ESAIM: COCV 25 (2019) 59
https://doi.org/10.1051/cocv/2018044

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

DESIGNING METRICS; THE DELTA METRIC FOR CURVES*

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Abstract. In the first part, we revisit some key notions. Let M be a Riemannian manifold. Let G be a group acting on M. We discuss the relationship between the quotient M/G, "horizontality" and "normalization". We discuss the distinction between *path-wise* invariance and *point-wise* invariance and how the former positively impacts the design of metrics, in particular for the mathematical and numerical treatment of geodesics. We then discuss a strategy to design metrics with desired properties.

In the second part, we prepare methods to normalize some standard group actions on the curve; we design a simple differential operator, called *the delta operator*, and compare it to the usual differential operators used in defining Riemannian metrics for curves.

In the third part we design two examples of Riemannian metrics in the space of planar curves. These metrics are based on the "delta" operator; they are "modular", they are composed of different terms, each associated to a group action. These are "strong" metrics, that is, smooth metrics on the space of curves, that is defined as a differentiable manifolds, modeled on the standard Sobolev space H^2. These metrics enjoy many important properties, including: metric completeness, geodesic completeness, existence of minimal length geodesics. These metrics properly project on the space of curves up to parameterization; the quotient space again enjoys the above properties.

Mathematics Subject Classification. 49Q10, 53A04,53C22, 58B20, 58D10, 58D15, 58E10.

Received March 12, 2018. Accepted August 5, 2018.

1. General introduction

We would like to find a Riemannian metric on the space of curves satisfying important properties. There is a wide literature on this subject; see Section 5.2 for a minimal review. Usually the approach is to write down a metric that seems suitable, then to try to prove that it satisfies some desired properties. We will change point of view.

In the first part of this paper, we discuss a general strategy to design a metric in a manifold when there are interesting groups acting on the manifold, so as to satisfy the desired properties. This strategy has often been covertly used: the purpose here is to analyze it in abstract, so as to identify some key elements, some *do's and don'ts*. In particular, we will stress the rôle of seminorms, the distinction between *path-wise* invariance and *point-wise* invariance[1] of the group action, the method of normalization.

* This work was supported by the PRIN *2015PA5MP7 (Calcolo delle Variazioni).*

Keywords and phrases: Riemannian metric, manifold of curves, shape space, Hilbert manifold, Sobolev space, geodesic, Fréchet mean.

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[1] These will be then respectively called *homotopy-wise* invariance and *curve-wise* invariance, when dealing with the space of curves in the second and third part of the paper.

The second part of this paper deals with spaces of immersed curves $c : [0,1] \to \mathbb{R}^n$, mostly in the case $n = 2$ of planar curves. In Section 5.2, we will review the current literature. In Section 6, we will define basic notions and notations; in Section 7, we will define the groups acting on the space of curves. In Section 9, we will properly define the differentiable structure, so that the spaces of curves to be studied will be differentiable manifolds, modeled on the standard Sobolev space H^2. In Section 8, we will prepare methods to normalize some standard actions on the curve, namely translation, rescaling and rotation. Following the ideas developed in the first part, we will design in Section 11 a simple differential operator, called *the delta operator*; to this end we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$; given an immersed curve $c : [0,1] \to \mathbb{C}$ and a vector field $h : [0,1] \to \mathbb{C}$, we define the *the delta operator* by $\Delta_c h \stackrel{\text{def}}{=} h'/c'$ where the division is in the sense of complex numbers. We denote by $D_c h$ the derivation by arc parameter $D_c h \stackrel{\text{def}}{=} h'/|c'|$ of h along c. The difference between $D_c h$ and $\Delta_c h$ is akin to the difference between Lagrangian coordinates and Eulerian coordinates: when using $\Delta_c h$, we are interested in the relative angle between h' and c', not in the angle between h' and a fixed reference versor in the space. In Section 12, we will discuss the properties of the delta operator, and of the seminorm associated to it, comparing it to the usual Sobolev-type Riemannian metrics for curves, that are based on the operator $D_c h$. In particular the kernel of the second order delta operator contains the infinitesimal action of translation, rotation and scaling; see Section 11.3; this implies that the second order delta seminorm $\|h\|_{\Delta^2,c} \stackrel{\text{def}}{=} \sqrt{\int_0^1 |\Delta_c^2 h|^2 \, \mathrm{d}s}$ is path-wise invariant for those actions (and point-wise invariant for reparameterization); this seminorm will be one of the building blocks of the Riemannian metrics presented in the third part.

In the third part, we will design two Riemannian metrics in the space of immersed curves, using as building blocks the normalizations and seminorms defined and studied in the second part. These Riemannian metrics enjoy many useful properties, both locally and globally.

- The first metric, discussed in Section 14, is apt for the space of open curves and for the space of closed curves as well.
- The second metric, discussed in Section 15, is designed for the space of open curves, where it enjoys better properties than the former.
- Each metric is "modular", in the sense that it is composed by terms, each associated to a different group action. (The precise way in which the terms interact is discussed in Section 14.5, see in particular Tab. 1)
- In particular, when considering quotient spaces (*e.g.* curves up translations), the quotient metric is obtained by deleting the corresponding term.
- Due to the aforementioned decomposition of the metric, along a geodesic the center of mass has constant speed, and the length has logarithmic speed; and there is an "angular velocity" that is constant.
- We will moreover discover in Section 14.3 that these metrics are invariant for an action that we call "curling"; so the space of open curves is a "principal homogeneous space" (in the category of smooth curves), see Section 14.4; this has important and useful consequences, as discussed in Section 5.1.
- These metrics are second-order Sobolev metrics; but there is only one term that is second-order (namely the seminorm $\|h\|_{\Delta^2,c}$ above defined, based on the "delta operator"); moreover this term, up to a log-transform, actually can be seen as a first-order seminorm; see Section 12.1.
  This simplifies the analysis; hopefully, it should also ease the numerical computation in applications (although this fact was not checked in this paper).
- For each of the two metrics, the Riemannian manifold is metrically complete (Thm. 14.18).
- In particular, it is geodesically complete, that is geodesics exist for all time.
- For any two immersed curves there is a minimal length geodesic connecting them (Thm. 14.28). For any finite collection of immersed curves there is a Fréchet mean (*a.k.a.* Karcher mean) (Thm. 14.29).
- Both metrics project to the space of *geometric curves* (*i.e.* curves up to reparameterization). This space can be seen as a metric space;[2] in this respect, it is a complete metric space; any pair of geometric curves can be connected by a minimal length geodesic; any finite family has a Fréchet mean. See in Section 16.

[2] In particular, it is proven that the quotient distance is nondegenerate, that is, different geometric curves have positive distance, see Section 16.3.

- Since translation and scale can be factored out of the metric, then when computing a minimal geodesic the approximating paths can be normalized at taste for translation and scale.
  When using the metric in Section 15, rotation can be factored out as well.

We wish to remark that, although the formula defining the metric seems more complex than other proposed models, we think that this metric is actually simpler, since it decomposes in terms, where the only "infinite dimensional" term is actually a first order metric. So, proof of the above stated properties is quite simpler than in other models, it mostly relays on elementary arguments.

For the above reasons, we think that these metrics may be useful in applications in Shape Optimization and Shape Analysis.

Incidentally, it is nice that this metric answers positively to many requests posed in Section 1.3 in [24]

Part 1. Designing metrics

2. Riemannian metrics and group actions

To design the metric on curves with desired properties, we will use a strategy. Since this strategy will be used over and over again, in this section we discuss it informally. All the results that we present informally in this section are rigorous under specific hypotheses; so the strategy will then become rigorous in the second part, when we will provide precise hypotheses and definitions, and we will use the strategy to design metrics in the space of curves.

Suppose that M is a connected Riemannian manifold with scalar product $\langle,\rangle_c$ and norm $\|\|_c$ on T_cM. (Since the scalar product can be recovered from the norm *via* polarization, we will mostly use the latter in formulas, to ease notations).

(Note that our main attention will be to the manifold Imm of immersed curves, but the discussion in this section applies to the general situation).

Let G be a Lie group acting on M. We assume that the action is free.[3]

Let M/G be the quotient space.[4] Let $\pi : M \to M/G$ be the projection, so that $(\pi, M, M/G)$ is a principal G-bundle.

We assume that the metric is invariant for the action of G.

Definition 2.1. Let $c \in M$ and $h \in T_cM$. Let $g \in G$, let $L_g(c) = gc$ be the action, and TL_g be its derivative, so that T_cL_g maps T_cM to $T_{gc}M$. We say that the metric is *invariant for the action of* G when

$$\|h\|_c = \|T_cL_gh\|_{gc}$$

for any $c \in M$ and $h \in T_cM$.

This is equivalent to saying that G acts isometrically on M, that is, L_g is an isometry for any given g.

This is the usual concept of "invariant metric" used in most papers. We will call it *point-wise invariance* to distinguish it from the *path-wise* invariance that we will define below.

If a metric $\|\|$ is point-wise invariant, then it projects to a metric $\pi^*\|\|$ in the quotient space M/G. If we associate the projected metric to M/G, then the projection π is a Riemannian submersion.

[3] Unfortunately in some cases of interest the action is faithful but not free.

[4] We are disregarding the fundamental question of regularity of M/G. This is a nontrivial problem, since in some notable cases, such as the quotient of immersed curves by the reparameterization group, the quotient fails to be a differential manifold. See [15] for a general discussion, or Theorem 2 in Section 4 in [2] for the case of second order metrics. We will provide an operative workaround in 16.4.

2.1. Quotient distance and Riemannian horizontality

Let $\mathbb{G}$ be the Lie algebra, that is $\mathbb{G} = T_1 G$, where 1 is the identity in G. For any $\xi \in \mathbb{G}$ there is a vector field[5] $\zeta = \zeta(\xi, c)$ on M that is the derivative of the action of G on $c \in M$; intuitively ζ is the infinitesimal action of G upon c in direction ξ; formally, if we denote by $R_c(g) = gc$ the action, then $\zeta = T_1 R_c(\xi)$.

Example 2.2. If $G = SO(n)$ then consider $\mathbb{Id} \in SO(n)$ and its variation $V \in \mathbb{R}^{n\times n}$ so that $\mathbb{Id} + V$ is orthogonal namely $(\mathbb{Id} + V)(\mathbb{Id} + V)^t = \mathbb{Id} = \mathbb{Id} + V + V^t + VV^t$ simplifying and discarding lower order terms $V + V^t = 0$ so the elements of $\mathbb{G}$ are the anti-symmetric matrixes. If $c \in M = \mathbb{R}^n$ then $\zeta = Vc$ is a "velocity vector".

We will denote by $[c] = \{gc : g \in G\}$ the *orbit* of c under the action of G. Since, we assumed that the action is free, then G and $[c]$ are diffeomorphic.

The vector space $V_c \stackrel{\text{def}}{=} \{\zeta(\xi, c), \xi \in \mathbb{G}\}$ is the tangent to the orbit $[c]$ in c; it is called *"the vertical space"*.

Having fixed a reference metric $\|\|$, then we can define the orthogonal to this space, that is called *"the horizontal space"* W_c.

In appropriate hypotheses, any path $\tilde\gamma : [0,1] \to M/G$ can be lifted to a path $\gamma : [0,1] \to M$ such that $\pi \circ \gamma = \tilde\gamma$. This lifting is unique if we fix $\gamma(0)$ and we decide that $\dot\gamma \in W_\gamma$ at all time. This is called *the horizontal lifting*.

Lemma 2.3. *If $\tilde\gamma : [0,1] \to M/G$ is a minimal length geodesic connecting $[x]$, $[y]$ then its horizontal lift $\gamma : [0,1] \to M$ is a minimal length geodesic connecting a point $x \in [x]$ to a point $y \in [y]$.*

Vice versa if g provides the minimum of $\inf_{g\in G} d_M(x, gy)$ and γ is the minimal geodesic connecting x to gy, then γ is horizontal.

In the above, any "geodesic" has constant speed.

The proofs are in Sections 26.9 to 26.12 in [14].

Another useful concept is the "horizontally projected" metric

$$\|h\|_{G^\perp,c} \stackrel{\text{def}}{=} \|p_c, h\|_c \tag{2.1}$$

where $p_c : T_c M \to W_c$ is the orthogonal projection.

2.1.1. *Quotient metric space*

Let d_M be the distance induced by the Riemannian metric on M. We assumed that the Riemannian metric is invariant for the group action, consequently the distance is invariant as well:

$$d_M(c_0, c_1) = d_M(gc_0, gc_1) \tag{2.2}$$

for any $g \in G$.

By definition to compute the distance $d_{M/G}$ in M/G we would minimize

$$d_{M/G}([x],[y]) = \inf \operatorname{len}_{M/G}(\tilde\gamma),$$

where $\tilde\gamma : [0,1] \to M/G$ connects $[x]$ to $[y]$ and $\operatorname{len}_{M/G}$ is the length associated to the metric $\pi^*\|\|$ projected on M/G.

But the distance can be defined more easily by

$$d_{M/G}([x],[y]) = \inf_{g\in G} d_M(x, gy). \tag{2.3}$$

[5]We call it "vector field" since often we consider ζ as a map mapping $c \in M$ to $\zeta \in T_c M$, by keeping ξ as a fixed parameter.

This provides a "metric space" approach to the study of the quotient spaces; this is quite important in cases when the quotient M/G does not have a smooth differential structure; this is the case *e.g.* of the quotient of immersed smooth curves by the smooth reparameterization group, that has an *"orbifold"* structure (see [15]). We will use the "metric space" approach in Section 16.

From (2.2) there follows that $d_{M/G}$ is symmetric and transitive; it may happen that $d_{M/G}([x],[y]) = 0$ for two different orbits $[x],[y]$, that is, in general it may happen that $d_{M/G}$ is a semidistance and not a distance.

Lemma 2.4. *$d_{M/G}$ is a distance iff the orbits of the action of G are closed in (M, d_M).*

In this case (M, d_M) is a true metric space, so this result will be useful.

Lemma 2.5. *If the metric space (M, d_M) is complete then the metric space $(M/G, d_{M/G})$ is complete.*

The proofs are in Appendix A.

2.2. Normalization

The *normalization* is a useful idea to represent the quotient in a more accessible way. (It is sometimes called *registration* in applied sciences).

It paramounts to finding a section M_0 of the bundle $\pi : M \to M/G$. In other words:

Definition 2.6. The *"normalization"* is a submanifold $M_0 \subset M$ that intersects each orbit in one point. So each orbit is represented by a point in M_0. We suppose moreover that M_0 is transversal to the orbits; that is,

$$T_c M = T_c M_0 \oplus V_c \tag{2.4}$$

at all $c \in M_0$.

Since G is a group, this implies that the bundle $\pi : M \to M/G$ can be trivialized. Indeed, by the above definition, we have a map

$$M \to M_0 \times G, \quad c \mapsto (\tilde{c}, g), \tag{2.5}$$

where $\tilde{c} \in M_0$ is the unique element in $[c] \cap M_0$ and $g \in G$ is the unique element such that $c = g\tilde{c}$.[6] (Here, we need that the action be free – if the action is faithful, then the map is not well defined). By transversality this map is a smooth diffeomorphism.

Intuitively, the idea is that M_0 represents M/G.[7]

Lemma 2.7. *For any smooth $\gamma : [0,1] \to M$ we can find another $\tilde{\gamma} : [0,1] \to M_0$ such that $\tilde{\gamma}(t)$ and $\gamma(t)$ are in the same orbit; i.e. there is a smooth path $g(t) \in G$ such that $\tilde{\gamma}(t) = g(t)\gamma(t)$.*

We will say that $\tilde{\gamma}(t)$ is the *normalization* of $\gamma(t)$.

The length of a path $\tilde{\gamma}$ in M_0 is not necessarily the length of the path $\pi\gamma$ projected in M/G.

Lemma 2.8. *When M_0 is orthogonal to each orbit, then a minimal geodesic in M/G corresponds to a minimal geodesic in M_0 (up to normalization).*

This follows from Lemma 2.3.

Since we are actually designing metrics, we look at this the other way around: if we can find an M_0 as above, we will then design a metric such that M_0 is orthogonal to the orbits.

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[6] Note that this map, when restricted to M_0, maps c to $(c, 1)$ where $1 \in G$ is the identity.

[7] Indeed the map that associates a class $[c]$ to the unique element $[c] \cap M_0$ is a section of the bundle $\pi : M \to M/G$, and is a diffeomorphism between M/G and M_0.

3. Point-wise and path-wise invariance

The following idea is expanded from ideas in [24] and in Section 11.5 in [13].

There is an important case when Lemma 2.8 holds, namely, when M_0 is orthogonal to each orbit: when the action is *path-wise invariant*.

Let $\gamma \in H^1([0,1] \to M)$ be path. The *geodesic action*, or *geodesic energy*, of γ is

$$\int_0^1 \|\dot\gamma\|_\gamma^2 \, \mathrm{d}t.$$

Definition 3.1. We say that a semimetric is *path-wise* invariant if

$$\int_0^1 \|\dot\gamma\|_\gamma^2 \, \mathrm{d}t = \int_0^1 \|\dot{\tilde\gamma}\|_{\tilde\gamma}^2 \, \mathrm{d}t$$

for any choice of smooth paths $\gamma : [0,1] \to M$ and $A : [0,1] \to G$ and where we define $\tilde\gamma(t) = A(t)\gamma(t)$.

If a semimetric is *path-wise* invariant then it is *point-wise* invariant; but more can be said.

Proposition 3.2. *These two facts are equivalent.*

- *the semimetric is path-wise invariant,*
- *the semimetric is point-wise invariant and, for any fixed $c \in M$, the null space of $\|\cdot\|_c$ contains V_c, namely, $\|v\|_c = 0$ for all $v \in V_c$. (Intuitively, $\|\cdot\|$ does not measure the infinitesimal action of $\mathbb{G}$).*

So a semimetric that is path-wise invariant cannot be a metric. So, when we will design a metric on M, then we will add other terms to $\|\|$ to create a true metric on the space.

We summarize these ideas.

Proposition 3.3. *We now consider a path $\gamma : [0,1] \to M$. When the action is path-wise invariant, given a path γ, the following lengths are equal.*

- *The length of γ in M,*
- *the length of $\tilde\gamma$ normalized in M_0,*
- *the length of $\pi\gamma$ projected in M/G;*

so, we have many equivalent ways to compute the length in the quotient space M/G. (This can be successfully exploited in designing algorithms to compute minimal geodesics).

Given a group, it is always possible to find a semimetric that is path-wise invariant. Indeed, given a metric $\|\|$ on M then the horizontally projected metric $\|h\|_{\perp,c}$ defined in (2.1) is always path-wise invariant. Unfortunately for metrics proposed in the past the computation of $\|h\|_{\perp,c}$ is quite cumbersome.

3.0.1. ... for curves

In the case of the manifold M of curves, we will also say *curve-wise* instead of *point-wise*, since each point in the manifold is actually a curve. The *path*[8] $\gamma : [0,1] \to M$ is represented by a homotopy $C : [0,1]^2 \to \mathbb{R}^n$, with independent variables $C(t, \theta)$; indeed we consider C as a path (in parameter t) of curves $C(t, \cdot)$, each in M. For this reason, we will say *homotopy-wise* invariant instead of *path-wise* invariant. We will also abbreviate this as *hom-wise*.

[8]We avoid referring to γ as a curve, because confusion arises with object of the manifold M of curves. So, we will always talk of paths in the infinite dimensional manifold M.

4. Designing

We suppose that there is a manifold M_0 that normalizes the action.

We use the map (2.5) that trivializes the bundle. We want to design a metric that splits orthogonally the map (2.5). To this end we define a metric $\|\|_G$ on G; possibly an invariant metric, but not necessarily. (See Rem. 4.4). We then define a metric on M_0. To this end we define a semimetric $\|h\|_0$ on M that is path-wise invariant, and projects to a metric in M_0. This is equivalent to asking that the null space of $\|h\|_0$ at $c \in M$ be exactly the vertical space V_c.

Note that, we view $\|h\|_0$ at the same time as a metric in M_0 and as a semi metric in M. This largely simplifies the analysis and the applications.

The norm on M is then defined by pullback as

$$\|h\| = \sqrt{\|\hat{h}\|_0^2 + \|\hat{g}\|_G^2}, \tag{4.1}$$

where the decomposition

$$T_c M \to T_{\tilde{c}} M_0 \times T_g G, \quad h \mapsto (\hat{h}, \hat{g}) \tag{4.2}$$

is the derivative of the map (2.5). In applications it is useful to pull back the two components separately, so that the metric on M is decomposed in the two components.

By definition M_0 is orthogonal to the orbits so Lemma 2.8 holds. Moreover M_0 is a totally geodesic submanifold of M.

If we wish that the norm satisfy an important property, such *e.g.* existence of minimal geodesic, then we will design $\|\|_0$ and $\|\|_G$ to satisfy these properties. (In particular, when G is finite dimensional then this is easily accomplished).

The above may seem complex but we will see, in the case of the space of curves, that it actually carries on quite naturally.

Remark 4.1. Suppose we are given a semimetric φ on M. We would like to check if it can be explained in terms of a "normalization", that is, φ is the pullback of a $\|\hat{g}\|_G$ as in equations (4.1) and (4.2). There is a local test to check this fact. Let $W_c = \{w \in T_c M : \varphi_c(w) = 0\}$ be the null space of φ. If φ derives from a normalization then W_c are the tangent bundles of a foliation of M in submanifolds; these submanifolds are indeed the translates of M_0 under the action of G. So, a necessary condition is that the subbundle W be involutive (by the Frobenious theorem). (Vice versa, if W is involutive, then we can at least conclude that φ is derived from normalization "in a local sense"; we do not detail this idea, since it will not be used in the following).

4.1. Geodesics

There is another benefit to the scheme.

Proposition 4.2. *Suppose for simplicity that each pair of points can be connected by an unique geodesic. Let $c_0, c_1 \in M$, let $g \in G$ and let $\tilde{c}_1 = gc_0$. Suppose that $C : [0,1] \to M$ is the geodesic connecting c_0 to c_1, and $\tilde{C} : [0,1] \to M$ is the geodesic connecting c_0 to $\tilde{c}_1$: then $\tilde{C}(t) = \xi(t)C(t)$ where $\xi(t)$ is the geodesic connecting the identity in G to g; and vice-versa. In particular, the projections of C and $\tilde{C}$ onto the quotient space M/G are identical.*

This is not true for generic metrics on M (even if they are point-wise invariant) as can be seen in this example.

