Lecture Notes on Partial Differential Equations

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1 Some basic facts regarding Sobolev spaces

In the sequel we will make constant use of Sobolev spaces. We will just summarize the basic facts needed in the sequel, referring for instance to [3] for a more detailed treatment of this topic. Actually, it is possible to define them in two different ways, whose (partial) equivalence is discussed below.

Definition 1.1. *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain and fix an exponent p with $1 \leq p < +\infty$. We can consider the class of regular functions $C^1\left(\overline{\Omega}\right)$ (i.e. the subset of $C^1(\Omega)$ consisting of functions u such that both u and ∇u admit a continuous extension on $\partial\Omega$) endowed with the norm*

$$\|u\|_{W^{1,p}} = \|u\|_{L^p} + \|\nabla u\|_{L^p}. \tag{1.1}$$

We define the space $H^{1,p}(\Omega)$ to be the completion with respect to the $W^{1,p}$ norm of $C^1(\overline{\Omega})$.

For unbounded domains, including the whole space $\mathbb{R}^n$, the definition is similar and based on the completion of

$$\left\{u \in C^1(\overline{\Omega}) : \ u \in L^p(\Omega), \ |\nabla u| \in L^p(\Omega)\right\}.$$

Note that $H^{1,p}(\Omega) \subset L^p(\Omega)$.

On the other hand, we can adopt a different viewpoint, inspired by the theory of distributions.

Definition 1.2. *Let $\Omega \subset \mathbb{R}^n$ be an open domain and consider the space $C_c^\infty(\Omega)$ whose elements will be called test functions. For $1 \leq p \leq \infty$ we say that $u \in L^p(\Omega)$ has i-th derivative in weak sense equal $g_i \in L^1_{\mathrm{loc}}(\Omega)$ if*

$$\int_\Omega u \partial_i \varphi \, dx = -\int_\Omega \varphi g_i dx \quad \forall \varphi \in C_c^\infty(\Omega). \tag{1.2}$$

Whenever such $g_1, \ldots, g_n$ exist, we say that is differentiable in weak sense. We define the space $W^{1,p}(\Omega)$ as the subset of $L^p(\Omega)$ whose elements u are weakly differentiable and such that the corresponding derivatives $\partial_i u$ also belong to $L^p(\Omega)$.

It is clear that if g_i exists, it must be uniquely determined, since $h \in L^1_{\mathrm{loc}}(\Omega)$ and

$$\int_\Omega h\varphi \, dx = 0 \qquad \forall \varphi \in C_c^\infty(\Omega)$$

implies $h = 0$. This implication can be easily proved showing that the property above is stable under convolution, namely $h_\epsilon = h * \rho_\epsilon$ satisfies $\int_{\Omega_\epsilon} h_\epsilon \varphi \, dx = 0$ for all $\varphi \in C_c^\infty(\Omega_\epsilon)$, where Ω_ϵ is the (slightly) smaller domain

$$\Omega_\epsilon := \{x \in \Omega : \ \mathrm{dist}(x, \partial\Omega) > \epsilon\}. \tag{1.3}$$

Here $\rho_\epsilon(x) = \epsilon^{-n}\rho(x/\epsilon)$ and ρ is smooth and compactly supported in the unit ball. Hence classical derivatives are weak derivatives and the notation $\partial_i(u)$ (or, equivalently, $D_i u$ or even $\frac{\partial u}{\partial x_i}$) is justified.

One classical way to relate weak and strong derivatives is via convolution: namely if u has weak i-th derivative g, then

$$\partial_i(u * \rho_\epsilon) = g * \rho_\epsilon \qquad \text{in } \Omega_\epsilon. \tag{1.4}$$

Identity (1.4) can be easily proved considering both sides as weak derivatives and applying Fubini's theorem; the smoothness of $u * \rho_\epsilon$ tells us that the derivative in the left hand side is (equivalent to) a classical one.

Notice also that Definition 1.2 covers the case $p = \infty$, while it is not immediately clear how to adapt Definition 1.1 to cover this case (and usually H Sobolev spaces are defined for $p < \infty$ only).

In the next proposition we consider the relation of $W^{1,\infty}$ with Lipschitz functions. We omit for brevity the simple proof, based on convolutions.

Proposition 1.3 (Lipschitz versus $W^{1,\infty}$ functions)**.** *If $\Omega \subset \mathbb{R}^n$ is open, then* $\mathrm{Lip}(\Omega) \subset W^{1,\infty}(\Omega)$ *and*

$$\|Du\|_{L^\infty(\Omega)} \le \mathrm{Lip}(u, \Omega) \ . \tag{1.5}$$

In addition, if Ω is convex then $\mathrm{Lip}(\Omega) = W^{1,\infty}(\Omega)$ *and equality holds in* (1.5)*.*

Since $H^{1,p}(\Omega)$ is defined by means of approximation by regular functions, for which (1.2) is just the elementary "integration by parts formula", it is clear that $H^{1,p}(\Omega) \subset W^{1,p}(\Omega)$. On the other hand, using convolutions and a suitable extension operator described below (in the case $\Omega = \mathbb{R}^n$ the proof is a direct application of (1.4)), one can prove the following result:

Theorem 1.4 ($H = W$)**.** *If either $\Omega = \mathbb{R}^n$ or Ω is a bounded regular domain, then*

$$H^{1,p}(\Omega) = W^{1,p}(\Omega) \qquad 1 < p < \infty. \tag{1.6}$$

With the word *regular* we mean that the boundary is locally the graph of a Lipschitz function of $(n-1)$-variables. However the equality $H = W$ is not true in general, as the following example shows.

Example 1.5. In the Euclidean plane $\mathbb{R}^2$, consider the open unit ball $x^2 + y^2 < 1$ deprived of one of its radii, say for instance the segment Σ given by $(-1, 0] \times \{0\}$. We can define on this domain Ω a function θ having values in $(-\pi, \pi)$ and representing the angle in polar coordinates. Fix an exponent $1 \le p < 2$. It is immediate to see that $\theta \in C^\infty(\Omega)$ and that its gradient is p-integrable, hence $\theta \in W^{1,p}$. On the other hand, $\theta \notin H^{1,p}(\Omega)$ because the definition we have given would require the existence of regular approximations for θ up to the boundary: more precisely, one can easily show using Fubini's theorem and polar coordinates that any $u \in H^{1,p}(\Omega)$ satisfies

$$u_r(\omega) := u(re^{i\omega}) \in W^{1,p}_{\mathrm{loc}}(\mathbb{R})$$

for $\mathscr{L}^1$-a.e. $r \in (0,1)$, a property not satisfied by θ (we shall see that $W^{1,1}_{\mathrm{loc}}$ functions on the real line have indeed continuous representatives).

Remark 1.6. Taking into account the example above, we mention the Meyers-Serrin theorem, ensuring for *any* open set $\Omega \subset \mathbb{R}^n$ and $1 \le p < +\infty$, the identity

$$\overline{C^\infty(\Omega) \cap W^{1,p}(\Omega)}^{W^{1,p}} = W^{1,p}(\Omega) \tag{1.7}$$

holds. The proof can be achieved by (1.4) and a partition of unity. Roughly speaking, the previous result underlines the crucial role played by the behaviour at the boundary in the approximation of a function in $W^{1,p}$. In the case $p = \infty$ the construction in the Meyers-Serrin theorem provides for all $u \in W^{1,\infty}(\Omega)$ a sequence $(u_n) \subset C^\infty(\Omega)$ converging to u uniformly in Ω, with $\sup_\Omega |\nabla u_n|$ convergent to $\|\nabla u\|_\infty$; considering this type of approximation in Definition 1.1 the validity of the theorem could be extended up to $p = \infty$.

As will be clear soon, we also need to define an appropriate subspace of $H^{1,p}(\Omega)$ in order to work with functions vanishing at the boundary.

Definition 1.7. *Given $\Omega \subset \mathbb{R}^n$ open, we define $H^{1,p}_0(\Omega)$ to be the completion of $C^1_c(\Omega)$ with respect to the $W^{1,p}$ norm.*

It is clear that $H^{1,p}(\Omega)$, being *complete*, is a *closed* subspace of $H^{1,p}(\Omega)$.
We now turn to some classical inequalities.

Theorem 1.8 (Poincaré inequality, first version)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open bounded subset with regular boundary and $1 \le p < +\infty$. Then there exists a constant $C(\Omega)$, only depending on Ω, such that*

$$\|u\|^p_{L^p} \le C(\Omega)\, \|\nabla u\|^p_{L^p} \qquad \forall u \in H^{1,p}_0(\Omega). \tag{1.8}$$

The proof of this result can be strongly simplified by means of these two following remarks:

- $H^{1,p}_0(\Omega) \subset H^{1,p}_0(\Omega')$ if $\Omega \subset \Omega'$ (*monotonicity property*)
- $C(\lambda\Omega) = \lambda^p C(\Omega)$. (*scaling property*)

The first fact is a consequence of the definition of the spaces $H^{1,p}$ in terms of regular functions, while the second follows by:

$$u_\lambda(x) = u(\frac{x}{\lambda}) \in H^{1,p}_0(\Omega) \quad \forall u \in H^{1,p}_0(\lambda\Omega). \tag{1.9}$$

Proof. By the monotonicity and scaling properties, it is enough to prove the inequality for $\Omega = Q_a \subset \mathbb{R}^n$ where Q_a is the cube centered at the origin, with sides parallel to the coordinate axes and having length $2a$. We write $x = (x_1, x')$ with $x' = (x_2, \ldots, x_n)$. By density, we may also assume $u \in C^1_c(\Omega)$ and hence use the following representation formula:

$$u(x_1, x') = \int_{-a}^{x_1} \frac{\partial u}{\partial x_1}(t, x')\, dt. \tag{1.10}$$

The Hölder inequality gives

$$|u|^p (x_1, x') \leq (2a)^{p-1} \int_{-a}^{a} \left| \frac{\partial u}{\partial x_1} \right|^p (t, x') dt \tag{1.11}$$

and hence we just need to integrate in x_1 to get

$$\int_{-a}^{a} |u|^p (x_1, x')\, dx_1 \leq (2a)^p \int_{-a}^{a} \left| \frac{\partial u}{\partial x_1} \right|^p (t, x') dt. \tag{1.12}$$

Now, integrating in x',repeating the previous argument for all the variables $x_j, j = 1, \ldots, n$ and summing all such inequalities we obtain the thesis with $C(Q_a) \leq (2a)^p/n$. $\square$

Remark 1.9. It should be observed that the previous proof, even though very simple, is far from giving the sharp constant for the Poincaré inequality. The determination of the sharp constant requires more refined methods.

Theorem 1.10 (Rellich)**.** *Let Ω be a domain as in the previous theorem and again $1 \leq p < \infty$. Then the immersion $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is continuous and compact.*

We do not give a complete proof of this result. We observe that it can be obtained using an appropriate linear and continuous *extension operator*

$$T : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n) \tag{1.13}$$

such that

$$\begin{cases} Tu = u & \text{in } \Omega; \\ \\ \text{supp}(Tu) \subset \Omega' & . \end{cases}$$

being Ω' a fixed bounded domain in $\mathbb{R}^n$ containing $\overline{\Omega}$. This construction is classical and relies on the fact that the boundary of $\partial\Omega$ is regular and so can be locally *straightened* by means of Lipschitz maps (we will use these ideas later on, when treating the boundary regularity of solutions to elliptic PDE's). The global construction is then obtained thanks to a partition of unity.

The operator T allows basically a reduction to the case $\Omega = \mathbb{R}^n$, considered in the next theorem.

Theorem 1.11. *The immersion $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^p_{\mathrm{loc}}(\mathbb{R}^n)$ is continuous and compact.*

Remark 1.12. It should be noted that the immersion $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$ is obviously continuous, but certainly not compact. To note this, just take a fixed element in $W^{1,p}(\mathbb{R}^n)$ and supported in the unit square and consider the sequence of its translates along vectors τ_h with $|\tau_h| \to \infty$. Of course this is a bounded sequence in $W^{1,p}(\mathbb{R}^n)$ but no subsequence converges in $L^p(\mathbb{R}^n)$.

Let us now briefly sketch the main points of the proof of this theorem, since some of the ideas we use here will be often considered in the sequel.

Proof. Basically, it is enough to prove that a bounded family $\mathcal{F} \subset W^{1,p}(\mathbb{R}^n)$ is totally bounded in $L^p_{loc}(\mathbb{R}^n)$. To obtain this, first observe that given any Borel domain $A \subset \mathbb{R}^n$ and any $\varphi \in C^1(A_{|h|})$ we have

$$\|\varphi - \tau_h \varphi\|_{L^p(A)} \leq |h| \, \|\nabla \varphi\|_{L^p(A_{|h|})} \tag{1.14}$$

where $A_{|h|}$ is the $|h|\,-$neighbourhood of the set A and $\tau_h\varphi(x) = \varphi(x+h)$. This follows by the elementary representation

$$(\tau_h \varphi - \varphi)(x) = \int_0^1 \langle \nabla \varphi(x+sh), h \rangle \, ds \tag{1.15}$$

since

$$\|\tau_h \varphi - \varphi\|^p_{L^p(A)} \leq \int_A \int_0^1 |\langle \nabla \varphi(x+sh), h\rangle|^p \, ds \, dx \tag{1.16}$$

$$\leq |h|^p \int_0^1 \int_{A_{|h|}} |\nabla \varphi(y)|^p \, dy \, ds = |h|^p \, \|\nabla \varphi\|^p_{L^p(A_{|h|})} \tag{1.17}$$

by means of the Minkowski integral inequality, the Cauchy-Schwarz inequality and finally by Fubini's theorem. Hence, denoting by $(\rho_\epsilon)_{\epsilon>0}$ any rescaled family of smooth mollifiers such that $\mathrm{supp}(\rho_\epsilon) \subset B(0,\epsilon)$, we have that for any $R > 0$

$$\sup_{\varphi \in \mathcal{F}} \|\varphi - \varphi * \rho_\epsilon\|_{L^p(B_R)} \to 0 \tag{1.18}$$

for $\epsilon \to 0$. In fact, by the previous result we deduce

$$\sup_{\varphi \in \mathcal{F}} \|\varphi - \varphi * \rho_\epsilon\|_{L^p(B_R)} \leq \epsilon \sup_{\varphi \in \mathcal{F}} (\int_{B_{R+\epsilon}} |\nabla \varphi|^p \, dx)^{1/p}. \tag{1.19}$$

To conclude we just need to observe that the regularised family $\{\varphi * \rho_\epsilon, \varphi \in \mathcal{F}\}$ is relatively compact in $L^p_{loc}(\mathbb{R}^n)$ for any fixed $\epsilon > 0$. But this is easy since the Young inequality implies

$$\sup_{B_R} |\varphi * \rho_\epsilon| \leq \|\varphi\|_{L^1(B_{R+\epsilon})} \, \|\rho_\epsilon\|_\infty \tag{1.20}$$

and similarly

$$\sup_{B_R} |\nabla(\varphi * \rho_\epsilon)| \leq \|\nabla\varphi\|_{L^1(B_{R+\epsilon})} \|\rho_\epsilon\|_\infty \tag{1.21}$$

so the claim is immediate by means of the Ascoli-Arzelá theorem. □

We also need to mention another inequality due to Poincaré.

Theorem 1.13. *Let us consider a bounded, regular and connected domain $\Omega \subset \mathbb{R}^n$ and an exponent $1 \leq p < \infty$ so that by Rellich's theorem we have the compact immersion $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$. Then, there exists a constant $C(\Omega)$ such that*

$$\int_\Omega |u - u_\Omega|^p \, dx \leq C(\Omega) \int_\Omega |\nabla u|^p \, dx \quad \forall u \in W^{1,p}(\Omega) \tag{1.22}$$

where $u_\Omega = \fint_\Omega u \, dx$.

Proof. By contradiction, if the desired inequality were not true, exploiting its homogeneity and translation invariance we could find a sequence (u_n) such that

- $(u_n)_\Omega = 0$ for all $n \in \mathbb{N}$;
- $\int_\Omega |u_n|^p \, dx = 1$ for all $n \in \mathbb{N}$;
- $\int_\Omega |\nabla u_n|^p \, dx \to 0$ for $n \to \infty$.

By Rellich's theorem there exists (up to estraction of a subsequence) a limit point $u \in L^p$ that is $u_n \to u$ in $L^p(\Omega)$. It is now a general fact that if $(\nabla\varphi_n)$ has some weak limit point g then necessarily $g = \nabla u$. Therefore, in this case we have by comparison $\nabla u = 0$ in $L^p(\Omega)$ and hence, by connectedness of the domain, we deduce that u must be equivalent to a constant (this last fact can be proved by a smoothing argument). By the required properties of the sequence (u_n) we must have at the same time

$$\int_\Omega u \, dx = 0 \tag{1.23}$$

and

$$\int_\Omega |u|^p \, dx = 1 \tag{1.24}$$

which is clearly impossible. □

Note that the previous proof is not constructive and crucially relies on the general compactness result by Rellich.

2 Variational formulation of some PDEs

After the first section, whose main purpose was to fix the notation and recall some basic tools, we are now ready to turn to some first facts concerning PDEs.

Let us consider the generalised Poisson equation

$$\begin{cases} -\Delta u = f - \sum_\alpha \partial_\alpha f_\alpha \quad \text{in } \Omega; \\ \\ u \in H_0^{1,2}(\Omega)\,. \end{cases}$$

with data $f, f_\alpha \in L^2(\Omega)$ for some fixed bounded and regular domain Ω. This equation has to be intended in a weak sense, that is, we look $u \in H_0^{1,2}(\Omega)$ satisfying

$$\int_\Omega \langle \nabla u, \nabla \varphi \rangle \; dx = \int_\Omega (f\varphi + \sum_\alpha f_\alpha \partial_\alpha \varphi)\, dx \qquad \forall \varphi \in C_c^\infty(\Omega). \tag{2.1}$$

Equivalently, by density, the previous condition could be requested for any $\varphi \in H_0^{1,2}(\Omega)$. In order to obtain existence we just need to apply Riesz's theorem to the associated *linear* functional $F(v) = \int_\Omega (fv + \sum_\alpha f_\alpha \partial_\alpha v)\; dx$ on the Hilbert space $H_0^{1,2}(\Omega)$ endowed withe the scalar product

$$(u, v) = \int_\Omega \langle \nabla u, \nabla v \rangle \; dx \tag{2.2}$$

which is equivalent to the usual one thanks to the Poincaré inequality proved in the beginning of the previous section.

We can consider many variants of the previous problem, basically by introduction of one or more of the following elements:

- more general linear operators instead of $-\Delta$;
- inhomogeneous or mixed boundary conditions;
- systems instead of single equations.

Our purpose now is to briefly discuss each of these situations.

2.1 Elliptic operators

The first variation is to consider scalar problems having the form

$$\begin{cases} -\sum_{\alpha,\beta} \partial_\alpha (A^{\alpha\beta} \partial_\beta u) u = f - \sum_\alpha \partial_\alpha f_\alpha \quad \text{in } \Omega; \\ \\ u \in H_0^{1,2}(\Omega)\,. \end{cases}$$

where A is a constant matrix satisfying the following requirements:

(i) $A^{\alpha\beta}$ is symmetric, that is $A^{\alpha\beta} = A^{\beta\alpha}$;

(ii) A has only strictly positive eigenvalues or, equivalently, $A \geq cI$ for some $c > 0$, in the sense of quadratic forms.

Here and in the sequel we use the capital letter I to denote the identity matrix on $\mathbb{R}^n$.

Actually, it is convenient to deal immediately with the case of a varying matrix $A(x)$ such that:

(i) A is a Borel and L^∞ function defined on Ω;

(ii) $A(x)$ is symmetric for a.e. $x \in \Omega$;

(iii) there exists a positive constant c such that $A(x) \geq cI$ for a.e. $x \in \Omega$.

As indicated above, the previous problem has to be intended in weak sense and precisely

$$\int_\Omega \langle A\nabla u, \nabla\varphi\rangle \, dx = \int_\Omega (f\varphi + \sum_\alpha f_\alpha \partial_\alpha \varphi) \, dx \qquad \forall \varphi \in C_c^\infty(\Omega) \tag{2.3}$$

with respect to any suitable class of test functions. In order to obtain existence we could modify the previous argument, but we prefer here to proceed differently and introduce some ideas that belong to the so-called *direct methods* of the Calculus of Variations. Let us consider the functional $F : H_0^{1,2}(\Omega) \to \mathbb{R}$

$$F(v) = \int_\Omega \frac{1}{2} \langle A\nabla v, \nabla v\rangle \, dx - \int_\Omega fv \, dx - \sum_\alpha \int_\Omega f_\alpha \partial_\alpha v \, dx. \tag{2.4}$$

First we note that, thanks to our third assumption on A, F is *coercive*, that is

$$\lim_{\|u\|_{H_0^{1,2}(\Omega)} \to +\infty} F(u) = +\infty \tag{2.5}$$

and consequently, in order to look for its minima it is enough to reduce to some closed ball of $H_0^{1,2}(\Omega)$. Now, take any minimizing sequence (u_n) of F: since $H_0^{1,2}(\Omega)$ is a separable Hilbert space we can assume, possibly extracting a subsequence, that $u_n \rightharpoonup u$ for some $u \in H_0^{1,2}(\Omega)$. Now, it is easy to see that F is continuous and convex (it is the sum of a linear and a convex functional) and so it also *weakly* lower semicontinous. Hence

$$F(u) \leq \liminf_{n\to\infty} F(u_n) = \inf_{H_0^{1,2}(\Omega)} F \tag{2.6}$$

and we conclude that u is a (global) minimum of F. Actually, the functional F is strictly convex and so u is its unique minimum.

Consequently, since F is a C^1 functional on $H_0^{1,2}(\Omega)$ we get $dF(u) = 0$, where dF is the differential in the Gateaux sense of F:

$$dF(u)\,[\varphi] := \lim_{\epsilon\to 0} \frac{F(u+\epsilon\varphi) - F(u)}{\epsilon} \qquad \forall \varphi \in H_0^{1,2}(\Omega).$$

Here a simple computation gives

$$dF(u)\,[\varphi] = \int_\Omega \langle A\nabla u, \nabla\varphi\rangle \; dx - \int_\Omega f\varphi \, dx - \sum_\alpha \int_\Omega f_\alpha \partial_\alpha h \, dx. \tag{2.7}$$

and the desired result follows.

2.2 Inhomogeneous boundary conditions

We now turn to study the boundary value problem for $u \in H^{1,2}(\Omega)$

$$\begin{cases} -\Delta u = f - \sum_\alpha \partial_\alpha f_\alpha & \text{in } \Omega; \\ \\ u = g & \text{su } \partial\Omega. \end{cases}$$

with $f,\ f_\alpha \in L^2(\Omega)$ and a suitable class of functions $g \in L^2(\partial\Omega)$. The boundary condition has to be considered in *weak sense* since the immersion $H^{1,2}(\Omega) \hookrightarrow C(\overline{\Omega})$ does not hold if $n \geq 2$. Here and in the sequel, unless otherwise stated, we indicate with Ω an open, bounded and regular subset of $\mathbb{R}^n$.

Theorem 2.1. *For any $1 \leq p < \infty$ the restriction operator*

$$T : C^1(\overline{\Omega}) \to C^0(\partial\Omega) \tag{2.8}$$

can be uniquely extended to a linear and bounded operator from $W^{1,p}(\Omega)$ to $L^p(\partial\Omega)$.

Consequently, we will interpret the boundary condition as

$$Tu = g. \tag{2.9}$$

It can also be easily proved that Tu is characterized by the identity

$$\int_\Omega u \frac{\partial\varphi}{\partial x_i} \, dx = -\int_\Omega \varphi \frac{\partial u}{\partial x_i} \, dx + \int_{\partial\Omega} \varphi \, Tu \, \nu_i \, d\sigma \qquad \forall \varphi \in C^1(\overline{\Omega}) \tag{2.10}$$

where $\nu = (\nu_1, \ldots, \nu_n)$ is the unit normal vector, pointing out of Ω. Indeed, using the equality $H^{1,p}(\Omega) = W^{1,p}(\Omega)$ one can start from the classical divergence theorem with $u \in C^1(\overline{\Omega})$ and then argue by approximation.

Remark 2.2. It is possible to show that the previously defined restriction operator T is not surjective if $p > 1$ and that its image can be described in terms of fractional Sobolev spaces. The borderline case $p = 1$ is special, and in this case Gagliardo proved the surjectivity of T.

We can now mimic the argument described in the previous section in order to achieve an existence result, provided the function g belongs to the image of T, that is there exists a function $\widetilde{u} \in W^{1,2}(\Omega)$ such that $T\widetilde{u} = g$. Indeed, if this is the case, our problem is reduced to show existence for the equation

$$\begin{cases} -\Delta v(x) = \widetilde{f} - \sum_\alpha \partial_\alpha \widetilde{f}_\alpha & \text{in } \Omega; \\ \\ u \in H_0^{1,2}(\Omega). \end{cases}$$

where $\widetilde{f} = f$ and $\widetilde{f}_\alpha = f_\alpha + \partial_\alpha \widetilde{u}$. This is precisely the first problem we have discussed above and so, denoted by v its unique solution, the function $u = v + \widetilde{u}$ will satisfy both our equation and the required boundary conditions. These methods can be applied, with minor changes, to problems having the form

$$\begin{cases} -\sum_{\alpha,\beta} \partial_\alpha (A^{\alpha\beta} \partial_\beta u) + \lambda u = f - \sum_\alpha \partial_\alpha f_\alpha & \text{in } \Omega; \\ \\ A^{\alpha\beta} \partial_\beta u \nu_\alpha = g & \text{on } \partial\Omega. \end{cases}$$

with $A^{\alpha\beta}$ a real matrix and $\lambda > 0$ a fixed constant. For the sake of brevity, we just discuss the case $A^{\alpha\beta} = \delta_{\alpha\beta}$ so that the problem above becomes

$$\begin{cases} -\Delta u + \lambda u = f & \text{in } \Omega; \\ \\ \frac{\partial u}{\partial \nu} = 0 & \text{su } \partial\Omega. \end{cases}$$

In order to give it a clear meaning, note that if u, $v \in C^1(\overline{\Omega})$ then

$$\int_\Omega \langle \nabla u, \nabla v \rangle \, dx = -\int_\Omega v \Delta u \, dx + \int_{\partial\Omega} v \frac{\partial u}{\partial \nu} \, d\sigma \tag{2.11}$$

and so in this case it is natural to ask that for any $v \in C^1(\overline{\Omega})$ the desired solution u satisfies

$$\int_\Omega \left[\langle \nabla u, \nabla v \rangle + \lambda uv \right] \, dx = \int_\Omega vf \, dx + \int_{\partial\Omega} vg \, d\sigma. \tag{2.12}$$

In order to obtain existence (and uniqueness) for this problem, it is enough to apply Riesz's theorem to the bilinear form on $H^{1,2}(\Omega)$

$$a(u,v) = \int_\Omega \left[\langle \nabla u, \nabla v \rangle + \lambda uv \right] \, dx \tag{2.13}$$

which is clearly equivalent to the standard Hilbert product on the same space.

2.3 Elliptic systems

In order to deal with systems, we first need to introduce some appropriate notation. We will consider functions $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ and, consequently, we will use Greek letters (say α, $\beta, \ldots$) in order to indicate the *starting domain* of such maps (so that α, $\beta \in \{1, 2, \ldots, n\}$), while we will use Latin letters (say $i, j, k, \ldots$) for the *target domain* (and hence i, $j \in \{1, 2, \ldots, m\}$). In many cases, we will need to work with *four indices matrices* (i.e. rank four tensors) like $A_{ij}^{\alpha\beta}$, whose meaning should be clear from the context. Finally, we will adopt Einstein convention and use it without explicit noticing.

Our first purpose now is to see whether it is possible to adapt some ellipticity condition (having the form $A \geq cI$ for some $c > 0$) to the vector-valued case. The first idea is to define the *Legendre condition*

$$A_{ij}^{\alpha\beta}\xi_\alpha^i\xi_\beta^j \geq c\,|\xi|^2 \quad \forall \xi \in M^{m\times n} \tag{2.14}$$

where $M^{m\times n}$ indicates the space of $m \times n$ real matrices. Let us apply it in order to obtain existence and uniqueness for the system

$$\begin{cases} -\partial_\alpha(A_{i,j}^{\alpha\beta}\partial_\beta u^j) = f_i - \partial_\alpha f_i^\alpha \quad i = 1, \ldots, m \\ \\ u \in H_0^1(\Omega; \mathbb{R}^m) \end{cases}$$

with data f_i, $f_i^\alpha \in L^2(\Omega)$.[1] The weak formulation of the problem is obviously

$$\int_\Omega A_{ij}^{\alpha\beta}\partial_\beta u^j \partial_\alpha \varphi^i \, dx = \int_\Omega \left[f_i\varphi^i + f_i^\alpha \partial_\alpha \varphi^i\right] \, dx \tag{2.15}$$

for every $\varphi \in [C_c^1(\Omega)]^m$ and again $i = 1, \ldots, m$. Now, if the matrix $A_{ij}^{\alpha\beta}$ is symmetric with respect to the transformation $(\alpha, i) \to (\beta, j)$ (which is implied for instance by the symmetries in (α, β) and (i, j)), then it defines a scalar product on $H_0^1(\Omega; \mathbb{R}^m)$ by the formula

$$(\varphi, \psi) = \int_\Omega A_{ij}^{\alpha\beta}\partial_\alpha\varphi^i\partial_\beta\varphi^j \, dx. \tag{2.16}$$

If, moreover, A satisfies the Legendre condition above for some $c > 0$, it is immediate to see that this scalar product is equivalent to the standard one (with $A_{i,j}^{\alpha\beta} = \delta^{\alpha\beta}\delta_{ij}$) and so we are led to apply again Riesz's theorem to conclude the proof.

Actually, it should be noted that here (and, in particular, in the scalar case) the symmetry hypothesis is not necessary, since we can exploit the following:

Theorem 2.3 (Lax-Milgram)**.** *Let H be a (real) Hilbert space and let $a : H \times H \to \mathbb{R}$ a bilinear, continuous and coercive form so that $a(u, u) \geq \lambda\,|u|^2 \quad \forall u \in H$ for some $\lambda > 0$. Then for all $F \in H'$ there exists $u_F \in H$ such that $a(u_F, v) = F(v)$ for all $v \in H$.*

[1] Note that we sometimes omit the Sobolev exponent when this is equal to two: for instance $H_0^1(\Omega)$ stands for $H_0^{1,2}(\Omega)$.

Proof. By means of the standard Riesz's theorem it is possible to define a linear operator $T : H \to H$ such that

$$a(u, v) = \langle Tu, v\rangle \quad \forall u,\, v \in H \tag{2.17}$$

and such T is continuous since

$$\|Tu\|^2 = \langle Tu, Tu\rangle = a(u, Tu) \leq C\,\|u\|\,\|Tu\| \tag{2.18}$$

where C is a constant of continuity for $a(\cdot,\cdot)$ and hence

$$\|T\| \leq C. \tag{2.19}$$

Now we introduce the auxiliary bilinear form

$$\widetilde{a}(u, v) = \langle TT^* u, v\rangle = \langle T^* u, T^* v\rangle \tag{2.20}$$

which is obviously symmetric and continuous. Moreover, thanks to the coercivity of a we have that $\widetilde{a}$ is coercive too because

$$\lambda\,\|u\|^2 \leq a(u, u) = \langle Tu, u\rangle = \langle u, T^* u\rangle \leq \|u\|\,\|T^* u\| = \|u\|\,\sqrt{\widetilde{a}(u, u)} \tag{2.21}$$

and so $\widetilde{a}(u, u) \geq \lambda^2\,\|u\|^2$. Since $\widetilde{a}$ determines an equivalent scalar product on H we can apply again Riesz theorem to obtain a vector $u'_F \in H$ such that $\widetilde{a}(u'_F, v) = F(v)\forall v \in H$. By the definitions of T and $\widetilde{a}$ this is the thesis once we just set $u_F = T^* u'_F$. □

As indicated above, we now want to formulate a different notion of ellipticity for the vector case. To this aim, it is useful to analyze the situation in the scalar case. We have the two following conditions:

(E) $A \geq \lambda I$ that is $\langle Au, u\rangle \geq \lambda\,|u|^2$ (*ellipticity*);

(C) $a_A(u, u) = \int_\Omega \langle A\nabla u, \nabla u\rangle\, dx \geq \lambda \int_\Omega |\nabla u|^2\, dx$ for all $u \in H^1_0(\Omega; \mathbb{R}^m)$ (*coercivity*).

It is obvious by integration that $(E) \Rightarrow (C)$ and we may wonder about the converse. As we will see below, this is also true in the scalar case ($m = 1$), while it is false when $m > 1$. It is convenient to work with functions having complex values and so let us define for $u, v \in H^1_0(\Omega, \mathbb{C})$

$$a_A(u, v) = \int_{\mathbb{R}^n} \langle A\nabla u, \overline{\nabla v}\rangle\, dx = \int_{\mathbb{R}^n} \sum_{\alpha,\,\beta=1}^{n} \sum_{i,\,j=1}^{m} A^{\alpha\beta}_{ij} \partial_{x_\alpha} u^i \overline{\partial_{x_\beta} u^j}\, dx. \tag{2.22}$$

A simple computation shows that our coercivity hypothesis implies that

$$\Re a_A(u, u) \geq \lambda \int_\Omega |\nabla u|^2\, dx. \tag{2.23}$$

Now consider a function $\varphi \in C_c^\infty(\Omega, \mathbb{R})$ to be fixed later and define $u_\tau(x) = \varphi(x)e^{i\tau x\cdot\xi}$ with $\tau > 0$ to be intended as a big positive parameter. We have that

$$\frac{1}{\tau^2}\Re a_A(u_\tau, u_\tau) = \int_\Omega \varphi^2 A^{\alpha\beta}\xi_\alpha\xi_\beta\, dx + o_\tau(1) \tag{2.24}$$

which is nothing but

$$\frac{1}{\tau^2}\int_\Omega |\nabla u_\tau|^2\, dx = \int_\Omega \varphi^2 |\xi|^2\, dx + o_\tau(1) \tag{2.25}$$

when A is the identity matrix. Hence, exploiting our coercivity assumption and letting $\tau \to +\infty$ we get

$$\int_\Omega \varphi^2 \left(A^{\alpha\beta}\xi_\alpha\xi_\beta - \lambda |\xi|^2\right) dx \geq 0 \tag{2.26}$$

which immediately implies the thesis (it is enough to choose φ not identically zero.) Actually, every single part of our discussion is still true in the case when $A^{\alpha\beta} = A^{\alpha\beta}(x)$ is Borel and L^∞ function in Ω and we can conclude that (E) holds for a.e. $x \in \Omega$: we just need to choose in the very last step for any Lebesgue point x_0 of A an appropriate sequence of rescaled and normalized mollifiers concentrating around x_0. The conclusion comes, in this situation, by Lebesgue differentiation theorem.

For the convenience of the reader we recall here some basic facts concerning Lebesgue points (see Section 13). Given $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x_0 \in \mathbb{R}^n$ we say that x_0 is a Lebesgue point for f if there exists $\lambda \in \mathbb{R}$ such that

$$\lim_{r\downarrow 0} \fint_{B_r(x_0)} |f(y) - \lambda|\, dy = 0. \tag{2.27}$$

In this case λ is unique and it is sometimes written

$$\lambda = \widetilde{f}(x_0) = \widetilde{\lim_{x\to x_0}} f(x). \tag{2.28}$$

The *Lebesgue differentiation theorem* says that for $\mathscr{L}^n$-a.e. $x_0 \in \mathbb{R}^n$ the following two properties hold: x_0 is a Lebesgue point and $\widetilde{f}(x_0) = f(x_0)$.

It is very interesting to note that the previous argument does not give a complete equivalence when $m > 1$: in fact, the coercivity condition

$$a_A(u,u) \geq \lambda \int |\nabla u|^2\, dx \qquad u \in \left[H^1(\mathbb{R}^n)\right]^m \tag{2.29}$$

can be applied to test functions having the form $u_\tau(x) = \varphi(x)be^{i\tau x\cdot a}$ with $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$ and implies the *Legendre-Hadamard condition*

$$A^{\alpha\beta}_{ij}\xi^i_\alpha\xi^j_\beta \geq \lambda |\xi|^2 \qquad \xi = a \otimes b \tag{2.30}$$

that is the *Legendre condition* restricted to rank one matrices $\xi^i_\alpha = a_\alpha b^i$. It is possible to show with explicit examples that the Legendre-Hadamard condition is in general *strictly weaker* than the Legendre condition.

It is possible to show with explicit examples that the Legendre-Hadamard condition is in general *strictly weaker* than the Legendre condition.

Example 2.4. *For instance, consider when $m = n = 2$ a tensor $A^{\alpha\beta}_{ij}$ such that*

$$A^{\alpha\beta}_{ij}\xi^i_\alpha\xi^j_\beta = \det(\xi) + \epsilon\,|\xi|^2 \tag{2.31}$$

for some fixed $\epsilon < 1/2$. Since rank one matrices have null determinant, the Legendre-Hadamard condition is fulfilled for any $\epsilon' \leq \epsilon$, while the Legendre condition fails for example by diagonal matrices $\Lambda(\sigma, -\sigma)$ because the sharp constant c in the arithmetic mean-geometric mean inequality $\sqrt{ab} \leq c(a+b)$ is $c = 1/2$.

Nevertheless, the Legendre-Hadamard condition is sufficient to imply coercivity:

Theorem 2.5 (Gårding)**.** *Assume that $A^{\alpha\beta}_{ij}$ satisfies the Legendre-Hadamerd condition for some positive constant λ. Then $a_A(u,u) \geq \lambda \int |\nabla u|^2\,dx$ for all $u \in H^1(\mathbb{R}^n)$.*

Remark 2.6. If A depends on x, we need some regularity to draw the same conclusion. Assume that Ω is bounded and regular, and that:

- $A \in C(\overline{\Omega})$;
- $A(x)$ satisfies $(LH)_\lambda$ $\lambda > 0$ independent of x.

Then, there exists a constant $\lambda' \in (0, \lambda)$ such that $a_A(u,u) \geq \lambda' \int_\Omega |\nabla u|^2\,dx$ for all $u \in H^1(\Omega)$.

In the following proof, we denote by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space and by $\widehat{\varphi}$ and $\widetilde{\varphi}$ the Fourier transform of φ and its inverse, respectively

$$\widehat{\varphi}(\xi) = (2\pi)^{-n/2}\int \varphi(x)e^{-ix\cdot\xi}\,dx \tag{2.32}$$

and

$$\widetilde{\varphi}(x) = (2\pi)^{-n/2}\int \varphi(x)e^{ix\cdot\xi}\,d\xi. \tag{2.33}$$

We will also make use of the Plancherel identity:

$$\int \widehat{\varphi}\overline{\widehat{\psi}}\,dx = \int \varphi\overline{\psi}\,dx \qquad \forall\varphi,\,\psi \in \mathcal{S}(\mathbb{R}^n). \tag{2.34}$$

Proof. By density it is enough to prove the result when $u \in [C_c^\infty(\mathbb{R}^n)]^m$. In this case we use the representation

$$u(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \varphi(x) e^{-ix\cdot\xi}\, dx \tag{2.35}$$

that is $u(\xi) = \widehat{\varphi}(\xi)$. Consequently,

$$\partial_\alpha u^j(\xi) = -i\widehat{x_\alpha \varphi^j} \tag{2.36}$$

and hence

$$\begin{aligned} a_A(u,u) &= \int_{\mathbb{R}^n} A_{jl}^{\alpha\beta} \frac{\partial u^j}{\partial x_\alpha} \overline{\frac{\partial u^l}{\partial x_\beta}}\, dx = A_{jl}^{\alpha\beta} \int_{\mathbb{R}^n} \widehat{x_\alpha \varphi^j}\, \overline{\widehat{x_\beta \varphi^l}}\, d\xi \\ &= A_{jl}^{\alpha\beta} \int_{\mathbb{R}^n} (x_\alpha \varphi^j)(\overline{x_\beta \varphi^l})\, dx, \end{aligned}$$

the last passage being due to Plancherel identity. But now we can apply our hypothesis

$$A_{jl}^{\alpha\beta} a_\alpha b^j a_\beta b^l \geq \lambda \left|a\right|^2 \left|b\right|^2 \tag{2.37}$$

to get

$$a_A(u,u) \geq \lambda \int_{\mathbb{R}^n} \left|x\right|^2 \left|\varphi(x)\right|^2\, dx. \tag{2.38}$$

If we perform the same steps with $\delta^{\alpha\beta}\delta_{jl}$ in place of $A_{jl}^{\alpha\beta}$ we see at once that

$$\int_{\mathbb{R}^n} \left|\nabla u\right|^2 (\xi)\, d\xi = \int_{\mathbb{R}^n} \left|x\right|^2 \left|\varphi(x)\right|^2\, dx \tag{2.39}$$

and this concludes the proof. □

Remark 2.7. This theorem by Gärding marks in some sense the difference between *pointwise* and *integral* inequalities. It is worth mentioning related results that are typically *non-local*. The first one is the *Korn inequality*: let $u \in [C_c^\infty(\mathbb{R}^n, \mathbb{R})]^n$ and $1 < p < \infty$. Then

$$\int_{\mathbb{R}^n} \left|\nabla u\right|^p\, dx \leq c(n,p) \int_{\mathbb{R}^n} \left|\frac{\nabla u + (\nabla u)^t}{2}\right|^p\, dx. \tag{2.40}$$

The second one is the *Korn-Poincaré inequality*: if Ω is an open, bounded domain with Lipschitz boundary in $\mathbb{R}^n$, then

$$\inf_{c\in\mathbb{R},\, {}^tA=-A} \int_\Omega \left|u(x) - Ax - c\right|^p\, dx \leq C(\Omega,p) \int_\Omega \left|\frac{\nabla u + (\nabla u)^t}{2}\right|^p\, dx. \tag{2.41}$$

2.4 Other variational aspects

The importance of the Legendre-Hadamard condition is also clear from a variational perspective. Indeed, let $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ be a locally Lipschitz function, that is $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$, fix a Lagrangian L and define a functional $F(u) = \int_\Omega L(x, u, \nabla u)\, dx$. We say that u is a *local minimum* for F if

$$F(u) \le F(v) \qquad \text{for all } v \in W^{1,\infty}_{\mathrm{loc}}(\Omega; \mathbb{R}^m) \text{ such that } \quad \{v \neq u\} \Subset \Omega. \tag{2.42}$$

We will make the following standard assumptions on the Lagrangian. We assume that $L : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ is Borel and, denoting the variables as (x, u, p), we assume that L is of class C^1 in (u, p) with

$$\sup_K |L| + |L_u| + |L_p| < +\infty \tag{2.43}$$

for any domain $K = \Omega' \times \{(u,p) |\, |u| + |p| \le R\}$ with $R > 0$. and $\overline{\Omega'} \subset \Omega$. In this case it is possible to show that

$$t \mapsto \int_{\Omega'} L(x, u + t\varphi, Du + tD\varphi)\, dx$$

is of class C^1 for all u, $\varphi \in W^{1,\infty}_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$ and $\overline{\Omega'} \subset \Omega$, and its derivative equals

$$\int_{\Omega'} L_u(x, u + t\varphi, Du + tD\varphi) \cdot \varphi + L_p(x, u + t\varphi, Du + tD\varphi) \cdot \nabla\varphi\, dx$$

(the assumption (19.2) is needed to differentiate under the integral sign). As a consequence, if u is a local minimizer, by looking at the derivative at $t = 0$ we obtain

$$\int_\Omega \left[\sum_i L_{u_i}(x, u, \nabla u)\varphi^i + \sum_{\alpha,i} L_{p_i^\alpha}(x, u, \nabla u) \frac{\partial \varphi^i}{\partial x_\alpha} \right] dx. \tag{2.44}$$

for any $\varphi \in W^{1,\infty}(\Omega; \mathbb{R}^m)$ with compact support. Hence, exploiting the arbitrariness of φ, we obtain the *Euler-Lagrange* equations in the weak sense:

$$\begin{cases} \frac{\partial}{\partial x_\alpha} L_{p_i^\alpha}(x, u, \nabla u) = L_{u^i}(x, u, \nabla u) \\ \\ i = 1, 2, \ldots, m. \end{cases}$$

Exploiting this idea, we can associate to many classes of PDEs appropriate energy functionals, so that the considered problem is nothing but the *Euler-Lagrange equation* for the corresponding functional. For instance, neglecting the boundary conditions (that can actually be taken into account by an appropriate choice of the ambient functional space), equations having the form

$$-\Delta u = g(x, u) \tag{2.45}$$

derive from the functional

$$L(x, s, p) = \frac{1}{2}|p|^2 - \int_0^u g(x, s)\, ds. \tag{2.46}$$

Adding stronger hypotheses on the Lagrangian L in analogy with what has been done above (i.e. requiring that

$$\sup_K |L_{uu}| + |L_{up}| + |L_{pp}| < +\infty$$

for any domain $K = \Omega' \times \{(u, p) | \, |u| + |p| \leq R\}$ with $R > 0$) we can find another *necessary* minimality condition corresponding to $\left[\frac{d^2}{dt^2}F(u + t\varphi)\right] \geq 0$, namely

$$0 \leq \Gamma(\varphi, \varphi) = \int_\Omega [A\nabla\varphi\nabla\varphi + B\nabla\varphi \cdot \varphi + C\varphi \cdot \varphi]\; dx \tag{2.47}$$

where the dependence on x and all indices are omitted for brevity and

$$\begin{cases} A(x) = L_{pp}(x, u(x), \nabla u(x)); \\ B(x) = L_{pu}(x, u(x), \nabla u(x)); \\ C(x) = L_{uu}(x, u(x), \nabla u(x)). \end{cases} \tag{2.48}$$

We can finally obtain *pointwise* conditions on the minimum u by means of the following theorem, whose proof can be obtained arguing as in the proof that coercivity implies ellipticity one can show the following result:

Theorem 2.8. *Consider the bilinear form on $H_0^1(\Omega; \mathbb{R}^m)$ defined by*

$$\Theta(u, v) = \int (A\nabla u\nabla v + B\nabla u \cdot v + Cu \cdot v)\, dx \tag{2.49}$$

where $A = A_{ij}^{\alpha\beta}(x)$, $B = B_{ij}^{\alpha}(x)$ and $C = C_{ij}(x)$ are Borel and L^∞ functions. If $\Theta(u, u) \geq 0$ for all $u \in H_0^1(\Omega; \mathbb{R}^m)$ then $A(x)$ satisfies the Legendre-Hadamard condition with $\lambda = 0$ for a.e. $x \in \Omega$.

