the difference between the dimension of the domain an the dimension of the codomain the highest is the regularity to be required. Actually, even if we do not explicitly require any sort of regularity for Φ, (3.75) is essentially telling us that the map is differentiable with vanishing derivatives up to the order n/k.

It is a rather classical fact in Riemannian geometry that on an n-dimensional compact Riemannian manifold with $n > 2$ the Green function behaves locally like the distance raised to the power $2 - n$ (see [**34**, Chapter 4]). The comparison is also global on a non compact manifold with non-negative Ricci curvature and Euclidean volume growth (see [**92**]) and in [**51**] we extended these results to Ahlfors regular RCD(K,N) metric measure spaces.
The aim of Lemma 3.35 below is to prove that the weak Ahlfors regularity result of Theorem 2.26 is enough to obtain an asymptotic version of this comparison on any RCD(K,N) m.m.s. satisfying Assumption 3.2.

Let us point out that in the work in progress [**50**], we sharpen these estimates catching the asymptotic behaviour of the Green function at regular points of an RCD(K,N) m.m.s. satisfying suitable volume growth assumptions via blow-up arguments.

Lemma 3.35. *Let $(X,\mathsf{d},\mathfrak{m})$ be an* RCD(K,N) *m.m.s. satisfying Assumption 3.2. Suppose that $x \in \mathcal{R}^*_k$ for some $k \geq 3$ and denote by $\theta_k(x) \in (0,+\infty)$ the value of the limit appearing in* (2.19)*. Then*

$$\lim_{r \to 0^+} \frac{F(x,r)}{\frac{1}{r^{k-2}}} = \frac{k-2}{\omega_k \theta_k(x)}.$$

Proof. Let us observe that

$$\frac{F(x,r)}{\frac{1}{r^{k-2}}} = (k-2)\frac{\int_r^{+\infty} \frac{s}{\mathfrak{m}(B_s(x))}\,\mathrm{d}s}{\int_r^{+\infty} \frac{1}{s^{k-1}}\,\mathrm{d}s}.$$

An application of De L'Hopital's rule yields now

$$\lim_{r \to 0^+} \frac{F(x,r)}{\frac{1}{r^{k-2}}} = \lim_{r \to 0^+} (k-2)\frac{\frac{r}{\mathfrak{m}(B_r(x))}}{\frac{1}{r^{k-1}}} = \frac{k-2}{\omega_k \theta_k(x)},$$

since, by the very definition of $\theta_k(x)$, it holds $\lim_{r \to 0^+} \frac{\mathfrak{m}(B_r(x))}{\omega_k r^k} = \theta_k(x)$. □

Let us assume up to the end of this section that $(X,\mathsf{d},\mathfrak{m})$ is an RCD(K,N) m.m.s. satisfying Assumption 3.17. It is not difficult to check that, under this assumption, the regular sets $\mathcal{R}_k$ of $(X,\mathsf{d},\mathfrak{m})$ associated to $k = 0, 1$ and 2 are empty.

In Theorem 3.36 we reach the same conclusion as in the corresponding statement of [**52**] under weakened assumptions. The main motivation for this generalization is that we wish to rely on the weaker regularity results we proved in Section 2.3.

Theorem 3.36. *Let $(X,\mathsf{d},\mathfrak{m})$ be as in the discussion above. Let $\Phi : X \to X$ be either a weak d_G-Lusin Lipschitz or a weak $\mathsf{d}_{\bar{G}}$-Lusin Lipschitz map (see Definition 3.28). Fix $\mu \in \mathcal{P}(X)$*

absolutely continuous w.r.t. $\mathfrak{m}$ *and assume that* $\nu := \Phi_\sharp\mu \ll \mathfrak{m}$*. If* μ *is concentrated on* $\mathcal{R}_n$ *for some* $n \geq 3$*, then* ν *is concentrated on* $\cup_{k\geq n}\mathcal{R}_k$*.*

Proof. We will divide the proof into two steps, let us briefly outline its strategy.

The first step consists in proving that, if we have a weak $\mathsf{d}_G/\mathsf{d}_{\bar{G}}$-Lusin Lipschitz map which maps a subset of $\mathcal{R}_n^*$ into $\mathcal{R}_k^*$ for some $n > k \geq 3$, then we essentially end up with a map which satisfies the assumptions of Proposition 3.33.
In the second step we use this information to prove that $\nu = \Phi_\sharp\mu$ is concentrated over $\cup_{k\geq n}\mathcal{R}_k$, a formal argument being the following one: suppose that $\mathfrak{m}(\Phi(\mathcal{R}_n^*)\cap\mathcal{R}_k^*) = 0$, then, neglecting the measurability issues, we could compute

$$\begin{aligned}\Phi_\sharp\mu(\mathcal{R}_k^*) =&\mu\left(\Phi^{-1}(\mathcal{R}_k^*)\right) = \mu\left(\Phi^{-1}(\mathcal{R}_k^*)\cap\mathcal{R}_n^*\right)\\ \leq&\mu\left(\Phi^{-1}(\mathcal{R}_k^*\cap\Phi(\mathcal{R}_n^*))\right) = \Phi_\sharp\mu(\mathcal{R}_k^*\cap\Phi(\mathcal{R}_n^*)) = 0.\end{aligned}$$

Step 1. We want to prove that, for any $3 \leq k < n$, if $P \subset \mathcal{R}_n^*$ is such that

i) $\Phi(P) \subset \mathcal{R}_k^*$;
ii) $\mathcal{H}^n(P) < \infty$;
iii) for any $x \in P$, x is a density point for $\mathfrak{m}$ and it holds

$$\lim_{r\to 0} \operatorname{ess\,sup}_{y\in P\cap B_r(x)} \frac{\mathsf{d}_G(\Phi(x),\Phi(y))}{\mathsf{d}_G(x,y)} < \infty; \tag{3.83}$$

then $\mathcal{H}^k(\Phi(P)) = 0$.
Moreover, an analogous conclusion holds with $\mathsf{d}_{\bar{G}}$ in place of d_G in (3.83). Since $\mathfrak{m}\llcorner\mathcal{R}_k^*$ and $\mathcal{H}^k\llcorner\mathcal{R}_k^*$ are mutually absolutely continuous it will follow that $\mathfrak{m}(\Phi(P)) = 0$.

In order to do so we first read (3.83) in terms of powers of the distance function. For any $x \in \mathcal{R}_n^*$ such that $\Phi(x) \in \mathcal{R}_k^*$, we claim that (3.83) yields

$$\lim_{r\to 0^+} \operatorname{ess\,sup}_{y\in B_r(x)\cap P} \frac{\mathsf{d}(\Phi(x),\Phi(y))}{\mathsf{d}(x,y)^{\frac{n-2}{k-2}}} < \infty. \tag{3.84}$$

To this aim we observe that, by the very definition of d_G and thanks to the two-sided bounds we obtained in Proposition 3.5, (3.83) can be turned into

$$\lim_{r\to 0^+} \operatorname{ess\,sup}_{y\in B_r(x)\cap P} \frac{F(x,\mathsf{d}(x,y))}{F(\Phi(x),\mathsf{d}(\Phi(x),\Phi(y)))} < \infty$$

and the same holds true in case we are working with $\mathsf{d}_{\bar{G}}$, thanks to (3.32). Observe now that Lemma 3.15 guarantees that, as $\mathsf{d}(x,y) \to 0$, also $\mathsf{d}_G(x,y) \to 0$ (and an analogous result holds for $\mathsf{d}_{\bar{G}}$, as we observed after (3.44)). Hence we can apply Lemma 3.35 to obtain, taking into account the fact that $x \in \mathcal{R}_n^*$ and $\Phi(x) \in \mathcal{R}_k^*$,

$$\lim_{r\to 0^+} \operatorname{ess\,sup}_{y\in B_r(x)\cap P} \frac{\mathsf{d}(\Phi(x),\Phi(y))^{k-2}}{\mathsf{d}(x,y)^{n-2}} < \infty,$$

which easily yields (3.84).

To get the claimed conclusion it is now sufficient to apply Proposition 3.33.

Step 2. Suppose by contradiction that

$$\nu\left(\bigcup_{k<n}\mathcal{R}_k\right) > 0.$$

Then we can find $k < n$ such that $\nu(\mathcal{R}_k) > 0$. Moreover, thanks to Theorem 2.26 and to the assumption $\nu \ll \mathfrak{m}$, we can also say that $\nu(\mathcal{R}_k^*) > 0$.
We claim that, if this is the case, we can find a measurable set $P \subset X$ such that

i) $P \subset \mathcal{R}_n^*$, $\Phi(P) \subset \mathcal{R}_k^*$;
ii) $\mu(P) > 0$, $\mathcal{H}^n(P) < \infty$;
iii) every $x \in P$ is a density point of P with respect to $\mathfrak{m}$ and satisfies

$$\lim_{r\to 0} \operatorname{ess\,sup}_{y\in P\cap B_r(x)} \frac{\mathsf{d}_G(\Phi(x),\Phi(y))}{\mathsf{d}_G(x,y)} < \infty. \tag{3.85}$$

In this way we reach a contradiction. Indeed in Step 1 we proved that, under the assumptions above, it holds $\mathcal{H}^k(\Phi(P)) = 0$. Therefore $\Phi(P)$ is $\mathfrak{m}$-measurable and we can compute

$$0 < \mu(P) \le \mu(\Phi^{-1}(\Phi(P))) = \Phi_\sharp\mu(\Phi(P)) = \nu(\Phi(P)),$$

which contradicts the fact that $\mathcal{H}^k(\Phi(P)) = 0$, since $\Phi(P) \subset \mathcal{R}_k^*$ and $\nu \llcorner \mathcal{R}_k^* \ll \mathcal{H}^k$.

Let us pass to the verification of the claim. We are assuming that $\nu(\mathcal{R}_k^*) = \Phi_\sharp\mu(\mathcal{R}_k^*) > 0$, hence $\mu(\Phi^{-1}(\mathcal{R}_k^*)) = \mu(\Phi^{-1}(\mathcal{R}_k^*) \cap \mathcal{R}_n^*) > 0$.
It follows from the weak Lusin Lipschitz property that we can find $Q \subset \Phi^{-1}(\mathcal{R}_k^*) \cap \mathcal{R}_n^*$ with positive $\mathfrak{m}$-measure and such that for all $x \in Q$

$$\lim_{r\to 0} \operatorname{ess\,sup}_{y\in Q\cap B_r(x)} \frac{\mathsf{d}_G(\Phi(x),\Phi(y))}{\mathsf{d}_G(x,y)} < \infty. \tag{3.86}$$

Up to restricting Q we can also assume that $\mathcal{H}^n(Q)$ is finite, in view of Theorem 2.26, and this restriction does not affect the validity of (3.86). Up to restrict again Q to the set P made of all the density points with respect to $\mathfrak{m}$ of Q we can also assume that all this points are density points with respect to $\mathfrak{m}$ without affecting (3.86), obtaining a set verifying all the sought properties i)-iii).

□

3.2. Regularity of vector fields drifting W_2-geodesics. In Theorem 3.37 below, which is [**119**, Theorem 3.13], we state a version of the so-called Lewy-Stampacchia inequality. It will be the key tool in order to apply the regularity theory of Lagrangian Flows we developed in Section 2 to vector fields drifting W_2-geodesics.
We refer to Section 1.2 for an overview on the basic results about W_2-geodesics and Kantorovich potentials.

Below we will indicate by $l_{K,N} : [0,+\infty) \to [0,+\infty)$ the continuous function, whose explicit expression will be of no importance for our purposes, appearing in the Laplacian comparison theorem (see [**110**] and [**119**, Theorem 3.5]).

Theorem 3.37 (Lewy Stampacchia inequality)**.** *Let $(X,\mathsf{d},\mathfrak{m})$ be an* RCD(K,N) *metric measure space for some $K \in \mathbb{R}$ and $1 < N < \infty$. Let $\mu_0, \mu_1 \in \mathcal{P}(X)$ be absolutely continuous w.r.t. $\mathfrak{m}$ and with bounded supports, $(\mu_t)_{t\in[0,1]}$ be the W_2-geodesic connecting them and $\varphi : X \to \mathbb{R}$ be a Kantorovich potential inducing it (which we can assume to be Lipschitz and with compact support).*

Then, for every $t \in (0,1)$, there exists $\eta_t \in \mathrm{Lip}(X)$ with compact support, uniformly w.r.t. time, and such that

$$-\mathcal{Q}_t(-\varphi) \le \eta_t \le \mathcal{Q}_{(1-t)}(-\varphi^c), \tag{3.87}$$

$$(t\eta_t)^{cc}(x) = t\eta_t(x) \quad \textit{and} \quad (-(1-t)\eta_t)^{cc}(x) = -(1-t)\eta_t(x) \quad \textit{for any } x \in \operatorname{supp}\mu_t$$

and $\eta_t \in D(\Delta)$ with

$$\|\Delta\eta_t\|_{L^\infty} \le \max\left\{ \frac{l_{K,N}(2\sqrt{t\,\|\varphi\|_{L^\infty}})}{t}, \frac{l_{K,N}(\sqrt{2(1-t)\,\|\varphi\|_{L^\infty}})}{1-t} \right\}. \tag{3.88}$$

Remark 3.38. We remark that, passing from the starting potentials to the regularized potentials η_t, we gain global regularity without modifying the potential in the support of μ_t, as it follows from (3.87) recalling that $-\mathcal{Q}_t(-\varphi) = \mathcal{Q}_{(1-t)}(-\varphi^c)$ on $\operatorname{supp}\mu_t$ (see Proposition 1.4).

In view of the applications of Section 3.3, in Proposition 3.39 below we collect some consequences of the improved regularity of Kantorovich potentials.

Proposition 3.39. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ m.m.s. for some $K \in \mathbb{R}$ and $1 < N < \infty$. Let $\mu_0, \mu_1 \in \mathcal{P}(X)$ be absolutely continuous w.r.t. $\mathfrak{m}$ with bounded densities and bounded supports. Then there exists a time dependent vector field $(b_t)_{t\in(0,1)}$ such that the following conditions are satisfied:*

(i) *for any $t \in (0, 1)$ it holds $b_t \in H^{1,2}_C(TX)$ and*

$$\int_\varepsilon^{1-\varepsilon} \left\{ \|\nabla_{\mathrm{sym}} b_s\|_{L^2(X,\mathfrak{m})} + \|\operatorname{div} b_s\|_{L^2(X,\mathfrak{m})} \right\} \mathrm{d}s < \infty \quad \text{for any } 0 < \varepsilon < 1; \tag{3.89}$$

(ii) *for any $0 < s < 1$, denoting by $(\mathbf{X}^t_s)_{t\in[s,1)}$ the Regular Lagrangian flow of $(b_t)_{t\in(s,1)}$, it holds that $\left(\mathbf{X}^t_s\right)_\sharp \mu_s = \mu_t$ for any $s \le t < 1$.*

Proof. We claim that the vector field $(\nabla\eta_s)_{s\in(0,1)}$ (where η_s are the regularized Kantorovich potentials we introduced in Theorem 3.37) does the right job.

Observe that, for any $s \in (0, 1)$, it holds that $\nabla\eta_t$ is bounded with bounded support, as it was stated in Theorem 3.37, Moreover, since $\eta_s \in D(\Delta)$, Corollary 1.81 implies that $\eta_s \in W^{2,2}(X, \mathsf{d}, \mathfrak{m})$ which yields, in turn, $\nabla\eta_s \in H^{1,2}_C(TX)$.
Let us check (3.89). To this aim we observe that the construction described in the proof of [**119**, Theorem 3.13] guarantees that the regularized potentials can be chosen to have all support contained in the same compact set $C \subset X$. Hence

$$\begin{aligned}\int_\varepsilon^{1-\varepsilon} &\|\operatorname{div} b_s\|_{L^2} \,\mathrm{d}s \\ &\le \int_\varepsilon^{1-\varepsilon} \max\left\{ \frac{l_{K,N}(2\sqrt{s\,\|\varphi\|_{L^\infty}})}{s}, \frac{l_{K,N}(\sqrt{2(1-s)\,\|\varphi\|_{L^\infty}})}{1-s} \right\} \mathfrak{m}(C)\,\mathrm{d}s < \infty.\end{aligned}$$

Dealing with the bound of the Sobolev norm we recall that Corollary 1.81 provides the quantitative bound

$$\int_X |\mathrm{Hess}\, f|^2 \,\mathrm{d}\mathfrak{m} \le \int_X \left\{ (\Delta f)^2 - K\,|\nabla f|^2 \right\} \mathrm{d}\mathfrak{m} \tag{3.90}$$

for any $f \in D(\Delta)$. Recalling that the regularized potentials can also be chosen uniformly Lipschitz on $(0, 1)$, the sought bound for $\int_\varepsilon^{1-\varepsilon} \|\nabla_{\mathrm{sym}} b_s\|_{L^2} \,\mathrm{d}s$ follows applying (3.90) to the functions η_s, taking into account the L^∞-bound for the Laplacian (3.88) and the uniform boundedness of the supports.

Passing to the proof of (ii), observe that the very construction of the regularized Kantorovich potentials (see Remark 3.38) η_s guarantees that $(\mu_s, b_s)_{s\in(0,1)}$ is a solution to the continuity equation with uniformly bounded density (see [**113**], the uniform bound for the densities is a consequence of Proposition 1.112). Moreover, (3.89) guarantees, via [**30**, Theorem 5.4, Theorem 8.3], that, for any $0 < s < t < 1$, there exists a unique Regular Lagrangian flow $(\mathbf{X}^r_s)_{r\in[s,t]}$ of $(b_r)_{r\in(s,t)}$. Observe that, by the very definition of RLF, also $r \mapsto (\mathbf{X}^r_s)_\sharp\mu_s$ is a solution, with uniformly bounded density and initial datum μ_s, to the continuity equation induced by $(b_r)_{r\in(s,t)}$. Hence $(\mathbf{X}^t_s)_\sharp\mu_s = \mu_t$ for any $0 < s \le t < 1$, since (i), coupled with the L^∞-bound on the divergence, implies that the continuity equation induced by $(b_r)_{r\in(s,t)}$

has a unique solution with uniformly bounded density (again by the results of [**30**, Theorem 5.4]). □

3.3. Conclusion.

Theorem 3.40 (Constancy of the dimension)**.** *Let* $(X,\mathsf{d},\mathfrak{m})$ *be an* $\mathrm{RCD}(K,N)$ *m.m.s. for some* $K\in\mathbb{R}$ *and* $1\le N<\infty$. *Then there is exactly one regular set* $\mathcal{R}_n$ *having positive* $\mathfrak{m}$*-measure in the Mondino-Naber decomposition of* $(X,\mathsf{d},\mathfrak{m})$.

Proof. As we already observed, the statement is not affected by tensorization with Euclidean factors. Thus we assume without loss of generality that $(X,\mathsf{d},\mathfrak{m})$ satisfies either Assumption 3.14 or Assumption 3.17.

Suppose by contradiction that there exist $3\le k<n$ such that $\mathfrak{m}(\mathcal{R}_k),\mathfrak{m}(\mathcal{R}_n)>0$. Then we can find $\theta_0,\theta_1\in\mathcal{P}(X)$, absolutely continuous w.r.t. $\mathfrak{m}$, with bounded densities and bounded supports, such that $\theta_0(\mathcal{R}_n)=1$ and $\theta_1(\mathcal{R}_k)=1$.
Let $(\theta_r)_{r\in[0,1]}$ be the W_2-geodesic joining them and recall from Proposition 1.112 that the measures θ_r are absolutely continuous w.r.t. $\mathfrak{m}$, with uniformly bounded densities and uniformly bounded supports. Applying the second conclusion in Proposition 1.112, we can also conclude that there exist $0<s<t<1$ such that $\theta_s(\mathcal{R}_n)>1/2$ and $\theta_t(\mathcal{R}_k)>1/2$. Calling $\Pi\in\mathcal{P}(\mathrm{Geo}(X))$ the unique geodesic plan lifting $(\theta_r)_{r\in[0,1]}$, it follows from what we just observed that

$$\Pi(\{\gamma\in\mathrm{Geo}(X):\gamma(s)\in\mathcal{R}_n\text{ and }\gamma(t)\in\mathcal{R}_k\})>0.$$

Hence, setting

$$A:=\{\gamma\in\mathrm{Geo}(X):\gamma(s)\in\mathcal{R}_n\text{ and }\gamma(t)\in\mathcal{R}_k\},\quad \bar{\Pi}:=\frac{1}{\Pi(A)}\Pi\llcorner A\quad\text{and } \mu_r:=(e_r)_\sharp\bar{\Pi},$$

for any $r\in[0,1]$, we obtain a W_2-geodesic $(\mu_r)_{r\in[0,1]}$ which joins probabilities with bounded support and bounded densities w.r.t. $\mathfrak{m}$ and such that μ_s is concentrated on $\mathcal{R}_n$ and μ_t is concentrated on $\mathcal{R}_k$.

Next we apply Proposition 3.39 to the W_2-geodesic $(\mu_r)_{r\in[0,1]}$ to obtain that, with the notation therein introduced, $\mathbf{X}_t^s$ is the RLF of a Sobolev time dependent vector field satisfying the assumptions of Theorem 3.26 (or Theorem 3.27). Hence $\mathbf{X}_t^s$ is a weak $\mathsf{d}_G/\mathsf{d}_{\bar{G}}$-Lusin Lipschitz map such that $\left(\mathbf{X}_s^t\right)_\sharp\mu_s=\mu_t$ and, applying Theorem 3.36, we eventually reach a contradiction. □

Remark 3.41. We point out that in [**135**] Honda constructs a family of spaces satisfying the weak (K,N)-Bochner inequality as in Theorem 1.103 ii), but not the *Sobolev to Lipschitz* property, having regular sets of different dimensions with positive measure. Therefore we realize that the $\mathrm{RCD}(K,N)$ condition has to be used in all its strength to get the constancy of the dimension.

Even though we obtained the constancy of the dimension for $\mathrm{RCD}(K,N)$ spaces without passing through the continuity of tangent cones along minimizing geodesics we conjecture that this property should be true also in this context, at least in the following weak form.

Conjecture 3.42. Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ m.m.s. and let Π be an optimal geodesic plan connecting probabilities with bounded densities and bounded supports (cf. Section 1.2). Then, for Π-a.e. $\gamma\in\mathrm{Geo}(X)$ and for any $\delta>0$ there exists a modulus of continuity $\omega=\omega_{\gamma,\delta}$ such that

$$\mathsf{d}_{pmGH}\left((X,\mathsf{d}/r,\mathfrak{m}_r^{\gamma(t)},\gamma(t)),(X,\mathsf{d}/r,\mathfrak{m}_r^{\gamma(s)},\gamma(s))\right)\le\omega_{\gamma,\delta}(|s-t|),$$

for any $0<\delta<s,t<1-\delta$ and for any $0<r<1$.

A first direct consequence of Conjecture 3.42 above would be that tangent cones arising from the same sequence of scaling are continuous in the interior of Π almost every geodesic, since the modulus of continuity is independent of r. We refer to [**77**] and [**145**] for the original stronger versions of the continuity of tangent cones on Ricci limit spaces, with modulus of continuity also independent of the geodesic.

Let us conclude this section stating and proving some corollaries of Theorem 3.40.

Definition 3.43 (Essential dimension)**.** Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ m.m.s. for some $K \in \mathbb{R}$ and $N \geq 1$. We shall indicate by *essential dimension* the unique n such that $\mathfrak{m}(\mathcal{R}_n) > 0$.

Theorem 3.44 (Structure theory reviewed)**.** *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *m.m.s. for some* $K \in \mathbb{R}$ *and* $1 \leq N < \infty$*. Then there exists a unique integer* $1 \leq n \leq N$*, called essential dimension of* $(X, \mathsf{d}, \mathfrak{m})$*, such that the following hold:*

i) $\mathfrak{m}(X \setminus \mathcal{R}_n) = 0$*, where*

$$\mathcal{R}_n := \Big\{ x \in X \;\Big|\; \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}) = \{(\mathbb{R}^n, \mathsf{d}_{\mathrm{eucl}}, c_n \mathcal{L}^n, 0^n)\} \Big\};$$

ii) $\mathfrak{m} = \theta \mathscr{H}^n \llcorner \mathcal{R}_n$*, for some density* $\theta \in L^1_{\mathrm{loc}}(\mathscr{H}^n)$*;*

iii) $(X, \mathsf{d}, \mathfrak{m})$ *is strongly* $(\mathfrak{m}, n)$*-rectifiable.*

In view of [**121**], the constancy of the dimension, that we stated and proved in Theorem 3.40 at the level of the Mondino-Naber decomposition, can be equivalently rephrased at the level of the dimensional decomposition of the tangent module $L^2(TX)$ (cf. Proposition 1.76) .

Corollary 3.45. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *m.m.s. for some* $K \in \mathbb{R}$ *and* $1 < N < \infty$*. Let* $1 \leq n \leq N$ *be the essential dimension of* $(X, \mathsf{d}, \mathfrak{m})$*. Then the tangent module* $L^2(TX)$ *has constant dimension equal to* n*.*

Proof. The result directly follows from [**121**, Theorem 3.3] and Theorem 3.40. □

Remark 3.46. With the notation introduced in Corollary 3.45, one has that n is the analytic dimension of $(X, \mathsf{d}, \mathfrak{m})$ (see Definition 1.77).

To let the picture about the different notions of dimension introduced in the literature so far be more complete, we also point out that n is also the dimension of $(X, \mathsf{d}, \mathfrak{m})$ according to [**152**, Definition 4.1]. Indeed, as it is observed in [**152**, Remark 4.14], if $\mathcal{R}_n$ is the unique regular set of positive measure, Theorem 2.19 guarantees that it is also the non empty regular set of maximal dimension.

Up to our knowledge, the problem of whether n is the Hausdorff dimension of (X, d) or not is still open also in the case of collapsed Ricci limit spaces (see [**77**, Remark 1.3]) essentially due to the lack of knowledge about the Hausdorff dimension of the singular set.

It might be interesting to sharpen the knowledge both of the singular set of an $\mathrm{RCD}(K, N)$ m.m.s., defined as $\mathcal{S} := X \setminus \cup_k \mathcal{R}_k$ and of the regular sets of dimension less than the essential dimension of $(X, \mathsf{d}, \mathfrak{m})$. We formulate a conjecture about this second problem.

Conjecture 3.47. Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ m.m.s. with essential dimension $1 \leq n \leq N$. Then $\mathcal{R}_n$ is the unique non empty regular set.

Eventually we give a positive answer to a conjecture raised in [**84**, Remark 1.13]. As it is therein observed, its validity follows from the fact that the tangent module has constant local dimension exploiting the results of [**133**].

Theorem 3.48. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *m.m.s. for some* $K \in \mathbb{R}$ *and* $1 \leq N < \infty$*. Assume that* $H^{2,2}(X, \mathsf{d}, \mathfrak{m}) = D(\Delta)$ *and*

$$\mathrm{tr}\,\mathrm{Hess}\, f = \Delta f, \quad \text{for any } f \in H^{2,2}(X, \mathsf{d}, \mathfrak{m}). \tag{3.91}$$

Then, there exists $n \in \mathbb{N}$, $1 \le n \le N$ *such that* $(X, \mathsf{d}, \mathfrak{m})$ *is a weakly non collapsed* $\mathrm{RCD}(K, n)$ *m.m.s..*

Proof. We wish to prove that the statement holds true with n equal to the dimension of the unique regular set with positive measure in the Mondino-Naber decomposition of $(X, \mathsf{d}, \mathfrak{m})$.

By the very definition of weakly non collapsed $\mathrm{RCD}(K, n)$ m.m.s. (cf. Definition 1.106), we just need to prove that $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(K, n)$ m.m.s.. To this aim, observe that the first and the second assumption in the statement of [**133**, Theorem 4.3] are fulfilled thanks to our choice of n and the validity of (3.91). To see that also the third one is satisfied, it suffices to observe that

$$\mathrm{Ric}_n(\nabla f, \nabla f) = \mathbf{\Gamma}_2(\nabla f, \nabla f) - |\mathrm{Hess}\, f|^2\, \mathfrak{m} \ge K\, |\nabla f|^2\, \mathfrak{m}, \tag{3.92}$$

for any $f \in H^{2,2}(X, \mathsf{d}, \mathfrak{m})$. We refer to [**112**, **133**] for the relevant notation about the measure valued Ricci tensor and the $\mathbf{\Gamma}_2$ operator. The equality in (3.92) follows from (3.91) and the inequality from the assumption that $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(K, N)$ m.m.s..

□

CHAPTER 4

Sets of finite perimeter over $\mathrm{RCD}(K,N)$ spaces: existence of Euclidean tangents

This is the first of two chapters dedicated to the theory of sets of finite perimeter over $\mathrm{RCD}(K,N)$ metric measure spaces as developed in [**8**], joint work with Ambrosio and Brué, and in the sequel [**48**], written in collaboration with Brué and Pasqualetto.

After [**52**, **85**, **120**, **144**, **170**] we have reached a good understanding of the structure of $\mathrm{RCD}(K,N)$ spaces *up to measure zero*, that we reviewed in Chapters 2 and 3 (see in particular Theorem 3.44). It sounded therefore quite natural to try to push the study further, investigating their structure up to sets of positive codimension, both from the analytic and from the geometric points of view.
In this perspective in the last three years there have been some independent and remarkable developments. We wish to mention a few of them below, without the aim of being complete in this list.

- In the setting of non collapsed Ricci limit spaces, Cheeger-Jiang-Naber have obtained in [**64**] rectifiability for the singular sets of any codimension. Let us also mention [**65**, **73**], both containing crucial developments for the study of singular sets over Ricci limits, and [**33**], joint work with Antonelli and Brué, where some of the estimates in [**73**] are proved for the singular strata of non collapsed RCD spaces.
- There have been some efforts aimed at defining a notion of boundary for metric measure spaces and relating it with the singular set of codimension 1. See [**142**–**144**].
- In [**88**] a notion of capacitary cotangent module has been proposed and the theory of vector fields defined Cap-almost everywhere on RCD spaces has been initiated.[1]

Apart from the increasing efforts devoted to the investigation of the fine structure in positive codimension, at this stage of the development of the RCD theory it is also natural to investigate the typical themes of Geometric Measure Theory, since it provides techniques for dealing with nonsmooth objects already when the ambient space is smooth.
One of the most fundamental results of Geometric Measure Theory, that eventually led to the Federer-Fleming theory of currents [**101**], is De Giorgi's structure theorem for sets $E \subset \mathbb{R}^n$ of finite perimeter. De Giorgi's theorem, established in [**81**, **82**], provides the representation of the perimeter measure $|D\chi_E|$ as the restriction of $\mathscr{H}^{n-1}$ to a suitable measure-theoretic boundary $\mathcal{F}E$ of E. In addition, it provides a description of E on small scales, showing that for all $x \in \mathcal{F}E$ the rescaled set $r^{-1}(E-x)$ is close, for $r>0$ sufficiently small, to an halfspace orthogonal to a unit normal vector $\nu_E(x)$ and that $\mathcal{F}E$ is $(|D\chi_E|, n-1)$-rectifiable.

Our goal in [**8**, **48**] has been to provide an extension of this result to the setting of $\mathrm{RCD}(K,N)$ metric measure spaces. Of course, part of the efforts have been aimed at the introduction of the right counterparts of the Euclidean notions of tangent (cf. Definition 4.30) and normal vector (cf. Theorem 5.6). In the statement below the non expert reader can understand $\mathrm{Tan}_x(X,\mathsf{d},\mathfrak{m},E)$, the space of tangents to the set of finite perimeter E at x, as the collection of *limits* of rescaled sets on rescaled spaces, for sequences of radii converging

[1] We point out however that [**84**], which has been crucial for the development of [**48**], has appeared after the publication of [**8**].

to 0. We refer to Remark 1.8 for the definition of Hausdorff-type measure induced by a given gauge function.

Theorem 4.1 (De Giorgi's theorem on $\mathrm{RCD}(K,N)$ spaces)**.** *Let* $(X,\mathsf{d},\mathfrak{m})$ *be an* $\mathrm{RCD}(K,N)$ *m.m.s. of essential dimension* $n\in\mathbb{N}$*, for some* $K\in\mathbb{R}$ *and* $1\le n\le N<\infty$*, and let* $E\subset X$ *be a set of locally finite perimeter. Then it holds that:*

i) $|D\chi_E|$ *is concentrated on the* reduced boundary $\mathcal{F}E=\cup_{i=1}^n\mathcal{F}_kE$*, where we set*

$$\mathcal{F}_kE:=\Big\{x\in X\ :\ \mathrm{Tan}_x(X,\mathsf{d},\mathfrak{m},E)=\Big\{(\mathbb{R}^k,\mathsf{d}_{eucl},c_k\mathcal{L}^k,0^k,\{x_k>0\})\Big\}\Big\};$$

ii) *for any* $k=1,\dots,n$*,* $\mathcal{F}_kE$ *is strongly* $(|D\chi_E|,k-1)$*-rectifiable;*

iii)

$$|D\chi_E|=\sum_{k=1}^n\frac{\omega_{k-1}}{\omega_k}\mathcal{S}^h\llcorner\mathcal{F}_kE,$$

where $\mathcal{S}^h$ *is the codimension one Hausdorff type measure built with gauge function* $h(B_r(x)):=\mathfrak{m}(B_r(x))/r$*.*

While in the general case the picture is not yet completely understood, since conjecturally one should have *constancy of the dimension* for the reduced boundary and better representation formulas for the perimeter measure with respect to the Hausdorff measure, in the non collapsed setting (cf. Definition 1.105) we reached a complete generalization of the Euclidean theorem.

Theorem 4.2 (De Giorgi's theorem on non collapsed $\mathrm{RCD}(K,N)$ spaces)**.** *Assume that* $(X,\mathsf{d},\mathscr{H}^N)$ *is an* $\mathrm{RCD}(K,N)$ *m.m.s. for some* $K\in\mathbb{R}$ *and* $1\le N<\infty$*, and let* $E\subset X$ *be a set of locally finite perimeter. Then it holds that*

i) $|D\chi_E|$ *is concentrated on the* reduced boundary $\mathcal{F}E=\mathcal{F}_NE$*;*

ii) *the reduced boundary* $\mathcal{F}_NE$ *is strongly* $(|D\chi_E|,N-1)$*-rectifiable;*

iii) $|D\chi_E|=\mathscr{H}^{N-1}\llcorner\mathcal{F}E$*.*

Theorem 4.1 and Theorem 4.2 are the main outcomes of the analysis pursued in [**8**, **48**]. In [**8**], whose contents are the main subject of the present chapter, we obtained existence of a Euclidean half-space in the tangent space to a set of locally finite perimeter almost everywhere with respect to the perimeter measure. This was the starting point of [**48**], where we got uniqueness of tangents, rectifiability of the reduced boundary and representation formulas for the perimeter, along with a Gauss-Green integration by parts formula of independent interest.

For the sake of motivating the contents of the present and of the forthcoming chapter, let us comment about the difficulties one meets trying to generalize the Euclidean theorem to this framework.

In De Giorgi's proof and its many extensions to currents and other weak objects, a crucial role is played by the normal direction ν_E coming out of the blow-up analysis, which is identified by looking at the polar decomposition $D\chi_E=\nu_E|D\chi_E|$ of the distributional derivative (choosing approximate continuity points of ν_E, relative to $|D\chi_E|$). In turn, the polar decomposition essentially depends on the particular structure of the tangent bundle of the Euclidean space.

At the moment of writing [**8**], [**88**] was not available yet. Therefore there was the necessity to find an approach to the study of blow-ups alternative to the classical one, since on a general metric measure space one is allowed to talk about vector fields only *up to* $\mathfrak{m}$*-negligible sets*, while any reasonable unit normal should be defined almost everywhere with respect to the perimeter measure, which has codimension one.[2]

[2]Let us also point out that, even after the development of a theory of vector fields defined capacity-almost everywhere in [**88**], it seems not clear how to adapt the Euclidean strategy to the RCD framework.

In the study of this problem, we realized the importance of the rigidity case in the Bakry-Émery inequality, namely the analysis of the implications of the condition

$$|\nabla P_t f| = P_t |\nabla f| \qquad \mathfrak{m}\text{-a.e. in } X, \text{ for every } t \geq 0 \tag{4.1}$$

for some nontrivial function f, if the ambient space is $\mathrm{RCD}(0, N)$.

Our rigidity result Theorem 4.3 shows that (4.1) is sufficiently strong to imply the splitting of the m.m.s. as $Z \times \mathbb{R}$, in addition with a monotonic dependence of f on the split real variable. This result could be considered as "dual" to the classical splitting theorem, since the basic assumption is not the existence of a curve with a special property (namely an entire geodesic), but rather the existence of a function satisfying (4.1).

Now, what is the relation between (4.1) and the fine structure of sets of finite perimeter?

To bypass the difficulties met in the development of a theory of tangents to sets of finite perimeter through the fine study of the unit normal, we establish this new principle: given a set of locally finite perimeter E over an $\mathrm{RCD}(K, N)$ m.m.s., at $|D\chi_E|$-a.e. point x, any tangent set F to E at x in any tangent, pointed, metric measure structure (Y, ϱ, μ, y) has to satisfy the condition

$$|\nabla P_t \chi_F| \mu = P_t^* |D\chi_F| \qquad \forall t \geq 0. \tag{4.2}$$

Notice that $|D\chi_F|$, the semigroup P_t and its dual P_t^* in (4.2) have, of course, to be understood in the tangent metric measure structure. The proof of this principle, given in Theorem 4.31, ultimately relies on the lower semicontinuity of the perimeter measure $|D\chi_E|$ (as it happens for the powerful principle that lower semicontinuity and locality imply asymptotic local minimality, see [**103**, **215**], and [**62**]) and gradient contractivity. From (4.2), gradient contractivity easily yields that all functions $f = P_s \chi_F$ satisfy (4.1); this leads to a splitting *both* of (Y, ϱ, μ) and F, and to the identification of a "tangent halfspace" F to E at x.

Combining (4.2) with the above mentioned characterization of the equality cases in the 1-Bakry-Émery inequality, we can prove that any set of finite perimeter E on an $\mathrm{RCD}(K, N)$ space admits a Euclidean half-space as tangent at x for $|D\chi_E|$-a.e. $x \in X$.

This chapter is organized as follows. In Section 1 we prove our rigidity result for the 1-Bakry-Émery inequality. We dedicate Section 2 to the study of the behaviour of sequences of sets E_i in m.m.s. $(X_i, \mathsf{d}_i, \mathfrak{m}_i)$ convergent in the measured Gromov-Hausdorff sense to $(X, \mathsf{d}, \mathfrak{m})$. In particular we adapt the study of [**18**] (see Section 4.2) to cover the case of converging sequences of sets of (locally) finite perimeter. We apply these results in Section 3, where we specialize our analysis to the case when $(X_i, \mathsf{d}_i, \mathfrak{m}_i)$ arise from the rescaling of a pointed m.m.s. This theme is also investigated in [**204**], but in our study we take advantage of the curvature-dimension bounds to establish the stronger rigidity property (4.2) satisfied by tangent sets F in the tangent metric measure structure. Then, using the splitting property and the principle that "tangents to a tangent are tangent", we are able to recover the first regularity result for tangents to sets of finite perimeter. Finally, Section 4 is devoted to a self-contained proof of a version of the iterated tangents principle suitable for this context, closely following the treatment of the analogous statement in the *codimension zero* case.

The presentation closely follows that of [**8**]. With respect to the original paper we decided to postpone the refined consequences of the existence of regular tangents in the non collapsed case after the proof of rectifiability of the reduced boundary in the next chapter.

1. Rigidity of the 1-Bakry-Émery inequality and splitting theorem

The splitting theorem for Riemannian manifolds with non-negative Ricci curvature [**72**] is one of the cornerstones of Riemannian Geometry and Geometric Analysis. After the seminal paper by Cheeger-Gromoll many efforts have been aimed at extending this result to different contexts, here we just mention the case of Ricci limit spaces, due to Cheeger-Colding [**68**] and that of $\mathrm{RCD}(0,N)$ spaces, due to Gigli [**108**]. In all these cases the perspective is *geometric*: the existence of a line together with suitable curvature assumptions implies the splitting. At the same time, also a dual *functional* perspective to the splitting problem has been investigated by several authors. Under the same curvature assumptions (non-negativity of the Ricci curvature), several functional inequalities have been established. The existence of a function verifying the equality in one of these inequalities has been seen to imply the geometric splitting in various circumstances. Here we just mention [**98**] and the very recent [**99**], pointing out that in both cases, also the functions enjoy some rigidity property.

Among the various functional inequalities valid for general $\mathrm{RCD}(0,\infty)$ metric measure spaces (and a fortiori by smooth Riemannian manifolds with non-negative Ricci curvature) there is the 1-Bakry-Émery gradient contractivity estimate (1.52). Our aim in this section is to prove a splitting type rigidity result for $\mathrm{RCD}(0,N)$ spaces admitting a non constant function satisfying the equality in this estimate. To the best of our knowledge the result is new even for smooth Riemannian manifolds.

Theorem 4.3. *Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(0,N)$ m.m.s.. Assume that there exist a non constant function $f\in\mathrm{Lip_b}(X)$ and $s>0$ satisfying*

$$|\nabla P_s f| = P_s|\nabla f| \qquad \mathfrak{m}\text{-a.e. in } X. \tag{4.3}$$

Then there exists a m.m.s. $(X',\mathsf{d}',\mathfrak{m}')$ such that X is isomorphic, as a metric measure space, to $X'\times\mathbb{R}$. Furthermore:

(i) *if $N\geq 2$ then $(X',\mathsf{d}',\mathfrak{m}')$ is an $\mathrm{RCD}(0,N-1)$ m.m.s.;*
(ii) *if $N\in[1,2)$ then X' is a point.*

Moreover, the function f written in coordinates $(x',t)\in X'\times\mathbb{R}$ depends only on the variable t and it is monotone.

Remark 4.4. As we already remarked, the action of the heat semigroup in $L^\infty(X,\mathfrak{m})$ can be defined by

$$P_t f(x) := \int_X f(y)p_t(x,y)\,\mathrm{d}\mathfrak{m}(y).$$

Using an approximation argument is it possible to see that, for any $f\in L^\infty(X,\mathfrak{m})$ and every $\varphi\in L^1(X,\mathfrak{m})$ the map $t\mapsto\int_X P_t f\varphi\,\mathrm{d}\mathfrak{m}$ is absolutely continuous with derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_X P_t f\varphi\,\mathrm{d}\mathfrak{m} = \int_X \Delta P_t f\varphi\,\mathrm{d}\mathfrak{m},$$

in other words $P_t f$ is still a solution of the heat equation.

Remark 4.5. The assumption $f\in\mathrm{Lip_b}(X)$ in Theorem 1.111 can be replaced with the more general $f\in\mathrm{Lip}(X)$, provided we extend the action of the heat semigroup to the class of Borel functions with at most linear growth at infinity, i.e.

$$|f(x)|\leq C(1+\mathsf{d}(x,x_0)) \qquad \text{for any } x\in X$$

for some $x_0\in X$ and $C\geq 0$. Even though under the $\mathrm{RCD}(0,N)$ condition the Gaussian estimates for the heat kernel (1.79) provide this extension, we shall consider only the case $f\in\mathrm{Lip_b}(X)$ that is enough for our purposes.

In order to better motivate Theorem 4.3 let us briefly address the rigidity case in the Bakry Émery inequality for $p=2$. Assume that $(M^n,\mathsf{d}_g,e^{-V}\mathrm{Vol}_g)$ is a smooth weighted Riemannian manifold with non-negative generalized N-Ricci tensor Ric_N, where

$$\mathrm{Ric}_N := \mathrm{Ric} + \mathrm{Hess}\,V - \frac{\nabla V\otimes\nabla V}{N-n},$$

and the last term is defined to be 0 when V is constant and $N = n$. Let $f : M \to \mathbb{R}$ be such that $|\nabla P_t f|^2 = P_t |\nabla f|^2$ for some $t > 0$. Then we can compute

$$\begin{aligned} 0 =& P_t |\nabla f|^2 - |\nabla P_t f|^2 = \int_0^t \frac{\mathrm{d}}{\mathrm{d}s} P_s |\nabla P_{t-s} f|^2 \, \mathrm{d}s \\ =& 2 \int_0^t P_s \left(|\mathrm{Hess}\, P_{t-s} f|^2 + \mathrm{Ric}_N(\nabla P_{t-s} f, \nabla P_{t-s} f) + \frac{(\nabla V \cdot \nabla P_{t-s} f)^2}{N-n} \right) \mathrm{d}s, \end{aligned}$$

where the second equality follows from the generalized Bochner identity and Δ is the weighted Laplacian. Therefore $\mathrm{Hess}\, f \equiv 0$, $(\nabla V \cdot \nabla f)^2 \equiv 0$. Thus $\Delta f \equiv 0$ since

$$\frac{(\Delta f)^2}{N} \leq |\,\mathrm{Hess}\, f|^2 + \frac{(\nabla V \cdot \nabla f)^2}{N-n} = 0.$$

Using a standard argument we obtain that M^n splits isometrically as $L \times \mathbb{R}$ for some Riemannian manifold L. Taking into account the fact that $\Delta f = 0$ we can prove that also the measure splits.

Furthermore, denoting by z, t the coordinates on L and $\mathbb{R}$ respectively, it holds that $P_s f(z,t) = f(z,t) = \alpha t$ for any $s \geq 0$ and for any $t \in \mathbb{R}$, for some constant $\alpha \neq 0$.

Passing to the study of the case $p = 1$, any function $f : \mathbb{R}^n \to \mathbb{R}$ such that $|\nabla P_t f| \equiv P_t |\nabla f|$ is of the form $f(z) = \varphi(z \cdot v)$ for some monotone function $\varphi : \mathbb{R} \to \mathbb{R}$ and some $v \in \mathbb{R}^n$. This is due to the commutation between gradient operator and heat flow on the Euclidean space and to the characterization of the equality case in Jensen's inequality. More in general, thanks to the tensorization property of the heat flow, it is possible to check that on any product m.m.s. $X = X' \times \mathbb{R}$, any function f depending only on the variable $t \in \mathbb{R}$ in a monotone way satisfies $|\nabla P_t f| = P_t |\nabla f|$ almost everywhere. Basically Theorem 4.3 is telling us that, in the setting of $\mathrm{RCD}(0, N)$ spaces, this is the only possible case.

Let us observe that, as the examples above show, in the rigidity case for $p = 1$ it is not necessarily true that the rigid function has vanishing Hessian. Therefore we cannot directly use $P_s f$ as a splitting function. Still our strategy relies on the properties of the normalized gradient $\nabla P_s f / |\nabla P_s f|$. First we will prove that it has vanishing symmetric covariant derivative and then that its flow lines are metric lines. The conclusion will be eventually achieved building upon the splitting Theorem 1.111.

Let us point out that in the very recent [**131**] Han has used the same approach to study the rigidity of some functional inequalities over $\mathrm{RCD}(K, \infty)$ metric measure spaces for positive K.

Let us start proving that if the rigidity condition (4.3) holds for some $s > 0$ then it must hold for any $s \geq 0$.

Lemma 4.6. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(0, N)$ metric measure space and $f \in \mathrm{Lip_b}(X)$. If there exists $s > 0$ such that*

$$|\nabla P_s f| = P_s |\nabla f| \qquad \mathfrak{m}\text{-a.e. in } X, \tag{4.4}$$

Then $|\nabla P_r f| = P_r |\nabla f|$ for any $r \geq 0$.

Proof. It is simple to check that $|\nabla P_r f| = P_r |\nabla f|$ for any $0 \leq r \leq s$. Indeed, using (4.4) and the Bakry-Émery inequality (1.52), we have

$$0 \leq P_{s-r}\left(P_r |\nabla f| - |\nabla P_r f|\right) = P_s |\nabla f| - P_{s-r} |\nabla P_r f| = |\nabla P_s f| - P_{s-r} |\nabla P_r f| \leq 0.$$

Let us now fix $\varphi \in \mathrm{Test}_c(X, \mathsf{d}, \mathfrak{m})$ and set

$$F(r) := \int_X ((P_r |\nabla f|)^2 - |\nabla P_r f|^2) \varphi \, \mathrm{d}\mathfrak{m}. \tag{4.5}$$

We claim that $F(r)$ is a real analytic function in $(0,\infty)$. Observe that the claim, together with the information $F \equiv 0$ in $(0,s)$, implies $F(r)=0$ for any $r\geq 0$ and thus our conclusion, due to the arbitrariness of the test function.
Integrating by parts the right hand side in (4.5) and relying on the validity of the heat equation, we can write

$$F(r)=\int_X (P_r|\nabla f|)^2\varphi\,\mathrm{d}\mathfrak{m}+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}r}\int_X (P_r f)^2\varphi\,\mathrm{d}\mathfrak{m}-\frac{1}{2}\int_X (P_r f)^2\Delta\varphi\,\mathrm{d}\mathfrak{m},$$

so the claim is a consequence of Lemma 4.7 below. □

Lemma 4.7. *Let $(X,\mathsf{d},\mathfrak{m})$ be an* $\mathrm{RCD}(K,N)$ *m.m.s.. For any $g\in L^\infty(X,\mathfrak{m})$ and any $\varphi\in L^1(X,\mathfrak{m})$ the map $t\mapsto \int_X (P_t g)^2\varphi\,\mathrm{d}\mathfrak{m}$ is real analytic in $(0,\infty)$.*

Proof. Exploiting a well-known analyticity criterion for real functions, it is enough to show, for any $[a,b]\subset(0,\infty)$, the existence of a constant $C=C(K,N,a,b)$ such that

$$\left|\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_X (P_t g)^2\varphi\,\mathrm{d}\mathfrak{m}\right|\leq C^n\,\|g\|^2_{L^\infty}\,\|\varphi\|_{L^1}\qquad \forall t\in(a,b),\quad \forall n\in\mathbb{N}. \tag{4.6}$$

Observe that (4.6) can be checked commuting the operators P_t and Δ and using iteratively the estimate

$$\|\Delta P_t g\|_{L^\infty}\leq C'\,\|g\|_{L^\infty}\qquad \forall t\in(a,b), \tag{4.7}$$

where $C'>0$ depends only on N, K, a and b.

