ON CRITICAL POINTS OF THE RELATIVE FRACTIONAL PERIMETER

ANDREA MALCHIODI, MATTEO NOVAGA, AND DAYANA PAGLIARDINI

ABSTRACT. We study the localization of sets with constant nonlocal mean curvature and prescribed small volume in a bounded open set, proving that they are sufficiently close to critical points of a suitable nonlocal potential. We then consider the fractional perimeter in half-spaces. We prove existence of minimizers under fixed volume constraint, and we show some properties such as smoothness and rotational symmetry.

CONTENTS

1. INTRODUCTION

Isoperimetric problems play a crucial role in several areas such as geometry, linear and nonlinear PDEs, probability, Banach space theory and others. The classical version consists in studying least-area sets contained in a fixed region (the Euclidean space or any given domain). If the ambient space is an N-dimensional manifold M^N with or without boundary, the goal would be to find, among all the compact hypersurfaces $\Sigma \subset M$ which bound a region Ω of given volume $V(\Omega) = m$ (for $0 < m < V(M)$), those of minimal area $A(\Sigma)$. Such a region Ω is called an *isoperimetric region* and its boundary Σ is called an *isoperimetric hypersurface.*

A first general existence and regularity result can be obtained for example combining the results in [2] with those in [22, 26]. In particular we have that if $N \leq 7$, Σ is smooth. We also refer the reader to the interesting survey [35].

Beyond the existence and the regularity problem, it is also interesting to study the geometry and the topology of the solutions, and to give a qualitative description of the isoperimetric regions. Concerning these issues, we recall that in [31] it was proved that a region of small prescribed volume in a smooth and compact Riemannian manifold has asymptotically (as the volume tends to zero) at least as much perimeter as a round ball.

Date: October 19, 2020.

Afterwards, regarding critical points of the perimeter relative to a given set, in [19] the existence of surfaces with the shape of half spheres was shown, surrounding a small volume near nondegenerate critical points of the mean curvature of the boundary of an open smooth set in $\mathbb{R}^3$. It was proved that the boundary mean curvature determines the main terms, studying the problem via a Lyapunov-Schmidt reduction. In [18], the same author showed that isoperimetric regions with small volume in a bounded smooth domain Ω are near global maxima of the mean curvature of Ω.

Results of this type were proven in [14] and [39]. These authors considered closed manifolds and proved that isoperimetric regions with small volume locate near the maxima of the scalar curvature. In [39] a viceversa was also shown: for every non-degenerate critical point p of the scalar curvature there exists a neighborhood of p foliated by constant mean curvature hypersurfaces. Moreover, in [38] the boundary regularity question for the capillarity problem was studied.

In recent years fractional operators have received considerable attention for both in pure and applied motivations. In particular, regarding perimeter questions, in [5] the link between the fractional perimeter and the classical De Giorgi's perimeter was analyzed, showing the equi-coercivity and the Γ-convergence of the fractional perimeter, up to a scaling factor $\omega_{N-1}^{-1}(1-2s)$, to the classical perimeter in the sense of De Giorgi and a local convergence result for minimizers was deduced.

Another relevant result about fractional perimeter was obtained in [20], generalizing a quantitative isoperimetric inequality to the fractional setting. Indeed, in the Euclidean space, it is known that among all sets of prescribed measure, balls have the least perimeter, i.e. for any Borel set $E \subset \mathbb{R}^N$ of finite Lebesgue measure, one has

$$N|B_1|^{\frac{1}{N}}|E|^{\frac{N-1}{N}} \leq P(E) \tag{1.1}$$

with B_1 denoting the unit ball of $\mathbb{R}^N$ with center at the origin and $P(E)$ is the distributional perimeter of E. The equality in (1.1) holds if and only if E is a ball.

In [21] a similar result for the fractional perimeter P_s (defined as in (2.3)) was obtained, improved then in [20] showing the following fact: for every $N \geq 2$ and any $s_0 \in (0, 1/2)$ there exists $C(N, s_0) > 0$ such that

$$P_s(E) \geq \frac{P_s(B_1)}{|B_1|^{\frac{N-2s}{N}}}|E|^{\frac{N-2s}{N}}\left\{1 + \frac{A(E)^2}{C(N,s)}\right\} \tag{1.2}$$

whenever $s \in [s_0, 1/2]$ and $0 < |E| < \infty$. Here

$$A(E) := \inf\left\{\frac{|E \triangle (B_{r_E}(x))|}{|E|} : x \in \mathbb{R}^N\right\}$$

stands for the *Fraenkel asymmetry* of E, measuring the L^1-distance of E from the set of balls of volume $|E|$ and $r_E = (|E|/|B_1|)^{1/N}$ so that $|E| = |B_{r_E}|$.

In the same spirit of extension of classical results to the fractional setting, we also mention [28]. Here the authors modify the classical Gauss free energy functional used in capillarity theory by considering surface tension energies of nonlocal type. They consider a family of problems including a nonlocal isoperimetric problem of geometric interest. More precisely, given $N \geq 2$, $s \in (0, 1/2)$, $\lambda \geq 1$ and $\varepsilon \in [0, \infty]$ they introduce the interaction kernels $\mathbf{K}(N,s,\lambda,\varepsilon)$, which are even functions $K:\mathbb{R}^N\setminus\{0\}\to[0,+\infty)$ such that

$$\frac{\chi_{B_\varepsilon}(z)}{\lambda|z|^{N+2s}}\le K(z)\le\frac{\lambda}{|z|^{N+2s}}\quad\forall\, z\in\mathbb{R}^N\setminus\{0\},$$

where $B_\varepsilon(x)$ is the ball of center x and radius ε. Taking $\Omega\subset\mathbb{R}^N$ and $\sigma\in(-1,1)$ the authors studied the nonlocal capillarity energy of $E\subset\Omega$ defined as

$$\mathcal{E}(E)=\int_E\int_{E^C\cap\Omega}K(x,y)\,\mathrm{d}x\,\mathrm{d}y+\sigma\int_E\int_{\Omega^C}K(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

with $K\in\mathbf{K}(N,s,\lambda,\varepsilon)$, giving existence and regularity results, density estimates and new equilibrium conditions with respect to those of the classical energy.

As it concerns constant nonlocal mean curvature, we mention the paper [9], where it was proved the existence of Delaunay type surfaces, i.e. a smooth branch of periodic topological cylinders with the same constant nonlocal mean curvature, and [30], where the author constructs two families of hypersurfaces with constant nonlocal mean curvature. Moreover, in [29] the axial symmetry of *smooth* critical points of the fractional perimeter in a half-space was shown, using a variant of the moving plane method.

Motivated by these results, our aim is to study sets with constant nonlocal mean curvature in an open bounded domain. Following [28], the notion of relative fractional perimeter $P_s(E,\Omega)$ and of relative fractional mean curvature $H^\Omega_{s,\partial E}$ that we shall use are given by formulas (2.3) and (2.4) in the next section.

We point out that these are not the only possible definitions of fractional perimeter and mean curvature localized in a set Ω (for instance, one could consider a nonlocal kernel depending on the set itself). However, on one hand it is a first natural choice, on the other hand the methods and results in this paper could probably be extended to more general kernels obtained by adding a smooth function possibly depending on Ω.

In the first part of this paper we show the following existence result for sets with constant nonlocal mean curvature and small volume.

Theorem 1.1. *Let $s\in(0,1/2)$ and $\Omega\subseteq\mathbb{R}^N$ be a bounded open set with smooth boundary. For $x\in\Omega$, we set*

$$V_\Omega(x):=\int_{\Omega^C}\frac{1}{|x-y|^{N+2s}}\,\mathrm{d}y. \tag{1.3}$$

Then for every strict local extremal or non-degenerate critical point x_0 of V_Ω, there exists $\overline{\varepsilon}>0$ such that for every $0<\varepsilon<\overline{\varepsilon}$ there exist embedded spherical-shaped surfaces S_ε with constant $H^\Omega_{s,S_\varepsilon}$ curvature and enclosing volume identically equal to ε, approaching x_0 as $\varepsilon\to0$.

One of the main tools for proving this result relies on the non-degeneracy of spheres with respect to the linearized non-local mean curvature equation, which follows from a result in [9]. After non-degeneracy is established, we can use a Lyapunov-Schmidt reduction to study a finite-dimensional problem, which is treated by carefully expanding the relative fractional perimeter of balls with small volume. Thanks to classical results in min-max theory, we obtain as a corollary a multiplicity result. Here and in the following, $\mathrm{cat}(\Omega)$ denotes the Lusternik-Schnirelman category of the set Ω (see [27] and Section 2 below for more details).

Corollary 1.2. *Let $s \in (0, 1/2)$ and $\Omega \subseteq \mathbb{R}^N$ be a bounded open set with smooth boundary. Then there exists $\overline{\varepsilon} > 0$ such that for every $0 < \varepsilon < \overline{\varepsilon}$ there exist at least cat(Ω) spherical-shaped surfaces S_ε with constant $H^{\Omega}_{s,S_\varepsilon}$ curvature and enclosing volume identically equal to ε.*

In the last part of this work we study the existence and some properties of sets minimizing the fractional perimeter in a particular domain, namely a half-space:

Theorem 1.3. *For all $m \in (0, +\infty)$, there exists a minimizer E for the problem*

$$\inf \Big\{ P_s(E, \mathbb{R}^N_+) : \ E \subset \mathbb{R}^N_+, \ |E| = m \Big\}, \tag{1.4}$$

where $\mathbb{R}^N_+ := \{x = (x', x_N) \in \mathbb{R}^N : \ x_N > 0\}$. Moreover, up to a horizontal translation, the set E is bounded, radially symmetric in x', and $\partial E \cap \{x_N > 0\}$ is of class C^∞ out of a singular set which is finite or accumulates at the origin.

This result is proved by showing first the existence of a properly rearranged minimizing sequence which is axially symmetric, and then employing some results from [6], [10], [28] to prove a diameter bound and smoothness of the limit set.

The paper is organized as follows: In Section 2 we introduce some notation on fractional perimeter and mean curvature, and we show some preliminary results, especially on the linearized fractional mean curvature. We prove in particular the minimal degeneracy for spheres, also relative to suitably large domains. In Section 3 we prove Theorem 1.1 via a Lyapunov-Schmidt reduction and Corollary 1.2 through a well known result about the Lusternik-Schnirelman category. Finally, in Section 4 we prove Theorem 1.3 in two steps: the existence of minimizers in a bounded domain is a standard consequence of the direct method of the Calculus of Variations. We then show the symmetry of minimizers and, using density estimates, we prove a bound on the diameter and hence the free minimality.

Acknowledgements

A.M. has been supported by the project *Geometric Variational Problems* from Scuola Normale Superiore, A.M. and D.P. by MIUR Bando PRIN 2015 2015KB9WPT$_{001}$, M.N. by the University of Pisa via the grant PRA-2017-23. The authors are all members of GNAMPA–INdAM.

2. Notation and preliminary results

In this section we introduce the notation that will be used throughout the paper. We first define fractional perimeter and fractional mean curvature, listing some of their properties.

For $0 < s < 1/2$ the *fractional perimeter* (or s-perimeter) of a measurable set $E \subset \mathbb{R}^N$ is defined as

$$P_s(E) := \int_E \int_{E^C} \frac{\mathrm{d}x \, \mathrm{d}y}{|x - y|^{N+2s}}, \tag{2.1}$$

where E^C is the complement of E. It has also a simple representation in terms of the usual seminorm in the fractional Sobolev space $H^s(\mathbb{R}^N)$, that is

$$P_s(E) = [\chi_E]^2_{H^s(\mathbb{R}^N)} := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\chi_E(x) - \chi_E(y)|^2}{|x - y|^{N+2s}} \, \mathrm{d}x \, \mathrm{d}y,$$

where χ_E denotes the characteristic function of E. We say that a set $E \subset \mathbb{R}^N$ has *finite s-perimeter* if (2.1) is finite. If E is an open set and ∂E is a smooth bounded surface, we have from [5, Theorem 2] that as $s \to 1/2$

$$(1-2s)P_s(E) \to \omega_{N-1}P(E), \tag{2.2}$$

where ω_{N-1} denote the volume of the unit ball in $\mathbb{R}^{N-1}$ for $N \geq 2$ and $P(E)$ is the perimeter in the sense of De Giorgi.