Example 4.3. Let $M = \mathbb{R}^2 \setminus \{0\}$ and $G = SO(2)$ be the group of rotation; endow M with the usual Euclidean Riemannian metric. Consider the points $c_0 = (1,0)$, $c_1 = (0,2)$ and then rotate the latter to obtain $\tilde{c}_1 = (-\sqrt{2}, \sqrt{2})$. Identifying $M/G = (0,\infty)$ (the half line), we have that the projected geodesics πC and $\pi\tilde{C}$ are quite different

$$\pi C(t) = t + 1 \neq \pi \tilde{C}(t) = \sqrt{1 - 4t + 7t^2}$$

(it even happens that the traces are different!).

To design a metric in M we define $M_0 = (0, \infty)$ embedded in M has the right half of the abscissa line; the map $M_0 \times SO(2) \to M$ is just the representation of a point of M in polar coordinates; the designed metric in M is the pullback of the flat metric in $M_0 \times SO(2)$ onto M.

4.2. Minimal geodesics

We show how this strategy affects the computation of geodesics.

Let $c_0, c_1 \in M$. We want to find a geodesic $C : [0, 1] \to M$ connecting c_0 to c_1.

We first decompose the endpoints using the map (2.5); so we find $g_0, g_1 \in G$ and $\tilde{c}_0, \tilde{c}_1 \in M_0$ such that $c_0 = g_0 \tilde{c}_0$ and $c_1 = g_1 \tilde{c}_1$.

We compute a minimal geodesic $g(t)$ connecting g_0 to g_1. If we carefully chose the metric in G, then this will be easy.

We then look for a geodesic $\xi(t)$ in M_0 connecting $\tilde{c}_0$ to $\tilde{c}_1$. Indeed then $g(t)\xi(t)$ will be a geodesic; this follows from the choice we made in (4.1), such that the two components are orthogonal.

By the definition, the geodesic minimizes the *geodesic energy*

$$\min \left\{ \int_0^1 \|\dot{\xi}(t)\|_{0,\xi(t)}^2 \, \mathrm{d}t : \xi : [0, 1] \to M_0 \right\} \tag{4.3}$$

in the family of all smooth paths $\xi : [0, 1] \to M_0$ connecting $\tilde{c}_0$ to $\tilde{c}_1$; note that $\xi(t) \in M_0$ at all times.

But at this point the Proposition 3.3 comes into play. Indeed, we can compute a minimum of the *geodesic energy*

$$\min \left\{ \int_0^1 \|\dot{\xi}(t)\|_{0,\xi(t)}^2 \, \mathrm{d}t : \xi : [0, 1] \to M \right\} \tag{4.4}$$

in the family of all smooth paths $\xi : [0, 1] \to M$ connecting $\tilde{c}_0$ to $\tilde{c}_1$. Note that we have dropped the constraint requiring that $\xi(t) \in M_0$ at all times.

Once we have computed it (or its numerical approximation), then we normalize it as prescribed in Lemma 2.7 to obtain $\tilde{\xi}$ and eventually $g(t)\tilde{\xi}(t)$ will be a minimal geodesic.

In Remark 14.25, we will show explicitly how all this works out the case of the group of rescalings on curves.

This is numerical advantageous. In the numerical minimization of (4.3), we should apply the constraint $\xi(t) \in M_0$ at any minimization step. (This would be unavoidable if the metric on M_0 was not the restriction of a semimetric on M). With the proposed approach (4.4) instead the constraint is dropped; the constraint is enforced only at the final normalization.

In practice, since the geodesic energy of a semimetric is not coercive, it may be useful to "normalize" ξ every few minimization steps – this has yet not been verified though, since this paper focuses on analysis and not on applications.

Another positive consequence is when there are multiple groups. Indeed the various part of the above process are computed independently. In particular if we write an algorithm to find approximate geodesic for the metric of immersed curves, then a part of it can be used to find approximate geodesic for "immersed curves up to translation", with no change. So in a sense "one algorithm fits all".

Remark 4.4. Let $c_0, c_1 \in M$, and let $C : [0, 1] \to M$ be a geodesic connecting c_0 to c_1. Let $g \in G$.

When the metric $\|\|_G$ on G is invariant for the action of G onto itself, then gC is a geodesic connecting gc_0 to gc_1.

If the metric $\|\|_G$ on G is not invariant, then this is not the case; but suppose that $\tilde{C}$ is a geodesic connecting gc_0 to gc_1, then Proposition 4.2 states that the projections of C and $\tilde{C}$ onto πM are equal. So if our fundamental interest is in the quotient space M/G, and the space M is a just a comfortable representation for M/G, then we may as well use a metric on G that is not invariant in the above design method.

In the second part of their paper, we will consider the space of parametric immersed curve, and the group of diffeomorphism, that acts on curves as reparameterization. It is well known that there are inherent difficulties in building a "good" invariant Riemannian metric onto the Lie manifold of diffeomorphism ("good" means: such that the Riemannian manifold would be complete and modeled on a Hilbert space, and the group action be smooth in the induced topology); so this may be a way to "bypass" these difficulties.

4.2.1. Multiple actions

If there are many groups $G_1, \dots, G_K$ acting on M we can proceed as follows. For simplicity we discuss the case of two groups.

Let G be the group of actions generated by G_1, G_2. We assume that $G_1 \cap G_2$ contains only the identity.

If the subgroup G_1 is normal in G, *i.e.* $G = G_1 \rtimes G_2$ is a semi-direct product, then we have a preferred order in the strategy: we first factor out G_1 then G_2.

The strategy is as follows.

(1) We seek a submanifold $M_1 \subset M$ that normalizes the action of G_1, and is invariant for the action of G_2; we design a metric on G_1 satisfying the required properties.
(2) We then seek a submanifold $M_2 \subset M_1$ that normalizes the action of G_2; we design a metric of G_1 satisfying the required properties.

The map 2.5 is then rewritten as

$$M \to M_2 \times G_2 \times G_1, \quad \Phi(c) = (c_2, g_2, g_1) \tag{4.5}$$

is the unique pairing such that $g_2 c_2 = c_1 \in M_1$ and $g_1 c_1 = c$.

This map has two interesting properties.

Lemma 4.5.
- *If $g \in G_1$, $c \in C$ and $\Phi(c) = (c_2, g_2, g_1)$ then $\Phi(gc) = (c_2, g_2, gg_1)$.*
- *If G_1 is normal, then the map will commute as follows, given $g \in G_2$ and $c \in C$ and $\Phi(c) = (c_2, g_2, g_1)$ then $\Phi(gc) = (c_2,\ gg_2,\ g^{-1} g_1 g)$*

Proof. The first statement is obvious. For the second we write $gc = (gg_1 g^{-1})(gg_2)c_2$ since G_1 is normal then $(gg_1 g^{-1}) \in G_1$; since M_1 is invariant for action of G_2 then $(gg_2)c_2 \in M_1$; obviously $(gg_2) \in G_2$. □

A consequence of the above, using the relation (2.4), is that the tangent $T_c M$ is the direct sum

$$T_c M = T_c M_2 \oplus V_{G_2,c} \oplus V_{G_1,c}, \tag{4.6}$$

where $V_{G_1,c}$ is the tangent to the orbit of the action of G_1 on M, and $V_{G_2,c}$ is the tangent to the orbit of the action of G_2 on M_1.

If the actions commute, then we can proceed in any order.

In some cases though, neither G_1 nor G_2 are normal in G. This is true for the reparameterization groups of curves.

Part 2. Building blocks; the delta operator

5. Previous contributions

5.1. Rôle of homogeneous spaces

Let M be a differential manifold. Let G be a Lie group acting transitively on M. In this case M is called a *homogeneous space for a group* G and G is *a group of symmetries* for M.

If M is a Riemannian manifold, we will moreover ask that the action of G be a Riemannian isometry.

The rôle of homogeneous spaces in Shape Theory is sometimes neglected. In synthesis, if M is a manifold of immersed curves, and it is a homogeneous space, then we can identify a template curve c and study the whole geometry of M by looking at M from the vantage point of c. Moreover local quantities (such as the covariant derivative, the curvatures...) need only be computed at c. Two natural choices are the circle $c(\theta) = (\cos(2\pi\theta), \sin(2\pi\theta))$ for spaces of closed planar curves, and the straight segment $c(\theta) = (\theta, 0)$ for open curves.

In the following, we will highlight which model shape spaces of curves are known to us to be homogeneous spaces.

5.2. Other approaches

We here present a minimal set of definitions. (A complete list will be in Sect. 6.1). We will denote mainly by $c = c(\theta)$ a C^1 immersed curve $c : [0,1] \to \mathbb{R}^n$; by h a vector field $h : [0,1] \to \mathbb{R}^n$ along the curve. We will write $c' = \frac{d}{d\theta}c$ for the derivative in θ. The symbol $D_c h$ will denote the derivation by arc parameter of h along c. We will say that a curve is *closed* when $c(0) = c(1)$ and $c'(0) = c'(1)$; so that the curve c is C^1 as a map from S^1 to $\mathbb{R}^n$. (For contrast, when we will consider the space of all immersed curves, we will sometimes call it *the space of open curves*).

5.2.1. Younes et al.

Let $\mathrm{St}(2, L^2)$ be the Stiefel manifold of ortho-normal pairs of vectors in $L^2 = L^2([0,1])$ (the usual Hilbert space of square-integrable functions $f : [0,1] \to \mathbb{R}$).

Younes *et al.* [26, 27] consider the space of closed curves up to translation and scaling. They consider the metric $\|h\| = \sqrt{\int |D_c h|^2 \,\mathrm{d}\theta}$ on curves. They define a transformation that we will call SQRT; by this transform the space of smooth immersed curves becomes $\mathrm{St}(2, L^2) \cap C^\infty$; and the metric of curves $\|h\|$ becomes the standard metric induced on the Stiefel manifold from the ambient space $L^2 \times L^2$.

We highlight some properties.

- Up to the SQRT, the space of smooth curves can be metrically completed. Its completion is represented by the Stiefel manifold $\mathrm{St}(2, L^2)$. See [8].
- Unfortunately the completed manifold of curves then contains absolutely continuous curves, not necessarily immersed.
- There is a closed form formula for geodesics [20].
- Any two points in the Stiefel manifold $\mathrm{St}(2, L^2)$ are connected by a minimal geodesic [10].
- For any given endpoints, the actual computation of the minimal geodesic in $\mathrm{St}(2, L^2)$ can be reduced to the computation of the minimal geodesic in $\mathrm{St}(2, \mathbb{R}^4)$; and this problem has 5 free parameters, so the minimal geodesic can be approximately computed with ease [20].
- The action of rotations in L^2 extends to an isometric transitive action on the Stiefel manifold, hence this Riemannian manifold of closed immersed curves is a homogeneous space.
- Unfortunately the problem of finding a minimal geodesic of geometric curves, (that is, of curves up to reparameterization), is ill posed. See [27].
- [20] expanded this metric to a metric defined on the space of all closed immersed curves. In this case a geodesic will move the center of mass with constant speed, and the scale of the curve with logarithmic

speed. (This was done by the "normalization" method, although in that paper this was not explained as is explained in this paper).

5.2.2. The elastic metric

The elastic metric [17]. We present it in the form summarized in [19]. Let $c : [0,1] \to \mathbb{R}^n$ be an immersed curve of length 1. We define $\phi : [0,1] \to \mathbb{R}$ by $\phi(t) = \log |c'(\theta)|$ and $\psi : [0,1] \to S^{n-1}$ by $\psi(\theta) = c'(\theta)/|c'(\theta)|$. The curve is then represented by the pair (ϕ, ψ). Fix $a, b > 0$. Let $u_1, u_2 : [0,1] \to \mathbb{R}$ and $v_1, v_2 : [0,1] \to \mathbb{R}^n$ with $v_1(\theta) \perp \psi(\theta), v_2(\theta) \perp \psi(\theta)$ for all θ, we consider (u_1, v_1) and (u_1, v_1) to be *tangent vectors* to the manifold of curves at (ϕ, ψ), the *elastic metric* is given by the scalar product

$$\int_0^1 (a^2 u_1 u_2 + b^2 v_1 \cdot v_2) e^{\phi} \, \mathrm{d}t. \tag{5.1}$$

The rotation group and reparameterization group act isometrically, so this metric projects to the space of curves up to translation, rotation, scaling, and reparameterization.

5.2.3. The square root representation

The SRV transform [19]. This corresponds to the previous metric when choosing $a = 1/2, b = 1$. This is most effective for open curves, since the space of open curves is mapped to the unit sphere in L^2. The geodesics are compute by minimizing the action, using a gradient descent method based on the Palais metric.

5.2.4. High order Sobolev metrics

Many authors studied metrics of the form

$$\langle h, k \rangle_G \stackrel{\text{def}}{=} \int_c \sum_{j=0}^{N} a_j \langle D_c^j h, D_c^j k \rangle \, \mathrm{d}s,$$

where $a_j \geq 0$ and $a_N > 0$. (Often, but not always, the coefficients a_j are assumed to be constant). They usually associate this metric to the space of immersed curves $c : S^1 \to \mathbb{R}^n$, seen as an open subset $\mathcal{I}^N$ of the Sobolev space H^N. The authors prove that, for $N \geq 2$, the metric above is a strong Riemannian metric on $\mathcal{I}^N$;[9] moreover

- Bruveris *et al.* [6] shown that the space of planar Sobolev immersions $\mathcal{I}^N$ is geodesically complete for a Sobolev metric with constant coefficients;
- Bauer and Harms [1] noted that the same method also implies metric completeness of the space of Sobolev immersions $\mathcal{I}^N$;
- Theorem 5.2 in [5] shows that any two curves may be connected by a minimizing geodesic.

5.2.5. Remarks

Finding geodesics up to reparameterization is a hard task, and is often ill posed in the case of first order metrics, see [27], or Section 3 in [2] for the case of the RSVT representation.

There are also examples of metrics that are hom-wise invariant wrt reparameterization. A simple way is to only allow deformations h that are orthogonal to the curve (so in this case we should talk of "sub-Riemannian" spaces of curves).

Another way is to only consider arc-parameterized curves. One such study is [22].

[9] That is, the map $\breve{G} : T^*M \to TM$ that represents derivatives as gradients by $\langle \breve{G}(\phi), k \rangle_G = \phi(k) \forall k$ is well defined and smooth.

6. Preliminary notions and definitions

6.1. Notation

We define some useful notations. Let $c : [0,1] \to \mathbb{R}^n$ be an immersed curve, and $f : [0,1] \to \mathbb{R}^m$ be a vector field along it. The *derivation by arc parameter* is the operator D_c defined[10] as

$$D_c f \stackrel{\text{def}}{=} \frac{f'}{|c'|}, \tag{6.1}$$

where $f' = \frac{d}{d\theta} f$, and similarly for c. The *integration by arc parameter* is

$$\int_c f \, \mathrm{d}s \stackrel{\text{def}}{=} \int_0^1 f(\theta)|c'(\theta)| \, \mathrm{d}\theta. \tag{6.2}$$

The *length* of a curve is

$$\mathrm{len}(c) \stackrel{\text{def}}{=} \int_0^1 |c'(\theta)| \, \mathrm{d}\theta = \int_c 1 \, \mathrm{d}s. \tag{6.3}$$

The *average integral* is

$$\fint_c f(s) \, \mathrm{d}s \stackrel{\text{def}}{=} \frac{1}{\mathrm{len}(c)} \int_c f(s) \, \mathrm{d}s$$

and we will sometimes denote this by $\mathrm{avg}_c(f)$.

6.2. Gâteaux differentials in the space of immersed curves

Let $E : M \to \mathbb{R}^k$ be a functional defined on a space of curves M; when this space is an open subset of a Banach space, then the formal definition of the *Gâteaux differential* is just

$$D_{c,h} E \stackrel{\text{def}}{=} \frac{\partial}{\partial t} E(c + th) \Big|_{t=0}. \tag{6.4}$$

The following rules (a subset of those in Prop. 4.5 in [13]) will be useful in the following

Proposition 6.1.

$$D_{c,h} \mathrm{len}(c) = \int_c (D_s h \cdot D_s c) \, \mathrm{d}s = - \int_c (h \cdot D_s^2 c) \, \mathrm{d}s, \tag{6.5}$$

where the last equality holds only for closed curves. Supposing that $O : M \to \mathbb{R}^k$ is a smooth functional:

$$D_{c,h}(D_s O) = -(D_s h \cdot D_s c)(D_s O) + D_s(D_{c,h} O), \tag{6.6}$$

$$D_{c,h} \int_c O \, \mathrm{d}s = \int_c D_{c,h} O + O.(D_s h \cdot D_s c) \, \mathrm{d}s, \tag{6.7}$$

$$D_{c,h} \fint_c O \, \mathrm{d}s = \fint_c D_{c,h} O + O.(D_s h \cdot D_s c) \, \mathrm{d}s - \fint_c O \, \mathrm{d}s \fint_c (D_s h \cdot D_s c) \, \mathrm{d}s. \tag{6.8}$$

[10] In other papers this was notated by D_s. The notation D_c was preferred since it stresses the dependency on the curve c.

For example, from (6.8) we easily obtain

$$D_{c,h}\operatorname{avg}_c(c) = \fint_c h + (c - \operatorname{avg}_c(c))(D_s h \cdot D_s c)\, \mathrm{d}s, \tag{6.9}$$

whereas from (6.5) we obtain

$$D_{c,h}\log\operatorname{len}(c) = \fint_c (D_s h \cdot D_s c)\, \mathrm{d}s = -\fint_c (h \cdot D_s^2 c)\, \mathrm{d}s. \tag{6.10}$$

If $C = C(t,\theta)$ is a homotopy, we can obtain a different interpretation of all previous equalities substituting formally $\frac{\partial}{\partial t}$ for $D_{c,h}$ and eventually $\frac{\partial}{\partial t}C$ for h.

7. Groups acting on curves

We denote by Imm the manifold of parameterized immersed curves $c : [0,1] \to \mathbb{R}^n$. (It is an open subspace of C^1; although we will not use the C^1 topology on the space of curves in the analytical treatment).[11]

By adding the constraint that $c(0) = c(1)$ and similarly for higher derivatives, we define the subset $\mathrm{Imm}_{\mathbb{f}}$ of closed parameterized curves. Equivalently we will consider closed curves as maps $c : S^1 \to \mathbb{R}^n$. Using an appropriate differentiable structure on Imm, the subset of closed curves is a submanifold of Imm; this will be clarified later.

Let S^1 be circle, represented by the quotient $\mathbb{R}/\mathbb{Z}$. It is an abelian Lie group, known as *"the circle group"*.

This is the main list of the groups that act on curves, that we will discuss in this paper. They act isometrically on any Riemannian manifold of curves that we will discuss.

- We will call $G_{\mathbb{r}} = SO(n)$ the group of rotations, $G_{\mathbb{t}} = \mathbb{R}^n$ the group of translations, and $G_{\mathbb{l}} = (0,\infty)$ the group of rescalings.
  So the whole group is
  $$(G_{\mathbb{l}} \times G_{\mathbb{r}}) \ltimes G_{\mathbb{t}},$$
  where rotations and rescalings commute.
  We will call this group "the Euclidean group" for simplicity (although many scholars usually assume that the Euclidean group does not include homothetic transformation).
  We do not include symmetries, because symmetries are not a connected Lie group, but rather a discrete group, so they would need a separate treatment.
- The reparameterizations that do not change the base-point, namely diffeomorphisms $\varphi : [0,1] \to [0,1]$ with $\varphi' > 0$. We denote this by Diff([0,1]) or $\mathbb{D}_0$ for brevity. We do not consider reparameterizations such that $\varphi' < 0$ for simplicity.
  (A precise definition of $\mathbb{D}_0$ is in Def. 16.1 in Sect. 16.1).
- For closed curves, we also consider the group of change of base-point that we will denote by $G_{\mathbb{bp}}$; it is the group of maps acting on $\mathrm{Imm}_{\mathbb{f}}$ by mapping $c(\theta)$ to $c(\theta + a)$, where $c : S^1 \to \mathbb{R}^n$ and $a \in S^1$.
  (In other papers where only closed curves are considered, this is considered a form of reparameterization).

Note that we may identify $G_{\mathbb{bp}} = S^1$ but in the case of planar closed curves also $G_{\mathbb{r}} = SO(2) = S^1$ so this may create confusion in some points: in that case we will prefer the notations $G_{\mathbb{bp}}$ and $G_{\mathbb{r}}$.

The above are standard group actions. For some specific Riemannian manifolds of curves, there is another hidden group action that we call *curling*, we will discuss in Section 14.3.

[11] In other papers Imm is considered to be a subset of C^∞.

Remark 7.1. The groups above are divided into two classes:

- Euclidean group
- reparameterization and change of base point (this last, only for closed curves).

Since the Euclidean group acts by composition on the left, while reparameterization and change of base point act by composition on the right, then they commute. So we will treat them independently.

In this paper, we will be mostly interested in the Euclidean group.

8. Normalizing Euclidean groups

From here on we will consider only planar curves.

We will now propose normalizations for translations, scaling and (in the case of open curves) rotation.

We present the corresponding "manifolds" informally, we will define them precisely in Section 9. In that section, we will see in Theorem 9.4 that the space Imm of immersed curves $c : [0, 1] \to \mathbb{C}$ is precisely defined as an open subspace of $H^2([0, 1]; \mathbb{C})$. We will see in Proposition 9.7 that the three "submanifolds" are differential submanifolds, and are mutually transversal, and each one is invariant for the other actions; so normalizations can be done in any order. Moreover they are transversal to the submanifold of closed curves.

We will use the log-transform.

Definition 8.1 (log-transform)**.** Let $c : [0, 1] \to \mathbb{R}^2$ be a C^1 immersed planar curve. The log-transform of c is given by two continuous functions $\tilde{e}, f : [0, 1] \to \mathbb{R}$ such that

$$c'(\theta) = e^{\tilde{e}(\theta)}\Big(\cos(f(\theta)), \sin(f(\theta))\Big) \tag{8.1}$$

for all θ.[12]

If we identify $\mathbb{R}^2 = \mathbb{C}$ then we can equivalently write $c' = e^{\tilde{e}+if}$. The choice of f is not unique; this will be addressed later (see Sect. 9.1). Obviously if (e, f) are known then c is known "up to translation".