Let us prove (4.7) arguing by duality. For any $\psi\in L^1\cap L^2(X,\mathfrak{m})$, we have

$$\begin{aligned}\left|\int_X \Delta P_t g\;\psi\,\mathrm{d}\mathfrak{m}\right| &=\left|\int_X \nabla P_{t/2}g\cdot\nabla P_{t/2}\psi\,\mathrm{d}\mathfrak{m}\right|\\ &\leq\left\|\nabla P_{t/2}g\right\|_{L^\infty}\left\|\nabla P_{t/2}\psi\right\|_{L^1}\\ &\leq C''\,\|g\|_{L^\infty}\;C''\,\|\psi\|_{L^1},\end{aligned}$$

where the last inequality is a consequence of the following general fact: there exists a constant $C''(N,K,a,b)>0$ such that

$$\|\nabla P_t h\|_{L^p}\leq C''\,\|h\|_{L^p}\qquad \forall t\in(a,b),\quad \forall h\in L^p(X,\mathfrak{m})\text{ with }1\leq p\leq\infty. \tag{4.8}$$

In order to check (4.8) we use the Gaussian estimates for the heat kernel and its gradient (1.79), (1.80) obtaining that there exists a constant $\alpha>1$ such that

$$|\nabla P_t h|(x)\leq C'' P_{\alpha t}\,|h|\,(x),\qquad \text{for }\mathfrak{m}\text{-a.e. }x\in X,\quad \forall t\in(a,b),$$

and we take the L^p norm at both sides. □

Let us introduce the most important object of our investigation. For any $s>0$ we consider the vector field

$$b_s:=\frac{\nabla P_s f}{P_s|\nabla f|}, \tag{4.9}$$

that, since $P_s\,|\nabla f|>0$ $\mathfrak{m}$-a.e., is well defined and satisfies

$$|b_s|=1\quad \mathfrak{m}\text{-a.e. in }X,\quad \forall s>0, \tag{4.10}$$

thanks to (4.3).

The first important ingredient of the proof of Theorem 4.3 is the following proposition. Its proof is inspired by an analogous result in [**108**].

Proposition 4.8. *For any $s>0$, $t\geq 0$ and any $g\in\mathrm{Test}(X,\mathsf{d},\mathfrak{m})$ it holds*

$$b_{t+s}\cdot\nabla P_t g=P_t(b_s\cdot\nabla g),\qquad \mathfrak{m}\text{-a.e. in }X. \tag{4.11}$$

We postpone the proof of Proposition 4.8 after the following lemma.

Lemma 4.9. *For any $s \geq 0$ the function $P_s f$ satisfies*

$$|\nabla P_{t+s} f| = P_t |\nabla P_s f|, \qquad \mathfrak{m}\text{-a.e. in } X, \quad \forall t \geq 0. \tag{4.12}$$

Proof. Using first the Bakry-Émery inequality (1.52) and then twice (4.3) we get

$$|\nabla P_{t+s} f| \leq P_t |\nabla P_s f| = P_{t+s} |\nabla f| = |\nabla P_{t+s} f|,$$

that proves our claim. □

Proof of Proposition 4.8. Let $s > 0$, $t \geq 0$ be fixed. The idea of the proof is to obtain (4.11) as the Euler equation associated to the functional

$$\Psi(h) := \int_X (P_t |\nabla h| - |\nabla P_t h|) \varphi \, \mathrm{d}\mathfrak{m} \qquad h \in \mathrm{Lip}(X),$$

where $\varphi \in \mathrm{Lip}_{\mathrm{bs}}$ is a fixed non-negative cut-off function. Indeed, thanks to Lemma 4.9 and the Bakry-Émery contraction estimate (1.52), we know that $P_s f$ is a minimum of Ψ. Thus

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}|_{\varepsilon=0} \Psi(P_s f + \varepsilon g) = 0 \qquad \forall g \in \mathrm{Test}(X, \mathsf{d}, \mathfrak{m}).$$

Notice that the differentiability of $\varepsilon \mapsto \Psi(P_s f + \varepsilon g)$ at $\varepsilon = 0$ can be easily checked using $|\nabla P_s f| = P_s |\nabla f| > 0$. Then we compute

$$\begin{aligned}
0 =& \frac{\mathrm{d}}{\mathrm{d}\varepsilon}|_{\varepsilon=0} \Psi(P_s f + \varepsilon g) \\
=& \frac{\mathrm{d}}{\mathrm{d}\varepsilon}|_{\varepsilon=0} \int_X (P_t |\nabla P_s f + \varepsilon \nabla g| - |\nabla (P_{t+s} f + \varepsilon P_t g)|) \varphi \, \mathrm{d}\mathfrak{m} \\
=& \int_X \left(P_t \left(\frac{\nabla P_s f}{|\nabla P_s f|} \cdot \nabla g \right) - \frac{\nabla P_{t+s} f}{|\nabla P_{t+s} f|} \cdot \nabla P_t g \right) \varphi \, \mathrm{d}\mathfrak{m} \\
=& \int_X (P_t(b_s \cdot \nabla g) - b_{t+s} \cdot \nabla P_t g) \varphi \, \mathrm{d}\mathfrak{m}.
\end{aligned}$$

The conclusion follows from the arbitrariness of φ. □

Proposition 4.10. *For any $s > 0$ it holds $\mathrm{div}\, b_s = 0$ and $D^{\mathrm{sym}} b_s = 0$ according to Definition 1.47.*

In particular, there exists a regular Lagrangian flow $\mathbf{X}^s : \mathbb{R} \times X \to X$ of b_s with

$$(\mathbf{X}_t^s)_\sharp \, \mathfrak{m} = \mathfrak{m}, \qquad \mathsf{d}(\mathbf{X}_t^s(x), \mathbf{X}_t^s(y)) = \mathsf{d}(x, y) \qquad \forall t \in \mathbb{R}, \quad \forall x, y \in X.$$

Proof. Let $g \in \mathrm{Test}_c(X, \mathsf{d}, \mathfrak{m})$ be fixed. Using (4.11) we obtain

$$\begin{aligned}
\left| \int_X b_s \cdot \nabla g(x) \, \mathrm{d}\mathfrak{m}(x) \right| &= \left| \int_X P_t(b_s \cdot \nabla g)(x) \, \mathrm{d}\mathfrak{m}(x) \right| \\
&= \left| \int_X b_{t+s} \cdot \nabla P_t g(x) \, \mathrm{d}\mathfrak{m}(x) \right| \\
&\leq \int_X |\nabla P_t g|(x) \, \mathrm{d}\mathfrak{m}(x).
\end{aligned}$$

To get $\mathrm{div}\, b_s = 0$ it suffices to show that

$$\lim_{t \to \infty} \int_X |\nabla P_t g|(x) \, \mathrm{d}\mathfrak{m}(x) = 0, \tag{4.13}$$

for any non-negative $g \in \mathrm{Test}_c(X, \mathsf{d}, \mathfrak{m})$. To this aim we use the Gaussian estimates for the heat kernel and its gradient (1.79), (1.80) concluding that there exist a constant $C = C(N) > 0$ and $\alpha > 1$ such that

$$|\nabla P_t g|(x) \le \frac{C}{\sqrt{t}} P_{\alpha t} g(x), \qquad \text{for } \mathfrak{m}\text{-a.e. } x \in X. \tag{4.14}$$

Let us prove that $D^{\mathrm{sym}} b_s = 0$ for any $s > 0$. First observe that, since b_s is divergence-free we have

$$\int_X b_{t+s} \cdot \nabla P_t g \; P_t g \, \mathrm{d}\mathfrak{m} = \frac{1}{2} \int_X b_{t+s} \cdot \nabla (P_t g)^2 \, \mathrm{d}\mathfrak{m} = 0, \tag{4.15}$$

for any $g \in \mathrm{Test}(X, \mathsf{d}, \mathfrak{m})$, for any $s > 0$ and $t \ge 0$. Using again (4.11) and (4.15) we deduce

$$\begin{aligned} 0 =& \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \int_X b_{t+s} \cdot \nabla P_t g \; P_t g \, \mathrm{d}\mathfrak{m} = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \int_X P_t \left(b_s \cdot \nabla g \right) P_t g \, \mathrm{d}\mathfrak{m} \\ =& \int_X \Delta (b_s \cdot \nabla g) \; g \, \mathrm{d}\mathfrak{m} + \int_X b_s \cdot \nabla g \Delta g \, \mathrm{d}\mathfrak{m} \\ =& 2 \int_X b_s \cdot \nabla g \; \Delta g \, \mathrm{d}\mathfrak{m}, \end{aligned}$$

that, by polarization, implies our claim.

The second part of the statement follows from Theorem 1.99. □

Lemma 4.11. *The vector field $b := b_s$ does not depend on $s > 0$. In particular, it holds*

$$b \cdot \nabla P_t g = P_t (b \cdot \nabla g) \qquad \mathfrak{m}\text{-}a.e., \tag{4.16}$$

for every $g \in \mathrm{Test}(X, \mathsf{d}, \mathfrak{m})$ and every $t \ge 0$.

The most important ingredient in the proof of Lemma 4.11 is the following lemma.

Lemma 4.12. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, \infty)$ m.m.s. and let $T : X \to X$ be a measure preserving isometry. Then, for any $f \in L^2(X, \mathfrak{m})$, it holds*

$$P_t (f \circ T)(x) = (P_t f) \circ T(x), \tag{4.17}$$

for any $t > 0$ and for $\mathfrak{m}$-a.e. $x \in X$.

Proof. We just provide a sketch of the proof since the result is quite standard in the field. First we observe that, since T is a measure preserving isometry, it holds that $f \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ if and only if $f \circ T \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ and in that case $\mathrm{Ch}(f \circ T) = \mathrm{Ch}(f)$. From this observation we deduce (4.17), since the heat flow is the gradient flow of the Cheeger energy in $L^2(X, \mathfrak{m})$. □

Proof of Lemma 4.11. Let $s > 0$ and let $\mathbf{X}^s$, the regular Lagrangian flow associated to b_s, be fixed.

We know from Proposition 4.10 that for any $t \in \mathbb{R}$ the flow map $\mathbf{X}^s_t$ is a measure preserving isometry of X. Therefore, for any $r \ge 0$ and any $g \in \mathrm{Test}(X, \mathsf{d}, \mathfrak{m})$, using (4.17) with $T = \mathbf{X}^s_t$ and (4.11), we get

$$\begin{aligned} (b_s \cdot \nabla P_r g) \circ \mathbf{X}^s_t =& \frac{\mathrm{d}}{\mathrm{d}t} P_r(g) \circ \mathbf{X}^s_t = \frac{\mathrm{d}}{\mathrm{d}t} P_r (g \circ \mathbf{X}^s_t) \\ =& P_r ((b_s \cdot \nabla g) \circ \mathbf{X}^s_t) = P_r (b_s \cdot \nabla g) \circ \mathbf{X}^s_t \\ =& (b_{r+s} \cdot \nabla P_r g) \circ \mathbf{X}^s_t. \end{aligned}$$

Since g is arbitrary, the first conclusion in the statement follows. The second one is a direct consequence of Proposition 4.8. □

Let us denote by $\mathbf{X}$ the regular Lagrangian flow of b from now on, choosing in particular the "good representative" of Theorem 1.99. Our next aim is to prove that for any $x \in X$ the curve $t \mapsto \mathbf{X}_t(x)$ is a line. This will yield the sought conclusion about the product structure of $(X, \mathsf{d}, \mathfrak{m})$ by the splitting Theorem 1.111.

Proposition 4.13. *For all $s > 0$ the identity*

$$P_s f(\mathbf{X}_{-t}(x)) = \min_{\overline{B}_t(x)} P_s f \tag{4.18}$$

holds true for any $t \geq 0$ and any $x \in X$.

Before then passing to the proof we wish to explain the heuristic standing behind it with a formal computation:

$$\frac{\mathrm{d}}{\mathrm{d}t} P_s f(\mathbf{X}_{-t}(x)) = -\nabla P_s f \cdot \frac{\nabla P_s f}{|\nabla P_s f|}(\mathbf{X}_{-t}(x)) = -\left|\nabla P_s f\right|(\mathbf{X}_{-t}(x)) = -\left|\nabla\left(P_s f \circ \mathbf{X}_t\right)\right|(x).$$

Therefore, setting $u(t,x) := P_s f(\mathbf{X}_{-t}(x))$, it holds that

$$\partial_t u(t,x) + |\nabla_x u(t,x)| = 0 \tag{4.19}$$

and it is well known that the Hopf-Lax semigroup[3]

$$\mathcal{Q}_t^{\infty} u_0(x) := \min_{\overline{B}_t(x)} u_0 \tag{4.20}$$

provides a solution of (4.19), and the unique viscosity solution (see [**174**]). Proposition 4.13 is just telling us that $u(t,x) = P_s f(X_{-t}(x))$ is precisely the Hopf-Lax semigroup solution. The proof is self contained but we have been strongly inspired by the analysis of the Hamilton-Jacobi equation on metric spaces pursued in [**12**, **174**].

Proof of Proposition 4.13. Let us denote by $u(t,x)$ the left hand side in (4.18). Observe that, since $\mathsf{d}(\mathbf{X}_{-t}(x), x) \leq t$, the inequality $\geq$ in (4.18) is obvious.

Now, we claim that for all $\gamma \in \mathrm{Lip}_1([0,\infty); X)$ the function $t \mapsto u(t, \gamma(t))$ is nonincreasing. In order to prove the claim, first we observe that $t \mapsto u(t,x) = P_s f(\mathbf{X}_{-t}(x))$ is of class C^1, since its derivative is $-P_s|\nabla f|(\mathbf{X}_{-t}(x))$ that is a continuous function. Indeed, the validity of this condition for $\mathfrak{m}$-a.e. $x \in X$ follows from the defining conditions of RLF and we can extend it to all $x \in X$ by continuity of the maps $(t,x) \mapsto u(t,x)$ and $(t,x) \mapsto -P_s\left|\nabla f\right|(\mathbf{X}_{-t}(x))$. Then by the Leibniz rule in [**14**, Lemma 4.3.4], it suffices to show that

$$\limsup_{h \to 0^+} \frac{|u(t, \gamma(t+h)) - u(t, \gamma(t))|}{h} \leq P_s|\nabla f|(\mathbf{X}_{-t}(\gamma(t))).$$

This inequality follows easily from Lemma 4.14 below and the inequality $|\nabla P_s f| \leq P_s|\nabla f|$, since

$$\frac{|u(t, \gamma(t+h)) - u(t, \gamma(t))|}{h} \leq \fint_t^{t+h} P_s|\nabla f|(\mathbf{X}_{-t}(\gamma(r)))\,\mathrm{d}r,$$

(here we also used that $r \mapsto \mathbf{X}_{-t}(\gamma(r))$ is 1-Lipschitz), by taking the limit as $h \downarrow 0$.

From the claim, the converse inequality in (4.18) follows easily, because for all $x \in X$ and all minimizers $\bar{x}$ of $P_s f$ in $\overline{B}_t(x)$ the geodesic property of (X, d) grants the existence of $\gamma \in \mathrm{Lip}_1([0,\infty); X)$ with $\gamma(t) = x$ and $\gamma(0) = \bar{x}$. It follows that

$$u(t,x) = u(t,\gamma(t)) \leq u(0,\gamma(0)) = u(0,\bar{x}) = P_s f(\bar{x}) = \min_{\overline{B}_t(x)} P_s f.$$

□

[3] Associated to the limit exponent $p = \infty$, cf.[**11**, Section 3].

Lemma 4.14. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, \infty)$ *m.m.s. and* $u \in \mathrm{Lip}(X)$. *Assume that* $|\nabla u|$ *has a continuous representative in* $L^\infty(X, \mathfrak{m})$. *Then*

$$|u(\gamma(t)) - u(\gamma(s))| \le \int_s^t |\nabla u|(\gamma(r)) \, |\gamma'| \, (r) \, \mathrm{d}r, \tag{4.21}$$

for any $s < t$ *and for any Lipschitz curve* $\gamma : \mathbb{R} \to X$ *(where we denoted by* $|\nabla u|$ *the continuous representative of the minimal relaxed slope of* u*).*

Proof. To get the sought conclusion we argue by regularization via heat flow as in the proof of [**16**, Theorem 6.2].
Let $\left(\mu_r^\lambda\right)_{r \in \mathbb{R}}$ be defined by $\mu_r^\lambda := (P_\lambda)^* \, \delta_{\gamma(r)}$. Contractivity yields now that

$$\begin{aligned} |P_\lambda u\,(\gamma(t)) - P_\lambda u(\gamma(s))| \le & \int_s^t \left(\int |\nabla u|^2 \, \mathrm{d}\mu_r^\lambda \right)^{\frac{1}{2}} \left| \dot{\mu}_r^\lambda \right| \mathrm{d}r \\ \le & e^{-K\lambda} \int_s^t \left(\int |\nabla u|^2 \, \mathrm{d}\mu_r^\lambda \right)^{\frac{1}{2}} |\dot{\gamma}_r| \, \mathrm{d}r \\ = & e^{-K\lambda} \int_s^t \left(P_\lambda \, |\nabla u|^2 \, (\gamma(r)) \right)^{\frac{1}{2}} |\dot{\gamma}_r| \, \mathrm{d}r, \end{aligned} \tag{4.22}$$

for any $\lambda > 0$ and for any $s, t \in \mathbb{R}$. Passing to the limit as $\lambda \downarrow 0$ both the first and the last expression in (4.22) and taking into account the continuity of u and $|\nabla u|$, we obtain (4.21). □

Corollary 4.15. *For any* $x \in X$ *the curve* $t \mapsto \mathbf{X}_t(x)$ *is a line, that is to say*

$$\mathsf{d}(\mathbf{X}_t(x), \mathbf{X}_s(x)) = |t - s| \qquad \forall s, t \in \mathbb{R}.$$

Proof. Let us start observing that any $x_t \in \overline{B}_t(x)$ such that

$$\min_{y \in \overline{B}_t(x)} P_s f(y) = P_s f(x_t)$$

has to satisfy $\mathsf{d}(x, x_t) = t$. Otherwise we might replace x_t with $\mathbf{X}_{-\varepsilon}(x_t)$ (that belongs to $B_t(x)$ for ε sufficiently small) and, since $P_s f$ is strictly increasing along the flow lines of $\mathbf{X}$, we would get a contradiction.
Furthermore $\mathbf{X}_t(x) \in \overline{B}_t(x)$ since $|b| = 1$. Thus it follows from (4.18) that $\mathsf{d}(\mathbf{X}_{-t}(x), x) = t$ for any $t \ge 0$. Using the semigroup property and the fact that $\mathbf{X}_t$ is an isometry for any $t \in \mathbb{R}$ (see Proposition 4.10) we get the sought conclusion. □

Proof of Theorem 4.3. As we anticipated the conclusion that X is isomorphic to $X' \times \mathbb{R}$ for some $\mathrm{RCD}(0, N-1)$ m.m.s. $(X', \mathsf{d}', \mathfrak{m}')$ follows from Corollary 4.15 applying Theorem 1.111.

Let us deal with the second part of the statement.
First of all we claim that all the flow lines of $\mathbf{X}$ are vertical lines in X, that is to say, denoting by $(z, s) \in X' \times \mathbb{R}$ the coordinates on X, $\mathbf{X}_t(z, s) = (z, t + s)$ for any $z \in X'$ and for any $s, t \in \mathbb{R}$. Indeed, since we proved that all integral curves of b are lines in (X, d), the construction provided by the splitting theorem shows that this is certainly true for a fixed $\bar{z} \in X'$. Let us consider any other $z \in X'$ and call $\mathbf{X}_t((z, 0)) = \left(\mathbf{X}_t^1((z, 0)), \mathbf{X}_t^2(z, 0)\right)$. Taking into account the semigroup property (1.72) and the fact that $\mathbf{X}_t$ is an isometry for any $t \in \mathbb{R}$, for any $\tau \in \mathbb{R}$ we can compute

$$\begin{aligned} \tau^2 + \mathsf{d}_Z^2(\bar{z}, z) = & \mathsf{d}^2\left(\mathbf{X}_\tau((\bar{z}, 0)), (z, 0)\right) = \mathsf{d}^2\left(\mathbf{X}_{t+\tau}((\bar{z}, 0)), \mathbf{X}_t((z, 0))\right) \\ = & \mathsf{d}^2\left((\bar{z}, t + \tau), (\mathbf{X}_t^1((z, 0)), \mathbf{X}_t^2((z, 0)))\right) \\ = & \left|(\mathbf{X}_t^2((z, 0)) - t) - \tau\right|^2 + \mathsf{d}_Z^2\left(\bar{z}, \mathbf{X}_t^1((z, 0))\right). \end{aligned}$$

Since τ is arbitrary, it easily follows that $\mathbf{X}_t^2((z,0)) = t$ for any $t \in \mathbb{R}$ and therefore $\mathbf{X}_t^1((z,0)) = z$ for any $t \in \mathbb{R}$, as we claimed.

From what we just proved it follows that $\nabla P_s f$ is trivial in the z variable and we can conclude that $P_s f$ depends only on the t-variable for any $s > 0$ thanks to the tensorization of the Cheeger energy (see Remark 1.110). Passing to the limit as $s \downarrow 0$ we obtain that the same holds true also for f.

Knowing that f depends only on the t-variable, the monotonicity in this variable can be immediately checked. □

2. Convergence and stability results for sets of finite perimeter

In this section we establish some useful compactness and stability results for sequences of sets of finite perimeter defined on a pmGH converging sequence of $\mathrm{RCD}(K,N)$ m.m. spaces. Most of the results adapt and extend to the case of our interest those of [**18**] that we partially reviewed in Section 4.2.

Until the end of this section we fix a sequence $((X_i, \mathsf{d}_i, \mathfrak{m}_i, x_i))_{i\in\mathbb{N}}$ of pointed $\mathrm{RCD}(K,N)$ m.m. spaces converging in the pmGH topology to (Y, ϱ, μ, y) and a proper metric space (Z, d_Z) where this convergence is realized.

Since in the rest of the chapter we will be mainly interested on the case of indicator functions. Let us observe that in that case we can rephrase the notion of L^1-strong convergence introduced in Definition 1.116 in the following way.

Definition 4.16. We say that a sequence of Borel sets $E_i \subset X_i$ such that $\mathfrak{m}_i(E_i) < \infty$ for any $i \in \mathbb{N}$ converges in L^1-strong to a Borel set $F \subset Y$ with $\mu(F) < \infty$ if $\chi_{E_i}\mathfrak{m}_i \rightharpoonup \chi_F \mu$ in duality with $\mathrm{C_{bs}}(Z)$ and $\mathfrak{m}_i(E_i) \to \mu(F)$.

We also say that a sequence of Borel sets $E_i \subset X_i$ converges in L^1_{loc} to a Borel set $F \subset Y$ if $E_i \cap B_R(x_i) \to F \cap B_R(y)$ in L^1-strong for any $R > 0$.

Remark 4.17. Let us remark that L^1-strong convergence implies L^1_{loc}-strong convergence as a consequence of Lemma 4.22 and the following observation:

$$\chi_{B_R(x_i)} \to \chi_{B_R(y)} \quad \text{in } L^1\text{-strong, for any } R > 0.$$

This convergence property follows from the fact that spheres have vanishing measure on $\mathrm{RCD}(K,N)$ spaces (cf. Remark 1.67).

Remark 4.18. It follows from the very definition of L^1-convergence that if a sequence of sets $E_i \to F$ in L^1 then $\chi_{E_i} \to \chi_F$ in L^2-strong.

Definition 4.19. We say that a sequence of sets with locally finite perimeter $E_i \subset X_i$ converges locally strongly in BV to a set of locally finite perimeter $F \subset Y$ if $E_i \to F$ in L^1_{loc} and $|D\chi_{E_i}| \rightharpoonup |D\chi_F|$ in duality with $\mathrm{C_{bs}}(Z)$.

Let us begin with a compactness result which adapts [**18**, Proposition 7.5] to the case of our interest (basically, we add the uniform L^∞ bound and this allows to remove the assumption on the existence of a common isoperimetric profile).

Proposition 4.20. *Let $(X_i, \mathsf{d}_i, \mathfrak{m}_i, x_i)$, (Y, ϱ, μ, y), and (Z, d_Z) be as above and fix $r > 0$. For any sequence of functions $f_i \in \mathrm{BV}(X_i, \mathfrak{m}_i)$ such that $\operatorname{supp} f_i \subset \overline{B}_r(x_i)$ for any $i \in \mathbb{N}$ and*

$$\sup_{i\in\mathbb{N}} \left\{ |Df_i|(X_i) + \|f_i\|_{L^\infty(X_i,\mathfrak{m}_i)} \right\} < \infty, \tag{4.23}$$

there exist a subsequence $i(k)$ and $f \in L^\infty(Y,\mu) \cap \mathrm{BV}(Y, \varrho, \mu)$ with $\operatorname{supp} f \subset \overline{B}_r(y)$ such that $f_{i(k)} \to f$ in L^1-strong.

Corollary 4.21. *For any sequence of Borel sets $E_i \subset X_i$ such that*

$$\sup_{i\in\mathbb{N}} |D\chi_{E_i}|\,(B_R(x_i)) < \infty \qquad \forall R > 0 \tag{4.24}$$

there exist a subsequence $i(k)$ and a Borel set $F \subset Y$ such that $E_{i(k)} \to F$ in L^1_{loc}.

We postpone the proof of Proposition 4.20 and Corollary 4.21 after a technical lemma.

Lemma 4.22. *Let $(X_i, \mathsf{d}_i, \mathfrak{m}_i, x_i)$, (Y, ϱ, μ, y), and (Z, d_Z) be as above and E_i, $\tilde{E}_i \subset X_i$ satisfy $\mathfrak{m}_i(E_i) + \mathfrak{m}_i(\tilde{E}_i) < \infty$. If $E_i \to F$ and $\tilde{E}_i \to \tilde{F}$ in L^1-strong, for some Borel sets F, $\tilde{F} \subset Y$, then $E_i \cap \tilde{E}_i \to F \cap \tilde{F}$ in L^1-strong.*

Proof. Observing that

$$\chi_{E_i \cap \tilde{E}_i} = \chi_{E_i} \cdot \chi_{\tilde{E}_i} = \frac{1}{4}\left[(\chi_{E_i} + \chi_{\tilde{E}_i})^2 - (\chi_{E_i} - \chi_{\tilde{E}_i})^2\right],$$

the conclusion follows from Proposition 1.118. □

Proof of Corollary 4.21. We claim that, possibly extracting a subsequence that we do not relabel, there exist radii $R_\ell \uparrow \infty$ as $\ell \to \infty$ with the following property

$$\sup_{i\in\mathbb{N}} \left|D\chi_{B_{R_\ell}(x_i)}\right|(X_i) < \infty \qquad \forall l \in \mathbb{N}. \tag{4.25}$$

Indeed, applying the coarea formula in the localized version of Corollary 1.41 to the functions $\mathsf{d}(x_i, \cdot)$ and recalling that $|\nabla \mathsf{d}(x_i, \cdot)|_i = 1$ $\mathfrak{m}_i$-a.e. for any i, we obtain

$$\int_0^R \left|D\chi_{B_r(x_i)}\right|(X_i)\,\mathrm{d}r = \mathfrak{m}_i(B_R(x_i)) \quad \text{for any } R > 0 \text{ and } i \in \mathbb{N}.$$

Observing that for any $R > 0$ it holds $\mathfrak{m}_i(B_R(x_i)) \to \mu(B_R(y))$, an application of Fatou's lemma yields now

$$\int_0^R \liminf_{i\to\infty} \left|D\chi_{B_r(x_i)}\right|(X_i)\,\mathrm{d}r \le \liminf_{i\to\infty} \mathfrak{m}_i(B_R(x_i)) = \mu(B_R(y)) \quad \text{for any } R > 0. \tag{4.26}$$

The claimed conclusion (4.25) can be obtained from (4.26) via a diagonal argument.
For any $\ell \in \mathbb{N}$ we can now estimate

$$\sup_{i\in\mathbb{N}} \left|D\chi_{E_i \cap B_{R_\ell}(x_i)}\right|(X) \le \sup_{i\in\mathbb{N}} |D\chi_{E_i}|\,(B_{R_\ell+1}(x_i)) + \sup_{i\in\mathbb{N}} \left|D\chi_{B_{R_\ell}(x_i)}\right|(X) < \infty,$$

thanks to the locality and subadditivity of perimeters (see [**5**, pg. 8]) for the first inequality and to (4.24), (4.25) for the second one. Thus for any $\ell \in \mathbb{N}$ we can apply Proposition 4.20 to the functions $f_i := \chi_{E_i \cap B_{R_\ell}(x_i)}$. Observing that L^1-strong limits of characteristic functions are characteristic functions by Proposition 1.118, we can use a diagonal argument together with Lemma 4.22 to recover the global limit set. □

Proof of Proposition 4.20. Let us fix $t > 0$. For any $i \in \mathbb{N}$ we write $f_i = P^i_t f_i + (f_i - P^i_t f_i)$ where, for any $i \in \mathbb{N}$, P^i_t denotes the heat semigroup on $(X_i, \mathsf{d}_i, \mathfrak{m}_i)$. Observe that, as a consequence of the regularizing estimates (1.27), it holds that

$$\sup_{i\in\mathbb{N}} \left\{\int_Z |P^i_t f_i|^2\,\mathrm{d}\mathfrak{m}_i + \mathrm{Ch}^i(P^i_t f_i)\right\} < \infty, \tag{4.27}$$

where Ch^i is the Cheeger energy on $(X_i, \mathsf{d}_i, \mathfrak{m}_i)$. Moreover, we claim that

$$\limsup_{R\to\infty} \sup_{i\in\mathbb{N}} \int_{Z\setminus B_R(x_i)} |P^i_t f_i|^2\,\mathrm{d}\mathfrak{m}_i = 0 \qquad \forall t > 0. \tag{4.28}$$

Indeed, using both the Gaussian estimates for the heat kernel in (1.79), we get

$$
\begin{aligned}
&\int_{Z\setminus B_R(x_i)} |P_t^i f_i|^2 \,\mathrm{d}\mathfrak{m}_i \\
&\qquad \le \|f_i\|_{L^\infty(X_i,\mathfrak{m}_i)} \int_{Z\setminus B_R(x_i)} P_t^i |f_i| \,\mathrm{d}\mathfrak{m}_i \\
&\qquad \le C\, \|f_i\|_{L^\infty(X_i,\mathfrak{m}_i)} \int_{Z\setminus B_R(x_i)} \int_{B_r(x_i)} \frac{e^{-\frac{\mathsf{d}^2(x,y)}{5t}+ct}}{\mathfrak{m}_i(B_{\sqrt{t}}(x))} |f_i(y)| \,\mathrm{d}\mathfrak{m}_i(y)\,\mathrm{d}\mathfrak{m}_i(x) \\
&\qquad \le C e^{-\frac{(R-r)^2}{10t}}\, \|f_i\|_{L^\infty(X_i,\mathfrak{m}_i)} \int_{Z\setminus B_R(x_i)} \int_{B_r(x_i)} \frac{e^{-\frac{\mathsf{d}^2(x,y)}{10t}+ct}}{\mathfrak{m}_i(B_{\sqrt{t}}(x))} |f_i(y)| \,\mathrm{d}\mathfrak{m}_i(y)\,\mathrm{d}\mathfrak{m}_i(x) \\
&\qquad \le C_t e^{-\frac{(R-r)^2}{10t}}\, \|f_i\|_{L^\infty(X_i,\mathfrak{m}_i)} \int_Z P^i_{\alpha t} |f_i| \,\mathrm{d}\mathfrak{m}_i \\
&\qquad \le C_t e^{-\frac{(R-r)^2}{10t}}\, \|f_i\|_{L^\infty(X_i,\mathfrak{m}_i)} \|f_i\|_{L^1(X_i,\mathfrak{m}_i)}\,,
\end{aligned}
$$

where $\alpha > 0$ is a constant depending only on K and N.

Taking into account (4.27) and (4.28), we can apply Theorem 1.124 to get that $P_t^i f_i$ admits a subsequence converging in L^1-strong. In order to conclude the proof it suffices to observe that

$$
\lim_{t\to 0^+} \sup_{i\in\mathbb{N}} \int_{X_i} |P_t^i f_i - f_i| \,\mathrm{d}\mathfrak{m}_i = 0,
$$

as it follows from the inequality

$$
\int_{X_i} |P_t^i f_i - f_i| \,\mathrm{d}\mathfrak{m}_i \le C(K,t)|Df_i|(X_i),
$$

with $C(K,t) \sim \sqrt{t}$ as $t \to 0$ (see for instance [**18**, Proposition 6.3]). □

Proposition 4.23. *Let $(X_i,\mathsf{d}_i,\mathfrak{m}_i,x_i)$ be $\mathrm{RCD}(K,N)$ m.m. spaces converging in the pmGH topology to (Y,ϱ,μ,y) and (Z,d_Z) realizing the convergence as above. Let $f_i \in \mathrm{BV}(X_i,\mathfrak{m}_i)$ converge in L^1-strong to $f \in L^1(Y,\mu)$. If $\sup_i |Df_i|(X_i) < \infty$ then $f \in \mathrm{BV}(Y,\varrho,\mu)$ and*

$$
\liminf_{i\to\infty} |Df_i|(X_i) \ge |Df|(Y). \tag{4.29}
$$

Furthermore, if

$$
\sup_{i\in\mathbb{N}} \|f_i\|_{L^\infty(X_i,\mathfrak{m}_i)} < \infty, \tag{4.30}
$$

then

$$
\liminf_{i\to\infty} \int_{X_i} g \,\mathrm{d}|Df_i| \ge \int_Y g\,\mathrm{d}|Df|, \qquad \textit{for all } g \in \mathrm{Lip}_{\mathrm{bs}}(Z) \textit{ non-negative.} \tag{4.31}
$$

Corollary 4.24. *Let $(X_i,\mathsf{d}_i,\mathfrak{m}_i,x_i)$ be $\mathrm{RCD}(K,N)$ m.m. spaces converging in the pmGH topology to (Y,ϱ,μ,y) and (Z,d_Z) realizing the convergence as above. Then, for any $f_i \in \mathrm{BV}(X_i,\mathsf{d}_i,\mathfrak{m}_i)$ convergent in energy in BV to $f \in \mathrm{BV}(Y,\varrho,\mu)$ such that $\sup_i \|f_i\|_{L^\infty(X_i,\mathfrak{m}_i)} < \infty$, it holds that $|Df_i| \rightharpoonup |Df|$ in duality with $\mathrm{C_{bs}}(Z)$.*

Proof. From (4.31) we can deduce with a standard measure theoretic argument that

$$
\liminf_{i\to\infty} |Df_i|(A) \ge |Df|(A) \qquad \forall A \subset Z \text{ open and bounded.} \tag{4.32}
$$

Let ν be any weak limit point of $|Df_i|$, in the weak topology induced by $\mathrm{C_{bs}}(Z)$, along some subsequence $i(k)$ (the sequence $|Df_i|(X_i)$ is bounded and therefore the family $\{|Df_i|\}_i$ is weakly compact). For any open and bounded set $A \subset Z$ such that $\nu(\partial A) = 0$, it holds

$\lim_k |Df_{i(k)}|(A) = \nu(A)$. Hence, taking into account also (4.32), we get $|Df|(A) \leq \nu(A)$. Thus $|Df| \leq \nu$, as measures in Z. On the other hand, since the evaluation on open sets is lower semicontinuous w.r.t. the weak convergence induced by $\mathrm{C_{bs}}(Z)$, by definition of convergence in energy in BV we have $\nu(Z) \leq \liminf_k |Df_{i(k)}|(Z) = |Df|(Z)$ and therefore $\nu = |Df|$. □

Proof of Proposition 4.23. The first part of the statement corresponds to [**18**, Theorem 6.4].

Let us deal with the second one. Fix any $t > 0$ and observe that $P_t^i f_i \to P_t f$ in $H^{1,2}$ according to Definition 1.119. Indeed, the L^1-strong convergence of f_i to f, combined with (4.30), yields that f_i converge in L^2-strong to f by Proposition 1.118. Therefore we can apply Proposition 1.122 to obtain the claimed conclusion. Hence Proposition 1.121 applies, yielding that

$$\liminf_{i\to\infty} \int_Z g|\nabla P_t^i f_i| \,\mathrm{d}\mathfrak{m}_i \geq \int_Z g|\nabla P_t f| \,\mathrm{d}\mu, \qquad \text{for all } g \in \mathrm{Lip_{bs}}(Z) \text{ non-negative.} \tag{4.33}$$

In order to prove (4.31) starting from its regularized version (4.33), we argue as in the proof of [**18**, Lemma 5.8]. Taking into account the Bakry-Émery contraction estimate $|\nabla P_t h| \leq e^{-Kt}P_t^* |Dh|$ (see (1.52)) and the estimate

$$\|P_t g - g\|_{L^\infty} \leq C(K,N,t)\mathrm{Lip}(g), \qquad \text{with } C(K,N,t) \sim \sqrt{t} \text{ as } t \to 0$$

which is available over any $\mathrm{RCD}(K,N)$ m.m.s. (and can be proved using the Gaussian estimates for the heat kernel (1.79)), we obtain

$$\begin{aligned}\liminf_{i\to\infty} \int_Z g \,\mathrm{d}\,|Df_i| \geq & \liminf_{i\to\infty} \int_Z P_t^i g \,\mathrm{d}\,|Df_i| - \limsup_{i\to\infty} \int_Z |P_t^i g - g| \,\mathrm{d}\,|Df_i| \\ \geq & e^{Kt} \liminf_{i\to\infty} \int_Z g \left|\nabla P_t^i f_i\right| \,\mathrm{d}\mathfrak{m}_i \\ & - C(K,N,t)\mathrm{Lip}(g) \limsup_{i\to\infty} |Df_i|(X_i) \\ \geq & e^{Kt} \int_Z g\,|\nabla P_t f| \,\mathrm{d}\mu - C(K,N,t)\mathrm{Lip}(g) \limsup_{i\to\infty} |Df_i|(X_i).\end{aligned} \tag{4.34}$$

The sought conclusion (4.31) can be obtained passing to the lim inf as $t \to 0$ in (4.34), recalling that $|\nabla P_t f|\,\mu \rightharpoonup |Df|$ in duality with $\mathrm{C_{bs}}(Z)$ as $t \downarrow 0$. □

The next result deals with the possibility of approximating in BV energy a set of finite perimeter in the limit space with a sequence of sets of finite perimeter defined on the approximating spaces.

Proposition 4.25. *Let $(X_i, \mathsf{d}_i, \mathfrak{m}_i, x_i)$ be $\mathrm{RCD}(K,N)$ m.m. spaces converging in the pmGH topology to (Y, ϱ, μ, y) and let (Z, d_Z) be realizing the convergence as above. Let $F \subset Y$ be a bounded set of finite perimeter. Then there exists a subsequence (i_k) and (uniformly bounded) sets of finite perimeter $E_{i_k} \subset X_{i_k}$ such that $\chi_{E_{i_k}} \to \chi_F$ in energy in* BV *as $k \to \infty$.*

Proof. Let us begin observing that the first part of [**18**, Theorem 8.1] provides existence of a sequence $(g_i) \subset \mathrm{BV}(X_i, \mathfrak{m}_i)$ strongly converging in BV to χ_F. Since by assumption $F \Subset B_R(y)$ for some $R > 0$, we can find a Lipschitz function $\eta : Z \to [0,1]$ with support contained in $B_{2R}(y)$ such that $\eta|_{B_R(y)} \equiv 1$ and it is easy to check, using the Leibniz rule, that the sequence $f_i := \eta g_i$ still converges in L^1-weak to χ_F and satisfies $|Df_i| \to |D\chi_F|$ as $i \to \infty$.

Furthermore, possibly composing with $\varphi(z) := (z \wedge 1) \vee 0$, using Proposition 1.118 and observing that $|D\varphi \circ f_i|(X_i) \leq |Df_i|(X_i)$ for any $i \in \mathbb{N}$ while $|D\varphi \circ \chi_F|(Y) = |D\chi_F|(Y)$, we can assume that $0 \leq f_i \leq 1$ for any $i \in \mathbb{N}$. In particular $\sup_{i\in\mathbb{N}} \|f_i\|_{L^\infty(X_i,\mathfrak{m}_i)} < \infty$. Therefore, Proposition 4.20 applies and we obtain that, possibly extracting a subsequence that we do not relabel, f_i converge in BV energy to χ_F.

Let us now assume, up to extract one more subsequence, that $(f_i)_\sharp(\chi_{B_{2R}(y)}\mathfrak{m}_i)$ weakly converge to some measure σ in $[0,1]$. Under this assumption, we claim that $\chi_{\{f_i>\lambda\}}$ still converge to χ_F in L^1-strong for $\mathcal{L}^1$-a.e. $\lambda \in (0,1)$.

In order to prove this claim, we fix $\lambda \in (0,1)$ that is not an atom of σ, so that

$$\lim_{\varepsilon\to 0}\lim_{i\to\infty} \mathfrak{m}_i(\{\lambda-\varepsilon < f_i \le \lambda\}) = 0. \tag{4.35}$$

From (4.35), using Proposition 1.118, it is immediate to get the L^1-strong convergence of $\chi_{\{f_i>\lambda\}}$ to χ_F: indeed, it suffices to observe that for all $\varepsilon \in (0,\lambda)$ the functions $\psi_\varepsilon \circ f_i$ still L^1-strongly converge to $\psi_\varepsilon \circ \chi_F = \chi_F$ for any ψ continuous, identically equal to 0 on $[0,\lambda-\varepsilon]$ and identically equal to 1 on $[\lambda,1]$. From the L^1-strong convergence we get, in particular,

$$\liminf_{i\to\infty} |D\chi_{\{f_i>\lambda\}}|(X_i) \ge |D\chi_F|(Y) \qquad \text{for a.e. } \lambda \in (0,1). \tag{4.36}$$

On the other hand, the coarea formula Theorem 1.40 and the strong convergence of f_i yield

$$\limsup_{i\to\infty} \int_0^1 |D\chi_{\{f_i>\lambda\}}|(X_i)\,\mathrm{d}\lambda = \limsup_{i\to\infty} |Df_i|(X_i) = |D\chi_F|(Y). \tag{4.37}$$

Thanks to Scheffè's lemma, the combination of (4.36) and (4.37) gives that $\left|D\chi_{\{f_i>\lambda\}}\right|(X_i)$ converge in $L^1(0,1)$ to the constant $|D\chi_F|(Y)$. Extracting a subsequence $(i(k))$ pointwise convergent on $(0,1)\setminus I$ with $\mathcal{L}^1(I)=0$ and setting $E_k = \{f_{i(k)} > \lambda\} \subset B_{2R}(y)$ with $\lambda \in (0,1)\setminus I$ and $\sigma(\{\lambda\})=0$, the conclusion is achieved. □

Proposition 4.26. *Let $E_i \subset X_i$ be sets of finite perimeter satisfying*

$$\sup_{i\in\mathbb{N}} |D\chi_{E_i}|(B_1(x_i)) < \infty.$$

Then there exists $F \subset Y$ of finite perimeter such that, up to extract a subsequence, $E_i \cap B_1(x_i) \to F \cap B_1(y)$ in L^1-strong and

$$\liminf_{i\to\infty} \int g \,\mathrm{d}|D\chi_{E_i}| \ge \int g\,\mathrm{d}|D\chi_F|, \tag{4.38}$$

for any $g \in C(Z)$, non-negative with $\mathrm{supp}\,(g) \subset \bar{B}_{1/2}(y)$.
If we assume also that

$$\lim_{i\to\infty} |D\chi_{E_i}|(B_{1/2}(x_i)) = |D\chi_F|(B_{1/2}(y)), \tag{4.39}$$

then (4.38) *improves to*

$$\lim_{i\to\infty}\int g\,\mathrm{d}|D\chi_{E_i}| = \int g\,\mathrm{d}|D\chi_F|, \quad \textit{for any } g\in C(Z) \textit{ with } \mathrm{supp}\,(g)\subset B_{1/2}(y). \tag{4.40}$$

Proof. The convergence $E_i \cap B_1(x_i) \to F \cap B_1(y)$ in L^1-strong up to subsequence can be obtained arguing as in the proof of Corollary 4.21.

Inequality (4.38) follows from Proposition 4.23 along with a localization argument that we sketch briefly. For any $i \in \mathbb{N}$, using Lemma 1.107 we build a good cut-off function $\eta_i \in \mathrm{Lip}(X_i,\mathsf{d}_i)$ satisfying $\eta_i = 1$ in $B_{1/2}(x_i)$ and $\eta_i = 0$ in $X_i \setminus B_{3/4}(x_i)$. By Proposition 1.29, up to extract a subsequence, we can assume that $\eta_i \to \eta_\infty \in \mathrm{Lip}(Y,\rho)$ uniformly and in L^2-strong. It is easily seen that $\eta_\infty = 1$ in $B_{1/2}(y)$ and $\eta_\infty = 0$ in $Y \setminus B_1(y)$. The sequence $(\eta_i\chi_{E_i})_i$ satisfies

$$\eta_i\chi_{E_i} \to \eta_\infty\chi_F \text{ in } L^1\text{-strong} \quad \text{and} \quad \sup_{i\in\mathbb{N}} |D(\eta_i\chi_{E_i})|(X_i) < \infty,$$

thanks to Proposition 1.118(ii) and standard calculus rules. Applying Proposition 4.23 to the sequence $(\eta_i\chi_{E_i})_i$ we get (4.38).

Inequality (4.40)) is a weak convergence result in the ball $B_{1/2}(y) \subset Z$, which can be proved arguing as in the proof of Corollary 4.24 taking into account (4.38) and (4.39). □

Let us conclude this section with a convergence result for quasi-minimal sets of finite perimeter. It will play a key role in the study of blow-ups of sets of finite perimeter we are going to perform in Section 3. The strategy of the proof is classical, see for instance [**3**, Theorem 4.8]. [4]

Proposition 4.27. *Let $(X_i, \mathsf{d}_i, \mathfrak{m}_i, x_i)$ be* $\mathrm{RCD}(K, N)$ *m.m. spaces converging in the pmGH topology to (Y, ϱ, μ, y) and let (Z, d_Z) be realizing the convergence as above. For any $i \in \mathbb{N}$, let $\lambda_i \geq 1$ and let $E_i \subset X_i$ be a set of finite perimeter satisfying the following λ_i-minimality condition: there exists $R_i > 0$ such that*

$$|D\chi_E|(B_{R_i}(x_i)) \leq \lambda_i |D\chi_{E'}|(B_{R_i}(x_i)) \qquad \forall E' \subset X_i \text{ such that } E_i\Delta E' \Subset B_{R_i}(x_i).$$

Assume that, as $i \to \infty$, $E_i \to F$ in L^1_{loc} for some set $F \subset Y$ of locally finite perimeter, $\lambda_i \to 1$ and $R_i \to \infty$. Then

(i) *F is an entire minimizer of the perimeter (relative to (Y, ϱ, μ)), namely*

$$|D\chi_F|(B_r(y)) \leq |D\chi_{F'}|(B_r(y)) \quad \text{whenever } F\Delta F' \Subset B_r(y) \Subset Y \text{ and } r > 0;$$

(ii) $|D\chi_{E_i}| \rightharpoonup |D\chi_F|$ *in duality with* $\mathrm{C_{bs}}(Z)$.

Proof. Let us fix $\bar{y} \in Y$ and let $F' \subset Y$ be a set of locally finite perimeter satisfying $F\Delta F' \Subset B_r(\bar{y})$. Let $\bar{x}_i \in X_i$ converging to $\bar{y}$ in Z and $R > 0$ be such that the following properties hold true:

$$\sup_{i\in\mathbb{N}} \left|D\chi_{B_R(x_i)}\right|(X_i) < \infty \qquad \text{and} \qquad B_r(\bar{x}_i) \Subset B_R(x_i) \qquad \forall i \in \mathbb{N}. \tag{4.41}$$

Using Proposition 4.25 we can find a sequence of sets of finite perimeter $E_i' \subset X_i$ converging to $F \cap B_R(y)$ in BV energy (note that $F \cap B_R(y)$ is a set of finite perimeter thanks to (4.41)).

Let ν be any weak limit of the sequence of measures with uniformly bounded mass $|D\chi_{E_i}|$. We claim that

$$\nu(B_s(\bar{y})) \leq |D\chi_{F'}|(B_s(\bar{y})) \ \text{ for } \mathcal{L}^1\text{-a.e. } s \in (r, , r), \text{ for some } 0 < r' < r. \tag{4.42}$$

Before proving (4.42) let us illustrate how to use it to conclude the proof. First of all, notice that (4.32) gives $\nu \geq |D\chi_F|$; if we apply (4.42) with $F' = F$ we conclude that $\nu = |D\chi_F|$ locally and then globally, achieving the conclusion (ii). The validity of (i) follows combining the identification $\nu = |D\chi_F|$ with (4.42), letting $s \uparrow r$.

Let us pass to the proof of (4.42). We first fix $0 < r' < r$ such that $F\Delta F' \subset B_{r'}(y)$. Then we fix a parameter $s \in (r', r)$ with $\nu(\partial B_s(\bar{y})) = 0$, $|D\chi_{F'}|(\partial B_s(\bar{y})) = 0$ and set

$$\tilde{E}_i^s := (E_i' \cap B_s(\overline{x}_i)) \cup (E_i \setminus B_s(\overline{x}_i)). \tag{4.43}$$

From now on, up to the end of the proof, we are going to adopt the notation $\mathrm{Per}(G, A)$ to denote $|D\chi_G|(A)$ whenever G has finite perimeter and A is a Borel set, to avoid multiple subscripts.