Following [28], the notion of fractional perimeter can be considered also relative to an open set $\Omega \subset \mathbb{R}^N$ by the formula

$$P_s(E,\Omega) := \int_E \int_{\Omega\setminus E} \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^{N+2s}}. \tag{2.3}$$

Let $s \in (0,1/2)$ and let $\Omega \subseteq \mathbb{R}^N$ be an open set. We recall that the nonlocal mean curvature of a set E at a point $x \in \partial E$ is defined as follows

$$H^{\Omega}_{s,\partial E}(x) := \int_{\Omega} \frac{\chi_{E^c\cap\Omega}(y) - \chi_E(y)}{|x-y|^{N+2s}}\,\mathrm{d}y, \tag{2.4}$$

(see [28, Theorem 1.3 and Proposition 3.2 with $\sigma = 0$ and $g = 0$]) where χ_E denotes the characteristic function of E, E^C is the complement of E, and the integral has to be understood in the principal value sense.

If E is smooth and compactly contained in Ω, let w be a smooth function defined on on ∂E, with small L^∞ norm. We call E_w the set whose boundary ∂E_w is parametrized by

$$\partial E_w = \{x + w(x)\nu_E(x) | x \in \partial E\} \tag{2.5}$$

where ν_E is a normal vector field to ∂E exterior to E.

The first variation of the s-perimeter (2.3) along normal perturbations is given by

$$\mathrm{d}_t P_s(E_{tw},\Omega)_{|t=0} = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} P_s(E_{tw},\Omega) = \int_{\partial E} H^{\Omega}_{s,\partial E} w\,\mathrm{d}\sigma, \tag{2.6}$$

for w of class $C^{1,\beta}$ with $\beta \in (2s,1)$, see [15].

In the following, we let $B_r(\xi)$ be a ball with center $\xi \in \mathbb{R}^N$ and radius $r > 0$. Given $w \in C^{1,\beta}(\partial B_1(0))$ we denote by $\mathbb{B}(\xi, w)$ the normal graph

$$\partial\mathbb{B}(\xi,w) := \{y \in \mathbb{R}^N : y = \xi + (1+w(\sigma))\sigma, \sigma \in \partial B_1(0)\}. \tag{2.7}$$

Then we let

$$S_\xi := \partial B_1(\xi) \qquad \text{and} \qquad P^{\Omega}_{s,\xi}(w) := P^{\Omega}_s(\mathbb{B}(\xi,w),\Omega). \tag{2.8}$$

Moreover, for $\varphi \in C^{1,\beta}(\partial\mathbb{B}(\xi,w))$, we set

$$\left(P^{\Omega}_{s,\xi}\right)'(w)[\varphi] := \int_{\partial\mathbb{B}(\xi,w)} H^{\Omega}_{s,\partial\mathbb{B}(\xi,w)}\varphi\,\mathrm{d}\sigma_w$$

where $d\sigma_w$ stands for the area element of $\partial\mathbb{B}(\xi, w_\varepsilon(\xi))$.

Consider next the *spherical fractional Laplacian*

$$L_s\varphi(\theta) := P.V. \int_S \frac{\varphi(\theta)-\varphi(\sigma)}{|\theta-\sigma|^{N+2s}}\,\mathrm{d}\sigma,$$

where $S = \partial B_1$ and the above integral is understood in the principal value sense.

It turns out that (see e.g. [9])

$$L_s : C^{1,\beta}(S) \to C^{\beta-2s}(S). \tag{2.9}$$

The operator L_s has an increasing sequence of eigenvalues $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$, whose explicit expression is given by

$$\lambda_k := \frac{\pi^{(N-1)/2}\Gamma((1-2s)/2}{(1+2s)2^{2s}\Gamma((N+2s)/2)}\left(\frac{\Gamma\Big(\dfrac{2k+N+2s}{2}\Big)}{\Gamma\Big(\dfrac{2k+N-2s-2}{2}\Big)} - \frac{\Gamma\Big(\dfrac{N+2s}{2}\Big)}{\Gamma\Big(\dfrac{N-2s-2}{2}\Big)}\right), \tag{2.10}$$

see [36, Lemma 6.26], where Γ is the Euler Gamma function. The eigenfunctions are the usual spherical harmonics, i.e. one has

$$L_s\psi = \lambda_k\psi \qquad \text{for every } k \in \mathbb{N} \text{ and } \psi \in \mathcal{E}_k,$$

where $\mathcal{E}_k$ are the spherical harmonics of degree k and dimension $n_k = N_k - N_{k-2}$, with

$$N_k = \frac{(n+k-1)!}{(n-1)!k!}, \quad k \geq 0, \qquad\qquad N_k = 0 \quad k < 0.$$

We recall that $n_0 = 1$ and that $\mathcal{E}_0$ consists of constant functions, whereas $n_1 = N$ and $\mathcal{E}_1$ is spanned by the restrictions of the coordinate functions in $\mathbb{R}^N$ to the unit sphere S.

For sets that are suitable graphs over the unit sphere S of $\mathbb{R}^N$, we have the following result concerning fractional mean curvature relative to the whole space, see [9, Theorem 2.1, Lemma 5.1 and Theorem 5.2] (see also formula (1.3) in the latter paper).

Proposition 2.1. *Given $\beta \in (2s, 1)$, consider the family of functions*

$$\Upsilon := \left\{\varphi \in C^{1,\beta}(S) \ : \ \|\varphi\|_{L^\infty(S)} < \frac{1}{2}\right\}.$$

Then the map $\varphi \mapsto H^{\mathbb{R}^N}_{s,\partial\mathbb{B}(0,\varphi)}$ is a C^∞ function from Υ into $C^{\beta-2s}(S)$. Moreover, its linearization at $\varphi \equiv 0$ is given by

$$\varphi \longmapsto 2d_{N,s}(L_s - \lambda_1)\varphi, \tag{2.11}$$

where λ_1 is defined in (2.10) *and $d_{N,s} := \frac{1-2s}{(N-1)|B_1^{N-1}|}$, B_1^{N-1} the unit ball in $\mathbb{R}^{N-1}$.*

In the above formula, with an abuse of notation, we are viewing $H^{\mathbb{R}^N}_{s,\partial\mathbb{B}(\xi,\varphi)}$ as a function defined on S via the formula

$$H^{\mathbb{R}^N}_{s,\partial\mathbb{B}(\xi,\varphi)}(\sigma) = H^{\mathbb{R}^N}_{s,\partial\mathbb{B}(0,\varphi)}(\xi + (1+\varphi(\sigma))\sigma); \qquad \sigma \in S.$$

As a consequence of the latter result we have that every function in the kernel of the above linearized nonlocal mean curvature is a linear combination of first-order spherical harmonics, i.e. if $w \in \text{Ker}\,(L_s - \lambda_1)$, we have

$$w = \sum_{i=1}^{N} \lambda_i Y_i, \tag{2.12}$$

where $\{Y_i\}_{i=1,\cdots,N} \in \mathcal{E}_1$ and $\lambda_i \in \mathbb{R}$. Let us define

$$W := \left\{w \in C^{1,\beta}(S) : \int_S w\, Y_i d\sigma = 0 \text{ for } i = 1, \ldots, N\right\}; \tag{2.13}$$

$$(2.14) \qquad R := \left\{ h \in C^{\beta-2s}(S) : \int_S h\, Y_i d\sigma = 0 \text{ for } i = 1, \dots, N \right\} :$$

it follows by Fredholm's theory that $L_s - \lambda_1$ is invertible from $W \subseteq C^{\beta-2s}(S)$ into R. Tautologically, if P_R is the orthogonal projection in $L^2(S)$ from $C^{\beta-2s}(S)$ onto R, the operator $P_R \circ (L_s - \lambda_1)$ is invertible.

As a consequence of the above proposition, using a perturbation argument, we deduce also the following result, for which we need to introduce some notation. Let Ω be a bounded set in $\mathbb{R}^N$, and for $\varepsilon > 0$ let $\Omega_\varepsilon := \frac{1}{\varepsilon}\Omega$. Fix a compact set Θ in Ω, and let $\xi \in \Theta_\varepsilon := \frac{1}{\varepsilon}\Theta$. Consider then the operator $L^{\Omega_\varepsilon}_{s,\xi}$ corresponding to the linearization of the s-mean curvature at $B_1(\xi)$ relative to Ω_ε, namely the non-local operator such that

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} H^{\Omega_\varepsilon}_{s,\partial\mathbb{B}(\xi,t\varphi)} = L^{\Omega_\varepsilon}_{s,\xi}\, \varphi,$$

see (2.7). Similarly to before, we are transporting $H^{\Omega_\varepsilon}_{s,\partial\mathbb{B}(\xi,t\varphi)}$ on S via the formula

$$H^{\Omega_\varepsilon}_{s,\partial\mathbb{B}(\xi,\varphi)}(\sigma) = H^{\Omega_\varepsilon}_{s,\partial\mathbb{B}(\xi,\varphi)}(\xi + (1+\varphi(\sigma))\sigma); \qquad \sigma \in S.$$

We have then the following result.

Proposition 2.2. *Let Ω, Θ, ξ and $L^{\Omega_\varepsilon}_{s,\xi}$ be as above, and let $\beta \in (2s, 1)$. Consider the family of functions*

$$\Upsilon := \left\{ \varphi \in C^{1,\beta}(S) \;:\; \|\varphi\|_{L^\infty(S)} < \frac{1}{2} \right\}.$$

Then $(\xi, \varphi) \mapsto H^{\Omega_\varepsilon}_{s,\partial\mathbb{B}(\xi,\varphi)}$ is a C^∞-map from $\Theta_\varepsilon \times \Upsilon$ into $C^{\beta-2s}(S)$, whose first- and second-order derivatives in ξ tend to zero uniformly as $\varepsilon \to 0$. Moreover, if W and R are as in (2.13)-(2.14), $P_R \circ L^{\Omega_\varepsilon}_{s,\xi}$ is invertible with uniformly bounded inverse from W into R.

Proof. Consider the expression in (2.4) for Ω_ε, which for $x \in \partial E$ can be written as

$$H^{\Omega_\varepsilon}_{s,\partial E}(x) := \int_{\mathbb{R}^N} \frac{\chi_{E^c \cap \Omega_\varepsilon}(y) - \chi_E(y) - \chi_{\Omega_\varepsilon^C}}{|x-y|^{N+2s}}\, \mathrm{d}y = H^{\mathbb{R}^N}_{s,\partial E}(x) - \int_{\Omega_\varepsilon^C} \frac{1}{|x-y|^{N+2s}}\, \mathrm{d}y.$$

If $\xi \in \Theta_\varepsilon$, $\varphi \in \Upsilon$ and $E = \mathbb{B}(\xi, \varphi)$, in the last integral $|x - y|$ is bounded below by a positive constant (depending on Θ) times $\frac{1}{\varepsilon}$. Notice also that the last integral is smooth in (ξ, φ) ($x \in \partial\mathbb{B}(\xi, \varphi)$) with zero-th, first- and second-order derivatives in ξ that tend to zero uniformly as $\varepsilon \to 0$. The conclusion then follows from Proposition 2.1 and the comments after (2.14). □

Given a topological space M and a subset $A \subseteq M$, we recall next the definition and some properties of the Lusternik-Schnirelman category.

Definition 2.3. [3, Definition 9.2] The category of A with respect to M, denoted by $\mathrm{cat}_M(A)$, is the least integer k such that $A \subseteq A_1 \cup \cdots \cup A_k$ with A_i closed and contractible in M for every $i = 1, \cdots, k$.

We set $\mathrm{cat}(\varnothing) = 0$ and $\mathrm{cat}_M(A) = +\infty$ if there are no integers with the above property. We will use the notation $\mathrm{cat}(M)$ for $\mathrm{cat}_M(M)$.

