EXISTENCE RESULTS FOR A SUPER-LIOUVILLE EQUATION ON COMPACT SURFACES

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Abstract. We are concerned with a super-Liouville equation on compact surfaces with genus larger than one, obtaining the first non-trivial existence result for this class of problems via min-max methods. In particular we make use of a Nehari manifold and, after showing the validity of the Palais-Smale condition, we exhibit either a mountain pass or linking geometry.

Keywords: super-Liouville equation, existence results, min-max methods.

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1. introduction

The Liouville equation in two dimensions, which has the form

$$-\Delta u = \widetilde{K} e^{2u} - K, \tag{1.1}$$

for some given functions $K, \widetilde{K}$ on a surface M, has been extensively studied and has wide applications in geometry and physics. A typical example is the prescription of curvature. Let g be a Riemannian metric on a surface M with Gaussian curvature $K = K_g$ and let $\widetilde{K}$ be a given function on M. The question is whether there exists a functions $u \in C^\infty(M)$ such that the conformal metric $\widetilde{g} = e^{2u} g$ has Gaussian curvature $\widetilde{K}$, see e.g. [10, 28].

Since the Gaussian curvature for $\widetilde{g}$ is given by $e^{-2u}(K_g - \Delta_g u)$, the problem is equivalent to the solvability of equation (1.1). Observe that the conformal factor u within the conformal class of $[g]$ can be found as a critical point of the following functional:

$$I(u) := \int_M \left(|\nabla u|^2 + 2K_g u - \widetilde{K} e^{2u} \right) \mathrm{d}v_g.$$

When M is a closed Riemann surface, which is the case we are interested in for this paper, the function $\widetilde{K}$ has to satisfy the Gauss–Bonnet formula with respect to the new metric $\widetilde{g}$. When $\widetilde{K}$ is constant with the sign compatible with the Gauss–Bonnet formula, the equation is always solvable, according to the *uniformization theorem*. For non-constant $\widetilde{K}$, though not being totally solved, we have a good understanding of the problem in most cases, see e.g. [38, Chapter 5] and [4, Chapter 6].

More recently, equation (1.1) has been studied in the context of hyperelliptic curves and of the Painlevé equations, see [9] and [11], respectively.

Equation (1.1) plays also an important role in mathematical physics. On one hand, it arises in Electroweak and Chern-Simons self-dual vortices, see [40, 44, 45]. On the other hand, it appears in the Liouville field theory with applications to string theory, see [33, 35, 36]. See also [39] for a recent connection between (1.1) and the Hawking mass.

Motivated by the supersymmetric extension of the Liouville theory, the following so-called *super-Liouville functional*:

$$\widetilde{I}(u, \psi) := \int_M \left(|\nabla u|^2 + 2K_g u - e^{2u} + 2 \left\langle (\not{D} + e^u)\psi, \psi \right\rangle \right) \mathrm{d}v_g,$$

was studied in [22], where u is a function on M, ψ is a spinor field on M, and $\not{D}$ is the Dirac operator acting on spinors ψ, see Subsection 2.1 for precise definitions. In a series of works they performed blow-up analysis and studied the compactness of the solution spaces under weak assumptions and in various setting; see e.g. [22, 23, 24, 25] and the references therein. For the role of the super-Liouville equations in physics we refer to [1, 12, 37]. One should note that the sign conventions adopted above are adapted to the sphere case.

In this paper we consider the problem posed on a closed Riemann surface of genus $\gamma > 1$. In this case the coefficients in the action functional need to be adapted to the Gauss–Bonnet formula. Let g be a Riemannian metric compatible with the given complex structure. We are going to consider the following functional:

$$J_\rho(u,\psi) \coloneqq \int_M \left(|\nabla u|^2 + 2K_g u + e^{2u} + 2\left\langle (\not{D} - \rho e^u)\psi, \psi \right\rangle \right) \mathrm{d}v_g, \tag{1.2}$$

where the parameter ρ is a positive constant. We are adopting a different notation from that in [22], making our choice compatible with equation (1.1). The Euler–Lagrange equation for J_ρ is

$$\begin{cases} \Delta_g u = & e^{2u} + K_g - \rho e^u |\psi|^2, \\ \not{D}_g \psi = & \rho e^u \psi, \end{cases} \tag{EL}$$

which takes the name of *super-Liouville equations*. The system (EL) clearly admits the *trivial solution* $(u_*, 0)$, where u_* satisfies

$$-\Delta u_* = -e^{2u_*} - K_g$$

and whose existence is given by the uniformization theorem. This is also a *ground state solution* in the sense that it has minimal critical level: this follows from the fact that the spinorial part does not affect the critical levels, while the scalar component of the functional is coercive and convex. The latter properties also yield uniqueness of such a trivial solution. The aim of the present paper is to find a solution with non-zero spinor part, a so-called *non-trivial solution*.

Notice that, if (u,ψ) solves (EL), then the pair $(u, \mathfrak{m}(\omega)\psi)$, where $\mathfrak{m}(\omega) \equiv \mathfrak{m}(e_1)\mathfrak{m}(e_2)$ denotes the multiplication by the real volume element $\omega = e_1 \cdot e_2$ (see e.g. [29]), satisfies

$$|\mathfrak{m}(\omega)\psi|^2 = |\psi|^2, \qquad \not{D}(\mathfrak{m}(\omega)\psi) = -\omega \not{D}\psi.$$

Therefore the solutions of (EL) corresponds bijectively to solutions of

$$\begin{cases} \Delta_g u = & e^{2u} + K_g - \rho e^u |\phi|^2, \\ \not{D}_g \phi = & -\rho e^u \phi. \end{cases}$$

The choice of the sign in the Dirac equation may be physically relevant. This also shows some generality of our treatment.

Conformal symmetry and reduction to uniformized case. System (EL) admits a conformal symmetry in the following sense. Suppose that (u,ψ) is a solution of (EL), let $v \in C^\infty(M)$ and consider the metric $\widetilde{g} \coloneqq e^{2v} g$. There exists an isometric isomorphism $\beta\colon S_g \to \widetilde{S}_{\widetilde{g}}$ of the spinor bundles corresponding to different metrics such that

$$\widetilde{\not{D}}_{\widetilde{g}} \left(e^{-\frac{v}{2}} \beta(\psi) \right) = e^{-\frac{3}{2}v} \beta(\not{D}_g \psi), \tag{1.3}$$

see e.g. [13, 16], where we are using the notation from [27]. Thus the pair

$$\begin{cases} \widetilde{u} = u - v, \\ \widetilde{\psi} = e^{-\frac{u}{2}} \beta(\psi), \end{cases}$$

solves the system

$$\begin{aligned}\Delta_{\widetilde{g}}\widetilde{u} &= e^{-2v}\Delta_g(u-v) = e^{-2v}(e^{2u}+K_g-\rho e^u|\psi|^2-\Delta_g v)\\ &= e^{2(u-v)}+e^{-2v}(K_g-\Delta_g v)-\rho e^{u-v}|e^{-\frac{v}{2}}\beta(\psi)|^2\\ &= e^{2\widetilde{u}}+K_{\widetilde{g}}-\rho e^{\widetilde{u}}|\widetilde{\psi}|^2,\\ \widetilde{\not{D}}_{\widetilde{g}}\widetilde{\psi} &= \rho e^{-\frac{3}{2}v}\beta(e^u\psi)=\rho e^{u-v}\left(e^{-\frac{1}{2}v}\beta(\psi)\right)=\rho e^{\widetilde{u}}\widetilde{\psi},\end{aligned}$$

analogous to (EL). Therefore, we can work with a convenient background metric inside the given conformal class. W.l.o.g., recalling that the genus is larger than one, we assume that the background metric g_0 is uniformized, meaning that $K_{g_0}\equiv -1$: notice that such a metric is unique. In this case the trivial solution is given by $(0,0)$: the main result of the paper is the existence of a non-trivial min-max solution obtained via a variational approach.

Theorem 1.1. *Let M be a closed Riemann surface of genus $\gamma>1$ with Riemannian metric g. Let $g_0\in[g]$ be a conformal uniformized metric, i.e. $K_{g_0}\equiv -1$, and suppose that the spin structure is chosen so that $0\notin \mathrm{Spec}(\not{D}_{g_0})$. Then for any $\rho\notin\mathrm{Spec}(\not{D}_{g_0})$, there exists a non-trivial solution to* (EL).

We stress that this is the first non-trivial existence result for this class of problems. Moreover, observe that by (1.3) $\dim\ker(\not{D}_g)$ is a conformal invariant, and the condition $0\notin\mathrm{Spec}(\not{D}_{[g]})$ is valid for many spin structures and conformal structures, as explained in Section 2. This condition is used to get equivalent norms on the suitable Sobolev spaces of spinors in terms of Dirac operators.

Remark 1.1. *Note that the spinor bundle $S\to M$ admits global automorphisms, e.g. the quaternionic structures, which form a group. These are parallel with respect to ∇^s and commute with the Clifford multiplications by tangent vectors, see* [26, Sect. 2]. *The functional J_ρ is thus invariant under the actions of such isometries. It follows that any non-trivial solution lies in a smooth family (of dimension no less than three) of non-trivial solutions. Given a solution (U,Ψ), an intuitive example is the antipodal solution $(U,-\Psi)$, which is in the orbit of the quaternionic structure group actions.*

Concerning the case of genus one, i.e. when the base surface is a torus, the problem might not be well-defined. Indeed, if we take $\widetilde{K}$ and K to be zero, then the system (EL) has only trivial solutions of the form $(a,0)$ where $a\in\mathbb{R}$. In this situation it might be interesting to consider the case of sign-changing $\widetilde{K}$, as in [28] for the prescribed Gaussian curvature problem. Meanwhile in the sphere case, where both $\widetilde{K}$ and K should be 1, the functional turns out to be even more strongly indefinite, and admits neither the classical mountain pass nor the linking geometry, see for example [21] for a recent result in this direction.