Using the locality of the perimeter (see [**4, 5**]) and the λ_i-minimality of E_i (notice that $R_i \geq r$ for i big enough), we get

$$\mathrm{Per}(E_i, \overline{B}_s(\bar{x}_i)) = \mathrm{Per}(E_i, B_r(\bar{x}_i)) - \mathrm{Per}(E_i, B_r(\bar{x}_i) \setminus \overline{B}_s(\bar{x}_i))$$

[4] We wish to point out that there are a couple of differences in the statement and the proof with respect to [**8**], where there is a typo in the statement and a small gap in the proof. We wish to thank Nicola Gigli and Camillo Brena for pointing this out to us.

$$
\begin{aligned}
&\le \lambda_i \mathrm{Per}(\tilde{E}_i^s, B_r(\bar{x}_i)) - \mathrm{Per}(E_i, B_r(\bar{x}_i) \setminus \overline{B}_s(\bar{x}_i)) \\
&= \lambda_i \mathrm{Per}(\tilde{E}_i^s, B_s(\bar{x}_i)) + \lambda_i \mathrm{Per}(\tilde{E}_i^s, \partial B_s(\bar{x}_i)) \\
&\quad + \lambda_i \mathrm{Per}(\tilde{E}_i^s, B_r(\bar{x}_i) \setminus \overline{B}_s(\bar{x}_i)) - \mathrm{Per}(E_i, B_r(\bar{x}_i) \setminus \overline{B}_s(\bar{x}_i)) \\
&= \lambda_i \mathrm{Per}(E_i', B_s(\bar{x}_i)) + \lambda_i \mathrm{Per}(\tilde{E}_i^s, \partial B_s(\bar{x}_i)) + (\lambda_i - 1)\mathrm{Per}(E_i, B_r(\bar{x}_i) \setminus \overline{B}_s(\bar{x}_i)).
\end{aligned}
$$

Observe that, taking the limit as $i \to \infty$, thanks to our choice of s, it holds that $(\lambda_i - 1)\mathrm{Per}(E_i, B_r(\bar{x}_i) \setminus \overline{B}_s(\bar{x}_i)) \to 0$, $\mathrm{Per}(E_i, \overline{B}_s(\bar{x}_i)) \to \nu(B_s(\bar{y}))$ and that $\lambda_i \mathrm{Per}(E_i', B_s(\bar{x}_i)) \to \mathrm{Per}(F', B_s(\bar{y}))$, since $\chi_{E_i'} \to \chi_{F' \cap B_R(y)}$ in BV energy and therefore Corollary 4.24 applies. It remains only to prove that

$$
\liminf_{i \to \infty} \mathrm{Per}(\tilde{E}_i^s, \partial B_s(\bar{x}_i)) = 0, \qquad \text{for a.e. } s \in (r', r). \tag{4.44}
$$

Applying (4.47) of Lemma 4.28 below with $f = \chi_{E_i'} - \chi_{E_i}$ we get

$$
\mathrm{Per}(\tilde{E}_i^s, X \setminus \overline{B}_s(\bar{x}_i)) \le \int_{X_i} |\chi_{E_i'} - \chi_{E_i}| \,\mathrm{d}|D\chi_{B_s(\bar{x}_i)}| + \mathrm{Per}(E_i, X \setminus \overline{B}_s(\bar{x}_i)) \quad \text{for a.e. } s \in (r', r),
$$

that, together with the strong locality of the perimeter, yields

$$
\mathrm{Per}(\tilde{E}_i^s, \partial B_s(\bar{x}_i)) \le \int_{X_i} |\chi_{E_i'} - \chi_{E_i}| \,\mathrm{d}|D\chi_{B_s(\bar{x}_i)}|, \quad \text{for } \mathcal{L}^1\text{-a.e. } s \in (r', r). \tag{4.45}
$$

Using Fatou's lemma, (4.45), the local version of the coarea formula of Corollary 1.41 and eventually Lemma 4.22 to prove that $\chi_{E_i'} - \chi_{E_i} \to \chi_F - \chi_{F'}$ in L^1-strong, we conclude that

$$
\begin{aligned}
\int_{r'}^{r} \liminf_{i \to \infty} \mathrm{Per}(\tilde{E}_i^s, \partial B_s(\bar{x}_i)) \,\mathrm{d}s &\le \liminf_{i \to \infty} \int_{r'}^{r} \mathrm{Per}(\tilde{E}_i^s, \partial B_s(\bar{x}_i)) \,\mathrm{d}s \\
&\le \liminf_{i \to \infty} \int_{r'}^{r} \int_{X_i} |\chi_{E_i'} - \chi_{E_i}| \,\mathrm{d}|D\chi_{B_s(\bar{x}_i)}| \\
&= \liminf_{i \to \infty} \int_{B_r(\bar{x}_i) \setminus B_{r'}(\bar{x}_i)} |\chi_{E_i'} - \chi_{E_i}| \,\mathrm{d}\mathfrak{m}_i = 0,
\end{aligned}
$$

therefore yielding (4.44). □

Lemma 4.28 (Leibniz rule in BV)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, \infty)$ m.m.s. and let $x \in X$. For any $f \in \mathrm{BV}(X, \mathsf{d}, \mathfrak{m}) \cap L^\infty(X, \mathfrak{m})$ and $\mathcal{L}^1$-a.e. $r \in (0, \infty)$ it holds*

$$
|D(f\chi_{B_r(x)})|(X) \le \int_X |f| \,\mathrm{d}|D\chi_{B_r(x)}| + |Df|(B_r(x)) \tag{4.46}
$$

and therefore locality gives

$$
|D(f\chi_{B_r(x)})|(X \setminus B_r(x)) \le \int_X |f| \,\mathrm{d}|D\chi_{B_r(x)}|, \quad \textit{for } \mathcal{L}^1\textit{-a.e. } r \in (0, \infty). \tag{4.47}
$$

Proof. Let us begin observing that the stated conclusion makes sense since, in view of the coarea formula Theorem 1.40, $\int |f| \,\mathrm{d}|D\chi_{B_r(x)}|$ is well defined for $\mathcal{L}^1$-a.e. $r \in (0, \infty)$.
We divide the proof into two intermediate steps. In the first one we are going to prove that (4.46) holds true under the assumption $f \in \mathrm{Lip_b}(X, \mathsf{d})$. In the second one we prove the sought inequality passing to the limit the inequalities for regularized functions that we obtained previously.

Step 1. More generally in this step we are going to prove, arguing by regularization on g, that for any $f \in \mathrm{Lip_b}(X, \mathsf{d})$ and for any non-negative function $g \in \mathrm{BV}(X, \mathsf{d}, \mathfrak{m}) \cap L^\infty(X, \mathfrak{m})$, it holds

$$
|D(fg)|(X) \le \int_X |f| \,\mathrm{d}|Dg| + \int_X |g| \,|\nabla f| \,\mathrm{d}\mathfrak{m}. \tag{4.48}
$$

Observe that, if $g \in \mathrm{Lip_b}(X, \mathsf{d})$ then (4.48) follows from the Leibniz rule. Hence, by the $L^\infty - \mathrm{Lip}$ regularization of the heat semigroup (1.49) it follows that, for any $t > 0$,

$$|D(fP_tg)|(X) \le \int_X |f|\,|\nabla P_t g| \,\mathrm{d}\mathfrak{m} + \int_X P_t g\,|\nabla f| \,\mathrm{d}\mathfrak{m}. \tag{4.49}$$

The convergence of P_tg to g in $L^1(X, \mathfrak{m})$ as $t \to 0$, the lower semicontinuity of the total variation and the Bakry-Émery contraction estimate allow us to pass to the lim inf at the left hand-side and to the limit at the right hand-side in (4.49) to get (4.48) (see also the proof of the second step for further details on the limiting procedure).

Step 2. It follows from what we just proved and from the $L^\infty - \mathrm{Lip}$ regularization property of the heat flow (1.49) that, for any $t > 0$,

$$|D(P_tf\chi_{B_r(x)})|(X) \le \int_X |P_tf| \,\mathrm{d}|D\chi_{B_r(x)}| + |DP_tf|(\overline{B}_r(x)) \quad \text{for } \mathcal{L}^1\text{-a.e. } r \in (0, \infty). \tag{4.50}$$

Next we observe that $P_tf\chi_{B_r(x)} \to f\chi_{B_r(x)}$ in $L^1(X, \mathfrak{m})$ as $t \to 0^+$ and therefore, by the lower semicontinuity of the total variation w.r.t. L^1 convergence it holds

$$|D(f\chi_{B_r(x)})|(X) \le \liminf_{t\to 0^+} |D(P_tf\chi_{B_r(x)})|(X). \tag{4.51}$$

Furthermore, the $L^1(X, \mathfrak{m})$ convergence of P_tf to f and the coarea formula Theorem 1.40 guarantee that we can find a sequence $t_i \downarrow 0$ in such a way that $P_{t_i}f$ converges to f in $L^1(X, |D\chi_{B_r(x)}|)$ for $\mathcal{L}^1$-a.e. $r \in (0, \infty)$. Eventually, let us observe that, due to the Bakry-Émery contraction estimate (1.52),

$$\limsup_{t\to 0^+} |DP_tf|\,(\overline{B}_r(x)) \le \limsup_{t\to 0^+} e^{-Kt} P_t^* \,|Df|\,(\overline{B}_r(x)) \le |Df|\,(\overline{B}_r(x)), \quad \forall r \in (0, \infty).$$

Passing to the lim inf as $t_i \downarrow 0$ at the left hand-side of (4.50) taking into account (4.51) and to the limit at the right hand-side taking into account what we observed above, we get the sought estimate (4.46). □

Remark 4.29. We wish to point out that in [**155**] Lahti proved a sharp Leibniz inequality for BV functions on PI spaces. On the one hand our result is more general since it does not rely on the doubling assumption. On the other hand our proof heavily relies on the regularity of $\mathrm{RCD}(K, \infty)$ spaces, while in [**155**] neither curvature assumptions nor infinitesimal Hilbertianity are necessary.

3. Tangents to sets of finite perimeter in $\mathrm{RCD}(K, N)$ spaces

In this section we begin the study of the structure of blow-ups of sets of finite perimeter over $\mathrm{RCD}(K, N)$ metric measure spaces. Inspired by the Euclidean theory developed by De Giorgi in the pioneering papers [**81, 82**], this can be seen as a first step in a program aimed at understanding the fine structure of sets of finite perimeter.

First we introduce a definition of tangent for sets of finite perimeter suitable for this abstract setting. The main difference with respect to the Euclidean theory is that the ambient space is not invariant under scaling: as it is natural, tangents to sets of finite perimeter are sets of finite perimeter in a tangent space to the ambient metric measure space.

Definition 4.30 (Tangents to a set of finite perimeter)**.** Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ m.m.s., $x \in X$ and let $E \subset X$ be a set of locally finite perimeter. We will denote by $\mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E)$ the collection of quintuples (Y, ϱ, μ, y, F) satisfying the following two properties:

(a) $(Y, \varrho, \mu, y) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m})$ and $r_i \downarrow 0$ are such that $(X, r_i^{-1}\mathsf{d}, \mathfrak{m}_x^{r_i}, x)$ converge to (Y, ϱ, μ, y) in the pointed measured Gromov-Hausdorff topology (cf. Definition 1.24);

(b) F is a set of locally finite perimeter in Y with $\mu(F) > 0$ and, if r_i are as in (a), then the sequence $f_i = \chi_E$ converges in L^1_{loc} to χ_F according to Definition 4.16.

It is clear that the following locality property of tangents holds:

$$\mathfrak{m}(A \cap (E\Delta F)) = 0 \qquad \Longrightarrow \qquad \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E) = \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, F) \qquad \forall x \in A, \tag{4.52}$$

whenever E, F are sets of locally finite perimeter and $A \subset X$ is open.

Theorem 4.31. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an* RCD(K,N) *m.m.s. and let $E \subset X$ be a set of locally finite perimeter. For $|D\chi_E|$-a.e. $x \in X$ the set* $\mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E)$ *is not empty and for all $(Y, \varrho, \mu, y, F) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E)$, one has*

$$|\nabla P_s \chi_F|\mu = P_s^*|D\chi_F| \qquad \forall s > 0, \tag{4.53}$$

where $P_s = P_s^Y$ is the heat semigroup relative to (Y, ϱ, μ). In particular, for all $t \geq 0$, all functions $f = P_t\chi_F$ satisfy

$$|\nabla P_s f| = P_s|\nabla f| \qquad \mu\text{-a.e. in } Y, \text{ for all } s > 0.$$

Moreover, for each $x \in X$ as above there exists a pointed m.m.s. $(Z, \mathsf{d}_Z, \mathfrak{m}_Z, \bar{z})$ such that

$$(Y, \varrho, \mu, y, F) = \Big((Z \times \mathbb{R}), \mathsf{d}_Z \times \mathsf{d}_{eucl}, \mathfrak{m}_Z \times \mathcal{L}^1, (\bar{z}, 0), \{t > 0\}\Big), \tag{4.54}$$

where we denoted by t the coordinate of the Euclidean factor in $Z \times \mathbb{R}$. Furthermore:

(i) *if $N \geq 2$ then Z is an* RCD$(0, N-1)$ *m.m.s.;*
(ii) *if $N \in [1,2)$ then Z is a point.*

A suitable version of the iterated tangent theorem, whose statement and proof are postponed to Section 4 (see in particular Theorem 4.41), implies also the following.

Theorem 4.32. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an* RCD(K,N) *m.m.s. and let $E \subset X$ be a set of locally finite perimeter. Then E admits a Euclidean half-space as tangent at x for $|D\chi_E|$-a.e. $x \in X$, that is to say*

$$\Big(\mathbb{R}^k, \mathsf{d}_{eucl}, c_k\mathcal{L}^k, 0^k, \{x_k > 0\}\Big) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E), \qquad \text{for some } k \in [1, N]. \tag{4.55}$$

Proof of Theorem 4.32. We claim that (4.55) holds true at all points $x \in X$ such that both the iterated tangent property of Theorem 4.41 and the rigidity property in Theorem 4.31 are satisfied (observe that $|D\chi_E|$-a.e. point satisfies these two properties).
Indeed, if $(Y, \varrho, \mu, y, F) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E)$, combining Theorem 4.31 with Theorem 4.3, we can say that (Y, ϱ, μ) is isomorphic to $Z \times \mathbb{R}$ for some RCD$(0, N-1)$ m.m.s. $(Z, \mathsf{d}_Z, \mathfrak{m}_Z)$. Furthermore, another consequence of Theorem 4.3 is that $F = \{t > t_0\}$ for some $t_0 \in \mathbb{R}$, where we denoted by t the coordinate on the Euclidean factor of Y. Up to a translation we can also assume that $y = (\overline{z}, 0)$ for some $\overline{z} \in Z$.
We go on observing that, if $i : Z \to Y$ denotes the canonical inclusion $i(z) := (z, 0)$, it holds $|D\chi_F| = i_\sharp \mathfrak{m}_Z$ and, for this reason, we shall identify in the sequel $|D\chi_F|$ and $\mathfrak{m}_Z$. Moreover, it is easy to check that, if $(W, \mathsf{d}_W, \mathfrak{m}_W, \bar{w}) \in \mathrm{Tan}_z(Z, \mathsf{d}_Z, \mathfrak{m}_Z)$, then

$$(W \times \mathbb{R}, \mathsf{d}_W \times \mathsf{d}_{eucl}, \mathfrak{m}_W \times \mathcal{L}^1, (\bar{w}, 0), \{t > 0\}) \in \mathrm{Tan}_{(z,0)}(Y, \varrho, \mu, F).$$

The sought conclusion can now be obtained by choosing z to be a regular point of $(Z, \mathsf{d}_Z, \mathfrak{m}_Z)$ (recall that $\mathfrak{m}_Z$-a.e. point of Z is regular by Theorem 2.18), so that W is a Euclidean space of dimension $k \in [0, N-1]$ and applying Theorem 4.41 to conclude that

$$(W \times \mathbb{R}, \mathsf{d}_W \times \mathsf{d}_{eucl}, \mathfrak{m}_W \times \mathcal{L}^1, (\bar{w}, 0), \{t > 0\}) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E).$$

□

The rest of this section is devoted to the proof Theorem 4.31. First, we are going to prove that tangents are non empty almost everywhere with respect to the perimeter measure, as a consequence of the compactness results developed in Section 2 and Proposition 4.36. Then, we will prove that they are rigid, in a suitable sense. This rigidity property will be achieved building mainly on two ingredients: lower semicontinuity and locality of the perimeter and the Bakry-Émery inequality, together with the characterization of its equality cases obtained in Section 1.

We start stating an asymptotic minimality result that stems from the lower semicontinuity of the perimeter. It has been proved in a slightly weaker form (namely with a smaller class of competitors E') first in [**4**] under Ahlfors regularity assumption and then in [**5**] for the general case. We refer to [**204**, Theorem 6.1] for the present form. The basic idea originates, up to our knowledge, in the work of Fleming [**103**] (see also [**62**, **215**] for variants of this idea in different contexts).

Proposition 4.33 (Asymptotic minimality and doubling)**.** *Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ m.m.s. and $E\subset X$ be a set of locally finite perimeter. For $|D\chi_E|$-a.e. $x\in X$ there exist $r_x>0$ and $\omega_x(r):(0,r_x)\to[0,\infty)$ such that $\omega_x(r)\to 0$ as $r\to 0^+$ and*

$$|D\chi_E|\,(B_r(x))\le(1+\omega_x(r))\,|D\chi_{E'}|\,(B_r(x)) \tag{4.56}$$

whenever $E\Delta E'\Subset B_r(x)$. In addition,

$$\limsup_{r\to 0^+}\frac{|D\chi_E|(B_{2r}(x))}{|D\chi_E|(B_r(x))}<\infty. \tag{4.57}$$

Remark 4.34. Let us emphasize that, with the terminology introduced in the preliminaries of the thesis, (4.57) means that $|D\chi_E|$ is asymptotically doubling over (X,d).

Let us also recall, since this is going to play a role in the forthcoming Chapter 5, that the codimension one measure introduced in Remark 1.8 plays a crucial role in the theory of sets of finite perimeter over PI spaces. Indeed $|D\chi_E|\,(\cdot)\ll\mathscr{H}^{h_1}$ for any set of finite perimeter E. This result has been proved by Ambrosio in [**5**, Lemma 5.2].

Lemma 4.35. *Let $(X,\mathsf{d},\mathfrak{m})$ be a PI space. For any set of locally finite perimeter $E\subset X$ it holds*

$$\mathscr{H}^{h_1}(B)=0\implies|D\chi_E|\,(B)=0\quad\textit{for any Borel set } B\subset X.$$

Also the following density estimates are important to prove that tangents are almost everywhere non empty. We refer again to [**4**, **5**] for its proof.

Proposition 4.36. *Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ m.m.s. and $E\subset X$ be a set of locally finite perimeter. For $|D\chi_E|$-a.e. $x\in X$ it holds*

$$0<\liminf_{r\to 0^+}\frac{r|D\chi_E|(B_r(x))}{\mathfrak{m}(B_r(x))}\le\limsup_{r\to 0^+}\frac{r|D\chi_E|(B_r(x))}{\mathfrak{m}(B_r(x))}<\infty, \tag{4.58}$$

and

$$\liminf_{r\to 0^+}\min\left\{\frac{\mathfrak{m}(E\cap B_r(x))}{\mathfrak{m}(B(x,r))};\frac{\mathfrak{m}(E^c\cap B_r(x))}{\mathfrak{m}(B(x,r))}\right\}>0. \tag{4.59}$$

Corollary 4.37. *Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ m.m.s. and let $E\subset X$ be a set of locally finite perimeter. Then, for $|D\chi_E|$-a.e. $x\in X$ one has $\mathrm{Tan}_x(X,\mathsf{d},\mathfrak{m},E)\neq\emptyset$. Moreover, if (Y,ϱ,μ,y,F) is as in Definition 4.30, the following properties hold true:*

(a) *F is an entire minimizer of the perimeter (relative to (Y,ϱ,τ)), i.e.*

$$|D\chi_F|(B_r(y))\le|D\chi_{F'}|(B_r(y))\qquad\textit{whenever } F\Delta F'\Subset B_r(y)\Subset Y;$$

(b) *realizing the convergence in a proper metric space* (Z, d_Z)*, the perimeters* $|D^i\chi_E|$ *relative to the rescaled spaces in condition (a) of Definition 4.30 weakly converge, in duality with* $\mathrm{C_{bs}}(Z)$*, to* $|D\chi_F|$*.*

Proof. Let us consider $x \in X$ such that the statements of Proposition 4.33 and Proposition 4.36 hold true and a sequence of radii $r_i \to 0$ such that $(X, r^{-1}\mathsf{d}, \mu_x^r, x) \to (Y, \varrho, \mu, y)$ in the pmGH topology. Thanks to (4.58) and Corollary 4.21 with $\chi_{E_i} = \chi_E$, possibly extracting a subsequence we can assume that there exists a set $F \subset Y$ with locally finite perimeter such that $\chi_E \to \chi_F$ in L^1_{loc}. Note that $\mu(F) > 0$ thanks to (4.59). This implies that $(Y, \varrho, \mu, y, F) \in \mathrm{Tan}(E, x)$. To achieve (a) and (b) it is enough to apply Proposition 4.27, recalling (4.56). □

Lemma 4.38. *Let* $(X_n, \mathsf{d}_n, \mathfrak{m}_n)$ *be* $\mathrm{RCD}(K,N)$ *m.m. spaces mGH converging to* (Y, ϱ, μ) *and assume that the convergence is realized into a proper metric space* (Z, d_Z)*. Let* η_n*,* η *be non-negative Borel measures giving finite mass to bounded sets, such that* $\operatorname{supp}\eta_n \subset \operatorname{supp}\mathfrak{m}_n$*,* $\operatorname{supp}\eta \subset \operatorname{supp}\mu$ *and* η_n *weakly converge to* η *in duality with* $\mathrm{C_{bs}}(Z)$*. Then*

$$P_t^Y \eta(x) \le \liminf_{n\to\infty} P_t^n \eta_n(x_n), \tag{4.60}$$

for any $t > 0$ *and for any* $\operatorname{supp}\mathfrak{m}_n \ni x_n \to x \in \operatorname{supp}\mu$*.*

Proof. In [**21**, Theorem 3.3], building on [**118**], it is proved that, denoting by p^n and p^Y the heat kernels of $(X_n, \mathsf{d}_n, \mathfrak{m}_n)$ and (Y, ϱ, μ) respectively, it holds

$$\lim_{n\to\infty} p_t^n(x_n, y_n, t) = p_t^Y(x, y), \quad \text{for any } t > 0, \tag{4.61}$$

whenever $\operatorname{supp}\mathfrak{m}_n \times \operatorname{supp}\mathfrak{m}_n \ni (x_n, y_n) \to (x, y) \in \operatorname{supp}\mu \times \mu$. Since

$$P_t^Y \eta(x) = \int p_t^Y(x, y)\,\mathrm{d}\eta(y) \quad \text{and} \quad P_t^n \eta_n(x_n) = \int p_t^n(x_n, y)\,\mathrm{d}\eta(y),$$

the validity of (4.60) follows from Lemma 1.3 and Fatou's lemma with the obvious choice for the weakly convergent sequence of measures and $f_n(\cdot) := p_t^n(x_n, \cdot)$, $f := p_t(x, \cdot)$, which satisfy the lower semicontinuity condition (1.2) in view of (4.61). □

Proposition 4.39. *Let* $E \subset X$ *be a set of finite perimeter and assume that* $(Y, \varrho, \mu, y, F) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E)$ *for some* $x \in X$*. Let* $r_i \downarrow 0$ *be a sequence of radii realizing the convergence in Definition 4.30. Then*

$$|\nabla^i P_t^i \chi_E| \mathfrak{m}_i \rightharpoonup |\nabla^Y P_t^Y \chi_F| \mu \qquad \textit{in duality with } \mathrm{C_{bs}}(Z), \textit{ for any } t > 0.$$

Proof. We wish to implement a strategy very similar to the one adopted in the proof of Proposition 1.122 (see [**18**, Theorem 5.4, Corollary 5.5] and [**118**]).

Let us begin proving that, for any $\operatorname{supp}\mathfrak{m}_i \ni x_i \to x \in \operatorname{supp}\mu$ and for any $t > 0$, it holds

$$\lim_{i\to\infty} P_t^i \chi_E(x_i) = P_t^Y \chi_F(x). \tag{4.62}$$

To this aim we first observe that, by the very definition of tangent, it holds that $\chi_E \mathfrak{m}_n \rightharpoonup \chi_F \mu$ in duality with $\mathrm{C_{bs}}(Z)$ and therefore Lemma 4.38 yields

$$P_t^Y \chi_F(x) \le \liminf_{i\to\infty} P_t^i \chi_E(x_i). \tag{4.63}$$

Moreover, since $(1 - \chi_E)\mathfrak{m}_n \rightharpoonup (1 - \chi_F)\mu$ in duality with $\mathrm{C_{bs}}(Z)$, applying Lemma 4.38 once more and with a simple algebraic manipulation, we obtain

$$\limsup_{i\to\infty} P_t^i \chi_E(x_i) \le P_t^Y \chi_F(x). \tag{4.64}$$

Combining (4.63) with (4.64) we obtain (4.62).

Let us proceed observing that, in view of the quantitative form of the L^∞-Lip regularization (1.49), for any $t > 0$ the functions $P^i_t \chi_E$ and $P^Y_t \chi_F$ are uniformly Lipschitz.

Fix now reference points $y \in Y$ and $X_i \ni x_i \to y$. Building upon Lemma 1.107, for any $R > 0$ it is possible to find Lipschitz cut-off functions $\eta_R : Y \to [0,1]$, $\eta^i_R : X_i \to [0,1]$ such that $\operatorname{supp} \eta_R \subset B^Y_{2R}(y)$, $\operatorname{supp} \eta^i_R \subset B^i_{2R}(x_i)$, $\eta_R|_{B^Y_R(y)} \equiv 1$, $\eta^i_R|_{B^i_R(x_i)} \equiv 1$, uniformly Lipschitz, with uniformly bounded laplacians and such that η^i_R converge to η_R both pointwise and L^2-strongly. We remark indeed that, in view of [**21**, Proposition 3.2], pointwise and L^2-strong convergence are equivalent for uniformly bounded, uniformly continuous and uniformly boundedly supported functions. Let us observe that, if we are able to prove that

$$f_i := \eta^i_R P^i_t \chi_E \to \eta_R P^Y_t \chi_F =: f \qquad \text{strongly in } H^{1,2} \text{ for all } R > 0,$$

the conclusion will follow from the locality of the minimal weak upper gradient and Theorem 1.123, which guarantees the L^1-strong convergence of $|\nabla^i(\eta^i_R P^i_t \chi_E)|^2$ to $|\nabla^Y \eta_R P^Y_t \chi_F|^2$ (that we can improve to L^1-strong convergence of $|\nabla^i(\eta^i_R P^i_t \chi_E)|$ to $|\nabla^Y \eta_R P^Y_t \chi_F|$ in view of the uniform Lipschitz bounds and of Proposition 1.118).

In order to prove the above claimed convergence, we begin observing that f_i converge pointwise to f by (4.62) and the very construction of the family of cut-off functions η^i_R. Therefore, taking into account the uniform Lipschitz bounds, the uniform boundedness and the uniform bounds on the supports, $f_i \to f$ strongly in L^2, again by [**21**, Proposition 3.2]. To improve the convergence from L^2-strong to $H^{1,2}$-strong we wish to apply Proposition 1.122. In order to do so, it remains to prove that Δf_i are uniformly bounded in L^2. To this aim we compute

$$\Delta f_i = \Delta \eta^i_R P^i_t \chi_E + 2\nabla \eta^i_R \cdot \nabla P^i_t \chi_E + \eta^i_R \Delta P^i_t \chi_E \tag{4.65}$$

and observe that all the terms at the right hand side in (4.65) are uniformly bounded in L^2 in view of the uniform L^∞ bounds on values, minimal weak upper gradients and laplacians of the cut-off functions, the uniform L^∞ and Lipschitz bounds on $P^i_t \chi_E$ and the regularizing estimate for the Laplacian under heat flow in (1.28). □

Proof of Theorem 4.31. Let us consider the case when E has finite perimeter. The generalization to sets of locally finite perimeter can be obtained building upon Lemma 4.28 and (4.52), arguing in a standard way.

Recall that the BV-version (1.52) of the 1-Bakry-Émery contraction estimate gives

$$|\nabla P_t \chi_E| \mathfrak{m} \le e^{-Kt} P^*_t |D\chi_E| \qquad \forall t > 0.$$

Let $h_t : X \to [0,1]$ be the density of $e^{Kt} |\nabla P_t \chi_E| \mathfrak{m}$ with respect to $P^*_t |D\chi_E|$. Then, one has

$$\int_X (1 - P_t h_t) \, \mathrm{d}|D\chi_E| = |D\chi_E|(X) - \int_X h_t dP^*_t |D\chi_E| = |D\chi_E|(X) - e^{Kt} \int_X |\nabla P_t \chi_E| \, \mathrm{d}\mathfrak{m}.$$

By lower semicontinuity, we get that $g_t := 1 - P_t h_t$ converges to 0 strongly in $L^1(X, |D\chi_E|)$.

Now, setting for simplicity of notation $\nu = |D\chi_E|$, we claim that

$$\lim_{t \downarrow 0} \frac{1}{\nu(B_{R\sqrt{t}}(x))} \int_{B_{R\sqrt{t}}(x)} g_t \, d\nu = 0 \qquad \forall R > 0, \text{ for } \nu\text{-a.e. } x \in X. \tag{4.66}$$

Thanks to the asymptotic doubling property (4.57), it is sufficient to prove the result ν-a.e. on a Borel set F with this property: for some $L > 0$, for all $x \in F$ and $0 < r < 1/L$ one has $\nu(B_{5r}(x)) \le L\nu(B_r(x))$. By Vitali's theorem (cf. Theorem 1.15), it follows that the localized maximal function

$$M|g|(x) := \begin{cases} \sup_{r\in(0,1/L)} \frac{\int_{B_r(x)} |g|\,\mathrm{d}\nu}{\nu(B_r(x))} & \text{if } x \in F; \\ 0 & \text{if } x \in X \setminus F; \end{cases}$$

satisfies

$$\nu(\{M|g| > \tau\}) \le \frac{L}{\tau}\int |g|\,\mathrm{d}\nu \qquad \forall \tau > 0.$$

Let us apply this estimate to the functions $g_t = 1 - P_t h_t$: given $\varepsilon > 0$, for $t < t(\varepsilon)$ one has $\int g_t\,\mathrm{d}\nu < \varepsilon^2$, and then $\nu(\{Mg_t > \varepsilon\}) \le L\varepsilon$. We obtain that

$$\int_{B_r(x)} g_t\,\mathrm{d}\nu \le \varepsilon\nu(B_r(x)) \qquad \text{for } r < \frac{1}{L},\ t < t(\varepsilon)$$

for all $x \in F_\varepsilon \subset F$, with $\mu(F \setminus F_\varepsilon)$ smaller than $L\varepsilon$. In particular, on F_ε one has

$$\limsup_{t\downarrow 0} \frac{1}{\nu(B_{R\sqrt{t}}(x))}\int_{B_{R\sqrt{t}}(x)} g_t\,d\nu \le \varepsilon \qquad \forall R > 0.$$

Since ε is arbitrary, we have proved that (4.66) holds ν-a.e. on F.

The claimed conclusion (4.53) will be achieved through two intermediate steps starting from (4.66).

First, let us observe that, for any R, s, $t > 0$ and for any $x \in X$, it holds

(4.67)

$$\frac{1}{\nu(B_{R\sqrt{t}}(x))}\int_{B_{R\sqrt{t}}(x)} g_{ts}\,\mathrm{d}\nu = \frac{1}{|D^t\chi_E|(B^t_R(x))}\int_{B^t_R(x)} P^t_s\left(1 - e^{Kt}\frac{|\nabla^t P^t_s\chi_E|}{(P^t_s)^*\,|D^t\chi_E|}\right)\mathrm{d}|D^t\chi_E|,$$

where we denoted by P^t, ∇^t, D^t and B^t the heat semigroup, the minimal weak upper gradients, the total variation measure and the balls associated to the rescaled metric measure structure $(X, \sqrt{t}^{-1}\mathsf{d}, \mathfrak{m}_x^{\sqrt{t}}, x)$ and we are identifying measures absolutely continuous w.r.t. the reference one with their densities.

Step 1. We claim that, if $(Y, \varrho, \mu, y, F) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E)$ and $t_i \downarrow 0$ is a sequence realizing the convergence in Definition 4.30, then

$$\int P_s\left(1 - \frac{|\nabla P_s\chi_F|}{P^*_s|D\chi_F|}\right)\mathrm{d}\eta_R \le \liminf_{i\to\infty}\int P^{t_i}_s\left(1 - e^{Kst_i}\frac{|\nabla^{t_i}P^{t_i}_s\chi_E|}{(P^{t_i}_s)^*|D^{t_i}\chi_E|}\right)\mathrm{d}\eta^i_R, \tag{4.68}$$

for $\mathcal{L}^1$-a.e. $R > 0$, where

$$\begin{cases} \eta_R := \frac{1}{|D\chi_F|(B_R(y))}|D\chi_F| \llcorner B_R(y), \\ \eta^i_R := \frac{1}{|D^{t_i}\chi_E|(B^{t_i}_R(x))}|D^{t_i}\chi_E| \llcorner B^{t_i}_R(x). \end{cases}$$

In order to prove (4.68), we begin observing that η^i_R weakly converges to η_R for $\mathcal{L}^1$-a.e. $R > 0$. Therefore, the validity of (4.68) will follow from Lemma 1.3 if we prove that

$$P_s\left(1 - \frac{|\nabla P_s\chi_F|}{(P_s)^*|D\chi_F|}\right)(w) \le \liminf_{i\to\infty} P^{t_i}_s\left(1 - e^{Kst_i}\frac{|\nabla^{t_i}P^{t_i}_s\chi_E|}{(P^{t_i}_s)^*|D^{t_i}\chi_E|}\right)(w_i), \tag{4.69}$$

whenever $w_i \in X_i \to w \in Y$. Let us observe that, for any $\varphi \in \mathrm{C_{bs}}(Z)$, it holds

$$\limsup_{i\to\infty} e^{Kst_i}\int \varphi\frac{|\nabla^{t_i}P^{t_i}_s\chi_E|}{(P^{t_i}_s)^*|D^{t_i}\chi_E|}\,\mathrm{d}\mathfrak{m}_i \le \int \varphi\frac{|\nabla P_s\chi_F|}{P^*_s|D\chi_F|}\,\mathrm{d}\mu. \tag{4.70}$$

Indeed, by Proposition 4.39, $|\nabla^{t_i} P_s^{t_i} \chi_E| \mathfrak{m}_i$ weakly converge to $|\nabla P_s \chi_F| \mu$ in duality with $\mathrm{C_{bs}}(Z)$, and the functions

$$f_i := \frac{\varphi}{(P_s^{t_i})^* |D^{t_i} \chi_E|} \qquad \text{and} \qquad f := \frac{\varphi}{P_s^* |D\chi_F|}$$

are continuous, have uniformly bounded supports and satisfy the upper semicontinuity property (1.1) thanks to Lemma 4.38 (recall that $|D^{t_i}\chi_E|$ weakly converge to $|D\chi_F|$ in duality with $\mathrm{C_{bs}}(Z)$). Hence (4.69) and then (4.68) follow from Lemma 1.1, taking into account also Remark 1.2.

Step 2. We can now prove (4.53). If we choose $x \in X$ such that (4.66) holds true (we proved above that $|D\chi_E|$-a.e. $x \in X$ has this property), combining (4.67) with (4.68), we obtain

$$\int_{B_R(y)} P_s \left(1 - \frac{|\nabla P_s \chi_F|}{P_s^* |D\chi_F|} \right) \mathrm{d}|D\chi_F| \le 0. \tag{4.71}$$

Observing that, by gradient contractivity on the $\mathrm{RCD}(0, N)$ space (Y, ϱ, μ), it holds

$$1 - \frac{|\nabla P_s \chi_F|}{P_s^* |D\chi_F|} \ge 0 \qquad \mu\text{-a.e. on } Y, \tag{4.72}$$

we can let $R \to \infty$ in (4.71) to get

$$\int P_s \left(1 - \frac{|\nabla P_s \chi_F|}{P_s^* |D\chi_F|} \right) \mathrm{d}|D\chi_F| = 0. \tag{4.73}$$

Then, using once more the sign property (4.72), we obtain (4.53).

Combining the just proved rigidity (4.53) with Theorem 4.3, we can say that (Y, ϱ, μ) is isomorphic to $Z \times \mathbb{R}$ for some $\mathrm{RCD}(0, N-1)$ m.m.s. $(Z, \mathsf{d}_Z, \mathfrak{m}_Z)$. Furthermore, another consequence of Theorem 4.3 is that $F = \{t > t_0\}$ for some $t_0 \in \mathbb{R}$, where we denoted by t the coordinate on the Euclidean factor of Y. Up to a translation we can also assume that $y = (\bar{z}, 0)$ for some $\bar{z} \in Z$.

□

During the proof of Theorem 4.31 we achieved the following relevant intermediate conclusion that we state separately for the sake of the forthcoming applications.

Lemma 4.40. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ space. Let $E \subset X$ be a set of finite perimeter. Then*

$$\lim_{t \searrow 0} \int \left| 1 - e^{Kt} \frac{|\nabla P_t \chi_E|}{P_t^* |D\chi_E|} \right| \mathrm{d}P_t^* |D\chi_E| = 0. \tag{4.74}$$

4. An iterated tangent theorem

In this last section we prove a version of the iterated tangent theorem by Preiss (see [**182**]) suitable for the applications to the theory of sets of finite perimeter over $\mathrm{RCD}(K, N)$ spaces. Its content is rather technical, and this is the reason for which we decided to postpone this part to the end of the chapter, even though we already made appeal to Theorem 4.41 in the proof of Theorem 4.32.

The proof is inspired by those of [**117**, Theorem 3.2] and [**24**, Theorem 6.4], dealing with pmGH tangents to $\mathrm{RCD}(K, N)$ spaces and tangents to sets of finite perimeters over Carnot groups, respectively (see also [**157**] for a previous result regarding pGH-tangents of metric spaces equipped with a doubling measure).

Theorem 4.41. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *m.m.s. and let* $E \subset X$ *be a set of finite perimeter. Then for* $|D\chi_E|$*-a.e.* $x \in X$ *the following property holds true: for every* $(Y, \varrho, \mu, y, F) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E)$ *one has*

$$\mathrm{Tan}_{y'}(Y, \varrho, \mu, F) \subset \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E) \qquad \text{for every } y' \in \operatorname{supp}|D\chi_F|.$$

Thanks to Corollary 4.37 we need only to prove the result at $|D\chi_E|$-a.e. $x \in X$ for all $(Y, \varrho, \mu, y, F) \in \mathrm{Tan}^*_x(X, \mathsf{d}, \mathfrak{m}, E)$, where $\mathrm{Tan}^*_x(X, \mathsf{d}, \mathfrak{m}, E)$ is defined adding to the conditions in Definition 4.30 the condition (b) of Corollary 4.37, namely that the perimeter measures of the rescaled spaces weakly converge, in the duality with $\mathrm{C_{bs}}(Z)$, to the perimeter measure of F.

Let us briefly recall the notion of outer measure and its main properties. Given a positive measure μ over a metric space (X, d) we set

$$\mu^*(A) := \inf\{\mu(B) : \ B \text{ Borel}, \ A \subset B\}, \qquad \forall A \subset X. \tag{4.75}$$

It is immediate to see that μ^* is countably sub-additive. Let us remark that if μ is asymptotically doubling then

$$\lim_{r\downarrow 0} \frac{\mu^*(A \cap B_r(x))}{\mu(B_r(x))} = 1 \qquad \text{for } \mu^*\text{-a.e. } x \in A. \tag{4.76}$$

Indeed, we can find a set $B \in \mathscr{B}(X)$ containing A such that $\mu(B) = \mu^*(A)$, so that $\mu^*(C \cap A) = \mu(C \cap B)$ for every $C \in \mathscr{B}(X)$. In particular, taking $C = B_r(x)$, we have

$$\lim_{r\downarrow 0} \frac{\mu^*(A \cap B_r(x))}{\mu(B_r(x))} = \lim_{r\downarrow 0} \frac{\mu(B \cap B_r(x))}{\mu(B_r(x))} = 1,$$

for every $x \in B$ of density 1 for the measure μ. Since μ is asymptotically doubling, μ-a.e $x \in B$ has this property and (4.76) follows.

Lemma 4.42. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *and let* $E \subset X$ *be as in the assumptions of Theorem 4.41. Let* $A \subset X$ *and* $x \in A$ *be such that*

$$\lim_{r\downarrow 0} \frac{|D\chi_E|^*(A \cap B_r(x))}{|D\chi_E|(B_r(x))} = 1,$$

where $|D\chi_E|^*$ *is the outer measure associated to* $|D\chi_E|$ *according to* (4.75)*. Assume that* $(Y, \varrho, \mu, F) \in \mathrm{Tan}^*_x(X, \mathsf{d}, \mathfrak{m}, E)$ *and consider*

$$\Psi_i : (X, r_i^{-1}\mathsf{d}) \to (Z, \mathsf{d}_Z) \qquad \forall i \in \mathbb{N},$$
$$\Psi : (Y, \mathsf{d}_Y) \to (Z, \mathsf{d}_Z),$$

a family of isometries realizing the pmGH convergence into a common metric space (Z, d_Z)*. Then, for any* $y' \in \operatorname{supp}|D\chi_F|$*, there exists a sequence* $(x_i) \subset A$ *such that*

$$\lim_{i\to\infty} \mathsf{d}_Z(\Psi_i(x_i), \Psi(y')) = 0.$$

Roughly speaking, Lemma 4.42 tells us that it is possible to approximate every point in the support of any tangent by means of points in A, whenever A is "large" in a measure-theoretic sense.

Proof of Lemma 4.42. As a first step we show the existence of an auxiliary sequence $(x_i) \subset X$, satisfying $\lim_i \mathsf{d}_Z(\Psi_i(x_i), \Psi(y')) = 0$ and

$$\lim_{i\to\infty} \frac{r_i|D\chi_E|(B_{rr_i}(x_i))}{C(x, r_i)} = |D\chi_F|(B_r(y')), \qquad \text{for } \mathcal{L}^1\text{-a.e. } r > 0, \tag{4.77}$$

where $C(x, r_i)$ was introduced in (1.16).

Let us set $X_i := \Psi_i(X)$, $E_i := \Psi_i(E)$ and, with a slight abuse of notation, identity F to $\Psi(F)$ and y' to $\Psi(y')$. Since by assumption it holds that $|D\chi_{E_i}| \rightharpoonup |D\chi_F|$, we have

$$\lim_{i\to\infty} |D\chi_{E_i}|(B_r^Z(y')) = |D\chi_F|(B_r^Z(y')), \qquad \text{for } \mathcal{L}^1\text{-a.e. } r > 0.$$

This implies that the distance of y' from X_i is infinitesimal as $i \to \infty$, hence we can find points $z_i \in X_i$ converging to y' in Z satisfying

$$\lim_{i\to\infty} |D\chi_{E_i}|(B_r^Z(z_i)) = |D\chi_F|(B_r^Z(y')), \qquad \text{for } \mathcal{L}^1\text{-a.e. } r > 0.$$

Let us set $x_i := \Psi_i^{-1}(z_i)$. Observe that $|D\chi_F|(B_r^Z(y')) = |D\chi_F|(B_r^Y(y'))$ and

$$|D\chi_{E_i}|(B_r^Z(z_i)) = \frac{r_i|D\chi_E|(B_{rr_i}(x_i))}{C(x, r_i)},$$

so that we get (4.77).

Let us now argue by contradiction. Assuming the conclusion of the lemma to be false we might find $\varepsilon > 0$ such that the limit in (4.77) holds with $r = \varepsilon$ and

$$B_{\varepsilon r_i}(x_i) \cap A = \emptyset \qquad \text{for } i \text{ sufficiently large,}$$

with x_i and r_i as in (4.77). Let $M > 0$ be large enough to grant that

$$B_{\varepsilon r_i}(x_i) \subset B_{Mr_i}(x) \tag{4.78}$$

(it is simple to see that such a constant exists, since the convergence in Z of $z_i = \Psi(x_i)$ ensures $\mathsf{d}(x, x_i) = O(r_i)$). Arguing as in the first part of the proof it is possible to see that

$$\lim_{i\to\infty} \frac{r_i|D\chi_E|(B_{Mr_i}(x))}{C(x, r_i)} = |D\chi_F|(B_M(y')) \qquad \text{for } \mathcal{L}^1\text{-a.e. } M > 0 \tag{4.79}$$

and from now on we assume, possibly increasing M, that both (4.78) and (4.79) hold true. Then, in view of (4.78), we have

$$\frac{|D\chi_E|^*(A \cap B_{Mr_i}(x))}{|D\chi_E|(B_{Mr_i}(x))} = \frac{|D\chi_E|^*(A \cap (B_{Mr_i}(x) \setminus B_{\varepsilon r_i}(x_i)))}{|D\chi_E|(B_{Mr_i}(x))} \le 1 - \frac{|D\chi_E|(B_{\varepsilon r_i}(x_i))}{|D\chi_E|(B_{Mr_i}(x))}.$$

Observe that the left hand side converges to 1 as $i \to \infty$, since x is of density 1 for A. Therefore, to get the sought contradiction, it suffices to show that

$$\liminf_{i\to\infty} \frac{|D\chi_E|(B_{\varepsilon r_i}(x_i))}{|D\chi_E|(B_{Mr_i}(x))} > 0.$$

Using (4.77) and (4.79), we get

$$\liminf_{i\to\infty} \frac{|D\chi_E|(B_{\varepsilon r_i}(x_i))}{|D\chi_E|(B_{Mr_i}(x))} = \frac{\lim_i \frac{r_i|D\chi_E|(B_{\varepsilon r_i}(x_i))}{C(x,r_i)}}{\lim_i \frac{r_i|D\chi_E|(B_{Mr_i}(x))}{C(x,r_i)}} \ge \frac{|D\chi_F|(B_\varepsilon(y'))}{|D\chi_F|\,(B_M(y'))} > 0,$$

where the last inequality holds true since we are assuming that $y' \in \operatorname{supp}|D\chi_F|$. □

Definition 4.43. We shall denote by $\mathcal{F}(K, N)$ the set of equivalence classes of quintuples $\mathfrak{X} = (X, \mathsf{d}, \mathfrak{m}, x, \nu)$ where $(X, \mathsf{d}, \mathfrak{m}, x)$ is a pointed $\mathrm{RCD}(K, N)$ m.m.s and ν is a non-negative and locally finite Borel measure with $\operatorname{supp}\nu \subset \operatorname{supp}\mathfrak{m}$, modulo the equivalence relation $\sim$ defined as follows. We say that $(X_1, \mathsf{d}_1, \mathfrak{m}_1, x_1, \nu_1) \sim (X_2, \mathsf{d}_2, \mathfrak{m}_2, x_2, \nu_2)$ if there exists an isometry $T : (\operatorname{supp}\mathfrak{m}_1, \mathsf{d}_1) \to (\operatorname{supp}\mathfrak{m}_2, \mathsf{d}_2)$ such that $T_\sharp\mathfrak{m}_1 = \mathfrak{m}_2$, $T(x_1) = x_2$ and $T_\sharp\nu_1 = \nu_2$. We shall denote by $\mathcal{F}$ the union of the sets $\mathcal{F}(K, N)$ for $K \in \mathbb{R}$, $1 \le N < \infty$. Observe that $\mathcal{F}$ can be realized as a countable union of sets $\mathcal{F}(K, N)$.

Let us introduce a distance in $\mathcal{F}$. Fix $\mathfrak{X}_1 = (X_1, \mathsf{d}_1, \mathfrak{m}_1, x_1, \nu_1)$, $\mathfrak{X}_2 = (X_2, \mathsf{d}_2, \mathfrak{m}_2, x_2, \nu_2)$ in $\mathcal{F}$, a proper metric measure space (Z, d_Z) and isometric embeddings $\Psi_i : (X_i, \mathsf{d}_i) \to (Z, \mathsf{d}_Z)$, $i = 1, 2$. For any integer $n \geq 1$ we define

$$\begin{aligned}
&\mathcal{D}_{n,\Psi_1,\Psi_2}(\mathfrak{X}_1, \mathfrak{X}_2) := \\
&\qquad \mathsf{d}_H(\Psi_1(X_1 \cap \overline{B}(x_1, n)), \Psi_2(X_2 \cap \overline{B}(x_2, n))) \wedge 1 \\
&\qquad + \left|\log\left(\frac{\mathfrak{m}_1(B(x_1, n))}{\mathfrak{m}_2(B(x_2, n))}\right)\right| \wedge 1 + W_1^Z\left((\Psi_1)_\sharp \frac{\chi_{B(x_1,n)}}{\mathfrak{m}_1(B(x_1, n))}\mathfrak{m}_1, (\Psi_2)_\sharp \frac{\chi_{B(x_2,n)}}{\mathfrak{m}_2(B(x_2, n))}\mathfrak{m}_2\right) \\
&\qquad + \left|\log\left(\frac{\nu_1(B(x_1, n))}{\nu_2(B(x_2, n))}\right)\right| \wedge 1 + W_1^Z\left((\Psi_1)_\sharp \frac{\chi_{B(x_1,n)}}{\nu_1(B(x_1, n))}\nu_1, (\Psi_2)_\sharp \frac{\chi_{B(x_2,n)}}{\nu_2(B(x_2, n))}\nu_2\right),
\end{aligned}$$

where d_H is the Hausdorff distance between compact subsets of Z and W_1^Z is the 1-Wasserstein distance in $(Z, \mathsf{d}_Z \wedge 1)$, namely

$$W_1^Z(\mu, \nu) := \inf\left\{\int_Z \mathsf{d}_Z(x, y) \wedge 1 \, \mathrm{d}\pi(x, y) : \ \pi \in \Gamma(\mu, \nu)\right\},$$

with $\Gamma(\mu, \nu) \subset \mathcal{P}(X \times X)$ the set of probability measures having μ and ν as marginals. We finally define

$$\mathcal{D}(\mathfrak{X}_1, \mathfrak{X}_2) := \inf_{\Psi_1,\Psi_2}\left\{\mathsf{d}_Z(\Psi_1(x_1), \Psi_2(x_2)) + \sum_{n=1}^{\infty} \frac{1}{2^n}\mathcal{D}_{n,\Psi_1,\Psi_2}(\mathfrak{X}_1, \mathfrak{X}_2)\right\}, \tag{4.80}$$

the infimum being taken among all possible proper metric spaces (Z, d_Z) and all isometric embeddings $\Psi_i : (X_i, \mathsf{d}_i) \to (Z, \mathsf{d}_Z)$ for $i = 1, 2$.