Hence, in our case, we find that $L_{pp}(x, u(x), \nabla u(x))$ satisfies the Legendre-Hadamard condition with $\lambda = 0$ for a.e. $x \in \Omega$.

3 Lower semicontinuity of integral functionals

The Morrey-Tonelli theorem is a first, powerful tool leading to an existence result for integral functionals of the form

$$F(u) := \int_\Omega L(x, u(x), Du(x))\, dx \ . \tag{3.1}$$

Before stating Morrey-Tonelli's Theorem, we recall some useful facts about uniformly integrable maps. A complete treatment of this subject can be found for instance in [24].

Theorem 3.1 (Dunford-Pettis)**.** *Let $(X, \mathcal{A}, \mu)$ be a finite measure space and $\mathcal{F} \subset L^1(X, \mathcal{A}, \mu)$. Then the following facts are equivalent:*

(i) the family $\mathcal{F}$ is sequentially relatively compact with respect to the weak-L^1 topology;

(ii) there exists a function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$, with

$$\frac{\phi(t)}{t} \longrightarrow +\infty \quad as \quad t \longrightarrow +\infty,$$

such that

$$\int \phi(|f|)\, d\mu \leq 1 \quad \forall\, f \in \mathcal{F} \ ;$$

(iii) $\mathcal{F}$ is uniformly integrable, i.e.

$$\forall\, \epsilon > 0 \quad \exists\, \delta > 0 \quad s.t. \quad \mu(A) < \delta \quad \Longrightarrow \quad \int_A |f|\, d\mu < \epsilon \quad \forall\, f \in \mathcal{F} \ .$$

Theorem 3.2 (Morrey-Tonelli)**.** *Let $L : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n}$ be a Borel Lagrangian with the following properties:*

(1) L is positive;

(2) L is continuous and its derivative L_p is continuous;

(3) $L(x, s, \cdot)$ is convex[2].

Then any sequence $(u_h) \subset W^{1,1}(\Omega; \mathbb{R}^m)$ converging to u in $L^1(\Omega; \mathbb{R}^m)$ and with uniformly integrable derivatives (Du_h) satisfies the lower semicontinuity inequality

$$F(u) \leq \liminf_{h\to\infty} F(u_h) \ .$$

[2]We will see that this assumption can be considerably weakened

Proof. Firstly we notice that there is a subsequence $u_{h(k)}$ such that

$$\liminf_{h\to\infty} F(u_h) = \lim_{k\to\infty} F(u_{h(k)})$$

and, possibly extracting one more subsequence,

$$u_{h(k)} \longrightarrow u \quad \text{a.e.}\ .$$

Thanks to Dunford-Pettis Theorem we can also assume that the weak-L^1 convergence

$$\nabla u_{h(k)} \rightharpoonup g \quad \text{in } L^1(\Omega;\mathbb{R}^{m\times n})\,;$$

holds. This immediately implies that u belongs to $W^{1,1}(\Omega;\mathbb{R}^m)$ and that $\nabla u = g$.

Thanks to Egorov's Theorem and the fact that u and ∇u are almost everywhere finite, for all $\epsilon > 0$ there exists a compact subset $K_\epsilon \subset \Omega$ such that

- $|\Omega \setminus K_\epsilon| < \epsilon$;
- $u_h \to u$ uniformly on K_ϵ;
- u and ∇u are bounded on K_ϵ.

Because of the convexity hypothesis (2) and the nonnegativity of L, we can estimate

$$\begin{aligned}
\liminf_{h\to\infty} F(u_h) &= \lim_{k\to\infty} \int_\Omega L(x, u_{h(k)}(x), \nabla u_{h(k)}(x))\, dx \\
&\geq \lim_{k\to\infty} \int_{K_\epsilon} L(x, u_{h(k)}(x), \nabla u_{h(k)}(x))\, dx \\
&\geq \lim_{k\to\infty} \int_{K_\epsilon} \left[L(x, u_{h(k)}(x), \nabla u(x)) + \left\langle L_p(x, u_{h(k)}(x), \nabla u(x)), \nabla u_{h(k)}(x) - \nabla u(x)\right\rangle\right] dx\ .
\end{aligned}$$

Thanks to the continuity of L, L_p and the uniform convergence of $u_{h(k)}$ to u on K_ϵ we have that

$$\begin{aligned}
&L(x, u_{h(k)}(x), \nabla u(x)) \to L(x, u(x), \nabla u(x)) \text{ in } L^1(K_\epsilon);\\
&L_p(x, u_{h(k)}(x), \nabla u(x)) \to L_p(x, u(x), \nabla u(x)) \text{ in } L^1(K_\epsilon, \mathbb{R}^{m\times n}).
\end{aligned}$$

Hence, the weak convergence $\nabla u_{h(k)} \rightharpoonup \nabla u$ ensures that

$$\begin{aligned}
&\lim_{k\to\infty} \int_{K_\epsilon} \left[L(x, u_{h(k)}(x), \nabla u(x)) + \left\langle L_p(x, u_{h(k)}(x), \nabla u(x), \nabla u_{h(k)}(x) - \nabla u(x)\right\rangle\right] dx \\
&= \int_{K_\epsilon} L(x, u(x), \nabla u(x))\, dx
\end{aligned}$$

and as $\epsilon \to 0$ we achieve the desired inequality

$$\liminf_{h\to\infty} F(u_h) \geq \int_\Omega L(x, u(x), \nabla u(x))\, dx\ .$$

$\square$

Before stating the following Corollary we recall Rellich's Theorem 1.10, which asserts that the inclusion $W^{1,1}(\Omega) \subset L^1(\Omega)$ is compact whenever $\Omega \subset \mathbb{R}^n$ is an open and bounded set with Lipschitz boundary.

Corollary 3.3. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set with Lipschitz boundary $\partial\Omega$ and L be a Borel Lagrangian satisfying hypotheses (2), (3) from Theorem 3.2 and*

(1') $L(x,u,p) \geq \phi(|p|) + c|u|$ for some $c > 0$ and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\lim_{t \to +\infty} \frac{\phi(t)}{t} = +\infty$.

Then the problem

$$\min \left\{ F(u) |\ u \in W^{1,1}(\Omega; \mathbb{R}^m) \right\}$$

admits a solution.

Proof. It is a classical application of the direct method of Calculus of Variations, where hypothesis $(1)'$ provides the sequentially-relative compactness of sublevels $\{F \leq t\}$ with respect to the so-called sequential weak-$W^{1,1}$ topology (i.e. convergence in L^1 of the functions and weak convergence in L^1 of the derivatives) and semicontinuity is given by Theorem 3.2. □

At this point one could ask whether the convexity assumption in Theorem 3.2 is natural. The answer is negative: as Legendre-Hadamard condition is weaker than Legendre condition, here we are in an analogous situation and the Example 2.4 fits again. Let's define a weaker, although less transparent, convexity condition, introduced by Morrey.

Definition 3.4 (Quasiconvexity)**.** *A Borel, locally bounded function $F : \mathcal{M}^{m \times n} \to \mathbb{R}$ is* quasiconvex *at $A \in \mathcal{M}^{m \times n}$ if*

$$\forall \varphi \in C_c^\infty(\Omega; \mathbb{R}^m) \qquad \fint_\Omega F(A + D\varphi)\, dx \geq F(A)\ . \tag{3.2}$$

We say that F is quasiconvex if it is quasiconvex at every point A.

Remark 3.5. Obviously we can replace the left hand side in (3.2) with the quantity $\fint_{\{D\varphi \neq 0\}} F(A + D\varphi)\, dx$; this follows from the equality

$$\fint_\Omega F(A + D\varphi)\, dx = \left(1 - \frac{|\{D\varphi \neq 0\}|}{|\Omega|}\right) F(A) + \frac{|\{D\varphi \neq 0\}|}{|\Omega|} \fint_{\{D\varphi \neq 0\}} F(A + D\varphi)\, dx\ .$$

Moreover, the dependence from Ω of this notion is only apparent: indeed, we can observe that whenever (3.2) is valid for Ω, then:

- it is valid for every $\Omega' \subset \Omega$;

- it is valid for $x + \lambda\Omega$, for $x \in \mathbb{R}^n$ and $\lambda > 0$ (the simple argument uses the change of variables $\varphi(\cdot) \mapsto \frac{1}{\lambda}\varphi(x + \lambda\cdot)$).

The definition of quasiconvexity is related to Jensen's inequality, which we briefly recall here.

Theorem 3.6 (Jensen)**.** *Let us consider a probability measure μ on a domain $X \subset \mathbb{R}^n$, with $\int_X |y| \, d\mu(y) < \infty$, and a convex, lower semicontinuous function $F : X \to \mathbb{R} \cup \{+\infty\}$. Then*

$$\int_X F(y) \, d\mu(y) \geq F\left(\int_X y \, d\mu(y)\right) .$$

Quasiconvexity should be considered as a weak version of convexity; indeed, if F were convex then the inequality holds for all maps, thanks to Jensen's inequality, while (3.2) corresponds to *gradient* maps.

Proposition 3.7. *Any convex function $F : \mathcal{M}^{m \times n} \to \mathbb{R}$ is quasiconvex.*

Proof. Fix $\varphi \in C_c^\infty$ and consider the law μ of the map $x \mapsto A + D\varphi(x)$ with respect to the rescaled Lebesgue measure $\mathscr{L}^n / \mathscr{L}^n(\Omega)$. Due to the compact support of φ one has

$$\int_{\mathbb{R}^n} y \, d\mu(y) = A + \int_\Omega D\varphi(x) \, dx = A \, .$$

and by a change of variable

$$⨍_\Omega F(A + D\varphi(x)) \, dx = \int_{\mathbb{R}^n} F(y) \, d\mu(y) \geq F\left(\int_{\mathbb{R}^n} y \, d\mu(y)\right) = F(A) \, .$$

□

Remark 3.8. The following chain of implications holds:

$$\text{convexity} \quad \Longrightarrow \quad \text{quasiconvexity} \quad \Longrightarrow \quad \text{Legendre-Hadamard with } \lambda = 0 \, .$$

All these notions are equivalent when either $n = 1$ or $m = 1$; in the other cases:

- the Example 2.4 implies that a quasiconvex function is not necessarily convex when $\min\{n, m\} \geq 2$;

- when $\max\{n, m\} \geq 3$ and $\min\{n, m\} \geq 2$, there exist non trivial examples showing that the Legendre-Hadamard condition does not imply quasiconvexity; the problem is still open for $n = m = 2$.

Let us recall that we introduced quasiconvexity as a "natural" hypothesis to improve Morrey-Tonelli's theorem. The following Theorem 3.11 confirms this fact.

Definition 3.9. *Let us consider an open bounded subset $\Omega \subset \mathbb{R}^n$ and a sequence (f_n) of real function on Ω. We write $f_n \to f$ w^*-$W^{1,\infty}$ if*

- *$f_n \to f$ uniformly in Ω;*
- *$\|\nabla f_n\|_{L^\infty}$ is uniformly bounded.*

Proposition 3.10. *If $f_n \to f$ w^*-$W^{1,\infty}$, then $f \in W^{1,\infty}$ and $\nabla f_n \stackrel{*}{\rightharpoonup} \nabla f$.*

This a direct consequence of the fact that (∇f_n) is sequentially compact in the w^*-topology of L^∞, and any weak* limit provides a weak distributional derivative of f (hence $f \in W^{1,\infty}$, the limit is unique and the whole sequence of derivatives w^*-converges).

Theorem 3.11. *Assume that the functional F in* (3.1) *is sequentially lower semicontinuous with respect to the w^*-$W^{1,\infty}$ topology at some point u. Then the Lagrangian $L(x, u(x), \cdot)$ is quasiconvex at $Du(x)$ for almost every $x \in \Omega$.*

Proof. It is sufficient to prove the result for any Lebesgue point $x_0 \in \Omega$ of Du. The main tool is a blow-up argument: if Q_1 is a unit cube and $v \in W_0^{1,\infty}(Q_1, \mathbb{R}^m)$, we set

$$F_r(v) := \int_{Q_1} L(x_0 + ry, u(x_0 + ry) + rv(y), Du(x_0 + ry) + Dv(y))\, dy\,.$$

The formal limit (as $r \downarrow 0$) of F_r

$$F_0(v) := \int_{Q_1} L(x_0, u(x_0), Du(x_0) + Dv(y))\, dy$$

is sequentially lower semicontinuous at $v = 0$ (with respect to the w^*-$W^{1,\infty}$ topology) because of the following two facts:

- each F_r is sequentially lower semicontinuous with respect to the w^*-$W^{1,\infty}$ topology, in fact

$$\begin{aligned} F_r(v) &= \int_{Q_r(x_0)} L\left(x, u(x) + rv\left(x - x_0/r\right), Du(x) + Dv\left(x - x_0/r\right)\right)\, dx \\ &= F\left(u + rv\left(x - x_0/r\right)\right) - \int_{\Omega \setminus Q_r(x_0)} L(x, u(x), Du(x))\, dx\,; \end{aligned}$$

- being x_0 a Lebesgue point for Du, for any sequence $(v_h) \subset W_0^{1,\infty}$ with $v_h \stackrel{*}{\rightharpoonup} 0$ in $W^{1,\infty}$ it is easily checked that

$$\lim_{r \to 0} \sup_h |F_r(v_h) - F_0(v_h)| = 0\,.$$

Let us introduce the auxiliary function

$$H(p) := L(x_0, u(x_0), Du(x_0) + p)\,.$$

Given a test function $\varphi \in C_c^\infty(Q_1, \mathbb{R}^m)$, we work with the 1-periodic function ψ such that $\psi_{|Q_1} = \varphi$ and the sequence of highly oscillating functions

$$v_h(x) := \frac{1}{h}\psi(hx)\ .$$

By construction $v_h \overset{*}{\rightharpoonup} 0$ in $W^{1,\infty}(Q_1)$ and $Dv_h(x) = D\psi(hx)$. Thanks to the lower semicontinuity of F_0 at 0 one has

$$\begin{aligned} H(0) &= F_0(0) \le \liminf_{h\to\infty} \int_{Q_1} H(Dv_h(x))\,dx \\ &= \liminf_{h\to\infty} h^{-n} \int_{Q_{1/h}} H(D\psi(y))\,dy \\ &= \int_{Q_1} H(D\psi(y))\,dy = \int_{Q_1} H(D\varphi). \end{aligned}$$

□

The previous result, due to Morrey, implies that quasiconvexity of the Lagrangian is equivalent to sequential lower semicontinuity of the integral functional in the weak*-$W^{1,\infty}$ topology. However, in many problems of Calculus of Variations only L^p bounds, with $p < \infty$, on the gradient are available. A remarkable improvement of Morrey's result is the following:

Theorem 3.12 (Acerbi-Fusco). *Suppose that the Lagrangian $L(x, s, p)$ is measurable in x, continuous in s and p and satisfies*

$$0 \le L(x, s, p) \le C(1 + |s|^\alpha + |p|^\alpha)$$

for some $\alpha > 1$ and some constant C; suppose also that the map $p \mapsto L(x, s, p)$ is quasiconvex for all (x, s). Then F is sequentially lower semicontinuous in the weak $W^{1,\alpha}$-topology.

4 Regularity Theory

We begin by studying the local behaviour of (weak) solutions of the system of equations

$$\begin{cases} -D_\alpha\big(A_{ij}^{\alpha\beta}(x)\partial_\beta u^j(x)\big) = f_i + \partial_\alpha F_i^\alpha \qquad i = 1, \dots, m \\ u \in H^1_{\mathrm{loc}}(\Omega; \mathbb{R}^m) \end{cases} \tag{4.1}$$

with $f_i \in L^2_{\mathrm{loc}}$ and $F_i^\alpha \in L^2_{\mathrm{loc}}$.

Theorem 4.1 (Caccioppoli-Leray inequality)**.** *If the Borel coefficients $A_{ij}^{\alpha\beta}(x)$ satisfy the Legendre condition $(L)_\lambda$ with $\lambda > 0$ and*[3]

$$\sup_x \left| A_{ij}^{\alpha\beta}(x) \right|_2 \leq \Lambda < +\infty \ ,$$

then there exists a constant $c = c(\lambda, \Lambda)$ such that for any ball $B_R(x_0) \Subset \Omega$ and any $k \in \mathbb{R}^m$

$$c \int_{B_{R/2}(x_0)} |Du|^2 \, dx \leq R^{-2} \int_{B_R(x_0)} |u(x) - k|^2 \, dx + R^2 \int_{B_R(x_0)} |f(x)|^2 \, dx + \int_{B_R(x_0)} |F(x)|^2 \, dx \ . \tag{4.2}$$

Before proceeding to the proof, some remarks are in order.

Remark 4.2. (1) The validity of (4.2) for all $k \in \mathbb{R}^m$ depends on the translation invariance of the PDE. Also, the inequality (and the PDE as well) has a natural scaling invariance: if we think of u as an adimensional quantity, then all sides have dimension length^{n-2}, because $f \sim$ length^{n-2} and $F \sim$ length^{n-1}.

(2) The Caccioppoli-Leray inequality is "unnatural", because we can't expect that for a general u the gradient is controlled by the variance! Precisely because of this fact we can expect that several useful (regularity) informations can be drawn from it. We will see indeed that CL inequalities are very "natural" and useful in the context of regularity theory.

Remark 4.3. In the regularity theory it often happens that one can estimate, for some $\alpha < 1$,

$$A \leq BA^\alpha + C \ .$$

The absorption scheme allows to bound A in terms of B, C and α only and acts as follows: by Young inequality

$$ab = \epsilon a \, \frac{b}{\epsilon} \leq \frac{\epsilon^p a^p}{p} + \frac{b^q}{\epsilon^q q} \qquad (p^{-1} + q^{-1} = 1)$$

for $p = 1/\alpha$ one obtains

$$A \leq BA^\alpha + C \leq \frac{\epsilon^p A}{p} + \frac{B^q}{\epsilon^q q} + C.$$

Now, if we choose ϵ sufficiently small, so that $\dfrac{\epsilon^p}{p} \leq \dfrac{1}{2}$, we get

$$A \leq 2 \frac{B^q}{\epsilon^q q} + 2C \ .$$

[3] we write $|A|_2$ for the Hilbert-Schmidt norm of a matrix

Let us prove Theorem 4.1.

Proof. Without loss of generality, we can consider $x_0 = 0$ and $k = 0$. As typical in regularity theory, we choose test functions depending on the solution u itself and namely

$$\Phi := u\eta^2$$

where $\eta \in C_c^\infty(B_R)$, $\eta \equiv 1$ in $B_{R/2}$, $0 \le \eta \le 1$ and $|\nabla\eta| \le 4/R$.
Since u solves (4.1), we have that

$$\int ADuD\Phi + \int f\Phi + \int F \cdot D\Phi = 0 \tag{4.3}$$

where integrations are understood to be on B_R. Moreover

$$D\Phi = \eta^2 Du + 2\eta u \otimes \nabla\eta\,, \tag{4.4}$$

so completing (4.3) with (4.4) we obtain

$$\int \eta^2 ADu\, Du + 2\int \eta ADu\, u \otimes \nabla\eta + \int f\Phi + \int \eta^2 FDu + 2\int \eta F\, u \otimes \nabla\eta = 0\,. \tag{4.5}$$

Let's deal with each addendum separately.

- By Legendre condition

$$\int_{B_R} \eta^2 A_{ij}^{\alpha\beta}\partial_\alpha u^i \partial_\beta u^j \ge \lambda \int_{B_R} \eta^2 |Du|^2 \ge \lambda \int_{B_{R/2}} |Du|^2\,.$$

- We have

$$2\int \eta ADu\, u \otimes \nabla\eta \le 2\int \eta |A||Du||u||\nabla\eta| \le \frac{8\Lambda}{R}\int (\eta|Du|)\,|u| \le \frac{8\Lambda\epsilon}{R}\int \eta^2|Du|^2 + \frac{8\Lambda}{R\epsilon}\int |u|^2\,,$$

  where the first estimate is due to Schwarz inequality, the second one relies on the boundedness of coefficients $A_{ij}^{\alpha\beta}$ and the estimate on $|\nabla\eta|$, and the third one is Young inequality.

- By Young inequality

$$\int_{B_R} |f_i u^i \eta^2| \le \int_{B_R} |f||u| \le \frac{1}{R^2}\int_{B_R} |u|^2 + R^2 \int_{B_R} |f|^2\,.$$

- Similarly

$$\int \eta^2 F_i^\alpha \partial_\alpha u^i \le \int |F|\,(\eta|Du|) \le \frac{\lambda}{4}\int \eta^2|Du|^2 + \frac{4}{\lambda}\int |F|^2\,.$$

- Again for the same arguments (Schwarz inequality, estimate on $\nabla\eta$ and Young inequality)

$$2\int_{B_R} |\eta F_i^\alpha u^i \partial_\alpha \eta| \leq \frac{2}{R}\int_{B_R} |F||u| \leq 2\int_{B_R} |F|^2 + \frac{2}{R^2}\int_{B_R} |u|^2 \ .$$

From (4.5) it follows that

$$\begin{aligned} \lambda\int_{B_R} \eta^2|Du|^2 &\leq \int_{B_R} \eta^2 ADu\, Du & (4.6)\\ &= -2\int_{B_R} \eta ADu\, u\otimes\nabla\eta - \int_{B_R} f\Phi - \int_{B_R}\eta^2 FDu - 2\int_{B_R}\eta F\, u\otimes\nabla\eta \\ &\leq \frac{8\Lambda\epsilon}{R}\int_{B_R}\eta^2|Du|^2 + \frac{\lambda}{4}\int \eta^2|Du|^2 & (4.7)\\ &+ \left(\frac{8\Lambda}{R\epsilon} + \frac{1}{R^2} + \frac{2}{R^2}\right)\int_{B_R}|u|^2 + R^2\int_{B_R}|f|^2 + \left(\frac{4}{\lambda}+2\right)\int_{B_R}|F|^2 \ . & (4.8) \end{aligned}$$

By choosing ϵ sufficiently small, in such a way that $8\Lambda\epsilon/R = \lambda/2$, one can absorb line (4.7), whence the thesis. □

Remark 4.4. The Legendre condition in the hypothesis can be replaced by a uniform Legendre-Hadamard condition, provided one assumes that $A_{ij}^{\alpha\beta} \in C(\overline{\Omega}, \mathcal{M}^{m\times n})$ (see Remark 2.6).

Remark 4.5 (Widman's technique)**.** There exists a sharper version of the Caccioppoli-Leray inequality, let's illustrate it in the simpler case $f = 0$, $F = 0$. Indeed, since

$$|\nabla\eta| \leq \frac{4}{R}\chi_{B_R\setminus B_{R/2}} \ ,$$

following the proof of Theorem 4.1 one obtains

$$\int_{B_{R/2}} |Du(x)|^2\, dx \leq \frac{c}{R^2}\int_{B_R\setminus B_{R/2}} |u(x)-k|^2\, dx \ . \tag{4.9}$$

Setting $k := \fint_{B_{R/2}} u$, the Poincaré inequality gives

$$\int_{B_{R/2}} |Du(x)|^2\, dx \leq c\int_{B_R\setminus B_{R/2}} |Du(x)|^2\, dx \ . \tag{4.10}$$

Adding to (4.10) the term $c\int_{B_{R/2}} |Du(x)|^2\, dx$, we get

$$(c+1)\int_{B_{R/2}} |Du(x)|^2\, dx \leq c\int_{B_R} |Du(x)|^2\, dx \ .$$

Setting $\theta := \frac{c}{c+1} < 1$, we obtained a decay inequality

$$\int_{B_{R/2}} |Du(x)|^2 \, dx \le \theta \int_{B_R} |Du(x)|^2 \, dx \ ;$$

by iterating (4.9), it is possible to infer

$$\int_{B_r} |Du(x)|^2 \, dx \le 2^\alpha (\rho/R)^\alpha \int_{B_R} |Du(x)|^2 \, dx \qquad 0 < r \le R$$

with $(1/2)^\alpha = \theta$. When $n = 2$, this implies that $u \in C^{0,\alpha/2}$, as we will see.

The following is another example of "unnatural" inequality.

Definition 4.6 (Reverse Hölder inequality). *A nonnegative function $f \in L^\alpha_{\mathrm{loc}}(\Omega)$ satisfies a* reverse Hölder inequality *if there exists a constant $c > 0$ such that*

$$\fint_{B_R(x)} f^\alpha \le c \left(\fint_{B_R(x)} f \right)^\alpha \quad \forall B_R(x) \subset \Omega \ .$$

At this point, for the sake of completeness, we recall the Sobolev inequalities. We will provide later detailed proofs, in the more general context of Morrey's theory, of the cases $p = n$ and $p > n$. We will also treat the case $p < n$ when dealing with De Giorgi's solution of Hilbert's XIX problem, since slightly more general versions of the Sobolev inequality are needed there.

Theorem 4.7 (Sobolev inequalities). *Let Ω be either the whole space $\mathbb{R}^n$ or a bounded regular domain.*

- *If $p < n$, denoting with $p^* = \frac{np}{n-p} > p$ the Sobolev conjugate exponent (characterized also by $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$), we have the continuous immersion*

$$W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega) \ .$$

- *If $p = n$, the inclusion of $W^{1,n}(\Omega)$ in $BMO(\Omega)$ provides exponential integrability in bounded subsets of Ω .*

- *If $p > n$,*

$$W^{1,p}(\Omega) \subset C^{0,\alpha}(\Omega) \qquad \text{with } \alpha = 1 - n/p. \ .$$

Remark 4.8. The Poincaré inequality tells us that

$$\int_{B_R} |u(x) - \overline{u}|^p \, dx \le cR^p \int_{B_R} |Du|^p \ ,$$

with $\overline{u} := \fint_{B_R} u$. So, as the immersion $W^{1,p} \hookrightarrow L^{p^*}$ is continuous,

$$\left(\int |u - \overline{u}|^{p^*} \right)^{1/p^*} \leq c \left(\int_{B_R} |Du|^p \right)^{1/p} . \tag{4.11}$$

Combining (4.11) with Caccioppoli-Leray inequality when $p^* = 2$ (that is, $p = \frac{2n}{n+2} < 2$), we write

$$\frac{c}{R} \left(\int_{B_R} |Du|^2 \right)^{1/2} \leq \left(\int_{B_R} |u - \overline{u}|^2 \right)^{1/2} \leq C \left(\int_{B_R} |Du|^p \right)^{1/p} .$$

Conveniently rescaling we discover that $|Du|^p$ satisfies reverse Hölder inequality with exponent $\alpha = 2/p > 1$, that is

$$\left(\fint_{B_R} |Du|^2 \right)^{1/2} \leq C \left(\fint_{B_R} |Du|^p \right)^{1/p} .$$

Remark 4.9. Together with Sobolev embedding theorem in Theorem 4.7 with $p > n$, another way to gain continuity is using Sobolev spaces $W^{k,p}$ with k high. In fact, we can arbitrarily expand the chain

$$W^{2,p} \hookrightarrow W^{1,p^*} \hookrightarrow L^{(p^*)^*} .$$

Iterating the $*$ operation k-times we get

$$\frac{1}{p^{*\dots*}} = \frac{1}{p} - \frac{k}{n} ,$$

therefore if $k > \left[\frac{n}{p}\right]$ (where $[\cdot]$ denotes the integer part) we obtain $W^{k,p} \subset C^{0,\alpha}$ with any $\alpha \in (0, 1 - n/p + [n/p])$.

4.1 Nirenberg method

For the moment let us consider a (local) solution u to the Poisson equation

$$-\Delta u = f \quad f \in L^2 .$$

Our aim is to prove that u belongs to H^2.

When we talk about an *a priori* estimate, we mean this reasoning: suppose that we already know that $\frac{\partial u}{\partial x_i} \in H^1$, then it is easy to check that this function solves

$$-\Delta \left(\frac{\partial u}{\partial x_i} \right) = \frac{\partial f}{\partial x_i}$$

in a weak sense. By the Caccioppoli-Leray inequality we get,

$$\int_{B_{R/2}} \left| D\frac{\partial u}{\partial x_i} \right|^2 \leq \frac{c}{R^2} \int \left| \frac{\partial u}{\partial x_i} \right|^2 + \int |f|^2 \ . \tag{4.12}$$

We have chosen the Poisson equation because constant coefficients commute with convolution, so in this case the *a priori* regularity assumption can be *a posteriori* removed. Indeed, estimate (4.12) applies to $u * \rho_\epsilon$ with $f * \rho_\epsilon$ in place of f, since $u * \rho_\epsilon$ satisfies

$$-\Delta(u * \rho_\epsilon) = f * \rho_\epsilon \ .$$

Passing to the limit as $\epsilon \to 0$ the same holds for u.

The situation is much more complex when the coefficients $A_{ij}^{\alpha\beta}$ are not constant and therefore convolution provides a much worse right hand side in the PDE. Nirenberg's idea is to introduce partial discrete derivatives

$$\Delta_{h,i} u(x) := \frac{u(x + he_i) - u(x)}{h} = \frac{\tau_{h,i} u - u}{h}(x) \ .$$

Remark 4.10. Some basic properties of derivation are still true and easy to prove, by translation invariance of Lebesgue measure:

- (sort of) Leibniz property

$$\Delta_{h,i}(ab) = (\tau_{h,i} a)\Delta_{h,i} b + (\Delta_{h,i} a) b \ ;$$

- integration by parts

$$\int \varphi(x) \Delta_{h,i} u(x)\, dx = -\int u(x) \Delta_{-h,i} \varphi(x)\, dx \quad \forall\, \varphi \in C_c^1 \ .$$

Lemma 4.11. *Consider* $u \in L^p_{\text{loc}}(\Omega)$*, with* $1 < p \leq \infty$ *and fix* $i \in \{1, \ldots, n\}$*. The partial derivative* $\frac{\partial u}{\partial x_i}$ *belongs to* $L^p_{\text{loc}}(\Omega)$ *if and only if*

$$\forall\, \Omega' \Subset \Omega \quad \exists\, c(\Omega') \quad s.t. \quad \int_\Omega (\Delta_{h,i} u)\, \varphi \leq c \|\varphi\|_{L^{p'}(\Omega)} \ .$$

Proof. The first implication has been proved in (1.14), because we know that $\Delta_{h,i} u$ is bounded in $L^p_{\text{loc}}(\Omega)$ when $h \to 0$, so we can conclude with Hölder inequality.

Now fix $\Omega' \Subset \Omega$,

$$\left| \int_{\Omega'} u \frac{\partial \varphi}{\partial x_i}\, dx \right| = \left| \lim_{h \to 0} \int_{\Omega'} u \Delta_{-h,i} \varphi\, dx \right| = \left| -\lim_{h \to 0} \int_{\Omega'} (\Delta_{h,i} u)\, \varphi\, dx \right| \leq c(\Omega') \|\varphi\|_{L^{p'}(\Omega')} \ ;$$

because of duality relation between $L^p(\Omega')$ and $L^{p'}(\Omega')$, there exists $\frac{\partial u}{\partial x_i} \in L^p_{\text{loc}}(\Omega)$. $\square$

Let us see how Lemma 4.11 contributes to regularity theory, still in the simplified case of the the Poisson equation. Suppose $f \in H^1_{\mathrm{loc}}$ in Poisson equation, then linearity and translation invariance allow to write

$$-\Delta \tau_{h,i} u = \tau_{h,i} f \quad \Longrightarrow \quad -\Delta(\Delta_{h,i} u) = \Delta_{h,i} f \ .$$

Thanks to Lemma 4.11, $\Delta_{h,i} f$ is bounded in L^2_{loc}, then by Caccioppoli-Leray inequality $\nabla \Delta_{h,i} u$ is bounded in L^2_{loc}. As $\Delta_{h,i}(\nabla u) = \nabla \Delta_{h,i} u$ is bounded in L^2_{loc}, thanks to Lemma 4.11 again we get

$$\frac{\partial}{\partial x_i}(\nabla u) \in L^2_{\mathrm{loc}} \ .$$

After these preliminaries about Nirenberg's method, we are now ready to prove the main result concerning H^2 regularity.

Theorem 4.12. *Let Ω be an open domain in $\mathbb{R}^n$. Consider a function $A \in C^{0,1}_{\mathrm{loc}}(\Omega; \mathbb{R}^{m^2 \times n^2})$ such that $A(x) := A^{\alpha\beta}_{ij}(x)$ satisfies the Legendre-Hadamard condition for a given constant $\lambda > 0$ and let $u \in H^1_{\mathrm{loc}}(\Omega)$ be a weak solution of the equation*

$$-\mathrm{div}(A(x)Du(x)) = f(x) - \mathrm{div}(F(x))$$

for data $f \in L^2_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$ and $F \in H^1_{\mathrm{loc}}(\Omega; \mathbb{R}^{m\times n})$. Then, for every subset $\Omega' \Subset \Omega$ there exists a constant $c := c(\Omega', A)$ such that

$$\int_{\Omega'} \left|D^2 u\right|^2 \, dx \leq c \left\{ \int_\Omega |u|^2 \, dx + \int_\Omega \left[|f|^2 + |F|^2 + |\nabla F|^2\right] \, dx \right\}.$$

In order to simplify the notation, let s denote in the following proof the unit vector corresponding to a given fixed direction and consequently $\tau_h := \tau_{h,s}$ and $\Delta_h := \Delta_{h,s}$.

Remark 4.13. Altough the thesis concerns a generic domain $\Omega' \Subset \Omega$, it is enough to prove it for *balls* inside Ω. More precisely, if $R < \mathrm{dist}(\Omega', \partial\Omega)$ we just need to prove the inequality

$$\int_{B_{R/2}(x_0)} \left|D^2 u\right|^2 \, dx \leq c \left\{ \int_{B_R(x_0)} |u|^2 \, dx + \int_{B_R(x_0)} \left[|f|^2 + |F|^2 + |\nabla F|^2\right] \, dx \right\}$$

for any $x_0 \in \Omega'$ since the general result can be easily obtained by a compactness and covering argument.

Proof. First note that the given equation is equivalent by definition to the identity

$$\int_\Omega ADuD\varphi \, dx = \int_\Omega f\varphi \, dx + \int_\Omega FD\varphi \, dx$$

for all $\varphi \in C^\infty_c(\Omega; \mathbb{R}^m)$. If we apply it to the test function $\tau_{-h}\varphi$ and do a change of variable, we find

$$\int_\Omega \tau_h(ADu)D\varphi \, dx = \int_\Omega \tau_h f\varphi \, dx + \int_\Omega \tau_h FD\varphi \, dx.$$

Subtracting these two equations, we get

$$\int_\Omega (\tau_h A)D(\Delta_h u)D\varphi \, dx = \int_\Omega (\Delta_h f)\varphi \, dx + \int_\Omega (\Delta_h F)D\varphi \, dx - \int_\Omega (\Delta_h A)DuD\varphi \, dx$$

which is nothing but the weak form of the equation

$$-\mathrm{div}((\tau_h A)Dv) = f' - \mathrm{div}(F') \tag{4.13}$$

with data $f' := \Delta_h f$ and $F' := \Delta F - (\Delta_h A D\varphi)$.

Now, the basic idea of the proof will be to use the *Caccioppoli-Leray inequality*. However, a direct application of the CL inequality would lead to an estimate having the L^2 norm of f' on the right hand side, and we know from Lemma 4.11 that this norm can be uniformly bounded in h only if $f \in H^1_{\text{loc}}$. Hence, rather than applying CL directly, we will revisit its proof, trying to get estimates depending only on the L^2 norm of f (heuristically, we view f' as a divergence). To this aim, take a cut-off function η compactly supported in B_R, with $0 \leq \eta \leq 1$, identically equal to 1 on $B_{R/2}$ and such that $|\nabla \eta| \leq 4/R$, and insert in (4.13) the test function $\varphi := \eta^2 \Delta_h u$. By means of simple computations we get

$$\begin{aligned} c\int_{B_{R/2}} \eta^2 |D\Delta_h u|^2 \, dx &\leq \frac{1}{R} \|\eta D(\Delta_h u)\|_2 \|\Delta_h u\|_2 + \|F'\|_2 (\|\eta D(\Delta_h u)\|_2 + \frac{1}{R}\|\Delta_h u\|_2) \\ &+ \int_{B_R} (\Delta_h f)\eta^2 (\Delta_h u)\, dx \end{aligned}$$

and using the Young inequality the previous becomes

$$c' \int_{B_{R/2}} \eta^2 |D\Delta_h u|^2 \, dx \leq \frac{1}{R}\int_{B_R} |F'|^2 \, dx + \frac{1}{R}\int_{B_R} |\Delta_h u|^2 \, dx + \int_{B_R} (\Delta_h f)\eta^2(\Delta_h u)\, dx$$

with c, c' appropriate constants. We need to study the different terms separately. First of all

$$\frac{1}{R}\int_{B_R} |\Delta_h u|^2 \, dx \leq \frac{1}{R}\int_{B_{R+h}} |Du|^2 \, dx$$

by means of (1.14). Hence, the Dirichlet integral can be treated again using the Caccioppoli inequality for u and gives an upper bound of the desired form. Let us then discuss the term

$$\begin{aligned} \left|\int_{B_R} (\Delta_h f)\eta^2(\Delta_h u)\, dx\right| &= \left|\int_{B_R} f \Delta_{-h}(\eta^2 \Delta_h u)\, dx\right| \\ &= \left|\int_{B_R} f \left[\tau_{-h}\eta \Delta_{-h}(\eta \Delta_h u) + (\Delta_{-h}\eta)(\eta \Delta_h u)\right] dx\right| \end{aligned}$$

modified by means of discrete by parts integration and discrete Leibniz rule respectively. The second summand in the last equality can be treated as follows:

$$\begin{aligned} \int_{B_R} |f(\Delta_{-h}\eta)\eta\Delta_h u| \, dx &\leq \frac{4}{R}\int_{B_R} |f|\,|\Delta_h u| \, dx \leq \frac{4}{R}\int_{B_R} |f|^2 \, dx + \frac{4}{R}\int_{B_R} |\Delta_h u|^2 \, dx \\ &\leq \frac{4}{R}\int_{B_R} |f|^2 \, dx + \frac{4}{R}\int_{B_{R+h}} |Du|^2 \, dx, \end{aligned}$$

where the last term admits an upper bound again by the CL inequality. The other summand requires the introduction of a parameter ϵ in the application of the Young inequality:

$$\begin{aligned}\int_{B_R} |f\,[\tau_{-h}\eta]\,\Delta_{-h}(\eta\Delta_h u)|\;dx &\leq \int_{B_R} |f\Delta_{-h}(\eta\Delta_h u)|\;dx \leq \frac{1}{\epsilon}\int_{B_R} |f|^2\;dx + \epsilon\int_{B_R} |\Delta_{-h}(\eta\Delta_h u)|^2\;dx \\ &\leq \frac{1}{\epsilon}\int_{B_R} |f|^2\;dx + 2\epsilon\int_{B_{R+h}} |D\eta|^2\,|\Delta_h u|^2\;dx + 2\epsilon\int_{B_{R+h}} \eta^2\,|\Delta_h(Du)|^2\;dx.\end{aligned}$$

But now, by definition of η

$$2\epsilon\int_{B_{R+h}} \eta^2\,|\Delta_h(Du)|^2\;dx \leq 2\epsilon\int_{B_R} |\Delta_h(Du)|^2\;dx$$

and hence we can fix ϵ sufficiently small so that we can absorb this term in the left-hand side of the inequality. The other terms and also the integral $\int_{B_R} |F'|^2\;dx$ can be studied in the very same way, so that finally we put together all corresponding estimates to get the thesis. □

Remark 4.14. It should be clear from the proof that the previous result basically concerns *inner* regularity and cannot be used in order to get information about the behaviour of the function u near the boundary $\partial\Omega$. In other terms, we can't guarantee that the constant $c(\Omega', A)$ remains bounded when Ω' invades Ω, that is for $R \to 0$, even if global regularity assumptions on A, u, f and F are made. The issue of boundary regularity requires different techniques that will be described later on.

5 Decay estimates for systems with constant coefficients

Our first target towards the development of a regularity theory is now to derive some decay estimates for constant coefficients differential operators. Let $A = A_{ij}^{\alpha\beta}$ be a matrix satisfying the Legendre-Hadamard condition for some $\lambda > 0$, let

$$\Lambda := \sqrt{\sum_{i,j}\sum_{\alpha\beta}(A_{ij}^{\alpha\beta})^2}$$

and consider the problem

$$\begin{cases} -\mathrm{div}(ADu) = 0 \\ u \in H^{1,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^m)\,. \end{cases}$$

Then, these two inequality hold for any $B_r(x_0) \subset B_R(x_0) \Subset \Omega$:

$$\int_{B_r(x_0)} |u|^2 \, dx \leq c(\lambda, \Lambda) \left(\frac{r}{R}\right)^n \int_{B_R(x_0)} |u|^2 \, dx \tag{5.1}$$

$$\int_{B_r(x_0)} |u - u_{r,x_0}|^2 \, dx \leq c(\lambda, \Lambda) \left(\frac{r}{R}\right)^{n+2} \int_{B_R(x_0)} |u - u_{R,x_0}|^2 \, dx \tag{5.2}$$

with $c(\lambda, \Lambda)$ depending only on λ and Λ. Here u_{r,x_0} and u_{R,x_0} denote the mean value of u respectively on $B_r(x_0)$ and $B_R(x_0)$.

Proof of (5.1)**.** By a standard rescaling argument, it is enough to study the case $R = 1$. For the sequel, let k be the smallest integer such that $k > \left[\frac{n}{2}\right]$ (and consequantly $H^k \hookrightarrow C^{0,\alpha}$ with $\alpha = k - \left[\frac{n}{2}\right]$) First of all, by the Caccioppoli-Leray inequality, we have that

$$\int_{B_{1/2}(x_0)} |\nabla v|^2 \, dx \leq c_1 \int_{B_1(x_0)} |v|^2 \, dx.$$

Now, for any $\alpha \in \{1, 2, \ldots, n\}$, we know that $D^\alpha v \in H^{1,2}_{\mathrm{loc}}$ by the previous H^2 regularity result, and since the matrix A has constant coefficients it will solve the same equation. Hence, we can iterate the argument in order to get an estimate having the form

$$\int_{B_{1/2^k}(x_0)} \sum_{|\sigma| \leq k} |D^\sigma v|^2 \leq c_k \int_{B_1(x_0)} |v|^2 \, dx$$

for some constant $c_k > 0$. Consequently, depending on our choice of the parameter k, we can find another constant κ such that

$$\sup_{B_{1/2^k}(x_0)} |v|^2 \leq \kappa \int_{B_1(x_0)} |v|^2 \, dx.$$

In order to conclude the proof, it is better to divide the problem into two cases. If $r \leq 1/2^k$, then

$$\int_{B_r(x_0)} |u|^2 \, dx \leq \omega_n r^n \sup_{B_{1/2^k}(x_0)} |u|^2 \leq \kappa \omega_n r^n \int_{B_1(x_0)} |u|^2 \, dx$$

where ω_n indicates the Lebesgue measure of the unit ball in $\mathbb{R}^n$. Hence, for this case we have the thesis provided we just let $c(\lambda, \Lambda)' = \kappa \omega_n$. If $r \in \left(1/2^k, 1\right)$, then it clear that $\int_{B_r(x_0)} |u|^2 \, dx \leq \int_{B_1(x_0)} |u|^2 \, dx$ and so, since we have a lower bound for r, we just need to choose $c(\lambda, \Lambda)''$ so that $c(\lambda, \Lambda)'' 2^{-kn} = 1$, that is $c(\lambda, \Lambda)'' = 2^{kn}$. Finally, $c(\lambda, \Lambda) := \max\{c(\lambda, \Lambda)'; c(\lambda, \Lambda)''\}$ is the constant needed to conclude.

We can now prove the second inequality, that concerns the notion of *variance* of the function u on a ball.