Remark 2.4. From Definition 2.3, it is easy to see that $\mathrm{cat}_M(A) = \mathrm{cat}_M(\bar{A})$. Moreover, if $A \subset B \subset M$, we have that $\mathrm{cat}_M(A) \leq \mathrm{cat}_M(B)$, see [3, Lemma 9.6].

Then assuming that

$$M = F^{-1}(0), \text{ where } F \in C^{1,1}(E, \mathbb{R}) \text{ with } E \supset M \text{ and } F'(u) \neq 0 \ \forall\, u \in M, \tag{2.15}$$

we set

$$\mathrm{cat}_k(M) = \sup\{\mathrm{cat}_M(A) : A \subset M \text{ and } A \text{ is compact}\}.$$

Note that if M is compact, $\mathrm{cat}_k(M)$ =cat(M). At this point we can state a useful result about the Lusternik-Schnirelman category, see e.g. [3] for the definition of Palais-Smale (PS) condition.

Theorem 2.5. [3, Theorem 9.10] *Let M be a Hilbert space or a complete Banach manifolds. Let* (2.15) *hold, let $J \in C^{1,1}(M, \mathbb{R})$ be bounded from below on M and let J satisfy (PS)-condition. Then J has at least $\mathrm{cat}_k(M)$ critical points.*

Remark 2.6. If M has boundary, under the same assumptions of Theorem 2.5 one can still find at least $\mathrm{cat}_k(M)$ critical points for J provided ∇J is non zero on ∂M and points in the outward direction.

3. Existence of critical points and asymptotic behavior for small volumes

In this section we prove Theorem 1.1 via a finite-dimensional reduction. This will determine the location of critical points of the relative s-perimeter depending on s and the geometry of the domain. One of the main tools is the following asymptotic expansion of the relative s-perimeter. Recall that for $\varepsilon > 0$ we set $\Omega_\varepsilon := \dfrac{1}{\varepsilon}\Omega$, and we aim to prove that the nonlocal mean curvature H_s^Ω of suitable small balls is *sufficiently close* to $H_s^{\mathbb{R}^N}$ on the same sets. Hereafter we will write simply H_s to denote $H_s^{\mathbb{R}^N}$.

Lemma 3.1. *Let $\Theta \subseteq \Omega$ be a fixed compact set. For all $\varepsilon > 0$ we consider $B_1(\bar{x})$ a ball of center $\bar{x} \in \Theta_\varepsilon := \frac{1}{\varepsilon}\Theta$ and with unit radius. Then, for the fractional perimeter, the following expansion holds*

$$P_s(B_1(\bar{x}), \Omega_\varepsilon) = P_s(B_1(\bar{x})) - \omega_N \varepsilon^{2s} V_\Omega(\varepsilon \bar{x}) + O(\varepsilon^{1+2s}) \qquad \text{as } \varepsilon \to 0, \tag{3.1}$$

where ω_N is the volume of the N-dimensional unit ball and V_Ω is defined in (1.3). *Moreover one has that*

$$\nabla_{\bar{x}} P_s(B_1(\bar{x}), \Omega_\varepsilon) = -\omega_N \varepsilon^{2s+1} \nabla V_\Omega(\varepsilon \bar{x}) + O(\varepsilon^{2+2s}). \tag{3.2}$$

Proof. Taking ε small enough, we can assume that $B_1(\bar{x}) \subset \Omega_\varepsilon$. From (2.3) we have

$$P_s(B_1(\bar{x}), \Omega_\varepsilon) - P_s(B_1(\bar{x})) = -\int_{B_1(\bar{x})} \int_{\mathbb{R}^N \setminus \Omega_\varepsilon} \frac{1}{|x-y|^{N+2s}} \, \mathrm{d}x \, \mathrm{d}y. \tag{3.3}$$

If we replace x with $\bar{x}$ in the last integrand, we obtain

$$\frac{1}{|x-y|^{N+2s}} = \frac{1}{|\bar{x}-y|^{N+2s}} + O\left(\frac{1}{|\bar{x}-y|^{N+2s+1}}\right); \quad x \in B_1(\bar{x}), \quad y \in \mathbb{R}^N \setminus \Omega_\varepsilon. \tag{3.4}$$

Therefore

$$\int_{B_1(\bar{x})}\int_{\mathbb{R}^N\setminus\Omega_\varepsilon}\frac{1}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y=\omega_N\int_{\mathbb{R}^N\setminus\Omega_\varepsilon}\frac{1}{|\bar{x}-y|^{N+2s}}\,\mathrm{d}y+\int_{\mathbb{R}^N\setminus\Omega_\varepsilon}\frac{O(1)}{|\bar{x}-y|^{N+2s+1}}\,\mathrm{d}y.$$

From the latter formulas and a change of variables one then finds

$$P_s(B_1(\bar{x}),\Omega_\varepsilon)-P_s(B_1(\bar{x}))=-\varepsilon^{2s}\omega_N\int_{\Omega^C}\frac{1}{|\varepsilon\bar{x}-y|^{N+2s}}\,\mathrm{d}y+O(\varepsilon^{1+2s}),$$

which concludes the proof of (3.1). To prove (3.2), we differentiate (3.3) in $\bar{x}$ and use the fact that $P_s(B_1(\bar{x}))$ is independent of $\bar{x}$, to obtain

$$\nabla_{\bar{x}}P_s(B_1(\bar{x}),\Omega_\varepsilon)=-\nabla_{\bar{x}}\int_{B_1(\bar{x})}\int_{\mathbb{R}^N\setminus\Omega_\varepsilon}\frac{1}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y.$$

We make the change of variables $x=\bar{x}+z$ to write the above formula as

$$\nabla_{\bar{x}}P_s(B_1(\bar{x}),\Omega_\varepsilon)=-\nabla_{\bar{x}}\int_{B_1(0)}\int_{\mathbb{R}^N\setminus\Omega_\varepsilon}\frac{1}{|\bar{x}+z-y|^{N+2s}}\,\mathrm{d}z\,\mathrm{d}y,$$

and the new changes of variable $\tilde{z}=\varepsilon z$, $\tilde{y}=\varepsilon y$ to transform it into

$$\nabla_{\bar{x}}P_s(B_1(\bar{x}),\Omega_\varepsilon)=-\varepsilon^{2s-N}\nabla_{\bar{x}}\int_{B_\varepsilon(0)}\int_{\mathbb{R}^N\setminus\Omega}\frac{1}{|\varepsilon\bar{x}+\tilde{z}-\tilde{y}|^{N+2s}}\,\mathrm{d}z\,\mathrm{d}y.$$

Since now the term $\bar{x}$ appears with a factor ε in the last integral, we obtain an extra power of ε when applying $\nabla_{\bar{x}}$. Performing the reverse changes of variables and using (3.4), formula (3.2) follows. □

Now we want to evaluate the deviation of the nonlocal mean curvature from a constant, when it is computed relatively to a large domain. To do that, we define

$$\tag{3.5}\begin{aligned}\tilde{H}_{s,\xi}&:S\to\mathbb{R}\\ \tilde{H}_{s,\xi}(x)&:=H^{\Omega_\varepsilon}_{s,S_\xi}(x+\xi).\end{aligned}$$

Lemma 3.2. *Let Θ, Θ_ε be as before, and let $\beta\in(2s,1)$. For the (relative) fractional mean curvature defined in* (2.4)*, the following expansion holds:*

$$\tag{3.6}\tilde{H}_{s,\xi}=c_{N,s}+O(\varepsilon^{2s})\qquad in\ C^{\beta-2s}(S)\qquad for\ \xi\in\Theta_\varepsilon,$$

where $c_{N,s}:=H_{s,S_\xi}$ and we recall that $S_\xi=\partial B_1(\xi)$ with $B_1(\xi)$ denoting the ball of center at ξ and unit radius. Moreover, one has that for all $i=1,\dots,N$,

$$\tag{3.7}\frac{\partial}{\partial\xi_i}\tilde{H}_{s,\xi}=O(\varepsilon^{2s+1})\qquad in\ C^{\beta-2s}(S)\qquad for\ \xi\in\Theta_\varepsilon.$$

Proof. Using the definition of (relative) fractional mean curvature (see (2.4)) and [37, Lemma 2], for $x\in\partial B_1$, we can write

$$\tag{3.8}\tilde{H}_{s,\xi}(x)=c_{N,s}+\int_{\mathbb{R}^N\setminus\Omega_\varepsilon}\frac{\mathrm{d}y}{|x+\xi-y|^{N+2s}}.$$

where $c_{N,s}:=H_{s,\xi}(\cdot+\xi)$.

Therefore we get that, for $x\in\partial B_1$,

$$\tag{3.9}\tilde{H}_{s,\xi}(x)=c_{N,s}+O(\varepsilon^{2s}).$$

Then, using (3.8) and differentiating with respect to ξ_i, we find that, for all $i = 1, \dots, N$,

$$\begin{aligned}\frac{\partial}{\partial \xi_i}\tilde{H}_{s,\xi} &= \frac{\partial}{\partial \xi_i}\left(c_{N,s} + \int_{\mathbb{R}^N\setminus\Omega_\varepsilon} \frac{\mathrm{d}y}{|x+\xi-y|^{N+2s}}\right) \\ &= O\left(\int_{\mathbb{R}^N\setminus\Omega_\varepsilon} \frac{\mathrm{d}y}{|x+\xi-y|^{N+2s+1}}\right) = O(\varepsilon^{2s+1}).\end{aligned} \tag{3.10}$$

Thus, we proved (3.6) and (3.7) in a pointwise sense. It is easy however to see that they also hold in the C^1 sense on the unit sphere S_ξ, and therefore also in $C^{\beta-2s}(S)$. □

We turn next to a finite-dimensional reduction of the problem, which is possible by the smallness of volume in the statement of Theorem 1.1. We refer to [4] for a general treatment of the subject.

Proposition 3.3. *Suppose that Ω is a smooth bounded set of $\mathbb{R}^N$, Θ a set compactly contained in Ω, and let $\beta \in (2s, 1)$. Then, for $\varepsilon > 0$ small, there exist $w_\varepsilon : \Theta_\varepsilon \times S \to \mathbb{R}$ and $\lambda = (\lambda_1, \cdots, \lambda_N) : \Theta_\varepsilon \times S \to \mathbb{R}^N$ such that*

$$|\mathbb{B}(\xi, w_\varepsilon)| = \omega_N; \qquad H^{\Omega_\varepsilon}_{s,\partial\mathbb{B}(\xi,w_\varepsilon)} = c + \sum_{i=1}^N \lambda_i Y_i,$$

where $c \to c_{N,s}$ as $\varepsilon \to 0$ and where $\{Y_i\}_{i=1,\cdots,N} \in \mathcal{E}_1$ (extended as zero-homogeneous function in a neighborhood of the unit sphere). Moreover, there exists $C > 0$ (depending on Θ, Ω, N and s) such that $\|w_\varepsilon\|_{C^{1,\beta}(S)} \le C\varepsilon^{2s}$ and $\|\partial_\xi w_\varepsilon\|_{C^{1,\beta}(S)} \le C\varepsilon^{2s+1}$.

Precisely, the above formula for $H^{\Omega_\varepsilon}_s$ means that

$$H^{\Omega_\varepsilon}_{s,\partial\mathbb{B}(\xi,w_\varepsilon)}(\xi + x(1+w_\varepsilon(x))) = c + \sum_{i=1}^N \lambda_i Y_i(x) \qquad \text{for every } x \in S.$$

Proof. Let us denote by $\overline{R}$ the family of functions in $C^{\beta-2s}(S)$ that are L^2-orthogonal, with respect to the <u>standard</u> volume element of S, to constants and to the first-order spherical harmonics. Notice that $\overline{R} \subseteq R$, see (2.14). Let us consider the two-component function $F_{\overline{R}} : \Theta_\varepsilon \times C^{1,\beta}(S) \to C^{\beta-2s}(S) \times \mathbb{R}$ defined by

$$F_{\overline{R}}(\xi, w) := \left(P_{\overline{R}}(H^{\Omega_\varepsilon}_{s,\partial\mathbb{B}(\xi,w)}), |\mathbb{B}(\xi,w)| - \omega_N\right); \qquad w \in W,$$

where $P_{\overline{R}} : C^{\beta-2s}(S) \mapsto \overline{R}$ the orthogonal L^2-projection onto the space $\overline{R}$, with respect to the <u>standard</u> volume element of S. With this notation, we want to find $w \in W$ such that $F_{\overline{R}}(\xi, w) = (0, 0)$.