Lemma 4.44. *$\mathcal{D}$ is a distance over $\mathcal{F}$ and a sequence $(X_i, \mathsf{d}_i, \mathfrak{m}_i, x_i, \nu_i) \subset \mathcal{F}$ converges to $(Y, \varrho, \mu, y, \nu)$ in the topology induced by $\mathcal{D}$ if and only if $(X_i, \mathsf{d}_i, \mathfrak{m}_i, x_i) \to (Y, \varrho, \mu, y)$ in the pmGH topology and $\nu_i \rightharpoonup \nu$ in duality with $\mathrm{C_{bs}}(Z)$, where (Z, d_Z) is a metric space where the pmGH convergence is realized. Moreover the subspace*

$$\overline{\mathcal{F}} := \{(X, \mathsf{d}, \mathfrak{m}, x, \nu) \in \mathcal{F} : \ \nu = h\mathfrak{m}, \ \textit{with } h \in L^\infty(X, \mathfrak{m})\} \tag{4.81}$$

is separable.

Proof. The verification that $\mathcal{D}$ is a distance is quite standard, see for instance [**118**]. The equivalence between the two notions of convergence can be proved following the same strategy in the proof of [**118**, Theorem 3.15], the only difference here being the addition to the quadruple of the measure ν. Let us prove that $\overline{\mathcal{F}}$ is separable. It is enough to prove that, given K and N, for any $k > 0$ the set

$$\overline{\mathcal{F}}_k(K, N) := \{(X, \mathsf{d}, \mathfrak{m}, x, \nu) \in \mathcal{F}(K, N) : \ \nu = h\mathfrak{m}, \ \text{with } \|h\|_{L^\infty(X,\mathfrak{m})} \leq k\}$$

is compact. Let us fix a sequence $(X_i, \mathsf{d}_i, \mathfrak{m}_i, x_i, \nu_i) \subset \overline{\mathcal{F}}_k(K, N)$. We can assume, up to extract a subsequence, that $(X_i, \mathsf{d}_i, \mathfrak{m}_i, x_i) \to (Y, \varrho, \mu, y)$ in the pmGH topology. Let us fix a proper metric space (Z, d_Z) realizing this convergence. Since $\nu_i \leq k\mathfrak{m}_i$ and $\mathfrak{m}_i \rightharpoonup \mu$ in duality with $\mathrm{C_{bs}}(Z)$ we deduce that the measures ν_i are locally bounded in Z, uniformly in $i \in \mathbb{N}$. Therefore, possibly extracting a subsequence, there exists a positive measure ν in Z such that $\nu_i \rightharpoonup \nu$ in duality with $\mathrm{C_{bs}}(Z)$. It is immediate to check that $\nu \ll \mu$, with density uniformly bounded by k. This concludes the proof. □

Proof of Theorem 4.41. Since tangents are invariant w.r.t. rescaling and closed w.r.t. $\mathcal{D}$-convergence, it is enough to prove that the set of points $x \in X$ such that there exist $(Y, \varrho, \mu, y, F) \in \mathrm{Tan}_x^*(X, \mathsf{d}, \mathfrak{m}, E)$ and $y' \in \operatorname{supp}|D\chi_F|$ such that

$$(Y, \varrho, \mu^1_{y'}, y', F) \notin \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E)$$

is $|D\chi_E|^*$-negligible, where $\mu_1^{y'} := C(y',1)^{-1}\mu$ (see Definition 4.30).

Let us fix positive integers k, m and a closed subset $\mathcal{U} \subset \overline{\mathcal{F}}$ with diameter, measured w.r.t. the distance $\mathcal{D}$ in (4.80), smaller than $(2k)^{-1}$. Since, according to Lemma 4.44, $\overline{\mathcal{F}}$ is separable, it is enough to prove that

$$A_{k,m} := \{x \in X \ : \ \exists\, (Y,\varrho,\mu,y,F) \in \mathrm{Tan}^*_x(X,\mathsf{d},\mathfrak{m},E) \cap \mathcal{U} \text{ and } y' \in \operatorname{supp}|D\chi_F| \text{ such that } \mathcal{D}((Y,\varrho,\mu^1_{y'},y',F),(X,r^{-1}\mathsf{d},\mathfrak{m}^r_x,x,E)) \geq 2k^{-1} \quad \forall r \in (0,1/m)\}$$

is $|D\chi_E|^*$-negligible, where we identified the set F with the measure $\chi_F\mu$.

If, by contradiction, $|D\chi_E|^*(A_{k,m}) > 0$, then, since $|D\chi_E|$ is asymptotically doubling by Proposition 4.33, we can find $x \in A_{k,m}$ such that

$$\lim_{r\downarrow 0} \frac{|D\chi_E|^*(A_{k,m} \cap B_r(x))}{|D\chi_E|(B_r(x))} = 1,$$

see (4.76). Since $x \in A_{k,m}$ there exist $(Y,\varrho,\mu,y,F) \in \mathrm{Tan}^*_x(X,\mathsf{d},\mathfrak{m},E) \cap \mathcal{U}$ and $y' \in \operatorname{supp}|D\chi_F|$ such that $\mathcal{D}((Y,\varrho,\mu^1_{y'},y',F),(X,r^{-1}\mathsf{d},\mathfrak{m}^r_x,x,E)) \geq 2k^{-1}$ for any $r \in (0,1/m)$ and Lemma 4.42 guarantees the existence of a sequence $(x_i) \subset A_{k,m}$ such that

$$\lim_{i\to\infty} \mathsf{d}_Z(\Psi_i(x_i),\Psi(y')) = 0,$$

where Ψ_i, Ψ are the embedding maps as in Lemma 4.42. Then, by definition of pmGH convergence, using the space (Z,d_Z) we deduce

$$(X, r_i^{-1}\mathsf{d}, \mathfrak{m}^{r_i}_x, x_i) \to (Y,\varrho,\mu,y').$$

Since $\chi_{B^Z(\bar z,1)}(1-\mathsf{d}_Z(\cdot,\bar z))$ belongs to $\mathrm{C_b}(Z)$ for every $\bar z \in Z$, it is immediate to check that

$$(X, r_i^{-1}\mathsf{d}, \mathfrak{m}^{r_i}_{x_i}, x_i) \to (Y,\varrho,\mu^1_{y'},y'), \qquad \text{in the pmGH topology,}$$

and $(\Psi)_\sharp\chi_E\mathfrak{m}^{r_i}_{x_i} \rightharpoonup \Psi_\sharp\chi_F\mu^1_{y'}$ in duality with $\mathrm{C_{bs}}(Z)$, that, thanks to (4.44), is equivalent to

$$\mathcal{D}((X,r_i^{-1}\mathsf{d},\mathfrak{m}^{r_i}_{x_i},x_i,E),(Y,\varrho,\mu^1_{y'},y',F)) \to 0, \tag{4.82}$$

see Definition 4.43. Since $x_i \in A_{k,m}$ we can find $(Y_i,\varrho_i,\mu_i,y_i,F_i) \in \mathrm{Tan}^*_{x_i}(X,\mathsf{d},\mathfrak{m},E) \cap \mathcal{U}$ and $y'_i \in \operatorname{supp}|D\chi_{F_i}|$ such that $\mathcal{D}((Y_i,\varrho_i,(\mu_i)^1_{y'_i},y'_i,F_i),(X,r^{-1}\mathsf{d},\mathfrak{m}^r_{x_i},x_i,E)) \geq 2k^{-1}$ for any $r \in (0,1/m)$.

Using (4.82) and taking into account that by construction $\operatorname{diam}\mathcal{U} < (2k)^{-1}$, we find the sought contradiction

$$\begin{aligned}
2k^{-1} &\leq \mathcal{D}((Y_i,\varrho_i,(\mu_i)^1_{y'_i},y'_i,F_i),(X,r_i^{-1}\mathsf{d},\mathfrak{m}^{r_i}_{x_i},x_i,E)) \\
&\leq \mathcal{D}((Y,\varrho,\mu^1_{y'},y',F),(X,r_i^{-1}\mathsf{d},\mathfrak{m}^{r_i}_{x_i},x_i,E)) + \mathcal{D}((Y_i,\varrho_i,(\mu_i)^1_{y'_i},y'_i,F_i),(Y,\varrho,\mu^1_{y'},y',F)) \\
&\leq \mathcal{D}((Y,\varrho,\mu^1_{y'},y',F),(X,r_i^{-1}\mathsf{d},\mathfrak{m}^{r_i}_{x_i},x_i,E)) + (2k)^{-1} \\
&\leq k^{-1},
\end{aligned}$$

for i large enough. □

CHAPTER 5

Sets of finite perimeter on $\mathrm{RCD}(K,N)$ spaces: rectifiability of the reduced boundary and Gauss-Green formula

In this second chapter dedicated to the theory of sets of finite perimeter over $\mathrm{RCD}(K,N)$ metric measure spaces we present the contents of [**48**], completing the picture about the generalization of De Giorgi's theorem to this setting and giving complete proofs of Theorem 4.1 and Theorem 4.2.

One of the main results presented in Chapter 4 is the existence of a Euclidean half-space as tangent space to a set of finite perimeter at almost every point (with respect to the perimeter measure). This conclusion could be improved to a uniqueness statement in [**8**] (up to negligible sets) only in the case of a non collapsed ambient space. Therefore the state of the theory of sets of finite perimeter was at that stage comparable to that of the structure theory after [**117**], where existence of Euclidean tangent spaces almost everywhere with respect to the reference measure was proved. Uniqueness of tangents in the possibly collapsed case and rectifiability for the reduced boundary were conjectured by analogy with the Euclidean theory, but left as open questions in [**8**].

In [**48**] we gave a positive answer to these questions, providing a counterpart in codimension one of [**170**] and of De Giorgi's theorem in this setting.
Together with uniqueness of tangents (cf. Theorem 5.14) and rectifiability (cf. Theorem 5.23) we also established a representation formula for the perimeter measure in terms of the codimension one Hausdorff measure (cf. Theorem 5.31). As an intermediate tool which, however, we find to have independent interest we proved a Gauss–Green integration-by-parts formula for Sobolev vector fields (cf. Theorem 5.6).

The proof of uniqueness for blow-ups of sets of finite perimeter follows a strategy quite similar to that of the uniqueness theorem for tangents to $\mathrm{RCD}(K,N)$ spaces presented in Chapter 2. As in that case, closeness to a rigid configuration (half-space in Euclidean space) at a certain location and along a certain scale, which is what we learn from Theorem 4.32, can be turned into closeness to the same configuration at almost any location and at any scale, yielding uniqueness.
To encode the "closeness information" in analytic terms we rely on the use of harmonic δ-splitting maps (cf. Section 2). Propagation of regularity almost at every location and at any scale, which was a consequence of a maximal function argument in Chapter 2 (see in particular Proposition 2.13), this time follows from a weighted maximal function argument suitably adapted to the codimension one framework. The argument heavily relies on the interplay between the fact that the perimeter measure is a codimension one measure (cf. Lemma 4.35) and the fact that harmonic functions satisfy L^2-Hessian bounds on $\mathrm{RCD}(K,N)$ spaces.

In order to explain the strategy and the difficulties in the proof of rectifiability for the reduced boundary, let us recall how things work on $\mathbb{R}^n$. Therein a crucial role is played by the exterior normal to the set of finite perimeter, which is an almost everywhere unit valued vector field providing the representation $D\chi_E = \nu_E\,|D\chi_E|$ for the distributional derivative of the set of finite perimeter E. In the Euclidean case, relying on the properties of the exterior normal one can obtain a characterization of blow-ups in a much simpler way than in [**8**] and even get rectifiability of the boundary, proving that sets where the unit normal is not oscillating too much are bi-Lipschitz to subsets of $\mathbb{R}^{n-1}$.

When trying to reproduce the Euclidean approach in the *non smooth* and *non flat* realm of RCD spaces, one faces two main difficulties. The first one due to the fact that the theory of tangent modules, as developed in [**112**], allows to talk about vector fields only up to negligible sets with respect to the reference measure (as the reduced boundary of a set of finite perimeter is not). The second one is that controlling the behaviour of the normal vector cannot be enough to control the behaviour of the set in this framework, since the space itself might "oscillate". This is a common feature of geometry on metric measure spaces (see also the introduction of [**64**]), which can be understood looking at the following example: let $(X, \mathsf{d}, \mathfrak{m})$ be any $\mathrm{RCD}(K, N)$ m.m.s. and take its product with the Euclidean line. Then consider the "generalized half-space" $\{t < 0\}$, where t denotes the coordinate along the line: it is easily seen that it is a set of locally finite perimeter and one can identify its reduced boundary with X. Moreover, whatever notion of unit normal we have in mind, this will be non oscillating in this case. Still, rectifiability of $(X, \mathsf{d}, \mathfrak{m})$ is highly non trivial and requires [**170**] to be achieved.

To handle the first difficulty we mentioned above, we rely on [**88**], where a notion of cotangent module with respect to the 2-capacity has been recently introduced and studied. Building upon the fact that the 2-capacity controls the perimeter measure in great generality, we introduce the notion of tangent module over the boundary of a set of finite perimeter (cf. Theorem 5.4).
Furthermore we prove that there is a well-defined unit normal to a set of finite perimeter as an element of this module, that it satisfies the Gauss–Green integration-by-parts formula and, relying on functional analysis tools, that it can be approximated by regular vector fields (cf. Theorem 5.6 for a rigorous statement).

The results obtained in the study of the unit normal are then combined in a new way with the theory of δ-splitting maps to prove rectifiability of the reduced boundary for sets of finite perimeter. We first introduce a notion of δ-orthogonality to the unit normal for δ-splitting maps. Then we prove on the one hand that δ-splitting maps δ-orthogonal to the unit normal control both the geometry of the space and that of the boundary of the set of finite perimeter (and vice-versa). On the other hand the combination of δ-orthogonality and δ-splitting is seen to be suitable for propagation at many locations and any scale with maximal function arguments (cf. Proposition 5.25 and Proposition 5.27).

This chapter is organised as follows: we dedicate Section 1 to the construction of the tangent module over the boundary of a set of finite perimeter and to establishing a Gauss–Green integration-by-parts formula. Uniqueness of blow-ups is the main outcome of Section 2, while rectifiability for the reduced boundary is obtained in Section 3. Eventually in Section 4 we obtain representation formulas for the perimeter in terms of codimension one Hausdorff (type) measures.

With respect to the presentation in [**48**] the main difference is in one of the steps of the proof of the rectifiability for the reduced boundary. In the approach adopted here there is a simplification with respect to the original one that we obtain relying on [**148**], where the study of quasi continuous representatives of Sobolev functions via Lebesgue points was pursued.

1. A Gauss-Green formula on RCD spaces

This section is dedicated to the construction of a module of vector fields defined almost everywhere with respect to the perimeter of a set of finite perimeter, to the establishment of some of its properties and to the proof of a Gauss Green integration by parts formula tailored for this setting.

Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ m.m. space and $E \subset X$ a set of finite perimeter. We recall that, by Lemma 4.35, one has $|D\chi_E| \ll \mathscr{H}^{h_1}$, so accordingly $|D\chi_E| \ll \mathrm{Cap}$ by Theorem 1.34. It thus makes sense to consider the projection $\pi_{|D\chi_E|} : L^0(\mathrm{Cap}) \to L^0(|D\chi_E|)$. Recall also that $\mathsf{QCR} : H^{1,2}(X) \to L^0(\mathrm{Cap})$ stands for the "quasi-continuous representative" operator. Then let us define

$$\mathrm{tr}_E : H^{1,2}(X) \to L^0(|D\chi_E|), \quad \mathrm{tr}_E := \pi_{|D\chi_E|} \circ \mathsf{QCR},$$

the trace operator over the boundary of E. Observe that $\mathrm{tr}_E(f) \in L^\infty(|D\chi_E|)$ holds for every test function $f \in \mathrm{Test}(X)$.

Remark 5.1. Let us point out that when $(X, \mathsf{d}, \mathfrak{m})$ is the Euclidean space of dimension n and $E \subset \mathbb{R}^n$ is open and smooth $\mathrm{tr}_E : H^1(\mathbb{R}^n) \to L^0(|D\chi_E|)$ coincides with the canonical trace operator. Indeed the two operators coincide on smooth functions and they are continuous. In the case of the canonical trace this is a standard result, while for tr_E this is a consequence of [**88**, Proposition 1.19] and the continuity of $\pi_{|D\chi_E|} : L^0(\mathrm{Cap}) \to L^0(|D\chi_E|)$.

Due to the finite dimensionality assumption, that is not in force in [**88**], we will also rely on the characterization of the quasi-continuous representative of a Sobolev function via Lebesgue points obtained for doubling metric measure spaces in [**148**] that we quote below in the simplified form we will need.

Theorem 5.2. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a doubling metric measure space. Let $u \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$. Then, for* Cap*-a.e. $x \in X$ the following limit exists*

$$u^*(x) := \lim_{r \to 0} \fint_{B_r(x)} u \, \mathrm{d}\mathfrak{m}. \tag{5.1}$$

Moreover, u^* is quasi continuous and for Cap*-a.e. $x \in X$ it holds*

$$\lim_{r \to 0} \fint_{B_r(x)} |u - u^*(x)|^2 \, \mathrm{d}\mathfrak{m} = 0. \tag{5.2}$$

Recall that in [**88**] a calculus for vector fields defined Cap-almost everywhere was developed on RCD spaces. Given the absolute continuity of the perimeter measure with respect to the capacity, we will introduce the module of vector fields defined almost everywhere with respect to the perimeter as a quotient of the capacitary module, roughly speaking. Before doing that we establish some auxiliary results.

Fix any Radon measure μ on a m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ and suppose that $\mu \ll \mathrm{Cap}$. Then there is a natural projection $\pi_\mu : L^0(\mathrm{Cap}) \to L^0(\mu)$. Given an $L^0(\mathrm{Cap})$-normed $L^0(\mathrm{Cap})$-module $\mathscr{M}_{\mathrm{Cap}}$, we define an equivalence relation $\sim_\mu$ on $\mathscr{M}_{\mathrm{Cap}}$ as follows: given any $v, w \in \mathscr{M}_{\mathrm{Cap}}$, we declare that

$$v \sim_\mu w \quad \Longleftrightarrow \quad |v - w| = 0 \text{ holds } \mu\text{-a.e. on } X.$$

Then the quotient $\mathscr{M}_\mu^0 := \mathscr{M}_{\mathrm{Cap}}/\sim_\mu$ inherits a natural structure of $L^0(\mu)$-normed $L^0(\mu)$-module. Call $\bar{\pi}_\mu : \mathscr{M}_{\mathrm{Cap}} \to \mathscr{M}_\mu^0$ the canonical projection. Moreover, for any exponent $p \in [1, \infty)$ we define

$$\mathscr{M}_\mu^p := \{v \in \mathscr{M}_\mu^0 \mid |v| \in L^p(\mu)\}. \tag{5.3}$$

It turns out that $\mathscr{M}_\mu^p$ is an $L^p(\mu)$-normed $L^\infty(\mu)$-module. Notice that $|\bar{\pi}_\mu(v)| = \pi_\mu(|v|)$ holds in the μ-a.e. sense for every $v \in \mathscr{M}_{\mathrm{Cap}}$.

Lemma 5.3. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be a m.m.s.,* $\mathscr{M}_{\mathrm{Cap}}$ *an* $L^0(\mathrm{Cap})$*-normed* $L^0(\mathrm{Cap})$*-module. Fix a finite Borel measure* $\mu \geq 0$ *on* X *such that* $\mu \ll \mathrm{Cap}$*. Let* V *be a linear subspace of* $\mathscr{M}_{\mathrm{Cap}}$ *such that* $|v|$ *admits a bounded* Cap*-a.e. representative for every* $v \in V$ *and*

$$\mathcal{V} := \left\{ \sum_{n\in\mathbb{N}} \chi_{E_n} v_n \;\middle|\; (E_n)_{n\in\mathbb{N}} \text{ Borel partition of } X,\ (v_n)_{n\in\mathbb{N}} \subset V \right\}$$

is dense in $\mathscr{M}_{\mathrm{Cap}}$*. Then for any* $p \in [1, \infty)$ *it holds that*

$$\mathcal{W} := \left\{ \sum_{i=1}^{n} \chi_{E_i} \bar{\pi}_\mu(v_i) \;\middle|\; n \in \mathbb{N},\ (E_i)_{i=1}^n \text{ Borel partition of } X,\ (v_i)_{i=1}^n \subset V \right\}$$

is dense in $\mathscr{M}^p_\mu$*.*

Proof. Fix $v \in \mathscr{M}^p_\mu$ and $\varepsilon > 0$. Since $|v|^p \in L^1(\mu)$, there is $\delta > 0$ such that $\left(\int_E |v|^p \, \mathrm{d}\mu \right)^{1/p} \leq \varepsilon/3$ holds for any Borel set $E \subset X$ with $\mu(E) < \delta$. Choose any $\bar{v} \in \mathscr{M}_{\mathrm{Cap}}$ such that $\bar{\pi}_\mu(\bar{v}) = v$. We can find $(\bar{v}_k)_k \subset \mathcal{V}$ so that $|\bar{v}_k - \bar{v}| \to 0$ in $L^0(\mathrm{Cap})$. Hence $\left|\bar{\pi}_\mu(\bar{v}_k) - \bar{\pi}_\mu(\bar{v})\right| = \pi_\mu(|\bar{v}_k - \bar{v}|) \to 0$ in $L^0(\mu)$. Thanks to Egorov theorem, there exists a compact set $K \subset X$ with $\mu(X \setminus K) < \delta$ such that (possibly taking a not relabeled subsequence) it holds that $\left|\bar{\pi}_\mu(\bar{v}_k) - v\right| \to 0$ uniformly on K. Consequently, by dominated convergence theorem we see that $\chi_K \bar{\pi}_\mu(\bar{v}_k) \to \chi_K v$ in $\mathscr{M}^p_\mu$. Then we can pick $k \in \mathbb{N}$ so that the element $\bar{w} := \bar{v}_k$ satisfies $\left\|\chi_K \bar{\pi}_\mu(\bar{w}) - \chi_K v\right\|_{\mathscr{M}^p_\mu} \leq \varepsilon/3$. If $\bar{w}$ is written as $\sum_{n\in\mathbb{N}} \chi_{E_n} \bar{w}_n$, then we have $\chi_K \bar{\pi}_\mu(\bar{w}) = \sum_{n\in\mathbb{N}} \chi_{K\cap E_n} \bar{\pi}_\mu(\bar{w}_n)$. By dominated convergence theorem we know that for $N \in \mathbb{N}$ sufficiently big the element $z := \sum_{n=1}^N \chi_{K\cap E_n} \bar{\pi}_\mu(\bar{w}_n) \in \mathcal{W}$ satisfies $\left\|z - \chi_K \bar{\pi}_\mu(\bar{w})\right\|_{\mathscr{M}^p_\mu} \leq \varepsilon/3$. Therefore, we conclude that

$$\|z - v\|_{\mathscr{M}^p_\mu} \leq \left\|z - \chi_K \bar{\pi}_\mu(\bar{w})\right\|_{\mathscr{M}^p_\mu} + \left\|\chi_K \bar{\pi}_\mu(\bar{w}) - \chi_K v\right\|_{\mathscr{M}^p_\mu} + \|\chi_{X\setminus K} v\|_{\mathscr{M}^p_\mu} \leq \varepsilon,$$

thus proving the statement. □

Let us state the two main results of this section. The first one gives existence and uniqueness of the tangent module over the boundary of a set of finite perimeter. The second theorem provides a Gauss–Green formula tailored for finite-dimensional RCD spaces along with a strong approximation result for the exterior normal of sets with finite perimeter. This approximation result, whose proof heavily relies on the abstract machinery of normed modules and on functional-analytic tools, plays a key role in the study of rectifiability properties for boundaries of sets with finite perimeter.

Theorem 5.4 (Tangent module over ∂E)**.** *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *space. Let* $E \subset X$ *be a set of finite perimeter. Then there exists a unique couple* $\left(L^2_E(TX), \bar{\nabla}\right)$ *– where* $L^2_E(TX)$ *is an* $L^2(|D\chi_E|)$*-normed* $L^\infty(|D\chi_E|)$*-module and* $\bar{\nabla} : \mathrm{Test}(X) \to L^2_E(TX)$ *is linear – such that:*

- i) *The equality* $|\bar{\nabla} f| = \mathrm{tr}_E(|\nabla f|)$ *holds* $|D\chi_E|$*-a.e. for every* $f \in \mathrm{Test}(X)$*.*
- ii) $\left\{ \sum_{i=1}^n \chi_{E_i} \bar{\nabla} f_i \;\middle|\; (E_i)_{i=1}^n \text{ Borel partition of } X,\ (f_i)_{i=1}^n \subset \mathrm{Test}(X) \right\}$ *is a dense subset of* $L^2_E(TX)$*.*

Uniqueness is intended up to unique isomorphism: given another couple $(\mathscr{M}, \bar{\nabla}')$ *satisfying* i), ii) *above, there exists a unique normed module isomorphism* $\Phi : L^2_E(TX) \to \mathscr{M}$ *such that* $\Phi \circ \bar{\nabla} = \bar{\nabla}'$*. The space* $L^2_E(TX)$ *is called* tangent module over the boundary of E *and* $\bar{\nabla}$ *is the* gradient*.*

We denote by $\bar{\mathsf{QCR}} : H^{1,2}_C(TX) \to L^0_{\mathrm{Cap}}(TX)$ the “quasi-continuous representative” map for Sobolev vector fields, whose existence has been proven in [**88**, Theorem 2.14] (see [**88**, Definition 2.12] for a notion of "quasi-continuous vector field" suitable for this context). Moreover we let

$$\mathrm{tr}_E : H^{1,2}_C(TX) \cap L^\infty(TX) \to L^2_E(TX), \quad \mathrm{tr}_E := \bar{\pi}_{|D\chi_E|} \circ \bar{\mathsf{QCR}}$$

and notice that $|\mathrm{tr}_E(v)| = \mathrm{tr}_E(|v|)$ holds in the $|D\chi_E|$-a.e. sense for every $v \in H^{1,2}_C(TX) \cap L^\infty(TX)$.

Remark 5.5. Arguing as in Remark 5.1 one can prove that the above defined operator tr_E coincides with the canonical trace in the case of smooth domains in $\mathbb{R}^n$.

Theorem 5.6 (Gauss–Green formula on RCD spaces)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be an* $\mathrm{RCD}(K, N)$ *space and $E \subset X$ be a set of finite perimeter such that $\mathfrak{m}(E) < \infty$. Then there exists a unique vector field $\nu_E \in L^2_E(TX)$ such that $|\nu_E| = 1$ holds $|D\chi_E|$-a.e. and*

$$\int_E \mathrm{div}\,(v)\, \mathrm{d}\mathfrak{m} = -\int \langle \mathrm{tr}_E(v), \nu_E\rangle\, \mathrm{d}|D\chi_E|, \tag{5.4}$$

for all $v \in H^{1,2}_C(TX) \cap D(\mathrm{div}\,)$ with $|v| \in L^\infty(\mathfrak{m})$. Moreover, there exists a sequence $(v_n)_n \subset$ $\mathrm{TestV}_E(X)$ *of test vector fields over the boundary of E (see Lemma 5.10 below for the precise definition of this class) such that $v_n \to \nu_E$ in the strong topology of $L^2_E(TX)$.*

Remark 5.7. In the case in which X is a Riemannian manifold and $E \subset X$ is a domain with smooth boundary, it holds that $L^2_E(TX)$ is the space of all Borel vector fields over X which are concentrated on the boundary of E and 2-integrable with respect to the surface measure and, in this case, $\bar{\nabla}$ is the classical gradient for smooth functions.

Remark 5.8. The tangent $L^0(\mathrm{Cap})$-module $L^0_{\mathrm{Cap}}(TX)$ is a Hilbert module; cf. [**88**, Proposition 2.8]. Therefore, it is immediate to see by passing to the quotient that $L^2_E(TX)$ is a Hilbert module as well.

The remaining part of this section is dedicated to the proofs of Theorem 5.4 and Theorem 5.6.

Proof of Theorem 5.4. UNIQUENESS. Call $\mathcal{W}$ the family of elements of $L^2_E(TX)$ considered in item ii). Given any $\omega = \sum_{i=1}^n \chi_{E_i} \bar{\nabla} f_i \in \mathcal{W}$, we are forced to set $\Phi(\omega) := \sum_{i=1}^n \chi_{E_i} \bar{\nabla}' f_i$. Well-posedness of such definition stems from the $|D\chi_E|$-a.e. identity

$$\Big|\sum_{i=1}^n \chi_{E_i} \bar{\nabla}' f_i\Big| = \sum_{i=1}^n \chi_{E_i} |\bar{\nabla}' f_i| = \sum_{i=1}^n \chi_{E_i} \mathrm{tr}_E(|\nabla f_i|) = \sum_{i=1}^n \chi_{E_i} |\bar{\nabla} f_i| = |\omega|,$$

which also shows that Φ preserves the pointwise norm. Then Φ is linear continuous, thus it can be uniquely extended to a linear continuous map $\Phi : L^2_E(TX) \to \mathscr{M}$ by density of $\mathcal{W}$ in $L^2_E(TX)$. By an approximation argument, it is easy to see that the extended Φ preserves the pointwise norm and it is an $L^\infty(|D\chi_E|)$-module morphism. Finally, the map Φ is surjective, because its image is dense (as $\mathscr{M}$ satisfies ii)) and closed (as Φ is an isometry). Consequently, we have proved that there exists a unique normed module isomorphism $\Phi : L^2_E(TX) \to \mathscr{M}$ such that $\Phi \circ \bar{\nabla} = \bar{\nabla}'$.

EXISTENCE. Let us consider the tangent $L^0(\mathrm{Cap})$-module $L^0_{\mathrm{Cap}}(TX)$ and the relative capacitary gradient operator $\tilde{\nabla} : \mathrm{Test}(X) \to L^0_{\mathrm{Cap}}(TX)$ associated to the space $(X, \mathsf{d}, \mathfrak{m})$; cf. Theorem 1.89. We define $L^0_E(TX)$ as $L^0_{\mathrm{Cap}}(TX)/\sim_{|D\chi_E|}$ and the $L^2(|D\chi_E|)$-normed $L^\infty(|D\chi_E|)$-module $L^2_E(TX)$ as in (5.3). Moreover, we define the differential $\bar{\nabla} : \mathrm{Test}(X) \to L^2_E(TX)$ as $\bar{\nabla} := \bar{\pi}_{|D\chi_E|} \circ \tilde{\nabla}$. Clearly, the map $\bar{\nabla}$ is linear by construction. Given any function $f \in \mathrm{Test}(X)$, it $|D\chi_E|$-a.e. holds

$$|\bar{\nabla} f| = |\bar{\pi}_{|D\chi_E|}(\tilde{\nabla} f)| = \pi_{|D\chi_E|}(|\tilde{\nabla} f|) = \pi_{|D\chi_E|}\big(\mathsf{QCR}(|\nabla f|)\big) = \mathrm{tr}_E(|\nabla f|),$$

which shows that i) is satisfied. We also set $V := \mathrm{Test}(X)$ and the associated space $\mathcal{V} \subset L^0_{\mathrm{Cap}}(TX)$ as in the statement of Lemma 5.3. By the defining property of the cotangent Cap-module we know that $\mathcal{V}$ is dense in $L^0_{\mathrm{Cap}}(TX)$, whence Lemma 5.3 ensures that $\mathcal{W}$ is dense in $L^2_E(TX)$. This means that property ii) holds. Therefore, the existence part of the statement is proven. □

Lemma 5.9. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ space. Let $E \subset X$ be a set of finite perimeter. Then*

$$\int f\, P_t^*|D\chi_E|\, \mathrm{d}\mathfrak{m} = \int \mathrm{tr}_E(P_t f)\, \mathrm{d}|D\chi_E| \quad \text{for every } f \in H^{1,2}(X) \cap L^\infty(\mathfrak{m}) \text{ and } t > 0. \tag{5.5}$$

Moreover, it holds that

$$\lim_{t \searrow 0} \int \mathrm{tr}_E(P_t f)\, \mathrm{d}|D\chi_E| = \int \mathrm{tr}_E(f)\, \mathrm{d}|D\chi_E| \quad \text{for every } f \in H^{1,2}(X) \cap L^\infty(\mathfrak{m}). \tag{5.6}$$

Proof. First of all, let us prove (5.5). Fix any $f \in H^{1,2}(X) \cap L^\infty(\mathfrak{m})$ and $t > 0$. We claim that there exists a sequence $(f_n)_n \subset \mathrm{Lip_{bs}}(X, \mathsf{d})$ bounded in $L^\infty(\mathfrak{m})$ and such that

$$f_n \to f \text{ strongly in } H^{1,2}(X), \text{ weakly}^* \text{ in } L^\infty(\mathfrak{m}). \tag{5.7}$$

To prove it, we argue as follows. Given any $s > 0$, the function $P_s f$ has a Lipschitz representative (still denoted by $P_s f$) thanks to the L^∞-Lip regularisation of the heat flow. Since $\{P_s f\}_{s>0}$ is bounded in $L^\infty(\mathfrak{m})$ by the weak maximum principle and $P_s|\nabla f|^2 \to |\nabla f|^2$ strongly in $L^1(\mathfrak{m})$, we can find a function $G \in L^1(\mathfrak{m})$ and a sequence $s_n \searrow 0$ such that $P_{s_n}|\nabla f|^2 \leq G$ holds $\mathfrak{m}$-a.e. for all n and $P_{s_n} f \to f$ weakly* in $L^\infty(\mathfrak{m})$. Fix $\bar{x} \in X$ and for any $n \in \mathbb{N}$ choose a compactly-supported 1-Lipschitz function $\eta_n : X \to [0, 1]$ such that $\eta_n = 1$ on $B_n(\bar{x})$. Therefore, standard computations (based on the Leibniz rule $\nabla(\eta_n P_{s_n} f) = \eta_n \nabla P_{s_n} f + P_{s_n} f \nabla \eta_n$, the dominated convergence theorem, and the Bakry-Émery contraction estimate) show that $f_n := \eta_n P_{s_n} f \in \mathrm{Lip_{bs}}(X, \mathsf{d})$ satisfy (5.7). Now observe that $P_t : H^{1,2}(X) \to H^{1,2}(X)$ is continuous, as a consequence of the Bakry-Émery contraction estimate and the continuity of $P_t : L^2(\mathfrak{m}) \to L^2(\mathfrak{m})$. This ensures that $P_t f_n \to P_t f$ strongly in $H^{1,2}(X)$ as $n \to \infty$, whence we know from [**88**, Propositions 1.12, 1.17 and 1.19] that (possibly passing to a not relabeled subsequence) $\mathsf{QCR}(P_t f_n) \to \mathsf{QCR}(P_t f)$ holds Cap-a.e., and accordingly $\mathrm{tr}_E(P_t f_n) \to \mathrm{tr}_E(P_t f)$ holds $|D\chi_E|$-a.e.. Moreover, since $|P_t f_n| \leq \sup_k \|f_k\|_{L^\infty(\mathfrak{m})} =: C$ in the $\mathfrak{m}$-a.e. sense for all $n \in \mathbb{N}$, we deduce that $|\mathsf{QCR}(P_t f_n)| \leq C$ holds Cap-a.e. for all $n \in \mathbb{N}$, and thus $\mathrm{tr}_E(P_t f_n) \leq C$ holds $|D\chi_E|$-a.e. for all $n \in \mathbb{N}$. All in all, we obtain (5.5) by letting $n \to \infty$ in $\int f_n\, P_t^*|D\chi_E|\, \mathrm{d}\mathfrak{m} = \int \mathrm{tr}_E(P_t f_n)\, \mathrm{d}|D\chi_E|$, which is satisfied thanks to the defining property of $P_t^*|D\chi_E|$; here we use the dominated convergence theorem and the L^∞-weak* convergence $f_n \to f$.

Let us now pass to the proof of (5.6). Fix $f \in H^{1,2}(X) \cap L^\infty(\mathfrak{m})$. By arguing as above, we see that $|\mathrm{tr}_E(P_t f)| \leq \|f\|_{L^\infty(\mathfrak{m})}$ holds $|D\chi_E|$-a.e. for all $t > 0$, and that any given sequence $t_n \searrow 0$ admits a subsequence $t_{n_i} \searrow 0$ such that $\mathrm{tr}_E(P_{t_{n_i}} f) \to \mathrm{tr}_E(f)$ holds $|D\chi_E|$-a.e.. Therefore, by dominated convergence theorem we conclude that $\lim_i \int \mathrm{tr}_E(P_{t_{n_i}} f)\, \mathrm{d}|D\chi_E| = \int \mathrm{tr}_E(f)\, \mathrm{d}|D\chi_E|$, which yields (5.6). □

Lemma 5.10 (Test vector fields over ∂E)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ space. Let $E \subset X$ be a set of finite perimeter and finite mass. We define the class $\mathrm{TestV}_E(X) \subset L^2_E(TX)$ of test vector fields over the boundary of E as*

$$\mathrm{TestV}_E(X) := \mathrm{tr}_E\big(\mathrm{TestV}(X)\big) = \left\{ \sum_{i=1}^n \mathrm{tr}_E(g_i) \bar{\nabla} f_i \;\middle|\; n \in \mathbb{N},\ (f_i)_{i=1}^n, (g_i)_{i=1}^n \subset \mathrm{Test}(X) \right\}.$$

Then $\mathrm{TestV}_E(X)$ is dense in $L^2_E(TX)$.

Proof. By item ii) of Theorem 5.4, it suffices to show that each $v \in L^2_E(TX)$ of the form $v = \chi_E \bar{\nabla} f$ – where $E \subset X$ is a Borel set and $f \in \mathrm{Test}(X)$ – can be approximated by elements of $\mathrm{TestV}_E(X)$ with respect to the strong topology of $L^2_E(TX)$. Fix $\varepsilon > 0$ and choose a function $h \in \mathrm{Lip}_c(X)$ such that $\|h - \chi_E\|_{L^2(|D\chi_E|)} \leq \varepsilon/(2\,\mathrm{Lip}(f))$. Moreover, by exploiting [**112**, eq. (3.2.3)] we can find a sequence $(g_n)_n \subset \mathrm{Test}(X)$ such that $\sup_n \|g_n\|_{L^\infty(\mathfrak{m})} < +\infty$ and $g_n \to h$ in $H^{1,2}(X)$. Hence, by using the results in [**88**] we see that (up to a not relabeled subsequence) it holds $\mathrm{tr}_E(g_n)(x) \to h(x)$ for $|D\chi_E|$-a.e. $x \in X$. Accordingly, by applying the dominated convergence theorem we conclude that $\big|(\mathrm{tr}_E(g_n) - h)\bar{\nabla} f\big| \to 0$ in $L^2(|D\chi_E|)$. Now choose $n \in \mathbb{N}$ so big that $g := g_n$ satisfies $\big\|(\mathrm{tr}_E(g) - h)\bar{\nabla} f\big\|_{L^2_E(TX)} < \varepsilon/2$. Hence, one has that

$$\begin{aligned}\big\|\mathrm{tr}_E(g)\bar{\nabla} f - v\big\|_{L^2_E(TX)} &\leq \big\|(\mathrm{tr}_E(g) - h)\bar{\nabla} f\big\|_{L^2_E(TX)} + \big\|(h - \chi_E)\bar{\nabla} f\big\|_{L^2_E(TX)} \\ &\leq \frac{\varepsilon}{2} + \|h - \chi_E\|_{L^2(|D\chi_E|)}\,\mathrm{Lip}(f) < \varepsilon.\end{aligned}$$

Given that $\mathrm{tr}_E(g)\bar{\nabla} f \in \mathrm{TestV}_E(X)$, the statement is achieved. □

The last ingredient we need is a representation formula for the total variation measure of a BV function in the special case of $\mathrm{RCD}(K,\infty)$ spaces.

We start recalling some useful consequences of the approach to BV functions via integration by parts studied in [**89**].

Remark 5.11. Given an infinitesimally Hilbertian space $(X, \mathsf{d}, \mathfrak{m})$ and any $f \in \mathrm{BV}(X, \mathsf{d}, \mathfrak{m})$, it holds

$$\int f \,\mathrm{div}\,(v)\,\mathrm{d}\mathfrak{m} \leq |Df|(X) \qquad \text{for every } v \in D(\mathrm{div}\,) \text{ with } |v| \leq 1 \;\; \mathfrak{m}\text{-a.e. and } \mathrm{div}\,(v) \in L^\infty(\mathfrak{m}).$$

Such inequality readily follows from an approximation argument, see [**89**, Theorem 3.3] and Proposition 1.51.

Theorem 5.12 (Representation formula for $|Df|$)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be an infinitesimally Hilbertian metric measure space. Let $f \in \mathrm{BV}(X, \mathsf{d}, \mathfrak{m})$ be given. Then for every open set $U \subset X$ it holds that*

$$|Df|(U) = \sup\left\{ \int_U f\,\mathrm{div}\,(v)\,\mathrm{d}\mathfrak{m} \;\middle|\; v \in D(\mathrm{div}\,),\ |v| \leq 1,\ \mathrm{div}\,(v) \in L^\infty(\mathfrak{m}),\ \mathrm{supp}(v) \Subset U \right\}.$$

Proof. Combine [**89**, Theorem 3.4] with Proposition 1.51 (recall that we have $\boldsymbol{b} \in \mathrm{Der}^{2,2}(X)$ for every $\boldsymbol{b} \in \mathrm{Der}^{\infty,\infty}(X)$ such that $\mathrm{supp}(\boldsymbol{b})$ is bounded, thanks to Remark 1.46). □

As we are going to see below, to obtain the total variation of a BV function on RCD spaces it is sufficient to restrict the attention only to those competitors that are Sobolev regular.

Theorem 5.13 (Representation formula for $|Df|$ on RCD spaces)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K,\infty)$ space and $f \in \mathrm{BV}(X)$. Then it holds that*

$$|Df|(X) = \sup\left\{ \int f\,\mathrm{div}\,(v)\,\mathrm{d}\mathfrak{m} \,:\, v \in H^{1,2}_C(TX) \cap D(\mathrm{div}\,),\ |v| \leq 1\,\mathfrak{m}\text{-a.e.},\ \mathrm{div}\,(v) \in L^\infty(\mathfrak{m}) \right\}.$$

Proof. Call S the right hand side of the above formula. We know by Remark 5.11 that $|Df|(X) \geq S$. In order to prove the converse inequality, fix any $\varepsilon > 0$. Theorem 5.12 guarantees the existence of a vector field $v \in D(\mathrm{div}\,)$ – with $|v| \leq 1$ in the $\mathfrak{m}$-a.e. sense and $\mathrm{div}\,(v) \in L^\infty(\mathfrak{m})$ – such that $\int f\,\mathrm{div}\,(v)\,\mathrm{d}\mathfrak{m} > |Df|(X) - \varepsilon/2$. Now define $v_t := e^{Kt}\,\mathsf{h}_{\mathrm{H},t}(v)$ for every $t > 0$. Notice that $v_t \in H^{1,2}_C(TX) \cap D(\mathrm{div}\,)$ by Proposition 1.91. Since $\mathrm{div}\,(v) \in L^\infty(\mathfrak{m})$ and $\mathrm{div}\,(v_t) = e^{Kt} P_t(\mathrm{div}\,(v))$, we deduce from the weak maximum principle that $\mathrm{div}\,(v_t) \in$

$L^\infty(\mathfrak{m})$ as well. More precisely, one has $\|\mathrm{div}\,(v_t)\|_{L^\infty(\mathfrak{m})} \le e^{Kt}\,\|\mathrm{div}\,(v)\|_{L^\infty(\mathfrak{m})}$ for all $t > 0$. Moreover, the weak maximum principle also guarantees that

$$|v_t| = e^{Kt}\,|\mathsf{h}_{\mathrm{H},t}(v)| \overset{(1.66)}{\le} \sqrt{P_t(|v|^2)} \le 1 \quad \text{in the } \mathfrak{m}\text{-a.e. sense.}$$

Given that $\lim_{t\searrow 0} \mathrm{div}\,(v_t) = \mathrm{div}\,(v)$ in $L^2(\mathfrak{m})$, we can find $t_n \searrow 0$ such that $\mathrm{div}\,(v_{t_n})(x) \to \mathrm{div}\,(v)(x)$ holds for $\mathfrak{m}$-a.e. $x \in X$. Being $\big(\mathrm{div}\,(v_{t_n})\big)_n$ a bounded sequence in $L^\infty(\mathfrak{m})$, we can finally conclude that $\lim_n \int f\,\mathrm{div}\,(v_{t_n})\,\mathrm{d}\mathfrak{m} = \int f\,\mathrm{div}\,(v)\,\mathrm{d}\mathfrak{m}$ by dominated convergence theorem. Therefore, there exists $n \in \mathbb{N}$ such that $w := v_{t_n}$ satisfies

$$\int f\,\mathrm{div}\,(w)\,\mathrm{d}\mathfrak{m} > \int f\,\mathrm{div}\,(v)\,\mathrm{d}\mathfrak{m} - \frac{\varepsilon}{2} > |Df|(X) - \varepsilon.$$

This shows that $|Df|(X) < S + \varepsilon$, whence $|Df|(X) \le S$ by arbitrariness of ε. □

Proof of Theorem 5.6. First of all, let us define $\mu_t := P_t^*|D\chi_E|$ for every $t > 0$. In the following we will tacitly identify the measure μ_t with its density with respect to $\mathfrak{m}$. Recall that $\mu_t \rightharpoonup |D\chi_E|$ in duality with $C_b(X)$ as $t \searrow 0$. Let us also set

$$\nu_t := \chi_{\{P_t^*|D\chi_E|>0\}} \frac{\nabla P_t\chi_E}{P_t^*|D\chi_E|} \in L^0(TX) \qquad \text{for every } t > 0.$$

It follows from the 1-Bakry-Émery estimate (1.52) that $|DP_t\chi_E| \le e^{-Kt}P_t^*|D\chi_E|$ holds $\mathfrak{m}$-a.e., thus accordingly $\nu_t \in L^\infty(TX)$ and $|\nu_t| \le e^{-Kt}$ is satisfied in the $\mathfrak{m}$-a.e. sense. Call

$$\mathcal{V} := \big\{v \in H^{1,2}_C(TX) \cap D(\mathrm{div}\,)\,\big|\,|v| \in L^\infty(\mathfrak{m})\big\}$$

and fix $v \in \mathcal{V}$. The Leibniz rule for the divergence ensures that $\varphi v \in D(\mathrm{div}\,)$ for any $\varphi \in \mathrm{Lip_b}(X)$, so the usual integration-by-parts formula yields

$$\int P_t\chi_E\,\mathrm{div}\,(\varphi v)\,\mathrm{d}\mathfrak{m} = -\int \varphi\,\langle\nabla P_t\chi_E, v\rangle\,\mathrm{d}\mathfrak{m} = -\int \varphi\,\langle v, \nu_t\rangle\,\mathrm{d}\mu_t, \tag{5.8}$$

for all $\varphi \in \mathrm{Lip_b}(X)$. Moreover, let us observe that $\langle v, \nu_t\rangle \in L^\infty(\mu_t)$ and $\|\langle v, \nu_t\rangle\|_{L^\infty(\mu_t)} \le e^{-Kt}\,\||v|\|_{L^\infty(\mathfrak{m})}$ for every $t > 0$. Let us call $\sigma_t := \langle v, \nu_t\rangle\mu_t$ for all $t > 0$. Fix any sequence $t_n \searrow 0$. Since $\mu_{t_n} \rightharpoonup |D\chi_E|$ in duality with $\mathrm{C_b}(X)$, we know that $(\mu_{t_n})_n$ is tight by Prohkorov theorem. Given that $\sup_n \|\langle v, \nu_{t_n}\rangle\|_{L^\infty(\mu_{t_n})}$ is finite, we deduce that $(\sigma_{t_n})_n$ is tight as well. By using Prohkorov theorem again, we can thus take a subsequence $(t_{n_i})_i$ such that $\sigma_{t_{n_i}} \rightharpoonup \sigma$ in duality with $\mathrm{C_b}(X)$ for some finite (signed) Borel measure σ on X. Since $\mathrm{Lip_b}(X)$ is dense in $\mathrm{C_b}(X)$ and the identity in (5.8) gives

$$\int \varphi\,\mathrm{d}\sigma = \lim_{i\to\infty}\int \varphi\,\mathrm{d}\sigma_{t_{n_i}} = -\int_E \mathrm{div}\,(\varphi v)\,\mathrm{d}\mathfrak{m} \qquad \text{for every } \varphi \in \mathrm{Lip_b}(X),$$

we see that σ is independent of the chosen sequence $(t_{n_i})_i$. Hence, $\sigma_t \rightharpoonup \sigma$ in duality with $\mathrm{C_b}(X)$ as $t \searrow 0$. Given any non-negative function $\varphi \in \mathrm{C_b}(X)$, it thus holds that

$$\begin{aligned}\left|\int \varphi\,\mathrm{d}\sigma\right| &\le \varliminf_{t\searrow 0}\int \varphi\,|\langle v, \nu_t\rangle|\,\mathrm{d}\mu_t \le e^{|K|}\,\||v|\|_{L^\infty(\mathfrak{m})}\,\varliminf_{t\searrow 0}\int \varphi\,\mathrm{d}\mu_t \\ &= e^{|K|}\,\||v|\|_{L^\infty(\mathfrak{m})}\int \varphi\,\mathrm{d}|D\chi_E|,\end{aligned}$$

whence $\sigma \ll |D\chi_E|$ and its Radon-Nikodým derivative $L(v) := \frac{\mathrm{d}\sigma}{\mathrm{d}|D\chi_E|}$ belongs to $L^\infty(|D\chi_E|)$. Consequently, taking into account (5.8) we deduce that

$$\int_E \mathrm{div}\,(\varphi v)\,\mathrm{d}\mathfrak{m} = -\int \varphi\,L(v)\,\mathrm{d}|D\chi_E| \qquad \text{for every } v \in \mathcal{V} \text{ and } \varphi \in \mathrm{Lip_b}(X). \tag{5.9}$$

Furthermore, one also has that

$$\lim_{t\searrow 0}\int \varphi\,\langle v,\nu_t\rangle\,\mathrm{d}\mu_t=\int \varphi\,L(v)\,\mathrm{d}|D\chi_E|\quad\text{for every } v\in\mathcal{V}\text{ and }\varphi\in\mathrm{Lip_b}(X).\tag{5.10}$$

Observe that for any $v\in\mathcal{V}$ and $\varphi\in\mathrm{Lip_b}(X)$, $\varphi\geq 0$ it holds that

$$\begin{aligned}\Big|\int \varphi\,L(v)\,\mathrm{d}|D\chi_E|\Big| &\overset{(5.10)}{=}\lim_{t\searrow 0}\Big|e^{Kt}\int \varphi\,\langle v,\nu_t\rangle\,\mathrm{d}\mu_t\Big|\\ &\leq \lim_{t\searrow 0}\Big(\|\varphi\|_{L^\infty(\mathfrak{m})}\,\||v|\|_{L^\infty(\mathfrak{m})}\int |1-e^{Kt}|\nu_t||\,\mathrm{d}\mu_t+\int\varphi\,\langle v,\frac{\nu_t}{|\nu_t|}\rangle\,\mathrm{d}\mu_t\Big)\\ &\overset{(4.74)}{\leq}\lim_{t\searrow 0}\int\varphi\,|v|\,\mathrm{d}\mu_t\overset{(5.5)}{=}\lim_{t\searrow 0}\int \mathrm{tr}_E(P_t(\varphi|v|))\,\mathrm{d}|D\chi_E|\\ &\overset{(5.6)}{=}\int\varphi\,\mathrm{tr}_E(|v|)\,\mathrm{d}|D\chi_E|.\end{aligned}$$

In the last two equalities we used the fact that $|v|\in H^{1,2}(X)$. By arbitrariness of φ, we obtain that $|L(v)|\leq \mathrm{tr}_E(|v|)$ holds $|D\chi_E|$-a.e. for all $v\in\mathcal{V}$. Let us now define $\omega:\,\mathrm{tr}_E(\mathcal{V})\to L^1(|D\chi_E|)$ as

$$\omega\big(\mathrm{tr}_E(v)\big):=L(v)\quad\text{for every } v\in\mathcal{V}.\tag{5.11}$$

The operator $L:\mathcal{V}\to L^\infty(|D\chi_E|)$ is linear by its very construction, whence by exploiting the inequality $|L(v)|\leq\mathrm{tr}_E(|v|)$ we can conclude that ω is well-posed, linear and satisfying

$$|\omega(v)|\leq|v|\quad|D\chi_E|\text{-a.e.}\quad\text{for every } v\in\mathrm{tr}_E(\mathcal{V}).$$

Since $\mathrm{TestV}(X)\subset\mathcal{V}$ and $\mathrm{TestV}_E(X)$ is dense in $L^2_E(TX)$, we infer from Lemma 5.10 that $\bar{\mathrm{tr}}_E(\mathcal{V})$ is a dense linear subspace of $L^2_E(TX)$. Therefore, we know from [**112**, Proposition 1.4.8] that ω can be uniquely extended to an element $\omega\in L^2_E(T^*X):=L^2_E(TX)^*$ satisfying $|\omega|\leq 1$ in the $|D\chi_E|$-a.e. sense. We denote by $\nu_E\in L^2_E(TX)$ the vector field corresponding to ω via the Riesz isomorphism. By combining (5.9) (with $\varphi\equiv 1$) and (5.11), we conclude that (5.4) is satisfied. It only remains to show that $|\nu_E|\geq 1$ holds $|D\chi_E|$-a.e.. In order to do it, just observe that Theorem 5.13 yields

$$\begin{aligned}|D\chi_E|(X)&\leq\sup_{\substack{v\in\mathcal{V},\\|v|\leq 1\ \mathfrak{m}\text{-a.e.}}}\int_E\mathrm{div}\,(v)\,\mathrm{d}\mathfrak{m}\overset{(5.4)}{=}\sup_{\substack{v\in\mathcal{V},\\|v|\leq 1\ \mathfrak{m}\text{-a.e.}}}-\int\langle\mathrm{tr}_E(v),\nu_E\rangle\,\mathrm{d}|D\chi_E|\\ &\leq\int|\nu_E|\,\mathrm{d}|D\chi_E|\leq|D\chi_E|(X),\end{aligned}$$

whence each inequality must be an equality. This clearly forces the $|D\chi_E|$-a.e. equality $|\nu_E|=1$. The element ν_E is uniquely determined by (5.4) as the space $\mathrm{tr}_E(\mathcal{V})$ is dense in $L^2_E(TX)$. Finally, the last part of the statement is an immediate consequence of Lemma 5.10. □

In the very recent paper [**53**] a Gauss Green formula for bounded divergence measure vector fields on locally compact $\mathrm{RCD}(K,\infty)$ metric measure spaces has been obtained. Let us briefly compare their result with the one we presented above.