Proof of (5.2)**.** Again, it is necessary to study two cases separately. If $r \leq R/2$, then by the Poincaré inequality there exists a constant $c > 0$, not depending on r such that

$$\int_{B_r(x_0)} |u - u_{x_0,r}|^2 \, dx \leq cr^2 \int_{B_r(x_0)} |Du|^2 \, dx$$

and so

$$\leq cr^2 \left(\frac{r}{R/2} \right)^n \int_{B_{R/2}(x_0)} |Du|^2 \, dx$$

$$\leq c2^n \left(\frac{r}{R} \right)^{n+2} \int_{B_R(x_0)} |u - u_{R,x_0}|^2 \, dx$$

respectively by the previous result applied to the gradient Du and finally by the Caccioppoli-Leray inequality. For the case $R/2 < r \leq R$ we need to use the following fact, that will be discussed below: let x_0, r and u be as above, then $u_{x_0,r}$ is a minimizer for the function

$$m \longmapsto \int_{B_r(x_0)} |u - m|^2 \, dx. \tag{5.3}$$

If we give this for granted, the conclusion is easy because

$$\int_{B_r(x_0)} |u - u_{r,x_0}|^2 \, dx \leq \int_{B_r(x_0)} |u - u_{R,x_0}|^2 \, dx \leq c' \left(\frac{r}{R} \right)^{n+2} \int_{B_R(x_0)} |u - u_{R,x_0}|^2 \, dx$$

for any c' such that $c \geq 2^{n+2}$.

Let us go back to the study of

$$\inf_{m \in \mathbb{R}} \int_\Omega |u - m|^p \, dx$$

for $1 \leq p < \infty$ and $u \in L^p(\Omega; \mathbb{R})$ where Ω is any open, bounded domain in $\mathbb{R}^n$. As we pointed out above, this problem is solved, when $p = 2$, by the mean value u_Ω, (it suffices to differentiate the integral with respect to m) but this is not true in general for $p \neq 2$. However, for the purpose of the previous result, it is enough to prove what follows. Of course

$$\inf_m \int_\Omega |u - m|^p \, dx \leq \int_\Omega |u - u_\Omega|^p \, dx$$

but we also claim that for any $m \in \mathbb{R}$ we also have

$$\int_\Omega |u - u_\Omega|^p \, dx \leq 2^p \int_\Omega |u - m|^p \, dx. \tag{5.4}$$

Since the problem is clearly translation invariant, it is sufficient to prove the thesis for $m = 0$. But in this case

$$\int_\Omega |u - u_\Omega|^p \, dx \le 2^{p-1} \int_\Omega |u|^p \, dx + 2^{p-1} \int_\Omega |u_\Omega|^p \, dx \le 2^p \int_\Omega |u|^p \, dx$$

thanks to the elementary inequality

$$|a + b|^p \le 2^{p-1} \left(|a|^p + |b|^p \right)$$

and to the fact that

$$\int_\Omega |u_\Omega|^p \, dx \le \int_\Omega |u|^p \, dx$$

which is a standard consequence of the Hölder inequality.

6 Regularity up to the boundary

Let us first consider a simple special case. Suppose we have to deal with the problem

$$\begin{cases} -\Delta u = f \\ \\ u \in H^1(R) \, . \end{cases}$$

where $R := (-a, a)^{n-1} \times (0, a)$ is a rectangle in $\mathbb{R}^n$ with sides parallel to the coordinate axes. Let us use coordinates $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$ and assume $f \in L^2(R)$. The rectangle $R' = (-a/2, a/2) \times (0, a/2)$ is not relatively compact in R, nevertheless via Nirenberg's method we may find estimates having the form

$$\int_{R'} |\partial_{x_s} \nabla u|^2 \, dx \le \frac{c}{a^2} \int_R |\nabla u|^2 \, dx$$

for $s = 1, 2, \ldots, n-1.$, provided $u = 0$ on $R \cap \{x_n = 0\}$. Indeed, we are allowed in this case to use test functions $\varphi = \eta \Delta_{h,s} u$ where the support of η can touch the hyperplane $\{x_n = 0\}$ (because of the homogeneous Dirichlet boundary condition on u). But now the equation may be rewritten as

$$\frac{\partial^2 u}{\partial x_n \partial x_n} = -\Delta_{x'} u + f$$

and here the right hand side $-\Delta_{x'} u + f$ is in $L^2(R')$. We conclude that also the missing second derivative in the x_n direction is in L^2, hence $u \in H^2(R')$. Now we want to use this idea in order to study the regularity up to the boundary for problems like

$$\begin{cases} -\mathrm{div}(ADu) = f + \mathrm{div} F \\ \\ u \in H^1_0(\Omega; \mathbb{R}^m) \end{cases}$$

under the following hypotheses

- $f \in L^2(\Omega; \mathbb{R}^m)$;
- $F \in H^1(\Omega; \mathbb{R}^{m\times n})$;
- $A \in C^{0,1}(\Omega; \mathbb{R}^{m^2\times n^2})$;
- $A(x)$ satisfies the Legendre-Hadamard condition uniformly in Ω;
- $\partial\Omega \in C^2$ in the sense that it is, up to a rigid motion, locally the graph of a C^2 function.

Theorem 6.1. *Under the previous assumptions, the function u belongs to $H^2(\Omega; \mathbb{R}^m)$.*

Since we already have the interior regularity result at our disposal, suffices to show that for any $x_0 \in \partial\Omega$ there exists a neighbourood U of x_0 in Ω such that $u \in H^2(U)$. Without loss of generality we assume $x_0 = 0$. There exist $h \in C^2(\mathbb{R}^{n-1})$ and $V = (-b, b)^n$ such that (up to a rigid motion, choosing the hyperplane $\{x_n = 0\}$ as the tangent one to $\partial\Omega$ at 0)

$$\Omega \cap V = \{x \in V : \; x_n < h(x')\} .$$

Consequently, we can define the change of variables $x'_n = x_n - h(x')$ and the function $H(x', x_n) = (x', x_n - h(x')))$ that maps $\Omega \cap V$ onto $H(\Omega \cap V)$, which contains a rectangle $R = (-a, a)^{n-1} \times (0, a)$. We set $\Omega' := H^{-1}(R) \subset V \cap \Omega$ and $U := H^{-1}(R')$, with $R' = (-a/2, a/2)^{n-1} \times (0, a/2)$.

It is clear that H is invertible and, called G its inverse, both H and G are C^2 functions. Moreover ∇H is a triangular matrix with $\det(\nabla H) = 1$. Besides, the maps G and H induce isomorphisms between H^1 and H^2 spaces (via change of variables in the definition of weak derivative, as we will see in a moment). To conclude, it suffices to show that $v = u \circ G$ belongs to $H^2(R'; \mathbb{R}^m)$. To this aim, we check that v solves in R the PDE

$$\begin{cases} -\mathrm{div}(\widetilde{A}Dv) = \widetilde{f} + \mathrm{div}\widetilde{F} \\[2ex] v = 0 \quad \text{on } \{x'_n = 0\} \cap R \end{cases}$$

where of course the boundary condition has to be interpreted in the weak sense and

$$\widetilde{f} = f \circ G, \quad \widetilde{F} = (F \cdot DH) \circ G, \quad \widetilde{A} = \left[DH \cdot A \cdot (DH)^t\right] \circ G$$

(here contractions are understood with respect to the greek indices, the only ones involved in the change of variables, see (6.1) below). These formulas can be easily derived by an elementary computation, starting from the weak formulation of the problem and applying a change of variables in order to express the different integrals in terms of the new coordinates. For instance

$$\int_{\Omega'} f_i(x)\varphi^i(x)\, dx = \int_R f_i \circ G(y)\varphi^i \circ G(y) \det(\nabla G(y))\, dy$$

just letting $x = G(y)$, but then $\det(\nabla G) = 1$ and we can set $\varphi = \psi \circ H$ so that equivalently $\psi = \varphi \circ H$ and hence

$$\int_{\Omega'} f_i(x)\varphi^i(x)\,dx = \int_R \widetilde{f}_i(y)\psi^i(y)\,dy.$$

Note that here the variable y is nothing but (x', x'_n). The computation for $\widetilde{F}$ or $\widetilde{A}$ is less trivial, but there is no conceptual difficulty. We just see the first one:

$$\begin{aligned}\int_{\Omega'} F_i^\alpha(x)\frac{\partial \varphi^i}{\partial x_\alpha}\,dx &= \int_R F_i^\alpha(G(y))\frac{\partial \varphi^i}{\partial x_\alpha}(G(y))\det(\nabla G(y))\,dy \\ &= \int_R F_i^\alpha(G(y))\frac{\partial \psi^i}{\partial y_\gamma}(y)\frac{\partial H^\gamma}{\partial x_\alpha}(G(y))\,dy\end{aligned}$$

which leads to the conclusion. Note that here and above the arbitrary test function φ has been replaced by the arbitrary test function ψ. However, we should ask whether the conditions on A (for instance, the Legendre-Hadamard condition) still hold true for $\widetilde{A}$. This is the case and we can verify it directly by means of the expression of $\widetilde{A}$ above. In fact,

$$\widetilde{A}_{ij}^{\alpha'\beta'} = \left(\frac{\partial H^{\alpha'}}{\partial x_\alpha}\widetilde{A}_{ij}^{\alpha\beta}\frac{\partial H^{\beta'}}{\partial x_\beta}\right)\circ G \tag{6.1}$$

and so, for any $\widetilde{a} \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$

$$\begin{aligned}\widetilde{A}_{ij}^{\alpha'\beta'}(y)\widetilde{a}_{\alpha'}\widetilde{a}_{\beta'}b^ib^j &= A_{ij}^{\alpha\beta}(G(y))\left(\frac{\partial H^{\alpha'}}{\partial x_\alpha}(G(y))\widetilde{a}_{\alpha'}\right)\left(\frac{\partial H^{\beta'}}{\partial x_\beta}(G(y))\widetilde{a}_{\beta'}\right)b^ib^j \\ &\geq \lambda\left|\nabla H(G(y))\widetilde{a}\right|^2|b|^2 \geq \lambda\left|(\nabla H(G(y)))^{-1}\right|^{-2}|\widetilde{a}|^2|b|^2\end{aligned}$$

since clearly

$$|\widetilde{a}|^2 \leq \left|(\nabla H(G(y)))^{-1}\right|^2\left|\nabla H(G(y))\widetilde{a}\right|^2.$$

Hence, $\widetilde{A}$ satisfies the Legendre-Hadamard condition for an appropriate constant $\lambda' > 0$ depending on λ and H and of course $\widetilde{A} \in C^{0,1}(R)$. Through this transformation of the domain, we can finally apply Nirenberg's method as described above and find that $\partial_{x_\alpha}v^i \in H^1(R')$ for $\alpha = 1, 2, \ldots, n-1$ and $i = 1, 2, \ldots, m$. Anyway, we cannot include in the previous conclusion the second derivatives $\partial^2_{x_nx_n}v^i$ and here we really need to refine the strategy seen above for the Poisson equation. Actually, this is not complicated because the equation readily implies that $\partial_n(\widetilde{A}_{ij}^{nn}\frac{\partial v^j}{\partial x_n}) \in L^2(R')$ for any $i \in \{1, 2, \ldots, m\}$ and we can apply Leibniz rule for distributional derivatives to get (since $\widetilde{A} \in C^{0,1}$) $\widetilde{A}_{ij}^{nn}\frac{\partial v^j}{\partial x_n\partial x_n} \in L^2(R)$. Then $\widetilde{A}_{ij}^{nn}$ is invertible (as a consequence of the Legendre-Hadamard condition) and so $\frac{\partial v^j}{\partial x_n\partial x_n} \in L^2(R')$.

If both the boundary and the data are sufficiently regular, this method can be iterated to get:

Theorem 6.2. *Assume, in addition to the hypotheses above, that $f \in H^k(\Omega;\mathbb{R}^m)$ and also $F \in H^{k+1}(\Omega;\mathbb{R}^{m\times n})$, $A \in C^{k,1}(\Omega, \mathbb{R}^{m^2\times n^2})$ with Ω such that $\partial\Omega \in C^{k+2}$. Then $u \in H^{k+2}(\Omega;\mathbb{R}^m)$.*

We are not going to present the detailed proof of the previous result, but the basic idea consists in differentiating the starting equation with respect to each fixed direction to get an equation having the same form:

$$-\mathrm{div}\left(AD\left(\frac{\partial u}{\partial x_s}\right)\right) = \frac{\partial f}{\partial x_s} + \frac{\partial \mathrm{F}}{\partial \mathrm{x_s}} + \mathrm{div}\left(\frac{\partial A}{\partial x_s}u\right)$$

which is the case, provided we set $\widetilde{F} = F + \frac{\partial A}{\partial x_s}u$.

7 Interior regularity for nonlinear problems

So far, we have just dealt with linear problems and the richness of different situations was only based on the possibility of varying the elliptic operator, the boundary conditions and the number of dimensions involved in the equations. We see now that Nirenberg's technique is particularly appropriate to deal also with nonlinear PDE's, as those arising from Euler-Lagrange equations.

Consider a function $F \in C^2(\mathbb{R}^{mn})$ and assume the following:

(i) There exists a constant $C > 0$ such that $|D^2F(\xi)| \leq C$ for any $\xi \in \mathbb{R}^{mn}$;

(ii) F satisfies a uniform Legendre condition, i.e. $\partial_{p_i^\alpha}\partial_{p_j^\beta}F(p)\xi_i^\alpha\xi_j^\beta \geq \lambda\,|\xi|^2$ for all $\xi \in \mathbb{R}^{mn}$, for some $\lambda > 0$ independent of $p \in \mathbb{R}^{mn}$.

Let $B_i^\alpha := \frac{\partial F}{\partial p_i^\alpha}$ and $A_{ij}^{\alpha\beta} := \frac{\partial^2 F}{\partial p_i^\alpha \partial p_j^\beta}$ and notice that $A_{ij}^{\alpha\beta}$ is symmetric with respect to the transformation $(\alpha, i) \to (\beta, j)$.

Let $\Omega \subset \mathbb{R}^n$ be an open domain and let $u \in H^1_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$ be a local minimum (in the sense recalled below) of the functional

$$w \longmapsto I(w) := \int_\Omega F(\nabla w)\,dx.$$

The implication

$$F \in C^\infty \Rightarrow u \in C^\infty$$

is strongly related to Hilbert's XIX problem (initially posed in 2 space dimensions and in the category of analytic functions). In the sequel we will first treat the case $n = 2$ and much later the case $n \geq 3$, which is significantly harder.

We say that w is a local minimum for I if for any $w \in H^1_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$ such that $\mathrm{spt}(w - w') \subset \Omega' \Subset \Omega$, we have that

$$\int_{\Omega'} F(\nabla w')\,dx \geq \int_{\Omega'} F(\nabla w)\,dx.$$

If this is the case, we can derive an appropriate Euler-Lagrange equation: considering perturbations of the form $w' = w + t\nabla\varphi$ with $\varphi \in C^\infty_c(\Omega,\mathbb{R}^m)$ we can prove (using the fact that the regularity assumptions on F allow differentiation under the integral sign) that

$$0 = \frac{d}{dt}\left[\int_\Omega F(\nabla w + t\nabla\varphi)\,dx\right]_{t=0} = \int_\Omega B_i^\alpha(\nabla w)\frac{\partial\varphi^i}{\partial x_\alpha}\,dx.$$

Now, suppose s is a fixed coordinate direction (and let e_s the corresponding unit vector) and $h > 0$ a "small" positive scalar: if we apply the previous argument to a test function having the form $\tau_{-h}\varphi$, we get

$$\int_\Omega \tau_h(B_i^\alpha(\nabla w))\frac{\partial\varphi^i}{\partial x_\alpha}\,dx = 0$$

and consequently, by subtracting this to the previous one

$$\int_\Omega \Delta_h(B_i^\alpha(\nabla w))\frac{\partial\varphi^i}{\partial x_\alpha}\,dx = 0.$$

However, as a consequence of the regularity of F, we can write

$$B_i^\alpha(\nabla u(x+he_s)) - B_i^\alpha(\nabla u(x)) = \int_0^1 \frac{d}{dt}B_i^\alpha(t\nabla u(x+he_s) + (1-t)\nabla u(x))\,dt$$

$$= \left[\int_0^1 A_{ij}^{\alpha\beta}(t\nabla u(x+he_s) + (1-t)\nabla u(x))\,dt\right]\left[\frac{\partial u^j}{\partial x_\beta}(x+he_s) - \frac{\partial u^j}{\partial x_\beta}(x)\right]$$

and setting

$$\widetilde{A}_{ij,h}^{\alpha\beta}(x) := \int_0^1 A_{ij}^{\alpha\beta}(t\nabla u(x+he_s) + (1-t)\nabla u(x))\,dt$$

we rewrite the previous condition as

$$\int_\Omega \widetilde{A}_{ij,h}^{\alpha\beta}(x)\frac{\partial \Delta_h u^j}{\partial x_\beta}(x)\frac{\partial\varphi^i}{\partial x_\alpha}(x)\,dx = 0.$$

Hence, $w = \Delta_h u$ solves the equation

$$-\mathrm{div}(\widetilde{A}_h Dw) = 0. \tag{7.1}$$

Now, it is obvious by the definition that $\widetilde{A}^{\alpha\beta}_{ij,h}(x)$ satisfies both the Legendre condition for the given constant $\lambda > 0$ and a uniform upper bound on the $L^\infty-$norm and therefore we can apply the Caccioppoli-Leray inequality to the problems (7.1) to obtain constants C_1 and C_2, not depending on h such that

$$\int_{B_R(x_0)} |D(\Delta_{h,s}u)|^2 \, dx \le \frac{C_1}{R^2} \int_{B_{2R}(x_0)} |\Delta_{h,s}u|^2 \, dx \le C_2$$

for any $B_R(x_0) \subset B_{2R}(x_0) \Subset \Omega$. Consequently, by Lemma 4.11 we deduce that

$$u \in H^2_{\mathrm{loc}}(\Omega; \mathbb{R}^m). \tag{7.2}$$

Moreover, we have that

- $\Delta_{h,s}u \to \frac{\partial u}{\partial x_s}$ in L^2_{loc} (this is clearly true if u is regular and then exploit the fact that the operators $\Delta_{h,s}$ are equibounded, still by Lemma 4.11);
- $\frac{\partial u}{\partial x_s}$ satisfies, in a weak sense, the equation $-\mathrm{div}\left(A(Du)D\frac{\partial u}{\partial x_s}\right) = 0$ (note that

$$A^{\alpha\beta}_{ij}(tDu(x+he_s) + (1-t)Du(x)) \overset{h\to 0}{\to} A^{\alpha\beta}_{ij}(x)$$

in L^p for any $1 \le p < \infty$ since they are uniformly bounded and converge $\mathscr{L}^n$-a.e.).

In order to solve Hilbert's XIX problem, we would like to apply a classical result by Schauder saying that if w is a weak solution of the problem $-\mathrm{div}(BDw) = 0$, then $B \in C^{0,\alpha} \Rightarrow w \in C^{2,\alpha}$,. But, we first need to improve the regularity of $B(x) = A(Du(x))$.. In fact, at this point we just know that $A(Du) \in H^1_{\mathrm{loc}}$, while we need $A(Du) \in C^{0,\alpha}$. The situation is much harder in the case $n > 2$, since this requires deep new ideas and the celebrated theory by De Giorgi-Nash-Moser.

8 Hölder, Morrey and Campanato spaces

In this section we introduce the Hölder spaces $C^{0,\alpha}$, the Morrey spaces $L^{p,\lambda}$ and the Campanato spaces $\mathcal{L}^{p,\lambda}$. All these spaces are relevant, besides the standard Lebesgue spaces, in the regularity theory, as we will see.

Definition 8.1. *Given $A \subset \mathbb{R}^n$, $u : A \to \mathbb{R}^m$ and $\alpha \in (0,1]$ we define the $\alpha-$Hölder semi-norm on A as*

$$\|u\|_{\alpha,A} := \sup_{x\neq y\in A} \frac{|u(x)-u(y)|}{|x-y|^\alpha}.$$

We say that u is α-Hölder in A, and write $u \in C^{0,\alpha}(A;\mathbb{R}^m)$, if $\|u\|_{\alpha,A} < \infty$.

If $\Omega \subset \mathbb{R}^n$ is open, we say that $u : \Omega \to \mathbb{R}^m$ is locally $\alpha-$Hölder if for any $x \in \Omega$ there exists a neighbourhood $U_x \Subset \Omega$ such that $\|u\|_{\alpha, U_x} < +\infty$. The corresponding vector space is denoted by $C^{0,\alpha}_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$.

If $k \in \mathbb{N}$, the space of functions of class $C^k(\Omega; \mathbb{R}^m)$ with all $i-$th derivatives with $|i| \le k$ in $C^{0,\alpha}(\Omega; \mathbb{R}^m)$ will be denoted by $C^{k,\alpha}(\Omega; \mathbb{R}^m)$.

Remark 8.2. With respect to the previous definition the spaces $C^{k,\alpha}(\Omega; \mathbb{R}^m)$ are Banach with the norm

$$\|u\|_{C^{k,\alpha}} = \|u\|_{C^k} + \sum_{|i|=k} \left\|D^i u\right\|_{C^{0,\alpha}} .$$

Definition 8.3 (Morrey spaces)**.** *Assume $\Omega \subset \mathbb{R}^n$ open, $\lambda \ge 0$ and $1 \le p < \infty$. We say that $f \in L^p(\Omega)$ belongs to $L^{p,\lambda}(\Omega)$ if*

$$\sup_{0<r<d_\Omega,\, x_0 \in \Omega} r^{-\lambda} \int_{\Omega(x_0,r)} |f|^p \, dx < +\infty$$

where $\Omega(x_0, r) := \Omega \cap B_r(x_0)$. It is easy to verify that

$$\|f\|_{L^{p,\lambda}} := \left(\sup_{0<r<d_\Omega,\, x_0 \in \Omega} r^{-\lambda} \int_{\Omega(x_0,r)} |f|^p \, dx \right)^{1/p}$$

is a norm on $L^{p,\lambda}(\Omega)$.

Remark 8.4. We mention here some of the basic properties of the Morrey spaces $L^{p,\lambda}$,:

(i) $L^{p,\lambda}(\Omega; \mathbb{R})$ are Banach spaces, for any $1 \le p < \infty$ and $\lambda \ge 0$;

(ii) $L^{p,0}(\Omega; \mathbb{R}) = L^p(\Omega; \mathbb{R})$;

(iii) $L^{p,\lambda}(\Omega; \mathbb{R}) = \{0\}$ if $\lambda > n$;

(iv) $L^{p,n}(\Omega; \mathbb{R}) \sim L^\infty(\Omega; \mathbb{R})$;

(v) $L^{q,\mu}(\Omega; \mathbb{R}) \subset L^{p,\lambda}(\Omega; \mathbb{R})$ if $q \ge p$ and $\frac{n-\lambda}{p} \ge \frac{n-\mu}{q}$.

Note that the condition $(n - \lambda)/p \ge (n - \mu)/q$ can also be expressed by asking $\lambda \le \lambda_c$ with the critical value λ_c defined by the equation $(n-\lambda_c)/p = (n-\mu)/q$. The proof of the first result is standard, the second statement is trivial, while the third and fourth ones are immediate applications of Lebesgue differentiation theorem. Finally the last one relies on Hölder inequality:

$$\begin{aligned}
\left(\int_{\Omega(x,r)} |f|^p \, dx \right) &\le C_n \left(\int_{\Omega(x,r)} |f|^q \, dx \right)^{p/q} r^{n(1-p/q)} \\
&\le C_n \|f\|_{L^{q,\mu}} r^{\mu p/q + n(1-p/q)} = C_n \|f\|_{L^{q,\mu}} r^{\lambda_c}.
\end{aligned}$$

Definition 8.5 (Campanato spaces)**.** *Suppose that Ω is a bounded open set. A function $f : \Omega \to \mathbb{R}$ belongs to the Campanato space $\mathcal{L}^{p,\lambda}$ if*

$$\|f\|^p_{\mathcal{L}^{p,\lambda}} := \sup_{x_0\in\Omega,\ 0<r<d_\Omega} r^{-\lambda} \int_{\Omega(x_0,r)} |f(x) - f_{x_0,r}|^p \, dx < \infty \ , \tag{8.1}$$

where d_Ω is the diameter of Ω and

$$f_{x_0,r} := \fint_{\Omega(x_0,r)} f(x)\, dx \ . \tag{8.2}$$

The mean $f_{x_0,r}$ defined in (8.2) is not perhaps the better object to calculate that sort of variance in (8.1), anyway it gives equivalent results, thanks to (5.4).

Remark 8.6. As in Remark 8.4, we briefly highlight the main properties of Campanato spaces.

(i) As defined in (8.1), $\|\cdot\|_{\mathcal{L}^{p,\lambda}}$ is merely a seminorm because constants have null $\mathcal{L}^{p,\lambda}$ norm. If Ω is connected, then $\mathcal{L}^{p,\lambda}$ modulo constants is a Banach space.

(ii) $\mathcal{L}^{q,\mu} \subset \mathcal{L}^{p,\lambda}$ when $p \le q$ and $(n-\lambda)/p \ge (n-\mu)/q$.

(iii) $C^{0,\alpha} \subset \mathcal{L}^{p,n+\alpha p}$, because

$$\int_{\Omega(x_0,r)} |f(x) - f_{x_0,r}|^p \, dx \le \|f\|^p_{C^{0,\alpha}} r^{\alpha p} |B(x_0,r)| = \|f\|^p_{C^{0,\alpha}} \omega_n r^{n+\alpha p} \ .$$

We will see that a converse statement holds (namely functions in these Campanato spaces have hölder continuous representative in their Lebesgue equivalence class), and this is very useful: we can replace the pointwise definition of Hölder spaces with an integral one.

Actually, Campanato spaces are interesting only when $\lambda \ge n$, exactly because of their relationship with Hölder spaces. On the contrary, if $\lambda < n$, Morrey spaces and Campanato spaces are perfectly equivalent. In the proof of this and other result we need a mild regularity assumption on Ω, namely the existence of $c > 0$ satisfying

$$\mathscr{L}^n\left(\Omega \cap B_r(x_0)\right) \ge c r^n \quad \forall x_0 \in \overline{\Omega},\ \forall r \in (0, d_\Omega)\,. \tag{8.3}$$

Basically, this assumption avoids domains with outer cusps.

Theorem 8.7. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded, regular set satisfying* (8.3) *and let $0 \le \lambda < n$. Then the spaces $L^{p,\lambda}$ and $\mathcal{L}^{p,\lambda}$ are equivalent, i.e.*

$$\|\cdot\|_{L^{p,\lambda}} \simeq \|\cdot\|_{\mathcal{L}^{p,\lambda}} + \|\cdot\|_{L^p} \ .$$

Proof. All through the proof we denote with c a generic constant depending from the constant c of regularity of the domain Ω and from n, p, λ.

Without using the hypothesis on λ, we easily prove that $L^{p,\lambda} \subset \mathcal{L}^{p,\lambda}$: trivially Jensen inequality ensures

$$\int_{\Omega(x_0,r)} |f_{x_0,r}|^p \, dx \leq \int_{\Omega(x_0,r)} |f(x)|^p \, dx \ ,$$

thus we can estimate

$$\int_{\Omega(x_0,r)} |f(x) - f_{x_0,r}|^p \, dx \leq 2^{p-1} \left(\int_{\Omega(x_0,r)} |f(x)|^p \, dx + \int_{\Omega(x_0,r)} |f_{x_0,r}|^p \, dx \right) \leq 2^p \int_{\Omega(x_0,r)} |f(x)|^p \, dx \ .$$

Conversely, we would like to estimate $r^{-\lambda} \int_{\Omega(x_0,r)} |f(x)|^p \, dx$ with $\|f\|_{\mathcal{L}^{p,\lambda}} + \|f\|_p$ for every $0 < r < d_\Omega$ and every $x_0 \in \Omega$. As a first step, by triangular inequality we separate

$$\int_{\Omega(x_0,r)} |f(x)|^p \, dx \leq 2^{p-1} \int_{\Omega(x_0,r)} |f(x) - f_{x_0,r}|^p \, dx + c r^n |f_{x_0,r}|^p \leq c \left(\|f\|^p_{\mathcal{L}^{p,\lambda}} + r^n |f_{x_0,r}|^p \right) \ ,$$

so we took out the problematic addendum $|f_{x_0,r}|^p$.

In order to estimate $|f_{x_0,r}|^p$, let us bring in an inequality involving means on concentric balls: when $x_0 \in \Omega$ is fixed and $0 < r < \rho < d_\Omega$, it holds

$$\begin{aligned} c\omega_n r^n |f_{x_0,r} - f_{x_0,\rho}|^p &\leq \int_{\Omega(x_0,r)} |f_{x_0,r} - f_{x_0,\rho}|^p \, dx \\ &\leq 2^{p-1} \left(\int_{\Omega(x_0,r)} |f_{x_0,r} - f(x)|^p \, dx + \int_{\Omega(x_0,r)} |f(x) - f_{x_0,\rho}|^p \, dx \right) \\ &\leq 2^{p-1} \|f\|^p_{\mathcal{L}^{p,\lambda}} \left(r^\lambda + \rho^\lambda \right) \leq 2^p \|f\|^p_{\mathcal{L}^{p,\lambda}} \rho^\lambda \ , \end{aligned}$$

thus we obtained that

$$|f_{x_0,r} - f_{x_0,\rho}| \leq c \|f\|_{\mathcal{L}^{p,\lambda}} r^{-\frac{n}{p}} \rho^{\frac{\lambda}{p}} = c \|f\|_{\mathcal{L}^{p,\lambda}} \left(\frac{\rho}{r} \right)^{\frac{n}{p}} \rho^{\frac{\lambda - n}{p}} \ . \tag{8.4}$$

Now fix a radius $R > 0$: if $r = 2^{-(k+1)} R$ and $\rho = 2^{-k} R$, inequality (8.4) means that

$$|f_{x_0,R/2^{k+1}} - f_{x_0,R/2^k}| \leq c \|f\|_{\mathcal{L}^{p,\lambda}} \left(\frac{R}{2^k} \right)^{\frac{\lambda - n}{p}} \ , \tag{8.5}$$

and, adding up when $k = 0, \ldots, N$, it means that

$$|f_{x_0,R/2^{N+1}} - f_{x_0,R}| \leq c \|f\|_{\mathcal{L}^{p,\lambda}} R^{\frac{\lambda - n}{p}} \frac{2^{N \cdot \frac{n-\lambda}{p}} - 1}{2^{\frac{n-\lambda}{p}} - 1} \leq c \|f\|_{\mathcal{L}^{p,\lambda}} \left(\frac{R}{2^N} \right)^{\frac{\lambda - n}{p}} \ . \tag{8.6}$$

Let us go back to our purpose of estimating $|f_{x_0,r}|^p$: we choose $R \in (d_\Omega/2, d_\Omega)$ and $N \in \mathbb{N}$ such that $r = R/2^{N+1}$. By triangular inequality

$$|f_{x_0,r}|^p \leq 2^{p-1}\left(|f_{x_0,r} - f_{x_0,R}|^p + |f_{x_0,R}|^p\right) ;$$

since

$$|f_{x_0,R}| \leq c(d_\Omega)\|f\|_{L^p} ,$$

the only thing left to conclude is to apply inequality (8.6) in this case:

$$|f_{x_0,r} - f_{x_0,R}|^p \leq c\|f\|_{\mathcal{L}^{p,\lambda}} r^{\lambda-n} ,$$

that is all we needed. □

Remark 8.8. When the dimension of the domain space is n, the Campanato space $\mathcal{L}^{1,n}$ is very important in harmonic analysis and elliptic regularity theory: after John-Nirenberg seminal paper, this space is called BMO (bounded mean oscillation). It consists of the space of all functions $f : \Omega \to \mathbb{R}$ such that there exists a constant C satisfying the inequality

$$\int_{\Omega(x_0,r)} |f(x) - f_{x_0,r}|\, dx \leq Cr^n \quad \forall\, 0 < r < d_\Omega,\ \forall\, x_0 \in \Omega .$$

Notice that $L^\infty(\Omega) \subsetneq BMO(\Omega)$: for example, consider $\Omega = (0,1)$ and $f(x) = \ln x$. For any a, $r > 0$ it is easy to check that

$$\int_a^{a+r} |\ln t - \ln(a+r)|\, dt = r + a\ln\left(\frac{a}{a+r}\right) \leq r ,$$

hence $\ln x \in BMO(\Omega)$. For simplicity, we replaced the mean $⨍_a^{a+r} \ln s\, ds$ with $\ln(a+r)$, but up to a multiplicative factor 2 this does not make a difference. On the contrary $\ln x \notin L^\infty(\Omega)$.

Theorem 8.9 (Campanato)**.** *With the previous notation, when $n < \lambda \leq n+p$ Campanato spaces $\mathcal{L}^{p,\lambda}$ are equivalent to Hölder spaces $C^{0,\alpha}$ with $\alpha = (\lambda - n)/p$. Moreover, if Ω is connected and $\lambda > n+p$, then $\mathcal{L}^{p,\lambda}$ is equivalent to the set of constants.*

Proof. As in the proof of Theorem 8.7, the letter c denotes a generic constant depending on the exponents, the space dimension n and the constant in (8.3).

Let $\lambda = n + \alpha p$. We already observed in Remark 8.6 that $C^{0,\alpha} \subset \mathcal{L}^{p,\lambda}$, so we need to prove the converse: given a function $f \in \mathcal{L}^{p,\lambda}$, we are looking for a representative which belongs to $C^{0,\alpha}$.

Recalling inequality (8.5) with fixed radius $R > 0$ and $x \in \Omega$, we obtain that the sequence $(f_{x,R/2^k})$ has the Cauchy property. Hence we define

$$\tilde{f}(x) := \lim_{k\to\infty} ⨍_{\Omega(x,R/2^k)} f(y)\, dy .$$

Clearly

$$\fint_{\Omega(x,R/2^k)} |f(y) - f_{x,R/2^k}|^p \, dy \longrightarrow 0 \quad \Longrightarrow \quad \fint_{\Omega(x,R/2^k)} |f(y) - \tilde{f}(x)|^p \, dy \longrightarrow 0 \ , \tag{8.7}$$

but since radii have a geometric behaviour, i.e.

$$\fint_{\Omega(x,r)} |f(y) - \tilde{f}(x)|^p \, dy \leq 2^n \fint_{\Omega(x,R/2^k)} |f(y) - \tilde{f}(x)|^p \, dy \ ,$$

then (8.7) implies that

$$\int_{\Omega(x,r)} |f(y) - \tilde{f}(x)|^p \, dy \longrightarrow 0$$

and in particular $\tilde{f}$ does not depend on the chosen radius R. Let us prove that

$$\tilde{f} \in C^{0,\alpha} \ .$$

We employ again an inequality from the proof of Theorem 8.7: letting $k \to \infty$ in (8.6), we get that

$$|\tilde{f}(x) - f_{x,R}| \leq c \|f\|_{\mathcal{L}^{p,\lambda}} R^\alpha$$

with $\alpha = (\lambda - n)/p$; consequently

$$|\tilde{f}(x) - \tilde{f}(y)| \leq |\tilde{f}(x) - f_{x,R}| + |f_{x,R} - f_{y,R}| + |f_{y,R} - \tilde{f}(y)| \leq c|x-y|^\alpha + |f_{x,R} - f_{y,R}| \ .$$

The theorem will be proved if we can estimate $|f_{x,R} - f_{y,R}|$. Choosing $R = 2|x-y|$, we integrate on a domain $\tilde{\Omega}$ with diameter $R/2$, such that

$$\Omega(x,R) \cap \Omega(y,R) \supset \tilde{\Omega} \ ,$$

in order to control

$$|\Omega(x,R) \cap \Omega(y,R)| \geq C 2^{-n} R^n \ .$$

So

$$\begin{aligned}
c 2^{-n} R^n |f_{x,R} - f_{y,R}|^p &\leq \int_{\Omega(x,R) \cap \Omega(y,R)} |f_{x,R} - f_{y,R}|^p \, ds \\
&\leq 2^{p-1} \left(\int_{\Omega(x,r)} |f(s) - f_{x,R}|^p \, ds + \int_{\Omega(y,R)} |f(s) - f_{y,R}|^p \right) \\
&\leq 2^p \|f\|^p_{\mathcal{L}^{p,\lambda}} R^{\lambda - n} \ ,
\end{aligned}$$

and finally

$$|f_{x,R} - f_{y,R}| \leq c \|f\|^p_{\mathcal{L}^{p,\lambda}} R^{\frac{\lambda-n}{p}} \leq c|x-y|^\alpha \ .$$

$\square$

Corollary 8.10 (Sobolev embedding for $p > n$). *With the same hypothesis on $\Omega \subset \mathbb{R}^n$, if $p > n$, then $W^{1,p}(\Omega) \subset C^{0,\alpha}(\Omega)$, with $\alpha = 1 - n/p$.*

Proof. An immediate consequence of Hölder inequality is

$$g \in L^p \quad \Longrightarrow \quad g \in L^{1,\frac{n}{p'}} .$$

For this reason

$$Du \in L^{1,\frac{n}{p'}}(\Omega) = L^{1,n-\frac{n}{p}}(\Omega) = L^{1,n-1+\alpha}(\Omega) . \tag{8.8}$$

Moreover, we have that

$$Du \in L^{p,\lambda}_{\text{loc}} \quad \Longrightarrow \quad u \in \mathcal{L}^{p,\lambda+p}_{\text{loc}} ,$$

because Poincaré inequality gives that

$$\int_{B(x,r)} |u(y) - u_{x,r}|^p \, dy \le cr^p \int_{B(x,r)} |Du|^p \le cr^{\lambda+p} . \tag{8.9}$$

Applying (8.9) to (8.8), we get

$$u \in \mathcal{L}^{1,n+\alpha}(\Omega)$$

and definitely

$$u \in C^{0,\alpha}(\Omega) .$$

□

9 XIX Hilbert problem and its solution in the two-dimensional case

Let $\Omega \subset \mathbb{R}^n$ open, let $F \in C^3(\mathcal{M}^{m\times n})$ and let us consider a local minimizer u of the functional

$$v \mapsto \int_\Omega F(Dv)\, dx \tag{9.1}$$

as in Section 2.4. We assume that $D^2F(p)$ satisfies the Legendre condition (**??**) with $\lambda > 0$ independent of p and is uniformly bounded.

We have seen that u satisfies the Euler-Lagrange equation (9.1) are

$$\frac{\partial}{\partial x_\alpha}\left(F_{p_i^\alpha}(Du)\right) = 0 \quad i = 1, \dots, m. \tag{9.2}$$

We have also seen in Section 7 how, differentiating (9.2) along the direction x_s, one can obtain

$$\frac{\partial}{\partial x_s}\left(\frac{\partial}{\partial x_\alpha}\left(F_{p_i^\alpha}(Du)\right)\right) = \frac{\partial}{\partial x_\alpha}\left(F_{p_i^\alpha p_j^\beta}(Du)\frac{\partial^2 u^j}{\partial x_\beta \partial x_s}\right) = 0 . \tag{9.3}$$

In the spirit of XIX Hilbert's problem, we are interested in the regularity properties of u. Fix $s \in \{1, \ldots, n\}$, let us call

$$U(x_1, \ldots, x_n) := \frac{\partial u}{\partial x_s}(x_1, \ldots, x_n) \in L^2(\Omega, \mathbb{R}^m), \qquad A(x_1, \ldots, x_n) := D^2F(Du(x_1, \ldots, x_n)) \,,$$

thus (9.3) means [4] that U satisfies the system of PDE

$$\operatorname{div}(ADU) = \sum_{\alpha} \sum_{j,\beta} \frac{\partial}{\partial x_\alpha} \left(F_{p_i^\alpha p_j^\beta}(Du) \frac{\partial^2 u^j}{\partial x_\beta \partial x_s} \right) = 0 \,. \tag{9.4}$$

Since $U \in H^1_{\text{loc}}(\Omega; \mathbb{R}^m)$ by (7.2), we can use Caccioppoli-Leray theorem for U, in the sharp version of Remark 4.5. Combining Caccioppoli-Leray inequality with Poincaré inequality, we obtain

$$\int_{B_{R/2}} |U(x)|^2 \, dx \le cR^2 \int_{B_{R/2}} |DU(x)|^2 \, dx \le c \int_{B_R \setminus B_{R/2}} |DU|^2 \,,$$

thus, adding $c \int_{B_{R/2}} |DU|^2$ to both sides, we get

$$\int_{B_{R/2}} |DU(x)|^2 \, dx \le \frac{c}{c+1} \int_{B_R} |DU(x)|^2 \, dx \,.$$

Now, if $\theta := \frac{c}{c+1} < 1$ and $\alpha = -\log_2 \theta$, we can write the previous inequality as

$$\int_{B_{R/2}} |DU(x)|^2 \, dx \le \left(\frac{1}{2}\right)^\alpha \int_{B_R} |DU(x)|^2 \, dx \,. \tag{9.5}$$

In order to get a power decay inequality from (9.5), we state this basic lemma and its useful improvement.

Lemma 9.1. *Consider an nondecreasing real function $f : (0, R_0] \to \mathbb{R}$ satisfying*

$$f\left(\frac{\rho}{2}\right) \le \left(\frac{1}{2}\right)^\alpha f(\rho) \qquad \forall \, \rho \le R_0 \,.$$

Then

$$f(r) \le 2^\alpha \left(\frac{r}{R}\right)^\alpha f(R) \qquad \forall \, 0 < r \le R \le R_0 \qquad .$$

[4] To be precise we should write in (9.4) $\operatorname{div}(ADU^i) = 0$ for all $i = 1, \ldots, m$

Proof. Fix $r < R \leq R_0$ and choose a number $N \in \mathbb{N}$ such that

$$\frac{R}{2^{N+1}} < r \leq \frac{R}{2^N} \ .$$

It is clear from the iteration of the hypothesis that

$$f\left(\frac{R}{2^N}\right) \leq \left(\frac{1}{2}\right)^{\alpha N} f(R) \ ,$$

thus, by monotonicity,

$$f(r) \leq f\left(2^{-N}R\right) \leq 2^{-\alpha N} f(R) = 2^{\alpha} 2^{-\alpha(N+1)} f(R) \leq 2^{\alpha}(r/R)^{\alpha} f(R) \ .$$

□

Lemma 9.2 (Iteration Lemma)**.** *Consider an increasing real function $f : (0, R_0] \to \mathbb{R}$ which satisfies for some positive coefficients A, $B \geq 0$ and exponents $\alpha > \beta$ the following inequality*

$$f(\rho) \leq A\left[\left(\frac{\rho}{R}\right)^{\alpha} + \epsilon\right] f(R) + BR^{\beta} \quad \forall\, 0 < \rho \leq R \leq R_0 \tag{9.6}$$

for some

$$\epsilon \leq \left(\frac{1}{2A}\right)^{\frac{\alpha}{\alpha-\gamma}} \ , \tag{9.7}$$

and some $\gamma \in (\alpha, \beta)$. Then

$$f(\rho) \leq c(\alpha, \beta, A)\left[\left(\frac{\rho}{R}\right)^{\beta} f(R) + B\rho^{\beta}\right] \ . \tag{9.8}$$

Proof. Having fixed the auxiliary exponent $\gamma \in (\beta, \alpha)$, define τ such that

$$2A\tau^{\alpha} = \tau^{\gamma} \ , \tag{9.9}$$

thus (9.7) gives the inequality

$$\epsilon\tau^{-\alpha} \leq 1 \ . \tag{9.10}$$

The following basic estimate uses the hypotheses (9.6) jointly with (9.9) and (9.10):

$$\begin{aligned} f(\tau R) &\leq A(\tau^{\alpha} + \epsilon) f(R) + BR^{\beta} = A\tau^{\alpha}(1 + \epsilon\tau^{-\alpha}) f(R) + BR^{\beta} && (9.11) \\ &\leq 2A\tau^{\alpha} f(R) + BR^{\beta} = \tau^{\gamma} f(R) + BR^{\beta} \ . && (9.12) \end{aligned}$$

The iteration of (9.11) easily gives

$$f(\tau^2 R) \leq \tau^{\gamma} f(\tau R) + B\tau^{\beta} R^{\beta} \leq \tau^{2\gamma} f(R) + \tau^{\gamma} BR^{\beta} + B\tau^{\beta} R^{\beta} = \tau^{2\gamma} f(R) + BR^{\beta}\tau^{\beta}(1 + \tau^{\gamma-\beta}) \ .$$

It can be proven by induction that

$$f(\tau^N R) \le \tau^{N\gamma} f(R) + BR^\beta \tau^{(N-1)\beta} \sum_{k=0}^{N-1} \tau^{k(\gamma-\beta)} = \tau^{N\gamma} f(R) + BR^\beta \tau^{(N-1)\beta} \frac{1-\tau^{N(\gamma-\beta)}}{1-\tau^{(\gamma-\beta)}} .$$

So, given $0 < \rho \le R \le R_0$, if N verifies

$$\tau^{N+1} R < \rho \le \tau^N R ,$$

we conclude choosing the constant $c(\alpha, \beta, A)$ in such a way that the last in the following chain of inequalities holds:

$$\begin{aligned} f(\rho) &\le f(\tau^N R) \le \tau^{N\gamma} f(R) + BR^\beta \tau^{(N-1)\beta} \frac{1-\tau^N \gamma - \beta}{1-\tau^{(\gamma-\beta)}} \\ &\le \tau^\beta \left(\tau^{(N+1)\beta} f(R)\right) + \frac{\tau^{-2\beta}}{1-\tau^{(\gamma-\beta)}} \left(BR^\beta \tau^{(N+1)\beta}\right) \\ &< \tau^\beta \left(\left(\frac{\rho}{R}\right)^\beta f(R) \right) + \frac{\tau^{-2\beta}}{1-\tau^{(\gamma-\beta)}} \left(B\rho^\beta\right) \\ &\le c(\alpha, \beta, A) \left(\left(\frac{\rho}{R}\right)^\beta f(R) + B\rho^\beta \right) . \end{aligned}$$

□

Remark 9.3. The fundamental gain in Lemma 9.2 is the passage from R^β to ρ^β and the removal of ϵ, provided that ϵ is small enough. These improvements can be obtained passing from the power α to the worse power $\beta < \alpha$.