Note that the quantity $\mathrm{d}s = |c'(\theta)| \,\mathrm{d}\theta$ that appears in *integration by arc parameter* (see Eq. (6.2)) is replaced by $e^{\tilde{e}(\theta)} \,\mathrm{d}\theta$ in log-coordinates; so this term will appear over and over again. In particular, the length of the curve may be written as

$$\mathrm{len}(c) = \int_0^1 e^{\tilde{e}(\theta)} \,\mathrm{d}\theta. \tag{8.2}$$

Remark 8.2. Consider the case of closed curves c. It is useful and convenient to consider the closed curve $c(\theta)$ as a map $c : S^1 \to \mathbb{R}^2$ (*cf.* Def. 9.6 and Rem. C.1); equivalently we can extend the map $c : [0, 1] \to \mathbb{R}^2$ to a map $c : \mathbb{R} \to \mathbb{R}^2$ by periodicity.

Problem is, the term $f(\theta)$ usually does not extend smoothly and periodically; *e.g.* for closed curves we have $f(1) - f(0) = 2\pi k$ with $k \in \mathbb{Z}$ the rotation index.

Note that all the derivatives of f instead can be extended periodically.

Similarly, suppose that $C = C(t, \theta)$ is a homotopy of class C^1 connecting two curves; since each curve $C(t, \cdot)$ is immersed then the representation

$$\frac{\partial}{\partial\theta} C(\theta) = e^{E(t,\theta)}\Big(\cos(F(t, \theta)), \sin(F(t, \theta))\Big) \tag{8.3}$$

will define two continuous functions E, F, where F is defined up to adding multiple integers of 2π.

[12] We choose the notation "$\tilde{e}$" for the real part, to avoid visual confusion with the Neper constant "e".

Our work will extensively peruse log-coordinates. A similar approach was proposed in [17] for a first order metric of curves (as we hinted in Sect. 5.2.2).

8.1. Translation

As a first step in designing a metric for the whole space of immersed curves, we want to factor out translation. Indeed translation is a normal subgroup of the Euclidean group, and it commutes with reparameterization (and change of base point).

Following the initial discussion, we will "normalize" translation.

One way to "normalize translation" would be to decide that for any curve we have $c(0) = 0$. This "manifold" though is not invariant for the action $G_{\mathbb{bp}}$ of *change of base-point*, so it is not good for closed curves.

A better approach is to decide that a "curve up to translation" is represented by a curve that has the center of mass

$$\operatorname{avg}_c(c) = \fint_c c \, \mathrm{d}s$$

in the origin. Let us formalize this idea.

Let Imm the space of all immersions; let M be the submanifold of immersed curves with center of mass in the origin.

This manifold M is invariant for all the group actions we listed in Section 7 (but translation, of course); so it is a perfect candidate for the first normalization step. The normalization manifold M is associated to the map

$$\mathrm{Imm} \to M \times \mathbb{R}^2, \quad c \mapsto (c - \operatorname{avg}_c(c),\ \operatorname{avg}_c(c)) \tag{8.4}$$

(that is the map (2.5) in this specific case).

We associate to the group $G_{\mathbb{t}} = \mathbb{R}^2$ of translations the standard metric; when we pull it back on Imm we obtain the semimetric

$$\|h\|_{\mathbb{t},c} \stackrel{\text{def}}{=} |D_{c,h} \operatorname{avg}_c(c)| \tag{8.5}$$

that corresponds to the pull back of $\|\hat{g}\|$ in (4.1). See equation (6.9) for an expanded expression of (8.5).

All of the (semi)metrics on M presented below are hom-wise translation invariant. Hence, if we combine the term (8.5) with one of the metrics presented below (as explained in (4.1)) then the map (8.4) will be an isometry. At the same time the metric (8.5) is hom-wi invariant for all the actions but translation.

With this decomposition, if C is a geodesic in Imm then the center of mass of $C(t, \cdot)$ moves with constant velocity.

We can consider M as "the manifold of immersed curves up to translation" or "the manifold of immersed curves with center of mass in the origin". That is, we can identify the quotient $\mathrm{Imm}/G_{\mathbb{t}}$ with the normalizing manifold M. Each seminorm in the following sections is hom-wi translation invariant, so it does not really make a difference.

This technique was already used in [20].

8.2. Scaling

Scaling commutes with rotation. So, we may factor them out in any order. We already normalized for translations, we now consider scaling.

We use as "normalization" the submanifold Imm_d of unit-length curves. The normalization map is just

$$\mathrm{Imm} \to \mathrm{Imm}_d \times (0, \infty) \quad , \quad c \mapsto (c/\operatorname{len}(c), \operatorname{len}(c)). \tag{8.6}$$

This submanifold Imm_d is invariant for all the group actions we listed in Section 7 (but excluding scaling, of course). (So, we may actually decide to factor out scaling before, and then translation).

The order we are following is though more apt to the log-transform, that we will use to provide a differentiable structure to M. For this reason, for convenience, we consider M_d to be a submanifold of M, that is, $M_d = M \cap \mathrm{Imm}_d$ is the manifold of curves of length one and with center in the origin. We will prove in Proposition 9.7 that this is a smooth submanifold of M. This submanifold M_d is invariant for all the group actions we listed in Section 7 – including translation (but excluding scaling, of course). The normalization map is

$$M \to M_d \times (0, \infty), \quad c \mapsto (c/\mathrm{len}(c), \mathrm{len}(c)) \tag{8.7}$$

(that is the map (2.5) in this specific case).

We associate to the multiplicative group $G_{\mathbb{l}} = (0, \infty)$ the metric dx/x so that it is complete. Equivalently, we can write the above map as

$$M \to M_d \times \mathbb{R}, \quad c \mapsto (c/\mathrm{len}(c), \log \mathrm{len}(c)). \tag{8.8}$$

The pullback of the standard metric on $\mathbb{R}$ is then

$$\|h\|_{\mathrm{len},c} \stackrel{\mathrm{def}}{=} \left| D_{c,h} \log \mathrm{len}(c) \right| \tag{8.9}$$

that is expanded in (6.10).

This map will be chained to the map (8.4) to provide a decomposition of Imm into "scale", "position" and M_d.

With this decomposition, if C is a geodesic in Imm then the logarithm of the length of $C(t, \cdot)$ is an affine map in t. This technique again was already used in [20].

8.3. Rotations

We now would wish to "normalize" the rotation of a curve.

Unfortunately, if we consider the space of closed curves, then the action of rotation and "change of base-point" interfere. The joint action of $G_{\mathbb{r}} \times G_{\mathbb{bp}}$ is not free. Moreover, if c is the circle then the orbit of rotations $G_{\mathbb{r}}$ is the same as the orbit of "change of base-point" $G_{\mathbb{bp}}$. (This implies that the quotient map $\mathrm{Imm}_{\mathbb{f}} \to \mathrm{Imm}_{\mathbb{f}}/G_{\mathbb{r}} \times G_{\mathbb{bp}}$ is not a principal G-bundle). But the design process discussed in Section 4.2.1 specifies that they should be transversal. So, we cannot normalize for both $G_{\mathbb{r}}$ and $G_{\mathbb{bp}}$.

Remark 8.3. One workaround would be to restrict the space and exclude all curves where the action is not free. This is similar in spirit to the idea in [15], where the authors defined a subset of the immersed closed curves where the whole group $\mathrm{Diff}(S^1)$ acts freely. We do not pursue this idea in this paper (but possibly in a future paper).

In the next Section 14, we will design a metric that will work well on immersed closed curves, and that projects to a metric on the space of geometric closed curves; so we will not normalize for rotations, instead we will define in Definition 14.1 a semimetric that "measures rotations" but is not associated to a normalization.

Instead if we consider the space of all open curves, with no special interest in its subspace of closed curves, then we can normalize for rotation. Let $\tilde{e}, f \in C([0,1])$ and

$$I_R(e, f) \stackrel{\mathrm{def}}{=} \frac{\int_0^1 f e^{\tilde{e}} \, \mathrm{d}\theta}{\int_0^1 e^{\tilde{e}} \, \mathrm{d}\theta} \tag{8.10}$$

for convenience. Since f is defined up to adding multiples of 2π, then $I_R(e,f)$ takes values in $S^1 = \mathbb{R}/(2\pi\mathbb{Z})$. For open curves we define the normalizing submanifold M_r to be given by the constraint

$$I_R(e,f) = 0 \tag{8.11}$$

in log-transform; in usual curve coordinates, it may be formally written as

$$\fint_c \arg c' \,\mathrm{d}s = 0. \tag{8.12}$$

This quantity is invariant for all actions, excluding rotation (obviously) and excluding change of base point (when considering closed curves). (This is easily proved by checking the rules in Appendix C).

A similar approach was proposed in [11] for a first order metric of arc parameterized curves.

The normalizing map, in log-transform, is

$$M \to M_r \times S^1, \quad (e,f) \mapsto \Big((e, f - I), I\Big), \tag{8.13}$$

(where we wrote $I = I_R(e,f)$ for convenience); this can be extended to a map for immersed curves.

In Section 15, we will shortly discuss a Riemannian metric that uses this normalization.

9. The differentiable manifolds

We now precisely define the differential structure of all the above "manifolds".

Let $c : [0,1] \to \mathbb{R}^2$ be a C^1 immersed planar curve. We define two continuous functions $\tilde{e}, f : [0,1] \to \mathbb{R}$ as in Definition 8.1.

Definition 9.1. We will say that c is part of the manifold Imm of immersed curves iff $\tilde{e}, f \in H^1$, where $H^1 = H^1([0,1])$ is the usual Sobolev space of functions.

This induces a differentiable structure, more details in the next section.

Note that if a curve c is arc parameterized and of length 1 then $\tilde{e} \equiv 0$, so we will consider the manifold of these curves to be $\{0\} \times H^1$ (identified with H^1 for simplicity of notations).

Remark 9.2. Usually H^1 is associated to the Hilbert norm

$$\|f\|_{H^1} = \sqrt{\int_0^1 |f'(x)|^2 + |f(x)|^2 \,\mathrm{d}x}, \tag{9.1}$$

(where $f : [0,1] \to \mathbb{R}^2, f = f(x)$) but it is easily proved that this is equivalent to

$$\|f\|_{\tilde{H}^1} = \sqrt{\int_0^1 |f'(x)|^2 \,\mathrm{d}x + \left|\int_0^1 f(x)\,\mathrm{d}x\right|^2}. \tag{9.2}$$

(The proof is based on a Poincaré type inequality, see Prop. 2.3 in [21].) This second metric is more apt to proving many following results, and we will use it extensively.[13]

[13] Note that this metric is again derived from a scalar product.

Lemma 9.3. *We recall that H^1 compactly embeds in $C^{0,1/2}$. This is particularly simple to prove using the equivalent norm (9.2). Precisely, if $g \in H^1$ then*

$$|g(\theta_0) - g(\theta_1)| \le \|g\|_{\tilde{H}^1} \sqrt{|\theta_0 - \theta_1|} \tag{9.3}$$

$$\max_\theta |g| \le \sqrt{2} \|g\|_{\tilde{H}^1} \tag{9.4}$$

9.1. Multiple representation and differentiable structure

Note that the "angle function" f is defined up to integer multiples of 2π; so there is a problem of "multiple representation" of a curve.

$H^1/(2\pi\mathbb{Z})$ is the manifold obtained from H^1 by identifying $f, \tilde{f} \in H^1$ when $f = \tilde{f} + 2k\pi$ for $k \in \mathbb{Z}$. Note that $H^1/(2\pi\mathbb{Z})$ is a smooth manifold modeled on H^1 since H^1 injects continuously in $C^{0,1/2}$.

$H^1/(2\pi\mathbb{Z})$ is not simply connected, and H^1 is the universal covering of $H^1/(2\pi\mathbb{Z})$.

We recall that M is submanifold of immersed curves with center of mass in the origin. Up to log transform, we identify M with $H^1 \times (H^1/(2\pi\mathbb{Z}))$.

Theorem 9.4. *$\mathbb{R}^2 \times H^1 \times (H^1/(2\pi\mathbb{Z}))$ is diffeomorphic with the space Imm of immersed curves $c : [0,1] \to \mathbb{C}$, seen as an open subspace of $H^2([0,1];\mathbb{C})$; where the diffeomorphism is the combination of the map (8.4) and of the log-transform on M.*

(A proof is in Appendix A).

Proposition 9.5. *The set $Imm_{\mathbb{f}}$ of closed curves is a smooth submanifold.*

This follows by pulling back the result in Proposition 9.7 using the map (8.4).

9.2. Submanifolds

Definition 9.6 (Closed planar curves)**.** We fix $k \in \mathbb{Z}$. We call $M_{\mathbb{f}.\mathbb{k}}$ the set of curves $c \in \mathrm{Imm}$ such that

$$\int_0^1 c' \, \mathrm{d}\theta = \int_0^1 e^{\tilde{e}} (\cos(f), \sin(f)) \, \mathrm{d}\theta = 0 \in \mathbb{R}^2, \tag{9.5}$$

$e(0) = e(1)$ and also[14] $f(1) = f(0) + 2\pi k$; so that the curve c is closed, is H^2 as a map from S^1 to $\mathbb{R}^2$, and has rotation index k.

We eventually prove that all "submanifolds" previously defined are indeed smooth submanifolds of M (that is the manifold of immersed curves up to translations).

Proposition 9.7.

- *The subset M_d of length one curves is a smooth submanifold of M.*
- *The subset M_r of rotationally normalized curves is a smooth submanifold of M.*
- *The subset $M_{\mathbb{f}.\mathbb{k}}$ of closed curves of index k is a smooth submanifold of M.*
- *Any subset defined by two or three of the above constraints is a smooth submanifold of M.*

Proof. The constraint for length one curves is

$$\mathrm{len}(c) = \int_0^1 e^{\tilde{e}} \, \mathrm{d}\theta = 1;$$

[14] To be precise, when f is considered as an element in $H^1/(2\pi\mathbb{Z})$, then the constraint $f(1) = f(0) + 2\pi k$ is applied to the lifting of f to H^1.

the constraint for rotational normalization is (8.10), that is

$$R(c) = \int f e^{\tilde{e}} \, \mathrm{d}\theta = 0; \tag{9.6}$$

the constraints for closed curves are

$$Z(c) \stackrel{\text{def}}{=} \int_0^1 c' \, \mathrm{d}\theta = \int_0^1 e^{\tilde{e}} (\cos(f), \sin(f)) \, \mathrm{d}\theta = 0 \in \mathbb{R}^2 \tag{9.7}$$

$$z_e(e) \stackrel{\text{def}}{=} e(0) - e(1) = 0 \in \mathbb{R} \tag{9.8}$$

$$z_f(f) \stackrel{\text{def}}{=} f(0) - f(1) = k2\pi \in \mathbb{R} \tag{9.9}$$

(as discussed at Eq. (9.5)).

The differentials are

$$D_{(\tilde{e},f),(\hat{e},0)} \mathrm{len}(c) = \int_0^1 \hat{e} e^{\tilde{e}} \, \mathrm{d}\theta \tag{9.10}$$

$$D_{(\tilde{e},f),(\hat{e},0)} R(c) = \int_0^1 \hat{e} f e^{\tilde{e}} \, \mathrm{d}\theta \tag{9.11}$$

$$D_{(\tilde{e},f),(\hat{e},0)} Z_1(c) = \int_0^1 \hat{e} \cos(f) e^{\tilde{e}} \, \mathrm{d}\theta \tag{9.12}$$

$$D_{(\tilde{e},f),(\hat{e},0)} Z_2(c) = \int_0^1 \hat{e} \sin(f) e^{\tilde{e}} \, \mathrm{d}\theta \tag{9.13}$$

$$D_{(\tilde{e},f),(\hat{e},0)} z_e(c) = \hat{e}(0) - \hat{e}(1) \tag{9.14}$$

for derivatives in direction $\hat{e}$ and

$$D_{(\tilde{e},f),(0,\hat{f})} \mathrm{len}(c) = 0 \tag{9.15}$$

$$D_{(\tilde{e},f),(0,\hat{f})} R(c) = \int_0^1 \hat{f} e^{\tilde{e}} \, \mathrm{d}\theta \tag{9.16}$$

$$D_{(\tilde{e},f),(0,\hat{f})} Z_1(c) = \int_0^1 -\hat{f} \sin(f) e^{\tilde{e}} \, \mathrm{d}\theta \tag{9.17}$$

$$D_{(\tilde{e},f),(0,\hat{f})} Z_2(c) = \int_0^1 \hat{f} \cos(f) e^{\tilde{e}} \, \mathrm{d}\theta \tag{9.18}$$

$$D_{(\tilde{e},f),(0,\hat{f})} z_f(c) = \hat{f}(0) - \hat{f}(1) \tag{9.19}$$

for derivatives in $\hat{f}$. Since e, f are continuous, then these differentials are well defined; moreover, the embedding $H^1 \to C^{0,1/2}$ shows that these are also continuous.

We consider as a first case the subset $M_{\mathbb{f}.\mathbb{k}} \cap M_r \cap M_d$ of closed rotationally normalized length-one curves.[15] We will show that the four above differentiable are maximal rank (in $\mathbb{R}^6$). Suppose that there are constants $a_1, a_2, a_3, a_4, a_5,$ and $a_6 \in \mathbb{R}$ such that

$$a_5(\hat{e}(0) - \hat{e}(1)) + a_6(\hat{f}(0) - \hat{f}(1))a + \int_0^1 \hat{e} e^{\tilde{e}} \Big(a_1 + a_2 f + a_3 \cos(f) + a_4 \sin(f) \Big)$$
$$+ \hat{f} e^{\tilde{e}} \Big(a_2 - a_3 \sin(f) + a_4 \cos(f) \Big) \, \mathrm{d}\theta = 0$$

[15]We noted that this is not a good normalization in applications since it is not invariant for the change of base point.

for all $\hat{e}, \hat{f} \in H^1$: this implies $a_5 = a_6 = 0$ and

$$a_1 + a_2 f + a_3 \cos(f) + a_4 \sin(f) = 0 \tag{9.20}$$

$$a_2 - a_3 \sin(f) + a_4 \cos(f) = 0 \tag{9.21}$$

for all θ. The relation is of the form $(a_1 + a_2 f, a_2) = A(a_3, a_4)$ where A is a rotation matrix, hence $(a_1 + a_2 f)^2 + a_2^2 = a_3^2 + a_4^2$: since the curve is closed then f cannot be constant, moreover f is continuous, so this last relation holds only if $a_2 = 0$. Since f is not constant then there are two different rotation matrixes A such that $(a_1, 0) = A(a_3, a_4)$ hence and this implies $a_1 = a_2 = a_3 = a_4 = 0$.

We consider as a second case the subset $M_r \cap M_d$ of rotationally normalized length-one curves. Suppose that there are constants $a_1, a_2 \in \mathbb{R}$ such that

$$\int_0^1 \hat{e} e^{\tilde{e}} \big(a_1 + a_2 f\big) + \hat{f} e^{\tilde{e}} a_2 \, \mathrm{d}\theta = 0$$

for all $\hat{e}, \hat{f}$: again this implies that $a_1 = a_2 = 0$.

All other cases are similar. □

A similar result can be stated in the space Imm of all immersions, where the manifolds of *"length one curves"*, *"curves with center of mass in the origin"*, *"closed curves"* and the *"rotationally normalized curves"* are all smooth submanifolds, and they are transversal. We do not detail, for sake of brevity.

10. Invariant operators

Following the strategy delineated in the first part we now need to find a simple semimetric on the space of immersed curves that is hom-wise invariant wrt the group actions.

We here present a simple semimetric for planar curves that is hom-wise invariant wrt the Euclidean group, and that is curve-wise reparameterization invariant (and base-point for closed curves).

Remark 10.1. To produce a semimetric hom-wise invariant for all group actions (Euclidean and reparameterization) we may consider the horizontal projection of the semimetric discussed below. Unfortunately the horizontal projection is too complex, so it defeats one of the objectives, namely to propose a model apt for numerical computations. Hopefully, in a future paper we will design and study a simpler semimetric that is hom-wise invariant for all group actions.

We start with some remarks.

There are many differential operators that are reparameterization and Euclidean (curve-wise) invariant.

Let $c, h : [0, 1] \to \mathbb{R}^n$, with c an immersed curve and $c, h \in C^1$. The most used and known differential operator D_c is defined in (6.1). Its square is

$$D_c D_c h = \frac{1}{|c'|} \frac{h''|c'| - h'|c'|'}{|c'|^2} = \frac{h''|c'| - h'|c'|'}{|c'|^3}. \tag{10.1}$$

Riemannian metrics based on these operators were studied in several papers.

The question is now... is there any other invariant "differential" operator that may be useful for our design strategy?

11. "Delta" operator

For planar curves indeed there is another interesting choice. In this section, we always identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$.

Definition 11.1. We propose the "delta" operator[16]

$$\Delta_c h \stackrel{\text{def}}{=} h'/c', \tag{11.1}$$

where the division is in the sense of complex numbers.

It is easily seen that it is reparameterization invariant. Moreover, we can write

$$T\Delta_c = D_c \tag{11.2}$$

to relate it to the classical D_c operator; where $T = c'/|c'|$ be the tangent vector, and the multiplication $T\Delta_c$ is the multiplication of complex numbers.

11.1. Intuitive idea

The difference between $D_c h$ and $\Delta_c h$ is akin to the difference between Lagrangian coordinates and Eulerian coordinates (but transported to the level of first derivatives).

When using $D_c h$ we are considering h to be positioned in the ambient space $\mathbb{R}^2$, and we are just renormalizing h' by $|c'|$, so that $D_c h$ will be reparameterization invariant.

When using $\Delta_c h$, we are considering h to be anchored to the curve, and so we are normalizing as above, and moreover we are interested in the relative angle between h' and c', not in the angle between h' and a fixed reference versor in the space.

11.2. Second order delta operator

The second order delta is

$$\Delta_c^2 h = \Delta_c \Delta_c h \stackrel{\text{def}}{=} \frac{1}{c'}\left(\frac{h'}{c'}\right)' = \frac{h''c' - h'c''}{(c')^3} \tag{11.3}$$

and it is again reparameterization invariant. Note again the similarity with $D_c D_c h$ (just delete the "absolute value" in Eq. (10.1)).

Using $T\Delta_c = D_c$ we may rewrite the second order operator as

$$\begin{aligned} \Delta_c \Delta_c h &= T^{-1} D_c (T^{-1} D_c h) = T^{-1}\big(T^{-1} D_c^2 h - (D_c T^{-1}) D_c h\big) &(11.4)\\ &= T^{-2} D_c^2 h - T^{-3} D_c T D_c h = T^{-3}(T D_c^2 h - D_c T D_c h), &(11.5)\end{aligned}$$

where all products are complex products.

Note that any combination of the differential operators, such as $\Delta_c^* D_c \Delta_c$ would be reparameterization invariant.