As we just pointed out the assumptions in [**53**] are more general, since they treat the general case of $\mathrm{RCD}(K,\infty)$ spaces without imposing upper dimension bounds. Moreover even the class of vector fields they allow for is more general, since their requirement is that the vector field is essentially bounded and it has measure valued divergence while we ask for the additional $H^{1,2}_C$ regularity. On the other hand our stronger assumptions allow to sharpen the representation formula of the normal trace of the vector field appearing in the integration by parts formula. While in [**53**] the term $\langle v,\nu_E\rangle$ is only identified by a suitable limiting procedure, in our case it gets a precise geometric meaning after the introduction of the module $L^2_E(TX)$.

2. Uniqueness of tangents for sets of finite perimeter

In this section we prove a uniqueness theorem (up to negligible sets) for blow-ups of sets with finite perimeter over $\mathrm{RCD}(K,N)$ metric measure spaces. This has to be considered as a further step in the direction of generalizing De Giorgi's theorem to the framework of $\mathrm{RCD}(K,N)$ spaces.

Let us point out that, as an additional condition with respect to those explicitly requested in Definition 4.30, up to a $|D\chi_E|$-negligible set, one also has that the perimeter measures on the rescaled spaces $|D^i\chi_E|$ weakly converge to $|D\chi_F|$ in duality w.r.t. $\mathrm{C_{bs}}$. This information, which is obtained in Corollary 4.37, will be helpful in the sequel.

Theorem 5.14. *Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ m.m.s. with essential dimension $1\le n\le N$, $E\subset X$ be a set of finite perimeter. Then, for $|D\chi_E|$-a.e. $x\in X$, there exists $k=1,\dots,n$ such that*

$$\mathrm{Tan}_x(X,\mathsf{d},\mathfrak{m},E)=\left\{(\mathbb{R}^k,\mathsf{d}_{\mathrm{eucl}},c_k\mathcal{L}^k,0^k,\{x_k>0\})\right\}.$$

After establishing Theorem 4.32 the state of the art in the theory of sets of finite perimeter was similar to that of the structure theory of $\mathrm{RCD}(K,N)$ spaces after [**117**] (cf. Theorem 2.15), where the authors proved existence of a Euclidean tangent space up to negligible sets. The content of this and of the next section instead can be seen as a counterpart in codimension 1 of the main results obtained by Mondino–Naber in [**170**] (cf. Chapter 2). Also the main ideas underlying the proofs of the uniqueness of tangents and the rectifiability result are quite similar to those leading to the metric measure rectifiability of $\mathrm{RCD}(K,N)$ spaces that we implemented in Chapter 2. As in that case, the existence of a Euclidean tangent along a fixed scale is a regularity information which can be propagated at any location and scale up to a set which is small w.r.t. the relevant measure, yielding uniqueness of tangents.

From a technical point of view, our construction heavily relies on the use of harmonic δ-splitting maps, whose basic theory in this framework has been presented in Section 2. With this tool at our disposal, the *propagation of regularity step* is a consequence of a weighted maximal argument which was suggested in [**65**]. Let us point out that, in order for the whole procedure to work, the fact that perimeter measures have codimension 1 (see Lemma 4.35) and the fact that harmonic functions satisfy L^2-Hessian bounds play a key role. The strategy would completely fail if perimeter measures had codimension bigger or equal than 2.

Below we are concerned with the propagation of the property of being a δ-splitting map. We are going to prove that, if $\alpha\in(0,2)$, outside a set of small codimension-α content any δ-splitting map at a given scale is a $C_{N,\alpha}\delta^{1/4}$ splitting map at any scale. In this way we sharpen Proposition 2.13, allowing for applications to the theory in codimension one.

Proposition 5.15. *Let $\alpha\in(0,2)$ and $N>1$. There exist constants $C_N>0$ and $C_{N,\alpha}>0$ such that, for any $0<\delta<1$, any $\mathrm{RCD}(-1,N)$ m.m.s. $(X,\mathsf{d},\mathfrak{m})$, any $p\in X$ and for any δ-splitting map $u:=(u_1,\dots,u_k):B_2(p)\to\mathbb{R}^k$, there exists a Borel set $G\subset B_1(p)$ with $\mathscr{H}^{h_\alpha}_5(B_1(p)\setminus G)<C_N\sqrt{\delta}\,\mathfrak{m}(B_2(p))$ such that for any $x\in G$ it holds*

$$\sup_{0<r<1} r^\alpha \fint_{B_r(x)} |\operatorname{Hess} u_a|^2\,\mathrm{d}\mathfrak{m}\le\sqrt{\delta}\qquad \textit{for any}\quad a=1,\dots,k, \tag{5.12}$$

and

$$u:B_r(x)\to\mathbb{R}^k\qquad \textit{is a } C_{N,\alpha}\delta^{1/4}\textit{-splitting map for any } 0<r<1/2. \tag{5.13}$$

Proof. Let us start proving (5.12). To this aim fix any $a = 1, \dots, k$ and denote by C_P and C_D the Poincaré and the doubling constants over balls of radius 10 of $(X, \mathsf{d}, \mathfrak{m})$. To be more precise C_P is a constant in the $(1,2)$-Poincaré inequality with $\lambda = 2$ as in (1.19). In particular, since $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(-1, N)$, C_P depends only on N. The same conclusion holds for C_D thanks to the Bishop-Gromov inequality (1.45).

Set

$$G := \left\{ x \in B_1(p) : \sup_{0<r<1} r^\alpha \fint_{B_r(x)} |\operatorname{Hess} u_a|^2 \, \mathrm{d}\mathfrak{m} \le \sqrt{\delta} \right\}.$$

We claim that $\mathscr{H}_5^{h_\alpha}(B_1(p) \backslash G) < C_N \sqrt{\delta} \mathfrak{m}(B_2(p))$. For any $x \in B_1(p) \backslash G$ we choose $\rho_x \in (0,1)$ satisfying

$$\rho_x^\alpha \fint_{B_{\rho_x}(x)} |\operatorname{Hess} u_a|^2 \, \mathrm{d}\mathfrak{m} > \sqrt{\delta}. \tag{5.14}$$

Observe that the family $\{B_{\rho_x}(x)\}_{x \in B_1(p) \backslash G}$ covers $B_1(p) \setminus G$. Using Vitali's covering Theorem 1.9 we can find a subfamily of disjoint balls $\{B_{\rho_i}(x_i)\}_{i \in \mathbb{N}}$ such that $B_1(p) \setminus G \subset \cup_{i \in \mathbb{N}} B_{5\rho_i}(x_i)$. This gives the sought conclusion

$$\begin{aligned}
\mathscr{H}_5^{h_\alpha}(B_1(p) \setminus G) &\le \sum_{i \in \mathbb{N}} h_\alpha(B_{5\rho_i}(x_i)) = \sum_{i \in \mathbb{N}} \frac{\mathfrak{m}(B_{5\rho_i}(x_i))}{(5\rho_i)^\alpha} \\
&\overset{(1.45)}{\le} C_N \sum_{i \in \mathbb{N}} \frac{\mathfrak{m}(B_{\rho_i}(x_i))}{\rho_i^\alpha} \overset{(5.14)}{\le} C_N \sum_{i \in \mathbb{N}} \frac{1}{\sqrt{\delta}} \int_{B_{\rho_i}(x_i)} |\operatorname{Hess} u_a|^2 \, \mathrm{d}\mathfrak{m} \\
&\le C_N \frac{1}{\sqrt{\delta}} \int_{B_2(p)} |\operatorname{Hess} u_a|^2 \, \mathrm{d}\mathfrak{m} \le C_N \sqrt{\delta} \mathfrak{m}(B_2(p)),
\end{aligned}$$

where we used the definition of $\mathscr{H}_5^{h_\alpha}$, and the fact that u is a δ-splitting map.

In order to verify (5.13) we just need to check that, for $a, b = 1, \dots, k$,

$$\fint_{B_r(x)} |\nabla u_a \cdot \nabla u_b - \delta_{a,b}| \, \mathrm{d}\mathfrak{m} < C_{N,\alpha} \delta^{1/4} \quad \text{for any } x \in G, \ 0 < r < 1.$$

We wish to get the estimate with a telescopic argument taken from [**139**, Lemma 5.9].

To this aim let us set $f_{a,b} := |\nabla u_a \cdot \nabla u_b - \delta_{a,b}|$ and note that $|\nabla f_{a,b}| \le C_N(|\operatorname{Hess} u_a| + |\operatorname{Hess} u_b|)$ as a consequence of Definition 2.5(i) and (1.61). Whence, the Poincaré inequality yields

$$\left| \fint_{B_r(x)} f_{a,b} \, \mathrm{d}\mathfrak{m} - \fint_{B_{r/2}(x)} f_{a,b} \, \mathrm{d}\mathfrak{m} \right| \le C_P r \left(\fint_{B_{2r}(x)} |\nabla f_{a,b}|^2 \, \mathrm{d}\mathfrak{m} \right)^{1/2}$$

and thanks to (5.12) we can continue the chain of inequalities with

$$\begin{aligned}
&\le C_N C_P \left(r^2 \fint_{B_{2r}(x)} |\operatorname{Hess} u_a|^2 \, \mathrm{d}\mathfrak{m} + r^2 \fint_{B_{2r}(x)} |\operatorname{Hess} u_b|^2 \, \mathrm{d}\mathfrak{m} \right)^{1/2} \\
&\le C_N C_P \delta^{1/4} r^{1-\alpha/2}, \quad \text{for any } 0 < r < 1/2,
\end{aligned}$$

where the assumption $\alpha \in (0,2)$ crucially enters into play here.

Applying a telescopic argument it is simple to see that

$$\left| \fint_{B_{2^{-1}}(x)} f_{a,b} \, \mathrm{d}\mathfrak{m} - \fint_{B_{2^{-k}}(x)} f_{a,b} \, \mathrm{d}\mathfrak{m} \right| \le C_\alpha C_N C_P \delta^{1/4}, \qquad \text{for any } k > 1. \tag{5.15}$$

Therefore, for any $0 < r < 1/2$ we take $k \in \mathbb{N}$ such that $2^{-k-1} < r \le 2^{-k}$ and, using that $u : B_2(p) \to \mathbb{R}^k$ is a δ-splitting map, we get

$$\begin{aligned}
\fint_{B_r(x)} f_{a,b}\, \mathrm{d}\mathfrak{m} \le & C_D 2^N \fint_{B_{2^{-k}}(x)} f_{a,b}\, \mathrm{d}\mathfrak{m} \\
\le & C_D 2^N \left| \fint_{B_{1/2}(x)} f_{a,b}\, \mathrm{d}\mathfrak{m} - \fint_{B_{2^{-k}}(x)} f_{a,b}\, \mathrm{d}\mathfrak{m} \right| + C_D 2^N \fint_{B_{1/2}(x)} f_{a,b}\, \mathrm{d}\mathfrak{m} \\
\overset{(5.15)+(1.45)}{\le} & 2^N C_D C_\alpha C_N C_P \delta^{1/4} + 8^N C_D^2 \fint_{B_2(p)} f_{a,b}\, \mathrm{d}\mathfrak{m} \\
\le & C_{N,\alpha} \delta^{1/4}.
\end{aligned}$$

□

For our purposes we just need to consider the case $\alpha = 1$ in Proposition 5.15. This is related to the fact that boundaries of sets with finite perimeter are codimension one objects. In order to shorten the notation in the sequel we will write h in place of h_1 when dealing with the codimension one Hausdorff measures and premeasures $\mathscr{H}^h, \mathscr{H}^h_\delta$.

Corollary 5.16. *Let $(X, \mathsf{d}, \mathfrak{m}, p)$ be an* RCD(K, N) *p.m.m.s. and $u : B_{4r}(p) \to \mathbb{R}^k$ a δ-splitting map for some $r > 0$ such that $|K|r^2 \le 4$ and $r < 1/2$. Then there exists $G \subset B_{2r}(p)$ with*

$$\mathscr{H}^h_5(B_{2r}(p) \setminus G) \le \mathscr{H}^h_{10r}(B_{2r}(p) \setminus G) \le C_N \sqrt{\delta} \frac{\mathfrak{m}(B_{2r}(p))}{2r}$$

such that $u : B_s(x) \to \mathbb{R}^k$ is a $C_N\delta^{1/4}$-splitting map such that $\fint_{B_s(x)} |\mathrm{Hess}\, u|^2\, \mathrm{d}\mathfrak{m} \le C_N s\delta$ for any $x \in G$ and any $0 < s < r$.

Proof. Apply Proposition 5.15 to the rescaled space $(X, (2r)^{-1}\mathsf{d}, \mathfrak{m}(B_{2r}(p))^{-1}\mathfrak{m}, p)$. □

Corollary 5.17. *Let us keep the notation of Corollary 5.16. Then for* Cap-*a.e. $x \in G$ it holds that*

$$\left| \bar\nabla u_a(x) \cdot \bar\nabla u_b(x) - \delta_{ab} \right| \le C_N \delta^{1/4}, \tag{5.16}$$

for any $a, b = 1, \dots, k$.

Proof. It is sufficient to observe that, thanks to Theorem 5.2, for Cap-a.e. $x \in X$ it holds that

$$\lim_{r \to 0} \fint_{B_r(x)} \left| \nabla u_a \cdot \nabla u_b - \bar\nabla u_a(x) \cdot \bar\nabla u_b(x) \right|\, \mathrm{d}\mathfrak{m} = 0.$$

The conclusion follows since by Corollary 5.16 we know that for every $x \in G$ it holds

$$\limsup_{r \to 0} \fint_{B_r(x)} |\nabla u_a \cdot \nabla u_b - \delta_{ab}|\, \mathrm{d}\mathfrak{m} \le C_N \delta^{1/4}.$$

□

Let $(X, \mathsf{d}, \mathfrak{m})$ be an RCD(K, N) metric measure space with essential dimension $n \le N$ (cf. Definition 3.43) and let $E \subset X$ be a set of locally finite perimeter. For any $k = 1, \dots, n$ we set

$$\begin{aligned}
A_k := \{x \in X \,:\, & \left(\mathbb{R}^k, \mathsf{d}_{\mathrm{eucl}}, c_k\mathcal{L}^k, 0^k, \{x_k > 0\}\right) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E), \text{ but for no } (Y, \varrho, \mu, y) \text{ s.t.} \\
& \mathrm{diam}(Y) > 0\ (Y \times \mathbb{R}^k, \varrho \times \mathsf{d}_{\mathrm{eucl}}, \mu \times \mathcal{L}^k, (y, 0^k), \{x_k > 0\}) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E)\}.
\end{aligned}$$

Let us point out that, with arguments analogous to those in [**170**, Lemma 6.1] (see also the first step in the proof o Theorem 2.18)one can show that A_k is a $|D\chi_E|$-measurable set for any $k = 1, \dots, n$. Moreover, thanks to Theorem 2.19 we know that it is impossible to find $x \in X$ for which the tangent cone at $\mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m})$ contains an element of the form $\mathbb{R}^n \times Y$ for some RCD m.m.s. $(Y, \mathsf{d}_Y, \mathfrak{m}_Y)$ with $\mathrm{diam}(Y) > 0$ (cf. [**152**]).

Lemma 5.18. *Under the assumptions above*

$$|D\chi_E| \left(X \setminus \bigcup_{k=1}^{n} A_k \right) = 0.$$

Proof. As a consequence of Theorem 4.32 we have

$$|D\chi_E| \left(X \setminus \bigcup_{k=1}^{n} A'_k \right) = 0,$$

where

$$A'_k := \{x \in X : \left(\mathbb{R}^k, \mathsf{d}_{\mathrm{eucl}}, c_k\mathcal{L}^k, 0^k, \{x_k > 0\}\right) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E) \text{ but}$$
$$(\mathbb{R}^m, \mathsf{d}_{\mathrm{eucl}}, c_m\mathcal{L}^m, 0^m, \{x_m > 0\}) \notin \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E) \text{ for any } m > k\}.$$

The measurability of the A'_k's can be verified as in the case of the A_k's.

It is clear that $A_k \subset A'_k$, let us prove $|D\chi_E|(A'_k \setminus A_k) = 0$. We argue by contradiction. If the claim is false we can find $x \in A'_k \setminus A_k$ such that the iterated tangent property of Theorem 4.41 holds true. Since $x \in A'_k \setminus A_k$ we can find $(Y, \varrho, \mu, y) \in \mathrm{RCD}(0, N - k)$ with $\mathrm{diam}(Y) > 0$ such that

$$(Y \times \mathbb{R}^k, \varrho \times \mathsf{d}_{\mathrm{eucl}}, \mu \times \mathcal{L}^k, (y, 0^k), \{x_k > 0\}) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E).$$

Moreover $\mathrm{Tan}_{(y',x,0)}(Y\times\mathbb{R}^k, \varrho\times\mathsf{d}_{\mathrm{eucl}}, \mu\times\mathcal{L}^k, \{x_k > 0\}) \subset \mathrm{Tan}(E, x)$ for any $(y', x) \in Y\times\mathbb{R}^{k-1}$, thanks to Theorem 4.41. Thus, choosing $(y', x, 0) \in Y \times \mathbb{R}^k$ such that Theorem 4.32 holds and y' is regular in Y we get the sought contradiction, since the essential dimension of Y is bigger or equal than one (otherwise $\mathrm{diam}(Y) = 0$). □

Proof of Theorem 5.14. In light of Lemma 5.18 it is enough to prove that A_k coincides up to a $|D\chi_E|$-negligible set with

$$\left\{x \in X : \ \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E) = \left\{(\mathbb{R}^k, \mathsf{d}_{\mathrm{eucl}}, c_k\mathcal{L}^k, 0^k, \{x_k > 0\})\right\}\right\}.$$

Let us assume without loss of generality that $A_k \subset B_2(p)$ for some $p \in X$. We claim that, for any $\eta > 0$, there exists $G^\eta \subset A_k$ with

$$\mathscr{H}^h_5(A_k \setminus G^\eta) \le C_N\eta\, |D\chi_E|\,(B_2(p)) \tag{5.17}$$

such that, for any $x \in G^\eta$ and for any $(Y, \varrho, \mu, y) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m})$, there exists a pointed $\mathrm{RCD}(0, N - k)$ m.m.s. $(Z, \mathsf{d}_Z, \mathfrak{m}_Z, z)$ satisfying

$$\mathsf{d}_{pmGH}((Y, \varrho, \mu, y), (\mathbb{R}^k \times Z, (0, z)) \le \eta. \tag{5.18}$$

Observe that the claim implies our conclusion. Indeed if we fix $\eta > 0$ and set $\eta_i := \eta 2^{-i}$ then $G_\eta := \cup_{i\in\mathbb{N}} G^{\eta_i}$ satisfies $\mathscr{H}^h_5(A_k \setminus G_\eta) = 0$ and thus $|D\chi_E|\,(A_k \setminus G_\eta) = 0$ thanks to Lemma 4.35. Moreover, for any $x \in G_\eta$, (4.5) holds. We conclude observing that $G := \cap_{k\in\mathbb{N}} G_{2^{-k}}$ still satisfies $|D\chi_E|\,(A_k \setminus G) = 0$ and any tangent cone at $x \in G$ splits off a factor $\mathbb{R}^k$. By definition of A_k we deduce that the only tangent at $x \in G$ is the Euclidean space of dimension k.

Let us pass to the verification of the claim. Fix $\delta \in (0, 1/2)$ and take $\varepsilon > 0$ as in Proposition 2.11. Of course we can assume $\varepsilon \le \delta$. We wish to prove that there exists a disjoint family of balls $\{B_{r_i}(x_i)\}_{i\in\mathbb{N}}$ such that $r_i^2|K| \le \varepsilon$ for any $i \in \mathbb{N}$ and

(i) $A_k \cap B_1(p) \subset \cup_{i\in\mathbb{N}} B_{5r_i}(x_i)$;

(ii) $\mathsf{d}_{pmGH}\left((X, r_i^{-1}\mathsf{d}, \mathfrak{m}_x^{r_i}, x_i), (\mathbb{R}^k, \mathsf{d}_{\mathrm{eucl}}, c_k\mathcal{L}^k, 0^k)\right) \le \varepsilon$;

(iii) $\frac{\omega_{k-1}}{\omega_k}(1-\varepsilon)\frac{\mathfrak{m}(B_{r_i}(x_i))}{r_i} \le |D\chi_E|(B_{r_i}(x_i)) \le \frac{\omega_{k-1}}{\omega_k}(1+\varepsilon)\frac{\mathfrak{m}(B_{r_i}(x_i))}{r_i}$.

Indeed, for any $x \in A_k$ there exists a sequence of radii $r_i \to 0$ such that

$$\lim_{i\to\infty} \mathsf{d}_{pmGH}((X, r_i^{-1}\mathsf{d}, \mathfrak{m}_x^{r_i}, x), (\mathbb{R}^k, \mathsf{d}_{\mathrm{eucl}}, \mathcal{L}^k, 0^k)) = 0 \quad \text{and} \quad \lim_{i\to\infty} \frac{r_i\,|D\chi_E|(B_{r_i}(x))}{\mathfrak{m}(B_{r_i}(x))} = \frac{\omega_{k-1}}{\omega_k},$$

as a consequence of Theorem 4.32, see also (2.5). Therefore, for any $x \in A_k$ we can choose $r_x^2|K| \le \varepsilon$ such that the pair (x, r_x) satisfies (ii) and (iii). In order to get a disjoint family of balls satisfying (i) we have just to apply Vitali's Lemma to $\{B_{r_x}(x)\}_{x\in A_k\cap B_1(p)}$.

Let us now focus the attention on a single ball $B_{20r_i}(x_i) \subset X$. Proposition 2.11 yields the existence of a δ-splitting map

$$u^i : B_{5r_i}(x_i) \to \mathbb{R}^k.$$

Thanks to Corollary 5.16 we can find $G_i \subset B_{5r_i}(x_i)$ with

$$\mathscr{H}_5^h(B_{5r_i}(x_i)\setminus G_i) \le C_N\sqrt{\delta}\,\frac{\mathfrak{m}(B_{5r_i}(x_i))}{5r_i} \tag{5.19}$$

and such that $u^i : B_s(x) \to \mathbb{R}^k$ is a $C_N\delta^{1/4}$-splitting map for any $x \in G_i$ and any $0 < s < 5r_i$. Applying Proposition 2.12, up to assuming δ small enough, we deduce that at any $x \in G_i$ (5.18) holds true.

To conclude let us verify that $G := \cup_{i\in\mathbb{N}} G_i$ satisfies (5.17):

$$\begin{aligned}\mathscr{H}_5^h(A_k\setminus G) &\le \sum_{i\in\mathbb{N}} \mathscr{H}_5^h(B_{5r_i}(x_i)\setminus G_i) \overset{(5.19)}{\le} \sum_{i\in\mathbb{N}} C_N\sqrt{\delta}\,\frac{\mathfrak{m}(B_{5r_i}(x_i))}{5r_i}\\ &\overset{(1.45)}{\le} C_N\sqrt{\delta}\sum_{i\in\mathbb{N}}\frac{\mathfrak{m}(B_{r_i}(x_i))}{r_i} \overset{(iii)}{\le} C_N\sqrt{\delta}\sum_{i\in\mathbb{N}}|D\chi_E|(B_{r_i}(x_i))\\ &\le C_N\sqrt{\delta}\,|D\chi_E|(B_2(p)).\end{aligned}$$

Since we can assume $\delta < \eta^2$ we get the sought estimate. □

As an intermediate step of the proof of uniqueness of tangents to sets of finite perimeter we have obtained the following result, that we point out explicitly since it will be relevant for the rest of the study in this chapter.

Corollary 5.19. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an* RCD(K,N) *m.m.s. of essential dimension $1 \le n \le N$ and let $E \subset X$ be a set of locally finite perimeter. Then, for any $\delta > 0$ and for any $1 \le k \le n$ there exists a countable family of (k,δ)-splitting maps $(u_i^k)_{i\in\mathbb{N}}$ defined on balls $B_{r_i}(x_i)$ such that for $|D\chi_E|$-a.e. $x \in \mathcal{F}_kE$ there exist $r_x > 0$ and $i \in \mathbb{N}$ for which $u_i^k : B_r(x) \to \mathbb{R}^k$ is an $r\delta$-splitting map for any $0 < r < r_x$ and*

$$\lim_{r\to 0} \fint_{B_r(x)} \left|\nabla(u_i^k)_a\cdot\nabla(u_i^k)_b - \delta_{ab}\right| \mathrm{d}\mathfrak{m} = \left|\bar\nabla(u_i^k)_a(x)\cdot\bar\nabla(u_i^k)_b(x) - \delta_{ab}\right| \le \delta,$$

for any $a, b = 1, \dots, k$.

Proof. The conclusion directly follows from the construction in the proof of Theorem 5.14 taking into account Corollary 5.17. □

Definition 5.20 (Reduced boundary)**.** Let $(X, \mathsf{d}, \mathfrak{m})$ be an RCD(K,N) metric measure space and $E \subset X$ a set of locally finite perimeter. For any $k = 1, \dots, n$, where n is the essential dimension of $(X, \mathsf{d}, \mathfrak{m})$, we set

$$\mathcal{F}_kE := \left\{x \in X \ : \ \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E) = \left\{(\mathbb{R}^k, \mathsf{d}_{\mathrm{eucl}}, c_k\mathcal{L}^k, 0^k, \{x_k > 0\})\right\}\right\}.$$

We know thanks to Theorem 5.14 that $|D\chi_E|(\cdot)$ is concentrated on $\mathcal{F}E := \cup_{k=1}^n \mathcal{F}_k E$ and, from now on, we shall call $\mathcal{F}E$ the *reduced boundary* of E.

In the study of the Gauss Green integration by parts formula on $\mathrm{RCD}(K,\infty)$ spaces pursued in [**53**], some conclusions have been obtained in the case of sets of finite perimeter E for which any weak* limit point of $P_t\chi_E$ in $L^\infty(X,|D\chi_E|)$ is constant (and equal to $1/2$). Here we wish to point out that, as a consequence of the uniqueness of tangents we proved above, this is the case for any set of finite perimeter on $\mathrm{RCD}(K,N)$ metric measure spaces.

Corollary 5.21. *Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ m.m.s. and let $E\subset X$ be a set of finite perimeter. Then $P_t\chi_E$ converge to $1/2$ as $t\to 0$ both in the $|D\chi_E|$-almost everywhere sense and in the weak* topology of $L^\infty(X,|D\chi_E|)$.*

Proof. In order to prove the sought conclusion, thanks to Theorem 5.14 it is sufficient to prove that

$$\lim_{t\to 0} P_t\chi_E(x) = 1/2 \tag{5.20}$$

for any $x\in\mathcal{F}E$. To this aim we just point out that the explicit computation of the evolution via heat flow of the indicator function of a half-space in the Euclidean space

$$P_t^{\mathbb{R}^n}\chi_{\mathbb{H}^n}(x) = \frac{1}{2}, \quad \text{for any } x\in\partial\mathbb{H}^n \text{ and for any } t>0,$$

together with the observations in the first part of the proof of Proposition 4.39 yield (5.20). □

Conjecture 5.22. Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ m.m.s. with essential dimension $1\le n\le N$ and let $E\subset X$ be a set of locally finite perimeter. Then $|D\chi_E|$ is concentrated of $\mathcal{F}_nE$, in particular the reduced boundary has constant dimension in the $|D\chi_E|$-a.e. sense.

Regarding the conjecture above let us stress that the fact that $\mathrm{RCD}(K,N)$ spaces have a well defined essential dimension played no role in the development of the theory of sets of finite perimeter so far. Moreover the validity of Conjecture 5.22 could be seen as a first step towards Conjecture 3.47.

3. Rectifiability of the reduced boundary

The main achievement of this section is a rectifiability result for the reduced boundary of sets with finite perimeter.

Theorem 5.23. *Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ m.m.s. and $E\subset X$ be a set of locally finite perimeter. Then, for any $k=1,\dots,n$, $\mathcal{F}_kE$ is $\big(|D\chi_E|,(k-1)\big)$-rectifiable.*

Let us recall that a set is $\big(|D\chi_E|,\ell\big)$-rectifiable if up to a $|D\chi_E|$-negligible set it can be covered by $\cup_{i\in\mathbb{N}}A_i$ where any A_i is biLipschitz equivalent to a Borel subset of $\mathbb{R}^\ell$, cf. Section 1.

Remark 5.24. We point out that, given any $\varepsilon>0$, the maps providing rectifiability of the reduced boundary in Theorem 5.23 can be taken $(1+\varepsilon)$-biLipschitz.

Let us outline the strategy of the proof of Theorem 5.23.
First of all, up to intersecting with a ball and thanks to the locality of perimeter and tangents, we can assume that E has finite measure and perimeter.
The biLipschitz maps from subsets of $\mathcal{F}_kE$ to $\mathbb{R}^{k-1}$ providing rectifiability are going to be *suitable approximations* of the $(k-1)$ coordinate maps over the hyperplane where the perimeter concentrates after the blow-up. Better said, they will be the first $(k-1)$ components of a (k,δ)-splitting map "δ-orthogonal to the exterior normal ν_E to the boundary of E". In the following, to simplify the notation, we shall write v in place of $\mathrm{tr}_E(v)$ for any $v\in H^{1,2}_C(TX)\cap D(\mathrm{div}\,)$.
The first step in order to obtain the rectifiability of the reduced boundary according to the

strategy we outlined above aims at proving that there exist indeed δ-splitting maps whose components are almost orthogonal in a suitable integral sense to the unit normal vector. The main difference of the presentation in this chapter with [**48**] is in its proof. Here we heavily rely on Theorem 5.2, which allows to follow a more natural strategy looking at the asymptotic behaviour of δ-splitting maps and avoiding some limiting procedures. The rigorous statement is as follows.

Proposition 5.25. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ m.m. space and $E \subset X$ a set of finite perimeter and measure. For any $\delta > 0$, $r_0 > 0$ and $|D\chi_E|$-a.e. $x \in \mathcal{F}_k E$ there exist $r = r_{x,\delta} < r_0$ and a δ-splitting map $v = (v_1, \ldots, v_{k-1}) : B_r(x) \to \mathbb{R}^{k-1}$ such that*

$$\frac{r}{\mathfrak{m}(B_r(x))} \int_{B_r(x)} |\nu \cdot \nabla v_\alpha| \,\mathrm{d}|D\chi_E| < \delta, \quad \textit{for } \alpha = 1, \ldots, k-1. \tag{5.21}$$

Proof. We wish to prove that the statement is verified choosing at almost every point of $\mathcal{F}_k E$ a $(k-1, \delta)$-splitting map whose components are obtained as linear combinations (with coefficients depending only on the given point) of the components of a (k, δ)-splitting map provided by Corollary 5.17. The idea underlying the proof can be explained in the following way, neglecting the regularity issues concerning the ambient space and the unit normal vector. First of all we observe that the limit as $r \to 0$ of the quantity we want to control in (5.21) should be comparable to the value of the scalar product between the unit normal ν_x at x and $\nabla u_a(x)$. Starting from this observation and taking into account the fact that $(\nabla u_1(x), \ldots, \nabla u_k(x))$ are *almost orthonormal* vectors, we can infer that there exists a linear transformation $A : \mathbb{R}^k \to \mathbb{R}^k$ such that, setting $v := A \circ u$, it holds $\nabla v_1(x) \cdot \nu(x) = \cdots = \nabla v_{k-1}(x) \cdot \nu(x) = 0$ and v is still a δ-spitting map at all sufficiently small scales.

Let us observe that, thanks to Corollary 5.19, it is sufficient to prove the statement on any $G \subset \mathcal{F}_k E$, where G verifies the following property. There exist a (k, δ)-splitting map $u : B_R(\bar{x}) \to \mathbb{R}^k$ and radii $r_x > 0$ for any $x \in G$ such that $u : B_r(x) \to \mathbb{R}^k$ is an $r\delta$-splitting map for any $0 < r < r_x$ and

$$\lim_{r \to 0} \fint_{B_r(x)} |\nabla u_a \cdot \nabla u_b - \delta_{ab}| \,\mathrm{d}\mathfrak{m} = |\nabla u_a(x) \cdot \nabla u_b(x) - \delta_{ab}| \le \delta, \tag{5.22}$$

for any $a, b = 1, \ldots, k$.

Recalling that $|D\chi_E|$ is asymptotically doubling, we can apply Theorem 1.16 (see also Remark 1.18) to conclude that there exists $\tilde{G} \subset G$ with $|D\chi_E|(G \setminus \tilde{G}) = 0$ and such that any $x \in \tilde{G}$ is a Lebesgue point of $x \mapsto \nabla u_a \cdot \nu_E$ with respect to $|D\chi_E|$ for any $a = 1, \ldots, k$, i.e.

$$\lim_{r \to 0} \fint_{B_r(x)} |\nabla u_a \cdot \nu - \nabla u_a(x) \cdot \nu(x)| \,\mathrm{d}\,|D\chi_E| = 0. \tag{5.23}$$

Next for any $x \in \tilde{G}$, we let B_x be the inverse matrix of $(\nabla u_a(x) \cdot \nabla u_b(x))_{a,b=1,\ldots,k}$. Observe that $|B_x - \mathbb{I}| \le C_k \delta$ thanks to (5.22). Then we can observe that for any orthogonal matrix $D : \mathbb{R}^k \to \mathbb{R}^k$, setting $v_D := D \circ B_x \circ u : B_{r_x}(x) \to \mathbb{R}^k$ it holds that v_D is a $C\delta$-splitting map for any $0 < r < r_x$,

$$\lim_{r \to 0} \fint_{B_r(x)} |\nabla (v_D)_a \cdot \nabla (v_D)_b - \delta_{ab}| \,\mathrm{d}\mathfrak{m} = 0$$

and

$$\lim_{r \to 0} \fint_{B_r(x)} |\nabla (v_D)_a \cdot \nu - \nabla (v_D)_a(x) \cdot \nu(x)| \,\mathrm{d}\,|D\chi_E| = 0, \tag{5.24}$$

for any $a, b = 1, \ldots, k$. By elementary linear algebra considerations we can find $D = D_x$ in such a way that $v := v_D$ verifies

$$\nabla v_a(x) \cdot \nu(x) = 0, \quad \text{for any } a = 1, \ldots, k-1. \tag{5.25}$$

We claim that the map $v = (v_1, \dots, v_{k-1})$ found with this construction satisfies the requirements of the statement at all points $x \in \tilde{G}$ such that

$$\limsup_{r\to 0} \frac{r\,|D\chi_E|\,(B_r(x))}{\mathfrak{m}(B_r(x)))} < \infty \tag{5.26}$$

and we observe that $|D\chi_E|$-a.e. point verifies this condition as we pointed out in the proof of Theorem 5.14. Indeed, as we already observed $v : B_r(x) \to \mathbb{R}^{k-1}$ is a $C\delta$-splitting map for any $0 < r < r_x$, moreover

$$\begin{aligned}
&\limsup_{r\to 0} \frac{r}{\mathfrak{m}(B_r(x))} \int_{B_r(x)} |\nabla v_a \cdot \nu| \, \mathrm{d}|D\chi_E| \\
&\quad = \limsup_{r\to 0} \frac{r\,|D\chi_E|\,(B_r(x))}{\mathfrak{m}(B_r(x)))} \lim_{r\to 0} \fint_{B_r(x)} |\nabla v_a \cdot \nu| \, \mathrm{d}\,|D\chi_E| \\
&\quad \overset{(5.25)}{=} \limsup_{r\to 0} \frac{r\,|D\chi_E|\,(B_r(x))}{\mathfrak{m}(B_r(x)))} \lim_{r\to 0} \fint_{B_r(x)} |\nabla v_a \cdot \nu - \nabla v_a(x) \cdot \nu(x)| \, \mathrm{d}\,|D\chi_E| \overset{(5.23),(5.26)}{=} 0.
\end{aligned}$$

Therefore $v = (v_1, \dots, v_{k-1}) : B_r(x) \to \mathbb{R}^{k-1}$ verifies the required properties up to choose r sufficiently small. □

The second step in the proof of Theorem 5.23 is showing that the map built in Proposition 5.25 is indeed biLipschitz with its image if restricted to suitable subsets of $\mathcal{F}_k E$ (see Proposition 5.27 below for the rigorous statement). These subsets are obtained collecting points $x \in \mathcal{F}_k E$ such that $B_s(x) \cap E$ is ε-close, in a suitable sense, to $B_s(0^k) \cap \{x_k > 0\}$ for any $s \le r_0$, where $r_0 > 0$ is a fixed radius.

Definition 5.26. Given $\varepsilon > 0$ and $r_0 > 0$, we define $(\mathcal{F}_k E)_{r_0,\varepsilon}$ as the set of points $x \in \mathcal{F}_k E$ satisfying

(i) $\mathsf{d}_{pmGH}\left(\left(X, s^{-1}\mathsf{d}, \frac{\mathfrak{m}}{\mathfrak{m}(B_s(x))}, x\right), \left(\mathbb{R}^k, \mathsf{d}_{\mathrm{eucl}}, \frac{1}{\omega_k}\mathcal{L}^k, 0^k\right)\right) < \varepsilon$ for any $s \le r_0$;

(ii)
$$\left|\frac{\mathfrak{m}(B_s(x) \cap E)}{\mathfrak{m}(B_s(x))} - \frac{1}{2}\right| + \left|\frac{s\,|D\chi_E|\,(B_s(x))}{\mathfrak{m}(B_s(x))} - \frac{\omega_{k-1}}{\omega_k}\right| < \varepsilon \quad \text{for any } s \le r_0.$$

Observe that, as a consequence of Theorem 5.14 and Remark 2.17, for any $\varepsilon > 0$ we have

$$\mathcal{F}_k E = \bigcup_{0<r<1} (\mathcal{F}_k E)_{r,\varepsilon} \quad \text{and} \quad (\mathcal{F}_k E)_{r,\varepsilon} \subset (\mathcal{F}_k E)_{r',\varepsilon} \text{ for } r' < r.$$

Hence for any $\eta > 0$ there exists $r = r(\eta) > 0$ such that

$$|D\chi_E|\,(\mathcal{F}_k E \setminus (\mathcal{F}_k E)_{s,\varepsilon}) < \eta, \quad \text{for any } \ 0 < s < r. \tag{5.27}$$

Proposition 5.27. *Let $N > 1$, $K \in \mathbb{R}$ and $k \in [1, N]$ be fixed. For any $\eta > 0$ there exists $\varepsilon = \varepsilon(\eta, N) < \eta$ such that, if $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(K, N)$ m.m.s., $E \subset X$ is a set of finite perimeter and finite measure, $p \in (\mathcal{F}_k E)_{2s,\varepsilon}$ for some $s \in (0, |K|^{-1/2})$ and there exists an ε-splitting map $u : B_{2s}(p) \to \mathbb{R}^{k-1}$ such that*

$$\frac{s}{\mathfrak{m}(B_{2s}(x))} \int_{B_{2s}(x)} |\nu \cdot \nabla u_a| \, \mathrm{d}|D\chi_E| < \varepsilon, \quad \textit{for any } a = 1, \dots, k-1, \tag{5.28}$$

then there exists $G \subset B_s(p)$ that satisfies:

(i) *$G \cap (\mathcal{F}_k E)_{2s,\varepsilon}$ is biLipschitz to a Borel subset of $\mathbb{R}^{k-1}$. More precisely,*

$$\big||u(x) - u(y)| - \mathsf{d}(x, y)\big| \le C_N \eta\, \mathsf{d}(x, y), \quad \forall x, y \in (\mathcal{F}_k E)_{2s,\varepsilon} \cap G; \tag{5.29}$$

(ii) $\mathscr{H}^h_5(B_s(p) \setminus G) < C_N \eta \frac{\mathfrak{m}(B_s(p))}{s}$.

Let us now prove Theorem 5.23 assuming Proposition 5.27.

Proof of Theorem 5.23. Assume without loss of generality that E has finite perimeter and measure, and that $\mathcal{F}_k E \subset B_2(p)$ for some $p \in X$. We claim that, for any $\eta > 0$, we can decompose $\mathcal{F}_k E = G^\eta \cup B^\eta \cup R^\eta$, where G^η is $(k-1)$-rectifiable and

$$\mathscr{H}_5^h(B^\eta) + |D\chi_E|(R^\eta) \le C_{N,K} |D\chi_E|(B_2(p))\eta + \eta. \tag{5.30}$$

Observe that the claim easily gives the sought conclusion. Indeed, setting $\eta_i := \eta 2^{-i}$, $G_\eta := \cup_i G^{\eta_i}$ and $R_\eta := \cup_{i\in\mathbb{N}} R^{\eta_i}$, G_η is still $(k-1)$-rectifiable and it holds

$$\mathscr{H}_5^h((\mathcal{F}_k E \setminus G_\eta) \setminus R_\eta) = 0,$$

hence, as a consequence of Lemma 4.35, $|D\chi_E|(\mathcal{F}_k E \setminus G_\eta) \setminus R_\eta) = 0$. Therefore

$$|D\chi_E|(\mathcal{F}_k E \setminus G_\eta) \le |D\chi_E|(R_\eta) \le C_N |D\chi_E|(B_2(p))\eta + \eta.$$

Setting $G := \cup_{i\in\mathbb{N}} G_{2^{-i}}$, we get that G is still $(k-1)$-rectifiable and coincides with $\mathcal{F}_k E$ up to a $|D\chi_E|$-negligible set.

Let us now prove the claim. To this aim fix $r > 0$ and $\varepsilon > 0$. We cover $(\mathcal{F}_k E)_{r,\varepsilon}$ with balls of radius smaller than $r/5$ with centre in $(\mathcal{F}_k E)_{r,\varepsilon}$ such that the assumptions of Proposition 5.27 are satisfied. The possibility of building such a covering is a consequence of Theorem 5.14 and of Proposition 5.25. By Vitali's Theorem 1.9, we can extract a disjoint family $\left\{B_{r_i/5}(x_i)\right\}_{i\in\mathbb{N}}$ such that $(\mathcal{F}_k E)_{r,\varepsilon} \subset \cup_i B_{r_i}(x_i)$. Applying Proposition 5.27, for any $i \in \mathbb{N}$ we can find $G_i \subset B_{r_i}(x_i)$ such that $G_i \cap (\mathcal{F}_k E)_{r,\varepsilon}$ is $(k-1)$-rectifiable and $\mathscr{H}_5^h(B_{r_i}(x_i) \setminus G_i) < C_N \eta \frac{\mathfrak{m}(B_{r_i}(x_i))}{r_i}$. Set $G_r^\eta := (\mathcal{F}_k E)_{r,\varepsilon} \cap (\cup_i G_i)$ and observe that

$$\begin{aligned}
\mathscr{H}_5^h((\mathcal{F}_k E)_{r,\varepsilon} \setminus G_r^\eta) &\le \sum_{i\in\mathbb{N}} \mathscr{H}_5^h(B_{r_i}(x_i) \setminus G_i) \le \sum_{i\in\mathbb{N}} C_N \eta \frac{\mathfrak{m}(B_{r_i}(x_i))}{r_i} \\
&\overset{(1.45)}{\le} C_N \eta \sum_{i\in\mathbb{N}} \frac{\mathfrak{m}(B_{r_i/5}(x_i))}{r_i/5} \le C_{N,K}\eta \sum_{i\in\mathbb{N}} |D\chi_E|(B_{r_i/5}(x_i)) \\
&\le C_{N,K}\eta |D\chi_E|(B_2(p)),
\end{aligned}$$

thanks to

$$\frac{\mathfrak{m}(B_{r_i/5}(x_i))}{r_i/5} \le C(k) |D\chi_E|(B_{r_i/5}(x_i)),$$

that holds true provided ε is small enough.
Setting $B_r^\eta := (\mathcal{F}_k E)_{r,\varepsilon} \setminus G_r^\eta$, the argument above gives the decomposition

$$(\mathcal{F}_k E)_{r,\varepsilon} = G_r^\eta \cup B_r^\eta,$$

where G_r^η is $(k-1)$-rectifiable and $\mathscr{H}_5^h(B_r^\eta) \le C_{N,K}\eta |D\chi_E|(B_2(p))$. Let us now choose $r > 0$ small enough to have (5.27). This allows us to write

$$\mathcal{F}_k E = G_r^\eta \cup B_r^\eta \cup (\mathcal{F}_k E \setminus (\mathcal{F}_k E)_{r,\varepsilon}) =: G^\eta \cup B^\eta \cup R^\eta$$

and to conclude the proof. □

3.1. Proof of Proposition 5.27. The proof is divided in three steps.
Aim of the first one is to provide a bridge between analysis and geometry suitable for this context: we prove that, whenever at a certain location and scale the set of finite perimeter is quantitatively close to a half-space in a Euclidean space and there is a $(k-1,\delta)$-splitting map which is also δ-orthogonal to the normal vector in the sense of (5.28), then the $(k-1,\delta)$-splitting map is an η-isometry (in the scale invariant sense) when restricted to the support of the perimeter.
The second step is analytic and dedicated to the propagation of the δ-orthogonality condition.