Now, thanks to Lemma 9.1, we are ready to transform (9.5) in

$$\int_{B_\rho} |DU(x)|^2 \, dx \le c \left(\frac{\rho}{R}\right)^\alpha \int_{B_R} |DU(x)|^2 \, dx \qquad \forall\, 0 < \rho \le R ,$$

therefore $|DU| \in L^{2,\alpha}$. So, as we remarked in the proof of Corollary 8.10, this gives

$$U \in \mathcal{L}^{2,\alpha+2} .$$

All these facts are true in any number n of space dimensions, but when $n = 2$ we can apply Campanato theorem to get

$$U \in C^{0,\alpha/2} ,$$

i.e. $u \in C^{1,\alpha/2}$ and $A = D^2F(Du) \in C^{0,\alpha/2}$.

The Schauder theory that we will consider in the next section will allow us to conclude that

$$u \in C^{2,\alpha/2}$$

and, as long as F is sufficiently regular, the iteration of this argument solves XIX Hilbert's regularity problem in the C^∞ category.

10 Schauder theory

We are treating Schauder theory in a local form in $\Omega \subset \mathbb{R}^n$, just because it would be too long and technical to deal also with boundary regularity (some ideas are analogous to those used in Section 6).
We recall the usual PDE we are studying, in a divergence form.

$$\begin{cases} \operatorname{div}(ADu) = \operatorname{div} F & \text{in } \Omega; \\ u \in H^1_{\mathrm{loc}}(\Omega; \mathbb{R}^m) \end{cases} \tag{10.1}$$

Theorem 10.1. *If $A^{\alpha,\beta}_{ij}$ are constant and satisfy the Legendre-Hadamard condition for some $\lambda > 0$, then*

$$F \in \mathcal{L}^{2,\mu}_{\mathrm{loc}} \qquad \Longrightarrow \qquad Du \in \mathcal{L}^{2,\mu}_{\mathrm{loc}} \ .$$

Proof. Since the estimates we make are local, we assume with no loss of generality that $F \in \mathcal{L}^{2,\mu}(\Omega)$. Let us fix a ball $B_R \Subset \Omega$ with center $x_0 \in \Omega$ and compare with u the solution v of the homogeneous problem

$$\begin{cases} -\operatorname{div}(ADv) = 0 & \text{in } B_R; \\ \\ v = u & \text{in } \partial B_R. \end{cases} \tag{10.2}$$

Since Dv belongs to H^1_{loc} for previous results concerning H^2 regularity and its components $D_i v$ solve the homogeneous problem (because we supposed to have constant coefficients), we can use the decay estimates (5.1) and (5.2).
So, if $0 < \rho < R$, (5.2) provides us with the following inequality:

$$\int_{B_\rho} |Dv(x) - (Dv)_\rho|^2 \, dx \le c \left(\frac{\rho}{R}\right)^{n+2} \int_{B_R} |Dv(x) - (Dv)_R|^2 \, dx \ . \tag{10.3}$$

Now we try to employ (10.3) to get some estimate for u, the original, "non-homogeneous", solution of (10.1). Obviously, we can write

$$u = w + v \ ,$$

where $w \in H^1_0(B_R; \mathbb{R}^m)$. Thus (first using $Du = Dv + Dw$, then the minimality of the mean and (10.3), eventually $Dv = Du - Dw$)

$$\begin{aligned} \int_{B_\rho} |Du(x) - (Du)_\rho|^2 \, dx &\le 2 \left(\int_{B_\rho} |Dw(x) - (Dw)_\rho|^2 \, dx + \int_{B_\rho} |Dv(x) - (Dv)_\rho|^2 \, dx \right) \\ &\le 2 \int_{B_\rho} |Dw(x) - (Dw)_R|^2 \, dx + c \left(\frac{\rho}{R}\right)^{n+2} \int_{B_R} |Dv(x) - (Dv)_R|^2 \, dx \\ &\le c' \int_{B_R} |Dw(x) - (Dw)_R|^2 \, dx + c \left(\frac{\rho}{R}\right)^{n+2} \int_{B_R} |Du(x) - (Du)_R|^2 \, dx \ . \end{aligned}$$

The auxiliary function

$$f(\rho) := \int_{B_\rho} |Du(x) - (Du)_\rho|^2 \, dx$$

is non decreasing because of the minimality property of the mean $(Du)_\rho$, when one minimizes $m \mapsto \int_{B_\rho} |Du(x) - m|^2 \, dx$. In order to get that f satisfies the hypothesis of Lemma 9.2, we have to estimate $\int_{B_r} |Dw|^2$. We can consider w as a function in $H^1(\mathbb{R}^n)$ (null out of Ω) so, by Gårding inequality (choosing the test function $\varphi = w$),

$$\int_{B_R} |Dw(x)|^2 \, dx \le c \int_{B_R} ADw(x)Dw(x) \, dx = c \int_{B_R} F(x)Dw(x) \, dx = c \int (F(x) - F_R)Dw(x) \, dx \tag{10.4}$$

because $\operatorname{div}(ADw) = \operatorname{div} F$ by linearity. Applying Young inequality to (10.4) and then absorbing $\int_{B_R} |Dw|^2$ in the left side of (10.4), we get

$$\int_{B_R} |Dw(x)|^2 \, dx \le c \int_{B_R} |F(x) - F_R|^2 \, dx \le c \|F\|^2_{\mathcal{L}^{2,\mu}} R^\mu \ ,$$

because $F \in \mathcal{L}^{2,\mu}$.

Therefore we obtained the decay inequality of Lemma 9.2 for f with $\alpha = n+2$, $\beta = \mu$ and $\epsilon = 0$, then

$$f(\rho) \le c \left(\frac{\rho}{R}\right)^\mu f(R) + c\rho^\mu \ ,$$

that is $|Du| \in \mathcal{L}^{2,\mu}$. □

Corollary 10.2. *With the previous notation, when $\mu = n + 2\alpha$, Theorem 10.1 and Campanato Theorem 8.9 yield that*

$$F \in C^{0,\alpha} \quad \Longrightarrow \quad Du \in C^{0,\alpha} \ .$$

Theorem 10.3. *Considering again* (10.1)*, suppose that $A_{ij}^{\alpha\beta} \in C(\Omega)$ and A satisfies a (locally) uniform Legendre-Hadamard condition for some $\lambda > 0$. If $F \in L^{2,\mu}_{\mathrm{loc}}$ with $\mu < n$, then $|Du| \in L^{2,\mu}_{\mathrm{loc}}$.*

Remark 10.4. Naturally, since $\mu < n$, Campanato spaces and Morrey spaces coincide. Nevertheless, we use Morrey spaces for simplicity reasons.

Proof. Here there is an example of Korn's technique of coefficient freezing.
Fix a point $x_0 \in \Omega$ and define

$$\tilde{F}(x) := F(x) + (A(x_0) - A(x))\, Du(x) \ ,$$

so that the solution u solves

$$\operatorname{div}(A(x_0)Du(x)) = \operatorname{div} F(x) + \operatorname{div}\left((A(x_0) - A(x))Du(x)\right) = \operatorname{div} \tilde{F} \ .$$

Using (5.1) for $u = v + w$, where v solves the homogeneous PDE (10.2) with frozen coefficients $A(x_0)$, we obtain

$$\begin{aligned}\int_{B_\rho} |Du(x)|^2\,dx &\leq c\left(\frac{\rho}{R}\right)^n \int_{B_R} |Du(x)|^2\,dx + c' \int |Dw(x)|^2\,dx \\ &\leq c\left(\frac{\rho}{R}\right)^n \int_{B_R} |Du(x)|^2\,dx + c' \int |\tilde{F}(x)|^2\,dx\ .\end{aligned}$$

Thanks to continuity property of A, there exists a continuity modulus ω which allows us to estimate

$$\int_{B_R} |\tilde{F}(x)|^2\,dx \leq 2\int_{B_R} |F(x)|^2\,dx + 2\omega^2(R)\int_{B_R} |Du(x)|^2\,dx\ . \tag{10.5}$$

Consequently, as $F \in L^{2,\mu}_{\text{loc}}$,

$$\int_{B_R} |\tilde{F}(x)|^2\,dx \leq cR^\mu + 2\omega^2(R)\int_{B_R} |Du(x)|^2\,dx\ .$$

We are ready to use Lemma 9.2 with $f(\rho) := \int_{B_\rho} |Du(x)|^2\,dx$, $\alpha = n$, $\beta = \mu < n$ and $\epsilon = \omega^2(R)$: it tells us that if R is under a threshold depending only on c, α, β, we have

$$f(\rho) \leq c\left(\frac{\rho}{R}\right)^\mu f(R) + c\rho^\mu\ ,$$

so that $|Du| \in L^{2,\mu}_{\text{loc}}$. □

Theorem 10.5 (Schauder)**.** *Suppose that the coefficients* $A^{\alpha,\beta}_{ij}(x)$ *of the PDE* (10.1) *belong to* $C^{0,\alpha}(\Omega)$ *and* A *satisfies a (locally) uniform Legendre-Hadamard in* Ω *for some* $\lambda > 0$. *Then the following implication holds*

$$F \in C^{0,\alpha}_{\text{loc}} \quad \Longrightarrow \quad Du \in C^{0,\alpha}_{\text{loc}}\ ,$$

that is to say

$$F \in \mathcal{L}^{2,n+2\alpha}_{\text{loc}} \quad \Longrightarrow \quad Du \in \mathcal{L}^{2,n+2\alpha}_{\text{loc}}\ .$$

Proof. With the same idea of freezing coefficients (and the same notation!), we estimate by (5.1)

$$\int_{B_\rho} |Du(x) - (Du)_\rho|^2\,dx \leq c\int \left(\frac{\rho}{R}\right)^{n+2} \int_{B_R} |Du(x) - (Du)_R|^2\,dx + c'\int_{B_R} |\tilde{F}(x)|^2\,dx\ . \tag{10.6}$$

Additionally, the Hölder propriety of A makes us rewrite (10.5) as

$$\int_{B_R} |\tilde{F}(x)|^2 \, dx \leq 2 \int_{B_R} |F(x)|^2 \, dx + cR^2\alpha \int_{B_R} |Du(x)|^2 \, dx \; . \tag{10.7}$$

If $F \in C^{0,\alpha}_{\text{loc}}$, we can complete (10.7) with

$$\int_{B_R} |\tilde{F}(x)|^2 \, dx \leq cR^{n+2\alpha} + cR^2\alpha \int_{B_R} |Du(x)|^2 \, dx \; .$$

Theorem 10.3 with $\mu = n - \alpha < n$ tells us that $Du \in L^{2,\mu}$, thus

$$\int_{B_R} |\tilde{F}(x)|^2 \, dx \leq cR^{n+2\alpha} + cR^{n+\alpha} \; . \tag{10.8}$$

Adding (10.8) to (10.6) and applying Lemma 9.2 with exponents $n+2$ and $n+\alpha$, we get

$$(Du \in \mathcal{L}^{2,n+\alpha} \quad (\Longrightarrow \quad Du \in C^{0,\alpha/2}) \; .$$

Replacing (10.8) with

$$\int_{B_R} |\tilde{F}(x)|^2 \, dx \leq cR^{n+2\alpha} + cR^{n+2\alpha} \; ,$$

we are allowed to reach the conclusion, again by Lemma 9.2 with exponents $n+2$ and $n+2\alpha$. □

11 Regularity in L^p spaces

In this section we deal with elliptic regularity in the category of L^p spaces, another natural class of spaces besides Morrey, Hölder and Campanato spaces.

Lemma 11.1. *In a measure space $(\Omega, \mathcal{F}, \mu)$ consider a $\mathcal{F}$-measurable function $f : \Omega \to [0, +\infty]$ and call*

$$F(t) := \mu\left(\{x \in \Omega : \; f(x) > t\}\right) \; .$$

The following equalities hold for $1 \leq p < +\infty$:

$$\int_\Omega f^p(x) \, d\mu(x) = p \int_0^\infty t^{p-1} F(t) \, dt \tag{11.1}$$

$$\int_{\{f>s\}} f^p(x) \, d\mu(x) = p \int_s^\infty t^{p-1} F(t) \, dt + s^p F(s) \; . \tag{11.2}$$

Proof. Since $f \geq 0$, we can neglect the modulus. It is a simple consequence of Fubini theorem that

$$\begin{aligned}
\int_\Omega f^p(x)\, d\mu(x) &= \int_\Omega p \left(\int_0^{f(x)} t^{p-1}\, dt \right) d\mu(x) = p \int_0^\infty t^{p-1} \left(\int \chi_{\{t<f(x)\}}\, d\mu(x) \right) dt \\
&= p \int_0^\infty t^{p-1} F(t)\, dt \ .
\end{aligned}$$

Equation (11.2) follows from (11.1) applied to the positive part of $f - s$. □

Theorem 11.2 (Markov inequality)**.** *In a measure space $(\Omega, \mathcal{F}, \mu)$ a function $f \in L^p(\Omega)$ satisfies*

$$t^p \mu\left(\{f > t\}\right) \leq \int_\Omega |f(x)|^p\, d\mu(x) \ . \tag{11.3}$$

Proof. We begin with the trivial inequality

$$t\chi_{\{|f|\geq t\}}(x) \leq |f(x)| \quad \forall\, x \in \Omega \ , \tag{11.4}$$

thus, integrating (11.4) in Ω we obtain

$$t\mu\left(\{|f| \geq t\}\right) \leq \int_\Omega |f(x)|\, d\mu(x) \ .$$

Substituting $t \mapsto t^p$ and $|f| \mapsto |f|^p$ we reach

$$t^p \mu\left(\{f > t\}\right) \leq t^p \mu\left(\{|f|^p \geq t^p\}\right) \leq \int_\Omega |f(x)|^p\, d\mu(x) \ .$$

□

Markov inequality inspires the definition of a space that is weaker than L^p, but it still keeps (11.3).

Definition 11.3 (Marcinkiewicz space)**.** *Given a measure space $(\Omega, \mathcal{F}, \mu)$ and an exponent $1 \leq p < +\infty$, the Marcinkiewicz space $L^p_w(\Omega, \mu)$ is defined by*

$$L^p_w(\Omega, \mu) := \{f : \Omega \to \mathbb{R}\ \mathcal{F}\text{-measurable} \,|\, \mu\left(\{f > t\}\right) \leq c/t^p\ \forall\, t > 0\} \ .$$

We denote[5] *with $\|f\|_{L^p_w}$ the smallest constant c for which*

$$\mu\left(\{f > t\}\right) \leq c/t^p \quad \forall\, t > 0 \ .$$

[5] Pay attention to the lack of subadditivity of $\|\cdot\|_{L^p_w}$: the notation is misleading and it is not a norm!

Remark 11.4. If μ is a finite measure, then

$$q < p \quad \Longrightarrow \quad L^p \subset L^p_w \subset L^q \ .$$

The first inclusion is due to Markov inequality (11.2), on the other hand, if $f \in L^p_w$, then

$$\begin{aligned}\int_\Omega |f|^q \, d\mu(x) &= q \int_0^\infty t^{q-1} F(t) \, dt \le q \left(\int_0^1 t^{q-1} F(t) \, dt + \int_1^\infty t^{q-1} F(t) \, dt \right) \\ &\le q\mu(\Omega) + q \int_1^\infty t^{q-1} \|f\|_{L^p_w} t^{-p} \, dt \le q\mu(\Omega) + \frac{q}{p-q} \|f\|_{L^p_w} \ .\end{aligned}$$

Definition 11.5 (Maximal operator)**.** *When $f \in L^1(\mathbb{R}^n)$ is a nonnegative function, we can define the maximal operator*

$$\mathcal{M}f(x) := \sup_{Q_r(x)} \fint_{Q_r(x)} f(y) \, dy \ , \tag{11.5}$$

where $Q_r(x)$ is the r-cube with center x.

It is important to remark that the maximal operator $\mathcal{M}$ does not map L^1 into L^1.

Example 11.6. In dimension $n = 1$, consider $f = \chi_{[0,1]} \in L^1$. Then

$$\mathcal{M}f(x) \sim \frac{1}{|x|} \qquad \text{when } |x| >> 1 \ ,$$

so $\mathcal{M}f \notin L^1$.

However, if $f \in L^1$, the maximal operator $\mathcal{M}f$ belongs to the weaker Marcinkiewicz space L^1_w, as we are going to see in Theorem 11.8. We recall first Vitali covering theorem, in a version valid in *any* metric space.

Lemma 11.7 (Vitali)**.** *Let $\mathcal{F}$ be a finite family of balls in a metric space (X, d). Then, there exists $\mathcal{G} \subset \mathcal{F}$ made of disjoint balls satisfying*

$$\bigcup_{B \in \mathcal{F}} \subset \bigcup_{B \in \mathcal{G}} \hat{B} \ .$$

Here, for B ball, $\hat{B}$ denotes the ball with the same center and triple radius.

Proof. The initial remark is that if B_1 and B_2 are intersecting balls then $B_1 \subset \widehat{B_2}$, provided the radius of B_2 is larger than the radius of B_1. Assume that the family of balls is ordered in such a way that their radii are nonincreasing. Pick the first ball B_1, then the first ball among those that do not intersect B_1 and continue in this way, until either there is no ball left or all the balls left intersect one of the chosen balls. The family $\mathcal{G}$ of chosen balls is, by construction, disjoint. If $B \in \mathcal{F} \setminus \mathcal{G}$, then B has not chosen because it intersects one of the balls in $\mathcal{G}$; the first of these balls B_f has radius larger than the radius of B (otherwise B would have been chosen before B_f), hence $B \subset \widehat{B_f}$. $\square$

Theorem 11.8 (Hardy-Littlewood theorem). *Consider $f \in L^1$, then the maximal operator $\mathcal{M}f$ defined in* (11.5) *satifies*

$$\|\mathcal{M}f\|_{L^1_w} \leq 3^n \|f\|_{L^1} \ .$$

Proof. Fix $t > 0$ and a compact set $K \subset \{\mathcal{M}f > t\}$: by inner regularity of the Lebesgue measure we will reach the conclusion showing that

$$|K| \leq \frac{3^n}{t} \|f\|_{L^1} \ .$$

Since $K \subset \{\mathcal{M}f > t\}$, for any $x \in K$ there exists a radius $r(x)$ such that

$$\int_{Q_{r(x)}(x)} f(y)\, dy \geq t r(x)^n \ .$$

Compactness allows us to cover K with a finite number of cubes

$$K \subset \bigcup_{i \in I} Q_{r(x_i)}(x_i) \ ,$$

then Vitali lemma states allows to find $J \subset I$ such that the cubes $Q_{r(x_j)}(x_j)$, $j \in J$, are pairwise disjoint and

$$\bigcup_{j \in J} Q_{3r(x_j)}(x_j) \supset \bigcup_{i \in I} Q_{r_i}(x_i) \supset K \ .$$

We conclude that

$$|K| \leq \sum_{j \in J} 3^n r(x_j)^n \leq 3^n \sum_{i \in I} \int_{Q_{r(x_i)}(x_i)} f(y)\, dy \leq \frac{3^n}{t} \|f\|_{L^1} \ .$$

□

12 Some classical interpolation theorems

In the sequel, we will make extensive use of some classical interpolation theorems, that are basic tools in Functional and Harmonic Analysis.

Assume $(X, \mathcal{F}\, \mu)$ is a measure space. For the sake of brevity, we will say that a linear operator U mapping a vector space $D \subset L^p(X, \mu)$ into $L^q(X, \mu)$ is of type (p, q) if it is continuous with respect to the $L^p - L^q$ topologies. If this happens, obviously U can be extended to a linear continuous operator from $L^p(X, \mu)$ to $L^q(X, \mu)$ and the extension is unique if D is dense.

The inclusion $L^p \cap L^q \subset L^r$ for $p \leq q$ and $r \in [p, q]$ can be better specified with the following result:

Theorem 12.1 (Riesz-Thorin interpolation theorem)**.** *Let p, $q \in [1, \infty]$ with $p \leq q$ and $T : D \subset L^p(X, \mu) \cap L^q(X, \mu) \to L^p(X, \mu) \cap L^q(X, \mu)$ a linear operator which is both of type (p, p) and (q, q). Then T is of type (r, r) for all $r \in [p, q]$.*

We do not give the proof of this theorem. A standard reference is [22]. In the sequel we shall consider operators T that are not necessarily linear, but Q-subadditive, namely

$$|T(f + g)| \leq Q(|T(f)| + |T(g)|) \qquad \forall f,\, g \in D\ .$$

We also say that a space D of real-valued functions is stable under truncations if $f \in D$ implies $f\chi_{\{|f|<k\}} \in D$ for all $k > 0$ (all L^p spaces are stable under truncations).

Definition 12.2 (Strong and weak (p, p) operators)**.** *Let $s \in [1, \infty]$ and $D \subset L^s(X, \mu)$ linear and let $T : D \subset L^s(X, \mu) \to L^s(X, \mu)$, not necessarily linear. We say that T is of strong type (s, s) if $\|T(u)\|_s \leq C\|u\|_s$ for all $u \in D$, for some constant C independent of u.*
If $s < \infty$, we say that T is of weak type (s, s) if

$$\mu\left(\{x :\ |Tu(x)| > \alpha\}\right) \leq C\left(\frac{\|u\|_s}{\alpha}\right)^s \quad \forall \alpha > 0,\, u \in D$$

for some constant C independent of u, α. Finally, by convention, T is called of weak type (∞, ∞) if it is of strong type (∞, ∞).

We can derive an appropriate interpolation theorem even in the case of weak continuity.

Theorem 12.3 (Marcinkiewicz interpolation theorem)**.** *Assume that p, $q \in [1, \infty]$, $D \subset L^p(X, \mu) \cap L^q(X, \mu)$ is a linear space stable under truncations and $T : D \to L^p(X, \mu) \cap L^q(X, \mu)$ is Q-subadditive, of strong type (p, p) and of weak type (q, q).*
Then T is of strong type (r, r) for all $r \in [p, q] \setminus \{q\}$.

Remark 12.4. The most important application of the previous result is perhaps the study of the boundedness of maximal operators, see the next Remark. In that case, one tipically works with $p = 1$ and $q = \infty$ and we limit ourselves to prove the theorem under these additional hypotheses.

Proof. We can truncate $f \in D$ as follows:

$$f = g + h, \quad g(x) = f(x)\chi_{\{|f| \leq \gamma s\}}, \quad h(x) = f(x)\chi_{\{|f| > \gamma s\}}$$

where γ is an auxiliary parameter to be fixed later. By assumption $g \in D \cap L^\infty(X, \mu)$ while $h \in D \cap L^1(X, \mu)$ by linearity of D. Hence

$$|T(f)| \leq Q|T(g)| + Q|T(h)| \leq Q\gamma s A_\infty + Q|T(h)|$$

with A_∞ as the operator norm of T acting from $D \cap L^\infty(X, \mu)$ into $L^\infty(X, \mu)$. Choose γ so that $QA_\infty\gamma = 1/2$, therefore

$$\{|T(f)| > s\} \subset \left\{|T(h)| > \frac{s}{2Q}\right\}$$

and so

$$\mu\left(\{|T(f)| > s\}\right) \leq \mu\left(\left\{|T(h)| > \frac{s}{2Q}\right\}\right) \leq \left(\frac{2A_1Q}{s}\right)\int_X |h|\,d\mu \leq \left(\frac{2A_1Q}{s}\right)\int_{|f|\geq\gamma s} |f|\,d\mu$$

where A_1 is the constant appearing in the (weak-L^1,L^1) estimate. By integration of the previous inequality, we get

$$p\int_0^\infty s^{p-1}\mu\left(\{|T(f)| > s\}\right)\,ds \leq 2A_1Qp\int_0^\infty \int_{|f|\geq\gamma s} s^{p-2}|f|\,d\mu\,ds$$

and by means of the Fubini-Tonelli theorem we finally get

$$\|T(f)\|_p^p \leq 2A_1Qp\int_X \left(\int_0^{\frac{|f(x)|}{\gamma}} s^{p-2}\,ds\right)|f(x)|\,d\mu(x) = \frac{2A_1Qp}{(p-1)\gamma^{p-1}}\|f\|_p^p$$

and the conclusion follows. □

Remark 12.5. As a byproduct of the previous result, we have that the maximal operator M defined in the previous section is of strong type (p, p) for any $p \in (1, \infty]$ (and only of weak type $(1, 1)$). These facts, that have been derived for simplicity in the standard Euclidean setting, can be easily generalized, for instance to *pseudo-metric spaces* (i.e. the distance fulfils only the triangle and symmetry assumption) endowed with a *doubling measure*, that is a measure μ such that $\mu(B_{2r}(x)) \leq \beta\mu(B_r(x))$ for some constant β not depending on the radius and the center of the ball. Notice that in this case the constant in the (weak-L^1,L^1) bound of the maximal operator does not exceed β^2, since $\mu(B_{3r}(x)) \leq \beta^2\mu(B_r(x))$.

13 Lebesgue differentiation theorem

In this section, we want to give a direct proof, based on the $(1,1)-$weak continuity of the maximal operator M, of the classical Lebesgue differentiation theorem.

Theorem 13.1. *Let (X,d,μ) a metric space with a finite doubling measure on its Borel $\sigma-$algebra and $p \in [1,+\infty)$. If $f \in L^p(\mu)$ then for μ-a.e. $x \in X$ we have that*

$$\lim_{r\downarrow 0} \fint_{B_r(x)} |f(y)-f(x)|^p \, d\mu(y) = 0.$$

Proof. Let

$$\Lambda_t := \left\{ x \in X |\limsup_{r\downarrow 0} \fint_{B_r(x)} |f(y)-f(x)|^p \, d\mu(y) > t \right\}.$$

The thesis can be achieved by showing that for any $t>0$ we have $\mu(\Lambda_t)=0$, since the stated property holds out of $\cup_n \Lambda_{1/n}$. Now, we can exploit the metric structure of X in order to approximate f in $L^1(\mu)$ norm by means of continuous and bounded functions: for any $\epsilon > 0$ we can write $f = g+h$ with $g \in C_b(X;\mathbb{R})$ and $\|h\|^p_{L^p} \leq t\epsilon$. Hence, it is enough to prove that for any $t>0$ we have $\mu(A_t)=0$ where

$$A_t := \left\{ x \in X |\limsup_{r\downarrow 0} \fint_{B_r(x)} |h(y)-h(x)|^p \, d\mu(y) > t \right\} .$$

This is easy, because by definition

$$A_t \subset \left\{ |h|^p > \frac{t}{2} \right\} \cup \left\{ M(|h|^p) > \frac{t}{2} \right\}$$

and if we consider the corresponding measures, we have (taking Remark 12.5 into account)

$$\mu(A_t) \leq \frac{2}{t} \|h\|^p_{L^p} + \frac{2}{t} M_{L^1_w;L^1} \|h\|^p_{L^p} \leq 2(1+M_{L^1_w;L^1})\epsilon$$

where $M_{L^1_w;L^1}$ is the constant in the (weak-L^1,L^1) bound. Since $\epsilon > 0$ is arbitrary we get the thesis. $\square$

Remark 13.2. All the previous results have been derived for the maximal operator defined in terms of *centered balls*, that is

$$Mf(x) = \sup_{r>0} \fint_{B_r(x)} f(y)\, dy$$

and the Lebesgue differentiation theorem has been stated according to this setting. However, it is clear that we can generalize everything to any metric space (X, d, μ) with a finite doubling measure and a family of sets $\mathcal{F} := \cup_{x \in X} \mathcal{F}_x$ with

$$M_{\mathcal{F}} f(x) = \sup_{A \in \mathcal{F}_x} \fint_A f(y)\, dy$$

provided there exists a universal constant $C > 0$ such that

$$\forall A \in \mathcal{F}_x \ \exists r > 0 \quad A \subset B_r(x), \quad \mu(A) \geq C\mu(B_r(x)).$$

More precisely, under these hypotheses $M_{\mathcal{F}} \leq \frac{1}{C} M$ and we may derive all estimates obtained above. Correspondingly, there is a version of the Lebesgue differentiation theorem referred to the family $\mathcal{F}$.

In Euclidean spaces an important example to which the previous remark applies, in connection with Calderón-Zygmund theory, is given by

$$\mathcal{F}_x := \{Q \text{ cube}, \, x \in Q\},$$

consequently Lebesgue theorem gives

$$\lim_{x \in Q,\, |Q| \to 0} \fint_Q |f(y) - f(x)|^p \, dy = 0$$

for a.e. $x \in \mathbb{R}^n$. Notice that requiring $|Q| \to 0$ (i.e. $\mathrm{diam}(Q) \to 0$) is essential to “factor” continuous functions as in the proof of Theorem 13.1.

14 Calderón-Zygmund decomposition

We need to introduce another powerful tool, that will be applied to the study of the BMO spaces. Here and below Q will indicate an open cube in $\mathbb{R}^n$ and similarly Q' or Q''.

Theorem 14.1. *Let $f \in L^1(Q)$, $f \geq 0$ and consider a real number α such that $\fint_Q f\, dx < \alpha$. Then, there exists a countable family of open cubes $\{Q_i\}_{i \in I}$ with $Q_i \subset Q$ and sides parallel to the ones of Q, such that*

(i) $Q_i \cap Q_j = \emptyset$ *if* $i \neq j$;

(ii) $\alpha < \fint_{Q_i} f\, dx \leq 2^n \alpha \quad \forall i$;

(iii) $f \leq \alpha$ *a.e. on* $Q \setminus \cup_i Q_i$.

Remark 14.2. The remarkable (and useful) aspect of this decomposition is that the "bad" set $\{f > \alpha\}$ is packed inside a family of cubes, carefully chosen in such a way that still the mean values inside the cubes is of order α. As a consequence of the existence of this decomposition, we have

$$\alpha \sum_i |Q_i| \leq \sum_i \int_{Q_i} f\, dx \leq \|f\|_1 .$$

The proof is based on a sort of *stopping-time* argument.

Proof. Divide the cube Q in 2^n subcubes by means of n bisections of Q with hyperplanes parallel to the sides of the cube itself. We will call this process *dyadic decomposition.* Then:

- If $⨍_{Q_i} f \geq \alpha$ we don't divide Q_i anymore;
- else we iterate the process on Q_i.

At each step we collect the cubes that verify the first condition and put together all such cubes, thus forming a countable family. The first two properties to be verified are obvious by construction. For the third one, note that if $x \in Q \setminus \cup_i Q_i$, then there exists a sequence of subcubes $(\widetilde{Q}_j)$ with $x \in \cap_j \widetilde{Q}_j$ and $|\widetilde{Q}_j| \to 0$, $⨍_{\widetilde{Q}_j} f\, dx \leq \alpha$. Thanks to the Lebesgue differentiation theorem we get $f(x) \leq \alpha$ for a.e. $x \in Q \setminus \cup_i Q_i$. $\square$

15 The BMO space

Given a cube $Q \subset \mathbb{R}^n$, we define

$$BMO(Q) := \left\{ u \in L^1(Q) | \sup_{Q' \subset Q} ⨍_{Q'} |u - u_{Q'}|\, dx < \infty \right\}.$$

An elementary argument replacing balls with concentric cubes shows that we have $BMO(Q) \sim \mathcal{L}^{1,n}$, that is the two spaces consist of the same elements and the corresponding semi-norms are equivalent. Here we recall the inclusion already discussed in Remark 8.8:

Theorem 15.1. *For any cube $Q \subset \mathbb{R}^n$ the following inclusion holds:*

$$W^{1,n}(Q) \hookrightarrow BMO(Q).$$

Proof. First, notice that $W^{1,n}(Q) \hookrightarrow \{u | |\nabla u| \in L^{1,n-1}(Q)\}$ (this is an immediate consequence of the Hölder inequality). Then, by the Poincaré inequality there exists a dimensional constant $C > 0$ such that for any $Q' \subset Q$ with sides of length h

$$\int_{Q'} |u - u_{Q'}|\, dx \leq Ch \int_{Q'} |\nabla u|\, dx \leq C\, |\nabla u|_{L^{1,n-1}} h^n .$$

$\square$

However, it should be clear that the previous inclusion is far from being an equality as elementary examples show, see Remark 8.8. We shall extend now to n-dimensional spaces the example in Remark 8.8, stating first a simple sufficient (and necessary, as we will see) condition for being BMO.

Proposition 15.2. *Let $u : Q \to \mathbb{R}$ be a measurable function such that, for some $b > 0$, $B \geq 0$, the following property holds:*

$$\text{for all cubes } C \subset Q \text{ there exists } a_C \in \mathbb{R} \text{ such that } |C \cap \{|u - a_C| \geq \sigma\}| \leq Be^{-b\sigma}|C|. \tag{15.1}$$

Then $u \in BMO$.

The proof of the Proposition is simple, since

$$\int_C |u - u_C|\, dx = \int_0^\infty |\{C \cap |u - u_C| > \sigma\}|\, d\sigma \leq \frac{B}{b}|C| \ .$$

Example 15.3. Thanks to Proposition 15.2 we verify that $\ln|x| \in BMO$, in fact $\ln|x|$ satisfies (15.1) for $\sigma \geq 1$ (the parameters b and B will be made precise later).

Fix a cube C, with h the length of the side of C. We define, respectively,

$$|\xi| := \max_{x\in C} |x| \quad |\eta| := \min_{x\in C} |x| \ ,$$

moreover

$$a_C := \ln|\xi| \ ,$$

so that

$$a_C - u = \ln\left(\frac{|\xi|}{|x|}\right) \ .$$

We estimate the Lebesgue measure of $C \cap \{|\xi| \geq |x|e^\sigma\}$: naturally we can assume that $|\xi| \geq |\eta|e^\sigma$, otherwise there is nothing to prove, so

$$|\xi|e^{-\sigma} \geq |\eta| \geq -|\eta - \xi| + |\xi| \geq -\sqrt{n}h + |\xi| \ ,$$

then

$$|\xi| \leq \frac{\sqrt{n}h}{1 - e^{-\sigma}} \ .$$

Finally

$$\frac{1}{h^n}\,|C \cap \{|u - a_C| \geq \sigma\}| \leq \frac{(\sqrt{n})^n\omega_n}{(1 - e^{-\sigma})^n}e^{-n\sigma} \ ,$$

so that (15.1) holds with $b = n$ and $B = (\sqrt{n})^n\omega_n\,(1 - e^{-1})^{-n}$.

The following theorem by John and Nirenberg was first presented in [19].

Theorem 15.4. *In the setting described above, there exist constants c_1, c_2 depending only on the dimension n such that*

$$\mathscr{L}^n\left(\{|u-u_Q|>t\}\right)\le c_1 e^{-c_2 t/\|u\|_{BMO}}\mathscr{L}^n(Q)\qquad \forall u\in BMO(Q).$$

Remark 15.5. In the proof we present here, we will find explicitly $c_1 = e^{1/e}$ and $c_2 = 1/(2^n e)$. However, these constants are not sharp.

Before presenting the proof, we discuss here two very important consequences of this result.

Corollary 15.6 (Exponential integrability of BMO functions)**.** *For any $c < c_2$ there exists $K(c, c_2)$ such that*

$$\int_Q e^{\frac{c\left|u-u_Q\right|}{\|u\|_{BMO}}}\,dx\le K(c,c_2).$$

Proof. It's a simple computation:

$$\int_Q e^{c\left|u-u_Q\right|}\,dx=c\int_0^\infty e^{ct}\mathscr{L}^n\left(\{|u-u_Q|>t\}\right)\,dt\le cc_1\int_0^\infty e^{(c-c_2)t}\,dt$$

where we assumed $\|u\|_{BMO(Q)}=1$ and we used the John-Nirenberg inequality. □

Remark 15.7 (Better integrability of $W^{1,n}$ functions)**.** The previous theorem basically tells that the class BMO and hence also $W^{1,n}$ has got exponential integrability properties. This result is partly specialized and refined by the celebrated *Moser-Trudinger inequality*, that we quote here without proof. Let $\Omega\subset\mathbb{R}^n$ be a bounded domain, $u\in W_0^{1,n}(\Omega)$ with $\int_\Omega|\nabla u|^n\,dx\le 1$. Then there exists a constant $C(\Omega)$ such that

$$\int_\Omega e^{\alpha_n|u|^{1^*}}\,dx\le C(\Omega)$$

where $\alpha_n := n\omega_{n-1}^{\frac{1}{n-1}}$. This inequality has first been presented in [23].

Theorem 15.8. *If $u\in BMO(Q)$ and $p\in[1,\infty)$ we have*

$$\left(⨍_Q |u-u_Q|^p\,dx\right)^{1/p}\le c(n,p)\,\|u\|_{BMO}\ .$$

Consequently the following isomorphisms hold:

$$\mathcal{L}^{p,n}\sim\mathcal{L}^{1,n}\sim BMO\ . \tag{15.2}$$

The proof relies on a simple and standard computation, similar to the one presented before in order to get exponential integrability.

We can now conclude this section, by proving the John-Nirenberg inequality.

Proof. By homogeneity, we can assume without loss of generality that $\|u\|_{BMO} = 1$. Let $\alpha > 1$ a parameter, to be specified later. We claim that it is possible to define, for any $k \geq 1$ a countable family of subcubes $\left\{Q_i^k\right\}_{i\in I_k}$ contained in Q such that

(i) $|u(x) - u_Q| \leq 2^n k\alpha$ a.e. on $Q \setminus \cup_{i\in I_k} Q_i^k$;

(ii) $\sum_{i\in I_k} \mathscr{L}^n(Q_i^k) \leq \alpha^{-k}\mathscr{L}^n(Q)$.

The combination of linear growth in (i) and geometric decay in (ii) leads to the exponential decay of the repartition function: indeed, choose k such that $2^n\alpha k \leq t < 2^n\alpha(k+1)$ and so

$$\mathscr{L}^n\left(\{|u - u_Q| > t\}\right) \leq \mathscr{L}^n\left(\{|u - u_Q| > 2^n\alpha k\}\right) \leq \alpha^{-k}\mathscr{L}^n(Q)$$

by the combined use of the previous properties. Now we want $\alpha^{-k} \leq c_1 e^{-c_2 t}$ for all $t \in (2^n\alpha k, 2^n\alpha(k+1))$ which is certainly verified if

$$\alpha^{-k} = c_1 e^{-c_2 2^n\alpha(k+1)}$$

and consequently we determine the constants c_1, c_2 by asking

$$e^{c_2 2^n\alpha} = \alpha, \qquad c_1 e^{-c_2 2^n} = 1\ .$$

By the first relation $c_2 = \frac{\log\alpha}{2^n\alpha}$ and we maximize with respect to $\alpha > 1$ to find

$$\alpha = e, \quad c_1 = e^{1/e}, \quad c_2 = \frac{1}{2^n e}\ .$$

So we just need to prove the claim. If $k = 1$ we simply apply the Calderón-Zygmund decomposition to $f = |u - u_Q|$ for the level α and get a collection $\{Q_i^1\}_{i\in I_1}$. We have to verify that the required conditions are verified. Condition (ii) follows by Remark 14.2, while (i) is obvious since $|u(x) - u_Q| \leq \alpha$ a.e. out of the union of Q_i^1 by construction. But, since $\|u\|_{BMO} = 1$, we also know that

$$\forall i \in I_1 \qquad \fint_{Q_i^1} |u - u_{Q_i^1}|\, dx \leq 1 < \alpha$$

and hence we can iterate the construction, by applying the Calderón-Zygmund decomposition to each of the functions $|u - u_{Q_i^1}|$ with respect to the corresponding cubes Q_i^1.

In this way, we find a family of cubes $\{Q^2_{i,l}\}$, each contained in one of the previous ones. Moreover $\|u(x) - u_{Q^1_i}| \le \alpha$ on $Q^1_i \setminus \cup_l Q^2_{i,l}$ and so

$$\begin{aligned}\sum_{i,l} \mathscr{L}^n(Q^2_{i,l}) &\le \sum_i \frac{1}{\alpha} \int_{Q^1_i} |u - u_{Q^1_i}| \, dx \\ &\le \sum_i \frac{1}{\alpha} \mathscr{L}^n(Q^1_i) \le \frac{1}{\alpha^2} \mathscr{L}^n(Q)\end{aligned}$$

which is (ii). In order to get (i), notice that

$$Q \setminus \cup Q^2_{i,l} \subset \big(Q \setminus \cup_i Q^1_i\big) \cup \big(\cup_i (Q^1_i \setminus \cup_l Q^2_{i,l})\big)$$

so for the first set in the inclusion the thesis is obvious by the case $k = 1$. For the second, we first observe that

$$|u_Q - u_{Q^1_i}| \le \fint_{Q^1_i} |u_Q - u| \, dx \le 2^n \alpha$$

and consequently

$$|u(x) - u_Q| \le |u(x) - u_{Q^1_i}| + |u_{Q_i} - u_Q| \le \alpha + 2^n \alpha \le 2^n \cdot 2\alpha.$$

With minor changes, we can deal with the general case $k > 1$ and this is what we need to conclude the argument and the proof. □

16 Stampacchia Interpolation Theorem

John-Nirenberg theorem stated in Theorem 15.4 can be extended considering the L^p norms, so that the case of BMO maps corresponds to the limit as $p \to \infty$.

Theorem 16.1 (John-Nirenberg)**.** *For any $p \in [1, +\infty)$, for $u \in L^p(Q)$ consider*

$$K_p^p(u) := \sup \left\{ \sum_i |Q_i| \left(\fint_{Q_i} |u(x) - u_{Q_i}| \, dx \right)^p \mid \{Q_i\}_i \text{ partition of } Q \right\} .$$

There exists a constant $c = c(p, n)$ such that

$$\|u - u_Q\|_{L^p_w} \leq c(p, n) K_p(u) .$$

The proof of Theorem 16.1 is basically the same as Theorem 15.4, the goal being to prove the polynomial decay

$$|\{|u| > t\}| \leq \frac{c(p, n)}{t^p} K_p^p(u)$$

instead of an exponential decay.

Theorem 16.2 (Stampacchia interpolation)**.** *Let $D \subset L^\infty(Q; \mathbb{R}^s)$ be a linear space and $p \in [1, \infty)$. Consider a linear operator $T : D \to BMO(Q_0)$, continuous with respect to the norms $(L^\infty(Q; \mathbb{R}^s), BMO(Q_0))$ and $(L^p(Q; \mathbb{R}^s), L^p(Q_0))$. Then for every $r \in [p, \infty)$ the operator T is continuous with respect to the $(L^r(Q; \mathbb{R}^s), L^r(Q_0))$ topologies.*

Proof. For simplicity we assume $s = 1$ (the proof is the same in the general case). We fix a partition $\{Q_i\}$ of Q and we regularize the operator T with respect to $\{Q_i\}$ (even if we do not write the dependence of $\tilde{T}$ from $\{Q_i\}$ for brevity):

$$\tilde{T}(u)(x) := \fint_{Q_i} |Tu(y) - (Tu)_{Q_i}| \, dy \quad \forall \, x \in Q_i .$$

We claim that $\tilde{T}$ satisfies the assumptions of Marcinkiewicz theorem. Indeed

(1) $\tilde{T}$ is obviously 1-subadditive;

(2) $L^\infty \to L^\infty$ continuity holds by the inequality

$$\|\tilde{T}\|_{L^\infty} = \sup_i \fint_{Q_i} |Tu(y) - (Tu)_{Q_i}| \, dy \leq \|Tu\|_{BMO} \leq c\|u\|_{L^\infty} ;$$

(3) $L^p \to L^p$ continuity holds too, in fact, by Jensen inequality,

$$\begin{aligned} \|\tilde{T}u\|^p_{L^p} &= \sum_i |Q_i| \left(\fint_{Q_i} |Tu(y) - (Tu)_{Q_i}|\, dy \right)^p \leq \sum_i \int_{Q_i} |Tu(y) - (Tu)_{Q_i}|^p \, dy \\ &\leq 2^{p-1} \sum_i \int_{Q_i} (|Tu(y)|^p + |(Tu)_{Q_i}|^p)\, dy \leq 2^p \|Tu\|^p_{L^p} \leq c2^p \|u\|^p_{L^p} \ . \end{aligned}$$

Thanks to Marcinkiewicz theorem the operator

$$\tilde{T} : D \subset L^r(Q) \longrightarrow L^r(Q_0) \tag{16.1}$$

is continuous for every $r \in [p, \infty]$, and its continuity constant can be bounded independently of the chosen partition $\{Q_i\}$.