The main difficulty in studying (EL) is that the Dirac operator is strongly indefinite: the spectrum of $\not{D}$ is real and symmetric with respect to the origin. The classical theory for variational problems involving Laplacians or Schrödinger operators, where the positive parts usually dominates the behavior of the functional, fails to work for Dirac type actions. There were methods developed for general strongly indefinite variational problems, see e.g. [6, 7, 17], but they are not directly applicable to Dirac operators. Dirac operators usually relates more closely to the geometry and topology of the spin manifolds. Recently several attempts have been made to attack such problems. With suitable nonlinearities as perturbation adding to the geometric equations, T. Isobe made remarkable progress in adapting the classical theory of calculus of variations to the Dirac setting [18, 19, 20]. Combined with the methods of Rabinowitz-Floer homology, A. Maalaoui and V. Martino also obtained existence results of some nonlinear Dirac type equations, see [30, 31, 32] and the references therein. In the case of super-Liouville equations we have to deal with an exponential nonlinearity, which does not fit in the above settings. Moreover, we are directly facing a geometric problem without auxiliary nonlinear perturbations, which is usually harder to deal with.

The article is organized in the following way. In the second section we introduce some preliminaries in spin geometry and discuss existence of harmonic spinors depending on the genus and on the conformal class. We also introduce suitable Sobolev spaces to work with and the Moser-Trudinger inequality. In the third section we tackle the strong-indefiniteness of the functional by building a natural constraint which defines a generalized Nehari manifold N. We then verify the Palais–Smale condition for $J_\rho|_N$ by showing first some a-priori bounds and then proving strong subsequential convergence. For suitable ρ we finally show either mountain pass or linking geometry on the Nehari manifold which yield the existence of a min-max critical point for J_ρ: the details of this construction are given in the last section.

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2. Preliminaries

We will assume some background in spin geometry and Sobolev spaces. For detailed material one can refer to [3, 13, 14, 29].

2.1. Spinor bundles and Dirac operator.

Here we introduce our setting and fix the notation. Let M be a closed Riemann surface with a fixed conformal structure and of genus γ. Let g be a Riemannian metric in the given conformal class and denote the Gaussian curvature by K_g. The orthonormal frame bundle $P_{\mathrm{SO}}(M,g) \to M$ is then a principal SO(2) bundle. Let $\mathrm{Spin}(2) = U(1) \to \mathrm{SO}(2)$ be the two-fold covering of the circle. A *spin structure* is given by a principal Spin(2) bundle $P_{\mathrm{Spin}}(M,g) \to M$ together with an equivariant two-fold covering

$$P_{\mathrm{Spin}}(M,g) \to P_{\mathrm{SO}}(M,g).$$

In dimension two such double coverings always exist; moreover they are in one-to-one correspondence with the elements in $H^1(M;\mathbb{Z}_2)$, see e.g. [29, Chapter 2]. This cohomology group has cardinality $2^{2\gamma}$.

Let $S \equiv S_g \to M$ be the associated spinor bundle with a real Riemannian structure g^s and induced spin connection ∇^s: sections of S are called *spinors*. Recall that the *Dirac operator* $\not{D}$ acting on spinors is defined as the composition of the following chain

$$\Gamma(S) \xrightarrow{\nabla^s} \Gamma(T^*M \otimes S) \xrightarrow{\cong} \Gamma(TM \otimes S) \xrightarrow{\mathfrak{m}} \Gamma(S),$$

where the second isometric isomorphism is given by the identification via the metric g, the third arrow $\mathfrak{m}$ denotes the Clifford multiplication, and the $\mathrm{End}(S)$-valued map $\mathfrak{m}\colon TM \to \mathrm{End}(S)$ satisfies the following Clifford relation:

$$\mathfrak{m}(X)\mathfrak{m}(Y) + \mathfrak{m}(Y)\mathfrak{m}(X) = -2g(X,Y), \qquad \forall X, Y \in \Gamma(TM).$$

Later, for simplicity, we will write $X \cdot \psi$ for $\mathfrak{m}(X)\psi$, where $X \in \Gamma(TM)$ and $\psi \in \Gamma(S)$. In terms of a local orthonormal frame $(e_i)_{i=1,2}$ we then have the *Dirac operator*

$$\not{D}\psi = \sum_i \mathfrak{m}(e_i)\nabla^s_{e_i}\psi, \quad \forall \psi \in \Gamma(S).$$

This is a self-adjoint elliptic operator of first order, and is a generalization of the Cauchy–Riemann operator in complex analysis. For a local expression of the Dirac operator in a local flat chart, see [25, Page 251]. The Dirac operator $\not{D}$ has a finite-dimensional kernel consisting of *harmonic spinors*. The dimension of the space of harmonic spinors is a conformal invariant, but it depends on the choice of spin structures and the conformal structures in general. The Bochner-Lichnerowicz formula

$$\not{D}^2 = (\nabla^s)^*\nabla^s + \frac{\mathrm{Scal}}{4}$$

where $(\nabla^s)^*$ denotes the metric adjoint of the spin connection and Scal is the scalar curvature of (M,g), implies that there is no non-trivial harmonic spinor if $\mathrm{Scal} \geq 0$ and $\mathrm{Scal} \not\equiv 0$. In particular, there is no harmonic spinor on the 2-sphere with arbitrary metric (since there is only one conformal structure on the 2-sphere). However, when the genus γ is greater than or equal to 1, there might exist non-trivial harmonic spinors for some choice of spin structures. The dimensions of the spaces of harmonic spinors have been computed in literature e.g. [16, 5, 8]. We summarize some facts here to have a picture of the different cases.

2.2. **Examples of Riemann surfaces with no non-trivial harmonic spinors.** Here we give some examples of Riemann surfaces having negative Euler characteristic $2\pi\chi(M) = 4\pi(1-\gamma) < 0$ but admitting no non-trivial harmonic spinors.

Any element $\alpha \in H^1(M, \mathbb{Z}_2)$ determines a spin structure $\xi(\alpha)$, as well as a holomorphic line bundle $\mathscr{L}_\alpha$ such that $\mathscr{L}_\alpha \otimes_{\mathbb{C}} \mathscr{L}_\alpha = \mathscr{K}_M$, where $\mathscr{K}_M$ denotes the canonical line bundle of M, see e.g. [16, 29]. Denote by $\mathscr{O}(\mathscr{L}_\alpha)$ the sheaf of germs of holomorphic sections of the holomorphic line bundle $\mathscr{L}_\alpha$, and set $h^0_{\alpha,g} = \dim H^0(M, \mathscr{O}(\mathscr{L}_\alpha))$. If the associated spinor bundle $S \equiv S(\alpha, g)$ admits a space of harmonic spinors of dimension $h_{\xi(\alpha),g}$, then

$$h_{\xi(\alpha),g} = 2h^0_{\alpha,g}.$$

It is known that, for a Riemann surface M of genus γ, there are precisely $2^{\gamma-1}(2^\gamma + 1)$ spin structures α on M for which $h^0_{\alpha,g}$ is an even number (such spin structures are called *even spin structures* on M), and for the other $2^{\gamma-1}(2^\gamma - 1)$ spin structures the number $h^0_{\alpha,g}$ is odd (*odd spin structures*).

For $\gamma = 1$ M is topologically a torus, and for any conformal structure $[g]$ we have four spin structures: three even spin structures with no non-trivial harmonic spinors and one odd spin structure (the trivial one $\alpha = 0$) with one-dimensional space of positive harmonic spinors (hence $h_{\xi(0),g} = 2$).

For $\gamma = 2$ the description is similar, namely for any conformal structure $[g]$ there are ten even spin structures with no non-trivial harmonic spinors and six odd spin structures with one-dimensional space of positive harmonic spinors (hence $h_{\xi(\alpha),g} = 2$).

These are the known cases where the dimension of $\ker(D)$ is independent of the choice of metric g (i.e. the choice of the Riemann surface structure on M). When the genera become larger, the dimension of the kernels generally depends on the conformal class. Even in this case we still have many examples where there are no non-trivial harmonic spinors.

Recall that a hyperelliptic Riemann surface is a complex projective curve admitting a rational surjective map onto $\mathbb{C}P^1$ which is 2-to-1 up to a finite set of branching points. All Riemann surfaces of genera $\gamma \leq 2$ are hyperelliptic, while there exist non-hyperelliptic surfaces of all genera $\gamma \geq 3$.

For the hyperelliptic case, C. Bär [5] showed that the spin structures correspond one-to-one to the pairwise inequivalent square roots of the canonical divisor, and in terms of a suitably defined *weight* of the divisors, he also clarified the dimensions h^0 of the kernels:

(1) if M is hyperelliptic with $\gamma = 2k + 1$,
   - there is exactly one spin structure of weight $\gamma - 1$ and in this case $h^0 = \frac{\gamma+1}{2} = k + 1$;
   - for $w = 1, 3, 5, \cdots, \gamma - 2$, there are exactly $\binom{2\gamma+2}{\gamma-w}$ spin structures of weight w and in this case $h^0 = \frac{w+1}{2}$;
   - there are exactly $\binom{2\gamma+1}{\gamma}$ spin structures of weight -1 and in this case $h^0 = 0$;

(2) if M is hyperelliptic with $\gamma = 2k$,
   - there is exactly $2\gamma + 2$ spin structure of weight $\gamma - 1$ and in this case $h^0 = [\frac{\gamma+1}{2}] = k$;
   - for $w = 1, 3, 5, \cdots, \gamma - 1$, there are exactly $\binom{2\gamma+2}{\gamma-w}$ spin structures of weight w and in this case $h^0 = \frac{w+1}{2}$;
   - there are exactly $\binom{2\gamma+1}{\gamma}$ spin structures of weight -1 and in this case $h^0 = 0$;

For non-hyperelliptic surfaces, there are also known examples where the dimensions of kernels are computed.