11.3. Kernel

The kernel of D_c is given by constant vector fields. The kernel of $D_c D_c$ is given by constant vector fields when we consider closed curves. When we consider open curves $D_c D_c h = 0$ iff $h(\theta) = a_1 s(\theta) + a_2$ where $a_1, a_2 \in \mathbb{R}^2$ and $s(\theta) = \int_0^\theta |c'(\tau)| \, d\tau$ is the arc parameter.

If $\Delta_c^2 h = 0$ then by (11.3) $h = \alpha c + \beta$ for two constants $\alpha, \beta \in \mathbb{C}$. This vector space coincides with the vector space of infinitesimal actions of

[16] For lack of a better name...

- rescalings (when $\beta = 0, \alpha > 0$),
- translations,
- rotations (when $\alpha \in i\mathbb{R}, \beta = 0$).

This is an important property, as explained in Proposition 3.2: it means that Δ_c^2 can be used as a building block for a semimetric that is hom-wise invariant for the above group actions.

This also means that $\Delta_C^2 \dot{C} \equiv 0$ for any homotopy consisting only of Euclidean motions. (Here, $C = C(t,\theta)$ and we write $\dot{C}$ for $\frac{\partial}{\partial t} C$, and C' for $\frac{\partial}{\partial \theta} C$).

The operator Δ^2 is moreover invariant for rotations and translations; it has a precise behavior *w.r.t.* rescalings.

Proposition 11.2. *Let $\alpha(t), \beta(t)$ be complex valued smooth functions; let $A(t)$ be the family of Euclidean actions given by $A(t)v = \alpha(t)v + \beta(t)$, for any fixed $v \in \mathbb{C}$. Let C a homotopy and $\tilde{C}(t,\theta) = A(t)C(t,\theta)$ then*

$$\Delta_{\tilde{C}}^2 \dot{\tilde{C}} = \alpha^{-1} \Delta_C^2 \dot{C}. \tag{11.6}$$

Proof. Indeed

$$\tilde{C}' = \alpha C', \quad \dot{\tilde{C}} = \alpha \dot{C} + \dot{\alpha} C + \dot{\beta}, \quad \dot{\tilde{C}}' = \alpha \dot{C}' + \dot{\alpha} C'$$

so

$$\Delta_{\tilde{C}}^2 \dot{\tilde{C}} = \Delta_{\tilde{C}} \Big(\frac{\alpha \dot{C}' + \dot{\alpha} C'}{\alpha C'} \Big) = \frac{1}{\alpha C'} \left(\frac{\dot{C}'}{C'} + \frac{\dot{\alpha}}{\alpha} \right)' = \alpha^{-1} \Delta_C^2 \dot{C}$$

□

12. Delta metrics

If we compare the first order norms associated to the operators D and Δ

$$\|h\|_{\Delta,c} \stackrel{\text{def}}{=} \sqrt{\int_0^1 \left| \frac{h'}{c'} \right|^2 |c'| \, d\theta}, \tag{12.1}$$

$$\|h\|_{D,c} \stackrel{\text{def}}{=} \sqrt{\int_c |D_c h|^2 \, ds}, \tag{12.2}$$

we see that there is nothing new since $\|h\|_{D,c}^2 = \|h\|_{\Delta,c}^2$.

Definition 12.1. We then define the second order seminorm

$$\|h\|_{\Delta^2,c} \stackrel{\text{def}}{=} \sqrt{\int_0^1 \left| \Delta_c^2 h \right|^2 \, ds} = \sqrt{\int_0^1 \left| \frac{1}{c'} \left(\frac{h'}{c'} \right)' \right|^2 |c'| \, d\theta}$$

$$= \sqrt{\int_0^1 \left| \left(\frac{h'}{c'} \right)' \right|^2 |c'|^{-1} \, d\theta} \tag{12.3}$$

$$= \sqrt{\int_c \left| T D_c^2 h - D_c T D_c h \right|^2 \, ds}, \tag{12.4}$$

where products are in $\mathbb{C}$ and the absolute value $||$ is the norm in $\mathbb{C}$.

Note that

$$\|h\|_{D^2,c} \stackrel{\text{def}}{=} \sqrt{\int_c |D_c^2 h|^2 \, \mathrm{d}s}, \tag{12.5}$$

in this case there is a clear difference. Obviously each (semi)norm in this section is reparameterization invariant (including change of base-point for closed curves). Note that $\|h\|_{\Delta^2,c} = \|h\|_{D\Delta,c}$, that is, we can equivalent use the operator $D_c\Delta_c$ in defining $\|h\|_{\Delta^2,c}$.

Polarizing (12.3) or (12.4) we obtain the Hermitian scalar product

$$\langle h,k\rangle_{\Delta^2,c} = \int_0^1 \frac{(h''c' - h'c'')(\overline{k}''\overline{c}' - \overline{k}'\overline{c}'')}{|c'|^5} \, \mathrm{d}\theta \tag{12.6}$$

$$= \int_c (TD_c^2 h - D_c T D_c h)\overline{(TD_c^2 k - D_c T D_c k)} \, \mathrm{d}s. \tag{12.7}$$

From the discussion in Section 11.3 there follows that $\|h\|_{\Delta^2}$ is actually a norm on the manifold of curves up to rotation, scaling and translation.

12.1. ... in log coordinates

The delta metric is especially interesting in log coordinates. Consider a homotopy of curves $C : [0,1]^2 \to \mathbb{C}$ that is represented by a pair $E, F : [0,1]^2 \to \mathbb{R}$ by the relation

$$C'(t,\theta) = e^{E(t,\theta)+iF(t,\theta)};$$

then

$$\|\dot C\|^2_{\Delta^2,C} = \int_C |\Delta_C^2 \dot C|^2 \, \mathrm{d}s = \int_0^1 \left(|\dot E'|^2 + |\dot F'|^2\right) e^{-E} \, \mathrm{d}\theta \tag{12.8}$$

and

$$\|\dot C\|^2_{l\Delta^2,C} = \mathrm{len}(C) \int_C |\Delta_C^2 \dot C|^2 \, \mathrm{d}s = \left(\int_0^1 e^E \, \mathrm{d}\theta\right) \int_0^1 \left(|\dot E'|^2 + |\dot F'|^2\right) e^{-E} \, \mathrm{d}\theta. \tag{12.9}$$

Indeed by equation (12.3), we have

$$\|\dot C\|^2_{\Delta^2,C} = \int_0^1 \left| \left(\frac{\dot C'}{C'}\right)' \right|^2 |C'|^{-1} \, \mathrm{d}\theta = \int_0^1 \left| \left(\frac{(\dot E + i\dot F)e^{E+iF}}{e^{E+iF}}\right)' \right|^2 e^{-E} \, \mathrm{d}\theta$$

$$= \int_0^1 \left| \left(\dot E + i\dot F\right)' \right|^2 e^{-E} \, \mathrm{d}\theta = \int_0^1 \left(|\dot E'|^2 + |\dot F'|^2\right) e^{-E} \, \mathrm{d}\theta.$$

13. Geodesic energy for delta metrics

13.1. Geodesic energy and rescaling

Let again C be a homotopy of class C^2. In this section, we consider the geodesic energies related to the proposed seminorms

$$\mathbb{E}_1(C) = \int_0^1 \|\dot{C}\|^2_{\Delta,c}\, \mathrm{d}t \ , \ \mathbb{E}_2(C) = \int_0^1 \|\dot{C}\|^2_{\Delta^2,c}\, \mathrm{d}t$$

and possibly linear combinations of the two.

We have a problem. We recall Proposition 11.2. Let $A(t)$ be given by $A(t)v = \alpha(t)v + \beta(t)$, for $v \in \mathbb{C}$. Let $\tilde{C}(t,\cdot) = A(t)C(t,\cdot)$ then

$$\Delta^2_{\tilde{C}}\dot{\tilde{C}} = \alpha^{-1}\Delta^2_C\dot{C};$$

consequently

$$\mathbb{E}_2(\tilde{C}) = \int_0^1 \frac{1}{|\alpha|}\|\dot{C}\|^2_{\Delta^2,c}\, \mathrm{d}t;$$

so if we rescale a homotopy to be larger and larger (keeping end points fixed), its action will converge to zero. (Indeed we already noted that the delta metric is not scale invariant).

A similar problem happens with D^2_s (although the formula is not as easy): when seeking a minimal length path between curves that are far enough, it would be convenient to blow up curves to infinity rather than connecting them (although this would be hardly defined a "geodesic").

A similar result holds for the first order seminorm $\mathbb{E}_1$: in this case though we assume that $\alpha(t) > 0$, that is, no rotation is allowed, and we obtain that

$$\mathbb{E}_1(\tilde{C}) = \int_0^1 \alpha\|\dot{C}\|^2_{\Delta,c}\, \mathrm{d}t;$$

so if we rescale a homotopy to be smaller and smaller (keeping end points fixed), its action will converge to zero.

Hence, if we consider a manifold of general curves M with any one of the seminorms above presented, the geodesic distance will vanish. (This is not a surprise though, this fact was already noted.)

13.2. Workarounds

There some possible workarounds.

(1) Add conformal terms, *e.g.* consider the seminorms

$$\sqrt{\mathrm{len}(c)}\|h\|_{\Delta^2,c} \quad \text{or} \quad \frac{1}{\sqrt{\mathrm{len}(c)}}\|h\|_{\Delta,c} \tag{13.1}$$

or linear combinations of the two. This is the approach that [25] already proposed for zero-th order norms. See also Section 5 in [24].

The length term is also a part of "almost local metrics", see Section 3 in [16].

(2) Add two seminorms, *i.e.* consider the seminorm

$$\sqrt{\|h\|^2_{\Delta^2,c} + a\|h\|^2_{\Delta,c}}, \tag{13.2}$$

where $a > 0$ is a fixed constant. Intuitively, the second order norm likes to enlarge curves in geodesics, the first order norm likes to shrink them, so they should balance.
This is approach is common in the literature, see *e.g.* [2] and references therein.
This seminorm though loses some useful properties. For example, it is not hom-wi rotation invariant. Moreover, the mathematical analysis is sometimes cumbersome, due to the complex interaction of the two terms.
We will design a metric in (14.9) that has a more complex formula but better properties and a simpler analysis.
Note that the metric (13.2) is locally equivalent to the metric (14.9) that we are studying in this paper. This may be proved by imitating the proofs in Section 14.8 (but is omitted for sake of brevity).

(3) Consider a space M_d of unit length curves. This is the approach in [20, 27], and many other papers.
(4) As a sub-case, consider the case of unit length curves parameterized by arc parameter. The Riemannian properties of the restriction of the elastic metric to this manifold was studied in [22].

We will use the first approach, to this end we define

$$\|h\|_{l\Delta^2,c} \stackrel{\text{def}}{=} \sqrt{\operatorname{len}(c)}\|h\|_{\Delta^2,c} \tag{13.3}$$

this semimetric is hom-wise Euclidean invariant; indeed for the associated energy of geodesics

$$\mathbb{E}_{l\Delta^2}(C) \stackrel{\text{def}}{=} \int_0^1 \|\dot{C}\|^2_{l\Delta^2,C}\,\mathrm{d}t = \int_0^1 \operatorname{len}(C)\|\dot{C}\|^2_{\Delta^2,C}\,\mathrm{d}t \tag{13.4}$$

we have this result.

Proposition 13.1. *As in Proposition 11.2, let $A(t)$ be a (smooth) family of Euclidean actions, given by $A(t)v = \alpha(t)v + \beta(t)$, for $v \in \mathbb{C}$. Let $\tilde{C}(t,\cdot) = A(t)C(t,\cdot)$ then*

$$\mathbb{E}_{l\Delta^2}(C) = \mathbb{E}_{l\Delta^2}(\tilde{C}).$$

The proof follows immediately from Proposition 11.2.

Remark 13.2. We may similarly define

$$\|h\|_{lD^2,c} = \sqrt{\operatorname{len}(c)}\|h\|_{D^2,c}$$

a conformal version of the standard second order seminorm, and then define

$$\mathbb{E}_{lD^2}(C) \stackrel{\text{def}}{=} \int_0^1 \operatorname{len}(C)\|\dot{C}\|^2_{D^2,C}\,\mathrm{d}t \tag{13.5}$$

the energy associated to it; but in this case it is not true in general that

$$\mathbb{E}_{lD^2}(C) = \mathbb{E}_{lD^2}(\tilde{C}).$$

(Indeed to obtain a result as in Prop. 11.2 it is needed that the rotation part of $A(t)$ be constant in t).

Part 3. The Riemannian manifolds

Due to the problem discussed in Section 8.3, we distinguish the case of closed and of open curves.

In the next Section 14, we will discuss a metric that works well for closed curves (although it may be used for the whole space of open curves), since there is no normalization wrt rotation.

In the Section 15, we will discuss a metric where we normalize for rotation. This normalization is not invariant for the change of base point, so that metric does not project to a metric on geometric closed curves. It has though better properties, so it may be preferred when studying open curves.

In the Section 16, we will discuss the same metrics but projected on "geometric curves" (*i.e.* curves up to reparameterizations).

14. A Riemannian manifold for parameterized (closed) curves

Consider the space of parameterized immersed closed curves and the normalization for rotations discussed in Section 8.3; the associated semimetric is not invariant for the action of changing base point, so that semimetric does not project on the "geometric space" of closed curves up to reparameterization.

We then design a specific Riemannian semimetric to deal with rotations (see Eq. (14.1)). With this we build the metric. This metric enjoys many important properties, as we will see in this section.

This metric is invariant for change of base point, hence it properly projects to a metric on the space of geometric closed curves (*i.e.* immersed closed curves up to parameterization), that we will discuss in Section 16.

This metric can be used on the whole space of parameterized "open" immersed curves, and it properly projects to a metric on the space of geometric (open) curves.

At the same time on the space of "open" curves we can use also a different metric, see Section 15.

14.1. Seminorm for rotations

We will deal with rotation using a specific seminorm.

Definition 14.1. Let $\tilde e, f, \hat e, \hat f \in H^1([0,1])$. We consider the pair $(\tilde e, f)$ to represent a curve in M, and $(\hat e, \hat f)$ to represent a tangent vector. We define the seminorm $\|h\|_{\mathbb{r}}$ by log-transform as

$$\|h\|_{\mathbb{r},c} = \left| \int_0^1 \hat f e^{\tilde e} \, \mathrm{d}\theta \right|. \tag{14.1}$$

In the proof of the following proposition, we will see that the formula (14.1) is well posed as a semimetric of immersed closed curves.

For convenience we define the seminorm

$$\|h\|_{\mathbb{r}/l,c} \stackrel{\text{def}}{=} \|h\|_{\mathbb{r},c} / \operatorname{len} c$$

that is corrected by rescaling.

Proposition 14.2. *This seminorm $\|h\|_{\mathbb{r}/l,c}$ is hom-wise invariant for scaling and translation, and is curve-wise invariant for rotation, reparameterization, change of base point.*

Proof. We compute $\|h\|_{\mathbb{r}/l,c}$ along a path C that is expressed in log-coordinates as (E,F):

$$\|\dot C\|_{\mathbb{r}/l,C} = \frac{\left| \int_0^1 \dot F e^E \, \mathrm{d}\theta \right|}{\int_0^1 e^E \, \mathrm{d}\theta}. \tag{14.2}$$

We first prove that the above formula is well posed.

If we choose a different representation for F (Sect. 9.1) then we would substitute F by $F + 2\pi k$, but the formula (14.2) is unaffected.

All curves $C(t,\cdot)$ have the same rotation index k, hence $\dot F(t,0) = \dot F(t,1)$ at all t so $F(t,\theta)$ can be seen as a map for $t \in [0,1], \theta \in S^1$. So all integrals in (14.2) can be read as integrals for $\theta \in S^1$. (Compare Rem. 8.2). In particular, this semimetric is invariant for change of base point.

We now prove the claimed invariances. In the following we use the formulas in Appendix C.

We apply an Euclidean transformation to the homotopy $C = C(t,\theta)$[17] to obtain

$$\tilde{C} = e^{l(t)+i\psi(t)} C + \beta(t); \tag{14.3}$$

where $l(t) \in \mathbb{R}$ is the rescaling and $\psi(t) \in \mathbb{R}$ is the rotation. Using the formulas (C.4) and (C.5) in appendix

$$\|\dot{\tilde{C}}\|_{\mathbb{r}/l,C} = \left| \dot{\psi} + \frac{\int_0^1 \dot{F} e^E \,\mathrm{d}\theta}{\int_0^1 e^E \,\mathrm{d}\theta} \right|, \tag{14.4}$$

(where $\dot{\tilde{C}} = \frac{\partial}{\partial t}\tilde{C}$) assuming moreover that the rotation ψ is constant in time this reduces again to (14.2), so the seminorm is hom-wise invariant for rescaling and translation, and curve-wise for rotation.

We then reparameterize the path as in (C.8), using diffeomorphisms $\varphi(t,\cdot)$ of S^1, then (14.2) becomes

$$\frac{\left| \int_0^1 (\dot{F} + F'\dot{\varphi}) e^E \varphi' \,\mathrm{d}\theta \right|}{\int_0^1 e^E \varphi' \,\mathrm{d}\theta}, \tag{14.5}$$

where E, F are evaluated at $(t, \varphi(t,\theta))$; assuming that φ is constant in t then $\dot{\varphi} \equiv 0$ so (changing parameter $\tau = \varphi(\theta)$) this becomes (14.2): so the seminorm is curve-wise invariant for reparameterizations of closed curves (including change of base point). □

Remark 14.3. The semimetric $\|h\|_{\mathbb{r},c}$ is not associated to a normalization (not even in a "local" sense). Indeed its null space does not satisfy the Frobenious theorem (see Rem. 4.1). Since the null space has codimension one, we use 1-forms: we define the 1-form $\Phi(x,v) = \int_0^1 v_2 e^{x_1} \,\mathrm{d}\theta$ on M then note that it is not a closed form, indeed $D_{x,w}\Phi(x,v) = \int_0^1 v_2 w_1 e^{x_1} \,\mathrm{d}\theta \neq D_{x,v}\Phi(x,w) = \int_0^1 v_1 w_2 e^{x_1} \,\mathrm{d}\theta$.

14.2. The metric

Definition 14.4. Let $m_l, m_r, m_t > 0$ be fixed.

We associate to the manifold Imm of all immersed curves the Riemannian metric

$$\|h\|^2_{(l\Delta^2+\mathrm{len}+r/l+t),c} \stackrel{\mathrm{def}}{=} \mathrm{len}(c)\|h\|^2_{\Delta^2,c} + m_l\|h\|^2_{\mathrm{len},c} + m_r\|h\|^2_{\mathbb{r},c}/\mathrm{len}(c)^2 + m_t\|h\|^2_{\mathbb{t},c}, \tag{14.6}$$

where the term $\|h\|_{\mathrm{len},c}$ derives from the length normalization, that was discussed in Section 8.2; while the term $\|h\|_{\mathbb{t},c}$ was discussed in Section 8.1.

Remark 14.5. The definition of this metric requires three constants m_l, m_r, m_t. This is common to many models in the literature. In this model though we have an important property: geodesics (and in particular minimal length geodesics) do not depend on the choice of m_l, m_t. This follows from the isometries discussed below in Propositions 14.9 and 14.10. (The choice of constants still affects other properties, such as the computation of gradients).

We will prove that this metric satisfies many useful properties: completeness, existence of geodesics, *etc.*

Since the metric is invariant for reparameterizations, then it projects to a metric for the space of geometric curves. Moreover if we wish to study *"geometric curves"*, that is, curves up to reparameterizations, then the only term affected will be the first term $\mathrm{len}(c)\|h\|_{\Delta^2,c}$. We will provide some results in Section 16.

[17] In (14.3) we identify the plane with the complex plane. Note that (14.3) is the same as (C.1).

14.3. Curling

All the seminorms that compose the proposed metric are invariant for an unusual group action, that we call "curve curling".

Let $\alpha : [0,1] \to \mathbb{C}$ smooth and such that $|\alpha(\theta)| = 1 \forall \theta$. A curve $c : [0,1] \to \mathbb{C}$ is mapped to $\tilde{c}$ by associating $\tilde{c}'(\theta) = \alpha(\theta) c'(\theta)$, and keeping the center of mass in the same position.

Note that this action does not map closed curves to closed curves.

In log coordinates this action is

$$G_{\mathbb{c}} \times M \to M, \quad (\rho, (\tilde{e}, f)) \mapsto (\tilde{e}, f + \rho),$$

where $G_{\mathbb{c}} = H^1/(2\pi\mathbb{Z})$ is the curling group.

The curling group in a sense contains the rotation group, whose action is

$$G_{\mathbb{r}} \times M \to M, \quad (a, (\tilde{e}, f)) \mapsto (\tilde{e}, f + a). \tag{14.7}$$

in log-coordinates; where $G_{\mathbb{r}} = S^1 = \mathbb{R}/(2\pi\mathbb{Z})$. Note though that the rotation group $G_{\mathbb{r}}$ was defined in Section 7 as a group acting on the plane $\mathbb{R}^2$, and as such it rotates the plane around its center – whereas the above action (14.7) rotates each curve around its center of mass. (This greatly simplifies the analysis).

"Curling" is in a sense a counterpart of "reparameterization", since it acts on the "f" component, whereas reparameterization acts mostly on the $\tilde{e}$ component.

Regarding the differentiable submanifolds discussed in Section 9.2, note that the normalizing submanifolds for translations and scaling are invariant for curling; but the manifold for rotations, and the submanifold of closed curves are not invariant.

14.4. Principal homogeneous space

Curling is a Riemannian isometry on the Riemannian manifold we are discussing.

Consider the combined group

$$G = (\mathbb{D}_0 \ltimes G_{\mathbb{c}}) \times G_{\mathbb{l}} \times G_{\mathbb{t}}$$

with group multiplication

$$(\varphi_2, \alpha_2, l_2, w_2)(\varphi_1, \alpha_1, l_1, w_1) = (\varphi_1 \circ \varphi_2, \alpha_2 + \alpha_1 \circ \varphi_2, l_2 + l_1, w_2 + w_1)$$

this acts on curves as follows: let us decompose Imm as $M \times \mathbb{R}^2$ using (8.4) and log coordinates, then G acts on immersed curves as

$$\begin{gathered} G \times (M \times \mathbb{R}^2) \to M \times \mathbb{R}^2 \\ (\varphi, \alpha, l, w)\ ,\ ((e, f), v) \mapsto ((l + \log \varphi' + e \circ \varphi, \alpha + f \circ \varphi), v + w) \end{gathered}$$

The combined action of reparameterization, translation, scaling and curling is free and transitive, so any open curve can be mapped to a reference curve *e.g.* a straight segment $c(\theta) = (\theta - 1/2, 0)$. If we restrict our attention to smooth curves and smooth actions, then there is a diffeomorphism

$$\mathrm{Imm} \sim (\mathbb{D}_0 \ltimes G_{\mathbb{c}}) \times G_{\mathbb{l}} \times G_{\mathbb{t}}.$$

Hence in the category of smooth objects, the Riemannian space of open curves that we are presenting is a principal homogeneous space.