In order to get information from Theorem 16.1, for $r \in [p, \infty)$, we estimate

$$K^r_r(Tu) = \sup_{\{Q_i\}} \sum_i |Q_i| \left(\fint_{Q_i} |Tu(y) - (Tu)_{Q_i}|\, dy \right)^r \leq \sup_{\{Q_i\}} \|\tilde{T}_{\{Q_i\}} u\|^r_{L^r} \leq c\|u\|_{L^r} \ ,$$

where the first inequality is Jensen inequality and the second one is due to the continuity property of $\tilde{T} : L^r(Q) \to L^r(Q_0)$ stated in (16.1). Therefore, by Theorem 16.1, we get

$$\|Tu - (Tu)_Q\|_{L^r_w} \leq c(r, n, T)\|u\|_{L^r} \qquad \forall u \in D \ .$$

Since $u \mapsto (Tu)_Q$ obviously satisfies a similar L^r_w estimate, we conclude that $\|Tu\|_{L^r_w} \leq c(r, n, T)\|u\|_{L^r}$ for all $u \in D$. Again, thanks to Marcinkiewicz theorem, with exponents p and r, we have that continuity $L^{r'} \to L^{r'}$ for every $r' \in [p, r)$. Since r is arbitrary, we got our conclusion. $\square$

We are now ready to employ these harmonic analysis tools to the study of regularity in L^p spaces for elliptic PDEs, considering first the case of constant coefficients. Suppose that $\Omega \subset \mathbb{R}^n$ is an open, bounded set with Lipschitz boundary $\partial\Omega$, suppose that the coefficients $A^{\alpha\beta}_{ij}$ satisfy Legendre-Hadamard condition with $\lambda > 0$ and consider the divergence form of the PDE

$$\begin{cases} -\mathrm{div}(ADu) = \mathrm{div}F \\ u \in H^1_0(\Omega; \mathbb{R}^m) \ . \end{cases} \tag{16.2}$$

In the spirit of Theorem 16.2, we define for each component

$$TF := \frac{\partial u^i}{\partial x_\alpha}, \qquad i = 1, \dots, m,\ \alpha = 1, \dots, n \ .$$

In the following arguments we omit for simplicity the dependence on i and α. Thanks to Campanato regularity theory, we already got the continuity of $T : \mathcal{L}^{2,\lambda} \to \mathcal{L}^{2,\lambda}$ when $0 \leq \lambda < n+2$, thus choosing $\lambda = n$ and using the isomorphism (15.2) we see that T is in particular continuous as an operator

$$T : L^\infty(\Omega; \mathbb{R}^{nm}) \longrightarrow BMO(\Omega) \ . \tag{16.3}$$

Remark 16.3. Let us remark the importance of weakening the norm in the target space in (16.3): we passed from $L^\infty(\Omega)$ (for which, as we will see, no estimate is possible) to $BMO(\Omega)$. For $BMO(\Omega)$ the regularity result for PDE is true and Theorem 16.2 allows us to interpolate between 2 and ∞.

We are going to apply Theorem 16.2 with $D = L^\infty(\Omega; \mathbb{R}^{nm})$ and $s = nm$. By Caccioppoli-Leray inequality (see Theorem 4.1 and Remark 4.4) we obtain the second hypothesis of Theorem 16.2: $T : L^2(\Omega; \mathbb{R}^{nm}) \to L^2(\Omega)$ is continuous. Therefore

$$T : D \to L^p(\Omega) \tag{16.4}$$

is (L^p, L^p)-continuous if $p \in [2, \infty)$. Since the unique extension of T to the whole of L^p still maps F into $\partial_{x_\alpha} u^i$, with u solution to (16.2), we have proved the following result:

Theorem 16.4. *For all $p \in [2, \infty)$ the operator $F \mapsto Du$ in* (16.2) *maps $L^p(\Omega; \mathbb{R}^{nm})$ into $L^p(\Omega; \mathbb{R}^{nm})$ continuously.*

Our intention is now to extend the previous result for $p \in (1, 2)$, by a duality argument.

Lemma 16.5 (Helmholtz decomposition)**.** *If $p \geq 2$ and B is a matrix satisfying the Legendre-Hadamard inequality, a map $G \in L^p(\Omega; \mathbb{R}^m)$ can always be written as a sum*

$$G = BD\phi + \tilde{G} \ , \tag{16.5}$$

where

$$\mathrm{div}\tilde{G} = 0$$

and, for some constant $c_* > 0$, it holds

$$\|D\phi\|_{L^p} \leq c_* \|G\|_{L^p} \ . \tag{16.6}$$

Proof. It is sufficient to solve in $H^1_0(\Omega; \mathbb{R}^m)$ the PDE

$$-\mathrm{div}(BD\phi) = \mathrm{div}G \ .$$

and put $\tilde{G} := G - BD\phi$. The estimate (16.6) is just a consequence of Theorem 16.4. □

Fix $p' \in (1, 2)$, so that $p > 2$, and set $D := L^2(\Omega)$. Our aim is to prove that $T : L^2 \to L^{p'}$ is continuous with respect to $(L^{p'}, L^{p'})$. We are showing that, for every $F \in D$, TF belongs to $(L^p)' \sim L^{p'}$. In the chain of inequalities that follows we are using A^*, that is the adjoint matrix of A, which certainly mantains the Legendre-Hadamard property. Lemma 16.5 is used in order to decompose the generic function $G \in L^p$ as in (16.5).

$$\begin{aligned}\sup_{\|G\|_{L^p}\le 1}\langle TF,G\rangle &= \sup_{\|G\|_{L^p}\le 1}\int TF(x)G(x)\,dx = \sup_{\|G\|_{L^p}\le 1}\int Du(x)\left(A^*D\phi(x)+\tilde{G}(x)\right)\,dx\\ &\le \sup_{\|D\phi\|_{L^p}\le c_*}\int (ADu(x))\,D\phi(x)\,dx = \sup_{\|D\phi\|_{L^p}\le c_*}\int F(x)D\phi(x)\,dx \le c_*\|F\|_{L^{p'}}\ .\end{aligned}$$

If we approximate now $F \in L^p$ in the L^p topology by functions $F_n \in L^2$ we can use the (L^p, L^p)-continuity to prove existence of weak solutions in $H^{1,p}_0$ to the PDE when the right hand side is L^p only. Notice that the solutions obtained in this way have no variational character anymore, since their energy $\int ADuDu\,dx$ is infinite (for this reason they are sometimes called *very weak* solutions). Since the variational characterization is lacking, uniqueness of these solutions needs a new argument, based on Helmholtz decomposition.

Theorem 16.6. *For all $p \in (1,2)$ there exists a continuous operator $T : L^p(\Omega;\mathbb{R}^{nm}) \to H^{1,p}_0(\Omega;\mathbb{R}^m)$ mapping F to the unique weak solution u to* (16.2)*.*

Proof. We already illustrated the construction of a solution u, by a density argument and uniform L^p bounds. To show uniqueness, suffices to show that $u \in H^{1,p}_0$ and $-\mathrm{div}(ADu) = 0$ implies $u = 0$. To this aim, we define $G = |Du|^{p-2}Du \in L^{p'}$ and apply Helmholtz decomposition $G = A^*D\phi + \tilde{G}$ with $\phi \in H^{1,p}_0$ and $\tilde{G} \in L^{p'}$ divergence-free. By a density argument first in u and then in ϕ (notice that the exponents are dual) we have $\int \tilde{G}Du\,dx = 0$ and $\int ADuD\phi\,dx = 0$, hence

$$\int_\Omega |Du|^p\,dx = \int_\Omega GDu\,dx = \int_\Omega A^*D\phi Du\,dx = \int_\Omega ADuD\phi\,dx = 0.$$

□

Remark 16.7 (General Helmholtz decomposition)**.** Thanks to Theorem 16.6, the Helmholtz decomposition showed above is possible for every $p \in (1,\infty)$.

Remark 16.8 ($W^{2,p}$ estimates)**.** By differentiating the equation and multiplying by cut-off functions, we easily see that Theorem 16.4 and Theorem 16.6 yield

$$-\mathrm{div}(ADu) = f,\quad |Du| \in L^p_{\mathrm{loc}},\ f \in L^p_{\mathrm{loc}} \qquad\Longrightarrow\qquad u \in W^{2,p}_{\mathrm{loc}}.$$

Remark 16.9 (No L^∞ bound is possible)**.** The regularity obtained by Stampacchia interpolation theorem is optimal: we show here that T does not map L^∞ into L^∞. To show this, argue by contradiction and assume that T has this property; then, we will show that necessarily T is discontinuous. Now, if $(\overline{B}_i)$ is a countable family of closed disjoint balls contained in B_1, by a scaling argument we can find (since T is discontinuous) functions

$F_i \in L^\infty(B_i)$ with $\|F_i\|_\infty = 1$ and solutions $u_i \in H^1_0(B_n)$ to the PDE with $\|\nabla u_i\|_\infty \geq i$. Then it is easily seen that the function

$$u(x) := \begin{cases} u_i(x) & \text{if } x \in B_i \\ 0 & \text{if } x \in B_1 \setminus \cup_i B_i \end{cases}$$

belongs to $H^1_0(B_1)$, solves the PDE, and its gradient is not bounded.

So, it remains to prove that T is necessarily discontinuous. By the same duality argument used before, if T were continuous we would get

$$\|Du\|_{L^1} \leq c\|F\|_{L^1}$$

whenever $u \in H^1_0(\Omega; \mathbb{R}^m)$ solves

$$\mathrm{div}\,(ADu) = \mathrm{div}(F)\ .$$

Hence, regularizing solutions by convolution, we could state that Du or, strictly speaking, the derivative in the sense of distributions, is a representable by vector-valued measure with finite total variation in Ω whenever F is a vector-valued measure and

$$\|Du\|_{\mathcal{M}(\Omega)} \leq c\|F\|_{\mathcal{M}(\Omega)}\ . \tag{16.7}$$

But, (16.7) is false. In fact, when $n = 2$ and $m = 1$, consider the trivial matrix $A^{\alpha\beta} := \delta_{\alpha\beta}$ and the related Poisson equation

$$-\mathrm{div}(ADu) = -\Delta u = \delta_0\ , \tag{16.8}$$

where δ_0 is the Dirac measure supported in 0. The well-known fundamental solution of the Laplace equation (16.8) is

$$u(x) = -\frac{\ln|x|}{2\pi}\ ,$$

so $\nabla u(x) = \frac{1}{2\pi}\frac{x}{|x|^2}$. Now, for any cut-off function η we have

$$-\Delta(\partial_{x_i}(u\eta)) = -\partial_{x_i}(\eta(0) + u\Delta\eta + \langle\nabla u, \nabla\eta\rangle =: -\partial_{x_i}F_i$$

and we conclude, by applying (16.7) to $\partial_{x_i}(\eta u)$ (with $F_j = 0$ for $j \neq i$), that the distributional derivative of ∇u is representable by a measure with locally finite total variation. But the pointwise derivative (which for sure coincides with the distributional derivative, out of the origin) is given by

$$D^2u(x) = \frac{1}{2\pi|x|^2}\left(I - \frac{x \otimes x}{|x|^2}\right)\ .$$

Since $|D^2u|(x) \sim 1/|x|^2$ and $1/|x|^2 \notin L^1_{\mathrm{loc}}$ we reach a contradiction.

Now we pass from constant to continuous coefficients, using Korn's technique.

Theorem 16.10. *In an open set $\Omega \subset \mathbb{R}^n$ let $u \in H^1_{\rm loc}(\Omega;\mathbb{R}^m)$ be a solution to the PDE*

$$-\mathrm{div}(ADu) = f + \mathrm{div}F$$

with coefficients $A \in C(\overline{\Omega};\mathbb{R}^{n^2m^2})$ which satisfy a uniform Legendre-Hadamard condition for some $\lambda > 0$. Moreover, if $p \in (1,\infty)$, let us suppose that $F \in L^p_{\rm loc}$ and $f \in L^q_{\rm loc}$, where the Sobolev conjugate exponent $q^* = qn/(n-q)$ coincides with p. Then $|Du| \in L^p_{\rm loc}(\Omega)$.

Proof. Let us fix $s \geq 2$ and let us show that

$$|Du| \in L^{s\wedge p}_{\rm loc}(\Omega) \quad \Longrightarrow \quad |Du| \in L^{s^*\wedge p}_{\rm loc}(\Omega)\ . \tag{16.9}$$

Proving (16.9) ends the proof because $|Du| \in L^2_{\rm loc}(\Omega)$ (case $s=2$) and in finitely many steps s^* becomes larger than p.

Fix a point $x_0 \in \Omega$ and a radius $R > 0$ such that $B_R(x_0) \Subset \Omega$: we choose a cut-off function $\eta \in C^\infty_c(B_R(x_0))$, with $0 \leq \eta \leq 1$ and $\eta \equiv 1$ in $B_{R/2}(x_0)$.
We claim that ηu belongs to $H^{1,s^*\wedge p}_0(B_R(x_0))$ if $R \ll 1$, as it is the unique fixed point of a contraction in $H^{1,s^*\wedge p}_0(B_R(x_0))$ that we are going to define and study in some steps. This implies in particular that $|Du| \in L^{s^*\wedge p}(B_{R/2}(x_0))$.
(1) We start localizing the equation. Replacing φ with $\eta\varphi$ in the PDE, by algebraic computations we obtain

$$\begin{aligned}
&\int_{B_R(x_0)} A(x)D(\eta u)(x)\nabla\varphi(x)\,dx\\
=\ &\int_{B_R(x_0)} A(x)\left(\eta(x)Du(x) + u(x)\otimes\nabla\eta(x)\right)\nabla\varphi(x)\,dx\\
=\ &\int_{B_R(x_0)} A(x)\left(Du(x)D(\eta\varphi)(x) + u(x)\otimes\nabla\eta(x)\nabla\varphi(x) - Du(x)\left(\nabla\eta(x)\varphi(x)\right)\right)\,dx\\
=\ &\int_{B_R(x_0)} f(x)\eta(x)\varphi(x)+F(x)\nabla(\eta\varphi)(x)+A(x)\left(u(x)\otimes\nabla\eta(x)\nabla\varphi(x)-Du(x)\nabla\eta(x)\varphi(x)\right)dx\\
=\ &\int_{B_R(x_0)} \tilde f(x)\varphi(x) + \tilde F(x)\nabla\varphi(x)\,dx\ ,
\end{aligned}$$

defining

$$\tilde f(x) := f(x)\eta(x) + F(x)\nabla\eta(x) - A(x)Du(x)\nabla\eta(x)$$

and

$$\tilde F(x) := F(x)\eta(x) + A(x)u(x)\otimes\nabla\eta(x)\ .$$

Thus ηu satisfies

$$-\mathrm{div}(A(x_0)D(\eta u)) = \tilde f + \mathrm{div}[\tilde F + (A - A(x_0))D(\eta u)]. \tag{16.10}$$

(2) In order to write $\tilde{f}$ in divergence form, let us consider the problem

$$\begin{cases} -\Delta w = \tilde{f} \\ w \in H_0^1(\Omega; \mathbb{R}^m) \end{cases} .$$

Thanks to the previous L^p regularity result for constant coefficients PDE's, since $\tilde{f} \in L^{s\wedge q}_{\mathrm{loc}}$ (because we assumed that $|Du| \in L^{s\wedge p}_{\mathrm{loc}}$), we have $|D^2 w| \in L^{s\wedge q}_{\mathrm{loc}}$ (see also Remark 16.8). By Sobolev immersion we get $|Dw| \in L^{(s\wedge q)^*}_{\mathrm{loc}}$, hence

$$|Dw| \in L^{s^*\wedge q^*}_{\mathrm{loc}} = L^{s^*\wedge p}_{\mathrm{loc}} .$$

Now we define

$$F^*(x) := \tilde{F}(x) + Dw(x) \in L^{s^*\wedge p}_{\mathrm{loc}} .$$

(3) Let $E = H_0^{1,s^*\wedge p}(B_R(x_0); \mathbb{R}^m)$ and let us define the operator $\Theta : E \to E$ which associates to each $V \in E$ the function $v \in E$ that solves

$$-\mathrm{div}\,(A(x_0)Dv) = \mathrm{div} F^* - \mathrm{div}\,((A(x_0) - A)DV) \ . \tag{16.11}$$

The operator Θ is well-defined because $|F^*| \in L^{s^*\wedge p}(B_R(x_0))$ (we saw that in step (2)) and we can take advantage of regularity theory for constant coefficients operators. The operator Θ is a contraction, in fact

$$\|D(v_1 - v_2)\|_E \leq c\| \left(A(x_0) - A\right) D(V_1 - V_2)\|_E \leq \frac{1}{2}\|D(V_1 - V_2)\|_{L^{s^*\wedge p}(B_R(x_0))}$$

if R is sufficiently small, according to the continuity of A. Here we use the fact that the constant c in the first inequality is scale invariant, so it can be "beaten" by the oscillation of A in $B_R(x_0)$, if R is small enough.

Let us call $v_* \in E$ the unique fixed point of (16.11). According to (16.10), ηu already solves (16.11), but in the larger space $H_0^{1,p\wedge s}$. Thus $\eta u \in H_0^{1,s^*\wedge p}$ if we are able to show that $v_* = \eta u$, and to see this it suffices to show that uniqueness holds in the larger space as well.

Consider the difference $v' := v_* - \eta u \in H_0^{1,s\wedge p}(B_R(x_0); \mathbb{R}^m) \subset H_0^1(B_R(x_0); \mathbb{R}^m)$: v' is a weak solution of

$$-\mathrm{div}\,(A(x)Dv') = 0 ,$$

hence $v' \equiv 0$ (we can indeed use the variational characterization of the solution). This concludes the proof. $\square$

17 De Giorgi's solution of Hilbert's XIX problem

17.1 The basic estimates

We briefly recall here the setting of Hilbert's XIX problem, that has already been described and solved in dimension 2.

We deal with local minima v of scalar functionals

$$v \longmapsto \int_\Omega F(Dv)\,dx$$

where $F \in C^{2,\beta}(\mathbb{R}^n)$ (at least, for some $\beta > 0$) satisfies the following ellipticity property: there exist two positive constants $\lambda \le \Lambda$ such that $\Lambda I \ge D^2F(p) \ge \lambda I$ for all $p \in \mathbb{R}^n$ (this implies in particular that $|D^2F|$ is uniformly bounded). We have already seen that under these assumptions it is possible to derive the Euler-Lagrange equations $\mathrm{div} F_p(Dv) = 0$. By differentiation, for any direction $s \in \{1, \dots, n\}$, the equation for $u := \partial v/\partial x_s$ is

$$-\mathrm{div}\,(F_{pp}(Dv)Du) = 0$$

that is, more explicitly

$$\frac{\partial}{\partial x_\alpha}\left(F_{p^\alpha p^\beta}(Dv)\frac{\partial}{\partial x_\beta}\left[\frac{\partial u}{\partial x_s}\right]\right) = 0\ .$$

Recall also the fact that in order to obtain this equation we needed to work with the approximation $\Delta_{h,s}v$ and with the interpolating operator

$$\widetilde{A}_h(x) := \int_0^1 F_{pp}(t\nabla v(x+he_s) + (1-t)\nabla v(x))\,dt$$

and to exploit the Caccioppoli-Leray inequality.

One of the striking ideas of De Giorgi was basically to split the problem, that is to deal with u and v separately, as Dv is only involved in the coefficients of the equation for u. The key point of the regularization procedure is then to show that under no regularity assumption on v (i.e. not more than measurability), if u is a solution of this equation, then $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$, with α depending only on n and on the ellipticity constants λ, Λ. If this is true, we can proceed as follows:

$$u \in C^{0,\alpha} \Rightarrow v \in C^{1,\alpha} \Rightarrow F_{pp}(Dv) \in C^{0,\alpha} \Rightarrow u \in C^{2,\alpha} \Rightarrow v \in C^{3,\alpha}$$

where the only non-trivial implication relies upon the Schauder estimates and on the fact that F_{pp} is Hölder continuous. If F is more regular, by iteration we can easily show that

$$F \in C^\infty \Rightarrow v \in C^\infty$$

and also, by the tools developed in [18], that

$$F \in C^{\omega} \Rightarrow v \in C^{\omega}$$

which is the complete solution of the problem raised by Hilbert.

Actually, we have solved this problem in the special case $n = 2$ since, by means of Widman's technique, we could prove that $|Du| \in L^{2,\alpha}$ and hence $|Du| \in \mathcal{L}^{2,\alpha+2}$ for some $\alpha > 0$. This is enough, *if* $n = 2$, to conclude $u \in C^{0,\alpha/2}$.

First of all, let us fix our setting. Let Ω be an open domain in $\mathbb{R}^n$, $0 < \lambda \leq \Lambda < \infty$ and let $A^{\alpha\beta}$ be a Borel symmetric matrix satisfying a.e. the condition $\lambda I \leq A(x) \leq \Lambda I$. We want to show that if $u \in H^1_{\mathrm{loc}}$ solves the problem

$$-\mathrm{div}\,(A(x)Du(x)) = 0$$

then $u \in C^{0,\alpha}_{\mathrm{loc}}$. Some notation is needed: for $B_\rho(x) \subset \Omega$ we define

$$A(k,\rho) := \{u > k\} \cap B_\rho(x)$$

where the dependence on the center x can be omitted. This should not create confusion, since we will often work with a fixed center. In this section, we will derive many functional inequalities, but typically we are not interested in finding the sharpest constants, but only on the functional dependence of these quantities. Therefore, in order to avoid complication of the notation we will use the same symbol (generally c) to indicate *different* constants, possibly varying from one passage to the next one. However we will try to indicate the functional dependence explicitly whenever this is appropriate and so we will use expressions like $c(n)$ or $c(n, \lambda, \Lambda)$ many times.

Theorem 17.1 (Caccioppoli inequality on level sets)**.** *For any $k \in \mathbb{R}$ and $B_\rho(x) \subset B_R(x) \Subset \Omega$ we have*

$$\int_{A(k,\rho)} |Du|^2\,dy \leq \frac{c}{(R-\rho)^2} \int_{A(k,R)} (u-k)^2\,dy \tag{17.1}$$

with $c = 16\Lambda^2/\lambda^2$.

Remark 17.2. It should be noted that the previous theorem generalizes the Caccioppoli-Leray inequality, since we don't ask $\rho = R/2$ and we introduce the sublevels.

Theorem 17.3 (Chain rule)**.** *If $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$, then for any $k \in \mathbb{R}$ the function $(u-k)^+$ belongs to $W^{1,1}_{\mathrm{loc}}(\Omega)$. Moreover we have that $D(u-k)^+ = Du$ a.e. on $\{u > k\}$, while $D(u-k)^+ = 0$ a.e. on $\{u \leq k\}$.*

Proof. Since this lemma is rather classical, we just sketch the proof. By the arbitrariness of u, the problem is clearly translation-invariant and we can assume without loss of generality $k = 0$. Consider the family of functions defined by $\varphi_\epsilon(t) := \sqrt{t^2+\epsilon^2} - \epsilon$ for $t \geq 0$ and identically zero elsewhere, whose derivatives are uniformly bounded and converge to $\chi_{\{t>0\}}$. Moreover, let $\{u_n\}$ a sequence of C^1_{loc} functions approximating u in $W^{1,1}_{\mathrm{loc}}$. We have that for any $n \in \mathbb{N}$ and $\epsilon > 0$ the classical chain-rule gives $D_j\left[\varphi_\epsilon(u_n)\right] = \varphi'_\epsilon(u_n)D_j(u_n)$. Passing to the limit as $n \to \infty$ gives $D_j\left[\varphi_\epsilon(u)\right] = \varphi'_\epsilon(u)D_j(u)$. Now, we can pass to the limit as $\epsilon \downarrow 0$ and use the dominated convergence theorem to conclude that $D_j u^+ = \chi_{\{u>0\}} D_j u$. $\square$

We can now come to the proof of the previous theorem.

Proof. Let η a cut-off function supported in $B_R(x)$, with $\eta = 1$ on $\overline{B}_\rho(x)$ and $|\nabla\eta| \leq 2/(R-\rho)$. If we apply the weak form of our equation to the test function $\varphi := \eta^2(u-k)^+$ we get

$$\begin{aligned}\int_{A(k,R)} \eta^2 ADuDu\,dx &= -2\int_{B_R(x)} \eta DuAD\eta(u-k)^+\,dx \\ &\leq \frac{\Lambda}{\epsilon}\int_{A(k,R)} \eta^2\left|Du\right|^2\,dx + \frac{4\epsilon\Lambda}{(R-\rho)^2}\int_{A(k,R)} (u-k)^2\,dx\end{aligned}$$

for any $\epsilon > 0$, by our upper bound and Young's inequality. Here we let $\epsilon = 2\Lambda/\lambda$ so that, thanks to the uniform ellipticity assumption, we obtain

$$\int_{A(k,R)} \eta^2 ADuDu\,dx \leq \frac{\lambda}{2}\int_{A(k,R)} \eta^2\left|Du\right|^2\,dx + \frac{8\Lambda^2}{\lambda(R-\rho)^2}\int_{A(k,R)} \left(u-k\right)^2\,dx\,.$$

Since on the smaller ball η is identically equal to 1 we eventually get

$$\int_{A(k,\rho)} \left|Du\right|^2\,dx \leq \frac{16\Lambda^2}{\lambda^2(R-\rho)^2}\int_{A(k,R)} \left(u-k\right)^2\,dx$$

which is our thesis. $\square$

The second great idea of De Giorgi was that (one-sided) regularity could be achieved for all functions satisfying the previous functional inequality, regardless the fact that these were solutions to an elliptic equation. For this reason he introduced a special class of objects.

Definition 17.4 (De Giorgi's class)**.** *We define the De Giorgi class $DG_+(\Omega)$ as follows:*

$$DG_+(\Omega) := \{u\ |u \text{ satisfies } (17.1) \text{ for all } k \in \mathbb{R} \text{ and } B_r(x) \Subset B_R(x) \Subset \Omega \text{ for some constant } c \in \mathbb{R}\}\,.$$

In this case, we also define $c^+_{DG}(u)$ to be the minimal constant larger than 1 for which the previous condition is verified.

Remark 17.5. From the previous proof, it should be clear that we don't really require u to be a solution, but just a *sub-solution* of our problem. In fact, we have proved that

$$-\mathrm{div}\,(ADu) \le 0 \Rightarrow u \in DG(\Omega),\ c^+_{DG}(u) \le \frac{16\Lambda^2}{\lambda^2}.$$

In a similar way, the class $DG_-(\Omega)$ (corresponding to supersolutions) and $c^-_{DG}(u)$ could be defined by

$$\int_{\{u<k\}\cap B_\rho(x)} |Du|^2\,dy \le \frac{c}{(R-\rho)^2}\int_{\{u<k\}\cap B_R(x)} (u-k)^2\,dy$$

and obviously $u \mapsto -u$ maps $DG_+(\Omega)$ in $DG_-(\Omega)$ bijectively, with $c^+_{DG}(u) = c^-_{DG}(-u)$.

The main part of the program by De Giorgi can be divided into two steps:

(i) If $u \in DG^+(\Omega)$, then it satisfies a strong *maximum principle* in a quantitative form (more precisely the L^2 to L^∞ estimate in Theorem 17.8);

(ii) If both u and $-u$ belong to $DG_+(\Omega)$ then $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$.

Let us start by discussing the first point. We define these two crucial quantities:

$$U(h,\rho) := \int_{A(h,\rho)} (u-h)^2\,dx, \qquad V(h,\rho) := \mathscr{L}^n\left(A(h,\rho)\right).$$

Theorem 17.6. *The following properties hold:*

(i) Both U and V are non-decreasing functions of ρ, but non-increasing functions of h.;

(ii) For any $h > k$ and $0 < \rho \le R$ the following inequalities hold:

$$V(h,\rho) \le \frac{1}{(h-k)^2}U(k,\rho), \quad U(k,\rho) \le c(n)\cdot c^+_{DG}(u)(R-\rho)^{-2}U(k,R)V(k,\rho)^{2/n}.$$

Proof. The first part of the theorem and the first inequality are trivial, since

$$\begin{aligned}(h-k)^2V(h,\rho) &= \int_{A(h,\rho)} (h-k)^2\,dx \le \int_{A(h,\rho)} (u-k)^2\,dx \\ &\le \int_{A(k,\rho)} (u-k)^2\,dx = U(k,\rho).\end{aligned}$$

For the second inequality, introduce a cut-off function η supported in $B_{(R+\rho)/2}(x)$ with $\eta = 1$ on $\overline{B}_\rho(x)$ and $|\nabla\eta| \le 4/R-\rho$. We need to note that

$$\int_{B_{(R+\rho)/2}} \eta^2|D(u-k)^+|^2\,dx \le \frac{4c^+_{DG}(u)}{(R-\rho)^2}\int_{A(k,R)} (u-k)^2$$

and

$$\int_{B_{(R+\rho)/2}} ((u-k)^+)^2 |D\eta|^2 \, dx \leq \frac{16}{(R-\rho)^2} \int_{A(k,R)} (u-k)^2 \, dx.$$

Combining these two inequalities, since $c^+_{DG}(u) \geq 1$, we get

$$\int_{B_{(R+\rho)/2}} |D(\eta(u-k)^+)|^2 \, dx \leq \frac{\kappa \cdot c^+_{DG}(u)}{(R-\rho)^2} \int_{A(k,R)} (u-k)^2 \, dx$$

and by the Sobolev embedding inequality this implies

$$\left(\int_{A(k,\rho)} (u-k)^{2^*} \, dx \right)^{2/2^*} \leq \frac{c(n) \cdot c^+_{DG}(u)}{(R-\rho)^2} \int_{A(k,R)} (u-k)^2 \, dx$$

for some constant $c(n)$ depending on the dimension n and on the numerical constant κ. In order to conclude, we just need to apply Hölder's inequality

$$U(k,\rho) = \int_{A(k,\rho)} ((u-k)^+)^2 \, dx \leq \left(\int_{A(k,\rho)} (u-k)^{2^*} \, dx \right)^{2/2^*} V(k,\rho)^{2/n}$$

with $p = 2^*/2 = n/n-2$, $p' = n/2$. □

The previous inequalities can be slightly weakened in order to get

$$V(h,\rho) \leq \frac{1}{(h-k)^2} U(k,R), \quad U(h,\rho) \leq c(n) \cdot c_{DG}(u)(R-\rho)^{-2} U(k,R) V(k,R)^{2/n}$$

and we shall use these to obtain the quantitative maximum principle.

We can view these inequalities as joint decay properties of U and V; in order to get the decay of a single quantity, it is convenient to define $\varphi := U^\xi V^\eta$ for some choice of the (positive) real parameters ξ, η to be determined. We obtain:

$$U^\xi(h,\rho) V^\eta(h,\rho) \leq \frac{C}{(h-k)^2} \frac{1}{(R-\rho)^{2\xi}} U^{\xi+\eta}(k,R) V^{2\xi/n}(k,R).$$

where $C := c(n) \cdot c^+_{DG}(u)$, a convention that will be systematically be adopted in the sequel. Since we are looking for some decay inequality for φ, we solve in (θ, ξ, η) the system

$$\xi + \eta = \theta\xi, \qquad \frac{2\xi}{n} = \theta\eta$$

getting $\eta = 1$ (by homogeneity this choice is not restrictive), $\xi = n\theta/2$ and

$$\theta = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2}{n}} \,. \tag{17.2}$$

Note that $\theta > 1$: this fact will play a crucial role in the following proof. In any case, we get the decay relation

$$\varphi(h,\rho) \leq \frac{c^\xi}{(h-k)^2}\frac{1}{(R-\rho)^{2\xi}}\varphi^\theta(k,R) \ .$$

Theorem 17.7. *Let $u \in DG_+(\Omega)$, $B_{R_0}(x) \Subset \Omega$. For any $h_0 \in \mathbb{R}$ there exists $d = d(h_0, R_0, c^+_{DG}(u))$ such that $\varphi(h_0 + d, R_0/2) = 0$. Moreover, we can take*

$$d^2 = c'(n)[c^+_{DG}(u)]^{\theta n/2}\varphi(h_0, R_0)^{\theta-1}/R_0^{n\theta},$$

the constant c'_n depending only on the dimension n. In particular $u \leq h_0 + d$ a.e. on $B_{R_0/2}(x)$.

Theorem 17.8 (L^2 to L^∞ estimate)**.** *If $u \in DG_+(u)$, then for any $B_{R_0}(x) \subset \Omega$ and for any $h_0 \in \mathbb{R}$*

$$\text{ess-}\sup_{B_{R_0/2}(x)} u \leq h_0 + c'(n)[c^+_{DG}(u)]^{\theta n/4}\left(R^{-n}\int_{A(h_0,R)}(u-k_0)^2\,dx\right)^{1/2}\left(\frac{V(h_0,R)}{R^n}\right)^{\frac{\theta-1}{2}} .$$

Proof. This corollary comes immediately from the theorem, once you express φ in terms of U and V and recall that $\xi + 1 = \theta\xi$ that is $\xi(\theta - 1) = 1$, by means of simple algebraic computations. □

Remark 17.9. From the corollary with $h_0 = 0$, we can get the *maximum principle* for u as anticipated above. In fact

$$\text{ess-}\sup_{B_{R_0/2}}(u^+)^2 \leq q(n)\fint_{B_{R_0}(x)} u^2\,dx$$

with $q(n)$ easily estimated in terms of c'_n, $c^+_{DG}(u)$, n and ω_n.

We are now ready to prove Theorem 17.8, the main result of this section.

Proof. Define $k_n := h_0 + d - d/2^n$ and $R_n := R_0/2 + R_0/2^{n+1}$, so that $k_n \uparrow (h_0 + d)$ while $R_n \downarrow R_0/2$. Here $d \in \mathbb{R}$ is a parameter to be fixed in the sequel. From the decay inequality for φ we get

$$\varphi(k_{n+1}, R_{n+1}) \leq \varphi(k_n, R_n)\left[\varphi(k_n,R_n)^{\theta-1}c^\xi\left(\frac{2^{n+2}}{R_0}\right)^{2\xi}\left(\left\{\frac{2^{n+2}}{d}\right\}\right)^2\right]$$

and letting $\psi_n := 2^{\mu n}\varphi(k_n, R_n)$ this becomes

$$\psi_{n+1} \leq \psi_n\left[2^\mu c^\xi 2^{2\xi+2}2^{n(2\xi+2)}R_0^{-2\xi}d^{-2}2^{-\mu n(\theta-1)}\psi_n^{\theta-1}\right].$$

This is true for any $\mu \in \mathbb{R}$ but we fix it so that $n(2\xi + 2) = \mu n(\theta - 1) = 0$, leading to a cancellation of two factors in the previous inequality. It should now be clear that if we are able to determine d such that

$$2^{\mu} c^{\xi} 2^{2\xi+2} \psi_0^{\theta-1} R_0^{-2\xi} d^{-2} \leq 1$$

then by induction $\psi_n \leq \psi_0$, $\forall n \in \mathbb{N}$. In that case, $\varphi(k_n, R_n) \leq 2^{-\mu n}$ and since by monotonicity

$$\varphi(h_0 + d, R_0/2) \leq \varphi(k_n, R_0/2) \leq \varphi(k_n, R_n)$$

we get the thesis. But the previous condition on d is satisfied if

$$d^2 \geq c'(n)[c_{DG}^{+}(u)]^{\theta n/2} R_0^{-2\xi}$$

and the desired claim follows. $\square$

We are now in position to discuss the notion of *oscillation*, which will be crucial for the conclusion of the argument by De Giorgi.

Definition 17.10. *Let $\Omega \subset \mathbb{R}^n$ be an open set, $B_r(x) \subset \Omega$ and $u : \Omega \to \mathbb{R}$ a measurable function. We define its oscillation on $B_r(x)$ as*

$$\omega(B_r(x))(u) := \text{ess-}\sup_{B_r(x)} u - \text{ess-}\inf_{B_r(x)} u.$$

When no confusion arises, we will omit the explicit dependence on the center of the ball, thus identifying $\omega(r) = \omega(B_r(x))$.

It is an immediate consequence of the previous results that if $u \in DG_+(u) \cap DG_-(\Omega)$, then

$$\text{ess-}\sup_{B_{r/2}(x)} u \leq \zeta \left(\fint_{B_r(x)} u^2 \, dx \right)^{1/2}, \qquad \text{ess-}\inf_{B_{r/2}(x)} u \leq \zeta \left(\fint_{B_r(x)} u^2 \, dx \right)^{1/2}$$

for a constant ζ, which is a function of the dimension n and of $c_{DG}(u)$. Here and in the sequel we shall denote by $c_{DG}(u)$ the maximum of $c_{DG}^{+}(u)$ and $c_{DG}^{-}(u)$ and by $DG(u)$ the intersection of the spaces $DG_+(u)$ and $DG_-(u)$.

Consequently, under the same assumptions,

$$\omega(B_{r/2}(x))(u) \leq 2\zeta \left(\fint_{B_r(x)} u^2 \, dx \right)^{1/2}.$$

Let us see the relation between the decay of the oscillation of u and the Hölder regularity of u.

Theorem 17.11. *Let $\Omega \subset \mathbb{R}^n$ be open, $c \geq 0$, $\alpha \in (0,1]$ and let $u : \Omega \to \mathbb{R}$ a measurable function such that for any $B_r(x) \subset \Omega$ we have $\omega(B_r(x)) \leq cr^\alpha$. Then $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$, that is there exists in the Lebesgue equivalence class of u a $C^{0,\alpha}_{\mathrm{loc}}$ representative.*

Proof. Clearly, by definition of essential extrema, we have that

$$\text{ess-}\inf_{B_r(x)} u \leq u(y) \leq \text{ess-}\sup_{B_r(x)} u \qquad \text{for } \mathscr{L}^n\text{-a.e. } y \in B_r(x)\ .$$

These inequalities imply $\text{ess-}\inf_{B_r(x)} u \leq u_{B_r(x)} \leq \text{ess-}\sup_{B_r(x)}$ and hence, by our hypothesis, $|u - u_{B_r(x)}| \leq c^2 r^{2\alpha}$. We have proved that $u \in \mathcal{L}^{2,n+2\alpha}(\Omega)$, but this gives $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ (regularity is local since no assumption is made on Ω), which is the thesis. □

This theorem motivates our interest in the study of oscillation of u, that will be carried on by means of some tools we haven't introduced so far.

17.2 Some useful tools

De Giorgi's proof of Hölder continuity is geometric in spirit and ultimately based on the isoperimetric inequalities. Notice that, as we will see, the isoperimetric inequality is also underlying the Sobolev inequalities used in the proof of the sup estimate for functions in $DG_+(\Omega)$.

We will say that a set $E \subset \mathbb{R}^n$ is regular if it is locally the epigraph of a C^1 function. In this case, it is well-known that by local parametrizations and a partition of unity, we can define $\sigma_{n-1}(\partial E)$, the $(n-1)-$dimensional surface measure of ∂E.

Of course, regular sets are a very unnatural (somehow too restrictive) setting for isoperimetric inequalities, but it is suffcient for our purpose. We state without proof two isoperimetric inequalities:

Theorem 17.12 (Isoperimetric inequality)**.** *Let $E \subset \mathbb{R}^n$ be regular and such that $\sigma_{n-1}(\partial E) < +\infty$. Then*

$$\min\left\{\mathscr{L}^n(E), \mathscr{L}^n(\mathbb{R}^n \setminus E)\right\} \leq c(n)\left[\sigma_{n-1}(\partial E)\right]^{1^*}$$

with $c(n)$ a dimensional constant.

It is also well-known that the best constant $c(n)$ in the previous inequality is $\mathscr{L}^n(B_1)/[\sigma_{n-1}(\partial B_1)]^{1^*}$, that is, balls have the best isoperimetric ratio.

Theorem 17.13 (Relative isoperimetric inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set, with $\partial\Omega$ Lipschitz. Let $E \subset \Omega$ with $(\partial E) \cap \Omega \in C^1$. Then*

$$\min\left\{\mathscr{L}^n(E), \mathscr{L}^n(\Omega \setminus E)\right\} \leq c(n,\Omega)\left[\sigma_{n-1}((\partial E) \cap \Omega)\right]^{1^*}\ .$$

Let's introduce another classical tool in Geometric Measure Theory.

Theorem 17.14 (Coarea formula)**.** *Let $\Omega \subset \mathbb{R}^n$ open and $u \in C^\infty(\Omega)$ nonnegative, then*

$$\int_\Omega |\nabla u|\, dx = \int_0^\infty \sigma_{n-1}\left(\Omega \cap \{u = t\}\right)\, dt\ .$$

Remark 17.15. It should be observed that the right-hand side of the previous formula is well-defined, since by the classical *Sard's theorem*

$$u \in C^\infty(\mathbb{R}^n) \Rightarrow \mathscr{L}^1\big(\{u(x) :\ \nabla u(x) = 0\}\big) = 0\ .$$

By the implicit function theorem this implies that almost every sublevel $\{u < t\}$ is regular.

Proof. A complete proof won't be described here since it is far from the main purpose of these lectures, however we sketch the main points. The interested reader may consult, for instance, [11].

We first prove $\int_\Omega |\nabla u|\, dx \le \int_0^\infty \sigma_{n-1}\left(\Omega \cap \{u = t\}\right)\, dt$. Consider the pointwise identity

$$u = \int_0^\infty \chi_{\{u>t\}}\, dt$$

that implies

$$\begin{aligned}
\int_\Omega |\nabla u|\, dx &= \sup_{\varphi \in C^1_c,\, |\varphi| \le 1} \int_\Omega \langle \nabla u, \varphi\rangle\, dx = \sup_{\varphi \in C^1_c,\, |\varphi| \le 1} \int_\Omega u \mathrm{div}\varphi\, dx \\
&= \sup_{\varphi \in C^1_c,\, |\varphi| \le 1} \int_0^\infty \Big(\int_\Omega (\mathrm{div}\varphi)\chi_{\{u>t\}}\, dx\Big)\, dt \\
&\le \int_0^\infty \left(\sup_{\varphi \in C^1_c,\, |\varphi| \le 1} \int_\Omega (\mathrm{div}\varphi)\chi_{\{u>t\}}\, dx \right)\, dt.
\end{aligned}$$

Hence, by the Gauss-Green theorem we obtain

$$\int_\Omega |\nabla u|\, dx \le \int_0^\infty \left(\sup_{\varphi \in C^1_c,\, |\varphi| \le 1} \int_{\Omega \cap \{u=t\}} \langle \varphi, \nu_t\rangle\, d\sigma_{n-1} \right)\, dt \le \int_0^\infty \sigma_{n-1}\left(\Omega \cap \{u = t\}\right)\, dt\,,$$

again exploiting the fact that for a.e. t the set $\{u = t\}$ is the (regular) boundary of $\{u > t\}$.

Let us consider the converse inequality, namely

$$\int_\Omega |\nabla u|\, dx \ge \int_{-\infty}^\infty \sigma_{n-1}\left(\Omega \cap \{u = t\}\right)\, dt\ .$$

This is trivial (with equality) if u is continuous and piecewise linear, since on each part of an adapted triangulation of Ω the coarea formula is just Fubini's theorem. The general case is obtained by approximation, choosing piecewise affine functions which converge to u in $W^{1,1}(\Omega)$ and using Fatou's lemma and the lower semicontinuity of $E \mapsto \sigma_{n-1}(\Omega \cap \partial E)$ (this, in turn, follows by the sup formula we already used in the proof of the first inequality). We omit the details. □

In order to deduce the desired Sobolev embeddings, we need a technical lemma.

Theorem 17.16. *Let $G : [0, +\infty) \to [0, +\infty)$ a nonincreasing measurable function. Then for any $\alpha \geq 1$ we have*

$$\alpha \int_0^\infty t^{\alpha-1} G(t)\, dt \leq \left(\int_0^\infty G^{1/\alpha}(t)\, dt \right)^\alpha .$$

Proof. It is sufficient to prove that for any $T > 0$ we have the finite time inequality

$$\alpha \int_0^T t^{\alpha-1} G(t)\, dt \leq \left(\int_0^T G^{1/\alpha}(t)\, dt \right)^\alpha$$

which is implied, by integration, by

$$T^{\alpha-1} G(T) \leq \left(\int_0^T G^{1/\alpha}(t)\, dt \right)^{\alpha-1} G^{1/\alpha}(T) .$$

This is equivalent to

$$G^{1/\alpha}(T) \leq \fint_0^T G^{1/\alpha}(s)\, ds$$

that is obvious, since G is nonincreasing. □

We are now ready to derive the desired embedding inequalities.

Theorem 17.17 (Sobolev embeddings, $p = 1$)**.** *We have*

$$\left(\int_{\mathbb{R}^n} |u|^{1^*}\, dx \right)^{1/1^*} \leq c(n) \int_{\mathbb{R}^n} |\nabla u|\, dx \qquad u \in W^{1,1}(\mathbb{R}^n).$$

Consequently, the have the following continuous embeddings:

(1) $W^{1,1}(\mathbb{R}^n) \hookrightarrow \text{Ł}^{1^*}(\mathbb{R}^n)$;

(2) for any $\Omega \subset \mathbb{R}^n$ *open, regular and bounded* $W_0^{1,1}(\Omega) \hookrightarrow L^{1^*}(\Omega)$.