For a genus $\gamma = 3$ non-hyperelliptic surface, among the $2^{2\gamma} = 64$ spin structures there are 28 odd ones with $h^0 = 1$ and 36 even ones with $h^0 = 0$.

The case for $\gamma = 4$ non-hyperelliptic surfaces is different: there are in total $2^{2\gamma} = 256$ spin structures, 120 of them are odd with $h^0 = 1$, and for the other 136 even spin structures, one of the followings may happen:

(I) there exists a unique even spin structure with $h^0 = 2$, while the other 135 even spin structures have $h^0 = 0$;

(II) all the 136 spin structures have $h^0 = 0$.

A non-hyperelliptic Riemann surface is called of type (I) or (II) if it satisfies the corresponding above conditions. Both classes are non-empty.

2.3. **Sobolev spaces for spinors.** The spinor bundle $S = S_g$ has a Riemannian structure g^s and a spin connection ∇^s induced from the Levi-Civita connection. Then we can define the usual Sobolev spaces with integer differentiability, namely $W^{k,p}(S)$ consists of the spinors whose k-th covariant derivatives are in L^p for $k \in \mathbb{N}$ and $p \in [1, +\infty]$ and $W^{-k,q}(S) := (W^{k,p}(S))^*$ where q is the Hölder conjugate of p. Here we will also consider fractional Sobolev exponents in the sequel.

Recall that $\not{D} = \not{D}_g$ is a first order elliptic operator which is essentially self-adjoint. The spectrum $\mathrm{Spec}(\not{D})$ is discrete and consists of real eigenvalues, $\mathrm{Spec}_0(\not{D}) \cup \{\lambda_k\}_{k\in\mathbb{Z}\setminus\{0\}}$, where $\mathrm{Spec}_0(\not{D})$ stands for the zero element in the spectrum (or the empty set) while the lambda's are the non-zero eigenvalues, indexed by $\mathbb{Z}_* \equiv \mathbb{Z}\setminus\{0\}$ in an increasing order (in absolute value) and counted with multiplicities:

$$-\infty \leftarrow \cdots \leq \lambda_{-l-1} \leq \lambda_{-l} \leq \cdots \leq \lambda_{-1} \leq 0 \leq \lambda_1 \leq \cdots \leq \lambda_k \leq \lambda_{k+1} \leq \cdots \to +\infty.$$

Moreover, the spectrum is symmetric with respect the the origin when $\dim M = 2$. Let φ_k be the eigenspinor corresponding to λ_k, $k \in \mathbb{Z}_*$ with $\|\varphi_k\|_{L^2(M)} = 1$, and let $\varphi_{0,j}$, $1 \leq j \leq h^0$, be an orthonormal basis of $\ker(\not{D})$. These together form a complete orthonormal basis of $L^2(S)$: any spinor $\psi \in \Gamma(S)$ can be expressed in terms of this basis as

$$\psi = \sum_{k\in\mathbb{Z}_*} a_k\varphi_k + \sum_{1\leq j\leq h^0} a_{0,j}\varphi_{0,j}, \tag{2.1}$$

and the Dirac operator acts as

$$\not{D}\psi = \sum_{k\in\mathbb{Z}_*} \lambda_k a_k \varphi_k.$$

For any $s > 0$, the operator $|\not{D}|^s \colon \Gamma(S) \to \Gamma(S)$ is defined as

$$|\not{D}|^s\psi = \sum_{k\in\mathbb{Z}_*} |\lambda_k|^s a_k\varphi_k,$$

provided that the right-hand side belongs to $L^2(S)$. The domain of $|\not{D}|^s$ is

$$H^s(S) := \left\{ \psi \in L^2(S) \mid \int_M \left\langle |\not{D}|^s\psi, |\not{D}|^s\psi \right\rangle \mathrm{d}v_g < \infty \right\},$$

which is a Hilbert space with inner product

$$\langle\psi, \phi\rangle_{H^s} = \langle\psi, \phi\rangle_{L^2} + \left\langle |\not{D}|^s\psi, |\not{D}|^s\phi \right\rangle_{L^2}.$$

For $s = k \in \mathbb{N}$, $H^k(S) = W^{k,2}(S)$ and the above norm is equivalent to the Sobolev $W^{k,2}$ norm. For $s < 0$, $H^s(S)$ is by definition the dual space of $H^{-s}(S)$.

Since S has finite rank, the general theory for Sobolev's embeddings on closed manifold continues to hold here. In particular, for $0 < s < 1$ and $q \leq \frac{2}{1-s}$, we have the continuous embeddings

$$H^s(S) \hookrightarrow L^q(S).$$

Furthermore, for $q < \frac{2}{1-s}$ the embedding is compact, see e.g. [3] for more details.

We will mainly be interested in the case $s = \frac{1}{2}$, for which $\frac{2}{1-s} = 4$. This is the largest space on which the Dirac action of the form

$$\psi \mapsto \int_M \left\langle \not{D}\psi, \psi \right\rangle \mathrm{d}v_g$$

is well-defined. Note that for $\psi \in H^{\frac{1}{2}}(S)$ we have $\not{D}\psi \in H^{-\frac{1}{2}}(S)$, which is defined in the distributional sense. Thus we can define the duality pairing

$$\left\langle \not{D}\psi, \psi \right\rangle_{H^{-\frac{1}{2}} \times H^{\frac{1}{2}}} \in \mathbb{R}.$$

On the other hand, by the expression (2.1) we see that the function

$$g^s_x\left(\not{D}\psi(x), \psi(x)\right)$$

is in $L^1(M)$, whose integral is exactly given by $\sum_{k\in\mathbb{Z}_*}\lambda_k a_k^2 < \infty$. By this we validate the Dirac action in the equivalent form

$$\left\langle \not{D}\psi,\psi\right\rangle_{H^{-\frac{1}{2}}\times H^{\frac{1}{2}}} = \int_M \left\langle \not{D}\psi(x),\psi(x)\right\rangle_{g^s(x)} \mathrm{d}v_g(x).$$

Suppose $h^0 = 0$, i.e. there are no non-trivial harmonic spinors. Then the Dirac operator $\not{D}$ is invertible. Splitting into the positive and negative parts of the spectrum $\mathrm{Spec}(\not{D})$, we have the decomposition

$$H^{\frac{1}{2}}(S) = H^{\frac{1}{2},+}(S)\oplus H^{\frac{1}{2},-}(S). \tag{2.2}$$

Let $\psi = \psi^+ + \psi^-$ be decomposed accordingly: then,

$$\int_M \left\langle \not{D}\psi^+,\psi^+\right\rangle \mathrm{d}v_g = \int_M \left\langle |\not{D}|^{\frac{1}{2}}\psi^+, |\not{D}|^{\frac{1}{2}}\psi^+\right\rangle \mathrm{d}v_g \geq \lambda_1(\not{D}_g)\|\psi^+\|^2_{L^2(M)},$$

where λ_1 is the first positive eigenvalue of $\not{D} = \not{D}_g$. Hence

$$\begin{aligned}\|\psi^+\|^2_{H^{\frac{1}{2}}} =& \|\psi^+\|^2_{L^2} + \||\not{D}|^{\frac{1}{2}}\psi^+\|^2_{L^2}\\ \leq& (\lambda_1(\not{D}_g)^{-1}+1)\||\not{D}|^{\frac{1}{2}}\psi^+\|^2_{L^2} \leq (\lambda_1(\not{D}_g)^{-1}+1)\|\psi^+\|^2_{H^{\frac{1}{2}}}.\end{aligned}$$

That is, for a given g, the integral $\int_M \left\langle \not{D}\psi^+,\psi^+\right\rangle \mathrm{d}v_g$ defines a norm on $H^{\frac{1}{2},+}(S)$ equivalent to the Hilbert's. Similarly, on $H^{\frac{1}{2},-}(S)$ there is an equivalent norm given by

$$-\int_M \left\langle \not{D}\psi^-,\psi^-\right\rangle \mathrm{d}v_g = \||\not{D}|^{\frac{1}{2}}\psi^-\|^2_{L^2}.$$

Consequently,

$$\int_M \left[\left\langle \not{D}\psi^+,\psi^+\right\rangle - \left\langle \not{D}\psi^-,\psi^-\right\rangle\right] \mathrm{d}v_g = \||\not{D}|^{\frac{1}{2}}\psi^+\|^2_{L^2} + \||\not{D}|^{\frac{1}{2}}\psi^-\|^2_{L^2}$$

defines a norm equivalent to the $H^{\frac{1}{2}}$-norm, which is quite convenient in our analysis. Note that this would fail if $h^0 \neq 0$.