TABLE 1. Invariances of the semimetrics wrt the group actions. Legenda: "CW" means curve-wise invariance, "HW" means homotopy-wise invariance. See end of Section 14.5 for further comments.

	$\|h\|_{l\Delta^2_e,c}$	$\|h\|_{l\Delta^2_f,c}$	$\|h\|_{\mathrm{len},c}$	$\|h\|_{\mathbb{r}/l,c}$	$\|h\|_{\mathbb{t},c}$
Reparameterization, change of base point	CW.	CW!	HW	CW!	HW
Curling	HW	CW.	HW	CW	HW
Scaling	HW	HW	CW.	HW	HW
Rotation	HW	HW	HW	CW.	HW
Translation	HW	HW	HW	HW	CW.

14.5. Decomposition of the metric according to the group actions

The metric (14.6) has a very nice structure. This metric is modular: *e.g.* if we wish to study "curves up to rotation" we just need to drop the third term and so on.

Equivalently, if we wish to study *curves up to translations and scaling* then we can restrict our attention to the manifold M_d, and so on. (But we cannot identify the space of *"curves up to translation and rotation"* with the normalizing manifold M_r defined in Sect. 8.3).

Each term of the metric has its meaning, and is related to the actions of the groups as follows.

- For the sake of this section we split the first term $\|h\|_{l\Delta^2} = \mathrm{len}(c)\|h\|_{\Delta^2}$ in two

$$\|h\|_{l\Delta^2,c} = \sqrt{\|h\|^2_{l\Delta^2_e,c} + \|h\|^2_{l\Delta^2_f,c}},$$

  where

$$\|h\|^2_{l\Delta^2_e,c} \overset{\mathrm{def}}{=} \mathrm{len}(c)\int_0^1 (\hat{e}')^2 e^{-\tilde{e}}\,\mathbb{d}\theta, \quad \|h\|^2_{l\Delta^2_f,c} \overset{\mathrm{def}}{=} \mathrm{len}(c)\int_0^1 (\hat{f}')^2 e^{-\tilde{e}}\,\mathbb{d}\theta$$

  then both $\|h\|_{l\Delta^2_e,c}$ and $\|h\|_{l\Delta^2_f,c}$ are hom-wise Euclidean invariant and curve-wise invariant for reparameterization and change of base point; moreover, $\|h\|_{l\Delta^2_e,c}$ is hom-wise invariant for curling, while $\|h\|_{l\Delta^2_f,c}$ is curve-wise invariant for curling.
- The second term $\|h\|_{\mathrm{len},c}$ controls the length of the curve, and is hom-wise invariant wrt rotation, translations reparameterization and change of base point.
- The third term $\|h\|_{\mathbb{r}/l,c} = \|h\|_{\mathbb{r},c}/\mathrm{len}(c)$ controls the rotation (in the sense expressed in Eq. (14.4)); it is hom-wise invariant for scaling and translation, and is curve-wise invariant for rotation, reparameterization, change of base point (see Prop. 14.2).
- The fourth term $\|h\|_{\mathbb{t},c}$ controls translations and is hom-wise invariant wrt scaling, rotations, reparameterization and change of base point.

The Table 1 summarizes these results.

We expect that a semimetric be curve-wise invariant for the group action that is related to it. So there are "CW." entries along the diagonal: these are spots where "CW" is the correct behavior.[18]

Outside of the diagonal, we would love to see only "HW" entries; any such entry means that a semimetric (say $\|h\|_{l\Delta^2,c}$) is hom-wise invariant for an action (say, translations): then this semimetric is, as to say, completely blind for that action. Unfortunately we have some "CW" entries out of the diagonal, marked as "CW!".

[18] Indeed for any group action (that is, a row in the table) there must be a semimetric that is not hom-wise invariant for that action – otherwise the sum of them would not be a metric (since its null space would not be zeroth-dimensional).

14.6. Decomposition of the energy according to the group actions

As a consequence we have this decomposition for the Euclidean action. Let $C : [0,1]^2 \to \mathbb{R}^2$ be a homotopy, let $E, F : [0,1]^2 \to \mathbb{R}$ a lifting such that

$$C = \left(e^E \cos(F), e^E \sin(F)\right)$$

(in writing this we do not identify $\mathbb{R}^2$ with $\mathbb{C}$). We define

$$\begin{aligned}
\mathbb{E}_{l\Delta^2}(C) &= \int_0^1 \operatorname{len}(C) \int_C |\Delta_C^2 \dot C|^2 \, ds \, dt = \int_0^1 \left(\left(\int_0^1 e^E \, d\theta \right) \int_0^1 \left(|\dot E'|^2 + |\dot F'|^2 \right) e^{-E} \, d\theta \right) dt, \\
\mathbb{E}_{\mathrm{len}}(C) &= \int_0^1 \left| \frac{\partial}{\partial t} \log \operatorname{len}(C) \right|^2 dt = \int_0^1 \left| \fint_c (D_s \dot C \cdot D_s C) \, ds \right|^2 dt \\
&= \int_0^1 \left| \int_0^1 \dot E e^E \, d\theta \right|^2 \left(\int_0^1 e^E \, d\theta \right)^{-2} dt, \\
\mathbb{E}_{\mathbb{r}/\ell}(C) &= \int_0^1 \left| \int_0^1 \dot F e^E \, d\theta \right|^2 \left(\int_0^1 e^E \, d\theta \right)^{-2} dt, \\
\mathbb{E}_{\mathbb{t}}(C) &= \int_0^1 \left| \frac{\partial}{\partial t} \operatorname{avg}_c(C) \right|^2 dt = \int_0^1 \left| \fint_C \dot C + (C - \operatorname{avg}_c(C))(D_s \dot C \cdot D_s c) \, ds \right|^2 dt
\end{aligned}$$

the above identities follow from equations (8.2), (12.9), and (13.4) for the first term, (8.2) and (6.10) for the second term, (14.1) for the third and (6.9) for the last term. Eventually

$$\mathbb{E}_{(l\Delta^2 + \mathrm{len} + r/l + t)}(C) \stackrel{\mathrm{def}}{=} \mathbb{E}_{l\Delta^2}(C) + m_l \mathbb{E}_{\mathrm{len}}(C) + m_r \mathbb{E}_{\mathbb{r}/\ell}(C) + m_t \mathbb{E}_{\mathbb{t}}(C) \tag{14.8}$$

will be the geodesic energy for the metric (14.6).

As we mentioned, each term "takes care" of a different action. We exemplify this fact.

Example 14.6. Let $c = c(\theta)$ be a closed curve with center of mass in the origin, and[19] let

$$C(t, \theta) = e^{l(t) + i\psi(t)} c(\theta) + \beta(t)$$

be a motion of c by translations $\beta(t) \in \mathbb{R}^2$, rotation $\psi(t) \in \mathbb{R}$ and rescalings $l(t) \in \mathbb{R}$; then

$$\begin{aligned}
\mathbb{E}_{l\Delta^2}(C) &= 0 \\
\mathbb{E}_{\mathrm{len}}(C) &= \int_0^1 |\dot l|^2 \, dt \\
\mathbb{E}_{\mathbb{r}/\ell}(C) &= \int_0^1 |\dot \psi|^2 \, dt \\
\mathbb{E}_{\mathbb{t}}(C) &= \int_0^1 |\dot \beta|^2 \, dt.
\end{aligned}$$

14.7. Metric in log-representation and isometries

To simplify the discussion we will often concentrate on the manifold M discussed in the Section 9. When restricting to the manifold M the last term is dropped (*i.e.* $m_t = 0$); moreover, we can represent the norm in log-transform.

[19] We identify the plane with the complex plane. A similar formula is also used in (C.1) and (14.3).

Proposition 14.7. *Let $m_l, m_r > 0$ be fixed. Let $\tilde e, f, \hat e, \hat f \in H^1([0,1])$. We consider the pair $(\tilde e, f)$ to represent a curve in M, and $(\hat e, \hat f)$ to represent a tangent vector. Then the norm has the form*

$$\|(\hat e,\hat f)\|^2_{(l\Delta^2+\mathrm{len}+r/l),(\tilde e,f)} \overset{\mathrm{def}}{=} \left(\int_0^1 e^{\tilde e}\,\mathrm{d}\theta\right)\int_0^1 \left(|\hat e'|^2+|\hat f'|^2\right)e^{-\tilde e}\,\mathrm{d}\theta + \frac{m_l\left|\int_0^1 \hat e e^{\tilde e}\,\mathrm{d}\theta\right|^2 + m_r\left|\int_0^1 \hat f e^{\tilde e}\,\mathrm{d}\theta\right|^2}{\left|\int_0^1 e^{\tilde e}\,\mathrm{d}\theta\right|^2} \tag{14.9}$$

that is Riemannian metric on $H^1\times(H^1/(2\pi\mathbb{Z}))$.

Again this follow from equations (8.2), (12.9), (8.9), and (14.1).

Proposition 14.8. *The formula (14.9) is a metric (and not a semimetric).*

Proof. Indeed if $\|(\hat e,\hat f)\|_{(l\Delta^2+\mathrm{len}+r/l),(\tilde e,f)} = 0$ then $|\hat e'|^2 \equiv |\hat f'|^2 \equiv 0$ so $\hat e, \hat f$ are constants, then the second and third term dictate that $\hat e = \hat f = 0$. □

Proposition 14.9. *The map*

$$\mathrm{Imm} \to M \times \mathbb{R}^2, \quad c \mapsto (c - \mathrm{avg}_c(c),\ \mathrm{avg}_c(c))$$

(that we saw in (8.4)) is an isometry of the manifold Imm (with the metric (14.6)) to $M\times\mathbb{R}^2$ (where M has the metric (14.9), and the Euclidean metric in $\mathbb{R}^2$ is rescaled by the factor m_t).

The above two propositions prove that the formula (14.6) is a metric on Imm, and not a semimetric.

Proposition 14.10. *We recall that M_d is the manifold of length-one curves (up to translation); up to log-transform, it is a submanifold of $H^1\times(H^1/(2\pi\mathbb{Z}))$, see Proposition 9.7; we associate to it (the restriction of) the metric $\|h\|_{l\Delta^2+\mathrm{len}+r/l,c}$, that can be simply expressed as*

$$\|(\hat e,\hat f)\|^2_{(\Delta^2+r),(\tilde e,f)} \overset{\mathrm{def}}{=} \int_0^1 \left(|\hat e'|^2+|\hat f'|^2\right)e^{-\tilde e}\,\mathrm{d}\theta + m_r\left|\int_0^1 \hat f e^{\tilde e}\,\mathrm{d}\theta\right|^2. \tag{14.10}$$

With this choice, the map

$$M \to M_d\times\mathbb{R}, \quad c\mapsto (c/\mathrm{len}(c), \log\mathrm{len}c)$$

is an isometry between the manifold M with the metric (14.9) and the product manifold $M_d\times\mathbb{R}$ (where we associate to $\mathbb{R}$ the usual metric but rescaled by m_l).

The proof is by straightforward computation.

14.8. Equivalence

We prove some important properties of the above (semi)metrics; we prove them on the space $H^1\times H^1$ for simplicity of presentation; all results below project to M.

We prove that the Riemannian metric $\|(\hat e,\hat f)\|_{(l\Delta^2+\mathrm{len}+r/l),(\tilde e,f)}$ defined in equation (14.9) in Definition 14.7 is equivalent to the standard metric in $H^1\times H^1$, in any bounded set; and similarly for the distances.

We remark that some results below are similar to results presented in Section 3 in [5] for a class Sobolev-type Riemannian metrics of order at least two.

For the sake of this section, let d be the distance induced by the metric (14.9) in $H^1\times H^1$.

As a first step we need to prove that the quantity

$$\log \frac{\max_\theta |c'|}{\min_\theta |c'|} \tag{14.11}$$

is Lipschitz (with constant 1) for the semidistance induced by the semimetric $\|h\|_{l\Delta^2} = \sqrt{\mathrm{len}(c)}\|h\|_{\Delta^2}$ in the space of immersed curves. But both (14.11) and this semidistance are Euclidean invariant. So the result is proved in this lemma, that states the above property in log-coordinates.

Lemma 14.11. *The quantity*

$$\max_\theta \tilde{e} - \min_\theta \tilde{e} \tag{14.12}$$

is Lipschitz (with constant 1) for the semidistance $d_{l\Delta^2}$ in $H^1 \times H^1$ generated by the semimetric

$$\|(\hat{e}, \hat{f})\|_{l\Delta^2} = \sqrt{\int_0^1 e^{\tilde{e}}\, \mathrm{d}\theta \ \int_0^1 (|\hat{e}'|^2 + |\hat{f}'|^2) e^{-\tilde{e}}\, \mathrm{d}\theta}. \tag{14.13}$$

Proof. For a generic function $g(\theta)$, we define for convenience the *oscillation*

$$\operatorname{osc} g = \max_\theta g - \min_\theta g. \tag{14.14}$$

Let us fix $c_0, c_1 \in M$. Let $a > d_{l\Delta^2}(c_0, c_1)$ and suppose that the energy of a homotopy C connecting c_0 to c_1 is less than a^2.

We rewrite the term (14.11) for the homotopy in log-coordinates as $\operatorname{osc} E = \max_\theta E - \min_\theta E$.

For any fixed t,

$$\sqrt{\int_0^1 (\dot{E}')^2 e^{-E}\, \mathrm{d}\theta \quad \int_0^1 e^E\, \mathrm{d}\theta} \ge \int_0^1 |\dot{E}'|\, \mathrm{d}\theta \ge (\max_\theta \dot{E}) - (\min_\theta \dot{E}), \tag{14.15}$$

then integrating the previous relation

$$\begin{aligned} \int_0^1 (\max_\theta \dot{E}) - (\min_\theta \dot{E})\, \mathrm{d}t &\le \int_0^1 \sqrt{\int_0^1 e^E\, \mathrm{d}\theta \quad \int_0^1 (\dot{E}')^2 e^{-E}\, \mathrm{d}\theta}\, \mathrm{d}t \\ &\le \sqrt{\int_0^1 \left(\int_0^1 e^E\, \mathrm{d}\theta \quad \int_0^1 (\dot{E}')^2 e^{-E}\, \mathrm{d}\theta\right) \mathrm{d}t} \le a \end{aligned} \tag{14.16}$$

But

$$E(1,\theta) - E(0,\theta) = \int_0^1 \dot{E}(\tau,\theta)\, \mathrm{d}\tau \le \int_0^1 \max_\xi \dot{E}(\tau,\xi))\, \mathrm{d}\tau$$

$$E(1,\theta) - E(0,\theta) = \int_0^1 \dot{E}(\tau,\theta)\, \mathrm{d}\tau \ge \int_0^1 \min_\xi \dot{E}(\tau,\xi))\, \mathrm{d}\tau$$

so

$$\max_\theta E(1,\theta) \le \max_\theta E(0,\theta) + \int_0^1 \max_\xi \dot{E}(\tau,\xi)\, \mathrm{d}\tau$$

$$\min_\theta E(1,\theta) \geq \min_\theta E(0,\theta) + \int_0^1 \min_\xi \dot{E}(\tau,\xi)\, \mathbb{d}\tau$$

subtracting

$$\max_\theta E(1,\theta) - \min_\theta E(1,\theta) \leq \max_\theta E(0,\theta) - \min_\theta E(0,\theta) + \int_0^1 \Big(\max_\theta \dot{E}(\tau,\theta) - \min_\theta \dot{E}(\tau,\theta) \Big)\, \mathbb{d}\tau$$

but the last integral is less than a by (14.16) so (by arbitrariness of $a > d_{l\Delta^2}$)

$$\operatorname{osc} E(1,\cdot) \leq \operatorname{osc} E(0,\cdot) + d_{l\Delta^2}(c_0, c_1). \tag{14.17}$$

□

Lemma 14.12. *The quantity*

$$\max_\theta \left| \tilde{e}(\theta) - \int_0^1 e^{\tilde{e}(\tau)}\, \mathbb{d}\tau \right|$$

is locally bounded wrt the semimetric $d_{l\Delta^2}$.

Proof. We first remark this fact: if $\tilde{c} : [0,1] \to \mathbb{R}^2$ is an immersed curve then there is a point such that $|\tilde{c}'(\theta)| = \operatorname{len}(\tilde{c})$. We express this idea in log-transform. Let $E \in H^1([0,1]^2)$, let $l(t) = \int_0^1 e^{E(t,\tau)}\, \mathbb{d}\tau$ then for any t there is a θ such that $E(t,\theta) = l(t)$ so

$$\max_\theta |E(1,\theta) - l(1)| \leq \operatorname{osc} E(1,\cdot) \leq \operatorname{osc} E(0,\cdot) + d_{l\Delta^2}, \tag{14.18}$$

by equation (14.17). □

The above lemma, when applied to curves, says that

$$\frac{\max_\theta |c'(\theta)|}{\operatorname{len}(c)} \quad \text{and} \quad \frac{\operatorname{len}(c)}{\min_\theta |c'(\theta)|}$$

are locally bounded for the semidistance induced by the semimetric $\|h\|_{l\Delta^2}$ in the space of immersed curves.

We now show that the proposed metric is locally equivalent to the standard metric.

Lemma 14.13. *Let $\mathbb{d}$ be the distance associated to the metric $\|\hat{e}, \hat{f}\|_{(l\Delta^2+\text{len}+r/l),(\tilde{e},f)}$. Let $a_1 > 0$. Let $(e_0, f_0) \in H^1 \times H^1$ be fixed; for any $(e_1, f_1) \in H^1 \times H^1$ with either*

$$\mathbb{d}\Big((e_0,f_0),(e_1,f_1)\Big) \leq a_1$$

or

$$\Big\|(e_0,f_0) - (e_1,f_1)\Big\|_{H^1 \times H^1} \leq a_1$$

we have

$$a_2 \|(\hat{e},\hat{f})\|_{(l\Delta^2+\text{len}+r/l),(e_1,f_1)} \leq \|(\hat{e},\hat{f})\|_{H^1\times H^1} \leq a_3 \|(\hat{e},\hat{f})\|_{(l\Delta^2+\text{len}+r/l),(e_1,f_1)} \tag{14.19}$$

where the constants $0 < a_2 < a_3$ depend only on e_0, f_0, a_1, m_l, m_r.

Proof. We first suppose that

$$\mathbb{d}\Big((e_0,f_0),(e_1,f_1)\Big) < a_1 .$$

Let $\tilde{E},F:[0,1]^2\to\mathbb{R}$ be homothopies connecting e_0,f_0 to e_1,f_1, respectively, and such that the geodesic energy is less than a_1^2. Let $l(t)=\log\int_0^1 e^{\tilde{E}(t,\theta)}\,\mathbb{d}\theta$. We set $E=\tilde{E}-l$.

Intuitively, $\tilde{E},F$ are the log representation of a homotopy $\tilde{C}$, e^l is the length of the curve at time t; while E,F are the log representation of the homotopy $\tilde{C}e^{-l}$ where each curve was rescaled to be unit length. So we can write

$$\int_0^1 e^E\,\mathbb{d}\theta = e^{-l}\int_0^1 e^{\tilde{E}}\,\mathbb{d}\theta = 1$$

at all t hence

$$\frac{\partial}{\partial t}\int_0^1 e^E\,\mathbb{d}\theta = \int_0^1 \dot{E}e^E\,\mathbb{d}\theta = 0.$$

Note that $\tilde{E}'=E'$. So, the geodesic energy of $\tilde{E},F$ wrt the metric $(l\Delta^2+\mathrm{len}+r/l)$ can be rewritten as

$$\begin{aligned}
&\int_0^1 \left\|(\dot{\tilde{E}},\dot{F})\right\|^2_{(l\Delta^2+\mathrm{len}+r/l)}\,\mathbb{d}t\\
&=\int_0^1\left(e^l\int_0^1\left(|\dot{E}'|^2+|\dot{F}'|^2\right)e^{-E-l}\,\mathbb{d}\theta+m_l e^{-2l}\left|\int_0^1\left(\dot{E}+\dot{l}\right)e^{E+l}\,\mathbb{d}\theta\right|^2+m_r e^{-2l}\left|\int_0^1\dot{F}e^{E+l}\,\mathbb{d}\theta\right|^2\right)\mathbb{d}t\\
&=\int_0^1\left(\int_0^1\left(|\dot{E}'|^2+|\dot{F}'|^2\right)e^{-E}\,\mathbb{d}\theta+m_l\left|\dot{l}\right|^2+m_r\left|\int_0^1\dot{F}e^{E}\,\mathbb{d}\theta\right|^2\right)\mathbb{d}t.
\end{aligned}$$

The previous formula transforms the geodesic energy of $\tilde{E},F$ according to the isometries seen in Propositions 14.9 and 14.10.

The fact that $\int_0^1 m_l\left|\dot{l}\right|^2\,\mathbb{d}t<a_1^2$ implies that $|l(0)-l(t)|<a_1/\sqrt{m_l}$ for all $t\in[0,1]$ so e^l and e^{-l} are bounded. Consequently we know by the previous lemma that E and $\tilde{E}$ are locally bounded. So we obtain that

$$\int_0^1(\dot{E}')^2e^{-E}\,\mathbb{d}\theta$$

is equivalent to

$$\int_0^1(\dot{E}')^2\,\mathbb{d}\theta=\int_0^1(\dot{\tilde{E}}')^2\,\mathbb{d}\theta$$

and similarly $\int_0^1(\dot{F}')^2e^{-E}\,\mathbb{d}\theta$ is equivalent to $\int_0^1(\dot{F}')^2\,\mathbb{d}\theta$; deriving $l(t)=\log\int_0^1 e^{\tilde{E}(t,\theta)}\,\mathbb{d}\theta$ we obtain

$$\dot{l}(t)=e^{-l}\int_0^1\dot{\tilde{E}}e^{\tilde{E}(t,\theta)}\,\mathbb{d}\theta$$

so $\dot{l}(t)$ is equivalent to $\int_0^1 \dot{\tilde{E}}\, \mathrm{d}\theta$; eventually $\int_0^1 \dot{F} e^E \, \mathrm{d}\theta$ is equivalent to $\int_0^1 \dot{F}\, \mathrm{d}\theta$. The above terms compose the norm $\|(\dot{\tilde{E}}, \dot{F})\|_{\tilde{H}^1}$ (defined in Eq. (9.2)) that is globally equivalent to the standard norm $\|(\dot{\tilde{E}}, \dot{F})\|_{H^1}$.