Proof. By Theorem 17.3 it is possible to reduce the thesis to the case $u \geq 0$, and smoothing reduces the proof to the case $u \in C^\infty$. Under these assumptions we have

$$\int_{\mathbb{R}^n} (u)^{1^*}\, dx = 1^* \int_0^\infty t^{1/(n-1)} \mathscr{L}^n(\{u > t\})\, dt \leq 1^* \left(\int_0^\infty \mathscr{L}^n(\{u > t\})^{1/1^*}\, dt \right)^{1^*}$$

thanks to the lemma above. Consequently, the isoperimetric inequality and the coarea formula give

$$\begin{aligned} \int_{\mathbb{R}^n} (u)^{1^*}\, dx &\leq c(n) \left(\int_0^\infty \sigma_{n-1}\left[\{u = t\}\right]\, dt \right)^{1^*} \\ &= c(n) \left(\int_{\mathbb{R}^n} |\nabla u|\, dx \right)^{1^*} . \end{aligned}$$

□

Theorem 17.18 (Sobolev embeddings, $1 < p < n$)**.** *We have*

$$\left(\int_{\mathbb{R}^n} |u|^{p^*}\, dx \right)^{1/p^*} \leq c(n,p) \left(\int_{\mathbb{R}^n} |\nabla u|^p\, dx \right)^{1/p} \qquad \forall u \in W^{1,p}(\mathbb{R}^n) .$$

Consequently, the have the following continuous embeddings:

(1) $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n)$;

(2) for any $\Omega \subset \mathbb{R}^n$ *open, regular and bounded* $W^{1,p}_0(\Omega) \hookrightarrow L^{p^*}(\Omega)$.

Proof. Again, it is enough to study the case $u \geq 0$. We can exploit the case $p = 1$ to get

$$\left(\int_{\mathbb{R}^n} u^{\alpha 1^*}\, dx \right)^{1/1^*} \leq c(n) \int_{\mathbb{R}^n} \alpha u^{\alpha - 1} |\nabla u|\, dx$$

and by Hölder the right hand side can be estimated from above with

$$c(n)\alpha \left[\int_{\mathbb{R}^n} u^{(\alpha-1)p'}\, dx \right]^{1/p'} \left[\int_{\mathbb{R}^n} |\nabla u|^p\, dx \right]^{1/p} .$$

Now, choose α such that $\alpha 1^* = (\alpha - 1)p'$. Consequently

$$\left(\int_{\mathbb{R}^n} u^{p^*}\, dx \right)^{1/1^* - 1/p'} \leq c(n,p) \left(\int_{\mathbb{R}^n} |\nabla u|^p\, dx \right)^{1/p} ,$$

but $1/1^* - 1/p' = 1/p^*$ and the claim follows. The continuous embedding in (2) follows by the global one in (1) applied to an extension of u (recall that regularity of Ω yields the existence of a continuous extension operator from $W^{1,p}(\Omega)$ to $W^{1,p}(\mathbb{R}^n)$). □

We will also make use of the following refinement, where no assumption is made on the behaviour of u at the boundary of the domain.

Theorem 17.19. *Let $u \in W^{1,1}(B_R)$ with $u \geq 0$ and suppose that $\mathscr{L}^n(\{u = 0\}) \geq \mathscr{L}^n(B_R)/2$. Then*

$$\left(\int_{B_R} u^{1^*}\,dx\right)^{1/1^*} \leq c(n)\int_{B_R} |\nabla u|\,dx\ .$$

Proof. This result is the local version of the embedding $W^{1,1} \hookrightarrow L^{1^*}$. Hence, in order to give the proof, it is just needed to mimic the previous argument substituting the isoperimetric inequality with the relative isoperimetric inequality, that is here

$$\mathscr{L}^n(\{u > t\}) \leq c(n)\sigma_{n-1}\left[\mathscr{L}^n(B_R \cap \{u = t\})\right]^{1^*}\ .$$

We leave the details to the reader. □

17.3 Proof of Hölder continuity

We will divide the final part of the proof in two parts.

Lemma 17.20 (Decay of V)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and let $u \in DG_+(\Omega)$. If $\overline{B}_{2R} \subset \Omega$ and $k_0 <$ ess-$\sup_{B_{2R}}(u) =: M$ with $V(k_0, R) \leq \mathscr{L}^n(B_R)/2$, then the sequence of levels $k_\nu = M - \frac{M-k_0}{2^\nu}$ for $\nu \geq 0$ has the property that*

$$V(k_\nu, R)^{2/1^*} \leq \frac{c(n)c^+_{DG}(u)R^{2n-2}}{\nu}\ .$$

Proof. Take two levels h, k such that $M \geq h \geq k \geq k_0$ and define $v := u \wedge h - u \wedge k = (u \wedge h - k)^+$. By construction $v \geq 0$ and since $u \in W^{1,1}(\Omega)$ we also have $v \in W^{1,1}(\Omega)$. It is also clear that $\nabla v \neq 0$ only on $A(k, R) \setminus A(h, R)$. Now, notice that

$$\mathscr{L}^n(\{v = 0\}) \geq \mathscr{L}^n(\{u \leq k\}) \geq \mathscr{L}^n(\{u \leq k_0\}) \geq \frac{1}{2}\mathscr{L}^n(B_R)$$

and so we can apply the relative version of the critical Sobolev embedding and Hölder inequality to get

$$\begin{aligned}(h-k)^{1^*}\mathscr{L}^n(A(h,R)) &= \int_{A(h,R)} v^{1^*}\,dx \leq c(n)\left(\int_{B_R} |\nabla v|\,dx\right)^{1^*} \\ &= \leq c(n)\left(\int_{A(k,R)} |\nabla u|^2\,dx\right)^{1^*/2} \mathscr{L}^n(A(k,R) \setminus A(h,R))^{1^*/2}.\end{aligned}$$

We can now exploit the De Giorgi property of u that is

$$\int_{A(k,R)} |\nabla u|^2 \, dx \leq \frac{c^+_{DG}(u)}{R^2} \int_{B_{2R}} (u-k)^2 \, dx \leq (M-k)^2 c_{DG}(u) R^{n-2}$$

in order to obtain

$$(h-k)^{1^*} \mathscr{L}^n(A(h,R)) \leq c(n) c^+_{DG}(u)^{1^*/2} (M-k)^{1^*} R^{(n-2)1^*/2} \mathscr{L}^n(A(k,R) \setminus A(h,R))^{1^*/2}$$

or equivalently

$$(h-k)^2 \, |A(h,R)|^{2/1^*} \leq c(n) c^+_{DG}(u) (M-k)^2 R^{n-2} (V(k,R) - V(h,R)) \,. \tag{17.3}$$

Here we can conclude the proof by applying (17.3) for $h = k_{i+1}$ and $k = k_i$, so that

$$\begin{aligned} \nu V(k_\nu, R)^{2/1^*} &\leq \sum_{i=1}^{\nu} V(k_i, R)^{2/1^*} \leq 2c(n) c^+_{DG}(u) R^{n-2} \sum_{i=1}^{\nu} [V(k_i, R) - V(k_{i+1}, R)] \\ &\leq 2c(n) \omega_n c^+_{DG}(u) R^{2n-2} \,. \end{aligned}$$

□

Theorem 17.21 ($C^{0,\alpha}$ regularity)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and let $u \in DG(\Omega)$. Then $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$, with $2\alpha = -\log(1 - 2^{-\nu-2})$,*

$$\nu = c(n) \, [c_{DG}(u)]^{\frac{\theta(n-1)}{(\theta-1)n}} + 1 \tag{17.4}$$

and θ given by (17.2).

Proof. Pick an $R > 0$ such that $\overline{B}_{2R}(x) \subset \Omega$ and consider for any $r \leq R$ the functions $m(r) := \text{ess-}\inf_{B_r(x)}(u)$ and $M(r) := \text{ess-}\sup_{B_r(x)}(u)$. Moreover, set $\omega(r) = M(r) - m(r)$ and $\mu(r) := (m(r) + M(r))/2$. We apply the previous lemma to the sequence $k_\nu := M(2r) - \frac{\omega(2r)}{2^{\nu+1}}$, but to do this we should have

$$\mathscr{L}^n(\{u > \mu(2r)\} \cap B_r(x)) \leq \frac{1}{2} \mathscr{L}^n(B_r(x)).$$

But, either $\mathscr{L}^n(\{u > \mu(2r)\} \cap B_r(x)) \leq \frac{1}{2}\mathscr{L}^n(B_r(x))$ or $\mathscr{L}^n(\{u < \mu(2r)\} \cap B_r(x)) \leq \frac{1}{2}\mathscr{L}^n(B_r(x))$. The second case is analogous, provided we work with $-u$ instead of u, and it is precisely here that we need the assumption that both u and $-u$ belong to $DG_+(\Omega)$.

Consequently, we can choose an index $\nu \in \mathbb{N}$, so large that

$$c(n) \left[c^+_{DG}(u)\right]^{\theta n/4} \left(\frac{V(k_\nu, r)}{r^n}\right)^{(\theta-1)/2} \leq \frac{1}{2} \,.$$

Using Lemma 17.20 it is easily seen that the choice of ν as in (17.4) provides the condition above, so we can choose ν independently of r (this is crucial for the validity of the scheme below).

Now apply the maximum principle in Theorem 17.8 to u with radii $r/2$ and r and $h_0 = M(2r) - \frac{1}{2^{\nu+1}}\omega(2r)$ to obtain

$$M\left(\frac{r}{2}\right) \leq h_0 + c(n)\left[c_{DG}^{+}(u)\right]^{\theta n/4}(M(2r) - h_0)\left(\frac{V(k_\nu, r)}{r^n}\right)^{(\theta-1)/2}$$

and by the appropriate choice of ν that has been described we deduce

$$M\left(\frac{r}{2}\right) \leq M(2r) - \frac{1}{2^{\nu+1}}\omega(2r) + \frac{1}{2}\frac{1}{2^{\nu+1}}\omega(2r) = M(2r) - \frac{1}{2^{\nu+2}}\omega(2r).$$

If we subtract the essential minima and use $m(r/2) \leq m(2r)$ we finally get

$$\omega(r/2) \leq \omega(2r)\left(1 - \frac{1}{2^{\nu+2}}\right)$$

which is the desired decay estimate. By the standard iteration argument, we find

$$\omega(r) \leq 4^{\alpha}\omega(R)\left(\frac{r}{R}\right)^{\alpha} \qquad 0 < r \leq R$$

for $2\alpha = -\log\left(1 - 2^{-\nu-2}\right)$ and the conclusion follows.

$\square$

18 Regularity for systems

18.1 De Giorgi's counterexample to regularity for systems

In the previous section we saw De Giorgi's regularity result for solutions $u \in H^1(\Omega)$ of the elliptic equation

$$\mathrm{div}\,(A(x)Du(x)) = 0$$

with bounded Borel coefficients A satisfying $\lambda I \leq A \leq \lambda I$. It turned out that $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$, with $\alpha = \alpha(n, \lambda, \Lambda)$.

It is natural to investigate about similar regularity properties for systems, still under no regularity assumption on A (otherwise, Schauder theory is applicable). In 1968, in [7], Ennio De Giorgi provided a counterexample showing that the scalar case is special. De Giorgi's example not also solves an elliptic PDE, but it is also the minimum of a convex variational problem.

When $m = n$, consider

$$u(x) := x|x|^\alpha \ . \tag{18.1}$$

We will show in (18.8), (18.9) and (18.7) that, choosing

$$\alpha = -\frac{n}{2}\left(1 - \frac{1}{\sqrt{(2n-2)^2+1}}\right) \ , \tag{18.2}$$

the function u is a solution of the Euler-Lagrange equation associated with the uniformly convex functional

$$L(u) := \int_{B_1} \left((n-2)\nabla \cdot u(x) + n\frac{x \otimes x}{|x|^2} Du(x)\right)^2 + |Du(x)|^2 \, dx \ . \tag{18.3}$$

If $n \geq 3$ then $|u| \notin L^\infty(B_1)$, because

$$-\alpha = \frac{n}{2}\left(1 - \frac{1}{\sqrt{(2n-2)^2+1}}\right) \geq \frac{3}{2}\left(1 - \frac{1}{\sqrt{17}}\right) > 1$$

and this provides a counterexample not only to Hölder regularity, but also to local boundedness of solutions.

Calling $B(x)$ the matrix such that $L(u) = \int_{B_1} \langle B(x)Du, Du\rangle \, dx$, we remark that $B(x)$ has a discontinuity at the origin (determined by the term $x \otimes x/|x^2|$).

The Euler-Lagrange equation associated to (18.3) is the following (in the weak distributional sense): for every $h = 1, \dots, n$ it must be

$$
\begin{aligned}
0 &= (n-2)\frac{\partial}{\partial x_h}\left((n-2)\sum_{t=1}^{n}\frac{\partial u^t}{\partial x_t} + n\sum_{s,t=1}^{n}\frac{x_s x_t}{|x|^2}\frac{\partial u^t}{\partial x_s}\right) && (18.4)\\
&+ n\sum_{k=1}^{n}\frac{\partial}{\partial x_k}\left[\frac{x_h x_k}{|x|^2}\left((n-2)\sum_{t=1}^{n}\frac{\partial u^t}{\partial x_t} + n\sum_{s,t=1}^{n}\frac{x_s x_t}{|x|^2}\frac{\partial u^t}{\partial x_s}\right)\right] && (18.5)\\
&+ \sum_{k=1}^{n}\frac{\partial^2 u^h}{\partial x_k^2}\ . && (18.6)
\end{aligned}
$$

We are going to prove in a few steps that u is the unique minimizer of L, with boundary data given by u itself, and that u solves the Euler-Lagrange equations. The steps are:

(i) u, as defined in (18.1), belongs to $C^\infty(B_1 \setminus \{0\}; \mathbb{R}^n)$ and solves in $B_1 \setminus \{0\}$ the Euler-Lagrange equations;

(ii) $u \in H^1(B_1; \mathbb{R}^n)$ and is also a weak solution in B_1 of the system.

Let us perform step (1). Fix $h \in \{1, \dots, n\}$, then

$$\Delta_h\left(x_h|x|^\alpha\right) = (\alpha n + \alpha^2)x_h|x|^{\alpha-2} \tag{18.7}$$

and this is what we need to put in (18.6) when u is given by (18.1). For both (18.4) and (18.5) we have to calculate

$$\sum_{t=1}^{n}\frac{\partial}{\partial x_t}\left(x_t|x|^\alpha\right) = (n+\alpha)|x|^\alpha\ ,$$

and

$$\sum_{s,t=1}^{n}\frac{x_s x_t}{|x|^2}\frac{\partial u^t}{\partial x_s} = \sum_{s,t=1}^{n}\alpha x_s^2 x_t^2|x|^{\alpha-4} = \alpha|x|^\alpha\ .$$

In order to complete the substitution in (18.4), since

$$\frac{\partial}{\partial x_h}|x|^\alpha = \alpha x_h|x|^{\alpha-2}\ ,$$

we obtain

$$(n-2)\frac{\partial}{\partial x_h}\left((n-2)\sum_{t=1}^{n}\frac{\partial u^t}{\partial x_t} + n\sum_{s,t=1}^{n}\frac{x_s x_t}{|x|^2}\frac{\partial u^t}{\partial x_s}\right) = \alpha(n-2)[(n-2)(n+\alpha)+n\alpha]x_h|x|^{\alpha-2}\ . \tag{18.8}$$

In order to complete the substitution in (18.5), since

$$\sum_{k=1}^{n} \frac{\partial}{\partial x_k} \left(x_h x_k |x|^{\alpha-2} \right) = (n+\alpha-1) x_h |x|^{\alpha-2} \ ,$$

we obtain

$$\begin{aligned} n \sum_{k=1}^{n} \frac{\partial}{\partial x_k} \left[\frac{x_h x_k}{|x|^2} \left((n-2) \sum_{t=1}^{n} \frac{\partial u^t}{\partial x_t} + n \sum_{s,t=1}^{n} \frac{x_s x_t}{|x|^2} \frac{\partial u^t}{\partial x_s} \right) \right] \\ = n(n+\alpha-1)[(n-2)(n+\alpha)+n\alpha] x_h |x|^{\alpha-2} \ . \end{aligned} \tag{18.9}$$

Putting together (18.8), (18.9) and (18.7), $u(x) = x|x|^\alpha$ solves the Euler-Lagrange equation if and only if

$$(2n-2)^2 \left(\alpha + \frac{n}{2} \right)^2 + \alpha n + \alpha^2 = 0 \ ,$$

which leads to the choice (18.2) of α.

Let us now perform step (ii), checking first that $u \in H^1$. As $|Du(x)| \sim |x|^\alpha$ and $2\alpha > -n$, it is easy to show that $|Du| \in L^2(B_1)$. Moreover, for every $\varphi \in C_c^\infty(B_1 \setminus \{0\})$ we have classical integration by parts formula

$$\int Du(x)\varphi(x)\, dx = -\int u(x) D\varphi(x)\, dx \ . \tag{18.10}$$

Thanks to Lemma 18.1 below, we are allowed to approximate in $H^1(B_1;\mathbb{R}^n)$ norm every $\varphi \in C_c^\infty(B_1)$ with a sequence $(\varphi_k) \subset C_c^\infty(B_1 \setminus \{0\})$. Then we can pass to the limit in (18.10) because $|Du| \in L^2(B_1)$ to obtain $u \in H^1(B_1;\mathbb{R}^m)$. Now, setting $A(x) = B(x)^* B(x)$ using the fact that the Euler-Lagrange PDE holds in the weak sense in $B_1 \setminus \{0\}$ (because it holds in the classical sense), we have

$$\int_{B_1} A(x) Du(x) D\varphi(x)\, dx = 0 \tag{18.11}$$

for every $\varphi \in C_c^\infty(B_1 \setminus \{0\};\mathbb{R}^n)$. Using Lemma 18.1 again, we can extend (18.11) to every $\varphi \in C_c^\infty(B_1;\mathbb{R}^n)$, thus obtaining the validity of the Euler-Lagrange PDE in the weak sense in the whole ball.

Finally, since the functional L in (18.3) is convex, the Euler-Lagrange equation is satisfied by u if and only if u is a minimizer of $L(u)$ with boundary condition

$$u(x) = x \quad \text{in } \partial B_1 \ .$$

This means that De Giorgi's counterexample holds not only for solution of the Euler-Lagrange equation, but also for minimizers.

Lemma 18.1. *Assume that $n > 2$. For every $\varphi \in C_c^\infty(B_1)$ there exists $\varphi_k \in C_c^\infty(B_1 \setminus \{0\})$ such that φ_k tends to φ strongly in $W^{1,2}(B_1)$.*

Proof. Consider $\psi \in C_c^\infty(\mathbb{R}^n)$ with $\psi \equiv 1$ on $\overline{B}_1$, then rescale ψ setting $\psi_k(x) := \psi(kx)$. Put $\varphi_k := \varphi(1 - \psi_k)$; in L^2 topology we have

$$\varphi - \varphi_k = \varphi\psi_k \longrightarrow \varphi, \qquad (D\varphi)\psi_k \to D\varphi$$

and

$$D(\varphi - \varphi_k) = (D\varphi)\psi_k + \varphi D\psi_k \ .$$

Hence, the thesis is equivalent to verify that

$$\int_{B_1} \varphi(x)^2 |D\psi_k(x)|^2 \, dx \longrightarrow 0 \ ,$$

but

$$\begin{aligned} \int_{B_1} \varphi(x)^2 |D\psi_k(x)|^2 \, dx &\le (\sup \varphi^2) k^2 \int_{B_1} |D\psi(kx)|^2 \, dx \\ &= (\sup \varphi^2) k^{2-n} \int_{\mathbb{R}^n} |D\psi(x)|^2 \, dx \longrightarrow 0 \, , \end{aligned}$$

where we used the fact that $n > 2$. □

We conclude noticing that the restriction $n \ge 3$ in the proof of Lemma 18.1, is not really needed. Indeed, when $n = 2$ we have

$$\mathrm{Cap}_2(\{0\}) := \inf \left\{ \int |D\psi(x)|^2 \, dx \,\middle|\, \psi \in C_c^\infty(\mathbb{R}^2), \ \psi(0) = 1 \right\} = 0 \ , \tag{18.12}$$

that is "points have null 2-dimensional capacity". Let us prove (18.12): considering radial functions $\psi(x) = a(|x|)$ the problem reduces to

$$\inf \left\{ \int_0^1 r |a'(r)|^2 \, dr \,|\, a(0) = 1, \ a(1) = 0 \right\} \ .$$

We can take $a_\gamma(r) := r^\gamma$, so

$$\int_0^1 r |a'_\gamma(r)|^2 \, dr = \frac{\gamma}{2} \xrightarrow{\gamma \to 0} 0 \ .$$

Using (18.12) to remove the point singularity also in the case $n = 2$, it follows that the functional $L(u)$ and its minimizer are a counterexample to Lipschitz regularity.

19 Partial regularity for systems

As we have seen with De Giorgi counterexample, it is impossible to expect an "everywhere" regularity result for elliptic systems: the main idea is to pursue a different goal, a "partial" regularity result, away from a singular set. This strategy goes back to De Giorgi's himself, and implemented for the first time in the study of minimal surfaces.

Definition 19.1 (Regular and singular set)**.** *For a generic function $u : \Omega \to \mathbb{R}$ we call* regular set *of u the set*

$$\Omega_{\mathrm{reg}}(u) := \{x \in \Omega \,|\, \textit{there exists } r > 0 \textit{ such that } B_r(x) \subset \Omega \textit{ and } u \in C^1(B_r(x)\} \ .$$

Analogously, the singular set *is*

$$\Sigma(u) := \Omega \setminus \Omega_{\mathrm{reg}}(u) \ .$$

The set $\Omega_{\mathrm{reg}}(u)$ is obviously the largest open subset A of Ω such that $u \in C^1(A)$. Briefly, let us recall here the main results we have already obtained for elliptic systems.

(a) If we are looking at the problem from the variational point of view, studying local minimizers $u \in H^1_{\mathrm{loc}}$ of $v \mapsto \int_\Omega F(Dv)\,dx$, with $F \in C^2(\mathbb{R}^{m\times n})$, $|D^2F(p)| \le \Lambda$, we have already the validity of the Euler-Lagrange equations. More precisely, if

$$\int_{\Omega'} F(Du(x))\,dx \le \int_{\Omega'} F(Dv(x))\,dx \qquad \forall\, v \text{ s.t. } \{u \ne v\} \Subset \Omega' \Subset \Omega \ ,$$

then

$$\frac{\partial}{\partial x_\alpha}\big(F_{p_i^\alpha}(Du)\big) = 0 \quad \forall\, i = 1,\dots,m \ .$$

(b) If F satisfies a uniform Legendre-Hadamard condition for some $\lambda > 0$, by Nirenberg method we have $Du \in H^1_{\mathrm{loc}}$ and (by differentiation of the (EL) equations with respect to x_γ)

$$\frac{\partial}{\partial x_\alpha}\left(F_{p_i^\alpha p_j^\beta}(Du)\frac{\partial^2 u^j}{\partial x_\alpha \partial x_\gamma}\right) = 0 \quad \forall\, i = 1,\dots,m, \gamma = 1,\dots,n \ . \tag{19.1}$$

Definition 19.2 (Uniform quasiconvexity)**.** *We say that F is λ-uniformly quasiconvex if*

$$\int_\Omega F(A + D\varphi(x)) - F(A)\,dx \ge \lambda \int_\Omega |D\varphi|^2\,dx \quad \forall\, \varphi \in C^\infty_c(\Omega;\mathbb{R}^m) \ .$$

Remark 19.3. Notice that F is λ-uniformly quasiconvex if and only if $F(p) - \lambda|p|^2$ is quasiconvex. In this case, by Remark 19.3, $(D^2F - \lambda I)$ satisfies the Legendre-Hadamard condition with parameter 0 or, equivalently, D^2F satisfies the Legendre-Hadamard condition with parameter λ. So, in this case local minimizers are H^2_{loc} and (19.1) holds.

In this section we shall provide a fairly complete proof of the following result, following with minor variants the original proof in [9].

Theorem 19.4 (Evans). *If $F \in C^2(\mathbb{R}^{m\times n})$ is λ-uniformly quasiconvex with $\lambda > 0$ and satisfies*

$$|D^2F(p)| \le \Lambda \qquad \forall p \in \mathbb{R}^{m\times n}\,, \tag{19.2}$$

then any local minimizer u belongs to $C^{1,\alpha}(\Omega_{\rm reg})$ for some $\alpha = \alpha(n,m,\lambda,\Lambda)$ and

$$\mathscr{L}^n\left(\Omega \setminus \Omega_{\rm reg}\right) = 0\ .$$

The following list summarizes some results in the spirit of Theorem 19.4. At this stage we should point out that the growth condition (19.2) is generally replaced in literature by the more general one

$$|D^2F(p)| \le C_0\left(1 + |p|^{q-2}\right) \qquad \text{with } q \ge 2\,, \tag{19.3}$$

which leads to the estimates $|DF(p)| \le C_1(1+|p|^{q-1})$ and $|F(p)| \le C_2(1+|p|^q)$.

(i) If $D^2F \ge \lambda I$ for some $\lambda > 0$, then Giaquinta and Giusti (see [14] and [16]) proved a much stronger estimate on the size of the singular set, namely (here $\mathscr{H}^k$ denotes the Hausdorff measure, that we will introduce later on)

$$\mathscr{H}^{n-2+\epsilon}\left(\Sigma(u)\right) = 0 \qquad \forall\, \epsilon > 0\ .$$

(ii) If $D^2F \ge \lambda I$ for some $\lambda > 0$ and it is globally uniformly continuous, then we have even $\mathscr{H}^{n-2}\left(\Sigma(u)\right) = 0$.

(iii) If u is locally Lipschitz, then Kristensen and Mingione proved in [20] that there exists $\delta > 0$ such that

$$\mathscr{H}^{n-\delta}\left(\Sigma(u)\right) = 0\ .$$

(iv) On the contrary, when $n = 2$ and $m = 3$, there exists a solution u for the system $\frac{\partial}{\partial x_\alpha}\left(F_{p_i^\alpha}(\nabla u)\right)$, provided in [21], such that

$$\Omega_{\rm reg}(u) = \emptyset\ .$$

This last result clarifies once for all that partial regularity can be expected for (local) minimizers only, and we will see how in the proof of Evans' result local minimality (and not only the Euler-Lagrange equations) plays a role.

We will start with a decay lemma relative to constant coefficient operators.

Lemma 19.5. *There exists a constant $\alpha = \alpha(n, m, \lambda, \Lambda)$ such that, for every constant matrix A satisfying the Legendre-Hadamard condition with λ and the inequality $|A| \leq \Lambda$, any solution $u \in H^1(B_r)$ of*

$$\operatorname{div}(ADu) = 0 \quad \text{in } B_r$$

satisfies

$$\fint_{B_{\alpha r}} |Du(x) - (Du)_{B_{\alpha r}}|^2 \, dx \leq \frac{1}{16} \fint_{B_r} |Du(x) - (Du)_{B_r}|^2 \, dx \; .$$

Proof. As a consequence of what we proved in the section about decay estimates for systems with constant coefficients, considering (5.2) with $\rho = \alpha r$ and $\alpha < 1$, we have that

$$\int_{B_{\alpha r}} |Du(x) - (Du)_{B_{\alpha r}}|^2 \, dx \leq c(\lambda, \Lambda, n, m) \left(\frac{\alpha r}{r}\right)^{n+2} \int_{B_r} |Du(x) - (Du)_{B_r}|^2 \, dx \; . \quad (19.4)$$

It is enough to consider the mean of (19.4), so that

$$\fint_{B_{\alpha r}} |Du(x) - (Du)_{B_{\alpha r}}|^2 \, dx \leq c(\lambda, \Lambda)\alpha^2 \fint_{B_r} |Du(x) - (Du)_{B_r}|^2 \, dx \; ;$$

we conclude choosing α such that $c(\lambda, \Lambda, n, m)\alpha^2 \leq 1/16$. Note that 1/16 could be replaced by an arbitrary positive constant however, 1/16 is already suitable for our purposes. □

Definition 19.6 (Excess)**.** *For any function $u \in H^1_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$ and any ball $B_r(x) \Subset \Omega$ the* excess *of u in $B_r(x)$ is defined by*

$$\operatorname{Exc}(u, B_\rho(x)) := \left(\fint_{B_\rho(x)} |Du(y) - (Du)_{B_\rho(x)}|^2 \, dy \right)^{1/2} \; .$$

When we consider functions F satisfying the more general growth condition (19.3), then we should modify the definition of excess as follows, see [9]:

$$\operatorname{Exc}(u, B_\rho(x))^2 = \fint_{B_\rho(x)} \left(1 + |Du(y) - (Du)_{B_\rho}(x)|^{q-2}\right) |Du(y) - (Du)_{B_\rho(x)}|^2 \; dy.$$

However, in our presentation we will cover only the case $q = 2$.

Remark 19.7 (Properties of the excess)**.** We list here the basic properties of the excess as defined above, all trivial to check.

(i) Any additive perturbation by an affine function $p(x)$ does not change the excess, that is

$$\operatorname{Exc}(u + p(x), B_\rho(x)) = \operatorname{Exc}(x, B_\rho(x)) \; .$$

(ii) The excess is positively 1-homogeneous, that is for any number $\lambda \geq 0$

$$\operatorname{Exc}(\lambda u, B_\rho(x)) = \lambda \operatorname{Exc}(u, B_\rho(x)) \ .$$

(iii) We have the following scaling property:

$$\operatorname{Exc}\left(\frac{u(\rho x + x_0)}{\rho}, B_1(0)\right) = \operatorname{Exc}(u, B_\rho(x_0)) \ .$$

Remark 19.8. The name "excess" is inspired by De Giorgi's theory of minimal surfaces, presented in [5] and [6], see also [13] for a modern presentation. The excess of a set E in a point is defined (for regular sets) by

$$\operatorname{Exc}(E, B_\rho(x)) := \fint_{B_\rho(x)\cap\partial E} |\nu_E(y) - \nu_E(x)|^2 \, d\sigma_{n-1}(y) \ ,$$

where ν_E is the inner normal of the set E. The correspondence between $\operatorname{Exc}(u, B_\rho(x))$ and $\operatorname{Exc}(E, B_\rho(x))$ can be made more evident ∂E as the graph associated to the function u, in a coordinate system where $\nabla u(x) = 0$.

The main ingredient in the proof of Evans' theorem will be the decay property of the excess: there exists a critical treshold such that, if the decay in the ball is below the treshold, then decay occurs in the smaller balls.

Theorem 19.9 (Excess decay)**.** *Let F be as in Theorem 19.4 and let $\alpha \in (0,1)$ be given by* (19.5)*. For every $M \geq 0$ there exists $\epsilon_0 = \epsilon_0(n, m, \lambda, \Lambda, M) > 0$ satisfying the following implication: if*

(a) $u \in H^1(B_r(x); \mathbb{R}^m)$ *is a local minimizer in* $B_r(x)$ *of* $v \mapsto \int F(Dv)\, dx$*,*

(b) $|(Du)_{B_r(x)}| \leq M$*,*

(c) $\operatorname{Exc}(u, B_r(x)) < \epsilon_0$*,*

then

$$\operatorname{Exc}(u, B_{\alpha r}(x)) \leq \frac{1}{2} \operatorname{Exc}(u, B_r(x)) \ .$$

In the case when D^2F is uniformly continuous, condition (b) is not needed for the validity of the implication and ϵ_0 is independent of M.

Proof. The proof is by contradiction: in step (ii) we will normalize the excesses, obtaining functions w_k with $\operatorname{Exc}(w_k, B_\alpha(0)) \geq 1/2$ while $\operatorname{Exc}(w_k, B_1(0)) = 1$. Each w_k is a solution of

$$\operatorname{div}\left(F_{p_i^\alpha}(Dw_k)\right) = 0 \ .$$

We will see in step (iii) that, passing through the limit as $k \to \infty$, the strong H^1_{loc}-limit w_∞ solves

$$\operatorname{div}\left(F_{p_i^\alpha p_j^\beta}(p_\infty) D w_\infty\right) = 0$$

with $\operatorname{Exc}(w_\infty, B_\alpha(0)) \geq 1/2$ and $\operatorname{Exc}(w_\infty, B_1(0)) \leq 1$: using Lemma 19.5, in step (iv) we will reach the contradiction.

For the sake of simplicity, we will prove the result under the additional assumption that

$$F(p) \geq \epsilon |p|^2 \tag{19.5}$$

for some $\epsilon > 0$.

(i) By contradiction, we have $M \geq 0$ and local minimizers $u_k : \Omega \to \mathbb{R}^m$ in $B_{r_k}(x_k)$ with

$$\epsilon_k := \operatorname{Exc}(u_k, B_{r_k}(x_k)) \longrightarrow 0$$

satisfying

$$\left|(Du_k)_{B_{r_k}(x_k)}\right| \leq M \tag{19.6}$$

but

$$\operatorname{Exc}(u_k, B_{\alpha r_k}(x_k)) > \frac{1}{2} \operatorname{Exc}(u_k, B_{r_k}(x_k)) \qquad \forall\, k \in \mathbb{N}\ .$$

(ii) Suitably rescaling and translating the functions u_k, we can assume that $x_k = 0$, $r_k = 1$ and $(u_k)_{B_1} = 0$ for all k. Setting $p_k := (Du_k)_{B_1}$, the hypothesis (19.6) gives, up to subsequences,

$$p_k \longrightarrow p_\infty \in \mathbb{R}^{m\times n}\ . \tag{19.7}$$

We start here a parallel and simpler path through this proof, in the case when D^2F is uniformly continuous: in this case no uniform bound on p_k is needed and we can replace (19.7) with

$$D^2F(p_k) \to A_\infty \in \mathbb{R}^{m^2\times n^2}\ . \tag{19.8}$$

Notice that (19.8) holds under (19.7), simply with $A_\infty = D^2F(p_\infty)$. Notice also that, in any case, A_∞ satisfies a (LH) condition with constant λ and $|A_\infty| \leq \Lambda$.

We do a second translation in order to annihilate the mean of the gradients of minimizers: let us define

$$v_k(x) := u_k(x) - p_k(x)\ ,$$

so that $(Dv_k)_{B_1} = 0$. According to property (i) of Remark 19.7 the excess does not change, so still

$$\operatorname{Exc}(v_k, B_1(0)) = \epsilon_k \longrightarrow 0$$

and

$$\operatorname{Exc}(v_k, B_\alpha(0)) > \frac{1}{2}\,\epsilon_k\ .$$

During these operations, we need not to lose sight of the variational problem we are solving, for example every function v_k minimizes the integral functional associated to

$$p \mapsto F(p+p_k) - F(p_k) - DF(p_k)p \ .$$

In order to get some contradiction, our aim is to find a "limit problem" with some decaying property. Let us define

$$w_k := \frac{v_k}{\epsilon_k} \qquad k \in \mathbb{N}.$$

It is trivial to check that $(w_k)_{B_1} = (Dw_k)_{B_1} = 0$, moreover

$$\mathrm{Exc}\,(w_k, B_1(0)) = 1 \qquad \text{and} \qquad \mathrm{Exc}\,(w_k, B_\alpha(0)) > \frac{1}{2} \ . \tag{19.9}$$

The key point of the proof is that w_k is a local minimizer of $v \mapsto \int F_k(Dv)\,dx$, where

$$F_k(p) := \frac{1}{\epsilon_k^2}\left[F(\epsilon_k p + p_k) - F(p_k) - DF(p_k)\epsilon_k p\right] \ .$$

(iii) We now study both the limit of F_k and the limit of w_k, as $k \to \infty$. Since $F_k \in C^2(\mathbb{R}^{m\times n})$, by Taylor expansion we are able to identify a limit Lagrangian, given by

$$F_\infty(p) = \frac{1}{2}\langle A_\infty p, p\rangle \ ,$$

to which $F_k(p)$ converge uniformly on compact subsets of $\mathbb{R}^{m\times n}$. Indeed, this is clear with $A_\infty = D^2F(p_\infty)$ in the case when $p_k \to p_\infty$; it is still true with A_∞ given by (19.8) when D^2F is uniformly continuous.

Once we have the limit problem defined by F_∞, we drive our attention to w_k: it is a bounded sequence in $H^{1,2}(B_1;\mathbb{R}^m)$ because the excesses are constant, so by Rellich theorem we have that (possibly extracting one more subsequence)

$$w_k \longrightarrow w_\infty \qquad \text{in } L^2(B_1;\mathbb{R}^m)$$

and, as a consequence,

$$Dw_k \rightharpoonup Dw_\infty \quad \text{in } L^2(B_1;\mathbb{R}^m) \ . \tag{19.10}$$

The analysis of the limit problem now requires the verification that w_∞ solves the Euler equation associated to F_∞. We need just to pass to the limit in the (EL) equation satisfied by w_k, namely

$$\int_{B_1} \langle D^2F(p_k + \epsilon_k Dw_k(x))Dw_k(x), D\varphi(x)\rangle\,dx = 0 \qquad \forall\,\varphi \in C_c^\infty(B_1;\mathbb{R}^m) \ .$$

Adding and subtracting $\langle D^2F(p_k)Dw_k, D\varphi\rangle$ and using the fact that $D^2F(p_k) \to A_\infty$ we obtain

$$\int_{B_1} \langle A_\infty Dw_\infty(x), D\varphi(x)\rangle\, dx = 0 \qquad \forall\, \varphi \in C_c^\infty(B_1;\mathbb{R}^m)\ , \tag{19.11}$$

provided we show that

$$\lim_{k\to\infty} \int_{B_1} |D^2F(p_k + \epsilon_k Dw_k) - A_\infty||Dw_k|\, dx = 0.$$

This can be obtained splitting the integral into the regions $\{|Dw_k| \le L\}$ and $\{|Dw_k| > L\}$, with L fixed. The first contribution goes to zero, thanks to the convergence of p_k to p_∞ or, when p_k is possibly unbounded, from the uniform continuity of D^2F. The second contribution tends to 0 as $L \uparrow \infty$ uniformly in k, since $|D^2F| \le \Lambda$ and $\|Dw_k\|_2 \le 1$.
(iv) Equality (19.11) means that

$$\mathrm{div}\,(A_\infty Dw_\infty) = 0$$

in a weak sense: since the equation has constant coefficients we can apply Lemma 19.5 to get

$$\fint_{B_\alpha} |Dw_\infty(x) - (Dw_\infty)_{B_\alpha}|^2\, dx \le \frac{1}{16} \fint_{B_1} |Dw_\infty(x)|^2\, dx\ . \tag{19.12}$$

Suppose we know how to improve (19.10) to a strong local convergence:

$$Dw_k \longrightarrow Dw_\infty \quad \mathit{in}\ L^2_{\mathrm{loc}}(B_1;\mathbb{R}^{m\times n})\ . \tag{19.13}$$

Since the excess is sequentially weakly lower semicontinuous we can use (19.9) and (19.13) to get

$$\mathrm{Exc}\,(w_\infty, B_\alpha(0)) \ge \frac{1}{2} \quad \text{and} \quad \mathrm{Exc}\,(w_\infty, B_1(0)) \le 1\ . \tag{19.14}$$

On the other hand (19.12) yields

$$\mathrm{Exc}\,(w_\infty, B_\alpha(0)) \le \frac{1}{4}\mathrm{Exc}\,(w_\infty, B_1(0)) \le \frac{1}{4},$$

so that we achieve the contradiction.

We stop here the proof, even if we didn't obtain yet the strong convergence property (19.13): this is a gain due to the variational character of the problem and, what remains of this section until Lemma 19.15, will be devoted to prove it. Notice that the strong convergence property will follow from the fact that w_k is a minimizer of a variational problem, not only the solution to a system of elliptic PDE: the counterexamples due to Müller and Šverak we already mentioned (see [21]) show that the decay of the excess does not hold for solutions, and therefore local strong convergence has to fail. $\square$

We will deal separately with two aspects of the convergence problem (19.13):

(1) on the one hand we will deduce the local *equi-integrability* of $|Dw_k|^2$ via quasiminima theory;

(2) on the other hand we will prove in Proposition 19.13 a kind of variational convergence of $\int F_k(Dv)\,dx$ to $\int F_\infty(v)\,dx$ which is the key ingredient for strong convergence of gradients.

We will put together these two results in Proposition 19.13 and then we will conclude in Lemma 19.15.

It will be useful for our purposes to explain the concept of quasiminimum, first introduced in the context of multiple integrals of calculus of variations by Giaquinta and Giusti in [14] and then developed in [15]. In a geometric context, similar ideas about quasiminima have been developed in [2].

Definition 19.10. *Let $Q \geq 1$. A function $u \in H^{1,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$ is a Q-quasiminimum of an integral functional $v \mapsto \int F(x, v(x), Dv(x))\,dx$ if*

$$\int_{B_r(x_0)} F(x, u(x), Du(x))\,dx \leq Q \int_{B_r(x_0)} F(x, u(x) + \varphi(x), Du(x) + D\varphi(x))\,dx$$

for all balls $B_r(x_0) \Subset \Omega$ and all $\varphi \in C^\infty_c(B_r(x_0);\mathbb{R}^m)$.

Our extra assumption (19.5) allows to read local minimizers of $v \mapsto \int F(Dv)$ as Q-quasiminima of the Dirichlet energy. The proof follows by a simple comparison argument.

Theorem 19.11. *Let F be satisfying $\epsilon|p|^2 \leq F(x,s,p) \leq \Lambda|p|^2$ with $0 < \epsilon \leq \Lambda$. If u is a local minimizer of $v \mapsto \int F(x, v, Dv)\,dx$, then u is a Q-quasiminimum of the Dirichlet integral*

$$v \mapsto \int_\Omega |Dv(x)|^2\,dx \tag{19.15}$$

with $Q = \Lambda/\epsilon$.

Some weak regularity property of quasiminima follows from the following proposition, whose proof can to be found in [15].

Proposition 19.12 (Higher integrability)**.** *If $u \in H^1_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$ is a Q-minimum of $v \mapsto \int |Dv|^2\,dx$, then there exist $q = q(n,m,Q) > 2$ and $C = C(n,m,Q)$ such that, for all balls $B_{2r}(x_0) \Subset \Omega$, it holds*

$$\left(\fint_{B_r(x_0)} |Du(x)|^q\,dx\right)^{1/q} \leq C \left(\fint_{B_{2r}(x_0)} |Du(x)|^2\,dx\right)^{1/2}.$$

Thanks to (19.5), we are able to apply Theorem 19.11 with $F = F_k$, which satisfy the assumption of the theorem with constants uniform in k, and then Proposition 19.12 to the sequence w_k studied in the proof of Theorem 19.9, for some Q independent of k. We obtain that

$$\sup_k \int_{B_\tau} |Dw_k(x)|^q \, dx < +\infty \qquad \forall \tau \in (0,1)$$

and, in particular, that $|Dw_k|^2$ is locally equi-integrable.

On the other hand, we prove a sort of local convergence of energies, in analogy with the techniques of Γ-convergence. Let us introduce a convenient notation: we set

$$\mathcal{F}_k(v, B_r) := \int_{B_r} F_k(Dv(x)) \, dx, \qquad \mathcal{F}_\infty(v, B_r) := \int_{B_r} F_\infty(Dv(x)) \, dx$$

with $F_\infty(p) = \frac{1}{2}\langle A_\infty p, p\rangle$, and

$$\mu_k := |Dw_k|^2 \chi_{B_1} \mathscr{L}^n \ .$$

Note that, up to subsequences, $\mu_k \rightharpoonup \mu_\infty$ (we consider the space of measures in the duality with $C^0_c(B_1)$). We recall classical and easy to prove semicontinuity properties of weak convergence of measures:

- for every open subset $A \subset B_1$ we have lower semicontinuity, i.e.

$$\liminf_{k \to \infty} \mu_k(A) \geq \mu_\infty(A) \ ;$$

- for every compact subset $C \subset B_1$ we have upper semicontinuity, i.e.

$$\limsup_{k \to \infty} \mu_k(C) \leq \mu_\infty(C) \ .$$

By lower semicontinuity, we have $\mu_\infty(B_\tau) \leq 1$. We shall also use the fact that the set $\{\tau \in (0,1) : \mu_\infty(\partial B_\tau) > 0\}$ is at most countable (because the sets are pairwise disjoint and the total mass is less than 1).