By Lemma 3.2 we have that

$$F_{\overline{R}}(\xi, 0) = (O(\varepsilon^{2s}), 0), \tag{3.11}$$

where the latter quantity is intended to be bounded by $C\varepsilon^{2s}$ in the $C^{\beta-2s}(S)$ sense. In our notation, the constant C is allowed to vary from one formula to the other.

By Proposition 2.2 and by the fact that

$$\frac{\mathrm{d}}{\mathrm{d}w}_{|w=0} |\mathbb{B}(\xi, w)|[\varphi] = \int_S \varphi \, d\sigma,$$

we have that $L_\xi := \nabla_w F_{\overline{R}}(\xi, 0) \in Inv(W, \overline{R} \times \mathbb{R})$ with $\|L_\xi^{-1}\|_{L(\overline{R}\times\mathbb{R},W)} \leq C$. Hence $F_{\overline{R}}(\xi, w) = (0,0)$ if and only if $F_{\overline{R}}(\xi, 0) + L_\xi[w] - L_\xi[w] + F_{\overline{R}}(\xi, w) - F_{\overline{R}}(\xi, 0) = (0,0)$, which can be written as

$$w = T_\xi(w) := -L_\xi^{-1}[F_{\overline{R}}(\xi, 0) - L_\xi[w] + F_{\overline{R}}(\xi, w) - F_{\overline{R}}(\xi, 0)].$$

Therefore $F_{\overline{R}}(\xi, w) = (0,0)$ if and only if w is a fixed point for T_ξ.

Let us show that T_ξ is a contraction on a ball $B_{\overline{C}\varepsilon^{2s}}$ or radius $\overline{C}\varepsilon^{2s}$ in $C^{1,\beta}(S)$ for $\overline{C}$ sufficiently large. From the definition of T_ξ, the above estimate (3.11) and the fact that

$$\|L_\xi^{-1}\|_{L(\overline{R}\times\mathbb{R},W)} \leq C,$$

we have

$$\|T_\xi(0)\|_{C^{1,\beta}(S)} = \|L_\xi^{-1}[F_{\overline{R}}(\xi, 0)]\|_{C^{1,\beta}(S)} \leq C^2\varepsilon^{2s}. \tag{3.12}$$

Then, taking w_1 and $w_2 \in B_{\bar{C}\varepsilon^{2s}}(\xi) \subseteq W$ it follows that

$$\|T_\xi(w_1) - T_\xi(w_2)\|_{C^{1,\beta}(S)} \leq C\|F_{\overline{R}}(\xi, w_1) - F_{\overline{R}}(\xi, w_2) - L_\xi[w_1 - w_2]\|_{C^{1,\beta}(S)}. \tag{3.13}$$

We notice that the map $w \mapsto |\mathbb{B}(\xi, w)|$ is a smooth function from the metric ball of radius $\frac{1}{2}$ in $C^{1,\beta}(S)$ into $\mathbb{R}$. Thanks also to the smoothness statement in Proposition 2.2, the right hand side in the latter formula can be bounded by

$$\begin{aligned} F_{\overline{R}}(\xi, w_1) - F_{\overline{R}}(\xi, w_2) - L_\xi[w_1 - w_2] &= \int_0^1 \Big(\nabla_w F_{\overline{R}}(\xi, w_2 + t(w_1 - w_2)) \\ &\quad - \nabla_w F_{\overline{R}}(\xi, 0)[w_1 - w_2] \Big) \,\mathrm{d}t \leq C\|w_1 - w_2\|^2_{C^{1,\beta}(S)}. \end{aligned} \tag{3.14}$$

Hence, in $B_{\bar{C}\varepsilon^{2s}} \subseteq W$ the Lipschitz constant of T_ξ is $C\bar{C}\varepsilon^{2s}$. So, choosing first any $\bar{C} \geq 2C^2$, and then $\varepsilon > 0$ small enough, we find therefore that T_ξ is a contraction in $B_{\bar{C}\varepsilon^{2s}} \subseteq W$. As a consequence, there exists $w_\varepsilon : \Theta_\varepsilon \times S \to W$ such that $\|w_\varepsilon\|_{C^{1,\beta}(S)} \leq \bar{C}\varepsilon^{2s}$ and such that $F_{\overline{R}}(\xi, w_\varepsilon) = (0,0)$.

We also recall that the fixed point w_ε is continuous and differentiable with respect to the parameter ξ, (see e.g. [7], Section 2.6). Recall that $w_\varepsilon = w_\varepsilon(\xi)$ solves

$$|\mathbb{B}(\xi, w_\varepsilon)| = \omega_N \quad \text{and} \quad P_{\overline{R}}(H^{\Omega_\varepsilon}_{s,\partial\mathbb{B}(\xi,w_\varepsilon)}) = 0 \qquad \text{for all } \xi \in \mathbb{R}^N.$$

We want next to differentiate the above relations with respect to ξ. For this purpose, it is convenient to fix an index i, and to consider the one-parameter family of centers

$$\xi(t) = (\xi_1, \ldots, \xi_i + t, \ldots, \xi_N) = \xi + t\mathbf{e}_i. \tag{3.15}$$

Our aim is to understand the variation of $\partial\mathbb{B}(\xi_t, w_\varepsilon(\xi_t))$ normal to $\partial\mathbb{B}(\xi, w_\varepsilon(\xi))$. The above variation is characterized by a translation in the i-th component and by a variation of w_ε, which is in the radial direction with respect to the center ξ. Therefore, letting ν_{w_ε} denote the unit outer normal vector to $\partial\mathbb{B}(\xi, w_\varepsilon(\xi))$, the normal variation of $\partial\mathbb{B}(\xi(t), w_\varepsilon(\xi(t)))$

with respect to $\partial\mathbb{B}(\xi, w_\varepsilon(\xi))$ (computed at $t = 0$) is the scalar product between the pointwise shift $\mathbf{e}_i + \frac{\partial w_\varepsilon(\xi)}{\partial \xi_i}$ and the outer unit normal ν_{w_ε} to $\partial\mathbb{B}(\xi, w_\varepsilon(\xi))$, that is,

$$\nu_{w_\varepsilon} \cdot \mathbf{e}_i + \frac{\partial w_\varepsilon(\xi)}{\partial \xi_i} x \cdot \nu_{w_\varepsilon}, \qquad x \in S. \tag{3.16}$$

Hence we have that

$$\frac{\partial}{\partial \xi_i}|\mathbb{B}(\xi, w_\varepsilon)| = 0 \quad \text{and} \quad P_{\overline{R}}\left(\frac{\partial}{\partial \xi_i} H^{\Omega_\varepsilon}_{s,\partial\mathbb{B}(\xi,w_\varepsilon(\xi))}\right)\left[\nu_{w_\varepsilon} \cdot \mathbf{e}_i + \frac{\partial w_\varepsilon(\xi)}{\partial \xi_i} x \cdot \nu_{w_\varepsilon}\right] = 0.$$

Using (3.7), the fact that $L_s - \lambda_1$ sends constants to constants (see (2.9)) and the proof of Proposition 2.2, one finds from the second equation in the latter formula that $\|v_{i,\varepsilon}\|_{C^{1,\beta}(S)} \leq C\varepsilon^{2s+1}$, where $v_{i,\varepsilon} = P_{\overline{W}}\partial_{\xi_i} w_\varepsilon$. Here $\overline{W}$ denotes the subspace of W orthogonal to constant functions on S and $P_{\overline{W}}$ the orthogonal projection onto it. Since $\dfrac{\partial w_\varepsilon}{\partial \xi_i} \in W$, it remains to control then the component of $\partial_{\xi_i} w_\varepsilon$ in the orthogonal complement of $\overline{W}$, namely its average.

Let us write

$$\partial_{\xi_i} w_\varepsilon = v_{i,\varepsilon} + c_{i,\varepsilon} \qquad \text{with } c_{i,\varepsilon} \in \mathbb{R}.$$

From a direct computation we have that

$$0 = \frac{\partial}{\partial \xi_i}|\mathbb{B}(\xi, w_\varepsilon)| = \int_S (1 + w_\varepsilon)^{N-1}\,(v_{i,\varepsilon} + c_{i,\varepsilon})\, d\sigma.$$

Since we know that $\|v_{i,\varepsilon}\|_{C^{1,\beta}(S)} \leq C\varepsilon^{2s+1}$, it follows from the latter formula that also $|c_{i,\varepsilon}| \leq C\varepsilon^{2s+1}$. Therefore one deduces

$$\|\partial_{\xi_i} w_\varepsilon\|_{C^{1,\beta}(S)} \leq C\varepsilon^{2s+1}, \tag{3.17}$$

which is the desired conclusion, possibly relabelling the constant C. □

We next show how to find ξ's so that the Lagrange multipliers λ_i in the statement of Proposition 3.3 vanish, thus obtaining surfaces with constant relative fractional mean curvature.

Proposition 3.4. *Let $w_\varepsilon : S \to \mathbb{R}$ given by Proposition 3.3. Recalling* (2.8)*, for $\xi \in \Theta_\varepsilon$, we define $\Phi_\xi := P^{\Omega_\varepsilon}_s(\mathbb{B}(\xi, w_\varepsilon))$. Then, for $\varepsilon > 0$ sufficiently small, if $\nabla_\xi \Phi_{\xi}{}_{|_{\xi=\bar\xi}} = 0$ for some $\bar\xi \in \Theta_\varepsilon$, one has*

$$H^{\Omega_\varepsilon}_{s,\partial\mathbb{B}(\bar\xi,w_\varepsilon)} \equiv c,$$

where $c = c(\varepsilon, \bar\xi)$.

Proof. Recall that $w_\varepsilon = w_\varepsilon(\xi)$ solves

$$|\mathbb{B}(\xi, w_\varepsilon)| = \omega_N \quad \text{and} \quad P_{\overline{R}}(H^{\Omega_\varepsilon}_{s,\partial\mathbb{B}(\xi,w_\varepsilon)}) = 0 \qquad \text{for all } \xi \in \mathbb{R}^N.$$

Since $|\mathbb{B}(\xi, w_\varepsilon)| = \omega_N$ for any choice of ξ, it follows that the integral over $\partial\mathbb{B}(\xi, w_\varepsilon(\xi))$ of the normal variation vanishes, i.e., recalling (3.16), we have for $\xi = \bar\xi$

$$\int_{\partial\mathbb{B}(\bar\xi,w_\varepsilon(\bar\xi))}\left[\nu_{w_\varepsilon} \cdot \mathbf{e}_i + \frac{\partial w_\varepsilon(\bar\xi)}{\partial \xi_i}\frac{x - \bar\xi}{|x - \bar\xi|} \cdot \nu_{w_\varepsilon}\right] d\sigma_{w_\varepsilon} = 0, \tag{3.18}$$

where $d\sigma_{w_\varepsilon}$stands for the area element of $\partial\mathbb{B}(\bar\xi, w_\varepsilon(\bar\xi))$ and where we are identifying $w_\varepsilon(\sigma)$ with $w_\varepsilon(\bar\xi + (1 + w_\varepsilon(\sigma))\sigma)$ for $\sigma \in S$.