The case when

$$\Big\|(e_0, f_0) - (e_1, f_1)\Big\|_{H^1 \times H^1} \le a_1$$

is simpler, using the standard embedding of H^1 in $C^{0,1/2}$, see Lemma 9.3, we conclude that $\max |E|$ is bounded by $\max_\theta |E(0, \theta)| + a_1$, and we proceed as before. □

Lemma 14.14. *Let* $\mathbb{d}$ *be the distance induced by the metric* $(l\Delta^2 + \mathrm{len} + r/l)$ *defined in* (14.9)*. The metrics spaces*

- $H^1 \times H^1$ *with the usual distance and*
- $H^1 \times H^1$ *with the distance* $\mathbb{d}$

have the same bounded sets.

Proof. Let $a_1 > 0$, $(e_0, f_0) \in H^1 \times H^1$ and consider the ball $B = B_{\mathbb{d}}\Big((e_0, f_0), a_1\Big)$ defined with the distance $\mathbb{d}$. We obtain from Lemma 14.13 constants $a_2, a_3 > 0$ such that (14.19) holds. Consider now a point $(e_1, f_1) \in B$, that is $\mathbb{d}\big((e_0, f_0), (e_1, f_1)\big) < a_1$ and let γ be a smooth path connecting (e_0, f_0) to (e_1, f_1) whose length (according to the metric $(l\Delta^2 + \mathrm{len} + r/l)$) is less than a_1. Due to (14.19) we obtain that the length of γ according to the standard metric $H^1 \times H^1$ is less than $a_3 a_1$; so $\Big\|(e_0, f_0) - (e_1, f_1)\Big\|_{H^1 \times H^1} \le a_3 a_1$. We conclude that B is contained in the standard ball $B_{H^1 \times H^1}\big((e_0, f_0), a_3 a_1\big)$.

Mutatis mutandi we can prove that each standard ball $B_{H^1 \times H^1}$ is contained in a ball $B_{\mathbb{d}}$. □

Due to this Lemma, in the following we will simply talk of *bounded sets.*

Corollary 14.15. *In any bounded set the distance* $\mathbb{d}$ *induced by the metric* (14.9) *is equivalent to the usual distance in* $H^1 \times H^1$*.*

Corollary 14.16 (Representation theorem, existence of gradients)**.** *Fix* $\tilde{e}, f \in H^1$*. For any* $\hat{e}, \hat{f} \in H^1$ *there are unique* $\check{e}, \check{f} \in H^1$ *such that*

$$\Big\langle \hat{e}_1, \phi \Big\rangle_{H^1} + \Big\langle \hat{f}_1, \psi \Big\rangle_{H^1} = \Big\langle (\check{e}, \check{f}), (\phi, \psi) \Big\rangle_{(l\Delta^2 + \mathrm{len} + r/l), (\tilde{e}, f)}$$

for all $\phi, \psi \in H^1$*; where the scalar product on the right is defined by polarizing* (14.9)*. Symmetrically for any* $\check{e}, \check{f} \in H^1$ *there are unique* $\hat{e}, \hat{f} \in H^1$ *such that the above holds.*

Proof. This follows from Lax–Milgram theorem (Cor. 5.8 in [4]). □

(In the language of other papers cited above, this result shows that the metric is a strong metric).

By exploiting the isometry in Proposition 14.9, this result extends to the whole manifold of closed immersed curves.

Corollary 14.17 (Completion of the space of smooth curves)**.** *The space of immersed curves Imm above defined is the closure/completion of the space of smooth immersed curves.*

We remark that the results obtained in this Section are quite similar to the results in Section 3 of [5] for the metrics studied there; and indeed we would expect that any Sobolev-type metric of curves of order 2 or more should enjoy these kind of properties; still the proofs for the metric here presented are simplified by exploiting the particular structure of the metric, and the log-transform.

14.9. Completeness

We now prove that the proposed Riemannian manifolds are metrically complete.

Theorem 14.18. *The Riemannian manifold Imm of all immersed curves with the metric* (14.6) *is metrically complete.*

Proof. We first prove that $H^1 \times H^1$ is metrically complete with $\mathbb{d}$, the distance induced by the metric (14.9). Let d_H be the standard distance in $H^1 \times H^1$. Suppose that c_n is a Cauchy sequence in M; up to a subsequence we assume that $\mathbb{d}(c_n, c_{n+1}) \le 2^{-n}$; then $\mathbb{d}(c_0, c_n) \le 2$; so by Lemma 14.13 (setting $a_1 = 3$) the distance $\mathbb{d}$ is equivalent to d_H, hence the sequence converges to a curve c_∞ in $H^1 \times H^1$. Using again the fact that the distances are equivalent, we obtain that the sequence converges to the curve c_∞ for the distance $\mathbb{d}$ induced by the metric (14.9).

Since $H^1 \times H^1$ is the universal covering of the manifold $M = H^1 \times (H^1/(2\pi\mathbb{Z}))$, and both distances are invariant for the action $(e, f) \mapsto (e, f + 2k\pi)$,[20] then M is complete as well.

We exploit the isometry seen in proposition 14.9 to conclude that the space of all immersed curves is complete. □

Since each manifold described in Section 9 is closed in Imm, then it is metrically complete.

Moreover, quotient manifolds (by Euclidean subgroups) are complete as well, since they can be associated to normalizing submanifolds. For example, the Riemannian space $M/SO(2)$ of (open) curves up to translations and rotations is complete, since (by normalization) it can be associated with the manifold M_r, that is closed in M.

14.10. Geodesics

Proposition 14.19. *The metric is smooth.*

Proof. We use the isometry 14.9 and prove that the metric $\|(\hat{e}, \hat{f})\|^2_{(l\Delta^2 + \mathrm{len} + r/l), (\tilde{e}, f)}$ defined in equation (14.9) is smooth on $M = H^1 \times (H^1/(2\pi\mathbb{Z}))$. The metric is composed of many terms. One important term is $\|(\hat{e}, \hat{f})\|^2_{\Delta^2} = \int_0^1 (|\hat{e}'|^2 + |\hat{f}'|^2) e^{-\tilde{e}} \,\mathbb{d}\theta$; this is smooth, indeed it is quadratic in $\hat{e}, \hat{f}$, it does not depend on f, and its k-th derivative in e in directions $m_1, m_2, \ldots, m_k \in H^1$ is

$$D^k_{e, m_1, m_2, \ldots m_k} \|(\hat{e}, \hat{f})\|^2_{\Delta^2} = \int_0^1 (|\hat{e}'|^2 + |\hat{f}'|^2) m_1 m_2, \ldots, m_k e^{-\tilde{e}} \,\mathbb{d}\theta$$

that is continuous (it follows from the embedding of H^1 into C^0). Similarly for all other terms. □

By standard results (see *e.g.* [12]) this implies local existence and uniqueness of a geodesic γ, given $\gamma(0)$ and $\dot{\gamma}(0)$. But we also proved that the Riemannian manifold is metrically complete, hence this result follows.

Theorem 14.20. *For any given $\gamma(0)$ and $\dot{\gamma}(0)$ there exists an unique geodesic $\gamma(t)$ defined for all $t \in \mathbb{R}$.*

This holds in the space of immersed (open) curves, in the space of closed curves, and in all other submanifold that are described in Proposition 9.7. The geodesic is smooth as a map of t in the appropriate space.

Remark 14.21. The metric $\|(\hat{e}, \hat{f})\|^2_{(l\Delta^2 + \mathrm{len} + r/l), (\tilde{e}, f)}$ does not depend on f; hence, for any fixed $q \in H^1$, along a geodesic the quantity

$$D_{\hat{f}, q} \|(\dot{E}, \dot{F})\|^2_{(l\Delta^2 + \mathrm{len} + r/l), (E, F)} = \Big(\int_0^1 e^E \,\mathbb{d}\theta \Big) \int_0^1 2q' \dot{F}' e^{-E} \,\mathbb{d}\theta + \frac{2m_r \int_0^1 q e^E \,\mathbb{d}\theta \int_0^1 \dot{F} e^E \,\mathbb{d}\theta}{\left| \int_0^1 e^E \,\mathbb{d}\theta \right|^2}$$

[20] See Section 9.1 for details.

will be constant. This holds only for unconstrained geodesics, that is, geodesics of open immersed curves.

14.11. Momenta

Suppose that $\gamma : [0,1] \to M$ is a geodesic, and G is a group acting isometrically on M. As in Section 2.1, let $\xi \in \mathbb{G}$ be an element in the Lie algebra, and $\zeta = \zeta(\xi, c)$ the vector field on M that is the derivative of the action of G on $c \in M$. By Emmy Noether's theorem, the following scalar product is constant

$$\langle \dot\gamma, \zeta(\xi, \gamma) \rangle.$$

By solving for arbitrariness of ξ this provides a conserved quantity, called a *momentum*.

We can then compute quantities that are conserved along geodesics of curves; this was pioneered in Section 2.5 in [16]. A tutorial is in Section 11.13 in [13].

We now compute conserved momenta for the metric of immersed curves ($l\Delta^2 + \text{len} + r/l + t$) defined in equation (14.6) in Definition 14.4. For all groups but translations, the action is factored into the manifold M, so we will, up to log-transform, equivalently work in $H^1 \times H^1$ with the metric ($l\Delta^2 + \text{len} + r/l$) defined in equation (14.9) in Definition 14.7; to this end we express the geodesic γ as (E, F); we will use the formulas in Appendix C.

We start with momenta related to Euclidean transformations.

- Translation (linear momentum). The center of mass $\text{avg}_c(\gamma(t))$ of the curve is an affine map of t. This can also be expressed using the center of mass of the starting and ending curves

$$\text{avg}_c(\gamma(t)) = t \,\text{avg}_c(\gamma(0)) + (1-t)\,\text{avg}_c(\gamma(1)). \tag{14.20}$$

  This result follows from the isometry discussed in Proposition 14.9.
  The center of mass is stationary iff the geodesic is *horizontal* wrt the action of translations (that is, $\dot\gamma$ is in the horizontal space at all times, see Sect. 2.1).
- Rescaling. Then $\xi \in \mathbb{G}_{\mathbb{I}} = \mathbb{R}$, and $\zeta(\xi, (E, F)) = (\xi, 0)$ by equations (C.4) and (C.5); the scalar product $\langle \dot\gamma, \zeta \rangle$ reduces to

$$m_l \frac{\xi \int_0^1 \dot E e^E \,\mathrm{d}\theta}{\int_0^1 e^E \,\mathrm{d}\theta}$$

  and, by Emmy Noether's theorem, this last term is constant; but

$$\frac{\int_0^1 \dot E e^E \,\mathrm{d}\theta}{\int_0^1 e^E \,\mathrm{d}\theta} = \frac{\mathrm{d}}{\mathrm{d}t} \log \left(\int_0^1 e^E \,\mathrm{d}\theta \right),$$

  where we recognize that $\log \int_0^1 e^E \,\mathrm{d}\theta$ is (in log-transform) $\log \text{len}\gamma$ the logarithm of the length of the curve; hence $\log \text{len}\gamma$ is an affine map of t, that is,

$$\int_0^1 e^E \,\mathrm{d}\theta = \text{len}\gamma = e^{a+bt} \tag{14.21}$$

  for $a, b \in \mathbb{R}$.
  If/when we wish to consider γ as a geodesic connecting two curves $\gamma(0)$ and $\gamma(1)$, this can also be expressed (setting $a = \log(\text{len}\gamma(0))$ and $b = \log(\text{len}\gamma(1)) - a$) using the length of the starting and ending curves

$$\text{len}\ \gamma(t) = (\text{len}\ \gamma(1))^t (\text{len}\ \gamma(0))^{1-t}. \tag{14.22}$$

This result also follows from the isometry discussed in Proposition 14.10.
The length is constant (*i.e.* $b = 0$) iff the geodesic is *horizontal* wrt the action of rescaling.

- Rotation (angular momentum). Then $\xi \in \mathbb{G}_{\mathbb{r}} = \mathbb{R}$, and $\zeta(\xi, (E, F)) = (0, \xi)$ by equations (C.4) and (C.5); the scalar product $\langle \dot\gamma, \zeta \rangle$ reduces to

$$m_r \frac{\xi \int_0^1 \dot F e^E \,\mathrm{d}\theta}{\int_0^1 e^E \,\mathrm{d}\theta}$$

that is constant; but the denominator is the length, so we obtain that

$$\int_0^1 \dot F e^E \,\mathrm{d}\theta = c e^{tb} \tag{14.23}$$

for appropriate constants c, b (where b is as before).
$c = 0$ iff the geodesic is *horizontal* wrt the action of rotation.

Since the manifold of closed curves is invariant for Euclidean actions, then the above momenta are invariant for geodesics of closed curves as well.

Corollary 14.22. *Along a geodesic γ the four "speeds"*

$$\sqrt{\mathrm{len}(\gamma)} \|\dot\gamma\|_{\Delta^2,\gamma}, \quad \|\dot\gamma\|_{\mathrm{len},\gamma}, \quad \|\dot\gamma\|_{\mathbb{r},\gamma}/\mathrm{len}(\gamma), \quad \|\dot\gamma\|_{\mathbb{t},\gamma} \tag{14.24}$$

are all constant.

Proof. The above discussion shows that the last three terms are constant. The sum of the squares of the four terms is the "total speed" squared $\|\dot\gamma\|^2_{(l\Delta^2+\mathrm{len}+r/l+t),\gamma}$ and this is known to be constant in any geodesic. □

Remark 14.23. The momenta related to "curling" was already presented in Remark 14.21. In particular if a geodesic of open curves is horizontal for rescaling and rotation, then it is horizontal for curling iff

$$\int_0^1 q' \dot F' e^{-E} \,\mathrm{d}\theta = 0$$

for any $t \in \mathbb{R}$ and $q \in H^1$; that is, iff $\left(\dot F' e^{-E}\right)' \equiv 0$.

The momenta associated to reparameterization is not defined on all possible geodesics, indeed the action of reparameterization is not smooth (in the category of H^2 maps).

Proposition 14.24. *Let (E, F) be a geodesic; we assume that it is smooth in (t, θ); up to rescaling we assume that all curves have length 1 (with no loss of generality, due to 14.25). The quantity*

$$B = B(t, \theta) = -\left(\dot E' e^{-E}\right)' E' + \left(\dot E' e^{-E}\right)'' - \left(\dot F' e^{-E}\right)' F' + m_r c F' e^E,$$

(where c is as in (14.23)) is conserved, in this sense: there is a function $\beta = \beta(\theta)$ such, for all t, $B = \beta$. β is zero iff the geodesic is horizontal *wrt the action of reparameterizations.*

Proof. An element of the Lie Algebra of reparameterizations is represented by a function $a : [0, 1] \to \mathbb{R}$ with null boundary conditions; we assume that a is smooth; $\zeta(\xi, (E, F)) = (aE' + a', aF')$ by equations (C.17) and

(C.18) (see Appendix, bellow); the scalar product reduces to

$$\int_0^1 \Big(\dot E'(E'a+a')' + \dot F'(F'a)'\Big) e^{-E}\,\mathrm{d}\theta + m_r c\int_0^1 F' a e^{E}\,\mathrm{d}\theta \tag{14.25}$$

where c is as in (14.23), and we set $b=0$.

Integrating by parts

$$\begin{aligned}&\int_0^1 -\Big(\dot E' e^{-E}\Big)'(E'a+a') - \Big(\dot F' e^{-E}\Big)' F'a\,\mathrm{d}\theta + m_r c\int_0^1 F'ae^E\,\mathrm{d}\theta\\ &= \int_0^1 -\Big(\dot E' e^{-E}\Big)' E'a + \Big(\dot E' e^{-E}\Big)'' a - \Big(\dot F' e^{-E}\Big)' F'a\,\mathrm{d}\theta + m_r c\int_0^1 F'ae^E\,\mathrm{d}\theta = \langle a, B\rangle\end{aligned}$$

and this scalar quantity is constant in t. The thesis then follows. □

If we consider the manifold of closed curves, then we must assume that a has periodic boundary conditions; the above result holds, and moreover the quantity

$$\int_0^1 \Big(\dot E' E'' + \dot F' F''\Big) e^{-E} + m_r c F' e^E\,\mathrm{d}\theta \tag{14.26}$$

is constant.

14.12. Minimal geodesic

Let us fix $c_0, c_1 \in M$. We will prove that there is a minimal geodesic connecting them. This is true in M, with the metric (14.9), as well as in any submanifold as described in Proposition 9.7. This extends to the space Imm of all immersions, due to the isometry seen in Proposition 14.9.

We start with a preliminary discussion. Suppose that c_0, c_1 are connected by a homotopy C. This provides boundary conditions

$$\log(c_0(\theta)) = E(0,\theta) + iF(0,\theta),\quad \log(c_1(\theta)) = E(1,\theta) + iF(1,\theta) + i2\pi j,$$

where j is integer and $E+iF=\log(C')$ is the log-representation. The geodesic energy corresponding to the metric (14.9) is

$$\mathbb{E}_{(l\Delta^2+\mathrm{len}+r/l)}(C) \stackrel{\mathrm{def}}{=} \int_0^1 \left(\left(\int_0^1 e^E\,\mathrm{d}\theta\right)\int_0^1 \Big(|\dot E'|^2 + |\dot F'|^2\Big) e^{-E}\,\mathrm{d}\theta + \frac{m_l\left|\int_0^1 \dot E e^E\,\mathrm{d}\theta\right|^2 + m_r\left|\int_0^1 \dot F e^E\,\mathrm{d}\theta\right|^2}{\left|\int_0^1 e^E\,\mathrm{d}\theta\right|^2}\right)\mathrm{d}t. \tag{14.27}$$

To compute the minimal length geodesic we should compute the minimum of the above energy. This is quite complex, but we can factor out scale, since the first two terms are hom-wise scaling invariant (or equivalently due to the isometry seen in Prop. 14.10). This follows from the discussion in the previous sections, and was already exploited in Lemma 14.13, but, for sake of simplicity, we show it explicitly.

Proposition 14.25. *We suppose that the homotopy is of the form Ce^l where each curve in C has unit length and $l=l(t)>0$; let $E+iF=\log(C')$ be the log-representation; then (14.27) becomes*

$$\mathbb{E}_{(l\Delta^2+\mathrm{len}+r/l)}(Ce^l) = \int_0^1 \left(\int_0^1 \Big(|\dot E'|^2 + |\dot F'|^2\Big) e^{-E}\,\mathrm{d}\theta + m_r\left|\int_0^1 \dot F e^E\,\mathrm{d}\theta\right|^2 + m_l\left|\dot l\right|^2\right)\mathrm{d}t. \tag{14.28}$$

Obviously in the minimum we will have $\dot{l}=0$ *and this means that the length along the geodesic will be* $\operatorname{len}(Ce^l)=e^l=\operatorname{len}(c_1)^t\operatorname{len}(c_0)^{(1-t)}$*; indeed this is the conserved momentum seen in* (14.22)*.*

We have then to minimize (14.28) *that becomes*

$$\mathbb{E}_{(\Delta^2+r)}(C)\stackrel{\text{def}}{=}\int_0^1\int_0^1(|\dot{E}'|^2+|\dot{F}'|^2)e^{-E}\,\mathrm{d}\theta\,\mathrm{d}t+m_r\int_0^1\left|\int_0^1\dot{F}e^E\,\mathrm{d}\theta\right|^2\,\mathrm{d}t \tag{14.29}$$

(that is the energy of the seminorm in Eq. (14.10)*) with boundary conditions*

$$\begin{aligned}\log(c_0(\theta))-\log(\operatorname{len}(c_0))&=E(0,\theta)+iF(0,\theta),\\ \log(c_1(\theta))-\log(\operatorname{len}(c_1))&=E(1,\theta)+iF(1,\theta)+i2\pi j\end{aligned} \tag{14.30}$$

(for $j\in\mathbb{Z}$*) with the constraint* $\forall t,\int_0^1 e^E\,\mathrm{d}\theta=1$*.*

To prove the desired theorem, we provide two lemmas.

Lemma 14.26. *Let* $a>0$*. Suppose that*

$$\mathbb{E}_{(\Delta^2+r)}(C)\le a^2$$

then E,F *are bounded in* $C^{0,1/2}([0,1]^2)$*, and the bound depends only on* a,m_r *and* $c_0=C(0,\cdot)$*.*

Proof. Since all curves $C(t,\cdot)$ are assumed to be length one, we can use Proposition 14.15 to switch to the standard metric in $H^1\times H^1$; then by Lemma A.1

$$\|E(t_1,\cdot)-E(t_2,\cdot)\|_{H_1([0,1])}\le a\sqrt{|t_2-t_1|}; \tag{14.31}$$

in particular for any $t\in[0,1]$

$$\|E(t,\cdot)\|_{H_1([0,1])}\le a+\|E(0,\cdot)\|_{H_1([0,1])}$$

(note that this last term depends only on c_0). Similarly for F. So using the usual compact embedding $H^1\to C^{0,1/2}$ (see Lem. 9.3) we obtain that

- E,F are uniformly bounded, and
- for any $t\in[0,1]$ the functions $E(t,\cdot)$ $F(t,\cdot)$ are Hölder continuous, with constant depending only on a_1,c_0.

At the same time by equations (14.31) and (9.4) in Lemma 9.3 we also obtain

$$\max_\theta |E(\theta,t_1) - E(\theta,t_2)| \le a_1 \sqrt{2|t_2-t_1|},$$

and similarly for F, so E, F are Hölder continuous in the t direction. □

We report this result, Theorem 3.23 in [7] in a much simplified form.

Lemma 14.27. *Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded. Suppose with $g = g(x,u,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is continuous, that $g \ge 0$ and for any fixed $u \in \mathbb{R}, x \in \mathbb{R}^n$ the map $\xi \mapsto g(x,u,\xi)$ is convex. Now for $u \in C^0(\Omega)$ and $\xi \in L^2(\Omega)$, let*

$$J(u,\xi) = \int_\Omega g(x, u(x), \xi(x))\, \mathrm{d}x.$$

Consider now sequences $u_0, \dots u_n \dots \in C^0(\Omega)$ $\xi_0, \dots, \xi_n, \dots, \in L^2$ and suppose that $u_n \to u_0$ uniformly whereas ξ_n converges weakly to ξ in L^2. Then $\liminf_n J(u_n, \xi_n) \ge J(u_0, \xi_0)$.