Proposition 19.13. *With the previous notation and the one used in proof of Theorem 19.9, we have:*

$$\limsup_{k \to \infty} \mathcal{F}_k(w_k, B_\tau) \leq \mathcal{F}_\infty(w_\infty, B_\tau) \tag{19.16}$$

for all $\tau \in (0,1)$ such that $\mu_\infty(\partial B_\tau) = 0$. In addition, (19.16) *implies,*

$$\limsup_{k \to \infty} \mathcal{F}_\infty(w_k, B_\tau) \leq \mathcal{F}_\infty(w_\infty, B_\tau) \ . \tag{19.17}$$

Proof. Fix $\tau \in (0,1)$ with $\mu_\infty(\partial B_\tau) = 0$, and fix τ_1, τ_2 such that $\tau_1 < \tau < \tau_2$. Fix also a function $\varphi \in C_c^\infty(B_{\tau_2})$ with $0 \le \varphi \le 1$ and $\varphi \equiv 1$ on $\overline{B}_{\tau_1}$. We start with the quasiconvexity property and positivity of F, so that

$$\mathcal{F}_k(w_k, B_\tau) \le \mathcal{F}_k(w_k, B_{\tau_2}) \le \mathcal{F}_k\left((1-\varphi)w_k + \varphi w_\infty, B_{\tau_2}\right) . \tag{19.18}$$

Carrying on with (19.18), we can split the limsup we are interested in (i.e., (19.16)) as a sum

$$\begin{aligned} \limsup_{k\to\infty} \mathcal{F}_k(w_k, B_\tau) \quad &\le \quad \limsup_{k\to\infty} \int_{B_{\tau_1}} F_k(Dw_\infty(x))\, dx &(19.19)\\ &+ \quad \limsup_{k\to\infty} \int_{B_{\tau_2}\setminus B_{\tau_1}} F_k\left(D\left((1-\varphi)w_k + \varphi w_\infty\right)(x)\right)\, dx\ . &(19.20) \end{aligned}$$

Since $\limsup_k \int_{B_{\tau_1}} F_k(Dw_\infty(x))\, dx = \mathcal{F}_\infty(w_\infty, B_{\tau_1}) \le \mathcal{F}_\infty(w_\infty, B_\tau)$, we can put aside (19.19) and focus on (19.20). Note that we can write

$$D\left((1-\varphi)w_k + \varphi w_\infty\right)(x) = D\varphi(x)(w_\infty(x) - w_k(x)) + (1-\varphi(x))Dw_k(x) + \varphi(x)Dw_\infty(x)\ ,$$

so that

$$\begin{aligned} &F_k\left(D\left((1-\varphi)w_k + \varphi w_\infty\right)(x)\right)\\ \le\ & C\left(1 + |D\left((1-\varphi)w_k + \varphi w_\infty\right)(x)|^2\right)\\ \le\ & C\left(1 + 2|D\varphi(x)|^2|w_k(x) - w_\infty(x)|^2 + 2|(1-\varphi(x))Dw_k(x) + \varphi(x)Dw_\infty(x)|^2\right)\\ \le\ & C\left(1 + 2|D\varphi(x)|^2|w_k(x) - w_\infty(x)|^2 + 4|Dw_k(x)|^2 + 4|Dw_\infty(x)|^2\right) \end{aligned}$$

and finally

$$\begin{aligned} &\limsup_{k\to\infty} \int_{B_{\tau_2}\setminus B_{\tau_1}} F_k\left(D\left((1-\varphi)w_k + \varphi w_\infty\right)(x)\right)\, dx\\ &\qquad\qquad \le C\int_{B_{\tau_1}\setminus B_{\tau_2}} \left(1 + 4|Dw_\infty(x)|^2\right)\, dx + 4C\mu_\infty\left(\overline{B}_{\tau_2} \setminus B_{\tau_1}\right)\ . \end{aligned}$$

Letting $\tau_1 \uparrow \tau$ and $\tau_2 \downarrow \tau$ we get $\mu_\infty\left(\overline{B}_{\tau_2} \setminus B_{\tau_1}\right) \downarrow \mu_\infty(\partial B_\tau) = 0$, and this concludes the proof of (19.16).

Now we prove (19.17). Since we already proved (19.16), it is sufficient to point out that, fixing $M > 0$,

$$\begin{aligned} \int_{B_\tau} |F_k(Dw_k(x)) - F_\infty(Dw_k(x))|\, dx \le &\int_{B_\tau\cap\{|Dw_k|\le M\}} |F_k(Dw_k(x)) - F_\infty(Dw_k(x))|\, dx &(19.21)\\ + &\int_{B_\tau\cap\{|Dw_k|> M\}} |F_k(Dw_k(x)) - F_\infty(Dw_k(x))|\, dx &(19.22) \end{aligned}$$

In fact, the first term (19.21) tends to zero as $k \to \infty$ when M is fixed and the second term (19.22) is arbitrarily small, uniformly in k, when $M >> 1$, using the local equi-integrability of $|Dw_k|^2$ in B_1. $\square$

Finally, we are able to complete the proof of Theorem 19.9 with Lemma 19.15. The following is just an elementary fact from real analysis that will be useful in Lemma 19.15.

Lemma 19.14. *Consider real sequences, (a_k) and (b_k), satisfying*

- $\liminf_{k\to\infty} a_k \geq a \in \mathbb{R}$;
- $\liminf_{k\to\infty} b_k \geq b \in \mathbb{R}$;
- $\limsup_{k\to\infty}(a_k + b_k) \leq a + b$;

then $a_k \longrightarrow a$ and $b_k \longrightarrow b$.

Lemma 19.15. *If a sequence (w_k) weakly converging to w_∞ in $H^1(B_\tau;\mathbb{R}^m)$ satisfies*

$$\limsup_{k\to\infty} \int_{B_\tau} \langle A_\infty Dw_k(x), Dw_k(x)\rangle\, dx \leq \int_{B_\tau} \langle A_\infty Dw_\infty(x), Dw\infty(x)\rangle\, dx \ , \tag{19.23}$$

then $Dw_k \longrightarrow Dw_\infty$ strongly in $L^2_{\mathrm{loc}}(B_\tau;\mathbb{R}^{n\times m})$.

Proof. The proof will be reduced to the classical statement that weak convergence in L^2 and convergence of L^2 norms implies weak convergence using the Fourier transform. Firstly we remark that

$$\Phi(u) := \int_{B_\tau} \langle A_\infty Du(x), Du(x)\rangle\, dx \qquad u \in H^1_0(B_\tau;\mathbb{R}^m)$$

is sequentially lower semicontinuous with respect to the weak H^1-convergence. In fact $\Phi \geq 0$ because of Gårding inequality, hence the quadratic form associated to Φ satisfies the Cauchy-Schwarz inequality and we can represent ϕ as follows:

$$\sqrt{\Phi(u)} = \sup\left\{ \int_{B_\tau} \langle A_\infty Du(x), Dv(x)\rangle\, dx \,\middle|\, v \in H^1_0(B_\tau;\mathbb{R}^m),\ \sqrt{\Phi(v)} \leq 1 \right\} \ . \tag{19.24}$$

When v is fixed, the application $u \mapsto \int ADuDv$ is continuous in the weak H^1-topology, then Φ is sequentially lower semicontinuous with respect to the weak H^1-convergence according to (19.24).
For every $\eta \in C^\infty_c(B_\tau)$, the lower semicontinuity of Φ means that

$$\liminf_{k\to\infty} \int_{B_\tau} \langle A_\infty D(w_k\eta)(x), D(w_k\eta)(x)\rangle\, dx \geq \int_{B_\tau} \langle A_\infty D(w_\infty\eta), D(w_\infty\eta)\rangle\, dx \ ,$$

so, cutting the terms which are automatically continuous,

$$\liminf_{k\to\infty} \int_{B_\tau} \langle A_\infty Dw_k(x), Dw_k(x)\rangle\eta^2(x)\, dx \geq \int_{B_\tau} \langle A_\infty Dw_\infty(x), Dw_\infty(x)\rangle\eta^2(x)\, dx \ .$$

By monotone approximation from below, the same property holds when η^2 is replaced by a nonnegative and lower semicontinuous function χ.

Call

$$a_k := \int_{B_\rho} \langle A_\infty Dw_k(x), Dw_k(x)\rangle \, dx, \qquad b_k := \int_{B_\tau \setminus \overline{B}_\rho} \langle A_\infty Dw_k(x), Dw_k(x)\rangle \, dx$$

and apply Lemma 19.14 and (19.23) to obtain that for every $\rho \in (0, \tau)$

$$\lim_{k\to\infty} \int_{B_\rho} \langle A_\infty Dw_k(x), Dw_k(x)\rangle \, dx = \int_{B_\rho} \langle A_\infty Dw_\infty(x), Dw_\infty(x)\rangle \, dx \ ,$$

and therefore for every $\eta \in C_c^\infty(B_\tau)$ radial we have

$$\lim_{k\to\infty} \int_{B_\tau} \langle A_\infty D(w_k\eta)(x), D(w_k\eta)(x)\rangle \, dx = \int_{B_\tau} \langle A_\infty D(w_\infty\eta)(x), D(w_\infty\eta)(x)\rangle \, dx \ .$$

Via Fourier transform, this means that

$$\sum_{\alpha=1}^{n} \|\xi_\alpha \widehat{w_k\eta}\|_{L^2}^2 \longrightarrow \sum_{\alpha=1}^{n} \|\xi_\alpha \widehat{w_\infty\eta}\|_{L^2}^2 \ ,$$

therefore a repeated application of Lemma 19.14 gives (since there is weak convergence of the Fourier transforms as well, and therefore lower semicontinuity) $\|\xi_\alpha \widehat{w_k\eta}\|_{L^2}^2 \to \|\xi_\alpha \widehat{w_\infty\eta}\|_{L^2}^2$ for all $\alpha \in \{1, \ldots, n\}$. It follows that for all α

$$\xi_\alpha \widehat{w_k\eta} \longrightarrow \xi_\alpha \widehat{w_\infty\eta}$$

strongly in L^2. Turning back with the Fourier transform we achieved the thesis because for every $\eta \in C_c^\infty(B_\tau)$ we have strong H^1-convergence $w_k\eta \longrightarrow w_\infty\eta$. $\square$

19.1 The first partial regularity result for systems: $\mathscr{L}^n(\Sigma(u)) = 0$

After proving Theorem 19.9 about the decay of the excess, we will see how it can be used to prove partial regularity for systems.

We briefly recall that $\Omega_{\mathrm{reg}}(u)$ denotes the largest open set contained in Ω where $u : \Omega \to \mathbb{R}^m$ admits a C^1 representative, while $\Sigma(u) := \Omega \setminus \Omega_{\mathrm{reg}}(u)$. Our aim is to show that for a solution of an elliptic system

- $\mathscr{L}^n(\Sigma(u)) = 0$;
- $\mathscr{H}^{n-2+\epsilon}(\Sigma(u)) = 0$ in the uniformly convex case and $\mathscr{H}^{n-2}(\Sigma(u)) = 0$ if D^2F is also uniformly continuous.

In order to exploit Theorem 19.9 and prove that $\mathscr{L}^n(\Sigma(u)) = 0$, fix $M \geq 0$, so that there is an associated $\epsilon_0 = \epsilon_0(n, m, \lambda, \Lambda, M)$ for which the decay property of the excess applies. Recall also that the constant α in the decay theorem depends only on the space dimensions and the ellipticity constants.

Definition 19.16. *We will call*

$$\Omega_M(u) := \left\{ x \in \Omega \,\middle|\, \exists \overline{B}_r(x) \subset \Omega \text{ with } \left|(Du)_{B_r(x)}\right| < M_1 \text{ and } \mathrm{Exc}\,(u, B_r(x)) < \epsilon_1 \right\}$$

where

$$M_1 := M/2 \tag{19.25}$$

and ϵ_1 verifies

$$2^{n/2}\epsilon_1 \leq \epsilon_0 \tag{19.26}$$

and, for the α given in Theorem 19.9,

$$(2^{n+1} + \alpha^{-n}2^{2+n/2})\epsilon_1 \leq M\ . \tag{19.27}$$

Remark 19.17. The set $\Omega_M(u) \subset \Omega$ of Definition 19.16 is easily seen to be open, since the inequalities are strict. Moreover, by Lebesgue approximate continuity theorem (that is, if $f \in L^p(\Omega)$, then for $\mathscr{L}^n$-almost every x one has $⨍_{B_r(x)} |f(y) - f(x)|^p\, dy \to 0$ as $r \downarrow 0$), it is easy to see that

$$\mathscr{L}^n(\{|Du| < M_1\} \setminus \Omega_M(u)) = 0\ . \tag{19.28}$$

Finally, using (19.28), we realize that

$$\mathscr{L}^n\left(\Omega \setminus \bigcup_{M\in\mathbb{N}} \Omega_M(u)\right) \leq \mathscr{L}^n\left(\Omega \setminus \bigcup_{M\in\mathbb{N}} \{|Du| < M_1\}\right) = 0\ . \tag{19.29}$$

By the previous remark, if we are able to prove that

$$\Omega_M(u) \subset \Omega_{\text{reg}} \qquad \forall M > 0, \tag{19.30}$$

we obtain $\mathscr{L}^n(\Sigma(u)) = 0$. So, the rest of this section will be devoted to the proof of the inclusion above, with M fixed.

Fix $x \in \Omega_M(u)$, according to Definition 19.16 there exists $r > 0$ such that $|(Du)_{B_r(x)}| < M_1$ and $\mathrm{Exc}\,(u, B_r(x)) < \epsilon_1$. We will prove that

$$B_{r/2}(x) \subset \Omega_{\text{reg}}(u) \ ,$$

so let us fix $y \in B_{r/2}(x)$.
(1). Thanks to our choice of ϵ_1 (see property (19.26) of Definition 19.16) we have

$$\begin{aligned}
\mathrm{Exc}\left(u, B_{r/2}(y)\right) &= \left(\fint_{B_{r/2}(y)} |Du(\xi) - (Du)_{B_{r/2}(y)}|^2 \, d\xi\right)^{1/2} \\
&\leq \left(\fint_{B_{r/2}(y)} |Du(\xi) - (Du)_{B_r(x)}|^2 \, d\xi\right)^{1/2} \\
&\leq 2^{n/2}\left(\fint_{B_r(x)} |Du(\xi) - (Du)_{B_r(x)}|^2 \, d\xi\right)^{1/2} = 2^{n/2}\mathrm{Exc}\,(u, B_r(x)) < \epsilon_0
\end{aligned}$$

so, momentarily ignoring the hypothesis that $\left|(Du)_{B_{r/2}(y)}\right|$ should be bounded by M (we are postponing this to point (ii) of this proof), Theorem 19.9 gives *tout court*

$$\mathrm{Exc}\left(u, B_{\alpha r/2}(y)\right) \leq \frac{1}{2}\mathrm{Exc}\left(u, B_{r/2}(y)\right) < \frac{1}{2}\epsilon_0 \ ,$$

thus, just iterating Theorem 19.9, we get

$$\mathrm{Exc}\left(u, B_{\alpha^k r/2}(y)\right) \leq 2^{-k}\mathrm{Exc}\left(u, B_{r/2}(y)\right) < 2^{n/2-k}\epsilon_1 \ . \tag{19.31}$$

As we have often seen through these notes, we can apply an interpolation argument to radii in the first inequality of (19.31) and then we obtain

$$\mathrm{Exc}\,(u, B_\rho(y)) \leq 2^\mu \left(\frac{\rho}{r/2}\right)^\mu \mathrm{Exc}\,(u, B_r(y)) \qquad \forall \rho \in (0, r/2]$$

with $\mu = (\log_2(1/\alpha))^{-1}$. We conclude that the components of Du belongs to the Campanato space $\mathcal{L}^{2,n+2\mu}(B_{r/2}(x))$ and then u belongs to $C^{1,\mu}(B_{r/2}(x))$.
(2). Now that we have explained how the proof runs through the iterative application of Theorem 19.9, we deal with the neglected hypothesis, that is $|(Du)_{B_{r/2}(y)}| < M$ and, at

each subsequent step, $|(Du)_{B_{\alpha^k r/2}(y)}| < M$. Remember that in part (i) of this proof we never used (19.25) and (19.27).
Since $x \in \Omega_M$ and r fulfills Definition 19.16, for the first step it is sufficient to use the triangular inequality in (19.32) and the Hölder inequality in (19.33): in fact we can estimate

$$\begin{aligned}
\left|(Du)_{B_{r/2}(y)}\right| &= \left|\fint_{B_{r/2}(y)} Du(\xi) - (Du)_{B_r(x)}\, d\xi + (Du)_{B_r(x)}\right| \\
&\leq \fint_{B_{r/2}(y)} \left|Du(\xi) - (Du)_{B_r(x)}\right|\, d\xi + \left|(Du)_{B_r(x)}\right| \qquad (19.32) \\
&\leq \left(\frac{2^n}{\omega_n r^n}\int_{B_r(x)} \left|Du(\xi) - (Du)_{B_r(x)}\right|\, d\xi\right) + \left|(Du)_{B_r(x)}\right| \\
&\leq 2^n \left(\fint_{B_r(x)} \left|Du(\xi) - (Du)_{B_r(x)}\right|^2\, d\xi\right)^{1/2} + \left|Du_{B_r(x)}\right| \qquad (19.33) \\
&\leq 2^n \mathrm{Exc}\,(u, B_r(x)) + \left|Du_{B_r(x)}\right| < 2^n\epsilon_1 + M_1 < M\ . \qquad (19.34)
\end{aligned}$$

We now show inductively that for every integer $k \geq 1$

$$\left|(Du)_{B_{\alpha^k r/2}(y)}\right| < M_1 + 2^n\epsilon_1 + \alpha^{-n}\epsilon_1 2^{n/2}\sum_{j=0}^{k-1} 2^{-j}\ . \qquad (19.35)$$

If we recall (19.25) and (19.27), it is clear that (19.35) implies

$$\left|(Du)_{B_{\alpha^k r/2}(y)}\right| < M$$

for every $k \geq 1$.
The first step ($k = 1$) follows from (19.34), because, estimating as in (19.32) and (19.33), we immediately get

$$\begin{aligned}
\left|(Du)_{B_{\alpha r/2}(y)}\right| &\leq \fint_{B_{\alpha r/2}(y)} \left|Du(\xi) - (Du)_{B_{r/2}(y)}\right|\, d\xi + \left|(Du)_{B_{r/2}(y)}\right| \\
&\leq \alpha^{-n}\mathrm{Exc}\left(u, B_{r/2}(y)\right) + \left|(Du)_{B_{r/2}(y)}\right| \\
&\leq \alpha^{-n}2^{n/2}\epsilon_1 + 2^n\epsilon_1 + M_1\ .
\end{aligned}$$

Being the first step already proved, we fix our attention on the $(k+1)^{\text{th}}$ step. Without the same procedure, we estimate again

$$\begin{aligned}\left|(Du)_{B_{\alpha^{k+1}r/2}(y)}\right| &\leq \fint_{B_{\alpha^{k+1}r/2}(y)} \left|Du(\xi)-(Du)_{B_{\alpha^k r/2}(y)}\right| d\xi + \left|(Du)_{B_{\alpha^k r/2}(y)}\right| \\ &\leq \alpha^{-n}\mathrm{Exc}\left(u, B_{\alpha^k r/2}(y)\right) + \left|(Du)_{B_{\alpha^k r/2}(y)}\right| \\ &\leq \alpha^{-n}2^{n/2-k}\epsilon_1 + M_1 + 2^n\epsilon_1 + \alpha^{-n}\epsilon_1 2^{n/2}\sum_{j=0}^{k-1} 2^{-j} \qquad (19.36)\end{aligned}$$

where (19.36) is obtained joining the estimate on the excess (19.31) with the inductive hypothesis (19.35).

In order to carry out our second goal, namely to prove that

$$\mathscr{H}^{n-2+\epsilon}(\Sigma(u)) = 0 \qquad \forall\, \epsilon > 0 \ ,$$

we need some additional results concerning Hausdorff measures.

19.2 Hausdorff measures

Definition 19.18. *Consider a subset $B \subset \mathbb{R}^n$, $k \geq 0$ and fix $\delta \in (0, \infty]$. We set*

$$\mathscr{H}^k_\delta(B) := c_k \inf\left\{\sum_{i=1}^{\infty}[\mathrm{diam}(B_i)]^k \middle| B \subset \bigcup_{i=1}^{\infty} B_i,\ \mathrm{diam}(B_i) < \delta\right\}$$

and

$$\mathscr{H}^k(B) := \lim_{\delta\to 0}\mathscr{H}^k_\delta(B) \ , \qquad (19.37)$$

the limit in (19.37) *being well defined because $\delta \mapsto \mathscr{H}^k_\delta(B)$ is nonincreasing. The constant $c_k \in [0, +\infty)$ will be conveniently fixed in Remark 19.20.*

It is easy to check that $\mathscr{H}^k$ is the counting measure when $k = 0$ (provided $c_0 = 1$) and $\mathscr{H}^k$ is identically 0 when $k > n$.

The *spherical* Hausdorff measure $\mathscr{S}^k$ has a definition analogous to Definition 19.18, but only covers made with balls are allowed, so that

$$\mathscr{H}^k_\delta \leq \mathscr{S}^k_\delta \leq 2^k \mathscr{H}^k_\delta \ . \qquad (19.38)$$

Remark 19.19. Simple but useful properties of Hausdorff measures are:

(i) The Hausdorff measures are translation invariant, that is

$$\mathscr{H}^k(B+h) = \mathscr{H}^k(B) \quad \forall\, B \subset \mathbb{R}^n, \forall\, h \in \mathbb{R}^n \ ,$$

and k-homogeneous, that is

$$\mathscr{H}^k(\lambda B) = \lambda^k \mathscr{H}^k(B) \quad \forall\, B \subset \mathbb{R}^n, \forall \lambda > 0 \ .$$

(ii) The Hausdorff measure is countably subadditive, which means that whenever we have a countable cover of a subset B, namely $B \subset \cup_{i\in I} B_i$, then

$$\mathscr{H}^k(B) \le \sum_{i\in I} \mathscr{H}^k(B_i) \ .$$

(iii) For every set $A \subset \mathbb{R}^n$ the map $B \mapsto \mathscr{H}^k(A\cap B)$ is σ-additive on Borel sets, which means that whenever we have a countable pairwise disjoint cover of a Borel B by Borel sets B_i, we have

$$\mathscr{H}^k(A\cap B) = \sum_{i\in I} \mathscr{H}^k(A\cap B_i) \ .$$

(iv) Having fixed the subset $B \subset \mathbb{R}^n$ and $\delta > 0$, we have that

$$k > k' \quad \Longrightarrow \quad \mathscr{H}^k_\delta(B) \le \delta^{k-k'} \mathscr{H}^{k'}_\delta(B) \ . \tag{19.39}$$

In particular, looking at (19.39) when $\delta \to 0$, we deduce that

$$\mathscr{H}^{k'}(B) < +\infty \quad \Longrightarrow \quad \mathscr{H}^k(B) = 0$$

or, equivalently,

$$\mathscr{H}^k(B) > 0 \quad \Longrightarrow \quad \mathscr{H}^{k'}(B) = +\infty \ .$$

Remark 19.20. The choice of c_k is meant to be consistent with the usual notion of k-dimensional area: if B is a Borel subset of a k-dimensional plane $\pi \subset \mathbb{R}^n$, $1 \le k \le n$, then we would like that

$$\mathscr{L}^k_\pi(B) = \mathscr{H}^k(B) \ . \tag{19.40}$$

It is useful to remember the isodiametric inequality: naming $\omega_n := \mathscr{L}^n(B_1(0))$, for every measurable subset $B \subset \mathbb{R}^n$ it is true that

$$\mathscr{L}^n(B) \le \omega_n \left(\frac{\mathrm{diam}(B)}{2}\right)^n \ . \tag{19.41}$$

Thanks to (19.41), it can be easily proved that equality (19.40) holds if we choose

$$c_k = \frac{\omega_k}{2^k} = \frac{\pi^{k/2}}{2^k\Gamma(1+k/2)} \ ,$$

where Γ is Euler's function:

$$\Gamma(t) := \int_0^\infty s^{t-1}e^{-s}\,ds \ .$$

More generally, with this choice of the normalization constant, if B is contained in an embedded C^1-manifold M of dimension k in $\mathbb{R}^n$, then

$$\mathscr{H}^k(B) = \sigma_k(B)$$

where σ_k is the classical k-dimensional surface measure defined on Borel subsets of M by local parametrizations and partitions of unity.

Proposition 19.21. *Consider a locally finite measure $\mu \geq 0$ on $\mathcal{B}(\mathbb{R}^n)$ and , fixing $t > 0$, put*

$$B := \left\{ x \,\middle|\, \limsup_{\rho \to 0} \frac{\mu(\overline{B}_\rho(x))}{\rho^k \omega_k} > t \right\} , \tag{19.42}$$

then B is a Borel set and

$$\mu(B) \geq t \mathscr{S}^k(B) .$$

Moreover, if μ vanishes on $\mathscr{H}^k$-finite sets, then $\mathscr{H}^k(B) = 0$.

A traditional proof of Proposition 19.21 is based on Besicovitch covering theorem, whose statement for completeness is included below. We present instead a proof based on a more general and robust covering theorem, valid in general metric spaces.

Theorem 19.22 (Besicovitch)**.** *There exists an integer $\xi(n)$ with the following property: if $A \subset \mathbb{R}^n$ is bounded and $\rho : A \to (0, +\infty)$, there exist sets $A_1, \ldots, A_{\xi(n)}$ such that*

(a) the balls $\{B_{\rho(x)}(x)\}_{x \in A_i}$ are pairwise disjoint;

(b) the new families still cover the set A, that is

$$A \subset \bigcup_{j=1}^{\xi(n)} \left(\bigcup_{x \in A_j} B_{\rho(x)}(x) \right) .$$

Let us introduce now the general covering theorem.

Definition 19.23 (Fine cover)**.** *A family $\mathcal{F}$ of closed balls in a metric space (X, d) is a* fine cover *of a set $A \subset X$ if*

$$\forall\, x \in A \quad \inf \left\{ r > 0 | \overline{B}_r(x) \in \mathcal{F} \right\} = 0 .$$

Theorem 19.24. *Fix $k \geq 0$, consider a fine cover $\mathcal{F}$ of $A \subset X$, with (X, d) metric space. Then there exists a countable pairwise disjoint subfamily $\mathcal{F}' = \{\overline{B}_i\}_{i \geq 1} \subset \mathcal{F}$ such that at least one of the following conditions holds:*

(i) $\sum_{i=1}^{\infty} [\mathrm{diam}(B_i)]^k = \infty$,

(ii) $\mathscr{H}^k\left(A \setminus \bigcup_{i=1}^{\infty} \overline{B}_i\right) = 0$.

Proof. The subfamily $\mathcal{F}'$ is chosen inductively, beginning with $\mathcal{F}_0 := \mathcal{F}$. Surely, there exists a closed ball, let us call it $\overline{B}_1$, such that

$$\operatorname{diam}\left(\overline{B}_1\right) > \frac{1}{2}\sup\left\{\operatorname{diam}(\overline{B}) |\, \overline{B} \in \mathcal{F}_0\right\} \ .$$

Now put

$$\mathcal{F}_1 := \{\overline{B} \in \mathcal{F}_0 |\, \overline{B} \cap \overline{B}_1 = \emptyset\} \ ,$$

and choose among them a ball $\overline{B}_2 \in \mathcal{F}_1$ such that

$$\operatorname{diam}(\overline{B}_2) > \frac{1}{2}\sup\left\{\operatorname{diam}(\overline{B}) |\, B \in \mathcal{F}_1\right\} \ .$$

If we try to go on analogously, the only chance by which the construction has to stop is that for some $l \in \mathbb{N}$ the family $\mathcal{F}_l = \emptyset$, so we are getting option (ii) in the statement. Otherwise, if suppose that the construction does not stop, we get a family $\mathcal{F}' = \{B_i\}_{i\geq 1}$. We prove that if (i) does not hold, and in particular $\operatorname{diam}(\overline{B}_i) \to 0$, then we have to find (ii) again.

Fix an index $i_0 \in \mathbb{N}$: for every $x \in A \setminus \bigcup_{i=1}^{i_0} \overline{B}_i$ there exists a ball $\overline{B}_{r(x)}(x) \in \mathcal{F}$ such that

$$\overline{B}_{r(x)}(x) \cap \bigcup_{i=1}^{i_0} \overline{B}_i = \emptyset \ ,$$

because $\mathcal{F}$ is a fine cover of A and the complement of $\cup_1^{i_0}\overline{B}_i$ is open in $\mathbb{R}^n$. On the other hand, we claim that there exists an integer $i(x) > i_0$ such that

$$\overline{B}_{r(x)}(x) \cap \overline{B}_{i(x)} \neq \emptyset \ . \tag{19.43}$$

In fact if

$$\forall\, i \in \mathbb{N} \quad \overline{B}_{r(x)}(x) \cap \overline{B}_i = \emptyset \ , \tag{19.44}$$

then

$$\forall\, i \in \mathbb{N} \quad \operatorname{diam}(\overline{B}_i) \geq \frac{1}{2}\operatorname{diam}(\overline{B}_{r(x)}(x)) \ ,$$

but $\operatorname{diam}(\overline{B}_i) \to 0$, so (19.44) is false. Without loss of generality, we can think that $i(x)$ is the first index for which (19.43) holds, too. Since, by construction, $\operatorname{diam}(\overline{B}_{i(x)}) > \frac{1}{2}\sup\{\operatorname{diam}(\overline{B})|\overline{B} \in \mathcal{F}_{i(x)-1}\}$ (and $\overline{B}_{r(x)}(x) \in \mathcal{F}_{i(x)-1}$), then $r(x) \leq \operatorname{diam}(\overline{B}_{i(x)})$.

If $\overline{B}_{i(x)} = \overline{B}_{r_{i(x)}}(y_{i(x)})$, then

$$\overline{B}_{r(x)}(x) \subset \overline{B}_{5r_{i(x)}}(y_{i(x)})$$

and

$$A \setminus \bigcup_{i=1}^{i_0} \overline{B}_i \subset \bigcup_{i=i_0+1}^{\infty} \overline{B}_{3r_i}(y_i) \ . \tag{19.45}$$

Choosing i_0 such that $10r_i < \delta$ for every $i > i_0$, (19.45) says that

$$\mathscr{H}^k_\delta\left(A \setminus \bigcup_{i=1}^{\infty} \overline{B}_i\right) \le \mathscr{H}^k_\delta\left(A \setminus \bigcup_{i=1}^{i_0} \overline{B}_i\right) \le \sum_{i=i_0+1}^{\infty} \omega_k (10r_i)^k \ .$$

We conclude remarking that when $\delta \to 0$, $i_0 \to +\infty$ and

$$\mathscr{H}^k\left(A \setminus \bigcup_{i=1}^{\infty} \overline{B}_i\right) \le \lim_{i_0 \to \infty} \omega_k \sum_{i=i_0+1}^{\infty} (10r_i)^k = 0 \ .$$

□

Now we are able to prove Proposition 19.21.

Proof. By intersecting B with balls, one easily reduces to the case of a bounded set B. Hence, we can assume B bounded and μ finite measure. Fix $\delta > 0$, an open set $A \supset B$ and consider the family

$$\mathcal{F} := \left\{ \overline{B}_r(x) \,\middle|\, r < \delta/2,\ \overline{B}_r(x) \subset A,\ \mu\left(B_r(x)\right) > t\omega_k r^k \right\} \ , \tag{19.46}$$

that is a fine cover of B. Applying Theorem 19.24, we get a subfamily $\mathcal{F}' \subset \mathcal{F}$ whose elements we will denote by

$$\overline{B}_i = \overline{B}_{r_i}(x_i) \ .$$

First we exclude possibility (i) of Theorem 19.24: as a matter of fact

$$\sum_{i=1}^{\infty} [\mathrm{diam}(\overline{B}_i)]^k \le 2^k \sum_{i=1}^{\infty} r_i^k < \frac{2^k}{t\omega_k} \sum_{i=1}^{\infty} \mu(\overline{B}_i) \le \frac{2^k \mu(A)}{t\omega_k} < \infty \ .$$

Since (ii) holds and we can compare $\mathscr{H}^k_\delta$ with $\mathscr{S}^k_\delta$ via (19.38),

$$\mathscr{S}^k_\delta(B) \le \mathscr{S}^k_\delta\left(\bigcup_{i=1}^{\infty} \overline{B}_i\right) \le \sum_{i=1}^{\infty} \omega_k r_i^k < \frac{1}{t} \sum_{i=1}^{\infty} \mu(\overline{B}_i) \le \frac{\mu(A)}{t} \ , \tag{19.47}$$

As $\delta \downarrow 0$ we get $t\mathscr{S}^k(B) \le \mu(A)$ and the outer regularity of μ gives $t\mathscr{S}^k(B) \le \mu(B)$.

Finally, the last statement of the proposition can be achieved noticing that the inequality gives that $\mathscr{S}^k(B)$ is finite; if we assume that μ vanishes with finite k-dimensional measure we obtain that $\mu(B) = 0$; applying once more the inequality we get $\mathscr{S}^k(B) = 0$.

□

19.3 The second partial regularity result for systems: $\mathscr{H}^{n-2+\epsilon}(\Sigma(u)) = 0$

Aware of the usefulness of Proposition 19.21 for our purposes, we are now ready to obtain that if $F \in C^2(\mathbb{R}^{m\times n})$ satisfies the Legendre condition for some $\lambda > 0$ and satisfies also

$$|D^2F(p)| \leq \Lambda < +\infty \qquad \forall\, p \in \mathbb{R}^{m\times n}$$

then any local minimizer u has a large regularity set of big measure indeed, in fact

$$\mathscr{H}^{n-2+\epsilon}(\Sigma(u)) = 0 \qquad \forall\, \epsilon > 0\ , \tag{19.48}$$

where, as usual, $\Sigma(u) := \Omega \setminus \Omega_{\text{reg}}(u)$.

Let us remark that, with respect to the first partial regularity result and with respect to Evans Theorem 19.4, we slightly but significantly changed the properties of the system, replacing the weaker hypothesis of uniform quasiconvexity with the Legendre condition for some positive λ (i.e. uniform convexity). In fact, thanks to the Legendre condition the sequence $\Delta_{h,s}(Du)$ satisfies an equielliptic family of systems, then, via Caccioppoli inequality the sequence $\Delta_{h,s}(Du)$ is uniformly bounded in L^2_{loc}. The existence of second derivatives in L^2_{loc} is useful to estimate the size of the singular set.
We will also obtain a stronger version of (19.48) for systems in which D^2F is uniformly continuous, we will see it in Corollary 19.27.

As for the strategy: in Proposition 19.25 we are going to split the singular set $\Sigma(u)$ in two other sets, $\Sigma_1(u)$ and $\Sigma_2(u)$, and then we are going to estimate separately the Hausdorff measure of each of them with the aid of Proposition 19.26 and Theorem 19.29.

Proposition 19.25. *Consider, as previously, a variational problem defined by $F \in C^2(\mathbb{R}^{m\times n})$ with $|D^2F| \leq \Lambda$, satisfying the Legendre condition for some $\lambda > 0$. If u is a local minimizer of such a problem, define the sets*

$$\Sigma_1(u) := \left\{ x \in \Omega \,\middle|\, \limsup_{r\to 0} r^{2-n} \int_{B_r(x)} |D^u(y)|^2\, dy > 0 \right\}$$

and

$$\Sigma_2(u) := \left\{ x \in \Omega \,\middle|\, \lim_{r\to 0} \left|(Du)_{B_r(x)}\right| = +\infty \right\}\ .$$

Then $\Sigma(u) \subset \Sigma_1(u) \cup \Sigma_2(u)$. If in addition D^2F is uniformly continuous, we have $\Sigma(u) \subset \Sigma_1(u)$.

Proof. Fix $x \in \Omega$ such that $x \notin \Sigma_1(u) \cup \Sigma_2(u)$, then

- there exists $M_1 < +\infty$ such that $\left|(Du)_{B_r(x)}\right| < M_1$ for arbitrarily small radii $r > 0$;

- thanks to Poincaré inequality

$$\mathrm{Exc}\,(u, B_r(x))^2 \leq C r^{n-2} \int_{B_r(x)} |D^2 u(y)|^2 \, dy \longrightarrow 0 \;;$$

thus for some $M > 0$ we have that $x \in \Omega_M(u)$, where $\Omega_M(u)$ has been specified in Definition 19.16, and $\Omega_M(u) \subset \Omega_{\mathrm{reg}}$ due to (19.30).

The second part of the statement can be achieved noticing that in the case when D^2F is uniformly continuous no bound on $|(Du)_{B_r(x)}|$ is needed in the decay theorem and in the characterization of the regular set. $\square$

Proposition 19.26. *If $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$, we have that*

$$\mathscr{H}^{n-2}\left(\Sigma_1(u)\right) = 0 \;.$$

Proof. Let us employ Proposition 19.21 with the absolutely continuous measure $\mu := |D^2 u|^2 \mathscr{L}^n$. Obviously we choose $k = n-2$ and we have that μ vanishes on sets with finite $\mathscr{H}^k$-measure. The thesis follows when we observe that

$$\Sigma_1(u) = \bigcup_{\nu=1}^{+\infty} \left\{ x \in \Omega \,\middle|\, \limsup_{r \to 0} \frac{\mu(B_r(x))}{\omega_k r^k} > \frac{1}{\nu} \right\} \;.$$

$\square$

By the second part of the statement of Proposition 19.25 we get:

Corollary 19.27. *If we add the uniform continuity of D^2F to the hypothesis of Proposition 19.26, we can conclude that*

$$\mathscr{H}^{n-2}\left(\Sigma(u)\right) = 0 \;. \tag{19.49}$$

The estimate on the Hausdorff measure of $\Sigma_2(u)$ is a bit more complex and passes through the estimate of the Hausdorff measure of the so-called *approximate discontinuity set* S_v of a function v.

Definition 19.28. *Given a function $v \in L^1_{\mathrm{loc}}(\Omega)$, we put*

$$\Omega \setminus S_v := \left\{ x \in \Omega \,\middle|\, \exists\, z \in \mathbb{R} \text{ s.t. } \lim_{r \downarrow 0} \fint_{B_r(x)} |v(y) - z| \, dy = 0 \right\} \;.$$

When such a z exists, it is unique and we will call it approximate limit *of v at the point x.*

Theorem 19.29. *If $v \in W^{1,p}(\Omega)$, $1 \le p \le n$, then*

$$\mathscr{H}^{n-p+\epsilon}(S_v) = 0 \qquad \forall\, \epsilon > 0\ .$$

Notice that the statement is trivial in the case $p > n$, by the Sobolev embedding theorem (i.e. $S_v = \emptyset$): as p increases the Hausdorff dimension of the approximate discontinuity set moves from $n-1$ to 0.

Applying this theorem to $v = Du \in H^{1,2}(\Omega)$, $p = 2$, we conclude that $\mathscr{H}^{n-2+\epsilon}(\Sigma_2(u)) = 0$.

Proof. (1) Fix $0 < \eta < \rho$, we claim that

$$n\omega_n \left|(v)_{B_\eta(x)} - (v)_{B_\rho(x)}\right| \le (n-1)\int_0^\rho t^{-n}\int_{B_t(x)} |Dv(y)|\, dy\, dt + \rho^{-(n-1)}\int_{B_\rho(x)} |Dv(y)|\, dy\ ; \tag{19.50}$$

we will show this in the third part of this proof.

Suppose that x is a point for which $\int_{B_t(x)} |Dv(y)|\, dy = o(t^{n-1+\epsilon})$ for some $\epsilon > 0$, then it is also true that $\rho^{n-1}\int_{B_\rho(x)} |Dv(y)|\, dy \to 0$ and the sequence $(v)_{B_r(x)}$ admits a limit z as $r \to 0$ because it is a Cauchy sequence. Thanks to the Poincaré inequality

$$\fint_{B_r(x)} |v(y) - (v)_{B_r(x)}|\, dy \le C r^{-(n-1)} \int_{B_r(x)} |Dv(y)|\, dy \overset{r\to 0}{\longrightarrow} 0\ ,$$

therefore

$$\fint_{B_r(x)} |v(y) - z|\, dy \overset{r\to 0}{\longrightarrow} 0\ ,$$

that is to say, $x \notin S_v$. This chain of implications means that, for all $\epsilon > 0$,

$$\Omega \setminus S_v \supset \left\{ x \in \Omega \,\middle|\, \int_{B_t(x)} |Dv(y)|\, dy = o(t^{n-1+\epsilon}) \right\}\ . \tag{19.51}$$

(2) In order to refine (19.51) suppose that

$$\int_{B_t(x)} |Dv(y)|\, dy = o(t^{n-p-\epsilon})$$

for some $\epsilon > 0$, then, by Hölder inequality,

$$\int_{B_t(x)} |Dv(y)|\, dy \le o(t^{n/p-1+\epsilon/p}) t^{n/p'} = o(t^{n-1+\epsilon/p})\ .$$

For this reason we can improve (19.51) with

$$\Omega \setminus S_v \supset \left\{ x \in \Omega \,\middle|\, \int_{B_t(x)} |Dv(y)|^p\, dy = o(t^{n-p+\epsilon}) \right\}\ . \tag{19.52}$$

in view of Proposition 19.21 the complement of the set $\left\{x \in \Omega \,\middle|\, \int_{B_t(x)} |Dv(y)|^p \, dy = o(t^{n-p+\epsilon})\right\}$ $\mathscr{H}^{n-p+\epsilon}$-negligible, hence the jump set S_v is $\mathscr{H}^{n-p+\epsilon}$-negligible, too.

(3) This third part is devoted to the proof of (19.50); for the sake of simplicity we put $x = 0$. Let us consider the characteristic function $\chi_{B_1(0)}$; since we would like to differentiate the map

$$\rho \mapsto \rho^{-n} \int \chi\left(\frac{y}{\rho}\right) v(y) \, dy \ ,$$

(in order to study its increment) a possible proof of (19.50) is based on a regularization of χ, differentiation and passage to the limit.

We produce instead a direct proof based on a *ad hoc* calibration: we need a vector field ϕ with Supp$\phi \subset \overline{B}_\rho(0)$ whose divergence almost coincides with the left member of (19.50), that is

$$\mathrm{div}\phi = n\left(\eta^{-n}\chi_{B_\eta} - \rho^{-n}\chi_{B_\rho}\right) \ . \tag{19.53}$$

Therefore,

$$\phi(x) := x\left(\left(\eta^{-n} \wedge |x|^{-n}\right) - \rho^{-n}\right)$$

verifies (19.53) and

$$\frac{n}{\eta^n}\int_{B_\eta(0)} v(y)\,dy - \frac{n}{\rho^n}\int_{B_\rho(0)} v(y)\,dy = \int v(y)\mathrm{div}\phi(y)\,dy \tag{19.54}$$

$$= -\int \phi(y)\cdot Dv(y)\,dy \le \int_{B_\rho(0)} |\phi(y)||Dv(y)|\,dy \le \int_{B_\rho} |y|^{-(n-1)}\,d\mu(y) \tag{19.55}$$

$$= \int_0^\infty \mu\left(|y|^{-(n-1)} > t\right)\,dt = (n-1)\int_0^\infty s^{-n}\mu(B_s(0))\,ds \tag{19.56}$$

$$= (n-1)\int_0^\rho s^{-n}\int_{B_s} |Dv(y)|\,dy\,ds + (n-1)\int_\rho^\infty s^{-n}\int_{B_\rho} |Dv(y)|\,dy\,ds$$

$$= (n-1)\int_0^\rho s^{-n}\int_{B_s} |Dv(y)|\,dy\,ds + \rho^{-(n-1)}\int_{B_\rho} |Dv(y)|\,dy \ ,$$

where we pass from (19.54) to (19.55) by the divergence theorem, from (19.55) to (19.56) by Cavalieri's principle and then it is all change of variables and Fubini's theorem. □

Remark 19.30. In the case $p = 1$ it is even possible to prove that S_v is σ-finite with respect to $\mathscr{H}^{n-1}$, so the measurement of the discontinuity set with the scale of Hausdorff measures is sharp. On the contrary, in the case $p > 1$ the right scale for the measurement of the approximate discontinuity set are the so-called capacities.

20 Some tools from convex and non-smooth analysis

20.1 Subdifferential of a convex function

In this section we briefly recall some classical notions and results from convex and non-smooth analysis, which will be useful in dealing with uniqueness and regularity results for viscosity solutions to partial differential equations.

In the sequel we consider a convex open subset Ω of $\mathbb{R}^n$ and a convex function $u : \Omega \to \mathbb{R}$. Recall that u is convex if and only if

$$f\big((1-t)x+ty\big) \le (1-t)f(x)+tf(y) \qquad \forall x,\, y \in \Omega,\, t \in [0,1]\ .$$

If $u \in C^2(\Omega)$ this is equivalent to say that $D^2u(x) \ge 0$ in the sense of symmetric operators for all $x \in \Omega$.

Definition 20.1 (Subdifferential)**.** *For each $x \in \Omega$, the subdifferential is the set*

$$\partial u(x) := \{p \in \mathbb{R}^n | u(y) \ge u(x) + \langle p, y-x\rangle \ \forall\, y \in \Omega\}\ \ .$$

Obviously $\partial u(x) = \{\nabla u(x)\}$ at any differentiability point.

Remark 20.2. According to Definition 20.1, it is easy to show that

$$\partial u(x) = \{p \in \mathbb{R}^n | \liminf_{t \to 0^+} \frac{u(x+tv)-u(x)}{t} \ge \langle p, v\rangle \ \forall\, v\}\ . \tag{20.1}$$

Indeed, when $p \in \partial u(x)$ the relation

$$\frac{u(x+tv)-u(x)}{t} \ge \langle p, v\rangle$$

passes through the limit. Conversely, after recalling the monotonicity property of difference quotients of a convex function, i.e.

$$\frac{u(x+t'v)-u(x)}{t'} \le \frac{(1-t'/t)\, u(x) + \frac{t'}{t}u(x+tv)-u(x)}{t'} = \frac{u(x+tv)-u(x)}{t} \quad 0 < t' < t\ , \tag{20.2}$$

we have that for every $y \in \Omega$ we have (choosing $t=1$, $v=y-x$)

$$u(y)-u(x) \ge \frac{u(x+t'v)-u(x)}{t'} \ge \langle p, y-x\rangle + \frac{o(t')}{t'}\ .$$

The same monotonicity property (20.2) yields that the $\liminf$ in (20.1) is a limit.

Remark 20.3. The following properties are easy to check:

(i) The graph of the subdifferential, i.e. $\{(x,p)|p \in \partial u(x)\} \subset \Omega \times \mathbb{R}^n$, is closed, in fact convex functions are continuous.