2.4. **Moser–Trudinger embedding.** Another space we use frequently is $H^1(M) = W^{1,2}(M,\mathbb{R})$. Consider the subspace in $H^1(M)$ of the functions with zero average

$$H^1_0(M) := \left\{ u \in H^1(M) \mid \int_M u\,\mathrm{d}v_g = 0\right\}.$$

Then $H^1(M) = \mathbb{R}\oplus H^1_0(M)$, and any $u \in H^1(M)$ can be written as $u = \bar{u} + \widehat{u}$ where $\bar{u} = \fint_M u\,\mathrm{d}v_g$ denotes the average of u. By Poincaré's inequality, $\|\nabla\widehat{u}\|_{L^2}$ defines a norm equivalent to $\|\widehat{u}\|_{H^1}$ on $H^1_0(M)$, and

$$|\bar{u}| + \|\nabla\widehat{u}\|_{L^2}$$

a norm equivalent to $\|u\|_{H^1}$. The Sobolev embedding theorems imply that for any $p < \infty$, $H^1(M)$ embeds into $L^p(M)$ continuously and compactly. Furthermore, the Moser–Trudinger inequality states that there exists $C > 0$ such that

$$\int_M \exp\left(\frac{4\pi|\widehat{u}|^2}{\|\nabla\widehat{u}\|^2_{L^2(M)}}\right)\mathrm{d}v_g \leq C.$$

As a consequence

$$8\pi\log\int_M e^{\widehat{u}}\,\mathrm{d}v_g \leq \frac{1}{2}\int_M |\nabla\widehat{u}|^2\,\mathrm{d}v_g + C.$$

This implies that e^u is L^p integrable for any $p > 0$. Moreover, the map

$$H^1(M) \ni u \mapsto e^u \in L^1(M)$$

is compact (see e.g. [4, Theorem 2.46]). It follows that the maps $H^1(M) \ni u \mapsto e^u \in L^p(M)$ are compact for all $p > 0$.

3. A natural constraint and the Palais-Smale condition

It is standard to prove that the functional

$$J_\rho \colon H^1(M) \times H^{\frac{1}{2}}(S) \to \mathbb{R}$$

defined in formula (1.2) is of class C^1. The critical points of J_ρ, which are weak solutions of (EL), are actually smooth. To see this we can use the argument from [22]. Note that, although the authors there are using different Banach spaces, the proof goes quite similarly and is omitted here. Alternatively, note that $u \in H^1(M)$ implies $e^u \in L^p(M)$ for any $p < \infty$, i.e. the equation is actually subcritical and we can appeal to a bootstrap argument to obtain the full regularity.

To obtain a non-trivial solution to the system (EL) we employ a min-max approach. As observed, thanks to the conformal covariance of the system, it is sufficient to consider the uniformized metric. From now on we assume that g has constant Gaussian curvature $K \equiv -1$. For this choice we then look for non-trivial critical points of the functional

$$J_\rho(u,\psi) = \int_M \left(|\nabla u|^2 - 2u + e^{2u} + 2\left(\langle \not{D}\psi, \psi \rangle - \rho e^u |\psi|^2 \right) \right) \mathrm{d}v_g,$$

which are non-trivial solutions of the system

$$(EL_0) \qquad \begin{cases} \Delta_g u = e^{2u} - 1 - \rho e^u |\psi|^2, \\ \not{D}_g \psi = \rho e^u \psi. \end{cases}$$

The argument in the sequel is simplified by this assumption, but it can be modified and adapted to a general metric. Note that in the uniformized case the Gauss-Bonnet formula yields

$$vol(M,g) = -2\pi\chi(M) = 4\pi(\gamma - 1).$$

Observe that in the functional J_ρ the first part is coercive and convex. The main difficulty is due to the spinorial part which is strongly indefinite. To overcome this issue we are inspired by an idea from [32] and we consider a natural constraint: in the next section we will find critical points of the restricted functional.

3.1. A Nehari type manifold.

Roughly speaking, the space $H^{\frac{1}{2},-}(S)$ defined in (2.2) contains infinitely-many directions decreasing the functional J_ρ to negative infinity and the usual variational approaches can not be applied. Hence we introduce a natural constraint in order to exclude most of these directions, obtaining a submanifold in $H^1(M) \times H^{\frac{1}{2}}(S)$, which we still call it a *Nehari manifold*, though it may not fit into the classical definition as in [2]. This may be considered to be a Nehari manifold in the generalized sense, as in [34, 42, 43].

Let $P^\pm \colon H^{\frac{1}{2}}(S) \to H^{\frac{1}{2},\pm}(S)$ be the orthonormal projection according to the splitting in (2.2). Consider the map

$$\begin{aligned} G \colon H^1(M) \times H^{\frac{1}{2}}(S) &\to H^{\frac{1}{2},-}(S), \\ (u,\psi) &\mapsto P^- \left[(1 + |\not{D}|)^{-1} (\not{D}\psi - \rho e^u \psi) \right]. \end{aligned}$$

Some explanations are in order. Recall that $H^{\frac{1}{2},-}$ is a Hilbert space, with inner product

$$\begin{aligned} \langle \psi, \varphi \rangle_{H^{1/2}} &= \langle \psi, \varphi \rangle_{L^2} + \left\langle |\not{D}|^{\frac{1}{2}} \psi, |\not{D}|^{\frac{1}{2}} \varphi \right\rangle_{L^2} \\ &= \left\langle (1 + |\not{D}|)\psi, \varphi \right\rangle_{H^{-\frac{1}{2}} \times H^{\frac{1}{2}}}. \end{aligned}$$

Now, let $G(u,\psi)$ be the element in $H^{\frac{1}{2}}(S)$ such that, for any $\varphi \in H^{\frac{1}{2}}(S)$,

$$\langle G(u,\psi), \varphi \rangle_{H^{1/2}} = \left\langle \not{D}\psi - \rho e^u \psi, P^-(\varphi) \right\rangle_{H^{-\frac{1}{2}} \times H^{\frac{1}{2}}}.$$

It follows that $G(u,\psi) \in H^{\frac{1}{2},-}(S)$, and it is given by the Riesz representation theorem as

$$G(u,\psi) = P^- \left[(1 + |\not{D}|)^{-1} (\not{D}\psi - \rho e^u \psi) \right].$$

Define the *Nehari manifold* $N = G^{-1}(0)$, which is non-empty since $(u,0)\in N$ for any $u \in H^1(M)$. Note that, for each u fixed, the subset

$$N_u \coloneqq \left\{ \psi \in H^{\frac{1}{2}}(S) \mid (u,\psi)\in N \right\} = \ker\left[P^- \circ (1+|\not{D}|)^{-1}\circ(\not{D} - \rho e^u)\right]$$

is a linear subspace (of infinite dimension). Hence we have a fibration $N \to H^1(M)$ with fiber N_u over $u \in H^1(M)$. The total space N is contractible.

Lemma 3.1. *The Nehari manifold N is a natural constraint for J_ρ, namely every critical point of $J_\rho|_N$ is an unconstrained critical point of J_ρ.*

Proof. To see that N is a manifold, we show for any (u,ψ) the surjectivity of the differential $\mathrm{d}G(u,\psi)$, which is given by

$$\mathrm{d}G(u,\psi)[v,\phi] = P^-\left[(1+|\not{D}|)^{-1}(\not{D}\phi - \rho e^u\phi - \rho e^u v\psi)\right].$$

Restricting to those vectors with $v=0$ and $\phi \in H^{\frac{1}{2},-}(S)$, we have

$$\begin{aligned}\langle \mathrm{d}G(u,\psi)[0,\phi],\phi\rangle_{H^{1/2}} &= \left\langle (1+|\not{D}|)^{-1}(\not{D}\phi - \rho e^u\phi),\phi\right\rangle_{H^{1/2}}\\ &= \int_M \left\langle \not{D}\phi - \rho e^u\phi,\phi\right\rangle \mathrm{d}v_g\\ &= -\left\||\not{D}|^{\frac{1}{2}}\phi\right\|^2_{L^2} - \rho\int_M e^u|\phi|^2\,\mathrm{d}v_g.\end{aligned}$$

Thus $\langle \mathrm{d}G(u,\psi)[0,\phi],\phi\rangle_{H^{1/2}}$ yields a negative-definite quadratic form on $H^{\frac{1}{2},-}(S)$. In particular, $\mathrm{d}G(u,\psi)$ is surjective onto $H^{\frac{1}{2},-}(S)$, for any (u,ψ). It follows from the regular value theorem (for an infinite dimensional version, see e.g. [15]) that $N = G^{-1}(0)$ is a submanifold of $H^1(M)\times H^{\frac{1}{2}}(S)$.

Next, we need to show that if (u_0,ψ_0) is a critical point of $J_\rho|_N$, then it is also a critical points of J_ρ on the full space $H^1(M)\times H^{\frac{1}{2}}(S)$.

Recall that the orthonormal basis (φ_k) for $H^{\frac{1}{2}}(S)$ consists of eigenspinors. Note that

$$\begin{aligned}(u,\psi)\in N \Leftrightarrow G(u,\psi)=0 &\Leftrightarrow \int_M \left\langle \not{D}\psi - \rho e^u\psi, h\right\rangle \mathrm{d}v_g = 0, \quad \forall h\in H^{\frac{1}{2},-}\\ &\Leftrightarrow G_j(u,\psi)\coloneqq \int_M \left\langle \not{D}\psi - \rho e^u\psi,\varphi_j\right\rangle \mathrm{d}v_g = 0,\quad \forall j<0,\end{aligned}$$

that is

$$N = G^{-1}(0) = \bigcap_{j<0} G_j^{-1}(0).$$

Now let (u_0,ψ_0) be a critical point of $J_\rho|_N$: $\nabla^N J(u_0,\psi_0)=0$. Then there exist $\mu_j\in\mathbb{R}$ such that[1]

$$\mathrm{d}J_\rho(u_0,\psi_0) = \sum_{j<0}\mu_j\,\mathrm{d}G_j(u_0,\psi_0). \tag{3.1}$$

Testing both sides with tangent vectors of the form $(0,h)$, we have

$$\int_M \left\langle \not{D}\psi_0 - \rho e^{u_0}\psi_0, h\right\rangle \mathrm{d}v_g = \sum_{j<0}\mu_j\int_M \left\langle \not{D}h - \rho e^{u_0}h,\varphi_j\right\rangle \mathrm{d}v_g.$$

[1]To see that such an infinite dimensional version of the Lagrange multiplier theory works, we note that

$$\nabla^N J(u_0,\psi_0) = \nabla J(u_0,\psi_0) - (\nabla J(u_0,\psi_0))^\perp$$

where $\nabla J(u_0,\psi_0)$ denotes the unconstrained gradient and $(\nabla J(u_0,\psi_0))^\perp$ denotes its normal component. Since the gradients $\{\nabla G_j(u_0,\psi_0) : j<0\}$ span the normal space, we can express $(\nabla J(u_0,\psi_0))^\perp$ in terms of them:

$$(\nabla J(u_0,\psi_0))^\perp = \sum_{j<0}\mu_j\nabla G_j(u_0,\psi_0)\in H^{\frac{1}{2}}(S)$$

for some $\mu_j\in\mathbb{R}$, $j<0$.