For the same reason, recalling (2.6) and (3.15), we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} P_s^{\Omega_\varepsilon}(\mathbb{B}(\xi(t), w_\varepsilon(\xi(t)))) = \int_{\partial\mathbb{B}(\bar\xi, w_\varepsilon(\bar\xi))} H^{\Omega_\varepsilon}_{s,\partial\mathbb{B}(\bar\xi, w_\varepsilon)} \left[\nu_{w_\varepsilon} \cdot \mathbf{e}_i + \frac{\partial w_\varepsilon(\bar\xi)}{\partial\xi_i} \frac{x-\bar\xi}{|x-\bar\xi|} \cdot \nu_{w_\varepsilon}\right] d\sigma_{w_\varepsilon}.$$

By our choice of $\bar\xi$ we have that, for all $i = 1, \dots, N$

$$\frac{\partial}{\partial\xi_i}|_{\xi=\bar\xi}\Phi_\xi = 0.$$

Recalling also that by Proposition 3.3, $H^{\Omega_\varepsilon}_{s,\partial\mathbb{B}(\xi,w_\varepsilon)} = c + \sum_{i=1}^N \lambda_i Y_i$ (see Section 2 for the definition of the first-order spherical harmonics Y_i and the formula after Proposition 3.3), from (3.18) we have that for all $i = 1, \dots, N$

$$0 = \int_{\partial\mathbb{B}(\bar\xi, w_\varepsilon(\bar\xi))} \left(\sum_{j=1}^N \lambda_j Y_j\right) \left[\nu_{w_\varepsilon} \cdot \mathbf{e}_i + \frac{\partial w_\varepsilon(\bar\xi)}{\partial\xi_i} \frac{x-\bar\xi}{|x-\bar\xi|} \cdot \nu_{w_\varepsilon}\right] d\sigma_{w_\varepsilon}. \tag{3.19}$$

Notice that by the estimates on w_ε and $\partial_\xi w_\varepsilon$ in Proposition 3.3 one has

$$\int_{\partial\mathbb{B}(\bar\xi, w_\varepsilon(\bar\xi))} Y_j \left[\nu_{w_\varepsilon} \cdot \mathbf{e}_i + \frac{\partial w_\varepsilon(\bar\xi)}{\partial\xi_i} \frac{x-\bar\xi}{|x-\bar\xi|} \cdot \nu_{w_\varepsilon}\right] d\sigma_{w_\varepsilon} = \delta_{ij} + o_\varepsilon(1); \qquad i, j = 1, \dots, N.$$

Therefore the system (3.19) implies the vanishing of all λ_j's, which gives the desired conclusion. □

The next step is to show that the fractional perimeter of $B_1(\xi)$ is sufficiently close to the fractional perimeter of the deformed ball $\mathbb{B}(\xi, w_\varepsilon)$, also when differentiating in ξ.

Proposition 3.5. *Let w_ε be as Proposition 3.4. The following Taylor expansion holds:*

$$P_s^{\Omega_\varepsilon}(\mathbb{B}(\xi, w_\varepsilon)) = P_s^{\Omega_\varepsilon}(B_1(\xi)) + O(\varepsilon^{4s}). \tag{3.20}$$

Moreover one has

$$\frac{\partial}{\partial\xi_i} P_s^{\Omega_\varepsilon}(\mathbb{B}(\xi, w_\varepsilon)) = \frac{\partial}{\partial\xi_i} P_s^{\Omega_\varepsilon}(B_1(\xi)) + O(\varepsilon^{1+4s}). \tag{3.21}$$

Proof. Thanks to the first statement of Lemma 3.2, following the notation in Section 2, we get that

(3.22)

$$\begin{aligned} P_s^{\Omega_\varepsilon}(\mathbb{B}(\xi, w_\varepsilon)) &= P_s^{\Omega_\varepsilon}(B_1(\xi)) + (P^{\Omega_\varepsilon}_{s,\xi})'(0)[w_\varepsilon] + P_s^{\Omega_\varepsilon}(\mathbb{B}(\xi, w_\varepsilon)) - (P^{\Omega_\varepsilon}_{s,\xi})'(0)[w_\varepsilon] - P_s^{\Omega_\varepsilon}(B_1(\xi)) \\ &= P_s^{\Omega_\varepsilon}(B_1(\xi)) + O(\varepsilon^{4s}) + \int_0^1 \Big((P^{\Omega_\varepsilon}_{s,\xi})'(t\, w_\varepsilon) - (P^{\Omega_\varepsilon}_{s,\xi})'(0)\Big)[w_\varepsilon]\, \mathrm{d}t, \end{aligned}$$

where $(P_s^{\Omega_\varepsilon})'$ is defined as in the formula after (2.6).

Using the fact that the nonlocal mean curvature is smooth, we deduce then that

$$\int_0^1 \Big((P^{\Omega_\varepsilon}_{s,\xi})'(t\, w_\varepsilon) - (P^{\Omega_\varepsilon}_{s,\xi})'(0)\Big)[w_\varepsilon]\, \mathrm{d}t = O(\varepsilon^{4s}),$$

so the last two formulas imply (3.20).

To prove (3.21), we use the estimate $\|\partial_\xi w_\varepsilon\|_{C^{1,\beta}(S)} \leq C\varepsilon^{2s+1}$ from Proposition 3.3. Calling τ_i the quantity in (3.16) and recalling the notation from Section 2, we write that

$$\frac{\partial}{\partial \xi_i} P_s^{\Omega_\varepsilon}(\mathbb{B}(\xi, w_\varepsilon)) = (P_{s,\xi}^{\Omega_\varepsilon})'(w_\varepsilon)[\tau_i].$$

Taylor-expanding the latter quantity we can write that

$$\begin{aligned}\frac{\partial}{\partial \xi_i} P_s^{\Omega_\varepsilon}(\mathbb{B}(\xi, w_\varepsilon)) &= (P_{s,\xi}^{\Omega_\varepsilon})'(0)[\tau_i] + \frac{1}{2}(P_{s,\xi}^{\Omega_\varepsilon})''(0)[\tau_i] + o(\varepsilon^{1+4s}) \\ &= \frac{\partial}{\partial \xi_i} P_s^{\Omega_\varepsilon}(B_1(\xi)) + O(\varepsilon^{1+4s}).\end{aligned} \tag{3.23}$$

This concludes the proof. □

Proof of Theorem 1.1. Suppose x_0 is a strict local extremal of V_Ω, without loss of generality a minimum. Then there exists an open set $\Upsilon \subset\subset \Omega$ such that $V_\Omega(x_0) < \inf_{\partial\Upsilon} V_\Omega - \delta$ for some $\delta > 0$. Let Φ_ξ be defined as in Proposition 3.4: by the estimates (3.1) and (3.20) it follows that for every $\bar{x} \in \frac{1}{\varepsilon}\Upsilon$

$$\Phi_{\bar{x}} = P_s^{\mathbb{R}^N}(B_1(\bar{x})) - \omega_N \varepsilon^{2s} V_\Omega(\varepsilon\bar{x}) + O(\varepsilon^{1+2s}). \tag{3.24}$$

Since $P_s^{\mathbb{R}^N}(B_1(\bar{x})) = P_s^{\mathbb{R}^N}(B_1(\frac{x_0}{\varepsilon}))$, we get

$$\begin{aligned}\Phi_{\frac{x_0}{\varepsilon}} - \Phi_{\bar{x}} &= \omega_N \varepsilon^{2s}(V_\Omega(\varepsilon\bar{x}) - V_\Omega(x_0)) + O(\varepsilon^{1+2s}) \\ &\geq \omega_N \varepsilon^{2s}(\inf_{\partial\Upsilon} V_\Omega(\varepsilon\bar{x}) - V_\Omega(x_0)) + O(\varepsilon^{1+2s}) \\ &> \delta\omega_N \varepsilon^{2s} + O(\varepsilon^{1+2s}) \geq \delta\omega_N\varepsilon^{2s} + C\varepsilon^{1+2s} > 0\end{aligned} \tag{3.25}$$

for $\varepsilon < \frac{\delta\omega_N}{C}$ where $C > 0$ is a constant.

Hence, for ε sufficiently small,

$$\Phi_{\frac{x_0}{\varepsilon}} > \sup_{\frac{1}{\varepsilon}\Upsilon} \Phi_\cdot.$$

As a consequence $\Phi_\cdot$ attains a maximum in the dilated domain $\frac{1}{\varepsilon}\Upsilon$, and the conclusion follows from Proposition 3.4.

Suppose now that x_0 is a non-degenerate critical point of V_Ω. From (3.2) and (3.21) one can find an open set $\Upsilon \subset\subset \Omega$ such that

$$\deg\left(\nabla\Phi, \frac{1}{\varepsilon}\Upsilon, 0\right) \neq 0.$$

This implies that Φ_ξ has a critical point in $\frac{1}{\varepsilon}\Upsilon$, and the conclusion again follows from Proposition 3.4.

Since in both cases the set Υ containing x_0 can be taken arbitrarily small, the localization statement in the theorem is also proved. □

Remark 3.6. From [4, Theorem 2.24] one has a relation between the Morse index of a critical point as found in Proposition 3.4 and the Morse index of the corresponding critical point of Φ. In our case, since round spheres are global minimizers for the s-perimeter relative to $\mathbb{R}^N$, these two indices coincide.

To prove Corollary 1.2, we need the following Lemma.

Lemma 3.7. *For all $x \in \partial\Omega$ one has*

$$\lim_{\Omega \ni y \to x} V_\Omega(y) = +\infty,$$

and

$$\lim_{\Omega \ni y \to x} \nabla V_\Omega(y) \cdot \nu_{\partial\Omega}(x) = +\infty,$$

where $\nu_{\partial\Omega}$ denotes the outer unit normal to $\partial\Omega$.

Proof. Letting $d := \mathrm{dist}(x, \partial\Omega)$ for $x \in \Omega$, thanks to the change of variables $y' = \frac{y}{d}$ we get

$$V_\Omega(x) = \int_{\Omega^C} \frac{1}{|x-y|^{N+2s}}\, \mathrm{d}y = d^{-2s} \int_{(\Omega/d)^C} \frac{1}{|x/d - y'|^{N+2s}}\, \mathrm{d}y'. \tag{3.26}$$

If $d \to 0$, since x/d has unit distance from the enlarged domain, setting $\mathbb{R}^N_+ = \{x \in \mathbb{R}^N : x > 0\}$ we have

$$\int_{(\Omega/d)^C} \frac{1}{|x/d - y'|^{N+2s}}\, \mathrm{d}y' \to \int_{(\mathbb{R}^N_+)^C} \frac{1}{|y' - (0,1)|^{N+2s}}\, \mathrm{d}y' < +\infty,$$

i.e. V_Ω behaves asymptotically as d^{-2s} when $d \to 0$. With a similar proof, one finds that the component of ∇V_Ω normal to $\partial\Omega$ behaves as d^{-2s-1}. □

Proof of Corollary 1.2. Given $\delta > 0$ small enough, let us define the set $\Omega^\delta \subseteq \Omega$ by

$$\Omega^\delta = \{x \in \Omega \ : \ d(x, \partial\Omega) > \delta\}.$$

From Lemma 3.7 we have

$$\nabla V_\Omega \cdot \nu_{\partial\Omega^\delta} > 0 \quad \text{on } \partial\Omega^\delta.$$

As in the proof of Theorem 1.1, it turns out that

$$\nabla\Phi_\cdot \cdot \nu_{\partial\frac{1}{\varepsilon}\Omega^\delta} > 0 \quad \text{on } \partial\frac{1}{\varepsilon}\Omega^\delta.$$

Clearly, since $\frac{1}{\varepsilon}\Omega^\delta$ is compact, the (PS)-condition holds. Hence the conclusion follows from Theorem 2.5 and Remark 2.6. □

Remark 3.8. It is interesting to see how the geometry of the domain (and not just the topology, as in Corollary 1.2) plays a role in order to obtain either uniqueness of multiplicity of solutions.

In the Appendix we will prove uniqueness for the unit ball B_1, i.e. we will show that V_{B_1} has a unique critical point at the origin which is a non-degenarate minimum.

Secondly, we will give an example of dumble-bell domain, topologically equivalent to a ball, such that the reduced functional Φ_ξ (defined as in Proposition 3.4) has at least three critical points, while Corollary 1.2 would give us only one solution.

4. Existence and regularity of minimizers in half-spaces

In this section we prove Theorem 1.3 on existence and regularity of minimizers for Problem (1.4). Similarly to [11,12] we shall first show the regularity and then the existence of minimizers.

4.1. **Regularity and axial symmetry of minimizers.** We first show the axial symmetry of minimizers.