In the last one we get the biLipschitz property relying on the observation that a map which is an η-isometry (in the scale invariant sense) at any location and scale is biLipschitz.

Step 1. Let $N > 0$, $K \in \mathbb{R}$ and $k \in [1, N]$ be fixed. We claim that, for any $\eta > 0$, there exists $\delta = \delta_{\eta,N} \leq \eta$ such that, for any pointed $\mathrm{RCD}(K, N)$ m.m.s. $(X, \mathsf{d}, \mathfrak{m}, x)$ and for any set of finite perimeter and finite measure $E \subset X$ such that, for some $0 < r < |K|^{-1/2}$,

(i) $\mathsf{d}_{pmGH}\left(\left(X, (2r)^{-1}\mathsf{d}, \frac{\mathfrak{m}}{\mathfrak{m}(B_{2r}(x))}, x\right), \left(\mathbb{R}^k, \mathsf{d}_{\mathrm{eucl}}, \frac{1}{\omega_k}\mathcal{L}^k, 0^k\right)\right) < \delta$;

(ii)
$$\left|\frac{\mathfrak{m}(B_t(x) \cap E)}{\mathfrak{m}(B_t(x))} - \frac{1}{2}\right| + \left|\frac{t\,|D\chi_E|\,(B_t(x))}{\mathfrak{m}(B_t(x))} - \frac{\omega_{k-1}}{\omega_k}\right| < \delta \quad \text{for any } t \leq 2r;$$

(iii) there exists $u := (u_1, \ldots, u_{k-1}) : B_{2r}(x) \to \mathbb{R}^{k-1}$ a δ-splitting map satisfying
$$\frac{r}{\mathfrak{m}(B_{2r}(x))} \int_{B_{2r}(x)} |\nu \cdot \nabla u_a| \;\mathrm{d}|D\chi_E| < \delta, \quad \text{for any } a = 1, \ldots, k-1,$$

then $u : \operatorname{supp}|D\chi_E| \cap B_r(x) \to B_r^{\mathbb{R}^{k-1}}(u(x))$ verifies
$$||u(y) - u(z)| - \mathsf{d}(y, z)| \leq \eta r$$
for any $y, z \in \operatorname{supp}|D\chi_E| \cap B_r(x)$.

By scaling it is enough to prove the claim when $r = 1/2$ and $|K| \leq 4$. Let us argue by contradiction. Then we could find $\eta > 0$, a sequence $(X_n, \mathsf{d}_n, \mathfrak{m}_n, E_n, x_n)$, points $z_1^n, z_2^n \in \operatorname{supp}|D\chi_{E_n}| \cap B_{1/2}(x_n)$, and $1/n$-splitting maps $u^n : B_1(x_n) \to \mathbb{R}^{k-1}$ satisfying (i), (ii) and (iii) with $\delta = 1/n$, $u^n(x_n) = 0$ and

$$||u^n(z_1^n) - u^n(z_2^n)| - \mathsf{d}_n(z_1^n, z_2^n)| \geq \eta, \quad \forall n \in \mathbb{N}. \tag{5.31}$$

Notice that $\mathsf{d}_n(z_1^n, z_2^n) \geq \min\{\eta/(C_N - 1), \eta\}$ since u^n is C_N-Lipschitz.

Observe that, by (i), $(X_n, \mathsf{d}_n, \mathfrak{m}_n, x_n)$ converge to $\left(\mathbb{R}^k, \mathsf{d}_{\mathrm{eucl}}, \frac{1}{\omega_k}\mathcal{L}^k, 0^k\right)$ in the pmGH topology. We let (Z, d_Z) be a proper metric space where this convergence is realized. Since E_n satisfies the bound

$$\left|\frac{\mathfrak{m}_n(E_n \cap B_t(x_n))}{\mathfrak{m}_n(B_t(x_n))} - \frac{1}{2}\right| + \left|\frac{t|D\chi_{E_n}|(B_t(x_n))}{\mathfrak{m}_n(B_t(x_n))} - \frac{\omega_{k-1}}{\omega_k}\right| < 1/n \quad \text{for any } t \leq 1, \tag{5.32}$$

up to extracting a subsequence, $E_n \cap B_1(x_n) \to F \cap B_1(0^k)$ in L^1-strong, where F is of locally finite perimeter in $B_1(0^k)$ thanks to Proposition 4.26.

Up to extracting again a subsequence we can assume $u^n \to u^\infty$ strongly in $H^{1,2}$ on $B_1(0^k)$, where $u^\infty : B_1^{\mathbb{R}^k}(0) \to \mathbb{R}^{k-1}$ is the restriction of an orthogonal projection, as a consequence of Proposition 1.29 and Theorem 1.126. We assume, without loss of generality, that $u^\infty(x) = (x_1, \ldots, x_{k-1})$ for any $x \in B_1(0^k)$.

We claim that $\mathcal{L}^k\left(\left(F \cap B_1(0^k)\right) \Delta \left(\{x_k > 0\} \cap B_1(0^k)\right)\right) = 0$ and

$$\int g \;\mathrm{d}|D\chi_{E_n}| \to \int g \;\mathrm{d}|D\chi_{\{x_k>0\}}| \quad \text{for any } g \in C(Z) \text{ with } \operatorname{supp}(g) \subset B_{1/2}(0^k). \tag{5.33}$$

This would imply that $z_1^\infty, z_2^\infty \in \{x_k = 0\}$, therefore $|u^\infty(z_1^\infty) - u^\infty(z_2^\infty)| = \mathsf{d}_{\mathrm{eucl}}(z_1^\infty, z_2^\infty)$ that contradicts (5.31).

In order to verify the claim let us choose a smooth function $\psi_\infty : \mathbb{R}^k \to \mathbb{R}$ with compact support in $B_1(0^k)$. Then we consider a sequence $\psi_n \in \mathrm{Lip}(X_n, \mathsf{d}_n)$ with $\operatorname{supp}(\psi_n) \subset B_1(x_n)$, $\|\psi_n\|_{L^\infty} + \||\nabla\psi_n|\|_{L^\infty} \leq 4$ and $\psi_n \to \psi_\infty$ strongly in $H^{1,2}$ along the sequence $(X_n, \mathsf{d}_n, \mathfrak{m}_n, x_n)$, whose existence is proved in Lemma 1.125. Observe now that

$$\nabla\psi_n \cdot \nabla u_a^n \to \nabla\psi_\infty \cdot e_a = \frac{\partial\psi_\infty}{\partial x_a} \quad \text{in } L^2\text{-strong, for any } a = 1, \ldots, k-1,$$

by Proposition 1.118(i) and Proposition 1.118(iii). This observation, along with Proposition 1.118(ii) and Remark 4.18, gives

$$\int_F \frac{\partial \psi_\infty}{\partial x_a} \, \mathrm{d}\frac{\mathcal{L}^k}{\omega_k} = \lim_{n\to\infty} \int_{E_n} \nabla\psi_n \cdot \nabla u_a^n \, \mathrm{d}\mathfrak{m}_n. \tag{5.34}$$

We can now use Theorem 5.6 and (iii) to conclude that

$$\begin{aligned} \left| \int_F \frac{\partial \psi_\infty}{\partial x_a} \, \mathrm{d}\frac{\mathcal{L}^k}{\omega_k} \right| &\overset{(5.34)}{=} \lim_{n\to\infty} \left| \int_{E_n} \nabla\psi_n \cdot \nabla u_a^n \, \mathrm{d}\mathfrak{m}_n \right| \\ &= \lim_{n\to\infty} \left| \int \psi_n \nabla u_a^n \cdot \nu_{E_n} \, \mathrm{d}|D\chi_{E_n}| \right| \\ &\le \lim_{n\to\infty} \int |\psi_n| \, |\nabla u_a^n \cdot \nu_{E_n}| \, \mathrm{d}|D\chi_{E_n}| = 0, \end{aligned}$$

for $a = 1, \dots, k-1$. Since $\psi_\infty \in C_c^\infty(B_1(0^k))$ is arbitrary we obtain that

$$\mathcal{L}^k \left(\left(F \cap B_1(0^k) \right) \Delta \left(\{x_k > \lambda\} \cap B_1(0^k) \right) \right) = 0 \quad \text{for some } \lambda \in \mathbb{R}.$$

Using again (5.32) we get $\mathcal{L}^k(F \cap B_1(0^k)) = \omega_k/2$ that forces $\lambda = 0$.
Let us finally prove (5.33). To this end we use again (5.32) with $t = 1/2$ obtaining that

$$\lim_{n\to\infty} |D\chi_{E_n}| (B_{1/2}(x_n)) = \frac{\omega_{k-1}}{2^{k-1}} = \left| D\chi_{\{x_k>0\}} \right| (B_{1/2}(0^k)).$$

We can now apply the third conclusion of Proposition 4.26 and conclude.

Step 2. By assumption there exists an ε-splitting map $u : B_{2s}(p) \to \mathbb{R}^{k-1}$ such that (5.28) holds true. We wish to propagate now both the ε-splitting condition and the orthogonality condition (5.28) at any scale and point outside a set of small $\mathscr{H}_5^h$-measure. More precisely we are going to prove that there exists a set $G \subset B_s(p)$ with $\mathscr{H}_5^h(B_s(p) \setminus G) \le C_N \sqrt{\varepsilon} \frac{\mathfrak{m}(B_s(p))}{s}$ such that

(i) for any $x \in G$, $0 < r < s$, $u : B_r(x) \to \mathbb{R}^{k-1}$ is a $C_N \varepsilon^{1/4}$-splitting map;
(ii) for any $x \in G$, $0 < r < s$, it holds

$$\frac{r}{\mathfrak{m}(B_r(x))} \int_{B_r(x)} |\nu \cdot \nabla u_a| \, \mathrm{d}|D\chi_E| < \sqrt{\varepsilon}, \quad \text{for } a = 1, \dots, k-1. \tag{5.35}$$

We can find a set G' satisfying the measure estimate and (i) applying Corollary 5.16. Hence it is enough to find a set G'' satisfying the measure estimate and (ii) and to take $G := G' \cap G''$. To do so we apply a weighted maximal argument. Let us fix $a = 1, \dots, k-1$ and set

$$M_E(x) := \sup_{0<r<s} \frac{r}{\mathfrak{m}(B_r(x))} \int_{B_r(x)} |\nu \cdot \nabla u_a| \, \mathrm{d}|D\chi_E|.$$

We claim that $G'' := \{x \in B_s(p) \,:\, M_E(x) < \sqrt{\varepsilon}\}$ has the sought properties.
Indeed, for any $x \in B_s(p) \setminus G''$, there exists $\rho_x \in (0, s)$ such that

$$\frac{\rho_x}{\mathfrak{m}(B_{\rho_x}(x))} \int_{B_{\rho_x}(x)} |\nu \cdot \nabla u_a| \, \mathrm{d}|D\chi_E| \ge \sqrt{\varepsilon}. \tag{5.36}$$

Applying Vitali's covering Theorem 1.9 to the family $\{B_{\rho_x}(x)\}_{x \in B_s(p)\setminus G''}$ we find a disjoint subfamily $\{B_{r_i}(x_i)\}_{i\in\mathbb{N}}$ such that $B_s(p) \setminus G'' \subset \cup_i B_{5r_i}(x_i)$. Taking into account the disjointedness of the covering, we can compute

$$\mathscr{H}_5^h(B_s(p) \setminus G'') \le \sum_{i\in\mathbb{N}} h(B_{5r_i}(x_i)) = \sum_{i\in\mathbb{N}} \frac{\mathfrak{m}(B_{5r_i}(x_i))}{5r_i}$$

$$\overset{(1.45)}{\le} C_N \sum_{i\in\mathbb{N}} \frac{\mathfrak{m}(B_{r_i}(x_i))}{r_i} \overset{(5.36)}{\le} C_N \sum_{i\in\mathbb{N}} \varepsilon^{-1/2} \int_{B_{r_i}(x_i)} |\nu\cdot\nabla u_a| \, \mathrm{d}|D\chi_E|$$
$$\le C_N \varepsilon^{-1/2} \int_{B_{2s}(p)} |\nu\cdot\nabla u_a| \, \mathrm{d}|D\chi_E| \overset{(5.28)}{\le} C_N\sqrt{\varepsilon}\,\frac{\mathfrak{m}(B_{2s}(p))}{s}.$$

Step 3. We claim now that for any $\eta > 0$ there exists $\varepsilon = \varepsilon_{\eta,N} > 0$ small enough such that for any $0 < r < s$ and $x \in G \cap (\mathcal{F}_k E)_{2s,\varepsilon}$ the map

$$u = (u_1, \dots, u_{k-1}) : \operatorname{supp}|D\chi_E| \cap B_r(x) \to \mathbb{R}^{k-1}$$

verifies

$$||u(y) - u(z)| - \mathsf{d}(y,z)| \le \eta r$$

for any $y, z \in \operatorname{supp}|D\chi_E| \cap B_r(x)$. The claim is a consequence of Step 1. Indeed, for any $x \in G \cap (\mathcal{F}_k E)_{2s,\varepsilon}$ and any $r \in (0, s)$, the conditions (i) and (ii) of Step 1 are satisfied by definition of $(\mathcal{F}_k E)_{2s,\varepsilon}$. Moreover u is a $C_N\varepsilon^{1/4}$-splitting map on $B_r(x)$ satisfying (5.35), hence also the assumption (iii) of Step 1 is satisfied for ε small enough.

In order to conclude the proof we just have to prove (i) in the statement of Proposition 5.27, since (ii) follows from Step 2 choosing ε small enough so that $\sqrt{\varepsilon} < \eta$. To this aim, take $x, y \in G \cap (\mathcal{F}_k E)_{2s,\varepsilon}$ and choose $r := \mathsf{d}(x, y)$. Our claim ensures that

$$\big||u(x) - u(z)| - \mathsf{d}(x,z)\big| \le r\eta \qquad \text{for any } z \in \operatorname{supp}|D\chi_E| \cap B_r(x),$$

therefore we can take $z = y$ and conclude.

4. Representation of the perimeter

In this last section we are concerned with some consequences of the results achieved in Section 2 and Section 3 at the level of representation formulas for the perimeter measure. Let us recall that in the classical Euclidean theory one can prove that if $E \subset \mathbb{R}^N$ is a set of locally finite perimeter then $|D\chi_E| = \mathscr{H}^{N-1} \llcorner \mathcal{F}E$. As we shall see below this is the case even in the setting of non collapsed $\mathrm{RCD}(K, N)$ metric measure spaces (cf. Theorem 5.29) an one can obtain a counterpart of the representation formula for the perimeter also for general $\mathrm{RCD}(K, N)$ spaces.

Remark 5.28. In general, even without the non collapsing assumption, it is easily seen that the reduced boundary $\mathcal{F}E$, that we introduced in Definition 5.20, is contained in the essential boundary $\partial^* E$, namely the complement of the sets of density and rarefaction. In the more general context of PI spaces it is known after [**5**] that $|D\chi_E|$ is representable as $\theta\mathcal{S} \llcorner \partial^* E$ for some density θ, where $\mathcal{S}$ denotes the measure induced by the gauge function $\zeta(B_r(x)) = \mathfrak{m}(B_r(x))/r$ with Carathéodory's construction. In particular in our context $\partial^* E$ and $\mathcal{F}E$ coincide up to $\mathcal{S}$-negligible sets.

Theorem 5.29. *Let $(X, \mathsf{d}, \mathscr{H}^N)$ be a non collapsed $\mathrm{RCD}(K, N)$ m.m.s. for some $K \in \mathbb{R}$ and $1 \le N < \infty$. Let $E \subset X$ be a set of locally finite perimeter. Then*

$$|D\chi_E| = \mathscr{H}^{N-1} \llcorner \mathcal{F}_N E. \tag{5.37}$$

Proof. Let us start with some preliminary observations about non collapsed RCD spaces (ncRCD spaces for short).

First, on any $\mathrm{ncRCD}(K, N)$ m.m.s. only the top dimensional regular set $\mathcal{R}_N$ is not empty. Then, for any $x \in \mathcal{R}_N$, thanks to the volume convergence theorem, it holds that

$$\lim_{r\to 0} \frac{\mathcal{H}^N(B_r(x))}{\omega_N r^N} = 1. \tag{5.38}$$

We refer to [**84**] for the proof of these results.
Moreover the Bishop-Gromov inequality yields that $(X, \mathsf{d}, \mathscr{H}^N)$ is N-Ahlfors regular under these assumptions. It follows that the codimension one Hausdorff type measure $\mathscr{H}^h$ and the $(N-1)$-dimensional Hausdorff measure $\mathscr{H}^{N-1}$ are mutually absolutely continuous.

Let us pass to the verification of (5.37). We know from [**5**] that $|D\chi_E| = \theta\mathcal{H}^h \llcorner \partial^* E$ for some density function θ, where $\partial^* E$ is the essential boundary of E. Thanks to Theorem 5.14 and to the the first observation above, we can improve this conclusion to $|D\chi_E| = \theta\mathscr{H}^{N-1} \llcorner \mathcal{F}_N E$.

Next, thanks to the rectifiability of the reduced boundary we can appeal to [**150**, Theorem 9] (see also [**23**, Theorem 5.4]) to conclude that

$$\theta(x) = \lim_{r\to 0} \frac{|D\chi_E|(B_r(x))}{\omega_{N-1} r^{N-1}}, \quad \text{for } \mathscr{H}^{N-1}\text{-a.e. } x \in \mathcal{F}_N E. \tag{5.39}$$

Therefore it is sufficient to prove that

$$\lim_{r\to 0} \frac{|D\chi_E|(B_r(x))}{\omega_{N-1} r^{N-1}} = 1, \quad \text{for } \mathscr{H}^{N-1}\text{-a.e. } x \in \mathcal{F}_N E. \tag{5.40}$$

We claim that (5.40) holds true at any $x \in \mathcal{F}E$ such that the perimeter measures on the rescaled space converge to the perimeter of the blow-up and we observe that, as we already remarked $\mathscr{H}^{N-1}$-a.e. point in $\mathcal{F}E$ has this property.
In order to prove the claim we compute

$$\begin{aligned}
\lim_{r\to 0} \frac{|D\chi_E|(B_r(x))}{\omega_{N-1} r^{N-1}} &= \frac{1}{\omega_{N-1}} \lim_{r\to 0} \frac{r\,|D\chi_E|(B_r(x))}{\mathscr{H}^N(B_r(x))} \cdot \frac{\mathscr{H}^N(B_r(x))}{r^N} \\
&= \frac{1}{\omega_{N-1}} \lim_{r\to 0} |D^r\chi_E|(B_1(x)) \cdot \lim_{r\to 0} \frac{\mathscr{H}^N(B_r(x))}{r^N} \\
&= \frac{1}{\omega_{N-1}} \cdot \frac{\omega_{N-1}}{\omega_N} \cdot \omega_N = 1,
\end{aligned}$$

where we denoted by $|D^r\chi_E|$ the perimeter measure on $(X, \mathsf{d}/r, \mathscr{H}^N/\mathscr{H}^N(B_r(x)), x)$ that converge to $(\mathbb{R}^N, \mathsf{d}_{\text{eucl}}, \mathcal{L}^N/\omega_N, 0)$ as $r \to 0$.

□

Remark 5.30. Let us point out, for the sake of the comparison with the result appearing in [**8**], that the rectifiability of the reduced boundary, together with the already mentioned results about differentiation of measures, allow in particular to obtain that the spherical codimension one Hausdorff measure and the classical codimension one Hausdorff measure coincide on the reduced boundary $\mathcal{F}_N E$.

More in general, without the non collapsing assumption, Theorem 5.14 allows to obtain a representation formula for the perimeter measure in terms of the codimension one spherical Hausdorff measure $\mathcal{S}^h$.
Before stating the representation result let us make a couple of comments. Even in the case of a weighted Euclidean space $(\mathbb{R}^N, \mathsf{d}_{\text{eucl}}, \theta\mathcal{L}^N)$, where $\theta : \mathbb{R}^N \to (0, \infty)$ is a smooth weight function, one can argue that the perimeter measure takes into account the presence of the weight. Indeed, if E has locally finite perimeter (in the weighted space), then its perimeter can be represented as $\theta\mathcal{H}^{N-1} \llcorner \mathcal{F}E$. When passing to metric measure spaces, one faces a new difficulty, due to the absence of a pointwise (or at least perimeter almost everywhere) defined weight function θ. Indeed, the density appearing in Theorem 2.26 is only defined $\mathfrak{m}$-a.e.. Therefore, with the aim of proving a representation formula in this more general context it is more appealing to interpret $\theta\mathscr{H}^{N-1}$ as codimension one measure built from $\theta\mathcal{L}^N$.

Theorem 5.31. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *m.m.s. with essential dimension* n*. Let* $E \subset X$ *be a set of locally finite perimeter. Then*

$$|D\chi_E| = \sum_{k=1}^{n} \frac{\omega_{k-1}}{\omega_k} \mathcal{S}^h \llcorner \mathcal{F}_k E. \tag{5.41}$$

Proof. We will rely on the very general differentiation formula [**164**, Theorem 3] which sharpens previous results in [**100**].

In order to do so we need to compute the *generalized density*

$$\lim_{r \to 0} \sup_{x \in B_s(y),\ s \le r} \frac{s\,|D\chi_E|\,(B_s(y))}{\mathfrak{m}(B_s(y))} \tag{5.42}$$

at points $x \in \mathcal{F}E$.

If $x \in \mathcal{F}_k E$ and verifies the additional conclusion of Corollary 4.37, then we can compute

$$\begin{aligned}\lim_{r \to 0} \frac{r\,|D\chi_E|\,(B_r(x))}{\mathfrak{m}(B_r(x))} &= \lim_{r \to 0} \frac{r\,|D\chi_E|\,(B_r(x))}{C(x,r)} \cdot \frac{C(x,r)}{\mathfrak{m}(B_r(x))} = \lim_{r \to 0} \frac{|D^r\chi_E|\,(B_1(x))}{\mathfrak{m}_x^r(B_1(x))} \\ &= \frac{\mathscr{H}^{k-1}(B_1(0))}{\mathscr{H}^k(B_1(0))} = \frac{\omega_{k-1}}{\omega_k},\end{aligned}$$

where the information $x \in \mathcal{R}_k$ and the weak convergence of the rescaled perimeter measures to the perimeter measure of a half-space (see Corollary 4.37) play a role.
Then we infer that

$$\lim_{r \to 0} \sup_{x \in B_s(y),\ s \le r} \frac{s\,|D\chi_E|\,(B_s(y))}{\mathfrak{m}(B_s(y))} \ge \frac{\omega_{k-1}}{\omega_k}. \tag{5.43}$$

We wish to prove that the inequality in (5.43) is an equality. In order to do so it suffices to show that, for any sequence of radii $r_i \to 0$ and points $x_i \in B_{r_i}(x)$, it holds

$$\limsup_{i \to \infty} \frac{r_i\,|D\chi_E|\,(B_{r_i}(x_i))}{\mathfrak{m}(B_{r_i}(x_i))} \le \frac{\omega_{k-1}}{\omega_k}.$$

Let us recall that, since $x \in \mathcal{F}_k E$, the sequence $(X, \mathsf{d}/r_i, \mathfrak{m}/C(x, r_i), x)$ is converging in the pmGH topology to $(\mathbb{R}^k, \mathsf{d}_{\mathrm{eucl}}, c_k\mathcal{L}^k, 0)$. Moreover we can assume, up to extract a subsequence, that x_i converge to $z \in B_1^{\mathbb{R}^k}(0)$.
Then we can compute

$$\begin{aligned}\limsup_{i \to \infty} \frac{r_i\,|D\chi_E|\,(B_{r_i}(x_i))}{\mathfrak{m}(B_{r_i}(x_i))} &= \lim_{i \to \infty} \frac{C(x, r_i)}{\mathfrak{m}(B_{r_i}(x_i))} \cdot \lim_{i \to \infty} \frac{r_i\,|D\chi_E|\,(B_{r_i}(x_i))}{C(x, r_i)} \\ &= \frac{c_k}{\mathcal{L}^k(B_1^{\mathbb{R}^k}(z))} \cdot \frac{|D\chi_{\mathbb{H}^k}|\,(B_1^{\mathbb{R}^k}(z))}{c_k} \\ &\le \frac{\omega_{k-1}}{\omega_k},\end{aligned} \tag{5.44}$$

since $|D\chi_{\mathbb{H}^k}|\,(B_1^{\mathbb{R}^k}(z)) \le \omega_{k-1}$ for any $z \in B_1^{\mathbb{R}^k}(0)$.

Combining (5.43) with (5.44) we get

$$\lim_{r \to 0} \sup_{x \in B_s(y),\ s \le r} \frac{s\,|D\chi_E|\,(B_s(y))}{\mathfrak{m}(B_s(y))} = \frac{\omega_{k-1}}{\omega_k},$$

for $|D\chi_E|$-a.e. $x \in \mathcal{F}_k E$.

An application of [**164**, Theorem 3] yields now

$$|D\chi_E| \llcorner \mathcal{F}_k E = \frac{\omega_{k-1}}{\omega_k} \mathcal{S}^h \llcorner \mathcal{F}_k E \tag{5.45}$$

and (5.41) follows.

□

We conclude the chapter with a conjecture concerning a different (and sharper) representation formula for the perimeter measure in the collapsed case.

Conjecture 5.32. There exists a constant $c = c_n > 0$ such that the following holds. For any $\mathrm{RCD}(K,N)$ m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ with essential dimension $1 \le n \le N$ and for any set of locally finite perimeter $E \subset X$ the limit

$$\theta(x) := \lim_{r \to 0} \frac{\mathfrak{m}(B_r(x))}{r^n} \tag{5.46}$$

exists for $|D\chi_E|$-a.e. $x \in X$. Moreover

$$|D\chi_E| = c_n \theta \mathcal{H}^{n-1} \llcorner \mathcal{F}_n E.$$

Let us remark that the tools used to establish the absolute continuity of the reference measure $\mathfrak{m}$ with respect to $\mathcal{H}^n$ in [**85**, **120**, **146**] seem to be not suitable for the study in codimension one.

CHAPTER 6

Polya-Szego inequality and Dirichlet p-spectral gap on $\mathrm{RCD}(N-1,N)$ spaces

This last chapter of the thesis is dedicated to some results about the Dirichlet p-spectral gap on $\mathrm{RCD}(N-1,N)$ metric measure spaces that we obtained in [**171**], joint work with Mondino.

Let us briefly introduce the problem of the Dirichlet p-spectral gap.
At the end of the Eighteenth century, Lord Rayleigh conjectured that, among all membranes of a given area, the disk has the lowest fundamental frequency of vibration. This was proven in 1920ies by Faber and Krahn for domains in the Euclidean plane and later extended by Krahn to higher dimensions.

Theorem 6.1 (Rayleigh-Faber-Krahn inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be a relatively compact open domain with smooth boundary. Then the first Dirichlet eigenvalue of Ω is bounded below by the first Dirichlet eigenvalue of a Euclidean ball having the same volume of Ω. Moreover the equality is attained if and only if Ω is a ball.*

The proof of Theorem 6.1 is based on two key facts:

- a variational characterisation for the first Dirichlet eigenvalue;
- the properties of *spherical decreasing rearrangements* of functions.

The variational characterisation of the first eigenvalue, originally due to Rayleigh, is given by

$$\lambda(\Omega) := \inf_{u \in C_c^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 \, \mathrm{d}\mathcal{L}^n}{\int_\Omega u^2 \, \mathrm{d}\mathcal{L}^n}. \tag{6.1}$$

Let us briefly introduce the notion of spherical decreasing rearrangement and the Polya-Szego inequality. Given an open subset $\Omega \subset \mathbb{R}^n$, the symmetrized domain $\Omega^* \subset \mathbb{R}^n$ is a ball with the same measure as Ω centred at the origin. If u is a real-valued Borel function defined on Ω, its spherical decreasing rearrangement u^* is a function defined on the ball Ω^* with the following properties: u^* depends only on the distance from the origin, it is decreasing along the radial direction and it is equi-measurable with u (i.e. the super-level sets have the same volume: $|\{u > t\}| = |\{u^* > t\}|$, for every $t \in \mathbb{R}$). Since the function and its spherical decreasing rearrangement are equi-measurable, their L^2-norms are the same. Faber and Krahn proved that the L^2-norm of the gradient of a function decreases under rearrangements. This last property was formalised, extended to every L^p, $1 < p < \infty$, and applied to several problems in mathematical physics by Polya and Szego in [**183**]. The Polya-Szego inequality, combined with the variational characterization (6.1), immediately gives Theorem 6.1.

Bérard-Meyer extended this idea to Riemannian manifolds (M^n, g) with $\mathrm{Ric}_g \geq (n-1)g$ in [**42**].

Theorem 6.2 (Bérard-Meyer inequality)**.** *Let (M^n, g) be a Riemannian manifold with $\mathrm{Ric}_g \geq (n-1)g$, and let $\Omega \subset M$ be an open subset with smooth boundary. Let $\mathbb{S}^n$ be the round n-dimensional sphere of radius 1 and let $\Omega^* \subset \mathbb{S}^n$ be a metric ball having the same renormalized volume of Ω, i.e $|\Omega|/|M| = |\Omega^*|/|\mathbb{S}^n|$. Then $\lambda(\Omega) \geq \lambda(\Omega^*)$ and equality is achieved if and only if M is isometric to $\mathbb{S}^n$ and Ω is a metric ball.*

The two key ideas in [**42**] are the following. First, for a function $u \in C^1_c(M)$ define a spherical decreasing rearrangement u^* on $\mathbb{S}^n$. Second, replace the Euclidean isoperimetric inequality by the Lévy-Gromov isoperimetric inequality [**127**, Appendix C] in the proof of the corresponding Polya-Szego type inequality. Let us finally mention that, arguing along the same lines, Theorem 6.2 was generalized to the first Dirichlet eigenvalue of the p-Laplacian for any $p \in (1,\infty)$ by Matei [**166**].

The spectral gap in $\mathrm{CD}(K,N)$ spaces for *Neumann boundary conditions*, called Lichnerowicz inequality, was established by Lott-Villani [**162**] in the case $p = 2$ (see also [**96**] and [**140**] for related results in $\mathrm{RCD}(K,N)$ spaces) and by Cavalletti-Mondino [**59**] for general $p \in (1,\infty)$.

As we have already pointed out, the coarea formula is a very robust tool that holds true for general metric measure spaces, see Theorem 1.40. Moreover in [**58**] Cavalletti and Mondino generalized the Lévy Gromov inequality to the setting of essentially non branching $\mathrm{CD}(K,N)$ metric measure spaces (verifying an additional upper bound on the diameter in the case $K \leq 0$) exploiting the so-called *localization technique* (see also [**153**]). Therefore the attempt to generalize Theorem 6.2 to this framework, following the same strategy of proof, seemed very natural.

In [**171**] we pursued such plan generalizing the Polya-Szego and Dirichlet p-spectral gap inequalities to the framework of essentially non branching $\mathrm{CD}(K,N)$ metric measure spaces, for positive K. Moreover, building upon the characterization of the equality in the Lévy-Gromov inequality proved in [**58**], we obtained a characterization of the equality both for the Polya-Szego and the Dirichlet p-spectral gap inequalities for $\mathrm{RCD}(N-1,N)$ spaces.
This last result, combined with the compactness of the class of $\mathrm{RCD}(N-1,N)$ metric measure spaces (with unit measure) allowed to obtained also an *almost rigidity* result for the Dirichlet p-spectral gap, which seems to be new even for smooth Riemannian manifolds, besides some particular cases (see [**43**]).

With respect to [**171**] here we limit the discussion to the infinitesimally Hilbertian case, for the sake of coherence with the rest of the thesis. Moreover, when treating rigidity, we focus only on the statements about the ambient spaces and not on the conclusion that can be achieved about the form of the eigenfunctions/extremizers for the Polya-Szego inequality.

Even though the topic of the present chapter is apparently more related with functional analysis than with geometric measure theory, the arguments leading to the almost rigidity result are very similar in nature to those we used in the development of the structure theory of $\mathrm{RCD}(K,N)$ metric measure spaces. We also remark that the almost rigidity for the Neumann spectral gap on $\mathrm{RCD}(N-1,N)$ spaces has led in [**64**] to deep geometric consequences on the singular sets of non collapsed Ricci limit spaces enlightening a connection between spectral gaps and the structure theory of spaces with Ricci curvature bounded from below.
Eventually we point out that the *non quantitative* almost rigidity result for the Dirichlet p-spectral gap inequality opens to the investigation of *quantitative* versions of this statement.

This chapter is organised as follows. In Section 1 we introduce the relevant one dimensional model spaces, the notion of rearrangement on model space and we establish some Polya-Szego type inequalities tailored for this setting. Section 2 is devoted to the proof of the Dirichlet p-spectral gap, obtained through the Polya-Szego inequality as in the classical case. In Section 3 we deal with the characterization of the equality cases in the Polya-Szego and spectral gap inequalities. Eventually in Section 4 we establish an almost rigidity result for the Dirichlet spectral gap relying on the theory of convergence and stability for functional spaces over sequences of $\mathrm{RCD}(N-1,N)$ spaces converging in the measured Gromov-Hausdorff sense.

1. Polya-Szego inequality

Below we recall the definition of the family of one dimensional model spaces for the curvature dimension condition $\mathrm{CD}(N-1,N)$ (cf. [**127**, Appendix C] and [**168**]). We remark that the discussion could be extended to the whole range of lower Ricci curvature bounds $K \in \mathbb{R}$ once we add an upper bound on the diameter on the space and up to using a family of model spaces in place of a single model space.

Definition 6.3 (One dimensional model spaces)**.** For any $1 < N < \infty$ we define the one dimensional model space $(I, \mathsf{d}_{\mathrm{eucl}}, \mathfrak{m}_N)$ for the curvature dimension condition $\mathrm{CD}(N-1,N)$ by

$$I := [0,\pi], \quad \mathfrak{m}_N := \frac{1}{c_N} \sin^{N-1}(t) \mathcal{L}^1 \llcorner [0,\pi], \tag{6.2}$$

where $\mathsf{d}_{\mathrm{eucl}}$ is the restriction to $[0,\pi]$ of the canonical euclidean distance over the real line and $c_N := \int_0^\pi \sin^{N-1}(t)\, \mathrm{d}\mathcal{L}^1(t)$ is the normalizing constant.
In order to shorten the notation, we set $h_N(t) := \frac{1}{c_N} \sin^{N-1}(t)$ for all $t \in [0,\pi]$.

Let us recall that, for any metric measure space $(X,\mathsf{d},\mathfrak{m})$ such that $\mathfrak{m}(X) = 1$, the isoperimetric profile $\mathcal{I}_{(X,\mathsf{d},\mathfrak{m})} : [0,1] \to [0,+\infty)$ is defined by

$$\mathcal{I}_{(X,\mathsf{d},\mathfrak{m})}(v) := \inf \left\{ \mathrm{Per}(E) : \, E \in \mathscr{B}(X), \quad \mathfrak{m}(E) = v \right\}.$$

Proposition 6.4. *Fix $N \in (1,\infty)$. Let $\left((X_n,\mathsf{d}_n,\mathfrak{m}_n)\right)_n$ be a sequence of normalized* $\mathrm{RCD}(N-1,N)$ *spaces converging to $(X,\mathsf{d},\mathfrak{m})$ in the measured Gromov-Hausdorff sense.*
Denote by $\mathcal{I}_n$ (resp. $\mathcal{I}$) the isoperimetric profile of $(X_n,\mathsf{d}_n,\mathfrak{m}_n)$ (resp. of $(X,\mathsf{d},\mathfrak{m})$).
Then, for any $t \in [0,1]$ and for any sequence $(t_n)_n$ with $t_n \to t$, it holds that

$$\mathcal{I}(t) \le \liminf_{n\to\infty} \mathcal{I}_n(t_n). \tag{6.3}$$

Proof. We refer to Section 4.2 for the basic definitions and statements about convergence of functions defined over mGH-converging sequences of metric measure spaces.
First of all note that in order to prove (6.3), without loss of generality we can assume that $\sup_n \mathcal{I}_n(t_n) < +\infty$.
For any $n \in \mathbb{N}$ let $E_n \subset X_n$ be a Borel set such that $\mathrm{Per}_n(E_n) = \mathcal{I}_n(t_n)$, whose existence follows as in the Euclidean case from standard lower semicontinuity and compactness arguments.
The sequence of the corresponding characteristic functions $(\chi_{E_n})_n$ satisfies the assumption of [**18**, Proposition 7.5], i.e.

$$\sup_{n\in\mathbb{N}} \left\{ \|\chi_{E_n}\|_{L^1(\mathfrak{m}_n)} + |D\chi_{E_n}|(X_n) \right\} = \sup_{n\in\mathbb{N}} \left\{ t_n + \mathcal{I}_n(t_n) \right\} < +\infty.$$

It follows from [**18**, Proposition 7.5] that, up to extracting a subsequence which we do not relabel, $(\chi_{E_n})_n$ strongly L^1-converges to a function $f \in L^1(X,\mathfrak{m})$. In particular we can say that

$$\|f\|_{L^1(\mathfrak{m})} = \lim_{n\to\infty} \|\chi_{E_n}\|_{L^1(\mathfrak{m}_n)} = \lim_{n\to\infty} t_n = t. \tag{6.4}$$

We now claim that f is the indicator function of a Borel set $E \subset X$, with $\mathfrak{m}(E) = t$. To this aim call $g_n := \chi_{E_n}(1-\chi_{E_n})$ and observe that $(g_n)_n$ strongly L^1-converges to $g := f(1-f)$ thanks to Proposition 1.118. Thus $g = 0$, since $g_n = 0$ for any $n \in \mathbb{N}$ and therefore g is the indicator function of a Borel set, as claimed.
We can now apply [**18**, Theorem 8.1] to get the Mosco convergence of the BV energies and conclude that

$$\mathrm{Per}(E) \le \liminf_{n\to\infty} \mathrm{Per}_n(E_n) = \liminf_{n\to\infty} \mathcal{I}_n(t_n).$$

The lower semicontinuity for the isoperimetric profiles (6.3) easily follows, since E is an admissible competitor in the definition of $\mathcal{I}(t)$. □

We will denote by $\mathcal{I}_N$ the isoperimetric profile of the model space $([0,\pi],\mathsf{d}_{\mathrm{eucl}},\mathfrak{m}_N)$.

In [**58**, **60**], exploiting the so-called localization technique (cf. [**153**]), the following version of the Lévy-Gromov isoperimetric inequality [**127**, Appendix C] for metric measure spaces was proven.

Theorem 6.5 (Lévy-Gromov inequality)**.** *Let* $(X,\mathsf{d},\mathfrak{m})$ *be an essentially non branching* CD$(N-1,N)$ *metric measure space for some* $1<N<\infty$*. Then, for any Borel set* $E\subset X$*, it holds*

$$\mathrm{Per}(E)\geq \mathcal{I}_N(\mathfrak{m}(E)).$$

In the same papers also the rigidity problem for the Lévy-Gromov inequality was addressed in the framework of RCD$(N-1,N)$ metric measure spaces. We refer to Section 4.1 for the construction of the warped product metric measure space.

Theorem 6.6 (Rigidity in Lévy-Gromov inequality)**.** *Let* $(X,\mathsf{d},\mathfrak{m})$ *be an* RCD$(N-1,N)$ *metric measure space for some* $N\in[2,+\infty)$ *with* $\mathfrak{m}(X)=1$*. Assume that there exists* $\bar v\in(0,1)$ *such that* $\mathcal{I}_{(X,\mathsf{d},\mathfrak{m})}(\bar v)=\mathcal{I}_N(\bar v)$*. Then* $(X,\mathsf{d},\mathfrak{m})$ *is a spherical suspension: there exists an* RCD$(N-2,N-1)$ *m.m.s.* $(Y,\mathsf{d}_Y,\mathfrak{m}_y)$ *with* $\mathfrak{m}_Y(Y)=1$ *such that* X *is isomorphic as a metric measure space to* $[0,\pi]\times_{\sin}^{N-1}Y$*.*

The working assumption of this section, unless otherwise stated, is that $(X,\mathsf{d},\mathfrak{m})$ is an RCD$(N-1,N)$ space for some $N\in(1,\infty)$, with $\mathfrak{m}(X)=1$ and $\mathrm{supp}\,(\mathfrak{m})=X$.

Definition 6.7 (Distribution function)**.** Given an open domain $\Omega\subset X$ and a non-negative Borel function $u:\Omega\to[0,\infty)$ we define its distribution function $\mu:[0,\infty)\to[0,\mathfrak{m}(\Omega)]$ by

$$\mu(t):=\mathfrak{m}(\{u>t\}). \tag{6.5}$$

Remark 6.8. Suppose that u is such that $\mathfrak{m}(\{u=t\})=0$ for any $0<t<\infty$. Then it makes no difference to consider closed superlevel sets or open superlevel sets in (6.5).

It is not difficult to check that the distribution function μ is non increasing and left-continuous. Moreover, if u is continuous then μ is strictly decreasing. We let $u^{\#}$ be the generalized inverse of μ, defined in the following way:

$$u^{\#}(s):=\begin{cases}\operatorname{ess\,sup} u & \text{if } s=0,\\ \inf\{t:\mu(t)<s\} & \text{if } s>0.\end{cases}$$

Definition 6.9 (Rearrangement on one dimensional model spaces)**.** Fix any $1<N<\infty$, and let $([0,\pi],\mathsf{d}_{\mathrm{eucl}},\mathfrak{m}_N)$ be the one-dimensional model space defined in (6.2). Let $\Omega\subset X$ be an open subset and consider $[0,r]\subset[0,\pi]$ such that $\mathfrak{m}_N([0,r])=\mathfrak{m}(\Omega)$.
For any Borel function $u:\Omega\to[0,\infty)$, the *monotone rearrangement* $u_N^*:[0,r]\to[0,\infty)$ is defined by

$$u_N^*(x):=u^{\#}(\mathfrak{m}_N([0,x])),\quad \forall x\in[0,r].$$

For simplicity of notation we will often write u^* in place of u_N^*.

Remark 6.10. We will consider for simplicity only monotone rearrangements of *non-negative* functions. Nevertheless, for an arbitrary Borel function $u:\Omega\to(-\infty,+\infty)$ the analogous statements hold by setting u^* the monotone rearrangement of $|u|$.

In the next proposition we collect some useful properties of the monotone rearrangement, whose proof in the Euclidean setting can be found for instance in [**147**, Chapter 1] and can be adapted with minor modifications to our framework.

Proposition 6.11. *Let* $(X,\mathsf{d},\mathfrak{m})$ *with* $\mathfrak{m}(X)=1$ *be an* RCD(K,N) *space for some* $N\in(1,\infty)$*. Let* $\Omega\subset X$ *be an open subset and consider* $[0,r]\subset[0,\pi]$ *such that* $\mathfrak{m}_N([0,r])=\mathfrak{m}(\Omega)$*.*

Let $u : \Omega \to [0, \infty)$ be Borel and let $u^* : [0, r] \to [0, \infty)$ be its monotone rearrangement. Then u and u^* have the same distribution function (we will often say that they are equimeasurable). Moreover,

$$\|u\|_{L^p(\Omega,\mathfrak{m})} = \|u^*\|_{L^p([0,r],\mathfrak{m}_N)}\,, \quad \forall 1 \le p < \infty, \tag{6.6}$$

and the monotone rearrangement operator $L^p(\Omega, \mathfrak{m}) \ni u \mapsto u^* \in L^p([0, r], \mathfrak{m}_N)$ is continuous.

Motivated by the working assumptions of Lemma 6.14 below, we state and prove a result about approximation via functions with non vanishing minimal weak upper gradient.

Lemma 6.12 (Approximation with non vanishing gradients)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be a locally compact geodesic metric measure space and let $\Omega \subset X$ be an open subset with $\mathfrak{m}(\Omega) < \infty$. Then for any non-negative $u \in \mathrm{Lip_c}(\Omega)$ with $\int \mathrm{lip}^p(u)\, \mathrm{d}\mathfrak{m} < \infty$, there exists a sequence $(u_n)_n$ with $u_n \in \mathrm{Lip_c}(\Omega)$ non-negative, $\mathrm{lip}(u_n) \neq 0$ $\mathfrak{m}$-a.e. on $\{u_n > 0\}$ for any $n \in \mathbb{N}$ and such that $u_n \to u$ in $H^{1,p}(X, \mathsf{d}, \mathfrak{m})$.*

If we additionally assume that $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(K, N)$ metric measure space then $\int |\nabla u_n|^p \, \mathrm{d}\mathfrak{m} \to \int |\nabla u|^p \, \mathrm{d}\mathfrak{m}$ as $n \to \infty$ and $|\nabla u_n| \neq 0$ $\mathfrak{m}$-a.e. on $\{u_n > 0\}$ for any $n \in \mathbb{N}$.

Proof. It is straightforward to check that there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ monotonically converging to 0 from above such that $\mathfrak{m}(\{\mathrm{lip}(u) = \varepsilon_n\}) = 0$ for any $n \in \mathbb{N}$.
Choose an open set Ω' containing the support of u and compactly contained in Ω. Let $v : \Omega \to [0, \infty)$ be the distance function from the complementary of Ω' in X, namely

$$v(x) := \mathsf{d}(x, X \setminus \Omega') \quad \text{for any } x \in \Omega.$$

Observe that $v \in \mathrm{Lip_c}(\Omega)$, moreover

$$\mathrm{lip}(v)(x) = 1 \quad \text{for any } x \in \Omega'. \tag{6.7}$$

Indeed it suffices to observe that the restriction of v to any geodesic connecting x with $y \in X \setminus \Omega'$ such that $v(x) = \mathsf{d}(x, y)$ has slope equal to 1 at x.

Next we introduce the approximating sequence $u_n := u + \varepsilon_n v$ and we claim that it has the desired properties. Indeed, if $u \in \mathrm{Lip_c}(\Omega)$ is non-negative, then also $u_n \in \mathrm{Lip_c}(\Omega)$ is so. From the inequality

$$\mathrm{lip}(u + \varepsilon_n v) \ge |\mathrm{lip}(u) - \varepsilon_n \mathrm{lip}(v)|$$

and from (6.7) it follows that $\{\mathrm{lip}(u_n) = 0\} \cap \{u_n > 0\} \subset \{\mathrm{lip}(u) = \varepsilon_n\}$. Since the ε_n are chosen in such a way that $\mathfrak{m}(\{\mathrm{lip}(u) = \varepsilon_n\}) = 0$, we infer that $\mathfrak{m}(\{\mathrm{lip}(u_n) = 0\} \cap \{u_n > 0\}) = 0$.
Clearly u_n converge uniformly to u as $n \to \infty$, guaranteeing in particular that $u_n \to u$ in $L^p(\Omega, \mathfrak{m})$. At the same time it holds that $\mathrm{lip}(u_n - u) = \varepsilon_n \mathrm{lip}(v)$. Therefore

$$\int_\Omega |\nabla (u_n - u)|^p \, \mathrm{d}\mathfrak{m} \le \varepsilon_n^p \int_\Omega \mathrm{lip}^p(v)\, \mathrm{d}\mathfrak{m} \to 0,$$

yielding that $u_n \to u$ in $H^{1,p}(X, \mathsf{d}, \mathfrak{m})$.

The last conclusion in the statement follows from the identification between slopes and minimal weak upper gradients on PI spaces, see Theorem 1.33. □

In Proposition 6.13 below we extend to the non smooth setting [**147**, Theorem 2.3.2]. The key idea is to replace the euclidean isoperimetric inequality with the Lévy-Gromov isoperimetric inequality.

Proposition 6.13 (Lipschitz to Lipschitz property of the rearrangement)**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(N-1, N)$ space with $\mathfrak{m}(X) = 1$, for some $N \in (1, \infty)$. Let $\Omega \subset X$ be an open subset and consider $[0, r] \subset [0, \pi]$ such that $\mathfrak{m}_N([0, r]) = \mathfrak{m}(\Omega)$.*
Let $u \in \mathrm{Lip}(\Omega)$ be non-negative with Lipschitz constant $L \ge 0$ and assume that $|\nabla u|(x) \neq 0$ for $\mathfrak{m}$-a.e. $x \in \{u > 0\}$. Then $u^* : [0, r] \to [0, \infty)$ is L-Lipschitz as well.