In particular, take $h = \varphi = \sum_{j<0} \mu_j \varphi_j \in H^{\frac{1}{2},-}$ to obtain

$$0 = \int_M \left\langle \not{D}\psi_0 - \rho e^{u_0}\psi_0, \varphi \right\rangle = \int_M \left\langle \not{D}\varphi - \rho e^{u_0}\varphi, \varphi \right\rangle \mathrm{d}v_g \le -C\|\varphi\|^2 - \int_M \rho e^{u_0}|\varphi|^2 \,\mathrm{d}v_g.$$

Thus $\varphi = 0$, i.e. $\mu_j = 0$ for all $j < 0$. Hence in (3.1) we have $\mathrm{d}J_\rho(u_0, \psi_0) = 0$. □

3.2. **Verification of the Palais-Smale condition.** This subsection is devoted to verifying the (PS) condition for the constrained functional $J_\rho|_N$. Note that

$$\mathrm{d}J_\rho(u,\psi)[v,\phi] = \int_M 2(-\Delta u - 1 + e^{2u} - \rho e^u|\psi|^2)v + 4\left\langle \not{D}\psi - \rho e^u\psi, \phi \right\rangle \mathrm{d}v_g$$

and for each $j < 0$, with G_j defined as in the above proof:

$$\mathrm{d}G_j(u,\psi)[v,\phi] = \int_M \left\langle \not{D}\phi - \rho e^u\phi, \varphi_j \right\rangle \mathrm{d}v_g - \int_M \rho e^u v \left\langle \psi, \varphi_j \right\rangle \mathrm{d}v_g.$$

For each $(u,\psi) \in N$, there exist constants $\mu_j(u,\psi)$ such that

$$\mathrm{d}^N J_\rho(u,\psi) = \mathrm{d}J_\rho(u,\psi) - \sum_{j<0} \mu_j(u,\psi)\,\mathrm{d}G_j(u,\psi),$$

that is such that for any $(v,\phi) \in H^1(M) \times H^{\frac{1}{2}}(S)$

$$\mathrm{d}^N J_\rho(u,\psi)[v,\phi] = \mathrm{d}J(u,\psi)[v,\phi] - \sum_{j<0} \mu_j(u,\psi)\,\mathrm{d}G_j(u,\psi)[v,\phi].$$

Formally writing $\varphi(u,\psi) \coloneqq \sum_{j<0} \mu_j \varphi_j$, then

$$\begin{aligned}\mathrm{d}^N J(u,\psi)[v,\phi] = &\int_M 2(-\Delta u - 1 + e^{2u} - \rho e^u|\psi|^2 + \rho e^u \left\langle \psi, \varphi(u,\psi) \right\rangle)v \,\mathrm{d}v_g \\ &+ \int_M 4\left(\left\langle \not{D}\psi - \rho e^u\psi, \phi \right\rangle - \left\langle \not{D}\varphi - \rho e^u\varphi, \phi \right\rangle\right) \mathrm{d}v_g.\end{aligned}$$

Note that this holds for arbitrary (v,ϕ), not only those tangent vectors to N.

Now let $(u_n,\psi_n) \in N$ be a $(PS)_c$ sequence for $J_\rho|_N$: this will satisfy

$$J_\rho(u_n,\psi_n) = \int_M \left[|\nabla u_n|^2 - 2u_n + e^{2u_n} + 2\left(\left\langle \not{D}\psi_n, \psi_n \right\rangle - \rho e^{u_n}|\psi_n|^2\right)\right] \mathrm{d}v_g \to c, \tag{3.2}$$

$$P^- \circ (1 + |\not{D}|)^{-1} \circ (\not{D}\psi_n - \rho e^{u_n}\psi_n) = 0; \tag{3.3}$$

moreover, since the differential of J_ρ is tending to zero only when applied to vectors tangent to N, there exists some $\varphi_n \in H^{\frac{1}{2},-}(S)$ such that

$$2(-\Delta u_n - 1 + e^{2u_n} - \rho e^{u_n}|\psi_n|^2) - \rho e^{u_n} \left\langle \psi_n, \varphi_n \right\rangle = \alpha_n \to 0 \text{ in } H^{-1}(M), \tag{3.4}$$

$$4(\not{D}\psi_n - \rho e^{u_n}\psi_n) - (\not{D}\varphi_n - \rho e^{u_n}\varphi_n) = \beta_n \to 0 \text{ in } H^{-\frac{1}{2}}(S). \tag{3.5}$$

Lemma 3.2. *With the same notation as above, we have*

1. *The auxiliary spinors φ_n satisfy $\|\varphi_n\|_{H^{\frac{1}{2}}} \to 0$ as $n \to \infty$.*
2. *The sequence (u_n,ψ_n) is uniformly bounded (with bounds depending on the level c) in $H^1(M) \times H^{\frac{1}{2}}(S)$.*

Proof. (1) Testing (3.3) against φ_n we find

$$\int_M \left\langle \not{D}\psi_n - \rho e^{u_n}\psi_n, \varphi_n \right\rangle \mathrm{d}v_g = 0,$$

while testing (3.5) against φ_n we get

$$-\int_M \left\langle \not{D}\varphi_n, \varphi_n \right\rangle \mathrm{d}v_g + \rho \int_M e^{u_n}|\varphi_n|^2 \,\mathrm{d}v_g = \left\langle \beta_n, \varphi_n \right\rangle.$$

Since φ_n lies in the span of the negative eigenspinors, we see that

$$C\|\varphi_n\|^2_{H^{\frac{1}{2}}} + \rho \int_M e^{u_n}|\varphi_n|^2 \, \mathrm{d}v_g = o(\|\varphi_n\|_{H^{\frac{1}{2}}}).$$

It follows that as $n \to \infty$,

$$\|\varphi_n\|_{H^{\frac{1}{2}}} \to 0, \qquad \int_M \rho e^{u_n}|\varphi_n|^2 \, \mathrm{d}v_g \to 0.$$

(2) Testing (3.4) against $v \equiv 1 \in H^1(M)$, we obtain

$$2\int_M e^{2u_n} \, \mathrm{d}v_g - 2\int_M \mathrm{d}v_g - 2\rho \int_M \left(e^{u_n}|\psi_n|^2 - e^{u_n}\langle \psi_n, \varphi_n\rangle\right) \mathrm{d}v_g = \langle \alpha_n, 1\rangle_{H^{-1}\times H^1},$$

which can be read as

$$\int_M e^{2u_n} \, \mathrm{d}v_g = 4\pi(\gamma - 1) + \rho \int_M e^{u_n}|\psi_n|^2 \, \mathrm{d}v_g + \frac{1}{2}\rho \int_M e^{u_n}\langle \psi_n, \varphi_n\rangle \, \mathrm{d}v_g + o(1). \tag{3.6}$$

Now we can control the second integral on the right-hand side by

$$\left|\frac{\rho}{2}\int_M e^{u_n}\langle \psi_n, \varphi_n\rangle \, \mathrm{d}v_g\right| \le \varepsilon \int_M \rho e^{u_n}|\psi_n|^2 \, \mathrm{d}v_g + \varepsilon \int_M e^{2u_n} \, \mathrm{d}v_g + C(\varepsilon, \rho)\|\varphi_n\|^4, \tag{3.7}$$

where $\varepsilon > 0$ is some small number. Substituting this into (3.6) and noting that $\|\varphi_n\| = o(1)$, we get

$$\int_M e^{2u_n} \ge \frac{4\pi(\gamma - 1)}{1+\varepsilon} + \frac{1-\varepsilon}{1+\varepsilon}\int_M \rho e^{u_n}|\psi_n|^2 \, \mathrm{d}v_g + o(1).$$

Testing (3.5) against ψ_n we deduce

$$4\int_M \left(\langle \not{D}\psi_n, \psi_n\rangle - \rho e^{u_n}|\psi_n|^2\right) \mathrm{d}v_g - \int_M \langle \not{D}\varphi_n - \rho e^{u_n}\varphi_n, \psi_n\rangle \, \mathrm{d}v_g = \langle \beta_n, \psi_n\rangle_{H^{-\frac{1}{2}}\times H^{\frac{1}{2}}}.$$

Since the second integral vanishes because of (3.3), we thus get

$$\int_M \left(\langle \not{D}\psi_n, \psi_n\rangle - \rho e^{u_n}|\psi_n|^2\right) \mathrm{d}v_g = o(\|\psi_n\|).$$

Combining these estimates with (3.2) we see that

$$\begin{aligned} c + o(1) &= \int_M |\nabla \widehat{u}_n|^2 \, \mathrm{d}v_g - 8\pi(\gamma - 1)\bar{u}_n + \int_M e^{2u_n} \, \mathrm{d}v_g + 2\int_M \left(\langle \not{D}\psi_n, \psi_n\rangle - \rho e^{u_n}|\psi_n|^2\right) \mathrm{d}v_g \\ &\ge \int_M |\nabla \widehat{u}_n|^2 \, \mathrm{d}v_g - 4\pi(\gamma - 1)(2\bar{u}_n - \frac{1}{1+\varepsilon}) + \frac{1-\varepsilon}{1+\varepsilon}\rho \int_M e^{u_n}|\psi_n|^2 \, \mathrm{d}v_g + C(\varepsilon, \rho)o(1) + o(\|\psi_n\|), \end{aligned}$$

which is to say,

$$\int_M |\nabla \widehat{u}_n|^2 + \frac{1-\varepsilon}{1+\varepsilon}\rho e^{u_n}|\psi_n|^2 \, \mathrm{d}v_g \le c + 4\pi(\gamma - 1)(2\bar{u}_n - \frac{1}{1+\varepsilon}) + C(\varepsilon, \rho)o(1) + o(\|\psi_n\|).$$