Theorem 14.28. *For any two curves $c_0, c_1 \in M$ there is a minimal geodesic connecting them.*
The same holds for any submanifold described in Proposition 9.7.

Proof. Let $\log(c_0) = \tilde e_0 + i f_0$ and $\log(c_1) = \tilde e_1 + i f_1$ be a choice of representation of c_0, c_1 in log-transform. For $j \in \mathbb{Z}$ we consider the minimization of the geodesic energy (14.27) subject to a choice of $j \in \mathbb{Z}$ and of $E, F \in H^1([0,1]^2)$ with boundary condition

$$E(0,\theta) = \tilde e_0(\theta), \quad F(0,\theta) = f_0(\theta), \quad E(1,\theta) = \tilde e_0(\theta), \quad F(1,\theta) = f_1(\theta) + 2\pi j,$$

and with the constraint $\forall t, \int e^E \,\mathrm{d}\theta = 1$.

Suppose that E_n, F_n, j_n is a minimizing sequence. By Lemma 14.26, E_n, F_n are bounded, E_n, F_n are equicontinuous; then necessarily j_n is bounded as well. By Lemma 14.13 $\dot E'_n, \dot F'_n$ are bounded in $L^2([0,1]^2)$.

By Ascoli–Arzelà and Banach–Alaoglu theorem, up to a subsequence, $j_n \to \tilde j$, $\dot E'_n, \dot F'_n$ weakly converge in L^2 to $\hat E, \hat F \in L^2([0,1]^2)$, and E_n, F_n uniformly converge to $\tilde E, \tilde F$. Consequently the weak derivative $\dot{\tilde E}'$ of $\tilde E$ is $\hat E$, and similarly for F.

By the Lemma 14.27 $\tilde E, \tilde F, \tilde j$ is the required geodesic.

If for any t and n the curve represented by $E_n(t,\cdot), F_n(t,\cdot)$ is in one of the submanifolds described in Proposition 9.7, then for any t $\tilde E(t,\cdot), \tilde F(t,\cdot)$ are in the same submanifold. Indeed the constraints that define the submanifolds (see in the proof of Prop. 9.7) are all continuous wrt uniform convergence. □

14.12.1. Karcher mean

The same method of proof can be generalized to other problems.

Theorem 14.29. *Fix $\tilde c_1, \dots, \tilde c_k \in \mathrm{Imm}$. Then the problem*

$$\inf_{c \in \mathrm{Imm}} \sum_{i=1}^k d\big(c, \tilde c_i\big)^2$$

has a minimum. Consider c be a minimum curve. Then the center of mass of c is the average of the center of masses of $\tilde c_1, \dots, \tilde c_k$; the length of c is the geometric mean of the lengths of $\tilde c_1, \dots, \tilde c_k$.

A minimum point c of this problem is called a Fréchet mean, or Karcher mean of the given points. When $k = 2$, c lies at the middle of a geodesic connecting c_1 to c_2.

15. A Riemannian manifold for open curves

As aforementioned, one strong point in this presentation is the "modular" character of the designed metric.

We now change a term in the metric presented in Section 14. The resulting metric is particularly well suited for open curves. It is not apt for closed curves, since it is not invariant for change of base point, so it does not project to a metric for geometric closed curves (that is, closed curves up to parameterization).

In Section 8.3, we defined a normalization for rotation as

$$I_R(e,f) \stackrel{\text{def}}{=} \frac{\int_0^1 f e^{\tilde{e}}\, d\theta}{\int_0^1 e^{\tilde{e}}\, d\theta} \tag{15.1}$$

(in log transform); hence, by design, we introduce the seminorm $\|h\|_R$ as the norm of the Gâteaux differential of I_R at the curve c in direction h; in log-coordinates this is expressed as

$$\|(\hat{e},\hat{f})\|_{R,(\tilde{e},f)} \stackrel{\text{def}}{=} \frac{\left|\int_0^1 (\hat{f}+f\hat{e})e^{\tilde{e}}\, d\theta \int_0^1 e^{\tilde{e}}\, d\theta - \int_0^1 f e^{\tilde{e}}\, d\theta \int_0^1 \hat{e} e^{\tilde{e}}\, d\theta\right|}{\left(\int_0^1 e^{\tilde{e}}\, d\theta\right)^2}. \tag{15.2}$$

Definition 15.1. Let $m_l, m_r, m_t > 0$ be fixed. We associate to the manifold Imm of all immmersed curves the Riemannian metric

$$\|h\|^2_{(l\Delta^2+\text{len}+R+t),c} \stackrel{\text{def}}{=} \text{len}(c)\|h\|^2_{\Delta^2,c} + m_l\|h\|^2_{\text{len},c} + m_r\|h\|^2_{R,c} + m_t\|h\|^2_{\mathfrak{t},c}, \tag{15.3}$$

where the terms $\|h\|^2_{\Delta^2,c}$, $\|h\|_{\text{len},c}$ and $\|h\|_{\mathfrak{t},c}$ are as in the norm defined in 14.4 in the previous section.

When restricting to the manifold M of curves normalized for translation, we can represent the norm in log-transform.

Proposition 15.2. *Let $m_l, m_r > 0$ be fixed. Let $\tilde{e}, f, \hat{e}, \hat{f} \in H^1([0,1])$. We consider the pair $(\tilde{e}, f)$ to represent a curve in M, and $(\hat{e},\hat{f})$ to represent a tangent vector. Then the norm has the form*

$$\begin{aligned}\|(\hat{e},\hat{f})\|^2_{(l\Delta^2+\text{len}+r/l),(\tilde{e},f)} &\stackrel{\text{def}}{=} \left(\int_0^1 e^{\tilde{e}}\, d\theta\right)\int_0^1 (|\hat{e}'|^2+|\hat{f}'|^2)e^{-\tilde{e}}\, d\theta \\ &+ \frac{m_l\left[\int_0^1 \hat{e}e^{\tilde{e}}\, d\theta\right]^2 + m_r\left[\int_0^1(\hat{f}+f\hat{e})e^{\tilde{e}}\, d\theta\int_0^1 e^{\tilde{e}}\, d\theta - \int_0^1 fe^{\tilde{e}}\, d\theta\int_0^1 \hat{e}e^{\tilde{e}}\, d\theta\right]^2}{\left|\int_0^1 e^{\tilde{e}}\, d\theta\right|^2}\end{aligned}$$

that is Riemannian metric on $H^1 \times (H^1/(2\pi\mathbb{Z}))$.

We briefly comment on the properties of this Riemannian metric.

Substituting the seminorm $\|h\|_{r/l}$ by the seminorm $\|h\|_R$ adds another interesting property. Indeed $\|h\|_R$ is hom-wise reparameterization invariant (this replaces a "CW!" with an "HW" in Table 1 in page 29).

15.1. Momenta

Translations and rescaling momenta are as in equations (14.20) and (14.22) in Section 14.11. Curling momentum is the same as in Remarks 14.21 and 14.23.

The quantity $I_R(E,F)$ is constant along geodesics, so it is the conserved angular momentum; consequently

$$\int_0^1 F e^E \, \mathrm{d}\theta = c e^{tb} \tag{15.4}$$

and this replaces the formula (14.23) seen in Section 14.11.

15.2. Geodesics

So when studying (or numerically computing) the geodesics, up to normalizing for rotation, translation and scaling, we can reduce to the metric

$$\|(\hat e, \hat f)\|^2_{(\Delta^2),(\tilde e, f)} \stackrel{\text{def}}{=} \int_0^1 \left(|\hat e'|^2 + |\hat f'|^2 \right) e^{-\tilde e} \, \mathrm{d}\theta;$$

note also that, up to curling and reparameterization, we may assume that the initial curve is just $c(\theta) = (\theta - 1/2, 0)$, that is, $E(0,\theta) = F(0,\theta) = 0$.

All results valid for the previous metric (completeness, existence of geodesics, etc.) hold for this metric as well.

Moreover, we have a stronger version of Remark 14.5: geodesics (and in particular minimal length geodesics) do not depend on the choice of m_l, m_t, m_r.

16. A "Riemannian manifold" for geometric curves

We now consider the "manifold" of geometric curves, that are immersed curves up to reparameterizations.[21] We distinguish two cases.

(1) The space of *"geometric closed curves"* that is the quotient space $\mathrm{Imm}_{\mathbb{f}}/\mathrm{Diff}(S^1)$.
(2) The space of *"geometric open curves"* is the quotient space $\mathrm{Imm}/\mathbb{D}_0$

(we recall that $\mathbb{D}_0$ is an abbreviation for $\mathrm{Diff}([0,1])$). An element in the above spaces will be denoted as $[c]$ that is the orbit of an immersed curve c by the action of reparameterization.

16.1. Topology on $\mathrm{Diff}(S^1)$ and $\mathbb{D}_0$

Since we properly defined Imm and $\mathrm{Imm}_{\mathbb{f}}$ as submanifolds of H^2, then we now properly define $\mathrm{Diff}(S^1)$ and $\mathbb{D}_0$.

It is well known that the family of H^2 diffeomorphisms of $[0,1]$ is a topological group [9]. In particular let φ_n, φ be diffeomorphisms; if $\varphi_n \to \varphi$ in H^2 then $\varphi_n^{-1} \to \varphi^{-1}$ in H^2.[22]

This suggests to explore a "symmetrized" definition of distance, to view $\mathbb{D}_0$ as a metric space (and similarly for $\mathrm{Diff}(S^1)$).

Definition 16.1. A diffeomorphism $\varphi : [0,1] \to [0,1]$ is in $\mathbb{D}_0$ if and only if both φ and φ^{-1} are in H^2. We view $\mathbb{D}_0$ as a metric space, with distance

$$d(\varphi, \psi) = \|\varphi - \psi\|_{H^2} + \|\varphi^{-1} - \psi^{-1}\|_{H^2} \tag{16.1}$$

that is φ_n converges to φ in $\mathbb{D}_0$ if and only if $\lim_n \varphi_n = \varphi$ and $\lim_n \varphi_n^{-1} = \varphi^{-1}$ in H^2.

Similarly a diffeomorphism $\varphi : S^1 \to S^1$ is in $\mathrm{Diff}(S^1)$ if and only if both φ and φ^{-1} are in H^2; we associate to $\mathrm{Diff}(S^1)$ the distance (16.1) as well.

[21] Since we consider only reparameterizations φ with $\varphi' > 0$ then this are actually *"oriented geometric curves"*.
[22] For convenience of the reader a straightforward proof is available as Lemma B.5.

Theorem 16.2.

- $\mathbb{D}_0$ *is a topological group.*
- *It is a complete metric space, and it is the metric completion of* $\mathbb{D}_0 \cap C^\infty$*.*
- *The action*

$$\varphi, c \in \mathbb{D}_0 \times \mathrm{Imm} \mapsto c \circ \varphi \in \mathrm{Imm}$$

  is continuous.
- *Diff*(S^1) *is a topological group.*
- *It is a complete metric space, and it is the metric completion of Diff*$(S^1) \cap C^\infty$*.*
- *The action*

$$\varphi, c \in \mathit{Diff}(S^1) \times \mathrm{Imm}_{\mathbb{f}} \mapsto c \circ \varphi \in \mathrm{Imm}_{\mathbb{f}}$$

  is continuous.

Proof. Completeness of $\mathbb{D}_0$ is trivial, if φ_n is a Cauchy sequence then $\varphi_n \to \varphi$ and $\varphi_n^{-1} \to \psi$ in H^2 hence uniformly hence $\psi = \varphi^{-1}$. All other results are proved in the Section B for convenience of the reader. □

16.2. Minimal geodesic

We now want to study the quotient spaces as metric spaces.

What follows holds when

- we consider the space of open curves and we endow it with the metric discussed in Section 14;
- we consider the space of closed curves and we endow it with the metric discussed in Section 14;
- we consider the space of open curves and we endow it with the metric discussed in Section 15.

We begin by proving existence of minimal length geodesics.

For simplicity we only present the case of open curves, endowed with the metric of Section 14.

We recall that the space $\mathrm{Imm}_{\mathbb{f}}$ of closed curves is decomposed in connected components, where each component $\mathrm{Imm}_{\mathbb{f},\mathbb{k}}$ contains only curves of rotational index k; moreover, $\mathrm{Imm}_{\mathbb{f},\mathbb{k}}$ is a closed submanifold of Imm. So, the theorems below hold (*mutatis mutandis*) for closed curves of same rotation number.

We know that Imm is diffeomorphic and isometric to $\mathbb{R}^2 \times (0,\infty) \times M_d$ (by the isometries seen in Props. 14.9 and 14.10); where the space $\mathbb{R}^2 \times (0,\infty) \times M_d$ decomposed the immersed curve in *"center of mass"*, *"length"*, and *"curve with center of mass in the origin and length 1"*. The reparameterizations act only on the infinite dimensional component M_d so we can study the problem of minimal geodesics in $(M_d/\mathbb{D}_0)$.

A similar decomposition holds for closed curves.

We use the method described in Section 2.1. Given $[c_0], [c_1]$ an initial and final geometric curve, we look for the minimum of

$$d_{\mathrm{Imm}/\mathbb{D}_0}([c_0],[c_1]) \stackrel{\text{def}}{=} \inf_{\varphi \in \mathbb{D}_0} d_{\mathrm{Imm}}(c_0, c_1 \circ \varphi) \tag{16.2}$$

(this is the definition we saw in (2.3), adapted for this specific case).

Using the isometries, as explained above, we can reduce the problem of finding a minimal geodesic in $M_d/\mathbb{D}_0$

$$d_{M_d/\mathbb{D}_0}([c_0],[c_1]) = \inf_{\varphi \in \mathbb{D}_0} d_{M_d}(c_0, c_1 \circ \varphi) \tag{16.3}$$

where the distance d_{M_d} is induced by the metric $\|\|_{\Delta^2 + r}$ (that was defined in Prop. 14.10).

Theorem 16.3. *Any two geometric open curves $[c_1], [c_2]$ are connected by a minimal geodesic. This geodesic is the projection of a geodesic connecting c_1 to $c_2 \circ \varphi$ in Imm where $\varphi \in \mathbb{D}_0$.*

Proof. We will prove that the infimum in (16.3) is a minimum.

Let φ_j be a sequence that approaches the infimum in (16.3).

By Theorem 14.28 for any given φ_j there is a minimal geodesic connecting c_0 to $c_j \overset{\text{def}}{=} c_1 \circ \varphi_j$. Let E_j, F_j be the minimizing geodesic in log-transform.

The distances $d_{M_d}(c_0, c_j)$ are a bounded sequence, this has important consequences.

Following the proof of Theorem 14.28, we can find geodesics E_j, F_j whose length in $H^1 \times H^1$ are bounded.

We so obtain that E_j, F_j are equicontinuous (by Lem. 14.26) and $\dot{E}_j', \dot{F}_j'$ are bounded in $L^2([0,1] \times [0,1])$. So, up to a subsequence, E_j, F_j converges uniformly to a geodesic E, F connecting (in log-coordinates) c_0 to a curve c; hence, this geodesic is contained in M_d.

Moreover, again up to a subsequence, $\dot{E}_j', \dot{F}_j'$ converge weakly in L^2, so using again Lemma 14.27 we obtain that

$$d_{M_d}(c_0, c) \leq \liminf_j d_{M_d}(c_0, c_j)$$

so c is the candidate minimum.

Moreover by 14.15 $E_j(1,\cdot), F_j(1,\cdot)$ are bounded in H^1, so in the limit $E(1,\cdot), F(1,\cdot)$ are in H^1, so the curve c is in M_d.

It also means that

$$\max_\theta |c_j'|, \frac{1}{\min_\theta |c_j'|}$$

are bounded, by Lemma 14.12. By chain rule

$$\max_\theta |c_1'(\varphi_j(\theta))\varphi_j'(\theta)|, \frac{1}{\min_\theta |c_1'(\varphi_j(\theta))\varphi_j'(\theta)|}$$

are bounded, since c_1 is fixed then

$$\max_\theta |\varphi_j'(\theta)|, \frac{1}{\min_\theta |\varphi_j'(\theta)|}$$

are bounded. So up to a subsequence the sequence φ_j will uniformly converge to a diffeomorphism φ, such that $c = c_1 \circ \varphi$.

By Lemma B.9, we obtain that $\varphi \in \mathbb{D}_0$, as we defined it in 16.1.

□

16.2.1. Fréchet mean

The Theorem 14.29 on existence of *Fréchet means* holds as well.

Theorem 16.4. *Fix $[\tilde{c}_1], \ldots [\tilde{c}_k] \in \mathrm{Imm}/\mathbb{D}_0$. Then the problem*

$$\inf_{[c] \in \mathrm{Imm}/\mathbb{D}_0} \sum_{i=1}^k d_{\mathrm{Imm}/\mathbb{D}_0}\left([c], [c_i]\right)^2$$

has minimum $[c]$.

This minimum can be computed remembering that, for any $c \in [c]$, the center of mass of c is the average of the center of masses of $\tilde{c}_1, \dots \tilde{c}_k$; the length of c is the geometric mean of the lengths of $\tilde{c}_1, \dots \tilde{c}_k$.

Hence, we can assume[23] that all curves are in M_d, and reduce the problem to finding $\varphi_1, \dots, \varphi_k \in \mathbb{D}_0$ and $c \in M_d$ that minimize

$$\sum_{i=1}^{k} d_{M_d}(c, c_i \circ \varphi)^2.$$

Theorem 16.5. *The above problem has minimum.*

We omit the proof, that is but a complicated repetition of the arguments used to prove Theorem 16.3.

16.3. True metric space

We now can answer a fundamental question. We indeed remarked in Section 2.1 that a quotient distance may be a semidistance, *i.e.* in general there may be two different orbits at zero distance. In this case, though, it is a true distance.

Theorem 16.6. *If* $[c_0] \neq [c_1]$ *then* $d_{\mathrm{Imm}/\mathbb{D}_0}([c_0], [c_1]) > 0$.

Proof. We proceed by contradiction. By 16.3 there exists a geodesic $C(t, \theta)$ and a $\varphi \in \mathbb{D}_0$ such that $C(0, \theta) = c_0(\theta)$ and $C(1, \theta) = c_0(\varphi(\theta))$ providing the minimum. If $d_{\mathrm{Imm}/\mathbb{D}_0}([c_0], [c_1]) = 0$ then the energy of C is zero so $C(t, \theta) = c_0(\theta)$ for all t hence $c_1 = c_0 \circ \varphi$ that means that $[c_0] = [c_1]$. □

Since, we now know that the quotients spaces are true metric spaces, we can then state this result.

Theorem 16.7. *The quotient space* $\mathit{Imm}_{\mathbf{f}}/\mathit{Diff}(S^1)$ *and* $\mathit{Imm}/\mathbb{D}_0$ *are complete metric spaces.*

This follows from Lemma 2.5 and Theorem 14.18.

16.4. Differential structure

We already remarked that there are some known problems in defining a differentiable structure on spaces of geometric curves. We propose a workaround.

Let M_1 be the family of all arc parameterized curves c *i.e.* all $c \in M_d$ such that $\forall t, |c'(t)| = 1$, and with center of mass in the origin. Obviously $M_1 \subseteq M_d$.

We know that the log-transform is a diffeomorphism that associates a curve $c \in M_d$ to a pair $\tilde{e}, f \in H^1$. So the log-transform of $\tilde{c}$ is just a choice of $f \in H^1/(2\pi\mathbb{Z})$ such that $\tilde{c}'(\theta) = e^{if(\theta)}$. So we associate M_1 to $H^1/(2\pi\mathbb{Z})$.

If we wish to define a differential structure on $(M_d/\mathbb{D}_0)$, then we will identify it with $H^1/(2\pi\mathbb{Z})$. With this choice M_1 is clearly a smooth submanifold of M_d.

In this sense, we can consider $\mathrm{Imm}/\mathbb{D}_0$ as a smooth submanifold of Imm. Up to translation and log transform the bundle structure $\mathrm{Imm} \to \mathrm{Imm}/\mathbb{D}_0$ is just the projection on the second component of $H^1 \times (H^1/2\pi\mathbb{Z})$.

Proposition 16.8. *Each fiber of* $\mathit{Imm} \to \mathit{Imm}/\mathbb{D}_0$ *is homeomorphic to* $\mathbb{D}_0$.

We do not claim though that this is a principal smooth G-bundle, since we prefer to view $\mathbb{D}_0$ as a metric space. (A further discussion of this subject may appear in a future paper).

17. Final remarks

17.1. Future developments

We acknowledge that there are many important points left to study.

[23] With no loss of generality, due to the isometries seen in Propositions 14.9 and 14.10.

- Computation of gradient. While we proved in 14.16 that the gradient exists, we did not provide any method to compute it; an explicit method is fundamental in applications to Shape Optimization.
- Regularity of minimal geodesics connecting smooth curves.
- and regularity of geodesics with smooth initial data.
- Numerical implementations.
- (Numerical) comparison with other models present in the literature.
- Probabilistic models.
- Full normalization for reparameterization.

17.2. Conclusions

There is still a lot of room for improvements.

We would like to design a metric that is designed for reparameterization; ideally the semimetric $\|h\|_{l\Delta^2,c}$ should be replaced by a term that decomposes into the sum of a "semimetric for reparameterizations", plus a "pure geometric semimetric" (that projects on the space of curves up to reparameterization); where the former should be hom-wise invariant for reparameterizations. It seems that there are ways to build such a structure, but the goal is to find a metric where the terms are also "simple" (both for easier analysis, and for effective numerical implementations). This will be hopefully the argument of a forthcoming paper.

It would be nice to replace the $\|h\|_{\mathfrak{r}/l,c}$ with a semimetric (well defined on closed curves) that is hom-wise invariant for reparameterizations and change of base-point. This seems currently harder.

(In a sense, the ultimate goal would be replace all "CW!" with "HW" in the Tab. 1).

Appendix A. Lemmas and proof

A.1 Proof of Lemma 2.4

Proof. Suppose that the orbits are closed in (M, d_M). Let $p, q \in M/G$ be such that $d_{M/G}(p, q) = 0$, by the relation (2.3) this means that there are points $x \in p$ and $y_n \in q$ such that $\lim_n d_M(x, y_n) = 0$; but we assumed that the orbits are closed, hence $x \in q$ so $p = q$.