Definition 4.1. Given a measurable set $E \subset \mathbb{R}^N_+$, we let $E^* \subset \mathbb{R}^N_+$ be the radially symmetric rearrangement of E with respect to x', which is defined by the following property: for almost every $t > 0$ the section $E^* \cap \{x_N = t\}$ is a ball centered at the origin of measure $|E \cap \{x_N = t\}|$ (see Figure 1).

Lemma 4.2. *For every $E \subset \mathbb{R}^N_+$, we have*

$$P_s(E^*, \mathbb{R}^N_+) \leq P_s(E, \mathbb{R}^N_+),$$

and equality holds if and only if $E = E^*$, up to translation and up to a negligible set.

Proof. Notice first that, given $\delta > 0$ and F, $G \subset \mathbb{R}^{N-1}$, by Riesz inequality [34] we have that

$$\begin{aligned}
\int_F \int_{\mathbb{R}^{N-1}\setminus G} \frac{dx'\,dy'}{||x'-y'|^2+\delta^2|^{\frac{N+2s}{2}}} &= \int_F \int_{\mathbb{R}^{N-1}} \frac{dx'\,dy'}{||x'-y'|^2+\delta^2|^{\frac{N+2s}{2}}} \\
&\quad - \int_F \int_G \frac{dx'\,dy'}{||x'-y'|^2+\delta^2|^{\frac{N+2s}{2}}} \\
&= c(\delta, N)|F| - \int_F \int_G \frac{dx'\,dy'}{||x'-y'|^2+\delta^2|^{\frac{N+2s}{2}}} \\
&\geq c(\delta, N)|F^*| - \int_{F^*} \int_{G^*} \frac{dx'\,dy'}{||x'-y'|^2+\delta^2|^{\frac{N+2s}{2}}} \\
&= \int_{F^*} \int_{\mathbb{R}^{N-1}\setminus G^*} \frac{dx'\,dy'}{||x'-y'|^2+\delta^2|^{\frac{N+2s}{2}}},
\end{aligned}$$

where

$$c(\delta, N) := \int_{\mathbb{R}^{N-1}} \frac{dx'\,dy'}{||x'-y'|^2+\delta^2|^{\frac{N+2s}{2}}},$$

and equality holds if and only if $G = G^*$ and $F = F^*$, up to negligible sets.

Writing $x = (x', x_N) \in \mathbb{R}^N$ and letting

$$E_t := \{x \in E : \ x_N = t\} \qquad \text{for } t > 0,$$

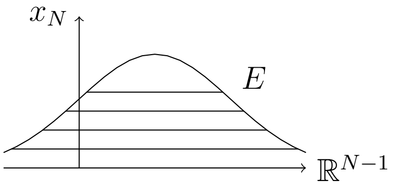
by Fubini–Tonelli's Theorem we then get

$$\begin{aligned}
P_s(E, \mathbb{R}^N_+) &= \int_0^\infty dx_N \int_0^\infty dy_N \int_{E_{x_N}} \int_{E_{y_N}} \frac{dx'\,dy'}{||x'-y'|^2+(x_N+y_N)^2|^{\frac{N+2s}{2}}} \\
&\geq \int_{\mathbb{R}} dx_N \int_{\mathbb{R}} dy_N \int_{E^*_{x_N}} \int_{E^*_{x_N}} \frac{dx'\,dy'}{||x'-y'|^2+(x_N+y_N)^2|^{\frac{N+2s}{2}}} \\
&= P_s(E^*, \mathbb{R}^N_+),
\end{aligned}$$

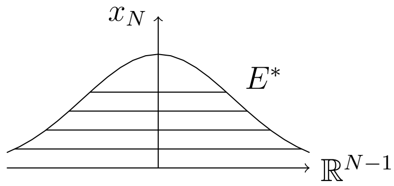
and equality holds if and only if $E = E^*$ (up to a negligible set). □

A direct consequence of Lemma 4.2 is the following

Corollary 4.3. *Let E be a minimizer of* (1.4)*, then $E = E^*$ (up to a negligible set).*



(a) Horizontal sections of E.



(b) Horizontal sections of E^*.

Figure 1. The radially symmetric rearrangement of E.

The regularity of minimizers follows from [28, Theorems 1.6 and 1.7] and [6].

Proposition 4.4. *Let E be a minimizer of* (1.4). *Then E is a bounded set and $\partial E \cap \{x_N > 0\}$ is of class C^∞ out of a countable singular set contained in its rotational axis, possibly accumulating at the origin. at $\{x_N = 0\}$.*

Proof. By [28, Theorem 1.7] E satisfies uniform lower density estimates, so that in particular it is a bounded set. Moreover, by [28, Theorem 1.6] $\partial E \cap \{x_N > 0\}$ is of class $C^{1,\alpha}$, for some $\alpha \in (0,1)$, out a closed singular set Σ of Hausdorff dimension at most $N-3$. Then, by a bootstrap argument as in [6, Theorem 1], which can be easily adapted to this setting, one obtains that $(\partial E \setminus \Sigma) \cap \{x_N > 0\}$ is of class C^∞.

Being ∂E rotationally symmetric, the singular set Σ is necessarily contained in the axis of rotation. Since the blow-up limit at each singular point of $\partial E \cap \{x_N > 0\}$ is a symmetric singular cone [10, Theorem 9.2], it is uniquely determined by [15, Theorem 3]. It follows that each singular point with $x_N > 0$ is isolated, so that Σ is a sequence of points possibly accumulating at $\{x_N = 0\}$. □

4.2. **Existence of minimizers.** Let us first show that the functional $P_s(\cdot, \mathbb{R}^N_+)$ satisfies an isoperimetric inequality.

Given $s \in (0, 1/2)$, $E \subset \mathbb{R}^N_+$ and an open set $\Omega \subset \mathbb{R}^N_+$ we let

$$\mathcal{F}_s(E,\Omega) := \int_{E\cap\Omega}\int_{\mathbb{R}^N_+\setminus E} \frac{dxdy}{|x-y|^{N+2s}} + \int_{E\setminus\Omega}\int_{\Omega\setminus E} \frac{dxdy}{|x-y|^{N+2s}}$$

be the localized fractional perimeter of E in Ω.

Notice that the functional $\mathcal{F}_s(\cdot,\Omega)$ is lower semicontinuous with respect to the L^1-topology, $\mathcal{F}_s(E,\Omega) \le P_s(E,\mathbb{R}^N_+)$ and $\mathcal{F}_s(E,\mathbb{R}^N_+) \le P_s(E,\mathbb{R}^N_+)$ for all sets E. Moreover, given two disjoint sets $\Omega_1, \Omega_2 \subset \mathbb{R}^N_+$, it holds (cf. [11, Eq. (9)])

$$\begin{aligned} \mathcal{F}_s(E,\Omega_1) + \mathcal{F}_s(E,\Omega_2) &= \mathcal{F}_s(E,\Omega_1\cup\Omega_2) + \int_{E\cap\Omega_1}\int_{\Omega_2\setminus E} \frac{dxdy}{|x-y|^{N+2s}} \\ &\quad + \int_{E\cap\Omega_2}\int_{\Omega_1\setminus E} \frac{dxdy}{|x-y|^{N+2s}}. \end{aligned} \tag{4.1}$$

Lemma 4.5. *There exists a constant $C = C(N,s)$ such that, for every unit cube $Q \subset \mathbb{R}^N_+$ and for every $E \subset \mathbb{R}^N_+$ such that $|E \cap Q| \le 1/2$, it holds*

$$\mathcal{F}_s(E,Q) \ge C\,|E\cap Q|^{\frac{N-2s}{N}}.$$

Proof. Given $E, Q \subset \mathbb{R}^N_+$ we let

$$\begin{aligned}\hat{E} &:= E \cup \{(x', x_N) \in \mathbb{R}^N : (x', -x_N) \in E\}, \\ \hat{Q} &:= Q \cup \{(x', x_N) \in \mathbb{R}^N : (x', -x_N) \in Q\}.\end{aligned}$$

Notice that $|\hat{E} \cap \hat{Q}| = 2|E \cap Q|$ and

$$2\mathcal{F}_s(E, Q) \le \overline{P}_s(\hat{E}, \hat{Q}) \le 4\mathcal{F}_s(E, Q),$$

where we set

$$\overline{P}_s(\hat{E}, \hat{Q}) := \int_{\hat{E}\cap\hat{Q}} \int_{\mathbb{R}^N \setminus \hat{E}} \frac{dxdy}{|x-y|^{N+2s}} + \int_{\hat{E}\setminus\hat{Q}} \int_{\hat{Q}\setminus\hat{E}} \frac{dxdy}{|x-y|^{N+2s}}.$$

Since $|E \cap Q| \le 1/2$, by the relative fractional isoperimetric inequality [12, Lemma 2.5] [1] we then get

$$\mathcal{F}_s(E, Q) \ge \frac{1}{4}\overline{P}_s(\hat{E}, \hat{Q}) \ge \frac{1}{4} C_{N,s} |\hat{E} \cap \hat{Q}|^{\frac{N-2s}{N}} = \frac{C_{N,s}}{2^{\frac{N+2s}{N}}} |E \cap Q|^{\frac{N-2s}{N}},$$

where the constant $C_{N,s}$ depends only on N, s. This gives the thesis. □

We also recall a technical lemma which will be useful in the sequel (see [11, Lemma 5.2]).

Lemma 4.6. *Let $s \in (0, 1/2)$, and let $\{x_i\}_i$ be a non-increasing sequence of positive real numbers such that*

$$\sum_{i=1}^{\infty} x_i^{\frac{N-2s}{N}} \le C \qquad \text{and} \qquad \sum_{i=1}^{\infty} x_i = \frac{1}{2},$$

for some $C > 0$. Then there exists $k_0 \in \mathbb{N}$ such that, for all $k \ge k_0$ it holds

$$\sum_{i=k+1}^{\infty} x_i \le (2Ck)^{-\frac{2s}{N}}.$$

We now show existence of minimizers, following closely the proof of [11, Theorem 4.2]. The main issue is controlling the loss of mass at infinity, as the ambient set $\mathbb{R}^N_+$ is not compact.

Proposition 4.7. *For each $m \in (0, +\infty)$ there exists a minimizer of* (1.4).

Proof. Without loss of generality we can assume that $m = 1/2$, since the argument is the same for all values of m. Moreover, since the problem is scaling invariant, if E is a minimizer with volume $1/2$ then the rescaled set $\widetilde{E} := (2m)^{1/N} E$ is a minimizer with volume m.

Let now E_n be a minimizing sequence for (1.4), that is,

$$\lim_{n\to\infty} P_s(E_n, \mathbb{R}^N_+) = \inf_{|E|=\frac{1}{2}} P_s(E, \mathbb{R}^N_+).$$

In particular, we have $P_s(E_n, \mathbb{R}^N_+) \le C$, where the constant C does not depend on n. Moreover, by Lemma 4.2, without loss of generality we can assume that the sets E_n are rotationally symmetric around the axis $\{(0, x_N) : x_N \in \mathbb{R}\}$.

[1]In [12, Lemma 2.5] the inequality is proved when $|E \cap Q| < 1/2$, but the same holds under our assumption.