Now we estimate the averages $\bar{u}_n$. Note that by (3.6) and (3.7) we also obtain

$$\int_M e^{2u_n} \, \mathrm{d}v_g \le \frac{4\pi(\gamma - 1)}{1-\varepsilon} + \frac{1+\varepsilon}{1-\varepsilon}\int_M \rho e^{u_n}|\psi_n|^2 \, \mathrm{d}v_g + C(\varepsilon, \rho)o(1).$$

Then by Jensen's inequality,

$$\begin{aligned} e^{2\bar{u}_n} &\le e^{2\bar{u}_n} \fint_M e^{2\widehat{u}_n} \, \mathrm{d}v_g = \frac{1}{4\pi(\gamma - 1)}\int_M e^{2u_n} \, \mathrm{d}v_g \\ &\le \frac{1}{1-\varepsilon} + \frac{1}{4\pi(\gamma - 1)}\frac{1+\varepsilon}{1-\varepsilon}\int_M \rho e^{u_n}|\psi_n|^2 \, \mathrm{d}v_g + C(\varepsilon, \rho, \gamma)o(1) \\ &\le \frac{1}{1-\varepsilon} + \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^2 \left(\frac{c}{4\pi(\gamma - 1)} + 2\bar{u}_n - \frac{1}{1+\varepsilon}\right) + C(\varepsilon, \rho, \gamma)o(1) + C(\varepsilon, \gamma)o(\|\psi_n\|). \end{aligned}$$

Thus there exists $C = C(\varepsilon, \rho, \gamma) > 0$ such that

$$|\bar{u}_n| \leq C\big(1 + c + o(\|\psi_n\|)\big).$$

The spinors can be controlled by the above growth estimates. Testing (3.5) against ψ_n^+, we find

$$4\int_M \left(\langle \not{D}\psi_n, \psi_n^+\rangle - \rho e^{u_n}\langle \psi_n, \psi_n^+\rangle\right) \mathrm{d}v_g - \int_M \langle \not{D}\varphi_n - \rho e^{u_n}\varphi_n, \psi_n^+\rangle \,\mathrm{d}v_g = \langle \beta_n, \psi_n^+\rangle_{H^{-\frac{1}{2}}\times H^{\frac{1}{2}}}.$$

It follows that

$$\begin{aligned}
C\|\psi_n^+\|^2 \leq & \int_M \langle \not{D}\psi_n, \psi^+\rangle \,\mathrm{d}v_g = \int_M \rho e^{u_n}\langle \psi_n, \psi_n^+\rangle \,\mathrm{d}v_g + \frac{1}{4}\int_M \langle \not{D}\varphi_n - \rho e^{u_n}\varphi_n, \psi_n^+\rangle \,\mathrm{d}v_g + o(\|\psi_n^+\|) \\
\leq & \left(\int_M e^{u_n}|\psi_n|^2 \,\mathrm{d}v_g\right)^{\frac{1}{2}} \left(\int_M e^{2u_n}\,\mathrm{d}v_g\right)^{\frac{1}{4}} \left(\int_M |\psi_n^+|^4 \,\mathrm{d}v_g\right) \\
& + \|\varphi_n\|\|\psi_n^+\| + \rho\left(\int_M e^{2u_n}\,\mathrm{d}v_g\right)^{\frac{1}{2}} \|\varphi_n\|\|\psi_n^+\| + o(\|\psi_n^+\|) \\
\leq & C\left(1 + c + o(\|\psi_n\|)\right)^{\frac{3}{4}} \|\psi_n^+\| + o(\|\psi_n^+\|).
\end{aligned}$$

For what concerns the other component ψ_n^-, we use (3.3) to get

$$\begin{aligned}
C\|\psi_n^-\|^2 \leq & -\int_M \langle \not{D}\psi_n^-, \psi_n\rangle \,\mathrm{d}v_g = -\rho\int_M e^{u_n}\langle \psi_n, \psi_n^-\rangle \,\mathrm{d}v_g \\
\leq & \rho\left(\int_M e^{2u_n}\,\mathrm{d}v_g\right)^{\frac{1}{4}} \left(\int_M e^{u_n}|\psi_n|^2\,\mathrm{d}v_g\right)^{\frac{1}{2}} \|\psi_n^-\| \\
\leq & C(1 + c + o(\|\psi_n\|))^{\frac{3}{4}}\|\psi_n^-\|.
\end{aligned}$$

Consequently,

$$\|\psi_n\|^2 = \|\psi_n^+\|^2 + \|\psi_n^-\|^2 \leq C\big(1 + c + o(\|\psi_n\|)\big)^{\frac{3}{4}}\|\psi_n\| + o(\|\psi_n\|).$$

Thus there exists some constant $C = C(c, \gamma, \rho) > 0$ such that

$$\|\psi_n\| \leq C(c, \gamma, \rho) < +\infty.$$

This uniform bound (depending on the level c) in turn gives bounds on $\bar{u}_n$ and thus

$$\int_M \left(|\nabla \widehat{u}_n|^2 + \rho e^{u_n}|\psi_n|^2\right) \mathrm{d}v_g \leq C'(c, \gamma, \rho) < \infty.$$

□

Now, passing to a subsequence if necessary, we may assume that there exist $u_\infty \in H^1(M)$ and $\psi_\infty \in H^{\frac{1}{2}}(S)$ such that

$$\begin{aligned}
u_n &\rightharpoonup u_\infty \text{ weakly in } H^1(M), \\
\psi_n &\rightharpoonup \psi_\infty \text{ weakly in } H^{\frac{1}{2}}(S).
\end{aligned}$$

Lemma 3.3. *The pair (u_∞, ψ_∞) is a smooth solution of (EL_0).*

Proof. According to the compactness of the Moser–Trudinger embedding ([4, theorem 2.46]), we see that

$$e^{u_n} \to e^{u_\infty} \text{ strongly in } L^p(M), \quad (p < \infty).$$

Meanwhile, thanks to Rellich-Kondrachov compact embedding Theorem (see e.g. [14])

$$\psi_n \to \psi_\infty \text{ strongly in } L^q(S), \quad (q < 4).$$

Hence $e^{u_n}|\psi_n|^2$ converges weakly in L^p to $e^{u_\infty}|\psi_\infty|^2$, for any $p < 2$.

It follows that (u_∞, ψ_∞) is a weak solution to (EL_0). As remarked before, any weak solution is a classical, hence smooth, solution. □

In particular, this implies that the weak limit (u_∞, ψ_∞) is in the Nehari manifold N. Consider the differences

$$v_n := u_n - u_\infty,$$
$$\phi_n := \psi_n - \psi_\infty.$$

Then $(v_n, \phi_n) \rightharpoonup (0,0)$ weakly in $H^1(M) \times H^{\frac{1}{2}}(S)$. The functions v_n satisfy

$$\Delta_g v_n = (e^{2u_n} - e^{2u_\infty}) - \rho\left(e^{u_n}|\psi_n|^2 - e^{u_\infty}|\psi_\infty|^2\right) - \frac{1}{2}\rho e^{u_n}\langle \psi_n, \varphi_n\rangle - \frac{1}{2}\alpha_n$$

where the right-hand side converges to 0 in $H^{-1}(M)$. Noting that $v_n \to 0$ strongly in $L^2(M)$, we conclude that $v_n \to 0$ strongly in $H^1(M)$, namely u_n converges strongly to u_∞ in $H^1(M)$. Similar argument works for the spinorial components. Indeed,

$$\not{D}_g \phi_n = \rho e^{u_n}\psi_n - \rho e^{u_\infty}\psi_\infty + \frac{1}{4}\left(\not{D}\varphi_n - \rho e^{u_n}\varphi_n\right) + \frac{1}{4}\beta_n$$

with right hand side converging to 0 in $H^{-\frac{1}{2}}(M)$, and $\phi_n = \psi_n - \psi_\infty$ converges to 0 in $L^2(S)$. Thus ψ_n converges strongly to ψ_∞ in $H^{\frac{1}{2}}(S)$. Since N is a submanifold of $H^1(M) \times H^{\frac{1}{2}}(S)$, the sequence (u_n, ψ_n) also converges inside N to (u_∞, ψ_∞); in other words, (u_∞, ψ_∞) lies in the closure of $\{(u_n, \psi_n)\}$ relative to N. Thus we verified the following

Proposition 3.4. *The functional $J_\rho|_N$ satisfies the Palais-Smale condition.*

4. Mountain pass and linking geometry

In this section we will show that the functional $J_\rho|_N$, for suitable ρ's, possesses either a mountain pass or linking geometry around the trivial solution $(0,0)$, which will yield existence of a non-trivial min-max critical point.

For later convenience let us introduce the notation

$$F(u) := \int_M \left(|\nabla u|^2 - 2u + e^{2u}\right) \mathrm{d}v_g, \qquad Q(u,\psi) := 2\int_M \left(\langle \not{D}\psi, \psi\rangle - \rho e^u |\psi|^2\right) \mathrm{d}v_g.$$

Then we have

(i) $J_\rho(u,\psi) = F(u) + Q(u,\psi)$.
(ii) $F(u) \geq 4\pi(\gamma - 1)$, and this lower bound is achieved by the unique minimizer $u_{min} \equiv 0$.
(iii) $Q(u,\psi)$ is quadratic in ψ and strongly indefinite.