Vice versa assume $d_{M/G}(p, q) = 0 \Rightarrow p = q$; let $y_n \in q$ be a sequence converging to a point $x \in M$; let $p = [x]$; since $\lim_n d_M(x, y_n) = 0$ then $d_{M/G}(p, q) = 0$ so by our assumption $p = q$ hence $x \in q$; all this implies that the orbit q is closed. □

A.2 Proof of Lemma 2.5

Proof. Let $p_n \in M/G$ be a Cauchy sequence; up to a subsequence assume *wlog* that $d_{M/G}(p_n, p_m) \le 2^{-n}$ for $m \ge n$; choose $y_0 \in p_0$; iteratively define $y_{n+1} \in p_{n+1}$ so that $d_M(y_n, y_{n+1}) \le 2^{1-n}$; then for $m \ge n$;

$$d_M(y_n, y_m) \le \sum_{k=n}^{m-1} d_M(y_k, y_{k+1}) \le \sum_{k=n}^{m-1} 2^{1-k} \le 2^{2-n}$$

for $m \ge n$; hence the sequence y_n converges to a point $\tilde{y} \in M$; in particular, passing to the limit, $d_M(y_n, \tilde{y}) \le 2^{2-n}$; let now $\tilde{p} = [\tilde{x}]$; from the definition of $d_{M/G}$ we obtain that $d_{M/G}(p_n, \tilde{p}) \le 2^{2-n}$. □

A.3 Proof of Theorem 9.4

Proof. Let $N \subseteq H^1([0, 1])$ be the open subset given by

$$N = \{T : [0, 1] \to \mathbb{C} : \forall\theta, T(\theta) \neq 0\}.$$

Consider the map $\Phi(\tilde{e}, f) = T$

$$(\tilde{e}, f) \in H^1 \times H^1/(2\pi\mathbb{Z}) \ \mapsto \ T \in N$$

given by

$$T(\tau) = e^{\tilde{e}(\tau)+if(\tau)};$$

it is well defined and known to be smooth (see [9]).

Fixing $(\tilde{e}, f)$ the directional derivative defines the linear operator $\mathbb{D}(\hat{e}, \hat{f}) = D_{(\tilde{e},f),(\hat{e},\hat{f})}\Phi$ from $H^1 \times H^1$ to $H^1([0,1];\mathbb{C})$; the operator norm of this linear operator $\mathbb{D}$ is bounded from above by $\sqrt{\int_0^1 e^{2|\tilde{e}(\tau)|}\,\mathrm{d}\tau}$. Since the operator is invertible, then by the open mapping theorem the inverse of $\mathbb{D}$ is as well continuous.

We conclude that the map Φ is a diffeomorphism.

Then consider the "integration map"

$$T \in N \ \mapsto \ \tilde{c} \in \mathrm{Imm} \cap \{\tilde{c}(0) = 0\}$$

given by

$$\tilde{c}(\theta) = \int_0^\theta T(\tau)\,\mathrm{d}\tau$$

that is a linear isomorphism from H^1 to $H^2 \cap \{\tilde{c}(0) = 0\}$, and hence a diffeomorphism from N to $\mathrm{Imm} \cap \{\tilde{c}(0) = 0\}$. Lastly, apply the map

$$(v, \tilde{c}) \in \mathbb{R}^2 \times (\mathrm{Imm} \cap \{\tilde{c}(0) = 0\}) \quad \mapsto \quad c \in \mathrm{Imm}$$

given by

$$c(\theta) = v + \tilde{c}(\theta) - \mathrm{avg}_c(\tilde{c})$$

that is easily proved to be a diffeomorphism.

The composition of the above three maps builds the required diffeomorphism. □

Lemma A.1. *Suppose that M is a Riemannian manifold with scalar product $\langle,\rangle_c$ and norm $\|\|_c$ on T_cM; let us call d the induced distance. Let $I \subseteq \mathbb{R}$ be an interval. Let $\gamma \in H^1(I \to M)$ be path whose energy*

$$\mathbb{E}(\gamma) = \int_I \|\dot{\gamma}(t)\|^2_{\gamma(t)}\,\mathrm{d}t$$

is finite. We recall that the length of γ is

$$\mathrm{len}(\gamma) = \int_I \|\dot{\gamma}(t)\|_{\gamma(t)}\,\mathrm{d}t$$

and by Cauchy-Schwarz inequality

$$\mathrm{len}(\gamma) \le \sqrt{\mathbb{E}(\gamma)}\sqrt{|I|}$$

where $|I|$ is the length of the interval. In particular, using this inequality on subintervals, the path γ is Hölder continuous, namely for $t_1, t_2 \in I$

$$d(\gamma(t_1), \gamma(t_2)) \le \sqrt{\mathbb{E}(\gamma)}\sqrt{|t_2 - t_1|} \quad .$$

Appendix B. .. for diffeomorphism groups

We present results for $\mathbb{D}_0$ and Imm. Results for Diff(S^1) and $\mathrm{Imm}_{\mathfrak{f}}$ are similarly proved. We recall that the precise definition of $\mathbb{D}_0$ is in 16.1 in Section 16.1.

Lemma B.1. *For any fixed $\varphi \in \mathbb{D}_0$ the map*

$$L^2([0,1]) \to L^2([0,1]), \quad g \mapsto g \circ \varphi$$

is a linear continuous invertible map. The operator norm is bounded by $\|(\varphi^{-1})'\|_\infty^{1/2} = \max_{s\in[0,1]} \sqrt{(\varphi^{-1})'(s)}$.

Proof. Let $\psi = \varphi^{-1}$ be the inverse. Note that $\varphi, \psi \in C^1([0,1])$. By the change of variable formula[24]

$$\int_0^1 |g(\varphi(t))|^2 \,\mathrm{d}t = \int_0^1 \left|g(s)\right|^2 \psi'(s)\,\mathrm{d}s \le \int_0^1 \left|g(s)\right|^2 \mathrm{d}s \max_{s\in[0,1]} \psi'(s).$$

□

Lemma B.2. *If $\varphi_n, \varphi : [0,1] \to [0,1]$ are C^1 diffeomorphisms and $\varphi_n \to_n \varphi$ in C^1 then $\varphi_n^{-1} \to_n \varphi^{-1}$ in C^1.*

Proof. Let $\psi = \varphi^{-1}$ and $\psi_n = \varphi_n^{-1}$ then

$$\psi_n'(s) - \psi'(s) = \frac{1}{\varphi_n'(\psi_n(s))} - \frac{1}{\varphi'(\psi(s))}$$

but

$$\max_s |\varphi_n'(\psi_n(s)) - \varphi'(\psi_n(s))| = \max_t |\varphi_n'(t) - \varphi'(\psi_n(\varphi(t)))|$$

and $\varphi_n' \to \varphi'$ uniformly, $\psi_n(\varphi(t)) \to t$ uniformly, and $\varphi'(\psi_n(\varphi(t))) \to \varphi'(t)$ uniformly. □

Lemma B.3. *Suppose $g \in L^2 = L^2([0,1])$, $\varphi_n, \varphi : [0,1] \to [0,1]$ are diffemorphisms suppose that $\varphi_n \to_n \varphi$ in C^1 then $g \circ \varphi_n \to_n g \circ \varphi$ in L^2.*

Proof. Let $\varepsilon > 0$. Let $\psi = \varphi^{-1}$ and $\psi_n = \varphi_n^{-1}$ be the inverses, by Lemma B.2 we know that $\psi_n' \to_n \psi'$ uniformly so there is a $L > 0$ such that

$$\max_{s\in[0,1]} \left|\psi'(s)\right| \le L, \quad \forall n \in I\!N \max_{s\in[0,1]} \left|\psi_n'(s)\right| \le L.$$

By density let $f \in C^0([0,1])$ such that $\|g - f\|_{L^2} \le \varepsilon/\sqrt{L}$ and (by uniform continuity of f) let $\delta > 0$ be such that

$$\forall s, t \in [0,1], |s - t| \le \delta \Rightarrow |f(s) - f(t)| \le \varepsilon.$$

[24] See *e.g.* Corollary 5.4.4 in [3]

We know that $\varphi_n \to_n \varphi$ uniformly so there is a $\overline{n}$ large such that $\forall n \geq \overline{n}$ we have $\|\varphi_n - \varphi\|_\infty \leq \delta$ so

$$\forall s \in [0,1], \forall n \geq \overline{n}, |f(\varphi_n(s)) - f(\varphi(s))| \leq \varepsilon$$

hence $\forall n \geq \overline{n}$ we have $\|f \circ \varphi_n - f \circ \varphi\|_{L^2} \leq \varepsilon$. Summarizing given $\varepsilon > 0$ we found $\overline{n}$ such that $\forall n \geq \overline{n}$

$$\|g \circ \varphi_n - g \circ \varphi\|_{L^2} \leq \|g \circ \varphi_n - f \circ \varphi_n\|_{L^2} + \|f \circ \varphi_n - f \circ \varphi\|_{L^2} + \|f \circ \varphi - g \circ \varphi\|_{L^2} \leq 3\varepsilon,$$

where for the first and third term we used Lemma B.1 and for the middle term we used the previous argument. $\square$

Lemma B.4. *Let $g_n, g \in L^2 = L^2([0,1])$ and $\varphi_n, \varphi : [0,1] \to [0,1]$ diffemorphisms suppose that $g_n \to g$ in L^2, and that $\varphi_n \to_n \varphi$ in C^1: then $g_n \circ \varphi_n \to_n g \circ \varphi$ in L^2.*

Proof.

$$\|g_n \circ \varphi_n - g \circ \varphi\|_{L^2} \leq \|g_n \circ \varphi_n - g \circ \varphi_n\|_{L^2} + \|g \circ \varphi_n - g \circ \varphi\|_{L^2}$$

and we use Lemmas B.1 and B.3. $\square$

Lemma B.5. *Let $\varphi_n, \varphi \in \mathbb{D}_0$ diffemorphisms; let $\psi = \varphi^{-1}$ and $\psi_n = \varphi_n^{-1}$ be the inverses. If $\varphi_n \to \varphi$ in H^2 then $\psi_n \to \psi$ in H^2.*

Proof. As in the proof of Lemma B.7, we know that

$$0 = (\psi \circ \varphi)'' = (\psi'' \circ \varphi)(\varphi')^2 + (\psi' \circ \varphi)\varphi''$$

(almost everywhere, and in the sense of distributions), so

$$\varphi''(t) = -(\psi'' \circ \varphi)(\varphi')^3$$

and similarly for ψ_n, φ_n. By the previous Lemma we have that $\psi_n'' \circ \varphi_n \to \psi'' \circ \varphi$ in L^2 and we know that $\varphi_n' \to \varphi'$ uniformly so $\varphi_n'' \to \varphi''$ in L^2. $\square$

Consequently

Lemma B.6. *The family of smooth diffeomorphisms is dense in $\mathbb{D}_0$.*

Lemma B.7. *The action*

$$\varphi, c \in \mathbb{D}_0 \times \mathrm{Imm} \mapsto c \circ \varphi \in \mathrm{Imm}$$

is well defined and continuous.

Proof. We sketch the proof. We know that $c, \varphi \in C^1$, so $(c \circ \varphi)' = (c' \circ \varphi)\varphi'$. The function c' is absolutely continuous, and φ is C^1 and monotone, so $c' \circ \varphi$ is absolutely continuous (this is exercise 5.8.59 in [3]). Consequently $(c' \circ \varphi)\varphi'$ that is the product of two absolutely continuous functions, is an absolutely continuous function, and its derivative (almost everywhere, and in the sense of distributions) is $(c \circ \varphi)'' = (c'' \circ \varphi)(\varphi')^2 + (c' \circ \varphi)\varphi''$. This derivative is in L^2 since $c'', \varphi'' \in L^2$ and all other terms are bounded and continuous. In the above we used Corollary 5.5.3, Theorems 5.3.6, 5.4.2 and Corollary 5.4.3 in [3].

We now prove continuity. Suppose that $c_n \to_n c$ in H^2 and $\varphi_n \to \varphi$ in $\mathbb{D}_0$. We consider the second order term

$$\|(c_n \circ \varphi_n)'' - (c \circ \varphi)''\|_{L^2} \leq \|(c \circ \varphi_n)'' - (c \circ \varphi)''\|_{L^2} + \|(c_n \circ \varphi_n)'' - (c \circ \varphi_n)''\|_{L^2}$$

for the first term we write

$$\begin{array}{ccccccc} (c\circ\varphi_n)'' = & (c''\circ\varphi_n) & (\varphi_n')^2 & + & (c'\circ\varphi_n) & (\varphi_n'') \\ & \downarrow \text{in } L^2 & \downarrow \text{unif.} & & \downarrow \text{unif.} & \downarrow \text{in } L^2 \\ (c\circ\varphi)'' = & (c''\circ\varphi) & (\varphi')^2 & + & (c'\circ\varphi) & (\varphi'') \end{array}$$

where for the first arrow we use Lemma B.3. For the second term we write

$$\begin{aligned} \|(c_n\circ\varphi_n)'' - (c\circ\varphi_n)''\|_{L^2} &\le \|(c_n''\circ\varphi_n) - (c''\circ\varphi_n)\|_{L^2}\|\varphi_n'\|_\infty^2 + \|(c_n'\circ\varphi_n) - (c'\circ\varphi_n)\|_\infty\|\varphi_n''\|_{L^2} \le \\ &\le \|c_n'' - c''\|_{L^2}\|(\varphi_n^{-1})'\|_\infty^2\|\varphi_n'\|_\infty^2 + \|c_n' - c'\|_\infty\|\varphi_n''\|_{L^2}, \end{aligned}$$

where we use Lemma B.1.

Reasoning similarly for lower order terms we obtain that $c_n\circ\varphi_n \to_n c\circ\varphi$ in H^2. □

Lemma B.8. *The group multiplication*

$$\varphi,\psi \in \mathbb{D}_0\times\mathbb{D}_0 \mapsto \psi\circ\varphi \in \mathbb{D}_0$$

is well defined and continuous.

Proof. Suppose that $\psi_n \to_n \psi$ and $\varphi_n \to \varphi$ in $\mathbb{D}_0$. By the previous lemma $\psi_n\circ\varphi_n \to_n \psi\circ\varphi$ in H^2. But also $\psi_n^{-1} \to_n \psi^{-1}$ and $\varphi_n^{-1} \to \varphi^{-1}$ in $\mathbb{D}_0$ so $(\psi_n\circ\varphi_n)^{-1} \to_n (\psi\circ\varphi)^{-1}$ in H^2. So $\psi_n\circ\varphi_n \to_n \psi\circ\varphi$ in $\mathbb{D}_0$. □

Lemma B.9. *Given $f,g \in \mathrm{Imm}$ and $\varphi : [0,1] \to [0,1]$ a C^1 diffeomorphism, if $f = g\circ\varphi$ then $\varphi \in \mathbb{D}_0$. Similarly for closed curves.*

Proof. We consider an open interval I where one of the two components (g_1, g_2) of g is monotone, say the first; then on that interval we can write $f_1\circ(g_1)^{-1} = \varphi$ and reasoning as in the beginning the proof of Lemma B.7 we obtain that $\varphi|_I \in H^2$. Since the curves are immersed then $[0,1]$ is covered by such intervals, hence $\varphi \in H^2([0,1])$. By symmetry we also obtain $\varphi^{-1} \in H^2([0,1])$. □

Appendix C. Group actions in log-transform

For convenience we write the actions of usual groups but in log-transform coordinates.

The scheme is as follows. Given a path $\gamma : [0,1] \to M$ in the manifold of curves, and a path $g : [0,1] \to G$ in the group G, we will write the general formula for $\tilde\gamma = g\gamma$ in log-coordinates, as well many derivatives. Then supposing that $\gamma(t) = c$ is constant, then $\dot{\tilde\gamma} = \dot g c$ will be the infinitesimal action. Instead supposing that $g(t)$ is constant, we will obtain formulas that may be used to check that a semimetric is invariant for the action of G.

In the following C and $\tilde C$ are homothopies, and E, F and $\tilde E, \tilde F$ are their representations in log–coordinates (see 8.1), so that

$$C' = e^{E+iF}, \quad \tilde C' = e^{\tilde E + i\tilde F}$$

(where we identified the plane $\mathbb{R}^2$ with the complex plane $\mathbb{C}$).

- **Euclidean group.** Suppose that a homotopy C is mapped[25] to

$$\tilde C = e^{l(t)+i\psi(t)} C + \beta(t); \tag{C.1}$$

[25] In (C.1) we identify the plane with the complex plane. Note that (C.1) is the same as (14.3).

(as in Eq. (14.3)) where $l(t) \in \mathbb{R}$ is the rescaling and $\psi(t) \in \mathbb{R}$ is the rotation. Then

$$\tilde{E} = E + l \tag{C.2}$$
$$\tilde{F} = F + \psi \tag{C.3}$$
$$\dot{\tilde{E}} = \dot{E} + \dot{l} \tag{C.4}$$
$$\dot{\tilde{F}} = \dot{F} + \dot{\psi} \tag{C.5}$$
$$\tilde{E}' = E' \tag{C.6}$$
$$\tilde{F}' = F' \tag{C.7}$$

and so on. In particular if l, ψ do not depend on time then $\dot{\tilde{E}} = \dot{E}$, $\dot{\tilde{F}} = \dot{F}$ and so on.
Vice versa if we set $\dot{E} = \dot{F} = 0$ and also $l(0) = 0 = \psi(0)$ then $\dot{l}, \dot{\psi}$ are in the Lie algebra hence those above reduce to the formulas for the infinitesimal action of the Euclidean group on curves in log-transform.

- **Reparameterization.** For the case of closed curves, suppose that $\varphi(t,\theta) : [0,1] \times S^1 \to S^1$ is smooth and $\varphi(t,\cdot)$ is a diffeomorphism of S^1 for each t. Similarly for the case of open curves $\varphi(t,\theta) : [0,1] \times [0,1] \to [0,1]$ is smooth and $\varphi(t,\cdot)$ is a diffeomorphism of $[0,1]$ for each t. Suppose that a homotopy C is mapped by reparameterization to $\tilde{C}(t,\theta) = C(t,\varphi(t,\theta))$ then

$$\tilde{C}'(t,\theta) = C'(t,\varphi(t,\theta))\varphi'(t,\theta) \tag{C.8}$$

so

$$\tilde{E} = E + \log \varphi' \tag{C.9}$$
$$\tilde{F} = F \tag{C.10}$$
$$\dot{\tilde{E}} = \dot{E} + E'\dot{\varphi} + \dot{\varphi}'/\varphi' \tag{C.11}$$
$$\dot{\tilde{F}} = \dot{F} + F'\dot{\varphi} \tag{C.12}$$
$$\tilde{E}' = E'\varphi' + \varphi''/\varphi' \tag{C.13}$$
$$\tilde{F}' = F'\varphi' \tag{C.14}$$
$$\dot{\tilde{E}}' = \dot{E}'\varphi' + E''\varphi'\dot{\varphi} + E'\dot{\varphi}' + \frac{(\dot{\varphi}''\varphi' - \dot{\varphi}'\varphi'')}{(\varphi')^2} \tag{C.15}$$
$$\dot{\tilde{F}}' = \dot{F}'\varphi' + F''\varphi'\dot{\varphi} + F'\dot{\varphi}' \tag{C.16}$$

where $\tilde{E}, \tilde{F}$ are evaluated at (t,θ) while E, F are evaluated at $(t,\varphi(t,\theta))$.

Remark C.1. We recall the problem already described in 8.2. Consider the case of closed curves, then term $F(t,\theta)$ (that was defined for $\theta \in [0,1]$ in 8.1) does not extend periodically; indeed we have $F(t,1) - F(t,0) = 2\pi k$ with k the rotation index. So the relation (C.10) should be used with care, and considered valid only when $\varphi(t,0) = 0$, that is, if $\varphi(t,\cdot)$ is a diffeomorphism of $[0,1]$. All derivatives of F instead can be extended periodically, so all other relations are safe to use.

There are two important subcases.

– One subcase is when $\varphi(0,\theta) = \theta$, so $\xi(\theta) \stackrel{\text{def}}{=} \dot{\varphi}(0,\theta)$ is in the Lie algebra of the reparameterization group. Setting $t = 0$ and $C(t,\theta) = c(\theta)$ then we obtain the formula for the infinitesimal action:

$$\dot{\tilde{E}} = E'\xi + \xi' \tag{C.17}$$
$$\dot{\tilde{F}} = F'\xi \tag{C.18}$$

$$\dot{\tilde{E}}' = E''\xi + E'\xi' + \xi'' \tag{C.19}$$
$$\dot{\tilde{F}}' = F''\xi + F'\xi' \tag{C.20}$$

– Another interesting case is when φ does not depend on t

$$\tilde{E} = E + \log\varphi' \tag{C.21}$$
$$\tilde{F} = F \tag{C.22}$$
$$\dot{\tilde{E}} = \dot{E} \tag{C.23}$$
$$\dot{\tilde{F}} = \dot{F} \tag{C.24}$$
$$\tilde{E}' = E'\varphi' + \varphi''/\varphi' \tag{C.25}$$
$$\tilde{F}' = F'\varphi' \tag{C.26}$$
$$\dot{\tilde{E}}' = \dot{E}'\varphi' \tag{C.27}$$
$$\dot{\tilde{F}}' = \dot{F}'\varphi' \tag{C.28}$$

This shows that the semimetric (14.9) is curve-wise reparameterization invariant.

- **Fixed point reparameterization** $\mathbb{D}_0$, the formulas are as above but we assume that $\varphi(t,k) = k$ and $\xi(k) = 0$ for any k integer.
- **Change of base-point**, for closed curves. In this case we assume that $\varphi(t,\theta) = \theta + a(t)$ so $\xi = \dot{a}$, and is contant in θ. Setting $t = 0$ and $C(t,\theta) = c(\theta)$ then we obtain the formula for the infinitesimal action:

$$\dot{\tilde{E}} = E'\dot{a} \tag{C.29}$$
$$\dot{\tilde{F}} = F'\dot{a} \tag{C.30}$$
$$\dot{\tilde{E}}' = E''\dot{a} \tag{C.31}$$
$$\dot{\tilde{F}}' = F''\dot{a} \tag{C.32}$$

- **Curve curling**. F is mapped to $F + \alpha(\theta)$ with $\alpha(\theta) \in \mathbb{R}$.

Acknowledgements. The author thanks Prof. Luigi Ambrosio and Dr. Luciano Mari (Scuola Normale Superiore) for the useful discussions. This work was supported by the PRIN *2015PA5MP7 (Calcolo delle Variazioni)*.

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