For $n \in \mathbb{N}$, we let $\{Q_{i,n}\}_{i\in\mathbb{N}}$ be a partition of $\mathbb{R}^N$ into disjoint unit cubes with vertices on $\mathbb{Z}^N$, such that the quantities $x_{i,n} = |E_n \cap Q_{i,n}|$ are non-increasing in i. In particular, there holds

$$\sum_{i=1}^{\infty} x_{i,n} = m = \frac{1}{2}. \tag{4.2}$$

Recalling the isoperimetric inequality in Lemma 4.5 and (4.1), we have

$$\begin{aligned} \sum_{i=1}^{\infty} x_{i,n}^{\frac{N-2s}{N}} &\leq c\sum_{i=1}^{\infty} \mathcal{F}_s(E_n, Q_{i,n}) \\ &\leq 2cP_s(E_n, \mathbb{R}^N_+) \leq c', \end{aligned}$$

for some constants $c, c' > 0$. By Lemma 4.6 we then obtain that

$$\sum_{i=k+1}^{\infty} x_{i,n} \leq c''\, k^{-\frac{2s}{N}}\,, \tag{4.3}$$

for some $c'' > 0$ and for all $k \in \mathbb{N}$. By a diagonal argument, up to extracting a subsequence, we can assume that $x_{i,n} \to \alpha_i$ as $n \to \infty$, for some $\alpha_i \in [0, 1/2]$. By (4.2) and (4.3) we then get

$$\sum_{i=1}^{\infty} \alpha_i = \frac{1}{2}\,. \tag{4.4}$$

Let now $z_{i,n} = (z'_{i,n}, z^N_{i,n})$ be the center of the cube $Q_{i,n}$. Up to extracting a further subsequence, we can suppose that $|z_{i,n} - z_{j,n}| \to c_{ij} \in [0, +\infty]$ as $n \to +\infty$, and that there exists $G_i \subset \mathbb{R}^N_+$ such that G_i is rotationally symmetric and

$$(E_n - z_{i,n}) \to G_i \quad \text{in the } L^1_{\mathrm{loc}}\text{-topology}, \tag{4.5}$$

for all $i \in \mathbb{N}$.

We say that $i \sim j$ if $c_{ij} < +\infty$. We denote by $[i]$ the equivalence class of i, and we let $\mathcal{A} := \{[i] : i \in \mathbb{N}\}$. Notice that G_i equals G_j up to a translation, if $i \sim j$. Up to horizontal translations, we can also assume that the sets G_i's are rotationally symmetric around the axis $\{(0, x_N) :\ x_N \in \mathbb{R}\}$. Moreover, by the rotational symmetry of the sets G_i, there exists at most one equivalence class $[\bar{i}]$ such that $|G_{\bar{i}}| > 0$ and the sequence $z^N_{i,n}$ remains bounded as $n \to +\infty$ for all $i \in [\bar{i}]$.

We claim that

$$P_s(G_{\bar{i}}, \mathbb{R}^N_+) + \sum_{[i]\in\mathcal{A}\setminus[\bar{i}]} P_s(G_i) \leq \lim_{n\to+\infty} P_s(E_n, \mathbb{R}^N_+) = \inf_{|E|=\frac{1}{2}} P_s(E, \mathbb{R}^N_+)\,. \tag{4.6}$$

To prove it, we fix $M \in \mathbb{N}$ and $R > 0$. Let $Q_R = [-R, R]^N$. We take different equivalence classes $i_0, i_1, \dots, i_M$ with $i_0 = \bar{i}$, and we notice that if $i_k \neq i_j$ then the set $z_{i_k,n} + Q_R$ is moving far apart from the set $z_{i_j,n} + Q_R$, and so we have

$$\lim_{n\to+\infty} \int_{z_{i_k,n}+Q_R} \int_{z_{i_j,n}+Q_R} \frac{dx\, dy}{|x-y|^{N+2s}} = 0.$$

By (4.5), the lower semicontinuity of the fractional perimeter and (4.1), we obtain

$$\begin{aligned}\mathcal{F}_s(G_{\bar{i}}, Q_R) + \sum_{k=1}^{M} P_s(G_{i_k}, Q_R) &\le \liminf_{n\to+\infty} \sum_{k=1}^{M} \mathcal{F}_s(E_n, (z_{i_k,n} + Q_R)) \\ &\le \liminf_{n\to+\infty} \mathcal{F}_s\left(E_n, \bigcup_{k=1}^{M} (z_{i_k,n} + Q_R)\right) + 2 \sum_{\substack{0\le k,j\le M\\ i_k\neq i_j}} \int_{z_{i_k,n}+Q_R} \int_{z_{i_j,n}+Q_R} \frac{dx\,dy}{|x-y|^{N+2s}} \\ &\le \liminf_{n\to+\infty} P_s(E_n, \mathbb{R}^N_+) = \inf_{|E|=\frac{1}{2}} P_s(E, \mathbb{R}^N_+)\,.\end{aligned}$$

By sending first $R \to +\infty$ and then $M \to +\infty$, this yields (4.6).

Now we claim that

$$\sum_{[i]\in\mathcal{A}} |G_i| = \frac{1}{2}. \tag{4.7}$$

Indeed, for every $i \in \mathbb{N}$ and $R > 0$ we have

$$|G_i| \ge |G_i \cap Q_R| = \lim_{n\to+\infty} |(E_n - z_{i,n}) \cap Q_R|.$$

If j is such that $j \sim i$ and $c_{ij} \le \frac{R}{2}$, possibly increasing R we have $Q_{j,n} - z_{i,n} \subset Q_R$ for all $n \in \mathbb{N}$, so that

$$\begin{aligned}|(E_n - z_{i,n}) \cap Q_R| &= \sum_{j=1}^{I_n} |(E_n - z_{i,n}) \cap Q_R \cap (Q_{j,n} - z_{i,n})| \\ &\ge \sum_{j:\, c_{ij}\le\frac{R}{2}} |(E_n - z_{i,n}) \cap Q_R \cap (Q_{j,n} - z_{i,n})| = \sum_{j:\, c_{ij}\le\frac{R}{2}} |(E_n - z_{i,n}) \cap (Q_{j,n} - z_{i,n})| \\ &= \sum_{j:\, c_{ij}\le\frac{R}{2}} |E_n \cap Q_{j,n}|,\end{aligned}$$

and so

$$|G_i| \ge \lim_{n\to+\infty} |(E_n - z_{i,n}) \cap Q_R| \ge \lim_{n\to+\infty} \sum_{j:\, c_{ij}\le\frac{R}{2}} |E_n \cap Q_{j,n}| = \sum_{j:\, c_{ij}\le\frac{R}{2}} \alpha_j.$$

Letting $R \to +\infty$ we then have

$$|G_i| \ge \sum_{j:\, i\sim j} \alpha_j = \sum_{j\in[i]} \alpha_j\,,$$

hence, recalling (4.4),

$$\sum_{[i]\in\mathcal{A}} |G_i| \ge \frac{1}{2},$$

thus proving (4.7) (since the other inequality is trivial).

Let now

$$E_n^{[i]} := E_n \cap \bigcup_{j\sim i} Q_{j,n},$$

and observe that we still have that the sets $(E_n^{[i]} - z_{i,n})$ converge to G_i as $n \to +\infty$, in the $L^1_{\rm loc}$-topology. From (4.6) we then get

$$\sum_{[i]\in\mathcal{A}} P_s(G_i, \mathbb{R}^N_+) \le \liminf_{n\to+\infty} P_s(E_n, \mathbb{R}^N_+) = \inf_{|E|=\frac{1}{2}} P_s(E, \mathbb{R}^N_+)\,. \tag{4.8}$$

This means that $G_{\bar{i}}$ (resp. G_i with $[i] \neq [\bar{i}]$) is a minimizer of $P_s(\cdot, \mathbb{R}^N_+)$ (resp. $P_s(\cdot)$) among sets of volume equal to $|G_{\bar{i}}|$ (resp. $|G_i|$). In particular, $G_{\bar{i}}$ is bounded thanks to Proposition 4.4, and G_i is a ball for $[i] \neq [\bar{i}]$, by the fractional isoperimetric inequality (1.2).

If the set $G_{\bar{i}}$ is nonempty, then the rescaled set $G = (2\,|G_{\bar{i}}|)^{-1/N} G_{\bar{i}}$ is a minimizer of (1.4) with volume 1/2, which would conclude the proof. If on the other hand $G_{\bar{i}} = \emptyset$, taking R such that $\omega_N R^N = 1/2$, recalling (4.7) we would have

$$\sum_{[i]\in\mathcal{A}\setminus[\bar{i}]} P_s(G_i) \geq P_s(B_R(0)) > \inf_{|E|=\frac{1}{2}} P_s(E, \mathbb{R}^N_+)\,,$$

contradicting (4.6). □

Proof of Theorem 1.3. Theorem 1.3 directly follows from Propositions 4.4 and 4.7. □

Eventually, we show that every minimizer necessarily touches the hyperplane $\{x_N = 0\}$ (possibly at a singular point of its boundary).

Lemma 4.8. *If E is a minimizer for* (1.4)*, then* $\mathrm{dist}(E, \{x_N = 0\}) = 0$.

Proof. By contradiction suppose that $\mathrm{dist}(E, \{x_N = 0\}) > 0$. Then, if $e := (e_1, \cdots, e_N)$ is the canonical basis of $\mathbb{R}^N$ and $\lambda := \mathrm{dist}(E, \{x_N = 0\}) > 0$, we consider the shifted set $E - \lambda e_N$. Using the following change of variables (i.e. translating downwards the set E by $\lambda \vec{e}_N$)

$$E \ni x \longmapsto x' = x - \lambda e_N \in E - \lambda e_N,$$
$$\mathbb{R}^N_+ \setminus E \ni y \longmapsto y' = y - \lambda e_N \in \mathbb{R}^N_+ \setminus (E - \lambda e_N),$$

we have

$$\begin{aligned} P_s(E, \mathbb{R}^N_+) &= \int_E \int_{\mathbb{R}^N_+\setminus E} \frac{\mathrm{d}x\,\mathrm{d}y}{|x - \lambda e_N - y + \lambda e_N|^{N+2s}} \\ &> \int_{E-\lambda e_N} \int_{\mathbb{R}^N_+\setminus(E-\lambda e_N)} \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^{N+2s}} = P_s(E - \lambda e_N, \mathbb{R}^N_+). \end{aligned} \tag{4.9}$$

This contradicts the minimality of E. □

Remark 4.9. If $\partial E \cap \{x_N = 0\}$ is not a singular point, that is, it is a disk of positive radius, it follows from [28, Theorem 1.4] that ∂E has a vertical contact angle with the hyperplane $\{x_N = 0\}$.

Remark 4.10. It would be interesting to know whether minimizers, or even critical points, of the functional in (1.4) are unique up to horizontal translations (see for instance [23–25] for similar uniqueness results). It would also be interesting to understand if minimizers are necessarily convex or at least everywhere regular, that is, if the singular set Σ is empty.

5. Appendix

We prove here the assertions in Remark 3.8.

Lemma 5.1. *If B_1 is the unit ball of $\mathbb{R}^N$, then $0 \in B_1$ is a non-degenerate global minimum of V_{B_1} and it is the unique critical point.*

Proof. First of all we note that V_{B_1} is a radial function, i.e. $V_{B_1}(x) = v_{B_1}(|x|)$. Hence, since V_{B_1} is smooth in the interior of the ball, it follows that $v'_{B_1}(0) = 0$. It is easily seen that

$$(\Delta V_{B_1})(0) = 2(1+s)(N+2s) \int_{B_1^C} \frac{1}{|y|^{N+2s+2}} \,\mathrm{d}y > 0,$$

where B_1^C denotes the complement of B_1. Therefore, since $v''_{B_1}(0) = \frac{1}{n}\Delta V_{B_1}(0)$, it follows that for fixed $\delta > 0$ one has $v''_{B_1}(t) > 0$ for $t \in [0,\delta]$, which implies the non-degeneracy of the origin as a critical point of V_{B_1}.

It remains to show the monotonicity of v_{B_1} in the whole interval $(0,1)$, but since Lemma 3.7 holds, it is sufficient to show that

$$\frac{\mathrm{d}}{\mathrm{d}t} V_{B_1}(t\vec{e}_1) \neq 0 \quad \text{for } t \in [\delta, 1-\delta]. \tag{5.1}$$

Recalling the definition (1.3), we get

$$\frac{\mathrm{d}}{\mathrm{d}t} V_{B_1}(t\vec{e}_1) = \tilde{c}_{N,s} \int_{B_1^C} \frac{y_1 - t}{|y - t\vec{e}_1|^{N+2s+2}} \,\mathrm{d}y, \tag{5.2}$$

where $\tilde{c}_{N,s}$ is a constant depending only on N and s, $y = (y_1, y') \in \mathbb{R} \times \mathbb{R}^{N-1}$.