4.1. **Local behavior near (0,0).** Let $(u,\psi) \in N$ be *close* to $(0,0)$. The constraint that defines N, i.e. $P^-(1 + |\not{D}|)^{-1}(\not{D}\psi - \rho e^u \psi) = 0$, implies

$$\int_M \left\langle \not{D}\psi - \rho e^u \psi, P^-\psi\right\rangle \mathrm{d}v_g = 0.$$

Hence we get

$$\begin{aligned} -\int_M \left\langle \not{D}\psi, \psi^-\right\rangle \mathrm{d}v_g &= -\rho\int_M e^u\left\langle \psi^+ + \psi^-, \psi^-\right\rangle \\ &= -\rho\int_M e^u|\psi^-|^2\,\mathrm{d}v_g - \rho\int_M e^u\left\langle \psi^+, \psi^-\right\rangle \mathrm{d}v_g. \end{aligned}$$

Since $\|e^u\|_{L^p} \leq C(1 + \|u\|_{H^1}) \leq C$ for $\|u\|$ uniformly bounded, we have

$$\|\psi^-\| \leq C\rho\|\psi^+\|. \tag{4.1}$$

Now consider the functional

$$J_\rho(u,\psi) = F(u) + Q(u,\psi) = F(u) + 2\int_M \left(\langle \not{D}\psi,\psi\rangle - \rho e^u|\psi|^2\right) \mathrm{d}v_g \tag{4.2}$$

$$= F(u) + 2\int_M \left\langle (\not{D}-\rho)\psi,\psi^+\right\rangle \mathrm{d}v_g + 2\int_M \rho(1-e^u)\left\langle \psi,\psi^+\right\rangle \mathrm{d}v_g.$$

The last integral is now of cubic order in (u,ψ), i.e.:

$$2\int_M \rho(1-e^u)\left\langle \psi,\psi^+\right\rangle \mathrm{d}v_g \le C\|u\|_{H^1}\|\psi\|\|\psi^+\|.$$

For the first term, if we take the equivalent norm $|\bar{u}|^2+\|\nabla\widehat{u}\|_{L^2}^2 \sim \|u\|_{H^1}^2$, then for $t^2 = |\bar{u}|^2+\|\nabla\widehat{u}\|_{L^2}^2 > 0$

- if $|\bar{u}|^2 \ge \frac{t^2}{2} \ge \|\nabla\widehat{u}\|_{L^2}^2$, then
$$F(u) \ge \int_M \left(e^{2\bar{u}} - 2\bar{u}\right) \mathrm{d}v_g \ge 4\pi(\gamma-1) + Ct^2,$$
- if $\|\nabla\widehat{u}\|_{L^2}^2 \ge \frac{t^2}{2} \ge |\bar{u}|^2$, then
$$F(u) \ge \int_M \left(|\nabla\widehat{u}|^2 + 1\right) \mathrm{d}v_g \ge 4\pi(\gamma-1) + \frac{1}{2}t^2,$$

thus in either case we have

$$F(u) \ge 4\pi(\gamma-1) + C^{-1}\|u\|_{H^1}^2.$$

It remains to analyze the middle integral term in the r.h.s. of (4.2). As before, we write $\psi = \sum_{j\in\mathbb{Z}_*} a_j\varphi_j$: then

$$2\int_M \left\langle (\not{D}-\rho)\psi,\psi^+\right\rangle \mathrm{d}v_g = \sum_{j>0} 2(\lambda_j-\rho)a_j^2.$$

From now on we assume that $\rho \notin \mathrm{Spec}(\not{D})$. Thus the above summation can be split into two parts

$$2\int_M \left\langle (\not{D}-\rho)\psi,\psi^+\right\rangle \mathrm{d}v_g = -\sum_{0<\lambda_j<\rho} 2(\rho-\lambda_j)a_j^2 + \sum_{\lambda_j>\rho} 2(\lambda_j-\rho)a_j^2.$$

4.1.1. *Mountain pass geometry.* First we consider the easier case $0<\rho<\lambda_1$, so the first part of the above summation vanishes. Then, locally near $(0,0)$ in N, we have

$$\begin{aligned} J_\rho(u,\psi) \ge& 4\pi(\gamma-1) + C^{-1}\|u\|_{H^1}^2 + C^{-1}\left(1-\frac{\rho}{\lambda_1}\right)\|\psi^+\|^2 - C\|u\|_{H^1}\|\psi\|\|\psi^+\| \\ \ge& 4\pi(\gamma-1) + C^{-1}\|u\|_{H^1}^2 + C^{-1}\left(1-\frac{\rho}{\lambda_1} - C^2\|\psi\|^2\right)\|\psi^+\|^2, \end{aligned}$$

where we have used Cauchy-Schwarz inequality for the last term, of cubic order. It follows that when $\|u\|_{H^1}^2 + \|\psi\|_{H^{\frac{1}{2}}}^2 = r^2 > 0$ is small, there exists a continuous function $\theta(r)>0$ such that

$$J_\rho(u,\psi) \ge J(0,0) + \theta(r).$$

On the other hand, we can choose a large constant $\bar{u}_1 \in H^1(M)$ such that $\rho e^{\bar{u}_1} > \lambda_1 + 1$ and then take $s>0$ large such that

$$\begin{aligned} J(\bar{u}_1, s\varphi_1) =& vol(M,g)(e^{2\bar{u}_1} - 2\bar{u}_1) + 2(\lambda_1 - \rho e^{\bar{u}_1})s^2 \\ =& 4\pi(\gamma-1)(e^{2\bar{u}_1} - 2\bar{u}_1) - 2(\rho e^{\bar{u}_1} - \lambda_1)s^2 \end{aligned}$$

is negative. Thus we have the mountain pass geometry locally near $(0,0)$ in the Nehari manifold N. Let Γ be the space of paths connecting $(0,0)$ and $(\bar{u}_1, s\varphi_1)$ inside N (notice that $\Gamma \ne \emptyset$ since N is contractible, and hence connected), parametrized by $t\in[0,1]$, and define

$$c_1 \coloneqq \inf_{\alpha\in\Gamma}\sup_{t\in[0,1]} J_\rho(\alpha(t)).$$

From the above arguments we have that $c_1 > 4\pi(\gamma-1)$. It follows that c_1 is a critical value for J_ρ with a critical point at this level, which is different from the trivial one. This concludes the proof of Theorem 1.1 in this case.

4.1.2. *Linking geometry.* Next we consider the case $\rho \in (\lambda_k, \lambda_{k+1})$ for some $k \geq 1$. Now there are more directions in which J_ρ becomes negative, but these are at most finitely-many, and we will apply a linking method to exploit the geometry of the functional.

Decomposing first the space $H^{\frac{1}{2}}(S)$ into two parts:

$$H^{\frac{1}{2},k+} := \left\{ \phi_1 \in H^{\frac{1}{2}}(S) \mid \phi_1 = \sum_{j>k} a_j \varphi_j \right\},$$

$$H^{\frac{1}{2},k-} := \left\{ \phi_2 \in H^{\frac{1}{2}}(S) \mid \phi_2 = \sum_{j\leq k} a_j \varphi_j \right\},$$

we have then the orthogonal decomposition

$$H^{\frac{1}{2}}(S) = H^{\frac{1}{2},k+} \oplus \left(H^{\frac{1}{2},k-} \cap H^{\frac{1}{2},+}(S) \right) \oplus H^{\frac{1}{2},-}(S).$$

Now consider the set

$$\mathscr{N}_k := \{0\} \times \left(H^{\frac{1}{2},k-} \cap H^{\frac{1}{2},+}(S) \right) \subset H^1(M) \times H^{\frac{1}{2}}(S).$$

It is easy to see that $\mathscr{N}_k$ is a linear subspace inside N, and along this subspace the functional J_ρ is not larger than the minimal critical value:

$$J_\rho(0,\phi_1) = 4\pi(\gamma-1) - 2\sum_{0<j\leq k} (\rho - \lambda_j) a_j^2 \leq 4\pi(\gamma-1).$$

For $\tau > 0$ let us consider the following cone around $\mathscr{N}_k$:

$$\mathcal{C}_\tau(\mathscr{N}_k) := \Big\{ (u,\psi) \in N \mid u \in H^1(M), \psi = \phi_1 + \phi_2 + \psi^- \in H^{\frac{1}{2},k+} \oplus (H^{\frac{1}{2},k-} \cap H^{\frac{1}{2},+}(S)) \oplus H^{\frac{1}{2},-}(S), \\ \|u\|_{H^1} + \|\phi_1\|^2 + \|\psi^-\|^2 < \tau \|\phi_2\|^2 \Big\}.$$

We claim that for τ suitably chosen this cone contains all the decreasing directions, in the sense that outside the cone we can find a region on which the functional is strictly above the ground state level.

Letting $(u,\psi) \in N \setminus \mathcal{C}_\tau(\mathscr{N}_k)$, i.e., with $\psi = \phi_1 + \phi_2$ decomposed as above, we have

$$\|u\|_{H^1}^2 + \|\phi_1\|^2 + \|\psi^-\|^2 \geq \tau \|\phi_2\|^2.$$

By (4.1), which can be now interpreted as

$$\|\psi^-\|^2 \leq C\rho^2 (\|\phi_1\|^2 + \|\phi_2\|^2),$$

we see that

$$\|\phi_2\|^2 \leq \frac{1}{\tau - C\rho^2} \left(\|u\|_{H^1}^2 + (1 + C\rho^2)\|\phi_1\|^2 \right). \tag{4.3}$$

Moreover, this also implies that

$$\|u\|_{H^1}^2 + \|\phi_1\|^2 \geq C(\|u\|^2 + \|\psi\|^2),$$

for some $C = C(\rho,\tau) > 0$.