Proof. Let μ be the distribution function associated to u and denote by $M := \sup u < \infty$. Observe that our assumptions guarantee continuity and strict monotonicity of μ (here the fact that $|\nabla u| \neq 0$ $\mathfrak{m}$-a.e. enters into play). Therefore for any $s, k \geq 0$ such that $s+k \leq \mathfrak{m}(\Omega)$ we can find $0 \leq t-h \leq t \leq M$ such that $\mu(t-h) = s+k$ and $\mu(t) = s$. Since u is L-Lipschitz

$$\int_{\{t-h\leq u\leq t\}} |\nabla u| \, \mathrm{d}\mathfrak{m} \leq L\left(\mu(t-h) - \mu(t)\right). \tag{6.8}$$

On the other hand, taking into account the fact that for a locally Lipschitz function v on an RCD(K,N) space it holds $|Dv| = |\nabla v|\,\mathfrak{m}$, an application of the coarea formula (1.25) yields

$$\int_{\{t-h\leq u\leq t\}} |\nabla u| \, \mathrm{d}\mathfrak{m} = \int_{t-h}^{t} \mathrm{Per}(\{u \geq r\})\, \mathrm{d}r. \tag{6.9}$$

Applying Theorem 6.5 we can estimate the right hand side of (6.9) in the following way:

$$\int_{t-h}^{t} \mathrm{Per}(\{u \geq r\})\, \mathrm{d}r \geq \int_{t-h}^{t} \mathcal{I}_N(\mu(r))\, \mathrm{d}r. \tag{6.10}$$

Recalling that the model isoperimetric profile $\mathcal{I}_N$ and μ are continuous, combining (6.8) with (6.10) and eventually applying the mean value theorem we get

$$Lk \geq \int_{t-h}^{t} \mathcal{I}_N(\mu(r))\, \mathrm{d}r = h\mathcal{I}_N(\mu(\xi_{t-h}^t)), \tag{6.11}$$

for some $t-h \leq \xi_{t-h}^t \leq t$. Calling $u^\#$ the inverse of the distribution function, (6.11) can be rewritten as

$$\left(u^\#(s) - u^\#(s+k)\right)\mathcal{I}_N(\mu(\xi_{t-h}^t)) \leq Lk. \tag{6.12}$$

Since $\mathcal{I}_N$ is strictly positive on $(0,1)$, it follows from (6.12) that $u^\#$ is locally Lipschitz. Moreover, at any differentiability point s of $u^\#$ (which in particular form a set of full $\mathcal{L}^1$-measure on $(0,1)$), it holds

$$-\frac{\mathrm{d}}{\mathrm{d}s}u^\#(s) \leq \frac{L}{\mathcal{I}_N(s)}. \tag{6.13}$$

Let $r : [0,1] \to [0,\pi]$ be such that $r(\mathfrak{m}_N([0,x])) = x$ for any $x \in [0,\pi]$. Differentiating in t the identity

$$\int_0^{r(t)} h_N(s)\, \mathrm{d}s = t,$$

we obtain that $1 = \frac{\mathrm{d}}{\mathrm{d}t}r(t)h_N(r(t))$ and, since $\mathcal{I}_N(s) = h_N(r(s))$,

$$\frac{\mathrm{d}}{\mathrm{d}t}r(t) = \frac{1}{\mathcal{I}_N(t)}. \tag{6.14}$$

By definition of the rearrangement u^*, for any $x \in [0,r]$ it holds that $u^*(x) = u^\#(\mathfrak{m}_N([0,x]))$. Combining the last identity with (6.13) and (6.14) we can estimate for $x \leq y$

$$\begin{aligned}
0 \leq u^*(x) - u^*(y) &= u^\#(\mathfrak{m}_N([0,x])) - u^\#(\mathfrak{m}_N([0,y])) \\
&= \int_{\mathfrak{m}_N([0,x])}^{\mathfrak{m}_N([0,y])} -\frac{\mathrm{d}}{\mathrm{d}s}u^\#(s)\, \mathrm{d}s \\
&\leq \int_{\mathfrak{m}_N([0,x])}^{\mathfrak{m}_N([0,y])} L\frac{\mathrm{d}}{\mathrm{d}s}r(s)\, \mathrm{d}s \\
&= Lr(\mathfrak{m}_N([0,y])) - Lr(\mathfrak{m}_N([0,x])) = Ly - Lx,
\end{aligned}$$

which gives the L-Lipschitz continuity of the monotone rearrangement u^*. □

The next lemma should be compared with [**147**], dealing with the case of smooth functions in Euclidean domains.

Lemma 6.14 (Derivative of the distribution function)**.** *Let* $(X,\mathsf{d},\mathfrak{m})$ *be an* $\mathrm{RCD}(N-1,N)$ *metric measure space and let* $\Omega\subset X$ *be an open subset. Assume that* $u\in\mathrm{Lip}_{\mathrm{loc}}(\Omega)$ *is non-negative and* $|\nabla u|(x)\neq 0$ *for* $\mathfrak{m}$*-a.e.* $x\in\{u>0\}$*. Then its distribution function* $\mu:[0,\infty)\to[0,\mathfrak{m}(\Omega)]$*, defined in* (6.5)*, is absolutely continuous. Moreover it holds*

$$\mu'(t)=-\int\frac{1}{|\nabla u|}\,\mathrm{dPer}(\{u>t\})\quad \textit{for } \mathcal{L}^1\textit{-a.e. } t, \tag{6.15}$$

where the quantity $1/|\nabla u|$ *is defined to be* 0 *whenever* $|\nabla u|=0$.

Proof. Fix any $\varepsilon>0$ and define

$$f_\varepsilon(x):=\frac{|\nabla u(x)|}{|\nabla u(x)|^2+\varepsilon}.$$

Fixing $t\geq 0$ and $h>0$, an application of the coarea formula (1.25) with $f=f_\varepsilon$ yields to

$$\int_{\{t\leq u\leq t+h\}}\frac{|\nabla u|^2}{|\nabla u|^2+\varepsilon}\,\mathrm{d}\mathfrak{m}=\int_t^{t+h}\left(\int\frac{|\nabla u|}{|\nabla u|^2+\varepsilon}\,\mathrm{dPer}(\{u>r\})\right)\mathrm{d}r. \tag{6.16}$$

Now we pass to the limit as $\varepsilon\to 0$ both at the right hand side and at the left hand side in (6.16). The assumption that $|\nabla u|\neq 0$ $\mathfrak{m}$-a.e. guarantees that the integrand at the left hand side monotonically converges $\mathfrak{m}$-a.e. to 1. Thus an application of the monotone convergence theorem yields that

$$\int_{\{t\leq u\leq t+h\}}\frac{|\nabla u|^2}{|\nabla u|^2+\varepsilon}\,\mathrm{d}\mathfrak{m}\to\mu(t)-\mu(t+h)\quad\text{as } \varepsilon\to 0. \tag{6.17}$$

With the above mentioned convention about the value of $1/|\nabla u|$ at points where $|\nabla u|=0$, applying the monotone convergence theorem twice at the right hand side of (6.16), we get

$$\int_t^{t+h}\left(\int\frac{|\nabla u|}{|\nabla u|^2+\varepsilon}\,\mathrm{dPer}(\{u>r\})\right)\mathrm{d}r\to\int_t^{t+h}\left(\int\frac{1}{|\nabla u|}\,\mathrm{dPer}(\{u>r\})\right)\mathrm{d}r \tag{6.18}$$

as ε goes to 0. Combining (6.16), (6.17) and (6.18), we get

$$\mu(t)-\mu(t+h)=\int_t^{t+h}\left(\int\frac{1}{|\nabla u|}\,\mathrm{dPer}(\{u>r\})\right)\mathrm{d}r.$$

It follows that the distribution function is absolutely continuous and therefore differentiable at almost all points with derivative given by (6.15). □

Before proceeding to the statement and the proof of the Polya-Szego inequality we need an identification result between slopes and 1-minimal weak upper gradients in the simplified setting of the model weighted interval $[0,\pi]$ (or a subinterval of $[0,\pi]$). The result would follow relying on the RCD theory (see in particular Theorem 1.33 and the discussion at the beginning of Section 4.1) but we chose to present an elementary argument. In this setting, for any $p\geq 1$, we say that $u\in H^{1,p}([0,\pi],\mathsf{d}_{\mathrm{eucl}},\mathfrak{m}_N)$ if the distributional derivative of u (defined through integration by parts) is in $L^p([0,\pi],\mathfrak{m}_N)$.

Lemma 6.15. *Let* $I\subset[0,\pi]$*,* $1<p<\infty$ *and let* $f\in H^{1,p}((I,\mathsf{d}_{\mathrm{eucl}},\mathfrak{m}_N))$ *be monotone. Then* $f\in H^{1,1}((I,\mathsf{d}_{\mathrm{eucl}},\mathfrak{m}_N))$ *and it holds*

$$|\nabla f|_1(x)=|f'|(x)=\mathrm{lip}(f)(x)\quad \textit{for } \mathcal{L}^1\textit{-a.e. } x\in I, \tag{6.19}$$

where we denoted by $|\nabla f|_1$ *the* 1*-minimal relaxed gradient of the abstract theory of Sobolev spaces on metric measure spaces, cf. Section 1.5.1.*

Proof. The fact that $f \in H^{1,1}((I, \mathsf{d}_{\mathrm{eucl}}, \mathfrak{m}_N))$ follows directly by Hölder inequality, since $\mathfrak{m}_N(I) \leq 1$. Since $\mathfrak{m}_N = h_N \mathcal{L}^1$ with h_N locally bounded away from 0 out of the two end-points of $[0,\pi]$, it follows that f is locally absolutely continuous in the interior of $[0,\pi]$. In particular it is differentiable $\mathcal{L}^1$-a.e. and $\mathrm{lip}(f)(x) = |f'|(x)$ at every differentiability point x. We are thus left to show the first equality in (6.19).
Note that the assumptions ensure that f is invertible onto its image, up to a countable subset of $f(I)$. The coarea formula in the 1-dimensional case reads as

$$\int_I (\varphi \cdot h_N)\, |\nabla f|_1 \,\mathrm{d}\mathcal{L}^1 = \int_I \varphi \,|\nabla f|_1 \,\mathrm{d}\mathfrak{m}_N = \int_{f(I)} (\varphi \cdot h_N)(f^{-1}(r))\,\mathrm{d}r, \quad \forall \varphi \in C_c(I). \tag{6.20}$$

On the other hand, the change of variable formula via a monotone absolutely continuous function gives

$$\int_I (\varphi \cdot h_N) |f'| \,\mathrm{d}\mathcal{L}^1 = \int_{f(I)} (\varphi \cdot h_N)(f^{-1}(r))\,\mathrm{d}r, \quad \forall \varphi \in C_c(I). \tag{6.21}$$

The combination of (6.20) with (6.21) then gives the first equality in (6.19). □

The following statement should be compared with [**147**], where the study of the monotone rearrangement on domains of $\mathbb{R}^n$ is performed.

Proposition 6.16. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an* $\mathrm{RCD}(N-1,N)$ *space for some $N \in (1,\infty)$. Let $\Omega \subset X$ be an open subset and consider $[0,r] \subset [0,\pi]$ such that $\mathfrak{m}_N([0,r]) = \mathfrak{m}(\Omega)$.*
Let $u \in \mathrm{Lip}(\Omega)$ be non-negative and assume that $|\nabla u|(x) \neq 0$ for $\mathfrak{m}$-a.e. $x \in \{u > 0\}$.
Then $u^* \in \mathrm{Lip}([0,r])$ and for any $1 < p < \infty$ it holds

$$\int_\Omega |\nabla u|^p \,\mathrm{d}\mathfrak{m} \geq \int_0^r |\nabla u^*|^p \,\mathrm{d}\mathfrak{m}_N. \tag{6.22}$$

Proof. Denote by $M := \sup u$. Since u is Lipschitz Proposition 6.13 guarantees that the monotone rearrangement u^* is still Lipschitz.
Introduce the functions $\varphi, \psi : [0,M] \to [0,\infty)$ defined by

$$\varphi(t) := \int_{\{u>t\}} |\nabla u|^p \,\mathrm{d}\mathfrak{m}, \quad \psi(t) := \int_{\{u>t\}} |\nabla u| \,\mathrm{d}\mathfrak{m}.$$

An application of the coarea formula Theorem 1.40 yields that φ and ψ are absolutely continuous and therefore $\mathcal{L}^1$-a.e. differentiable with derivatives given $\mathcal{L}^1$-a.e. by the expressions

$$\varphi'(t) = -\int |\nabla u|^{p-1} \,\mathrm{d}\mathrm{Per}(\{u>t\}) \text{ and } \psi'(t) = -\mathrm{Per}(\{u>t\}),$$

respectively. An application of Hölder's inequality yields that for any $0 \leq t-h \leq t \leq M$

$$\int_{\{t-h<u\leq t\}} |\nabla u| \,\mathrm{d}\mathfrak{m} \leq \left(\int_{\{t-h<u\leq t\}} |\nabla u|^p \,\mathrm{d}\mathfrak{m} \right)^{\frac{1}{p}} (\mu(t-h) - \mu(t))^{\frac{p-1}{p}}, \tag{6.23}$$

where μ denotes the distribution function associated to u. It follows from the discussion above and from Lemma 6.14 that $\mathcal{L}^1$-a.e. point $t \in (0,M)$ is a differentiability point of both μ, φ and ψ. In view of (6.23), at any such point it holds that

$$-\psi'(t) \leq (-\varphi'(t))^{\frac{1}{p}} (-\mu'(t))^{\frac{p-1}{p}}. \tag{6.24}$$

Applying the Lévy-Gromov inequality Theorem 6.5 we obtain that $\mathrm{Per}(\{u>t\}) \geq \mathcal{I}_N(\mu(t))$. Therefore, taking into account the strict monotonicity of μ, (6.24) turns into

$$-\varphi'(t) \geq \frac{(\mathcal{I}_N(\mu(t)))^p}{(-\mu'(t))^{p-1}} \quad \text{for } \mathcal{L}^1\text{-a.e. } t. \tag{6.25}$$

Thus

$$\int_\Omega |\nabla u|^p \, \mathrm{d}\mathfrak{m} = \int_0^M -\varphi'(t)\,\mathrm{d}t \geq \int_0^M \frac{(\mathcal{I}_N(\mu(t)))^p}{(-\mu'(t))^{p-1}}\,\mathrm{d}t. \tag{6.26}$$

It follows from the very definition of the monotone rearrangement and from the properties of the model isoperimetric profile that $\mathrm{Per}(\{u^* > t\}) = \mathcal{I}_N(\mu(t))$ (recall that u and u^* have the same distribution function). Moreover, since we already know that u^* is Lipschitz, we are in position to apply Lemma 6.14 to conclude (taking also into account Lemma 6.15) that

$$-\mu'(t) = \frac{\mathrm{Per}(\{u^* > t\})}{|(u^*)'((u^*)^{-1}(t))|} \quad \text{for } \mathcal{L}^1\text{-a.e. } t. \tag{6.27}$$

Applying the coarea formula to the function u^* and taking into account (6.27) and Lemma 6.15 we conclude that

$$\begin{aligned}\int_0^r |\nabla u^*|^p \, \mathrm{d}\mathfrak{m}_N = \int_0^r \left|(u^*)'\right|^p \, \mathrm{d}\mathfrak{m}_N &= \int_0^{\sup u^*} \left|(u^*)'((u^*)^{-1}(t))\right|^{p-1} \mathrm{Per}(\{u^* > t\})\,\mathrm{d}t \\ &= \int_0^{\sup u^*} \frac{(\mathcal{I}_N(\mu(t)))^p}{(-\mu'(t))^{p-1}}\,\mathrm{d}t.\end{aligned} \tag{6.28}$$

Comparing (6.26) with (6.28) we can conclude that

$$\int_\Omega |\nabla u|^p \, \mathrm{d}\mathfrak{m} \geq \int_0^r |\nabla u^*|^p \, \mathrm{d}\mathfrak{m}_N,$$

giving (6.22). □

Theorem 6.17 (Polya-Szego inequality). *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(N-1, N)$ space for some $N \in (1, \infty)$. Let $\Omega \subset X$ be an open subset and consider $[0, r] \subset I$ such that $\mathfrak{m}_N([0, r]) = \mathfrak{m}(\Omega)$. Then the monotone rearrangement maps $H_0^{1,p}(\Omega)$ into $H^{1,p}(([0, r], \mathsf{d}_{\mathrm{eucl}}, \mathfrak{m}_N))$ for any $1 < p < \infty$. Moreover for any $u \in H_0^{1,p}(\Omega)$ it holds $u^*(r) = 0$ and*

$$\int_0^r |\nabla u^*|^p \, \mathrm{d}\mathfrak{m}_N \leq \int_\Omega |\nabla u|^p \, \mathrm{d}\mathfrak{m}. \tag{6.29}$$

Proof. By the very definition of $H_0^{1,p}(\Omega)$ we can find a sequence $(u_n)_n$ with $u_n \in \mathrm{Lip_c}(\Omega)$ for any $n \in \mathbb{N}$ and u_n converging to u in $H^{1,p}(X, \mathsf{d}, \mathfrak{m})$, Moreover, thanks to Lemma 6.12, we can assume that $|\nabla u_n| \neq 0$ for $\mathfrak{m}$-a.e. $x \in \{u_n > 0\}$ for any $n \in \mathbb{N}$, so that we can apply Proposition 6.16 to each u_n obtaining

$$\int_0^r |\nabla u_n^*|^p \, \mathrm{d}\mathfrak{m}_N \leq \int_\Omega |\nabla u_n|^p \, \mathrm{d}\mathfrak{m}. \tag{6.30}$$

Observe now that the strong $L^p(X, \mathfrak{m})$-convergence of u_n to u and the strong L^p-continuity of the monotone rearrangement (see Proposition 6.11) guarantee that $u_n^* \to u^*$ in $L^p([0, r], \mathfrak{m}_N)$. From the lower semicontinuity of the p-energy w.r.t. $L^p([0, r], \mathfrak{m}_N)$-convergence it follows that

$$\int_0^r |\nabla u^*|^p \, \mathrm{d}\mathfrak{m}_N \leq \liminf_{n\to\infty} \int_0^r |\nabla u_n^*|^p \, \mathrm{d}\mathfrak{m}_N.$$

Hence, taking into account (6.30) and the strong convergence in $H^{1,p}(X, \mathsf{d}, \mathfrak{m})$ of u_n to u, we conclude that

$$\int_0^r |\nabla u^*|^p \, \mathrm{d}\mathfrak{m}_N \leq \int_\Omega |\nabla u|^p \, \mathrm{d}\mathfrak{m},$$

which is the desired conclusion. □

In the following we will need an improved version of the Polya-Szego inequality. To this aim, for any non-negative $u \in H^{1,p}_0(\Omega)$ we introduce a function $f_u : [0, \sup u^*) \to [0,\infty]$ by

$$f_u(t) := \int |\nabla u^*|^{p-1} \, \mathrm{dPer}(\{u^* > t\}). \tag{6.31}$$

Observe that this definition makes sense thanks to Theorem 6.17 and the coarea formula, which also yields

$$\int_0^{\sup u^*} f_u(t)\, \mathrm{d}t = \int_0^r |\nabla u^*|^p \, \mathrm{d}\mathfrak{m}_N, \tag{6.32}$$

for any $u \in H^{1,p}_0(\Omega)$.

We are now in position to state and prove our improved Polya-Szego inequalities.

Proposition 6.18 (Improved Polya-Szego Inequalities)**.** *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* RCD$(N-1,N)$ *space for some* $N \in (1,\infty)$. *Let* $\Omega \subset X$ *be an open subset and consider* $[0,r] \subset [0,\pi]$ *such that* $\mathfrak{m}_N([0,r]) = \mathfrak{m}(\Omega)$.

Suppose that $u \in H^{1,p}_0(\Omega)$ *is such that* u^* *has non vanishing derivative* $\mathcal{L}^1$*-a.e. on* $(0,r)$. *Then*

$$\int_\Omega |\nabla u|^p \, \mathrm{d}\mathfrak{m} \geq \int_0^{\sup u^*} \left(\frac{\mathrm{Per}(\{u > t\})}{\mathcal{I}_N(\mu(t))} \right)^p f_u(t)\, \mathrm{d}t. \tag{6.33}$$

As a consequence, under the same assumptions, it holds that

$$\int_\Omega |\nabla u|^p \, \mathrm{d}\mathfrak{m} \geq \int_0^{\sup u^*} \left(\frac{\mathcal{I}_{(X,\mathsf{d},\mathfrak{m})}(\mu(t))}{\mathcal{I}_N(\mu(t))} \right)^p f_u(t)\, \mathrm{d}t. \tag{6.34}$$

Proof. In order to prove (6.33) we just need to observe that our assumptions, even though being weaker than those of Proposition 6.16, put us in position to make its proof work.

Indeed, with the same notation therein introduced, we observe that the monotone rearrangement u^* has the same distribution function of u. Moreover, Theorem 6.17 implies in particular that $u^* \in \mathrm{AC}_{\mathrm{loc}}((0,r))$. Therefore, since we are assuming that $|\nabla u^*|(t) \neq 0$ for $\mathcal{L}^1$-a.e. t, it follows from Lemma 6.14 (taking into account also Lemma 6.15) that μ is absolutely continuous and therefore differentiable $\mathcal{L}^1$-a.e. with the explicit expression for the derivative given (for $\mathcal{L}^1$-a.e. t) by

$$-\mu'(t) = \frac{\mathrm{Per}(\{u^* > t\})}{|\nabla u^*|\,((u^*)^{-1}(t))} = \frac{\mathcal{I}_N(\mu(t))}{|\nabla u^*|\,((u^*)^{-1}(t))}. \tag{6.35}$$

The second equality is a consequence of the very construction of the monotone rearrangement.

Following verbatim the beginning of the proof of Proposition 6.16 we obtain that (6.25) is still valid in the present setting. Taking into account (6.35) we obtain that

$$\begin{aligned} -\varphi'(t) \geq \frac{(\mathrm{Per}(\{u > t\}))^p}{(-\mu'(t))^{p-1}} &= \frac{(\mathrm{Per}(\{u > t\}))^p}{(\mathcal{I}_N(\mu(t)))^{p-1}} |\nabla u^*|\,((u^*)^{-1}(t))^{p-1} \\ &= \left[\frac{\mathrm{Per}(\{u > t\})}{\mathcal{I}_N(\mu(t))} \right]^p |\nabla u^*|\,((u^*)^{-1}(t))^{p-1} \mathcal{I}_N(\mu(t)) \\ &= \left[\frac{\mathrm{Per}(\{u > t\})}{\mathcal{I}_N(\mu(t))} \right]^p f_u(t) \end{aligned}$$

for $\mathcal{L}^1$-a.e. $t \in (0, \sup u^*)$. The desired inequality (6.33) follows now recalling that

$$\int_\Omega |\nabla u|^p \, \mathrm{d}\mathfrak{m} = \int_0^{\sup u^*} (-\varphi'(t))\, \mathrm{d}t.$$

The conclusion (6.34) is a consequence of (6.33) after observing that $\{u > t\}$ is an admissible competitor in the definition of $\mathcal{I}_{(X,\mathsf{d},\mathfrak{m})}(\mu(t))$ since by the very definition it holds that $\mathfrak{m}(\{u > t\}) = \mu(t)$. □

Remark 6.19. In order to prove the forthcoming Theorem 6.30 we will need to slightly enlarge the class of functions where (6.33) and (6.34) hold true. In particular, we claim that (6.33) holds true for any $u \in H_0^{1,p}(\Omega)$ such that u^* is C^1 and strictly decreasing. Indeed for any such u it holds that the set of critical values of u^* is $\mathcal{L}^1$-negligible. Moreover, the distribution function μ of u (which coincides with the distribution function of u^* by equimeasurability, as we already observed), is differentiable at any regular point of u^*, with derivative given by (6.35). Hence the whole proof of Proposition 6.18 can be carried over without modifications.

2. Spectral gap with Dirichlet boundary conditions

We wish to bound from below the p-spectral gap of an $\mathrm{RCD}(N-1,N)$ space with the one of the corresponding one dimensional model space, for any $N \in (1,\infty)$ and $p \in (1,\infty)$. This extends to the non-smooth setting the celebrated result of Bérard-Meyer [**42**] (see also [**166**]) proved for smooth Riemannian manifolds with $\mathrm{Ric} \geq K, K > 0$. Let us point out that analogous statements have been obtained in [**140**] with different techniques for $p = 2$. The advantage of the present method is that it can be used with minor modifications in the general case of essentially non branching $\mathrm{CD}(N-1,N)$ spaces, as done in [**171**]. Moreover, it allows to handle the almost rigidity problem, as we shall see in Section 4.

For every $N \in (1,\infty)$, let $([0,\pi], \mathsf{d}_{\mathrm{eucl}}, \mathfrak{m}_N)$ be the one dimensional model space defined in (6.2). For every $v \in (0,1)$, let $r(v) \in [0,\pi]$ be such that $v = \mathfrak{m}_N([0, r(v)])$.
To let the notation be more compact, for any fixed $1 < p < \infty$, for any $v \in (0,1)$ and for any choice of $1 < N < \infty$, we define

$$\lambda^p_{N,v} := \inf\left\{\frac{\int_0^{r(v)} |u'|^p \,\mathrm{d}\mathfrak{m}_N}{\int_0^{r(v)} u^p \,\mathrm{d}\mathfrak{m}_N} : \ u \in \mathrm{Lip}([0,r(v)];[0,\infty)), \ u(r(v)) = 0 \text{ and } u \not\equiv 0\right\}$$

and we call $\lambda^p_{N,v}$ the *comparison first eigenvalue for the p-Laplacian with Dirichlet boundary conditions for Ricci curvature bounded from below by $N-1$, dimension bounded from above by N and volume v.*

Moreover, for any metric measure space $(X,\mathsf{d},\mathfrak{m})$ with $\mathfrak{m}(X) = 1$, for any open subset $\Omega \subset X$ and for any $1 < p < \infty$, we define

$$\lambda^p_X(\Omega) := \inf\left\{\frac{\int_\Omega |\nabla u|^p \,\mathrm{d}\mathfrak{m}}{\int_\Omega u^p \,\mathrm{d}\mathfrak{m}} : \quad u \in \mathrm{Lip_c}(\Omega;[0,\infty)) \text{ and } u \not\equiv 0\right\}, \tag{6.36}$$

and we call $\lambda^p_X(\Omega)$ the *first eigenvalue of the p-Laplacian on Ω with Dirichlet boundary conditions.*
Observe that for any $2 \leq N \in \mathbb{N}$, $\lambda^p_{N,v} = \lambda^p_{\mathbb{S}^N}(B_v)$, where $\mathbb{S}^N$ is the round N-dimensional sphere or radius 1 and $B_v \subset \mathbb{S}^N$ is a metric ball (i.e. a spherical cap) with volume v.

Theorem 6.20 (p-Spectral gap with Dirichlet boundary conditions)**.** *Let $(X,\mathsf{d},\mathfrak{m})$ be an* $\mathrm{RCD}(N-1,N)$ *space for some $1 < N < \infty$ and assume that $\mathfrak{m}(X) = 1$. Let $\Omega \subset X$ be an open domain with $\mathfrak{m}(\Omega) = v \in (0,1)$. Then, for any $1 < p < \infty$, it holds*

$$\lambda^p_X(\Omega) \geq \lambda^p_{N,v}.$$

Proof. For any $u \in \mathrm{Lip_c}(\Omega;[0,\infty))$ not identically zero we introduce the notation

$$\mathcal{R}_p(u) := \frac{\int_\Omega |\nabla u|^p \, \mathrm{d}\mathfrak{m}}{\int_\Omega u^p \, \mathrm{d}\mathfrak{m}}$$

for the p-Rayleigh quotient of u.

It follows from the combination of Proposition 6.11 and Proposition 6.16 that for any $u \in \mathrm{Lip_c}(\Omega;[0,\infty))$ such that $|\nabla u| \neq 0$ $\mathfrak{m}$-a.e. on $\{u > 0\}$ it holds

$$\mathcal{R}_p(u) \geq \mathcal{R}_p(u^*),$$

where $u^* : [0, r(v)] \to [0,\infty)$ is the monotone rearrangement of u on the model space. Observe now that $u \in \mathrm{Lip}_c(\Omega)$ implies, by construction of the monotone rearrangement, that u^* vanishes at $r(v)$. We thus get

$$\mathcal{R}_p(u^*) \geq \lambda^p_{N,v}.$$

The desired conclusion follows from Lemma 6.12 yielding that for any $u \in \mathrm{Lip_c}(\Omega;[0,\infty))$ we can find a sequence $(u_n)_n \subset \mathrm{Lip_c}(\Omega;[0,\infty))$ such that $|\nabla u_n| \neq 0$ $\mathfrak{m}$-a.e. on $\{u_n > 0\}$ for any $n \in \mathbb{N}$ and

$$\mathcal{R}_p(u_n) \to \mathcal{R}_p(u), \quad \text{as } n \to \infty.$$

□

In order to let the picture be more complete we collect here some known result about the p-Laplace equation with homogeneous Dirichlet boundary conditions on metric measure spaces (verifying the curvature dimension condition) that will be useful in the next section about rigidity. We refer to [**156**] and [**116**] for a more detailed discussion about this topic and equivalent characterizations of first eigenfunctions.

Recall that $H^{1,p}_0(\Omega)$ is defined to be the closure w.r.t. the $H^{1,p}$-norm of $\mathrm{Lip_c}(\Omega)$. In the fairly general context of metric measure spaces it makes sense to talk about the first eigenfunction of the p-Laplace equation if the notion is understood in the following weak sense.

Definition 6.21 (First eigenfunction)**.** Let $(X, \mathsf{d}, \mathfrak{m})$ be an RCD(K,N) metric measure space for some $K \in \mathbb{R}$ and $1 < N < \infty$ and let $\Omega \subset X$ be an open domain. We say that $u \in H^{1,p}_0(\Omega)$ is a first eigenfunction of the p-Laplacian on Ω (with homogeneous Dirichlet boundary conditions) if $u \not\equiv 0$ and it minimizes the Rayleigh quotient

$$\mathcal{R}_p(v) = \frac{\int_\Omega |\nabla v|^p \, \mathrm{d}\mathfrak{m}}{\int_\Omega |v|^p \, \mathrm{d}\mathfrak{m}},$$

among all functions $v \in H^{1,p}_0(\Omega)$ such that $v \not\equiv 0$.

Remark 6.22. Let us observe that if $u \in H^{1,p}_0(\Omega)$ is a first eigenfunction of the p-Laplacian then $\mathcal{R}_p(u) = \lambda^p_X(\Omega)$ (that is the first eigenvalue of the p-Laplace equation defined in (6.36)), since by the very definition of the space $H^{1,p}_0(\Omega)$ it makes no difference to minimize the Rayleigh quotient over $\mathrm{Lip_c}(\Omega)$ or over $H^{1,p}_0(\Omega)$. As we will see below, the advantage of considering the minimization over $H^{1,p}_0(\Omega)$ is to gain existence of minimizers.

We conclude this section with a general existence result for first eigenfunctions of the p-laplacian. The main ingredient for its proof, as in the smooth case, is the Sobolev inequality which implies in turn that also Rellich compactness theorem holds true in this setting. A good reference for this part is [**2**, Chapter 5].

Theorem 6.23 (Existence of minimizers)**.** *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* RCD$(N-1,N)$ *space, for some* $1 < N < \infty$*. Let* $\Omega \subset X$ *be an open subset, fix* $1 < p < \infty$*, and assume that* $\lambda^p_X(\Omega) < \infty$*. Then there exists a first eigenfunction of the* p*-Laplace equation (with homogeneous Dirichlet boundary conditions) on* Ω*.*

Proof. If $\lambda_X^p(\Omega) < \infty$, we can find a sequence $(u_n)_n \subset H_0^{1,p}(\Omega)$ such that $\|u_n\|_{L^p} = 1$ for any $n \in \mathbb{N}$ and $\||\nabla u_n|\|_{L^p}^p \to \lambda_X^p(\Omega)$ as $n \to \infty$.

Since $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(N-1, N)$ space it is compact and doubling. Hence we can apply [**2**][Theorem 5.4.3] (which is a general version of Rellich theorem for metric measure spaces) to the sequence $(u_n)_n$ to find a limit function $u \in H_0^{1,p}(\Omega)$ such that $u_n \to u$ in $L^p(\Omega, \mathfrak{m})$ as $n \to \infty$ and hence $\|u\|_{L^p} = 1$. It follows from the lower semicontinuity of the p-energy w.r.t. $L^p(\Omega, \mathfrak{m})$-convergence that

$$\int_\Omega |\nabla u|^p \, \mathrm{d}\mathfrak{m} \le \liminf_{n\to\infty} \int_\Omega |\nabla u_n|^p \, \mathrm{d}\mathfrak{m} = \lambda_X^p(\Omega),$$

thus u is a first eigenfunction of the p-laplacian with homogeneous Dirichlet boundary conditions on Ω. □

Remark 6.24. Let us remark that the definition of Sobolev space adopted in [**2**] is different with respect to the working one of this thesis. However, as a consequence of [**10**][Lemma 8.2], if $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(K, N)$ m.m.s. and $f \in H^{1,p}(X, \mathsf{d}, \mathfrak{m})$ according to Section 1.5.1, then f is a Sobolev function according to [**2**][Definition 5.1.1].

3. Rigidity

This section is devoted to prove some rigidity statements associated to the Polya-Szego and spectral gap inequalities. The rough idea here is that if equality occurs in the Polya-Szego inequality then it occurs in the Lévy-Gromov inequality too. Hence one can build on top of the rigidity statements in the Lévy-Gromov isoperimetric inequality established in [**58**, **60**].

Theorem 6.25. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(N-1, N)$ *space for some* $N \in [2, \infty)$ *with* $\mathfrak{m}(X) = 1$.
Assume that there exists a nonnegative function $u \in \mathrm{Lip}(X)$ *achieving equality in the Polya-Szego inequality* (6.22)*, with* $|\nabla u|(x) \neq 0$ *for* $\mathfrak{m}$*-a.e.* $x \in \mathrm{supp}(u)$*. Then* $(X, \mathsf{d}, \mathfrak{m})$ *is a spherical suspension, namely there exists an* $\mathrm{RCD}(N-2, N-1)$ *space* $(Y, \mathsf{d}_Y, \mathfrak{m}_Y)$ *with* $\mathfrak{m}_Y(Y) = 1$ *such that* $(X, \mathsf{d}, \mathfrak{m})$ *is isomorphic as a metric measure space to* $[0, \pi] \times_{\sin}^{N-1} Y$.

Remark 6.26. Before discussing the proof, let us stress that Theorem 6.25 is stated for a *non-negative* function u just for uniformity of notation with the previous sections. Nevertheless, such a non-negativity assumption can be suppressed, once the rearrangement u^* in the Polya-Szego inequality (6.22) is understood as the decreasing rearrangement of $|u|$ (see also Remark 6.10). The same holds for Theorem 6.28 below.

Proof of Theorem 6.25. If equality occurs in (6.22), it follows from the proof of Proposition 6.16 that equality must occur in (6.25) for $\mathcal{L}^1$-a.e. $t \in (0, M)$, where $M := \max u$. Hence for $\mathcal{L}^1$-a.e. $t \in (0, M)$ it holds:

$$\mathrm{Per}(\{u > t\}) = \mathcal{I}_N(\mu(t)). \tag{6.37}$$

Since, by the very definition of the distribution function, we have $\mathfrak{m}(\{u > t\}) = \mu(t)$, it follows that $\mathcal{I}_{(X,\mathsf{d},\mathfrak{m})}(\mu(t)) = \mathcal{I}_N(\mu(t))$ for $\mathcal{L}^1$-a.e. $t \in (0, M)$. Thus we by Theorem 6.6 to conclude that $(X, \mathsf{d}, \mathfrak{m})$ is isomorphic to a spherical suspension $[0, \pi] \times_{\sin}^{N-1} Y$ for some $\mathrm{RCD}(N-2, N-1)$ space $(Y, \mathsf{d}_Y, \mathfrak{m}_Y)$.

□

Remark 6.27. A natural question is whether the condition $|\nabla u| \neq 0$ $\mathfrak{m}$-a.e. is sharp in Theorem 6.25. Clearly, if u is a constant function, also the decreasing rearrangement u^* is constant. Hence u, u^* achieve equality in the Polya-Szego inequality but one cannot expect to infer anything on the space. However in Theorem 6.28 we show that, as soon as u is non constant, the equality in Polya-Szego forces the space to be a spherical suspension. The proof of such a statement is more delicate than that of Theorem 6.25 and builds on top of the almost rigidity for Lévy-Gromov inequality.

Theorem 6.28 (Space rigidity in the Polya-Szego inequality)**.** *Let* $(X,\mathsf{d},\mathfrak{m})$ *be an* RCD$(N-1,N)$ *space for some* $N\in[2,+\infty)$ *and assume that* $\mathfrak{m}(X)=1$.
Let $\Omega\subset X$ *be an open set such that* $\mathfrak{m}(\Omega)=v\in(0,1)$ *and assume that there exists a nonnegative function* $u\in H^{1,p}_0(\Omega)$*,* $u\not\equiv 0$*, achieving equality in the Polya-Szego inequality* (6.29)*. Then* $(X,\mathsf{d},\mathfrak{m})$ *is a spherical suspension, namely there exists an* RCD$(N-2,N-1)$ *space* $(Y,\mathsf{d}_Y,\mathfrak{m}_Y)$ *with* $\mathfrak{m}_Y(Y)=1$ *such that* $(X,\mathsf{d},\mathfrak{m})$ *is isomorphic as a metric measure space to* $[0,\pi]\times^{N-1}_{\sin}Y$.

Proof. Let $(u_n)_n$ be a sequence of Lipschitz functions with compact support in Ω such that $|\nabla u_n|\neq 0$ $\mathfrak{m}$-a.e. on $\{u_n>0\}$ for any $n\in\mathbb{N}$ approximating u in $L^p(\Omega,\mathfrak{m})$ and in $H^{1,p}$ energy given by Lemma 6.12. Let u_n^* and u^* be the decreasing rearrangements of u_n and u respectively. The L^p-continuity of the decreasing rearrangement, together with the lower semicontinuity of the p-energy and the Polya-Szego inequality, yield

$$\begin{aligned}\int_0^{r(v)}|\nabla u^*|^p\,\mathrm{d}\mathfrak{m}_N &\leq \liminf_{n\to\infty}\int_0^{r(v)}|\nabla u_n^*|^p\,\mathrm{d}\mathfrak{m}_N\\ &\leq \limsup_{n\to\infty}\int_\Omega|\nabla u_n|^p\,\mathrm{d}\mathfrak{m}=\int_\Omega|\nabla u|^p\,\mathrm{d}\mathfrak{m}.\end{aligned}\tag{6.38}$$

It follows that $(u_n^*)_n$ converges in $H^{1,p}$-energy to u^*, since by assumption u achieves the equality in the Polya-Szego inequality.
Up to extracting a subsequence, that we do not relabel, we can assume that $(u_n^*)_n$ converges to u^* both locally uniformly on $(0,r(v)]$ and in $H^{1,p}(([0,r(v)],\mathsf{d}_{\text{eucl}},\mathfrak{m}_N))$, and moreover that both the $\liminf$ and the $\limsup$ in (6.38) are full limits. Denoting by μ_n and μ the distribution functions of u_n and u respectively, it follows that, for any $t\in(0,\sup u^*)$ such that $\mathfrak{m}_N(\{u^*=t\})=0$, it holds $\mu_n(t)\to\mu(t)$ as $n\to\infty$.
Moreover, if we let $f_n:=f_{u_n}$ be as in (6.31), then the improved Polya-Szego inequality (6.34) guarantees that

$$\int_\Omega|\nabla u_n|^p\,\mathrm{d}\mathfrak{m}\geq\int_0^{\sup u_n^*}\left(\frac{\mathcal{I}_{(X,\mathsf{d},\mathfrak{m})}(\mu_n(t))}{\mathcal{I}_N(\mu_n(t))}\right)^p f_n(t)\,\mathrm{d}t\geq\int_0^{\sup u_n^*}f_n(t)\,\mathrm{d}t=\int_{[0,\pi]}|\nabla u_n^*|^p\,\mathrm{d}\mathfrak{m}_N,$$

which, combined with the equality in the equality in (6.38), gives

$$\lim_{n\to\infty}\int_0^{\sup u_n^*}\left(\left(\frac{\mathcal{I}_{(X,\mathsf{d},\mathfrak{m})}(\mu_n(t))}{\mathcal{I}_N(\mu_n(t))}\right)^p-1\right)f_n(t)\,\mathrm{d}t=0.\tag{6.39}$$

Let us argue by contradiction and suppose that $(X,\mathsf{d},\mathfrak{m})$ is not isomorphic to a spherical suspension. It follows from Theorem 6.5 that $\mathcal{I}_{(X,\mathsf{d},\mathfrak{m})}(v)>\mathcal{I}_N(v)$ for any $v\in(0,1)$. By Proposition 6.4 we know that $\mathcal{I}_{(X,\mathsf{d},\mathfrak{m})}$ is lower semicontinuous on $[0,1]$ and $\mathcal{I}_N$ is continuous on $[0,1]$ and positive on $(0,1)$. Hence for any $0<\varepsilon<1/2$ there exists $c_\varepsilon>0$ such that

$$\inf_{v\in[\varepsilon,1-\varepsilon]}\left\{\left(\frac{\mathcal{I}_{(X,\mathsf{d},\mathfrak{m})}(v)}{\mathcal{I}_N(v)}\right)^p-1\right\}>c_\varepsilon>0.\tag{6.40}$$

Thanks to the assumption that u is non constant and to what we already observed, we can find $0<t_0<t_1<t_2<t_3<\sup u^*$, $0<\varepsilon<1$ and $n_0\in\mathbb{N}$ such that the following hold true:

$$\int_{\{t_1<u^*<t_2\}}|\nabla u^*|^p\,\mathrm{d}\mathfrak{m}_N>0,\tag{6.41}$$

$$\{t_1 < u^* < t_2\} \subset \{t_0 < u_n^* < t_3\} \quad \text{for any } n \geq n_0 \tag{6.42}$$

and

$$\mu_n(t) \in [\varepsilon, 1-\varepsilon] \quad \text{for any } t \in [t_0, t_3] \text{ and } n \geq n_0. \tag{6.43}$$

Combining (6.42) with the $L^p(\mathfrak{m}_N)$ convergence of $|\nabla u_n^*|$ to $|\nabla u^*|$ and the coarea formula, we obtain that

$$\liminf_{n\to\infty} \int_0^{\sup u_n^*} f_n(t)\,\mathrm{d}t \geq \liminf_{n\to\infty} \int_{\{t_0<u_n^*<t_3\}} |\nabla u_n^*|^p \,\mathrm{d}\mathfrak{m}_N \geq \int_{\{t_1<u^*<t_2\}} |\nabla u^*|^p \,\mathrm{d}\mathfrak{m}_N. \tag{6.44}$$

Eventually, putting (6.40) together with (6.41) and (6.44), we obtain

$$\liminf_{n\to\infty} \int_0^{\sup u_n^*} \left(\left(\frac{\mathcal{I}_{(X,\mathsf{d},\mathfrak{m})}(\mu_n(t))}{\mathcal{I}_N(\mu_n(t))} \right)^p - 1 \right) f_n(t)\,\mathrm{d}t \geq c_\varepsilon \int_{\{t_1<u^*<t_2\}} |\nabla u^*|^p \,\mathrm{d}\mathfrak{m}_N > 0,$$

contradicting (6.39). □

Corollary 6.29 (Rigidity in the Polya-Szego inequality-Smooth Setting)**.** *Let (M,g) be an N-dimensional Riemannian manifold, $N \geq 2$, with $\mathrm{Ric}_g \geq (N-1)g$ and denote by $\mathfrak{m}$ the normalized Riemannian volume measure. Let $\Omega \subset X$ be an open subset with $\mathfrak{m}(\Omega) \in (0,1)$. Assume that for some $p \in (1,\infty)$ there exists $u \in H_0^{1,p}(\Omega)$, $u \not\equiv 0$, achieving equality in the Polya-Szego inequality* (6.29)*.*
Then (M,g) is isometric to the round sphere $\mathbb{S}^N$ of constant sectional curvature one.

Theorem 6.30 (Rigidity for the p-spectral gap)**.** *Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(N-1,N)$ space. Let $\Omega \subset X$ be an open set with $\mathfrak{m}(\Omega) = v$ for some $v \in (0,1)$ and suppose that $\lambda_X^p(\Omega) = \lambda_{N,v}^p$. Then $(X,\mathsf{d},\mathfrak{m})$ is isomorphic to a spherical suspension.*

Proof. Suppose that $\lambda_X^p(\Omega) = \lambda_{N,v}^p$. Let $u \in H_0^{1,p}(\Omega)$ be a non-negative eigenfunction with $\|u\|_{L^p} = 1$ associated to the first eigenvalue $\lambda_X^p(\Omega)$, whose existence is guaranteed by Theorem 6.23. Then Theorem 6.17 gives

$$\lambda_{N-1,N,v}^p = \int_\Omega |\nabla u|^p \,\mathrm{d}\mathfrak{m} \geq \int_0^r |\nabla u^*|^p \,\mathrm{d}\mathfrak{m}_N \geq \lambda_{N,v}^p,$$

where, as before, r is defined by $\mathfrak{m}_N([0,r]) = \mathfrak{m}(\Omega) = v$. Hence equality holds true in all the inequalities so that u^* is an eigenfunction of the p-Laplacian associated to the first eigenvalue on the one dimensional model space $([0,r], \mathsf{d}_{\mathrm{eucl}}, \mathfrak{m}_N)$. It follows from the corresponding ODE that $u^* \in C^0([0,r]) \cap C^1((0,r))$ and it is strictly decreasing.

Hence, taking into account Remark 6.19, (6.33) holds true so that

$$\lambda_{N,v}^p = \int_\Omega |\nabla u|^p \,\mathrm{d}\mathfrak{m} \geq \int_0^{\sup u^*} \left(\frac{\mathrm{Per}(\{u>t\})}{\mathcal{I}_N(\mu(t))} \right)^p f_u(t)\,\mathrm{d}t \geq \int_0^{\sup u^*} f_u(t)\,\mathrm{d}t = \lambda_{N,v}^p.$$

Therefore

$$\mathrm{Per}(\{u>t\}) = \mathcal{I}_N(\mu(t)), \tag{6.45}$$

for $\mathcal{L}^1$-a.e. t such that $f_u(t) \neq 0$.

In particular there exists at least one level t_0 such that the super-level set $\{u > t\}$ is optimal for the Lévy-Gromov inequality. Thus by Theorem 6.6 we obtain that $(X,\mathsf{d},\mathfrak{m})$ is isomorphic, as a metric measure space, to a spherical suspension.

□

4. Almost rigidity in the Dirichlet p-spectral gap

This section is dedicated to an almost-rigidity result which seems interesting even for smooth Riemannian manifolds. The idea is to argue by contradiction, exploiting on the one hand the compactness of the class of RCD$(N-1,N)$ spaces with respect to measured Gromov Hausdorff convergence (cf. Remark 1.26) and, on the other hand, the lower-semicontinuity of the functionals involved.

Theorem 6.31 (Almost rigidity in the p-spectral gap)**.** *Fix $2 \le N < \infty$ and $v \in (0,1)$. Then, for any $\varepsilon > 0$, there exists $\delta = \delta(v,N) > 0$ with the following property: let $(X, \mathsf{d}, \mathfrak{m})$ be an* RCD$(N-1,N)$ *m.m.s. with $\mathfrak{m}(X) = 1$ and $\Omega \subset X$ be an open domain with $\mathfrak{m}(\Omega) = v$ and $\lambda^p_X(\Omega) < \lambda^p_{N,v} + \delta$.*
Then there exists a spherical suspension $(Y, \mathsf{d}_Y, \mathfrak{m}_Y)$ (i.e. there exists an RCD$(N-2,N-1)$ *space $(Z, \mathsf{d}_Z, \mathfrak{m}_Z)$ with $\mathfrak{m}_Z(Y) = 1$ such that Y is isomorphic as a metric measure space to $[0,\pi] \times^{N-1}_{\sin} Z$) such that*

$$\mathsf{d}_{mGH}\left((X, \mathsf{d}, \mathfrak{m}), (Y, \mathsf{d}_Y, \mathfrak{m}_Y)\right) < \varepsilon.$$

The following result will play a key role in the compactness argument.

Lemma 6.32. *Let $(v_n)_n$ be a sequence of functions in $H^{1,p}\left(([0,r], \mathsf{d}_{\mathrm{eucl}}, \mathfrak{m}_N)\right)$ such that $v_n(r) = 0$ for any $n \in \mathbb{N}$. Assume that $(v_n)_n$ converge in $L^p([0,r], \mathfrak{m}_N)$ and in energy to $v \in H^{1,p}\left(([0,r], \mathsf{d}_{\mathrm{eucl}}, \mathfrak{m}_N)\right)$. Define*

$$f_n(t) := \int |\nabla v_n|^{p-1} \,\mathrm{dPer}(\{v_n > t\}), \quad f(t) := \int |\nabla v|^{p-1} \,\mathrm{dPer}(\{v > t\})$$

and let $\eta_n := f_n \mathcal{L}^1$ and $\eta := f \mathcal{L}^1$. Then $\eta_n \rightharpoonup \eta$ in duality with bounded and continuous functions.

Proof. We begin by observing that any function in $H^{1,p}\left(([0,r], \mathsf{d}_{\mathrm{eucl}}, \mathfrak{m}_N)\right)$ is continuous in $(0,r]$. Indeed this result is well known in the case when, instead of $\mathfrak{m}_N$, the interval is equipped with the Lebesgue measure. In the case of our interest it suffices to observe that the density of $\mathfrak{m}_N$ w.r.t. $\mathcal{L}^1$ is uniformly bounded from below on $[\varepsilon, r]$ for any $\varepsilon > 0$. Moreover, by an analogous argument, functions in $H^{1,p}(([0,r], \mathsf{d}_{\mathrm{eucl}}, \mathfrak{m}_N))$ with uniformly bounded p-energies are uniformly Hölder continuous on $[\varepsilon, r]$ for any $\varepsilon > 0$.

In view of what we remarked above, up to extracting a subsequence we can assume that $(v_n)_n$ converges to v uniformly on $[\varepsilon, r]$ for any $\varepsilon > 0$ (recall that $v_n(r) = 0$ for any $n \in \mathbb{N}$). Moreover we can assume that the measures $\gamma_n := |\nabla v_n|^p \,\mathrm{d}\mathfrak{m}_N$ weakly converge to $\gamma := |\nabla v|^p \,\mathrm{d}\mathfrak{m}_N$.