By Fubini–Tonelli Theorem

$$\int_{B_1^C} \frac{y_1 - t}{|y - t\vec{e}_1|^{N+2s+2}} \,\mathrm{d}y = \int_{\mathbb{R}^{N-1}} \mathrm{d}y' \int_{\{y_1 : (y_1,y') \in B_1^C\}} \frac{y_1 - t}{|y - t\vec{e}_1|^{N+2s+2}} \,\mathrm{d}y. \tag{5.3}$$

Since $(y_1, y') \in B_1^c \times \mathbb{R}^{N-1}$, we have two cases:

1) if $|y'| \geq 1 \quad \Rightarrow \quad y_1 \in \mathbb{R}$;
2) if $|y'| < 1 \quad \Rightarrow \quad y_1 \leq -\sqrt{1 - |y'|^2} \ \vee \ y_1 \geq \sqrt{1 - |y'|^2}$.

In the first case we obtain by oddness

$$\int_{\{y_1 : (y_1,y') \in B_1^C\}} \frac{y_1 - t}{|y - t\vec{e}_1|^{N+2s+2}} \,\mathrm{d}y = \int_{\{y_1 \in \mathbb{R}\}} \frac{y_1 - t}{((y_1 - t)^2 + |y'|^2)^{(N+2s+2)/2}} \,\mathrm{d}y = 0. \tag{5.4}$$

In the second case, using the changes of variables $y_1 - t = s$ and $z = t - y_1$, we get

$$\begin{aligned} &\int_{\{y_1 : (y_1,y') \in B_1^C\}} \frac{y_1 - t}{|y - t\vec{e}_1|^{N+2s+2}} \,\mathrm{d}y \\ &= \int_{\{y_1 \leq -\sqrt{1-|y'|^2}\}} \frac{y_1 - t}{|y - t\vec{e}_1|^{N+2s+2}} \,\mathrm{d}y + \int_{\{y_1 \geq \sqrt{1-|y'|^2}\}} \frac{y_1 - t}{|y - t\vec{e}_1|^{N+2s+2}} \,\mathrm{d}y \\ &= \int_{\{z \geq t + \sqrt{1-|y'|^2}\}} \frac{z}{(z^2 + |y'|^2)^{(N+2s+2)/2}} \,\mathrm{d}z \\ &+ \int_{\{s \geq \sqrt{1-|y'|^2} - t\}} \frac{s}{(s^2 + |y'|^2)^{(N+2s+2)/2}} \,\mathrm{d}y > 0, \end{aligned} \tag{5.5}$$

since $\{z : z \geq t + \sqrt{1 - |y'|^2}\} \subseteq \{z : z \geq \sqrt{1 - |y'|^2} - t\}$ and since the first integral is negative.

Putting together (5.2), (5.3), (5.4) and (5.5) we obtain (5.1) which concludes the proof. □

Lemma 5.2. *Let Φ_ξ be defined as in Proposition 3.4. There exist a dumble-bell domain (as in Figure 2) with the same topology of the ball, such that Φ_ξ has at least three critical points.*

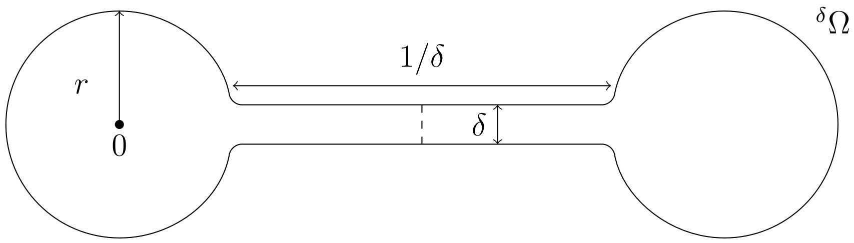


Figure 2. A dumb-bell domain $^\delta\Omega$.

Sketch of the Proof. We consider a sequence of domains $^\delta\Omega$ as in Figure 2. Fixed $r \in (0,1)$, it is easy to see that

$$V_{^\delta\Omega} \to V_{B_1} \quad \text{in } C^2(B_r(0)) \quad \text{as } \delta \to 0. \tag{5.6}$$

For δ small, by Lemma 5.1, we get that $V_{^\delta\Omega}$ has a unique non-degenerate minimum x_1 in $B_{r/2}(0)$ and there exists $\gamma > 0$ such that

$$\inf_{\partial B_r(0)} V_{^\delta\Omega} > \sup_{B_{r/2}(0)} V_{^\delta\Omega} + \gamma.$$

By symmetry, we have a non-degenerate minimum point x_2 in the other ball with the same properties. Recall also that from Lemma 3.7 that if $x \in \partial(^\delta\Omega)$, it holds

$$\lim_{^\delta\Omega \ni y \to x} V_{^\delta\Omega}(y) = +\infty.$$

Hence, from (3.24) (with a similar formula for the gradient in ξ) and the above observations, there exists a critical point of Φ other that x_1 and x_2, by Mountain Pass Theorem. $\square$

References

[1] N. Abatangelo and E. Valdinoci, *A notion of nonlocal curvature*, Numer. Funct. Anal. Optim. **35** (2014), no. 7-9, 793–815.

[2] F. J. Almgren Jr., *Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints*, Mem. Amer. Math. Soc. **4** (1976), no. 165, viii+199.

[3] A. Ambrosetti and A. Malchiodi, *Nonlinear analysis and semilinear elliptic problems*, Cambridge Studies in Advanced Mathematics, vol. 104, Cambridge University Press, Cambridge, 2007.

[4] ———, *Perturbation methods and semilinear elliptic problems on* $\mathbf{R}^n$, Progress in Mathematics, vol. 240, Birkhäuser Verlag, Basel, 2006.

[5] L. Ambrosio, G. De Philippis, and L. Martinazzi, *Gamma-convergence of nonlocal perimeter functionals*, Manuscripta Math. **134** (2011), no. 3-4, 377–403.

[6] B. Barrios, A. Figalli, and E. Valdinoci, *Bootstrap regularity for integro-differential operators and its application to nonlocal minimal surfaces*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **13** (2014), no. 3, 609–639.

[7] A. Bressan, *Hyperbolic systems of conservation laws*, Oxford Lecture Series in Mathematics and its Applications, vol. 20, Oxford University Press, Oxford, 2000. The one-dimensional Cauchy problem.

[8] F. Brock, *Weighted Dirichlet-type inequalities for Steiner symmetrization*, Calc. Var. Partial Differential Equations **8** (1999), no. 1, 15–25.

[9] X. Cabré, M. M. Fall, and T. Weth, *Curves and surfaces with constant nonlocal mean curvature: meeting Alexandrov and Delaunay*, Math. Ann. **370** (2018), no. 3-4, 1513–1569.

[10] L. Caffarelli, J.-M. Roquejoffre, and O. Savin, *Nonlocal minimal surfaces*, Comm. Pure Appl. Math. **63** (2010), no. 9, 1111–1144.

[11] A. Cesaroni and M. Novaga, *Volume constrained minimizers of the fractional perimeter with a potential energy*, Discrete Contin. Dyn. Syst. Ser. S **10** (2017), no. 4, 715–727.

[12] A. Di Castro, M. Novaga, B. Ruffini, and E. Valdinoci, *Nonlocal quantitative isoperimetric inequalities*, Calc. Var. Partial Differential Equations **54** (2015), no. 3, 2421–2464.

[13] M. Cozzi, *On the variation of the fractional mean curvature under the effect of $C^{1,\alpha}$ perturbations*, Discrete Contin. Dyn. Syst. **35** (2015), no. 12, 5769–5786.

[14] O. Druet, *Sharp local isoperimetric inequalities involving the scalar curvature*, Proc. Amer. Math. Soc. **130** (2002), no. 8, 2351–2361.

[15] J. Davila, M. Del Pino, and J. Wei, *Nonlocal minimal Lawson cones*, J. Differential Geom. **109** (2018), no. 1, 111–175.

[16] E. Di Nezza, G. Palatucci, and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136** (2012), no. 5, 521–573.

[17] A. Ehrhard, *Inégalités isopérimétriques et intégrales de Dirichlet gaussiennes*, Ann. Sci. École Norm. Sup. (4) **17** (1984), no. 2, 317–332 (French).

[18] M. M. Fall, *Area-minimizing regions with small volume in Riemannian manifolds with boundary*, Pacific J. Math. **244** (2010), no. 2, 235–260.

[19] ______, *Embedded disc-type surfaces with large constant mean curvature and free boundaries*, Commun. Contemp. Math. **14** (2012), no. 6, 1250037, 35.

[20] A. Figalli, N. Fusco, F. Maggi, V. Millot, and M. Morini, *Isoperimetry and stability properties of balls with respect to nonlocal energies*, Comm. Math. Phys. **336** (2015), no. 1, 441–507.

[21] N. Fusco, V. Millot, and M. Morini, *A quantitative isoperimetric inequality for fractional perimeters*, J. Funct. Anal. **261** (2011), no. 3, 697–715.

[22] E. Gonzalez, U. Massari, and I. Tamanini, *On the regularity of boundaries of sets minimizing perimeter with a volume constraint*, Indiana Univ. Math. J. **32** (1983), no. 1, 25–37.

[23] M. Grossi, *Uniqueness of the least-energy solution for a semilinear Neumann problem*, Proc. Amer. Math. Soc. **128** (2000), no. 6, 1665–1672.

[24] ______, *Uniqueness results in nonlinear elliptic problems*, Methods Appl. Anal. **8** (2001), no. 2, 227–244. IMS Workshop on Reaction-Diffusion Systems (Shatin, 1999).

[25] ______, *A uniqueness result for a semilinear elliptic equation in symmetric domains*, Adv. Differential Equations **5** (2000), no. 1-3, 193–212.

[26] M. Grüter, *Boundary regularity for solutions of a partitioning problem*, Arch. Rational Mech. Anal. **97** (1987), no. 3, 261–270.

[27] I. M. James, *On category, in the sense of Lusternik-Schnirelmann*, Topology **17** (1978), no. 4, 331–348.

[28] F. Maggi and E. Valdinoci, *Capillarity problems with nonlocal surface tension energies* **42** (2017), no. 9, 1403–1446.

[29] C. Mihaila, *Axial symmetry for fractional capillarity droplets*, Comm. Partial Differential Equations **43** (2018), no. 12, 1673–1701.

[30] I. A. Minlend, *Solutions to Serrin's overdetermined problem on Manifolds* (2017), preprint.

[31] F. Morgan and D. L. Johnson, *Some sharp isoperimetric theorems for Riemannian manifolds*, Indiana Univ. Math. J. **49** (2000), no. 3, 1017–1041.

[32] S. Nardulli, *The isoperimetric profile of a smooth Riemannian manifold for small volumes*, Ann. Global Anal. Geom. **36** (2009), no. 2, 111–131.

[33] L. Nirenberg, *Topics in nonlinear functional analysis*, Courant Lecture Notes in Mathematics, vol. 6, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2001.

[34] F. Riesz, *Sur une inégalité intégrale*, Journ. London Math. Soc. **5** (1930), 162–168.

[35] A. Ros, *The isoperimetric problem*, Global theory of minimal surfaces, Clay Math. Proc., vol. 2, Amer. Math. Soc., Providence, RI, 2005, pp. 175–209.

[36] S. G. Samko, *Hypersingular integrals and their applications*, Analytical Methods and Special Functions, vol. 5, Taylor & Francis, Ltd., London, 2002.

[37] M. Sáez and E. Valdinoci, *On the evolution by fractional mean curvature*, Comm. Anal. Geom. **27** (2019), no. 1, 211–249.

[38] J. E. Taylor, *Boundary regularity for solutions to various capillarity and free boundary problems*, Comm. Partial Differential Equations **2** (1977), no. 4, 323–357.
[39] R. Ye, *Foliation by constant mean curvature spheres*, Pacific J. Math. **147** (1991), no. 2, 381–396.

Andrea Malchiodi
Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa, Italy
Email address: andrea.malchiodi@sns.it

Matteo Novaga
Dipartimento di Matematica, Università? di Pisa, Largo B. Pontecorvo 5, 56217 Pisa, Italy
Email address: matteo.novaga@unipi.it

Dayana Pagliardini
Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa, Italy
Email address: dayana.pagliardini@sns.it