Then in (4.2) for the scalar component we have as before the control

$$F(u) \geq 4\pi(\gamma-1) + C\|u\|_{H^1}^2.$$

For the spinorial part, since $\psi = \phi_1 + \phi_2 + \psi^-$ is an orthogonal decomposition, we have

$$\begin{aligned}
Q(u,\psi) =& 2\int_M \left\langle (\not{D} - \rho)\psi, \psi^+ \right\rangle \mathrm{d}v_g + 2\rho \int_M (1-e^u) \left\langle \psi, \psi^+ \right\rangle \mathrm{d}v_g \\
=& 2\int_M \left\langle (\not{D} - \rho)\phi_1, \phi_1 \right\rangle \mathrm{d}v_g + 2\int_M \left\langle (\not{D} - \rho)\phi_2, \phi_2 \right\rangle \mathrm{d}v_g \\
&+ 2\rho \int_M (1-e^u) \left\langle \psi, \phi_1 + \phi_2 \right\rangle \mathrm{d}v_g \\
\geq& C\left(1 - \frac{\rho}{\lambda_{k+1}}\right) \|\phi_1\|^2 - C\left(\frac{\rho}{\lambda_k} - 1\right) \|\phi_2\|^2 - C\rho \|u\|_{H^1} \|\psi\| (\|\phi_1\| + \|\phi_2\|).
\end{aligned}$$

Assuming $\|u\|_{H^1}^2 + \|\psi\|^2 = r^2$ is small and noting (4.3), we get

$$\begin{aligned}
J_\rho(u, \phi_1 + \phi_2) \geq& 4\pi(\gamma - 1) + C\|u\|_{H^1}^2 + C\left(1 - \frac{\rho}{\lambda_{k+1}} - Cr^2\right) \|\phi_1\|^2 \\
&- C\left(\frac{\rho}{\lambda_k} - 1 - Cr^2\right) \|\phi_2\|^2 \\
\geq& 4\pi(\gamma - 1) + C\|u\|_{H^1}^2 + C\left(1 - \frac{\rho}{\lambda_{k+1}} - Cr^2\right) \|\phi_1\|^2 \\
&- C\left(\frac{\rho}{\lambda_k} - 1\right) \frac{\|u\|_{H^1}^2 + (1 + C\rho^2)\|\phi_1\|^2}{\tau - C\rho^2}.
\end{aligned}$$

We can first choose r small enough and then choose τ large enough such that

$$\begin{aligned}
J_\rho(u, \phi_1 + \phi_2) \geq& 4\pi(\gamma - 1) + C(\|u\|^2 + \|\phi_1\|^2) \\
\geq& 4\pi(\gamma - 1) + Cr^2
\end{aligned}$$

outside the cone $\mathcal{C}(\mathscr{N}_k)$. Thus the claim is confirmed.

For r as above, consider the set

$$L_1 := \big(\partial B_r(0,0) \backslash \mathcal{C}(\mathscr{N}_k)\big) \cap N,$$

which is non-empty since $(0, r\varphi_{k+1}) \in L_1$. Recall that N is locally modeled by a Hilbert space, e.g. $T_{(0,0)}N$. We can assume that in a local chart, $\mathscr{N}_k$ is some *coordinate subspace*, while L_1 is homeomorphic to a collar neighborhood of the sphere (of infinite dimension) which lies in a subspace complementary to $\mathscr{N}_k$ and intersects $\mathcal{C}_\tau(\mathscr{N}_k)$ only at $\{(0,0)\}$.

Next we introduce a set L_2 on which the functional attains low values and such that it links with L_1, see Figure 1. The construction of such a set is performed in several steps. First we take the ball

$$B_R^{0,k}(0) := \Big\{(0, \phi_2) \in \mathscr{N}_k \mid \|\phi_2\| \leq R\Big\} \subset \mathscr{N}_k$$

with $R > 0$ a large constant to be fixed later. Note that for any $(0, \phi_2)$, $J_\rho(0, \phi_2) \leq 4\pi(\gamma - 1)$ and for $(0, \phi_2) \in \partial B_R^{0,k}(0)$,

$$J_\rho(0, \phi_2) \leq 4\pi(\gamma - 1) - C\left(\frac{\rho}{\lambda_k} - 1\right) \|\phi_2\|^2 \leq 4\pi(\gamma - 1) - C\left(\frac{\rho}{\lambda_k} - 1\right) R^2.$$

For any $(0, \phi_2) \in \partial B_R^{0,k}(0)$, we consider the following curves. First let

$$\sigma_1 \colon [0, T] \to N, \quad \sigma_1(t) := (t, \phi_2 + At\varphi_{k+1}),$$

where $A > 0$ is again a constant to be fixed later. One easily sees that this is a curve in N and

$$\begin{aligned}
J_\rho(t, \phi_2 + At\varphi_{k+1}) &= vol(M,g)(e^{2t} - 2t) + 2\int_M \left\langle (\not{D} - \rho e^t)\phi_2, \phi_2 \right\rangle \mathrm{d}v_g \\
&\quad + 2A^2t^2 \int_M \left\langle (\not{D} - \rho e^t)\varphi_{k+1}, \varphi_{k+1} \right\rangle \mathrm{d}v_g \\
&\leq 4\pi(\gamma - 1)(e^{2t} - 2t) + 2\int_M \left\langle (\not{D} - \rho)\phi_2, \phi_2 \right\rangle \mathrm{d}v_g + 2A^2t^2(\lambda_{k+1} - \rho e^t) \\
&\leq 4\pi(\gamma - 1)(e^{2t} - 2t) - C\left(\frac{\rho}{\lambda_k} - 1\right) R^2 + 2A^2t^2(\lambda_{k+1} - \rho e^t).
\end{aligned}$$

Now we fix some constants:

- we choose $T > 0$ such that $\rho e^T - \lambda_{k+1} \geq 1$;
- then we choose $A > 0$ such that

$$4\pi(\gamma - 1)(e^{2T} - 2T) - 2A^2T^2(\rho e^T - \lambda_{k+1}) < 4\pi(\gamma - 1);$$

- finally, choose $R > 0$ such that for any $t \in [0, T]$

$$4\pi(\gamma - 1)(e^{2t} - 2t) - C\left(\frac{\rho}{\lambda_k} - 1\right) R^2 + 2A^2t^2(\lambda_{k+1} - \rho e^t) < 4\pi(\gamma - 1).$$

Then we consider the curve

$$\sigma_2 \colon [-1, 1] \to N, \quad \sigma_2(r) \coloneqq \big(T, (-r)\phi_2 + AT\varphi_{k+1}\big),$$

which joins $(T, \phi_2 + AT\varphi_{k+1})$ to $(T, -\phi_2 + AT\varphi_{k+1})$ inside N. Thereafter we can come back to $\mathscr{N}_k$ via the curve

$$\sigma_3 \colon [0, T] \to N, \quad \sigma_3(t) \coloneqq \big((T - t), \phi_2 + A(T - t)\varphi_{k+1}\big).$$

Finally, consider the subset

$$\mathcal{D} \coloneqq \left\{ (t, At\varphi_{k+1} + \phi_2) \mid t \in [0, T], (0, \phi_2) \in B_R^{0,k}(0) \right\},$$

which is compact and homeomorphic to a finite-dimensional cylindrical segment

$$[0, T] \times B_R^{0,k}(0).$$

Note that $\mathcal{D} \subset N$ and let $L_2 = \partial\mathcal{D}$, see Figure 1. The curves $\sigma_1, \sigma_2, \sigma_3$ constructed above pass through every point of $L_2 \backslash B_R^{0,k}(0)$. It follows that on L_2 the functional attains low values. One can shrink L_2 (in an homotopically equivalent way) into the coordinate chart to see that L_1 and L_2 actually link, see e.g. [2, 41] for a rigorous definition of this concept.

Now we define the linking level. Let Γ be the space of continuous maps

$$\alpha \colon \mathcal{D} \to N,$$

such that $\alpha(v, h) = (v, h)$ for any $(v, h) \in L_2 = \partial\mathcal{D}$. This set Γ is clearly non-empty since $\mathrm{Id}_{\mathcal{D}} \in \Gamma$. Then we define the linking level

$$c_1 \coloneqq \inf_{\alpha \in \Gamma} \max_{(v,h) \in \mathcal{D}} J_\rho(\alpha(v, h)).$$

As L_1 and L_2 link, we have from the above arguments that

$$c_1 \geq 4\pi(\gamma - 1) + \theta(r).$$

It follows that c_1 is a critical value for J_ρ, and again we obtain a critical point for J_ρ which is different from the trivial one. This concludes the proof of Theorem 1.1 in this case as well.

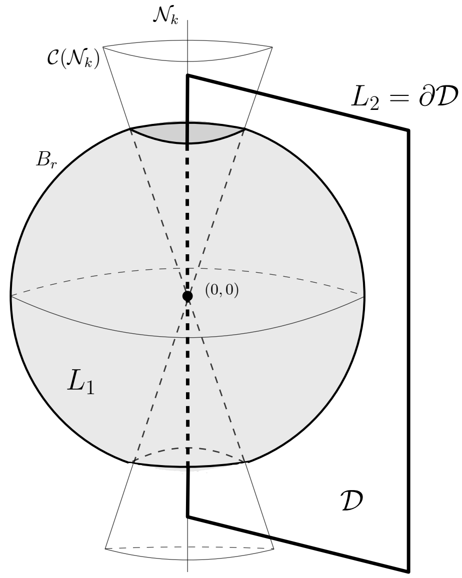

Figure 1.

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