We need to prove that for any bounded and continuous function $\phi : [0,\infty) \to \mathbb{R}$ it holds

$$\lim_{n\to\infty} \int \phi(t) f_n(t) \,\mathrm{d}t = \int \phi(t) f(t) \,\mathrm{d}t. \tag{6.46}$$

To this aim we observe that, thanks to the coarea formula, it holds

$$\begin{aligned}\int \phi(t) f_n(t) \,\mathrm{d}t &= \int \phi(t) \left(\int |\nabla v_n|^{p-1} \,\mathrm{dPer}(\{v_n > t\}) \right) \mathrm{d}t \\ &= \int \phi(v_n(x)) \left|\nabla v_n\right|^p (x) \,\mathrm{d}\mathfrak{m}_N(x)\end{aligned}$$

for any $n \in \mathbb{N}$ (and an analogous identity holds true for f). Thus, in order to prove (6.46), it remains to prove that

$$\lim_{n\to\infty} \int_0^r \phi(v_n(x)) \left|\nabla v_n\right|^p (x) \,\mathrm{d}\mathfrak{m}_N(x) = \int_0^r \phi(v(x)) \left|\nabla v\right|^p (x) \,\mathrm{d}\mathfrak{m}_N(x). \tag{6.47}$$

To this aim we observe that for any $\varepsilon > 0$ it holds that $\phi \circ v_n$ converge uniformly to $\phi \circ v$ on $[\varepsilon, r]$, hence

$$\lim_{n\to\infty} \int_\varepsilon^r \phi(v_n(x)) \left|\nabla v_n\right|^p(x) \, \mathrm{d}\mathfrak{m}_N(x) = \int_\varepsilon^r \phi(v(x)) \left|\nabla v\right|^p(x) \, \mathrm{d}\mathfrak{m}_N(x). \tag{6.48}$$

Moreover, calling $M := \max \phi$, it holds that

$$\limsup_{n\to\infty} \left| \int_0^\varepsilon \phi \circ v_n \left|\nabla v_n\right|^p \, \mathrm{d}\mathfrak{m}_N - \int_0^\varepsilon \phi \circ v \left|\nabla v\right|^p \, \mathrm{d}\mathfrak{m}_N \right| \le 2M \int_0^\varepsilon \left|\nabla v\right|^p \, \mathrm{d}\mathfrak{m}_N \tag{6.49}$$

and the right hand-side in (6.49) goes to 0 as ε goes to 0. Therefore, in order to prove (6.47), it is sufficient to split the interval of integration into $[0, \varepsilon]$ and $[\varepsilon, r]$, pass to the lim sup as $n \to \infty$ taking into account (6.48) and (6.49) and then to let $\varepsilon \downarrow 0$. □

Proof of Theorem 6.31. Let us argue by contradiction. If the conclusion is false there exist $\varepsilon > 0$, a sequence $(X_n)_{n\in\mathbb{N}}$ of $\mathrm{RCD}(N-1, N)$ spaces with $\mathfrak{m}_n(X_n) = 1$ and open domains $\Omega_n \subset X_n$ such that $\mathfrak{m}_n(\Omega_n) = v$, $\lambda_X^p(\Omega_n) \le \lambda_{N,v}^p + \frac{1}{n}$ and

$$\mathsf{d}_{mGH}\left((X_n, \mathsf{d}_n, \mathfrak{m}_n), (X, \mathsf{d}, \mathfrak{m})\right) \ge \varepsilon \tag{6.50}$$

for any spherical suspension $(X, \mathsf{d}, \mathfrak{m})$.

By the very definition of $\lambda_X^p(\Omega)$ and thanks to Lemma 6.12, for any $n \in \mathbb{N} \setminus \{0\}$ we can find a nonnegative function $u_n \in \mathrm{Lip_c}(\Omega_n)$ with $\left|\nabla u_n\right|(x) \ne 0$ for $\mathfrak{m}_n$-a.e. $x \in \{u_n > 0\}$ such that $\|u_n\|_{L^p(\mathfrak{m}_n)} = 1$ and

$$\int_{\Omega_n} \left|\nabla u_n\right|^p \, \mathrm{d}\mathfrak{m}_n \le \lambda_X^p(\Omega_n) + \frac{1}{n} \le \lambda_{N,v}^p + \frac{2}{n}.$$

Call μ_n (respectively f_n) the distribution function of u_n (respectively the function associated to u_n as in (6.31)). Recalling (6.31), (6.32) and applying (6.34) to the function u_n we obtain

$$\int_0^r \left|\nabla u_n^*\right|^p \, \mathrm{d}\mathfrak{m}_N \le \int_0^{\sup u_n^*} \left(\frac{\mathcal{I}_{(X_n, \mathsf{d}_n, \mathfrak{m}_n)}(\mu_n(t))}{\mathcal{I}_N(\mu_n(t))} \right)^p f_n(t) \, \mathrm{d}t \le \lambda_{N,v}^p + \frac{2}{n}, \tag{6.51}$$

where, as usual, r is given by $\mathfrak{m}_N([0, r]) = v$. As a first consequence of (6.51) we obtain that, up to extracting a subsequence, u_n^* weakly converges in $H^{1,p}\left(([0, r], \mathsf{d}_{\mathrm{eucl}}, \mathfrak{m}_N)\right)$ to a function u^*. Moreover the convergence is uniform on $[\varepsilon, r]$ for any $\varepsilon > 0$ so that in particular $u^*(r) = 0$. By the lower semicontinuity of the p-energy

$$\int_0^r \left|\nabla u^*\right|^p \, \mathrm{d}\mathfrak{m}_N \le \liminf_{n\to\infty} \int_0^r \left|\nabla u_n^*\right|^p \, \mathrm{d}\mathfrak{m}_N \le \lambda_{N,v}^p.$$

Hence u^* is the first eigenfunction of the p-Laplacian on the model space $([0, r], \mathsf{d}_{\mathrm{eucl}}, \mathfrak{m}_N)$ with unit L^p-norm satisfying $u^*(r) = 0$. In particular u_n^* converges to u^* in L^p and in $H^{1,p}$-energy.

It follows that u^* has negligible level sets. Hence taking into account the local uniform convergence of the functions u_n^* to u^*, we obtain the pointwise convergence of the distribution functions μ_n to the distribution function μ of u^*.

Moreover, using Lemma 6.32 we get that the sequence of measures $\eta_n := f_n \mathcal{L}^1$ weakly converges to $\eta := f_{u^*} \mathcal{L}^1$ in duality with bounded and continuous functions.

By compactness there exists an $\mathrm{RCD}(N-1, N)$ space $(X, \mathsf{d}, \mathfrak{m})$ with $\mathfrak{m}(X) = 1$ and such that (a subsequence of) $(X_n)_n$ converges to it in the measured Gromov Hausdorff sense.

Introduce now functions g_n and g by

$$g_n(t) := \left(\frac{\mathcal{I}_{(X_n, \mathsf{d}_n, \mathfrak{m}_n)}(\mu_n(t))}{\mathcal{I}_N(\mu_n(t))} \right)^p, \quad g := \left(\frac{\mathcal{I}_{(X, \mathsf{d}, \mathfrak{m})}(\mu(t))}{\mathcal{I}_N(\mu(t))} \right)^p,$$

for any $t \in [0,\infty)$. Proposition 6.4, together with the pointwise convergence of the distribution functions, yields that

$$g(t) \leq \liminf_{n\to\infty} g_n(t_n) \tag{6.52}$$

for any $t \in [0,\infty)$ and for any sequence $(t_n)_{n\in\mathbb{N}}$ such that $t_n \to t$ as $n \to \infty$.
Applying Lemma 1.3 with functions g_n, g and measures η_n and η, we conclude that

$$\int_0^{\sup u^*} \left(\frac{\mathcal{I}_{(X,\mathsf{d},\mathfrak{m})}(\mu(t))}{\mathcal{I}_N(\mu(t))}\right)^p f_{u^*}(t)\,\mathrm{d}t \leq \liminf_{n\to\infty} \int_0^{\sup u_n^*} \left(\frac{\mathcal{I}_{(X_n,\mathsf{d}_n,\mathfrak{m}_n)}(\mu_n(t))}{\mathcal{I}_N(\mu_n(t))}\right)^p f_n(t)\,\mathrm{d}t \leq \lambda_{N,v}^p,$$

where the last inequality follows from (6.51).
Summarizing, we proved that

$$\lambda_{N,v}^p = \int_0^{\sup u^*} f_{u^*}(t)\,\mathrm{d}t \leq \int_0^{\sup u^*} \left(\frac{\mathcal{I}_{(X,\mathsf{d},\mathfrak{m})}(\mu(t))}{\mathcal{I}_N(\mu(t))}\right)^p f_{u^*}(t)\,\mathrm{d}t \leq \lambda_{N,v}^p.$$

Hence

$$\mathcal{I}_{(X,\mathsf{d},\mathfrak{m})}(\mu(t)) = \mathcal{I}_N(\mu(t))$$

for at least one value of t such that $\mu(t) \neq 0, 1$. Therefore $(X,\mathsf{d},\mathfrak{m})$ is isomorphic to a spherical suspension by Theorem 6.6. This is in contradiction with (6.50) since the sequence $(X_n,\mathsf{d}_n,\mathfrak{m}_n)_n$ is converging to $(X,\mathsf{d},\mathfrak{m})$ in the mGH sense. □

Bibliography

[1] A. D. Aleksandrov, *A theorem on triangles in a metric space and some of its applications*, Trudy Mat. Inst. Steklov., v 38, 1951, pp. 5–23. MR0049584

[2] L. Ambrosio and P. Tilli, *Topics on Analysis in Metric Spaces*, Oxford University Press, 2004. Oxford Lecture Series in Mathematics and Its Applications.

[3] L. Ambrosio, *Corso introduttivo alla teoria geometrica della misura ed alle superfici minime*, Appunti dei Corsi Tenuti da Docenti della Scuola. [Notes of Courses Given by Teachers at the School], Scuola Normale Superiore, Pisa, 1997. MR1736268

[4] ———, *Some fine properties of sets of finite perimeter in Ahlfors regular metric measure spaces*, Advances in Mathematics **159** (2001), no. 1, 51 –67.

[5] ———, *Fine properties of sets of finite perimeter in doubling metric measure spaces*, Set-Valued Anal. **10** (2002), no. 2-3, 111–128.

[6] ———, *Transport equation and Cauchy problem for BV vector fields*, Invent. Math. **158** (2004), no. 2, 227–260. MR2096794 (2005f:35127)

[7] ———, *Calculus, heat flow and curvature-dimension bounds in metric measure spaces*, Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. I. Plenary lectures, 2018, pp. 301–340. MR3966731

[8] L. Ambrosio, E. Brué, and D. Semola, *Rigidity of the 1-Bakry-Émery inequality and sets of finite perimeter in RCD spaces*, Geom. Funct. Anal. **29** (2019), no. 4, 949–1001. MR3990192

[9] L. Ambrosio, E. Bruè, and D. Trevisan, *Lusin-type approximation of Sobolev by Lipschitz functions, in Gaussian and* $\mathrm{RCD}(K,\infty)$ *spaces*, Adv. Math. **339** (2018), 426–452. MR3866903

[10] L. Ambrosio, M. Colombo, and S. Di Marino, *Sobolev spaces in metric measure spaces: reflexivity and lower semicontinuity of slope*, Variational methods for evolving objects, 2015, pp. 1–58. MR3587446

[11] L. Ambrosio and S. Di Marino, *Equivalent definitions of BV space and of total variation on metric measure spaces*, J. Funct. Anal. **266** (2014), no. 7, 4150–4188. MR3170206

[12] L. Ambrosio and J. Feng, *On a class of first order Hamilton–Jacobi equations in metric spaces*, Journal of Differential Equations **256** (2014), no. 7, 2194 –2245.

[13] L. Ambrosio, N. Gigli, A. Mondino, and T. Rajala, *Riemannian Ricci curvature lower bounds in metric measure spaces with σ-finite measure*, Trans. Amer. Math. Soc. **367** (2015), no. 7, 4661–4701.

[14] L. Ambrosio, N. Gigli, and G. Savaré, *Gradient flows in metric spaces and in the space of probability measures*, Second, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2008. MR2401600

[15] L. Ambrosio, N. Gigli, and G. Savaré, *Calculus and heat flow in metric measure spaces and applications to spaces with Ricci bounds from below*, Invent. Math. **195** (2014), no. 2, 289–391. MR3152751

[16] ———, *Metric measure spaces with Riemannian Ricci curvature bounded from below*, Duke Math. J. **163** (2014), no. 7, 1405–1490. MR3205729

[17] ———, *Bakry-Émery curvature-dimension condition and Riemannian Ricci curvature bounds*, The Annals of Probability **43** (2015), no. 1, 339–404. MR3298475

[18] L. Ambrosio and S. Honda, *New stability results for sequences of metric measure spaces with uniform Ricci bounds from below*, Measure theory in non-smooth spaces, 2017, pp. 1–51. MR3701735

[19] ———, *Local spectral convergence in* $\mathrm{RCD}^*(K,N)$ *spaces*, Nonlinear Anal. **177** (2018), no. part A, 1–23. MR3865185

[20] L. Ambrosio, S. Honda, J. W. Portegies, and D. Tewodrose, *Embedding of* $\mathrm{RCD}^*(K,N)$ *spaces in L^2 via eigenfunctions*, 2018. Preprint, arXiv:1812.03712.

[21] L. Ambrosio, S. Honda, and D. Tewodrose, *Short-time behavior of the heat kernel and Weyl's law on* $\mathrm{RCD}^*(K,N)$ *spaces*, Ann. Global Anal. Geom. **53** (2018), no. 1, 97–119. MR3746517

[22] L. Ambrosio and B. Kirchheim, *Currents in metric spaces*, Acta Math. **185** (2000), no. 1, 1–80. MR1794185 (2001k:49095)

[23] ———, *Rectifiable sets in metric and Banach spaces*, Math. Ann. **318** (2000), no. 3, 527–555. MR1800768

[24] L. Ambrosio, B. Kleiner, and E. Le Donne, *Rectifiability of sets of finite perimeter in Carnot groups: existence of a tangent hyperplane*, J. Geom. Anal. **19** (2009), no. 3, 509–540. MR2496564

[25] L. Ambrosio, M. Lecumberry, and S. Maniglia, *Lipschitz regularity and approximate differentiability of the DiPerna-Lions flow*, Rend. Sem. Mat. Univ. Padova **114** (2005), 29–50 (2006). MR2207860

[26] L. Ambrosio, A. Mondino, and G. Savaré, *On the Bakry-Émery condition, the gradient estimates and the Local-to-Global property of* $RCD^*(K,N)$ *metric measure spaces*, The Journal of Geometric Analysis **26** (2014), no. 1, 1–33. MR3441502

[27] L. Ambrosio, A. Mondino, and G. Savaré, *Nonlinear diffusion equations and curvature conditions in metric measure spaces*, Mem. Amer. Math. Soc. **262** (2019), no. 1270, v+121. MR4044464

[28] L. Ambrosio, A. Pinamonti, and G. Speight, *Tensorization of Cheeger energies, the space* $H^{1,1}$ *and the area formula for graphs*, Adv. Math. **281** (2015), 1145–1177. MR3366861

[29] L. Ambrosio, F. Stra, and D. Trevisan, *Weak and strong convergence of derivations and stability of flows with respect to MGH convergence*, J. Funct. Anal. **272** (2017), no. 3, 1182–1229. MR3579137

[30] L. Ambrosio and D. Trevisan, *Well-posedness of Lagrangian flows and continuity equations in metric measure spaces*, Anal. PDE **7** (2014), no. 5, 1179–1234. MR3265963

[31] ______, *Lecture notes on the DiPerna-Lions theory in abstract measure spaces*, Ann. Fac. Sci. Toulouse Math. (6) **26** (2017), no. 4, 729–766. MR3746641

[32] M. T. Anderson, *On the topology of complete manifolds of nonnegative Ricci curvature*, Topology **29** (1990), no. 1, 41–55. MR1046624

[33] G. Antonelli, E. Brué, and D. Semola, *Volume bounds for the quantitative singular strata of non collapsed RCD metric measure spaces*, Anal. Geom. Metr. Spaces **7** (2019), no. 1, 158–178. MR4015195

[34] T. Aubin, *Some nonlinear problems in Riemannian geometry*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998. MR1636569

[35] K. Bacher and K.-T. Sturm, *Localization and tensorization properties of the curvature-dimension condition for metric measure spaces*, J. Funct. Anal. **259** (2010), no. 1, 28–56. MR2610378 (2011i:53050)

[36] ______, *Ricci bounds for Euclidean and spherical cones*, Singular phenomena and scaling in mathematical models, 2014, pp. 3–23. MR3205034

[37] D. Bakry, *On Sobolev and logarithmic Sobolev inequalities for Markov semigroups*, New trends in stochastic analysis (Charingworth, 1994), 1997, pp. 43–75. MR1654503

[38] D. Bakry, *Transformations de Riesz pour les semi-groupes symétriques. II. Étude sous la condition* $\Gamma_2 \geq 0$, Séminaire de probabilités, XIX, 1983/84, 1985, pp. 145–174. MR889473 (89h:42023)

[39] D. Bakry and M. Émery, *Diffusions hypercontractives*, Séminaire de probabilités, XIX, 1983/84, 1985, pp. 177–206. MR889476 (88j:60131)

[40] D. Bakry, I. Gentil, and M. Ledoux, *Analysis and geometry of Markov diffusion operators*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 348, Springer, Cham, 2014. MR3155209

[41] J.-D. Benamou and Y. Brenier, *A computational fluid mechanics solution to the Monge-Kantorovich mass transfer problem*, Numer. Math. **84** (2000), no. 3, 375–393. MR1738163

[42] P. Bérard and D. Meyer, *Inégalités isopérimétriques et applications*, Ann. Sci. École Norm. Sup. (4) **15** (1982), no. 3, 513–541. MR690651

[43] J. Bertrand, *Pincement spectral en courbure de Ricci positive*, Comment. Math. Helv. **82** (2007), no. 2, 323–352. MR2319931

[44] J. Bertrand, C. Ketterer, I. Mondello, and T. Richard, *Stratified spaces and synthetic Ricci curvature bounds*, 2018.

[45] N. Bouleau and F. Hirsch, *Dirichlet forms and analysis on Wiener space*, De Gruyter Studies in Mathematics, vol. 14, Walter de Gruyter & Co., Berlin, 1991. MR1133391

[46] H. Brézis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973. North-Holland Mathematics Studies, No. 5. Notas de Matemática (50). MR0348562

[47] E. Brué, A. Naber, and D. Semola, *Boundary regularity for spaces with lower Ricci curvature bounds*, 2020. Work in progress.

[48] E. Brué, E. Pasqualetto, and D. Semola, *Rectifiability of the reduced boundary for sets of finite perimeter over RCD(K,N) spaces*, 2019. Submitted, arXiv:1909.00381.

[49] E. Brué, E. Pasqualetto, and D. Semola, *Rectifiability of* $\mathrm{RCD}(\mathrm{K},\mathrm{N})$ *spaces via* δ*-splitting maps*, 2020. Accepted for publication on Acc. Sci. Acc. Fenn., preprint arXiv:2001.07911.

[50] E. Brué and D. Semola, *Improved regularity for flows on spaces with lower Ricci curvature bounds*, 2020. Work in progress.

[51] E. Brué and D. Semola, *Regularity of lagrangian flows over RCD*(K, N) spaces*, Journal für die reine und angewandte Mathematik **0** (2019).

[52] ______, *Constancy of the dimension for RCD(K,N) spaces via Regularity of Lagrangian Flows*, Communications on Pure and Applied Mathematics **73** (2020), no. 6, 1141–1204, available at https://onlinelibrary.wiley.com/doi/pdf/10.1002/cpa.21849.

[53] V. Buffa, G. E. Comi, and M. Miranda Jr, *On BV functions and essentially bounded divergence-measure fields in metric spaces*, 2019. Preprint, arXiv:1906.07432.

[54] D. Burago, Y. Burago, and S. Ivanov, *A course in metric geometry*, Graduate Studies in Mathematics, vol. 33, American Mathematical Society, Providence, RI, 2001. MR1835418 (2002e:53053)

[55] Yu. Burago, M. Gromov, and G. Perelman, *A. D. Aleksandrov spaces with curvatures bounded below*, Uspekhi Mat. Nauk **47** (1992), no. 2(284), 3–51, 222. MR1185284

[56] F. Cavalletti, F. Maggi, and A. Mondino, *Quantitative isoperimetry à la Levy-Gromov*, Comm. Pure Appl. Math. **72** (2019), no. 8, 1631–1677. MR3974951

[57] F. Cavalletti and E. Milman, *The globalization theorem for the curvature dimension condition*, 2016. Preprint, arXiv:1612.07623.

[58] F. Cavalletti and A. Mondino, *Sharp and rigid isoperimetric inequalities in metric-measure spaces with lower Ricci curvature bounds*, Invent. Math. **208** (2017), no. 3, 803–849. MR3648975

[59] ______, *Sharp geometric and functional inequalities in metric measure spaces with lower Ricci curvature bounds*, Geom. Topol. **21** (2017), no. 1, 603–645. MR3608721

[60] ______, *Isoperimetric inequalities for finite perimeter sets under lower Ricci curvature bounds*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. **29** (2018), no. 3, 413–430. MR3819097

[61] F. Cavalletti, A. Mondino, and D. Semola, *Quantitative Obata's theorem*, 2019. Preprint arXiv:1910.06637.

[62] J. Cheeger, *Differentiability of Lipschitz functions on metric measure spaces*, Geom. Funct. Anal. **9** (1999), no. 3, 428–517. MR2000g:53043

[63] J. Cheeger, T. H. Colding, and G. Tian, *On the singularities of spaces with bounded Ricci curvature*, Geom. Funct. Anal. **12** (2002), no. 5, 873–914. MR1937830

[64] J. Cheeger, W Jiang, and A Naber, *Rectifiability of Singular Sets in Noncollapsed Spaces with Ricci Curvature bounded below*, 2018. Preprint, arXiv:1805.07988.

[65] J. Cheeger and A. Naber, *Regularity of Einstein manifolds and codimension 4 conjecture*, Ann. of Math. **182** (2015), 1–73 (English (US)).

[66] J. Cheeger, *Structure theory and convergence in Riemannian geometry*, Milan J. Math. **78** (2010), no. 1, 221–264. MR2684779

[67] ______, *Quantitative differentiation: a general formulation*, Comm. Pure Appl. Math. **65** (2012), no. 12, 1641–1670. MR2982637

[68] J. Cheeger and T. H. Colding, *Lower bounds on Ricci curvature and the almost rigidity of warped products*, Ann. of Math. (2) **144** (1996), no. 1, 189–237. MR1405949 (97h:53038)

[69] ______, *On the structure of spaces with Ricci curvature bounded below. I*, J. Differential Geom. **46** (1997), no. 3, 406–480. MR1484888 (98k:53044)

[70] ______, *On the structure of spaces with Ricci curvature bounded below. II*, J. Differential Geom. **54** (2000), no. 1, 13–35. MR1815410 (2003a:53043)

[71] ______, *On the structure of spaces with Ricci curvature bounded below. III*, J. Differential Geom. **54** (2000), no. 1, 37–74. MR1815411 (2003a:53044)

[72] J. Cheeger and D. Gromoll, *The splitting theorem for manifolds of nonnegative Ricci curvature*, J. Differential Geometry **6** (1971/72), 119–128. MR0303460 (46 #2597)

[73] J. Cheeger and A. Naber, *Lower bounds on ricci curvature and quantitative behavior of singular sets*, Inventiones Mathematicae **191** (2013), no. 2, 321–339 (English (US)).

[74] ______, *Quantitative stratification and the regularity of harmonic maps and minimal currents*, Comm. Pure Appl. Math. **66** (2013), no. 6, 965–990. MR3043387

[75] T. H. Colding, *New monotonicity formulas for Ricci curvature and applications. I*, Acta Math. **209** (2012), no. 2, 229–263. MR3001606

[76] T. H. Colding and W. P. Minicozzi II, *On uniqueness of tangent cones for Einstein manifolds*, Invent. Math. **196** (2014), no. 3, 515–588. MR3211041

[77] T. H. Colding and A. Naber, *Sharp Hölder continuity of tangent cones for spaces with a lower Ricci curvature bound and applications*, Ann. of Math. (2) **176** (2012), no. 2, 1173–1229. MR2950772

[78] D. Cordero-Erausquin, R. J. McCann, and M. Schmuckenschläger, *A Riemannian interpolation inequality à la Borell, Brascamp and Lieb*, Invent. Math. **146** (2001), no. 2, 219–257. MR1865396

[79] G. Crippa and C. De Lellis, *Estimates and regularity results for the DiPerna-Lions flow*, J. Reine Angew. Math. **616** (2008), 15–46. MR2369485

[80] S. Daneri and G. Savaré, *Eulerian calculus for the displacement convexity in the Wasserstein distance*, SIAM J. Math. Anal. **40** (2008), no. 3, 1104–1122. MR2452882

[81] E. De Giorgi, *Su una teoria generale della misura $(r-1)$-dimensionale in uno spazio ad r dimensioni*, Ann. Mat. Pura Appl. (4) **36** (1954), 191–213. MR62214

[82] ______, *Nuovi teoremi relativi alle misure $(r-1)$-dimensionali in uno spazio ad r dimensioni*, Ricerche Mat. **4** (1955), 95–113. MR74499

[83] G. De Philippis and N. Gigli, *From volume cone to metric cone in the nonsmooth setting*, Geom. Funct. Anal. **26** (2016), no. 6, 1526–1587. MR3579705

[84] ______, *Non-collapsed spaces with Ricci curvature bounded from below*, Journal de l'École polytechnique — Mathématiques **5** (2018), 613–650.

[85] G. De Philippis, A. Marchese, and F. Rindler, *On a conjecture of Cheeger*, Measure Theory in Non-Smooth Spaces (201701).

[86] G. De Philippis and F. Rindler, *On the structure of $\mathcal{A}$-free measures and applications*, Ann. of Math. (2) **184** (2016), no. 3, 1017–1039. MR3549629

[87] G. De Philippis and J. N. Zimbron, *The behaviour of harmonic functions at singular points of RCD spaces*, 2019. Preprint arXiv:1909.05220v1.

[88] C. Debin, N. Gigli, and E. Pasqualetto, *Quasi-continuous vector fields on* RCD *spaces*, Potential Anal. (2020). Submitted, arXiv:1903.04302.

[89] S. Di Marino, *Sobolev and BV spaces on metric measure spaces via derivations and integration by parts*, 2014. Submitted, arXiv:1409.5620.

[90] S. Di Marino, N. Gigli, E. Pasqualetto, and E. Soultanis, *Infinitesimal Hilbertianity of locally* $\mathrm{CAT}(\kappa)$*-spaces*, 2018. Submitted, arXiv:1812.02086.

[91] S. Di Marino and G. Speight, *The p-weak gradient depends on p*, Proc. Amer. Math. Soc. **143** (2015), no. 12, 5239–5252. MR3411142

[92] Y. Ding, *Heat kernels and Green's functions on limit spaces*, Comm. Anal. Geom. **10** (2002), no. 3, 475–514. MR1912256

[93] R. J. DiPerna and P.-L. Lions, *Ordinary differential equations, transport theory and Sobolev spaces*, Invent. Math. **98** (1989), no. 3, 511–547. MR1022305 (90j:34004)

[94] E. Durand-Cartagena, L. Ihnatsyeva, R. Korte, and M. Szumańska, *On Whitney-type characterization of approximate differentiability on metric measure spaces*, Canad. J. Math. **66** (2014), no. 4, 721–742. MR3224261

[95] M. Erbar, *The heat equation on manifolds as a gradient flow in the Wasserstein space*, Ann. Inst. Henri Poincaré Probab. Stat. **46** (2010), no. 1, 1–23. MR2641767

[96] M. Erbar, K. Kuwada, and K.-T. Sturm, *On the equivalence of the entropic curvature-dimension condition and Bochner's inequality on metric measure spaces*, Inventiones mathematicae **201** (2014), no. 3, 1–79. MR3385639

[97] L. C. Evans and R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, Studies in Advanced Mathematics, Taylor & Francis, 1991.

[98] A. Farina, L. Mari, and E. Valdinoci, *Splitting theorems, symmetry results and overdetermined problems for Riemannian manifolds*, Comm. Partial Differential Equations **38** (2013), no. 10, 1818–1862. MR3169764

[99] A. Farina and J. Ocariz, *Splitting theorems on complete Riemannian manifolds with nonnegative Ricci curvature*, 2020. Preprint arXiv:2001.02468.

[100] H. Federer, *Geometric measure theory*, Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag New York Inc., New York, 1969. MR0257325

[101] H. Federer and W. H. Fleming, *Normal and integral currents*, Ann. of Math. (2) **72** (1960), 458–520. MR123260

[102] A. Figalli, *A simple proof of the Morse-Sard theorem in Sobolev spaces*, Proc. Amer. Math. Soc. **136** (2008), no. 10, 3675–3681. MR2415054

[103] W. H. Fleming, *Flat chains over a finite coefficient group*, Trans. Amer. Math. Soc. **121** (1966), 160–186. MR185084

[104] K. Fukaya, *Collapsing of Riemannian manifolds and eigenvalues of Laplace operator*, Invent. Math. **87** (1987), no. 3, 517–547. MR874035

[105] M. Fukushima, Y. Oshima, and M. Takeda, *Dirichlet forms and symmetric Markov processes*, extended, De Gruyter Studies in Mathematics, vol. 19, Walter de Gruyter & Co., Berlin, 2011. MR2778606

[106] N. Garofalo and A. Mondino, *Li-Yau and Harnack type inequalities in* $\mathrm{RCD}^*(K,N)$ *metric measure spaces*, Nonlinear Anal. **95** (2014), 721–734. MR3130557

[107] N. Gigli, *Optimal maps in non branching spaces with Ricci curvature bounded from below*, Geom. Funct. Anal. **22** (2012), no. 4, 990–999. MR2984123

[108] ______, *The splitting theorem in non-smooth context*, 2013. Preprint, arXiv:1302.5555.

[109] ______, *An Overview of the Proof of the Splitting Theorem in Spaces with Non-Negative Ricci Curvature*, Analysis and Geometry in Metric Spaces **2** (2014), 169–213. MR3210895

[110] ______, *On the differential structure of metric measure spaces and applications*, Mem. Amer. Math. Soc. **236** (2015), no. 1113, vi+91. MR3381131

[111] ______, *Lecture notes on differential calculus on RCD spaces*, Publ. Res. Inst. Math. Sci. **54** (2018), no. 4, 855–918. MR3869373

[112] ______, *Nonsmooth differential geometry—an approach tailored for spaces with Ricci curvature bounded from below*, Mem. Amer. Math. Soc. **251** (2018), no. 1196, v+161. MR3756920

[113] N. Gigli and B.-X. Han, *The continuity equation on metric measure spaces*, Calc. Var. Partial Differential Equations **53** (2015), no. 1-2, 149–177. MR3336316

[114] N. Gigli and B. Han, *Independence on p of weak upper gradients on* RCD *spaces*, Journal of Functional Analysis **271** (201407).

[115] N. Gigli, K. Kuwada, and S.-i. Ohta, *Heat flow on Alexandrov spaces*, Communications on Pure and Applied Mathematics **66** (2013), no. 3, 307–331.

[116] N. Gigli and A. Mondino, *A PDE approach to nonlinear potential theory in metric measure spaces*, J. Math. Pures Appl. (9) **100** (2013), no. 4, 505–534. MR3102164

[117] N. Gigli, A. Mondino, and T. Rajala, *Euclidean spaces as weak tangents of infinitesimally Hilbertian metric measure spaces with Ricci curvature bounded below*, J. Reine Angew. Math. **705** (2015), 233–244. MR3377394

[118] N. Gigli, A. Mondino, and G. Savaré, *Convergence of pointed non-compact metric measure spaces and stability of Ricci curvature bounds and heat flows*, Proc. Lond. Math. Soc. (3) **111** (2015), no. 5, 1071–1129. MR3477230

[119] N. Gigli and S. Mosconi, *The abstract Lewy-Stampacchia inequality and applications*, J. Math. Pures Appl. (9) **104** (2015), no. 2, 258–275. MR3365829

[120] N. Gigli and E. Pasqualetto, *Behaviour of the reference measure on* RCD *spaces under charts*, 2016. Accepted at Communications in Analysis and Geometry, arXiv: 1607.05188.

[121] ______, *Equivalence of two different notions of tangent bundle on rectifiable metric measure spaces*, 2016. Accepted at Communications in Analysis and Geometry, arXiv:1611.09645.

[122] ______, *On the notion of parallel transport on RCD spaces*, Rev. Mat. Iberoam. **36** (2020), no. 2, 571–609. MR4082919

[123] N. Gigli and L. Tamanini, *Second order differentiation formula on RCD*(K,N) spaces*, 2018. Accepted at J. Eur. Math. Soc., preprint available at arXiv:1802.02463.

[124] ______, *Second order differentiation formula on* $\mathrm{RCD}(K,N)$ *spaces*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. **29** (2018), no. 2, 377–386. MR3797990

[125] A. Grigor′ yan, *Heat kernels on weighted manifolds and applications*, The ubiquitous heat kernel, 2006, pp. 93–191. MR2218016

[126] M. Gromov, *Sign and geometric meaning of curvature*, Rend. Sem. Mat. Fis. Milano **61** (1991), 9–123 (1994). MR1297501

[127] M. Gromov, *Structures métriques pour les variétés riemanniennes*, Textes Mathématiques [Mathematical Texts], vol. 1, CEDIC, Paris, 1981. Edited by J. Lafontaine and P. Pansu. MR682063

[128] M. Gromov, *Metric structures for Riemannian and non-Riemannian spaces*, English, Modern Birkhäuser Classics, Birkhäuser Boston Inc., Boston, MA, 2007. Based on the 1981 French original, With appendices by M. Katz, P. Pansu and S. Semmes, Translated from the French by Sean Michael Bates. MR2307192 (2007k:53049)

[129] P. Hajł asz and P. Koskela, *Sobolev met Poincaré*, Mem. Amer. Math. Soc. **145** (2000), no. 688, x+101. MR1683160

[130] P. Hajłasz, *Sobolev spaces on an arbitrary metric space*, Potential Analysis **5** (1996), 403–415.

[131] B.-X. Han, *Rigidity of some functional inequalities on RCD spaces*, 2020. Preprint available at arXiv:2001.07930.

[132] B.-X. Han, *Characterizations of monotonicity of vector fields on metric measure spaces*, Calc. Var. Partial Differential Equations **57** (2018), no. 5, Paper No. 113, 35. MR3829199

[133] ______, *Ricci tensor on* $\mathrm{RCD}^*(K,N)$ *spaces*, J. Geom. Anal. **28** (2018), no. 2, 1295–1314. MR3790501

[134] J. Heinonen, *Lectures on analysis on metric spaces*, Universitext, Springer-Verlag, New York, 2001. MR1800917 (2002c:30028)

[135] S. Honda, *Bakry-Émery conditions on almost smooth metric measure spaces*, Anal. Geom. Metr. Spaces **6** (2018), no. 1, 129–145. MR3877323

[136] R. Jiang, *Cheeger-harmonic functions in metric measure spaces revisited*, J. Funct. Anal. **266** (2014), no. 3, 1373–1394. MR3146820

[137] ———, *The Li-Yau inequality and heat kernels on metric measure spaces*, J. Math. Pures Appl. (9) **104** (2015), no. 1, 29–57. MR3350719

[138] R. Jiang, H. Li, and H. Zhang, *Heat kernel bounds on metric measure spaces and some applications*, Potential Anal. **44** (2016), no. 3, 601–627. MR3489857

[139] W. Jiang and A. Naber, *L^2 curvature bounds on manifolds with bounded Ricci curvature*, 2016. Preprint arXiv:1605.05583.

[140] Y. Jiang and H.-C. Zhang, *Sharp spectral gaps on metric measure spaces*, Calc. Var. Partial Differ. Equ. **55** (2016), no. 1.

[141] R. Jordan, D. Kinderlehrer, and F. Otto, *The variational formulation of the Fokker-Planck equation*, SIAM J. Math. Anal. **29** (1998), no. 1, 1–17. MR1617171 (2000b:35258)

[142] V. Kapovitch, M. Kell, and C. Ketterer, *On the structure of RCD spaces with upper curvature bounds*, 2019. Preprint available at arXiv:1908.07036.

[143] V. Kapovitch, A. Lytchak, and A. Petrunin, *Metric-measure boundary and geodesic flow on Alexandrov spaces*, 2017. Accepted by J. Eur. Math. Soc., preprint,at arXiv:1705.04767.

[144] V. Kapovitch and A. Mondino, *On the topology and the boundary of N-dimensional* $\mathrm{RCD}(K,N)$ *spaces*, 2019. Accepted by Geometry and Topology, preprint available at arXiv:1907.02614.

[145] V. Kapovitch and N. Li, *On dimensions of tangent cones in limit spaces with lower Ricci curvature bounds*, J. Reine Angew. Math. **742** (2018), 263–280. MR3849628

[146] M. Kell and A. Mondino, *On the volume measure of non-smooth spaces with Ricci curvature bounded below*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **18** (2018), no. 2, 593–610. MR3801291

[147] S. Kesavan, *Symmetrization & applications*, extended, Series in Analysis, vol. 3, World Scientific Publishing Co. Pte. Ltd., Hacken- sack, NJ, 2006.

[148] J. Kinnunen and V. Latvala, *Lebesgue points for Sobolev functions on metric spaces*, Rev. Mat. Iberoamericana **18** (2002), no. 3, 685–700. MR1954868

[149] J. Kinnunen and O. Martio, *The Sobolev capacity on metric spaces*, Ann. Acad. Sci. Fenn. Math. **21** (1996), no. 2, 367–382. MR1404091

[150] B. Kirchheim, *Rectifiable metric spaces: local structure and regularity of the Hausdorff measure*, Proc. Amer. Math. Soc. **121** (1994), no. 1, 113–123. MR1189747

[151] Y. Kitabeppu, *A Bishop-type inequality on metric measure spaces with Ricci curvature bounded below*, Proc. Amer. Math. Soc. **145** (2017), no. 7, 3137–3151. MR3637960

[152] ———, *A Sufficient Condition to a Regular Set Being of Positive Measure on* RCD *Spaces*, Potential Analysis **51** (2019Aug), no. 2, 179–196.

[153] B. Klartag, *Needle decompositions in Riemannian geometry*, Mem. Amer. Math. Soc. **249** (2017), no. 1180, v+77. MR3709716

[154] K. Kuwada, *Duality on gradient estimates and Wasserstein controls*, J. Funct. Anal. **258** (2010), no. 11, 3758–3774. MR2606871 (2011d:35109)

[155] P. Lahti, *A sharp leibniz rule for bv functions on metric measure spaces*, 2018. Preprint arXiv:1811.07713.

[156] V. Latvala, N. Marola, and M. Pere, *Harnack's inequality for a nonlinear eigenvalue problem on metric spaces*, Journal of Mathematical Analysis and Applications **321** (2006), no. 2, 793 –810.

[157] E. Le Donne, *Metric spaces with unique tangents*, Ann. Acad. Sci. Fenn. Math. **36** (2011), no. 2, 683–694. MR2865538

[158] P. Li and L.-F. Tam, *Green's functions, harmonic functions, and volume comparison*, J. Differential Geom. **41** (1995), no. 2, 277–318. MR1331970

[159] P. Li, L.-F. Tam, and J. Wang, *Sharp bounds for the Green's function and the heat kernel*, Math. Res. Lett. **4** (1997), no. 4, 589–602. MR1470428

[160] P. Li and S.-T. Yau, *On the parabolic kernel of the Schrödinger operator*, Acta Math. **156** (1986), no. 3-4, 153–201. MR834612

[161] G. Liu, *3-manifolds with nonnegative Ricci curvature*, Invent. Math. **193** (2013), no. 2, 367–375. MR3090181

[162] J. Lott and C. Villani, *Weak curvature conditions and functional inequalities*, J. Funct. Anal. **245** (2007), no. 1, 311–333. MR2311627 (2008f:53039)

[163] ———, *Ricci curvature for metric-measure spaces via optimal transport*, Ann. of Math. (2) **169** (2009), no. 3, 903–991. MR2480619 (2010i:53068)

[164] V. Magnani, *On a measure theoretic area formula*, Proc. Roy. Soc. Edinburgh Sect. A **145** (2015), 885–891.

[165] C. Mantegazza, G. Mascellani, and G. Uraltsev, *On the distributional Hessian of the distance function*, Pacific J. Math. **270** (2014), no. 1, 151–166. MR3245852

[166] A.-M. Matei, *First eigenvalue for the p-Laplace operator*, Nonlinear Anal. **39** (2000), no. 8, Ser. A: Theory Methods, 1051–1068. MR1735181

[167] R. J. McCann, *A convexity principle for interacting gases*, Adv. Math. **128** (1997), no. 1, 153–179. MR1451422

[168] E. Milman, *Sharp isoperimetric inequalities and model spaces for the Curvature-Dimension-Diameter condition*, Journal of the European Mathematical Society **017** (2015), no. 5, 1041–1078 (eng).

[169] M. Miranda Jr., *Functions of bounded variation on "good" metric spaces*, J. Math. Pures Appl. (9) **82** (2003), no. 8, 975–1004. MR2005202

[170] A. Mondino and A. Naber, *Structure theory of metric measure spaces with lower Ricci curvature bounds*, J. Eur. Math. Soc. (JEMS) **21** (2019), no. 6, 1809–1854. MR3945743

[171] A. Mondino and D. Semola, *Polya-Szego inequality and Dirichlet p-spectral gap for non-smooth spaces with Ricci curvature bounded below*, J. Math. Pures Appl. (9) **137** (2020), 238–274. MR4088507

[172] A. Naber and D. Valtorta, *Rectifiable-Reifenberg and the regularity of stationary and minimizing harmonic maps*, Ann. of Math. (2) **185** (2017), no. 1, 131–227. MR3583353

[173] ———, *Energy identity for stationary Yang Mills*, Invent. Math. **216** (2019), no. 3, 847–925. MR3955711

[174] A. Nakayasu, *Metric viscosity solutions for Hamilton-Jacobi equations of evolution type*, Adv. Math. Sci. Appl. **24** (2014), no. 2, 333–351. MR3362772

[175] S.-i. Ohta, *Finsler interpolation inequalities*, Calc. Var. Partial Differential Equations **36** (2009), no. 2, 211–249. MR2546027 (2011m:58027)

[176] S.-i. Ohta and K.-T. Sturm, *Heat flow on Finsler manifolds*, Comm. Pure Appl. Math. **62** (2009), no. 10, 1386–1433. MR2547978 (2010j:58058)

[177] ———, *Non-contraction of heat flow on Minkowski spaces*, Arch. Ration. Mech. Anal. **204** (2012), no. 3, 917–944. MR2917125

[178] F. Otto and C. Villani, *Generalization of an inequality by Talagrand and links with the logarithmic Sobolev inequality*, J. Funct. Anal. **173** (2000), no. 2, 361–400. MR1760620

[179] F. Otto, *The geometry of dissipative evolution equations: the porous medium equation*, Comm. Partial Differential Equations **26** (2001), no. 1-2, 101–174. MR1842429 (2002j:35180)

[180] P. Petersen, *On eigenvalue pinching in positive Ricci curvature*, Invent. Math. **138** (1999), no. 1, 1–21. MR1714334

[181] A. Petrunin, *Alexandrov meets Lott-Villani-Sturm*, Münster J. Math. **4** (2011), 53–64. MR2869253 (2012m:53087)

[182] D. Preiss, *Geometry of measures in* $\mathbf{R}^n$*: distribution, rectifiability, and densities*, Ann. of Math. (2) **125** (1987), no. 3, 537–643. MR890162

[183] G. Pólya and G. Szegö, *Isoperimetric inequalities in mathematical physics*, Annals of Mathematics Studies, vol. 27, University Press, Princeton, NJ, 1951.

[184] T. Rajala, *Interpolated measures with bounded density in metric spaces satisfying the curvature-dimension conditions of Sturm*, J. Funct. Anal. **263** (2012), no. 4, 896–924. MR2927398

[185] ———, *Local Poincaré inequalities from stable curvature conditions on metric spaces*, Calc. Var. Partial Differential Equations **44** (2012), no. 3-4, 477–494. MR2915330

[186] ———, *Failure of the local-to-global property for* $CD(K,N)$ *spaces*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **15** (2016), 45–68. MR3495420

[187] T. Rajala and K.-T. Sturm, *Non-branching geodesics and optimal maps in strong* $CD(K,\infty)$*-spaces*, Calc. Var. Partial Differential Equations **50** (2014), no. 3-4, 831–846. MR3216835

[188] ———, *Non-branching geodesics and optimal maps in strong* $CD(K,\infty)$*-spaces*, Calc. Var. Partial Differential Equations **50** (2014), no. 3-4, 831–846. MR3216835

[189] S Rigot, *Differentiation of measures in metric measure spaces*, 2018. preprint version available on arXiv:1802.02069.

[190] A. Sard, *The measure of the critical values of differentiable maps*, Bull. Amer. Math. Soc. **48** (1942), 883–890. MR7523

[191] G. Savaré, *Self-improvement of the Bakry-Émery condition and Wasserstein contraction of the heat flow in* $\mathrm{RCD}(K,\infty)$ *metric measure spaces*, Discrete Contin. Dyn. Syst. **34** (2014), no. 4, 1641–1661. MR3121635

[192] R. Schoen and S. T. Yau, *Complete three-dimensional manifolds with positive Ricci curvature and scalar curvature*, Seminar on Differential Geometry, 1982, pp. 209–228. MR645740

[193] N. Shanmugalingam, *Newtonian spaces: an extension of Sobolev spaces to metric measure spaces*, Rev. Mat. Iberoamericana **16** (2000), no. 2, 243–279. MR1809341 (2002b:46059)

[194] K. Shiohama, *An introduction to the geometry of Alexandrov spaces*, Lecture Notes Series, vol. 8, Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1993. MR1320267

[195] M. Simon, *Ricci flow of non-collapsed three manifolds whose Ricci curvature is bounded from below*, J. Reine Angew. Math. **662** (2012), 59–94. MR2876261

[196] M. Simon and P. Topping, *Local mollification of riemannian metrics using ricci flow, and ricci limit spaces*, 2017. To appear, Geometry and Topology. https: //arxiv.org/abs/1706.09490.

[197] K.-T. Sturm, *Analysis on local Dirichlet spaces. I. Recurrence, conservativeness and L^p-Liouville properties*, J. Reine Angew. Math. **456** (1994), 173–196. MR1301456 (95i:31003)

[198] ______, *Analysis on local Dirichlet spaces. II. Upper Gaussian estimates for the fundamental solutions of parabolic equations*, Osaka J. Math. **32** (1995), no. 2, 275–312. MR1355744 (97b:35003)

[199] ______, *Analysis on local Dirichlet spaces. III. The parabolic Harnack inequality*, J. Math. Pures Appl. (9) **75** (1996), no. 3, 273–297. MR1387522 (97k:31010)

[200] ______, *On the geometry of metric measure spaces. I*, Acta Math. **196** (2006), no. 1, 65–131. MR2237206

[201] ______, *On the geometry of metric measure spaces. II*, Acta Math. **196** (2006), no. 1, 133–177. MR2237207 (2007k:53051b)

[202] ______, *Gradient flows for semiconvex functions on metric measure spaces—existence, uniqueness, and Lipschitz continuity*, Proc. Amer. Math. Soc. **146** (2018), no. 9, 3985–3994. MR3825851

[203] ______, *Ricci tensor for diffusion operators and curvature-dimension inequalities under conformal transformations and time changes*, J. Funct. Anal. **275** (2018), no. 4, 793–829. MR3807777

[204] E.-B. Sylvester, T.-G. James, P. Lahti, and N. Shanmugalingam, *Asymptotic behavior of BV functions and sets of finite perimeter in metric measure spaces*, 2018. Preprint arXiv:1810.05310.

[205] G. Tian and F. Wang, *Cheeger-Colding-Tian theory for conic Kahler-Einstein metrics*, 2018. Preprint arXiv:1807.07209.

[206] P. Topping, *Ricci flow and Ricci limit spaces*, 2018. To appear, Springer Lecture Notes. https://arxiv.org/abs/1904.11375.

[207] N. Th. Varopoulos, *Green's functions on positively curved manifolds*, J. Functional Analysis **45** (1982), no. 1, 109–118. MR645648

[208] ______, *Green's functions on positively curved manifolds. II*, J. Functional Analysis **49** (1982), no. 2, 170–176. MR680657

[209] N. Th. Varopoulos, *The Poisson kernel on positively curved manifolds*, J. Functional Analysis **44** (1981), no. 3, 359–380. MR643040

[210] C. Villani, *Optimal transport. old and new*, Grundlehren der Mathematischen Wissenschaften, vol. 338, Springer-Verlag, Berlin, 2009. MR2459454

[211] C. Villani, *Synthetic theory of Ricci curvature bounds*, Jpn. J. Math. **11** (2016), no. 2, 219–263. MR3544918

[212] M.-K. von Renesse and K.-T. Sturm, *Entropic measure and Wasserstein diffusion*, Ann. Probab. **37** (2009), no. 3, 1114–1191. MR2537551 (2010k:60185)

[213] N. Weaver, *Lipschitz algebras*, World Scientific Publishing Co., Inc., River Edge, NJ, 1999. MR1832645 (2002g:46002)

[214] ______, *Lipschitz algebras and derivations. II. Exterior differentiation*, J. Funct. Anal. **178** (2000), no. 1, 64–112. MR1800791 (2002g:46040a)

[215] B. White, *A new proof of the compactness theorem for integral currents*, Comment. Math. Helv. **64** (1989), no. 2, 207–220. MR997362

[216] H.-C. Zhang and X.-P. Zhu, *Ricci curvature on Alexandrov spaces and rigidity theorems*, Comm. Anal. Geom. **18** (2010), no. 3, 503–553. MR2747437 (2012d:53128)