(ii) Convex functions are locally Lipschitz in Ω; to see this, fix a point $x_0 \in \Omega$ and $x,\ y \in B_r(x_0) \Subset B_R(x_0) \Subset \Omega$, for the sake of simplicity suppose that $|x-x_0| \le |y-x_0| < r$ and $u(y)-u(x) > 0$. Thanks to monotonicity of difference quotients seen in (20.2), we can estimate

$$\frac{|u(y)-u(x)|}{|y-x|} \le \frac{u(y_R)-u(x)}{|y_R-x|} \le \frac{\mathrm{Osc}(u, B_R(x_0))}{R-r} ,$$

where $y_R \in \partial B_R(x_0)$ is on the segment between x and y. Thus

$$\mathrm{Lip}(u, B_r(x_0)) \le \frac{\mathrm{Osc}(u, B_R(x_0))}{R-r} .$$

Moreover,

$$\text{ess-sup}_{B_r(x_0)}|Du| \le \frac{\mathrm{Osc}(u, B_R(x_0))}{R-r} ,$$

because of (1.5).

(iii) As a consequence of (ii) and Rademacher's theorem, $\partial u(x) \neq \emptyset$ for all $x \in \Omega$. In addition, a convex function u belongs to C^1 if and only if $\partial u(x)$ is a singleton for every $x \in \Omega$. Indeed, if x_h are differentiability points of u such that $x_h \to x$ and $\nabla u(x_h)$ has two distinct points, then $\partial u(x)$ is not a singleton. Hence ∇u has a continuous extension to the whole of Ω and $u \in C^1$.

(iv) Given convex functions $f_k : \Omega \to \mathbb{R}$, locally uniformly converging in Ω to f, and $x_k \to x \in \Omega$, any sequence (p_k) with $p_k \in \partial f_k(x_k)$ is bounded (by the local Lipschitz condition) and any limit point p of (p_k) satisfies

$$p \in \partial f(x) .$$

In fact, it suffices to pass to the limit as $k \to \infty$ in the inequalities

$$f_k(y) \ge f_k(x_k) + \langle p_k, y - x_k \rangle \quad \forall y \in \mathbb{R}^n.$$

As a first result of non-smooth analysis, we state the following theorem.

Theorem 20.4 (Nonsmooth mean value theorem)**.** *Consider a convex function $f : \Omega \to R$ and a couple of points $x,\ y \in \Omega$. There exist z in the segment between x and y and $p \in \partial f(z)$ such that*

$$f(x) - f(y) = \langle p, x-y \rangle .$$

Proof. Choose a positive convolution kernel ρ with support contained in $\overline{B}_1$ and define the sequence of functions $f_\epsilon := f * \rho_\epsilon$, which are easily seen to be convex in the set Ω_ϵ in (1.3), because

$$\begin{aligned} f_\epsilon((1-t)x_0 + tx_1) &= \int_\Omega f((1-t)x_0 + tx_1 - \epsilon\xi)\rho(\xi)\,d\xi \\ &\leq \int_\Omega ((1-t)f(x_0 - \epsilon\xi) + tf(x_1 - \epsilon\xi))\,\rho(\xi)\,d\xi \\ &= (1-t)f_\epsilon(x_0) + tf_\epsilon(x_1)\ ; \end{aligned}$$

moreover $f_\epsilon \to f$ locally uniformly. Thanks to the classical mean value theorem for regular functions, for every $\epsilon > 0$ there exists $\theta_\epsilon \in (0,1)$ such that

$$f_\epsilon(x) - f_\epsilon(y) = \langle p_\epsilon, x - y\rangle\ .$$

with $p_\epsilon = \nabla f_\epsilon(x_\epsilon) \in \partial f_\epsilon(x_\epsilon)$. Since $(\theta_\epsilon, p_\epsilon)$ are uniformly bounded as $\epsilon \to 0$, we can find $\epsilon_k \to 0$ with $\theta_{\epsilon_k} \to \theta \in [0,1]$ and $p_{\epsilon_k} \to p$. Remark 20.3(iv) allows us to conclude that $p \in \partial f((1-\theta)x + \theta y)$ and

$$f(x) - f(y) = \langle p, x - y\rangle\ .$$

□

Proposition 20.5. *Given a convex function $f : \Omega \to \mathbb{R}$, its subdifferential ∂f satisfies the monotonicity property:*

$$\langle p - q, x - y\rangle \geq 0 \qquad \forall p \in \partial f(x),\ \forall q \in \partial f(y).$$

Proof. It is sufficient to sum the inequalities satisfied, respectively, by p and q, i.e.

$$\begin{aligned} f(y) - f(x) &\geq \langle p, y - x\rangle \\ f(x) - f(y) &\geq \langle q, x - y\rangle. \end{aligned}$$

□

Remark 20.6. Recall that continuous function $f : \Omega \to \mathbb{R}$ is convex if and only if its weak second derivative $D^2 f$ is non-negative, i.e. for all nonnegative $\varphi \in C_c^\infty(\Omega)$ and all $\xi \in \mathbb{R}^n$ we have

$$\int_\Omega f(x)\frac{\partial^2\varphi}{\partial\xi^2}(x)\,dx \geq 0\ .$$

This result is easily obtained passing through approximation by convolution, because, still in the weak sense,

$$D^2(f * \rho_\epsilon) = \big(D^2 f\big) * \rho_\epsilon\ .$$

Although we shall not need this fact in the sequel, except in Remark 20.15, let us mention for completeness that the positivity condition on the weak derivative $D^2 f$ implies that this derivative is representable by a symmetric matrix-valued measure. To see this, it suffices to apply the following result to the pure second derivatives $D^2_{\xi\xi} f$:

Lemma 20.7. *Consider a positive distribution $T \in \mathscr{D}'(\Omega)$, i.e.*

$$\forall \varphi \in C_c^\infty(\Omega), \varphi \geq 0 \qquad \langle T, \varphi \rangle \geq 0 \ .$$

Then there exists a locally finite nonnegative measure μ in Ω such that

$$\langle T, \psi \rangle = \int_\Omega \psi \, d\mu \quad \forall \psi \in C_c^\infty(\Omega) \ .$$

Proof. Fix an open set $\Omega' \Subset \Omega$, define $K := \overline{\Omega'}$ and a cut-off function $\varphi \in C_c^\infty(\Omega)$ with $\varphi|_K \equiv 1$. For every test function $\psi \in C_c^\infty(\Omega')$, since $\|\psi\|_{L^\infty}\varphi - \psi \geq 0$ and T is a positive (and, of course, linear) distribution, we have

$$\langle T, \psi \rangle \leq \langle T, \|\psi\|_{L^\infty}\varphi \rangle = C(\Omega')\|\psi\|_{L^\infty} \ ,$$

where $C(\Omega') := \langle T, \varphi \rangle$. Replacing ψ by $-\psi$, the same estimate holds with $|\langle T, \psi \rangle|$ in the left hand side. By Riesz Representation Theorem we obtain the existence of μ. $\square$

Definition 20.8 (λ-convexity, uniform convexity, semiconvexity)**.** *Given $\lambda \in \mathbb{R}$ and a function $f : \Omega \to \mathbb{R}$ satisfying the inequality*

$$\int_\Omega f(x) \frac{\partial^2 \varphi}{\partial \xi^2}(x) \, dx \geq \lambda \int_\Omega \varphi(x) \, dx$$

for every non-negative $\varphi \in C_c^\infty(\Omega)$ and for every $\xi \in \mathbb{R}^n$ (in short $D^2 f \geq \lambda I$), in the spirit of Remark 20.6, we say that f is λ-convex. We say also that

- *f is* uniformly convex *if $\lambda > 0$;*
- *f is* semiconvex *if $\lambda \leq 0$.*

Notice that, with the notation of Definition 20.8, a function f is λ-convex if and only if $f(x) - \lambda|x|^2/2$ is convex.

Analogous concepts can be given in the concave case, namely λ-concavity (i.e. $D^2 f \leq \lambda I$), uniform concavity, semiconcavity. An important class of semiconcave functions is given by squared distance functions:

Example 20.9. Given a closed set $E \subset \mathbb{R}^n$, the square of the distance from E is λ-concave. Indeed,

$$\operatorname{dist}^2(x, E) - |x|^2 = \inf_{y \in E} (x-y)^2 - |x|^2 = \inf_{y \in E} \left(|y|^2 - 2\langle x, y \rangle \right) \ , \tag{20.3}$$

since the functions $x \mapsto |y|^2 - 2\langle x, y\rangle$ are affine, their infimum over $y \in E$, that is (20.3), is concave.

Remark 20.10 (Inverse of the subdifferential)**.**

(i) If $f : \Omega \to \mathbb{R}$ is λ-convex, Proposition 20.5 proves that for every $p \in \partial f(x)$ and every $q \in \partial f(y)$, we have

$$\langle p - q, x - y \rangle \geq \lambda |x-y|^2 \ . \tag{20.4}$$

(ii) If $\lambda > 0$, for each $p \in \mathbb{R}^n$ no more than one $x \in \Omega$ can satisfy $p \in \partial f(x)$, because through (20.4) we get

$$p \in \partial f(x) \cap \partial f(y) \implies 0 = \langle p - p, x - y \rangle \geq \lambda |x-y|^2 \implies x = y \ .$$

In particular, setting

$$L := \bigcup_{x \in \Omega} \partial f(x) \ ,$$

there exists a unique, well-defined map $\Psi : L \to \Omega$ such that $p \in \partial f(\Psi(p))$.

(iii) Moreover Ψ is a Lipschitz map: rewriting (20.4) for Ψ we get

$$\lambda |\Psi(p) - \Psi(q)|^2 \leq \langle p - q, \Psi(p) - \Psi(q) \rangle \leq |p-q| |\Psi(p) - \Psi(q)| \ ,$$

thus $\operatorname{Lip}(\Psi) \leq 1/\lambda$.

(iv) The conjugate of a convex function f is generally defined as

$$f^*(x^*) := \sup_{x \in \Omega} \left(\langle x^*, x \rangle - f(x) \right) \ ;$$

we immediately point out that f^* is a convex function, because it is the supremum of a family of affine functions.

(iv) We now show that ∂f^* is single-valued (hence $f^* \in C^1$) and

$$Df^*(x^*) = \Psi(x^*) \ .$$

Indeed, fix x^*, as f is λ-convex there exists a maximizing x such that $f^*(x^*) = \langle x^*, x \rangle - f(x)$, consequently for every y^*

$$f^*(y^*) = \sup_{y \in \Omega} \left(\langle y^*, y \rangle - f(y) \right) \geq \langle y^*, x \rangle - f(x) = f^*(x^*) + \langle y^* - x^*, x \rangle \ . \tag{20.5}$$

Hence $x \in \partial f^*(x^*)$. In order to prove that $\nabla f^*(x^*) = \Psi(x^*)$ it remains to show that $x = \Psi(x^*)$. The point $x \in \Omega$ has been chosen because of its maximizing property: for every $y \in \Omega$

$$\langle x^*, x\rangle - f(x) = f^*(x^*) \geq \langle x^*, y\rangle - f(y) \ ,$$

thus

$$f(y) \geq f(x) + \langle x^*, y - x\rangle \ ,$$

that is $x^* \in \partial f(x)$ or, by the definition of Ψ, $x = \Psi(x^*)$.

20.2 Convex functions and Measure Theory

Now we recall some classical results in Measure Theory in order to have the necessary tools to prove Alexandrov Theorem 20.14 on differentiability of convex functions.

Thanks to the next classical result we can, with a slight abuse of notation, keep the same notation ∇f for the pointwise gradient and the weak derivative, at least for locally Lipschitz functions.

Theorem 20.11 (Rademacher). *Any Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\mathscr{L}^n$-almost every point and the pointwise gradient coincides $\mathscr{L}^n$-a.e. with the distributional derivative ∇f.*

Proof. Fix a point x_0 which is a Lebesgue point of Df, i.e.

$$⨍_{B_r(x_0)} |Df(y) - Df(x_0)|\, dy \stackrel{r\to 0}{\longrightarrow} 0 \ . \tag{20.6}$$

Defining

$$f_r(y) := \frac{1}{r}\left(f(x_0 + ry) - f(x_0)\right)$$

and noticing that $Df_r(y) = Df(x_0 + ry)$ (still in the distributional sense), we are able to rewrite (20.6) as

$$⨍_{B_1(0)} |Df_r(y) - Df(x_0)|\, dy \stackrel{r\to 0}{\longrightarrow} 0 \ ,$$

where f_r sequence of functions with equibounded Lipschitz constant and $f_r(0) = 0$ for every $r > 0$. Thanks to the Ascoli-Arzelà theorem, as $r \downarrow 0$ this family of functions has limit points in the uniform topology. Any limit point g obviously satisfies $g(0) = 0$, and since Dg is a limit point of Df_r in the weak* topology, the strong convergence of Df_r to $Df(x_0)$ gives $Dg \equiv Df(x_0)$, still in the weak sense. We conclude that $g(x) = Df(x_0)x$, so that g is uniquely determined and

$$f_r(y) = \frac{1}{r}\left(f(x_0 + ry) - f(x_0)\right) \stackrel{r\to 0}{\longrightarrow} Df(x_0)y$$

uniformly in $\overline{B}_1(0)$. This convergence property is immediately seen to be equivalent to the classical differentiability of f at x_0, with gradient equal to $Df(x_0)$. $\square$

The proof of the following classical result can be found, for instance, in [10] and [11].

Theorem 20.12 (Area formula)**.** *Consider a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}^n$ and a Borel set $A \subset \mathbb{R}^n$. Then the function*

$$N(y, A) := \operatorname{card}\big(f^{-1}(y) \cap A\big)$$

is (and, in particular, $f(A) = \{N > 0\}$) is $\mathscr{L}^n$-measurable and

$$\int_A |\det \nabla f(x)|\, dx = \int_{\mathbb{R}^n} N(y, A)\, dy \geq \mathscr{L}^n(f(A)) \ .$$

Definition 20.13 (Pointwise second-order differentiability)**.** *A function $f : \mathbb{R}^n \to \mathbb{R}$ is pointwise second-order differentiable at $x \in \mathbb{R}^n$ if there exist $p \in \mathbb{R}^n$ and $S \in \mathrm{Sym}^{n\times n}$ such that*

$$f(y) = f(x) + \langle p, y - x\rangle + \frac{1}{2}\langle S(y-x), y-x\rangle + o(|y-x|^2) \ .$$

Notice that pointwise second-order differentiability implies first-order differentiability, and that $p = Df(x)$ (here understood in the pointwise sense). Also, the symmetry assumption on S is not restrictive, since in the formula S can also be replaced by its symmetric part.

We are now ready to prove the main result of this section, Alexandrov Theorem.

Theorem 20.14 (Alexandrov)**.** *A convex function $f : \mathbb{R}^n \to \mathbb{R}$ is $\mathscr{L}^n$-a.e. pointwise second-order differentiable.*

Proof. The proof is based on the inverse function Ψ, introduced in Remark 20.10. Obviously, there is no loss of generality supposing that f is λ-convex for some positive $\lambda > 0$.

We briefly recall, from Remark 20.10, that ∂f associates to each $x \in \mathbb{R}^n$ the subdifferential set, on the contrary Ψ is a single-valued map which associates to each $p \in \mathbb{R}^n$ the point x such that $p \in \partial f(x)$. Let us call the set of "bad" points

$$\Sigma := \{p \,|\, \nexists \nabla\Psi(p) \text{ or } \exists \nabla\Psi(p) \text{ and } \det \nabla\Psi(p) = 0\} \ .$$

Since Ψ is Lipschitz, Rademacher Theorem 20.11 and the Area Formula 20.12 give

$$\mathscr{L}^n\left(\Psi(\Sigma)\right) \leq \int_\Sigma |\det \nabla\Psi|\, dp = 0 \ .$$

We shall prove that the stated differentiability property holds at all points $x \notin \Psi(\Sigma)$. Let us write $x = \Psi(p)$ with $p \notin \Sigma$, so that there exists the derivative $\nabla\Psi(p)$ and, since it is invertible, we can name

$$S(x) := (\nabla\Psi(p))^{-1} \ .$$

Notice also that S is symmetric, since Ψ is a gradient map, hence there is no need to symmetrize it. If $y = \Psi(q)$, we get

$$\begin{aligned} S(x)^{-1}\left(q - p - S(x)(y-x)\right) &= \left(y - x - \nabla\Psi(p)(q-p)\right) \\ &= -\left(\Psi(q) - \Psi(p) - \nabla\Psi(p)(q-p)\right) \\ &= o(|p-q|) = o(|x-y|)\ . \end{aligned}$$

Therefore

$$\lim_{\substack{y \to x \\ q \in \partial f(y)}} \frac{|q - \nabla f(x) - S(x)(y-x)|}{|y-x|} = 0\ . \tag{20.7}$$

The result got in (20.7), together with the non-smooth mean value Theorem 20.4, give us the second order expansion. In fact, there exist $\theta \in (0,1)$ and $q \in \partial f((1-\theta)y + \theta x)$ such that

$$\begin{aligned} f(y) - f(x) &= \langle q, y-x\rangle \\ &= \langle q - \nabla f(x) - S(x)(y-x), y-x\rangle + \langle \nabla f(x), y-x\rangle + \langle S(x)(y-x), y-x\rangle \\ &= \langle \nabla f(x), y-x\rangle + \langle S(x)(y-x), y-x\rangle + o(|y-x|^2)\ . \end{aligned}$$

$\square$

Remark 20.15 (Characterization of S)**.** A blow-up analysis, analogous to the one performed in the proof of Rademacher's theorem, shows that the matrix $S(x)$ in Alexandrov's theorem is the density of the measure $D^2 f$ with respect to $\mathscr{L}^n$, see [1] for details.

21 Viscosity solutions

21.1 Basic definitions

In this section we want to give the notion of *viscosity solution* for general equations having the form

$$E(x, u(x), Du(x), D^2u(x)) = 0 \tag{21.1}$$

where u is defined on some *locally compact* domain $A \subset \mathbb{R}^n$. This topological assumptions is actually very useful, because we can deal at the same time with open and closed domains.

We first need to recall two classical ways to regularize a function.

Definition 21.1 (u.s.c. and l.s.c. regularizations). *Let $A' \subset A$ a dense subset and $u : A' \to \overline{\mathbb{R}}$. We define its upper regularization u^* on A by one of the following equivalent formulas:*

$$\begin{aligned} u^*(x) &:= \sup\left\{\limsup_h u(x_h) | (u_h) \subset A' \ x_h \to x\right\} \\ &= \inf_{r>0} \sup_{B_r(x)\cap A'} u \\ &= \min\left\{v | v \text{ is u.s.c. and } v \geq u\right\}. \end{aligned}$$

Similarly we can define the lower regularization u_*

$$\begin{aligned} u_*(x) &:= \inf\left\{\liminf_h u(x_h) | (u_h) \subset A' \ x_h \to x\right\} \\ &= \sup_{r>0} \inf_{B_r(x)\cap A'} u \\ &= \max\left\{v | v \text{ is l.s.c. and } v \leq u\right\} \end{aligned}$$

which is also characterized by the identity $u_* = -(-u)^*$.

Remark 21.2. It is clear that pointwise $u_* \leq u \leq u^*$. In fact, u is continuous at a point $x \in A$ (or, more precisely, has a continuous extension in case $x \in A \setminus A'$) if and only if $u_*(x) = u^*(x)$.

We now assume that $E : L \subset A \times \mathbb{R} \times \mathbb{R}^n \times \mathrm{Sym}^{n\times n} \to \mathbb{R}$, with L dense. Here and in the sequel we denote by $\mathrm{Sym}^{n\times n}$ the space of symmetric $n \times n$ matrices.

Definition 21.3 (Subsolution). *A function $u : A \to \mathbb{R}$ is a subsolution for the equation* (21.1) *(and we write $E \leq 0$) if the two following conditions hold:*

(i) u^* is a real-valued function;

(ii) For any $x \in A$, if φ is C^∞ in a neighbourhood of x and $u^* - \varphi$ has a local maximum at x, then

$$E_*(x, u^*(x), D\varphi(x), D^2\varphi(x)) \leq 0 \ . \tag{21.2}$$

It is obvious from the definition that the property of being a subsolution is invariant under u.s.c. regularization, i.e. u is a subsolution if and only if u^* is a subsolution.

The geometric idea in this definition is to use a local *comparison principle*, since assuming that $u^* - \varphi$ has a maximum at x implies, if u is smooth, that $Du^*(x) = D\varphi(x)$ and $D^2u^*(x) \leq D^2\varphi(x)$. So, while in the classical theory of PDE's an integration by parts formula allows to transfer derivatives from u to the test function ϕ, here the comparison principle allows to transfer (to some extent, since only an inequality holds for second-order derivatives) the derivatives from u to the test function ϕ.

Similarly, we give the following:

Definition 21.4 (Supersolution). *A function $u : A \to \mathbb{R}$ is a supersolution for the equation* (21.1) *(and we write $E \geq 0$) if the two following conditions hold:*

(i) u_* is a real-valued function;

(ii) For any $x \in A$, if φ is C^∞ in a neighbourhood of x and $u_* - \varphi$ has a local minimum at x, then

$$E^*(x, u_*(x), D\varphi(x), D^2\varphi(x)) \geq 0. \tag{21.3}$$

We finally say that u is a solution of our problem if it is both a subsolution and a supersolution.

Remark 21.5. Without loss of generality, we can always assume in the definition of subsolution that the value of the local maximum is zero, that is $u^*(x) - \varphi(x) = 0$. This is true because the test function φ is arbitrary and the value of ϕ at x does not appear in (21.2). Also, possibly adding $|y - x|^4$ to ϕ (so that first and second derivatives of ϕ at x remain unchanged), we can assume with no loss of generality that the local maximum is strict. Analogous remarks hold for subsolutions.

Remark 21.6. A trivial example of viscosity solution is given by the Dirichlet function $\chi_\mathbb{Q}$ on $\mathbb{R}$, which is easily verified to solve the equation $u' = 0$ in the sense above. This example shows that some continuity assumption is needed, to hope for reasonable existence and uniqueness results.

Remark 21.7. Rather surprisingly, a solution of $E = 0$ in the viscosity sense does not necessarily solve $-E = 0$ in the viscosity sense. To show this, consider the equations $|f'| - 1 = 0$ and $1 - |f'| = 0$ and the function $f(t) = \min\{1 - t, 1 + t\}$. In this case, it is immediate to see that f is a subsolution of the first problem (and actually a solution, as we will see), but it is not a subsolution of the second problem, since we can choose identically $\varphi = 1$ to find that the condition $1 - |\varphi'(0)| \leq 0$, corresponding to (21.3), is violated.

We have instead the following parity properties:

(a) Let E be *odd* in (u, p, S). If u verifies $E \leq 0$, then $-u$ verifies $E \geq 0$.

(b) Let E be *even* in (u, p, S). If u verifies $E \leq 0$, then $-u$ verifies $-E \geq 0$.

We now spend some words on the ways of simplifying the conditions that have to be checked in order prove the subsolution or supersolution property. We just examine the case of subsolutions, the case of supersolutions being the same (with obvious variants).

We have already seen in Remark 21.5 that we can assume without loss of generality that $u^* - \varphi$ has a strict local maximum, equal to 0, at x. We can also work equivalently with the larger the class of C^2 functions φ in a neighbourhood of x. One implication is trivial, let us see the converse one. Let $\varphi \in C^2$ and assume $u^*(y) - \varphi(y) \leq 0$ for $y \in \overline{B}_r(x)$, with equality only when $y = x$. By appropriate mollifiers, we can build a sequence $(\varphi_n) \subset C^\infty(\overline{B}_r(x))$ with $\varphi_n \to \varphi$ in $C^2(\overline{B}_r(x))$. Let then x_n be a maximum in $\overline{B_r(x)}$ of the function $u^* - \varphi_n$. Since $\phi_n \to \phi$ uniformly, it is easy to see that any limit point of (x_n) has to be a maximum for $u^* - \varphi$, hence it must be x; in addition the convergence of the maximal values yields $u^*(x_n) \to u^*(x)$. The subsolution property, applied with φ_n at x_n, gives

$$E_*(x_n, u^*(x_n), D\varphi_n(x_n), D^2\varphi_n(x_n)) \leq 0$$

and we can now let $n \to \infty$ and use the lower semicontinuity of E_* to get the thesis.

Actually, it is rather easy now to see that the subsolution property is even equivalent to

$$E_*(x, u^*(x), p, S) \leq 0 \quad \forall\ (p, S) \in J_2^+ u^*(x)$$

where

$$J_2^+ u^*(x) := \left\{ (p, S) | u^*(y) \leq u^*(x) + \langle p, y - x\rangle + \frac{1}{2}\langle S(y-x), y-x\rangle + o(|y-x|^2) \right\}.$$

Indeed, let $P(y) := u^*(x) + \langle p, y-x\rangle + \frac{1}{2}\langle A(y-x), y-x\rangle$, so that $u^*(y) \leq P(y) + o(|y-x|^2)$, with equality when $y = x$. Hence, for any $\epsilon > 0$ we have $u^*(y) \leq P(y) + \epsilon|y-x|^2$ on a sufficiently small neighbourhood of x with equality at $y = x$ and we can apply (**??**) to this smooth function to get

$$E_*(x, u^*(x), p + 2\epsilon(y-x), S + 2\epsilon I) = E_*(x, u^*(x), DP(y) + 2\epsilon(y-x), D^2P(y) + 2\epsilon I) \leq 0$$

and by lower semicontinuity we can let $\epsilon \to 0$ and prove the claim.

Remark 21.8. After these preliminary facts, it should be clear that this theory, despite its elegance, has two main restrictions: on the one hand it is only suited to first or second order equations (since no information on third derivatives comes from local comparison), on the other hand it cannot be generalized to vector-valued functions.

21.2 Viscosity solution versus classical solutions

We first observe that a classical solution is *not* always a viscosity solution. To see this, consider on $\mathbb{R}$ the problem $u''-2=0$. The function $f(t)=t^2$ is clearly a classical solution, but it is not a viscosity solution, because it is not a viscosity supersolution (take $\varphi \equiv 0$ and study the situation at the origin).

Since we can always take $u=\phi$ if u is at least C^2, the following theorem is trivial:

Theorem 21.9 (C^2 viscosity solutions are classical solutions)**.** *Let $\Omega \subset \mathbb{R}^n$ be open, $u \in C^2(\Omega)$ and E continuous. If u is a viscosity solution of* (21.1) *on Ω, then it is also a classical solution of the same problem.*

The converse holds if $S \mapsto E(x,u,p,S)$ is nonincreasing in $\mathrm{Sym}^{n\times n}$.

Theorem 21.10 (Classical solutions are viscosity solutions)**.** *If u is a classical subsolution (resp. supersolution) of* (21.1)*, then it is also a viscosity subsolution (resp. supersolution) of the same problem whenever $E(x,u,p,\cdot)$ is nonincreasing in* $\mathrm{Sym}^{n\times n}$.

Proof. We just study the case of subsolutions. For a test function φ, if $u-\varphi$ has a local maximum at a point x then we know by elementary calculus that $Du(x) = D\varphi(x)$ and $D^2u(x) \le D^2\varphi(x)$ and by definition $E_*(x,u(x),Du(x),D^2u(x)) \le 0$. Consequently, exploiting our monotonicity assumption (which is inherited by E_*) we obtain $E_*(x,u(x),D\varphi(x),D^2\varphi(x)) \le 0$ and the conclusion follows. □

Before going further, we need to spend some words on conventions. First of all, it should be clear that this theory also applies to parabolic equations such as $(\partial_t-\Delta)u-g=0$ if we let $x := (y,t) \in \mathbb{R}^n \times (0,+\infty)$ with $A = \mathbb{R}^n \times (0,+\infty)$. Secondly, it is worth remarking that many authors adopt a different conventions, which we might call *elliptic convention*, which is "opposite" to the one we gave before. Indeed, according to this convention, if (for instance) we deal with a problem of the form $F(D^2u)=0$, we require for a subsolution that $u^*-\varphi$ has a maximum at x implies $F(\nabla^2\varphi(x)) \ge 0$ (i.e. a subsolution of $-F(D^2u)=0$ in our terminology). As a consequence, in the previous theorem, we should replace "nonincreasing" with "nondecreasing."

Now, we are ready to introduce the first important tool for the following theorems.

Theorem 21.11. *Let $\mathcal{F}$ be a family of subsolutions of* (21.1) *in A and let $u : A \to \overline{\mathbb{R}}$ be defined by*

$$u := \sup\{v | v \in \mathcal{F}\}.$$

Then u is a subsolution of the same problem on the domain $A \cap \{u^* < \infty\}$.

Proof. Assume as usual $u^*-\varphi$ has a strict local maximum at x, equal to 0, and denote by K the compact set $\overline{B}_r(x) \cap A$ for some r to be chosen sufficiently small, so that x is the unique maximum of $u^*-\varphi$ on K.

We can find a sequence (x_h) inside K convergent to x and a family of functions $(v_h) \subset \mathcal{F}$ such that $u^*(x) = \lim_h u(x_h) = \lim_h v_h(x_h)$. Hence, if we call y_h the maximum of $v_h^* - \varphi$ on K, then

$$u^*(y_h) - \varphi(y_h) \geq v_h^*(y_h) - \varphi(y_h) \geq v_h^*(x_h) - \varphi(x_h) \geq v_h(x_h) - \varphi(x_h).$$

Since by our construction we have $v_h(x_h) - \varphi(x_h) \to 0$ for $h \to \infty$, we get that every limit point z of (y_h) satisfies

$$u^*(z) - \varphi(z) \geq 0.$$

Hence z is a maximum in K of $u^* - \varphi$, $u^*(z) - \varphi(z) = 0$ and z must coincide with x. Consequently $y_h \to x$, $u^*(y_h) - \varphi(y_h) \to 0$ and, by comparison, the same is true for the intermediate terms, so that $v_h^*(y_h) \to u^*(x)$. In order to conclude, we just need to consider the viscosity condition at the points y_h which reads

$$E_*(y_h, v_h^*(y_h), D\varphi(y_h), D^2\varphi(y_h)) \leq 0$$

and let $h \to \infty$ to get

$$E_*(x, u^*(x), D\varphi(x), D^2\varphi(x)) \leq 0.$$

□

We can now state a first existence result.

Theorem 21.12 (Perron)**.** *Let f and g be respectively a subsolution and a supersolution of* (21.1)*, such that $f_* > -\infty$ and $g^* < +\infty$ on A. If $f \leq g$ on A and the function $E(x, u, p, \cdot)$ is non-increasing, then there exists a solution u of* (21.1) *satisfying $f \leq u \leq g$.*

Proof. Call

$$\mathcal{F} := \{v | v \text{ is a subsolution of (21.1) and } v \leq g\} .$$

We know that $f \in \mathcal{F}$, so that this set is not empty. Hence, we can define $u := \sup\{v |\ v \in \mathcal{F}\}$. By our definition of $\mathcal{F}$, we have that $u \leq g$ and therefore $u^* \leq g^* < +\infty$. Since $u^* \geq u_* \geq f_* > -\infty$, in A, by Theorem 21.11 u is a subsolution on A. Consequently, we just need to prove that it is also a supersolution on the same domain.

Pick a test function φ such that $u_* - \varphi$ has a relative minimum, equal to 0, at x_0. We know that, without loss of generality we can assume that

$$u_*(x) - \varphi(x) \geq |x - x_0|^4 \qquad \text{on } A \cap \overline{B}_r(x) \tag{21.4}$$

for some sufficiently small $r > 0$. Assume by contradiction that

$$E^*(x_0, u_*(x_0), D\varphi(x_0), D^2\varphi(x_0)) < 0 \tag{21.5}$$

and define a function $w := \max\{\varphi + \delta^4, u\}$ for some parameter $\delta > 0$. We claim that:

(a) w is a subsolution of (21.1);

(b) $w \le g$ (and hence $w \in \mathcal{F}$);

(c) $\{w > u\} \neq \emptyset$,

provided we choose δ sufficiently small.

It is easily proved, again by contradiction and exploiting the fact that E^* is upper semicontinuous, that for $\delta > 0$ sufficiently small we have

$$E^*(x, \varphi(x) + \delta^4, D\varphi(x), D^2\varphi(x)) \le 0 \qquad \text{on } \overline{B}_{2\delta}(x_0) \cap A \ .$$

This means that $\varphi + \delta^4$ is a classical subsolution of (21.1) on this domain and hence, by our monotonicity hypothesis, it has to be also a viscosity subsolution. Consequently, by a very special case of the previous theorem, we get that the function w is a viscosity subsolution of (21.1) on $\overline{B}_{2\delta}(x_0) \cap A$. Moreover, by (21.4), we know that $w = u$ on $A \setminus \overline{B}_\delta(x_0)$. Since the notion of viscosity solution is clearly local, so that w is a global subsolution on A.[6]

To prove that $\{w > u\} \neq \emptyset$ we just need to observe that any $\delta > 0$ $u_*(x_0) = \varphi(x_0) < \varphi(x_0) + \delta^4$, and on any sequence (x_n) such that $u(x_n) \to u_*(x_0)$, we must have for n sufficiently large the inequality $u(x_n) < \varphi(x_n) + \delta^4$.

Finally, we have to show that $w \le g$: this completes the proof of the claim a gives the desired contradiction in order to conclude. To this aim, it is enough to prove that there exists $\delta > 0$ such that $\varphi + \delta^4 < g$ on $A \cap \overline{B}_\delta(x_0)$. But this readily follows, by an elementary argument, by showing that $\varphi(x_0) = u_*(x_0) < g_*(x_0)$. Again, assume by contradiction that $u_*(x_0) = g_*(x_0)$: if this were the case, the function $g_* - \varphi$ would have a local minimum at x_0 and so, since g_* is a viscosity supersolution, we would get

$$E^*(x_0, g_*(x_0), D\varphi(x_0), D^2\varphi(x_0)) \ge 0,$$

which is in contrast with (21.5). □

21.3 The distance function

Our next goal is now to study the uniqueness problem, which is actually very delicate as the previous examples show. We begin here with a special case.

Let $C \subset \mathbb{R}^n$ be a closed set, $C \neq \emptyset$ and let $u(x) := \mathrm{dist}(x, C)$. We claim that the distance function is a viscosity solution of the equation $|p|^2 - 1 = 0$ on $A := \mathbb{R}^n \setminus C$.

First of all, it is clearly a viscosity supersolution in A. This follows by Theorem 21.11 (in the obvious version for supersolutions), once we observe that $u(x) = \inf_{y \in C} |x - y|$

[6]We mean that if $A = A_1 \cup A_2$ and we know that u is a subsolution both on A_1 and A_2, relatively open in A, then it is also a subsolution on A.

and that for any $y \in C$ the function $x \to |x-y|$ is a classical supersolution in A (because $y \notin A$) and hence a viscosity supersolution of our problem.

The fact that u is also a subsolution follows by the general implication:

$$\mathrm{Lip}(f) \le 1 \;\Rightarrow\; |Df|^2 - 1 \le 0 \qquad \text{in the sense of viscosity solutions.}$$

Indeed, let x be a local maximum for $f - \varphi$, so that $f(y) - \varphi(y) \le f(x) - \varphi(x)$ for any $y \in B_r(x)$ (and r small enough). This is equivalent, on the same domain, to $\varphi(y) - \varphi(x) \ge f(y) - f(x) \ge -|y-x|$ and by the Taylor expansion we finally get

$$\langle D\varphi(x), y-x \rangle + o(|y-x|) \ge -|y-x| \ .$$

This readily implies the claim.

Less trivial, but still true, is the converse implication, namely

$$|Df|^2 - 1 \le 0 \text{ in the sense of viscosity solutions} \quad \Rightarrow \quad \mathrm{Lip}(f) \le 1$$

for f continuous (or at least upper semicontinuous), which is proved by means of the regularizations $f^\epsilon(x) := \sup_y \bigl(f(y) - |x-y|^2/(2\epsilon)\bigr)$ that we will study more in detail later on. We just sketch here the structure of the argument:

(1) still $|Df^\epsilon|^2 - 1 \le 0$ in the sense of viscosity solutions;

(2) $|Df^\epsilon|^2 - 1 \le 0$ pointwise $\mathscr{L}^n$-a.e., because f^ϵ is semiconcave and we can apply Alexandrov's theorem (here, since the equation is of first order, Rademacher's theorem would be sufficient);

(3) by Proposition 1.3 one obtains $\mathrm{Lip}(f^\epsilon) \le 1$;

(4) $f^\epsilon \downarrow f$ and hence $\mathrm{Lip}(f) \le 1$.

We now come to our uniqueness result.

Theorem 21.13. *Let $C \subset \mathbb{R}^n$ be a closed set as above, $A = \mathbb{R}^n \setminus C$ and let $u \in C(\overline{A})$ be a nonnegative viscosity solution of $|p|^2 - 1 = 0$ on A with $u = 0$ on ∂A. Then $C \neq \emptyset$ and $u(x) = \mathrm{dist}(x, C)$.*

Proof. By our assumptions we can clearly extend u continuously to $\mathbb{R}^n$, so that $u = 0$ identically on C. It is immediate to verify $|\nabla u|^2 - 1 \le 0$ in the sense of viscosity solutions on $\mathbb{R}^n$. Consequently, thanks to the previous regularization argument, $\mathrm{Lip}(u) \le 1$ and hence, for any $y \in C$ we have that $u(x) \le |x-y|$ which means $u(x) \le \mathrm{dist}(x, C)$. In the sequel, in order to simplify the notation, we will write $w(x)$ for the distance function $\mathrm{dist}(x, C)$.

So, it remains to show that $w \le u$. Assume first that A is bounded: we will show later on that this is not restrictive. By contradiction, assume that $w(x_0) > u(x_0)$ for some x_0; in this case there exist $\lambda_0 > 0$ and $\gamma_0 > 0$ such that

$$\sup_{x,y} \left\{ w(x) - (1+\lambda)u(y) - \frac{1}{2\epsilon}|x-y\|^2 \right\} \ge \gamma_0$$

for all $\epsilon > 0$ and $\lambda \in (0, \lambda_0)$. Indeed, it suffices to bound from below the supremum with $w(x_0) - (1+\lambda)u(x_0)$, which is larger than $\gamma_0 := (w(x_0) - u(x_0))$ for $\lambda > 0$ small enough.

Moreover, for $\epsilon > 0$ and $\lambda \in (0, \lambda_0)$ the supremum is actually a maximum because it is clear that we can localize x in A (otherwise the whole sum is nonpositive) and y in a bounded set of $\mathbb{R}^n$ (because w is bounded on A, and again for $|y - x|$ large the whole sum is nonpositive). So, call $(\overline{x}, \overline{y})$ a maximizing couple, omitting for notational simplicity the dependence on the parameters ϵ, λ. The function $x \to w(x) - \frac{1}{2\epsilon}\left|x - \overline{y}\right|^2$ has a maximum at $x = \overline{x}$ and so we can exploit the fact that $w(\cdot)$ is a viscosity solution of our equation (with respect to the test function $\varphi(x) = |x - \overline{y}|^2/(2\epsilon)$) to derive $|D\varphi|^2(\overline{x}) \le 1$, that is

$$\frac{|\overline{x} - \overline{y}|}{\epsilon} \le 1 \ .$$

We also claim that necessarily $\overline{y} \in A$. if ϵ is sufficiently small, so that $\epsilon < 2\gamma_0$. Indeed, assume by contradiction that $\overline{y} \notin A$, so that $w(\overline{y}) = 0$, then by the triangle inequality

$$\begin{aligned} \gamma_0 &\le w(\overline{x}) - \frac{1}{2\epsilon}|\overline{x} - \overline{y}|^2 \le |\overline{x} - \overline{y}| - \frac{1}{2\epsilon}|\overline{x} - \overline{y}|^2 \\ &= |\overline{x} - \overline{y}| \left(1 - \frac{1}{2\epsilon}|\overline{x} - \overline{y}|\right) \le \frac{|\overline{x} - \overline{y}|}{2} \end{aligned}$$

since we have shown that $|\overline{x} - \overline{y}| \le \epsilon$. As a consequence, we get $2\gamma_0 \le |\overline{x} - \overline{y}| \le \epsilon$, which gives a contradiction.

Now, choosing ϵ so that $\overline{y} \in A$, the function $y \to (1+\lambda)u(y) + \frac{1}{2\epsilon}\left|\overline{x} - y\right|^2$ has a minimum at $y = \overline{y}$ and arguing as above we obtain

$$\left| \frac{\overline{x} - \overline{y}}{\epsilon} \right| \ge (1 + \lambda) \ ,$$

which is not compatible with $|\overline{x} - \overline{y}| \le \epsilon$. Hence, at least when A is bounded, we have proved that $w = u$.

In the general case, fix a constant $R > 0$ and define $u_R(x) := u(x) \wedge \mathrm{dist}(x, \mathbb{R}^n \setminus \overline{B_R})$: this is a supersolution of our problem on $A \cap B_R$, since $u(x)$ is a supersolution on A and $\mathrm{dist}(x, \mathbb{R}^n \setminus \overline{B_R})$ is a supersolution on B_R (by the infimum property). Moreover, $\mathrm{Lip}(u_R) \le 1$ implies that u_R is a global subsolution and we can apply the previous result (special case) to the function u_R to get

$$u_R(x) = d(x, (\mathbb{R}^n \setminus A) \cup (\mathbb{R}^n \setminus B_R)).$$

Letting $R \to \infty$ we first exclude $C = \emptyset$ since in that case $u_R \uparrow \infty$ which is not admissible since $u_R \le u$ and then (by $C \neq \emptyset$) we obtain $u(x) = \mathrm{dist}(x, C)$. $\square$

Remark 21.14. We can also give a different interpretation of the result above. In the spirit of the classical Liouville's theorems we can say that "the equation $|Du|^2 - 1 = 0$ does not have entire viscosity solutions on $\mathbb{R}^n$ that are bounded form below". Nevertheless, there exist trivial examples of functions that solve this equation in the viscosity sense and are unbounded from below (e.g. take $u(x) = x_i$ for some $i \in \{1, \dots, n\}$.)

21.4 Maximum principle for semiconvex functions

We now turn to the case of second-order problems having the form $F(Du, D^2u) = 0$ on an open domain $A \subset \mathbb{R}^n$. We will always assume that $F(p, S)$ is non-increasing in its second variable S, so that classical solutions are viscosity solutions.

Let us begin with some heuristics. Let $f, g \in C^2(A) \cap C(\overline{A})$ and assume that f is a subsolution on A, g is a supersolution on A and $f \le g$ on ∂A and that one of the inequalities $F(Df, D^2f) \le 0$, $F(Dg, D^2g) \ge 0$ is always strict. Then $f \le g$ in A. Indeed, assume by contradiction $\sup_A(f - g) > 0$, then there exists a $x_0 \in A$ which is a maximum for $f - g$. Consequently $\nabla f(x_0) = \nabla g(x_0)$ and also $D^2f(x_0) \le D^2g(x_0)$. These two facts imply by the monotonicity of F that

$$F(Df(x_0), D^2f(x_0)) \ge F(Dg(x_0), D^2g(x_0)) \tag{21.6}$$

On the other hand, f (resp. g) is also a regular subsolution (resp. supersolution) so that

$$F(Df(x_0), D^2f(x_0)) \le 0 \quad F(Dg(x_0), D^2g(x_0)) \ge 0 \,. \tag{21.7}$$

Hence, if we compare (21.6) with (21.7), we find a contradiction as soon as one of the two inequalities in (21.7) is strict.

In order to hope for a comparison principle, this argument shows the necessity to approximate subsolutions (or supersolutions) with strict subsolutions, and this is always linked to some form of strict monotonicity of the equation, variable from case to case (of course in the case $F \equiv 0$ no comparison principle is possible). To clarify this point, let us consider the following example. Consider the space-time coordinates $x = (y, t)$ and a parabolic problem

$$F(D_{y,t}u, D^2_{y,t}u) = \partial_t u - G(D^2_y u)$$

with G nondecreasing, in the appropriate sense. In this case, we can reduce ourselves to strict inequalities by performing the transformation $u \to e^{\lambda t}u$.

In order to get a general uniqueness result for viscosity solution, we cannot just argue as in the case of the distance function and we need to follow a strategy introduced by Jensen. The first step is to obtain a refined versions of the maximum principle. We start with an elementary observation.

Remark 21.15. If $(p, S) \in J^2_+ u(x)$ and u has a relative maximum at x, then necessarily $p = 0$ and $S \leq 0$. To see this, it is enough to apply the definitions: by our two hypotheses

$$0 \geq u(y) - u(x) = \langle p, y - x\rangle + \frac{1}{2}\langle S(y-x), y-x\rangle + o(|y-x|^2)$$

and hence

$$\langle p, \frac{y-x}{|y-x|}\rangle \leq o(|y-x|) \ \Rightarrow p = 0,$$
$$\frac{\langle S(y-x), y-x\rangle}{|y-x|^2} \leq o(1) \ \Rightarrow S \leq 0.$$

We are now ready to state and prove Jensen's maximum principle for semiconvex functions.

Theorem 21.16 (Jensen's maximum principle)**.** *Let $u : \Omega \to \mathbb{R}$ be semiconvex and let $x_0 \in \Omega$ a local maximum for u. Then, there exist a sequence (x_n) convergent to x_0 and $\epsilon \downarrow 0$ such that u is pointwise second-order differentiable at x_n and*

$$Du(x_n) \to 0 \qquad\qquad D^2 u(x_n) \leq \epsilon_n I \ .$$

The proof is based on the following lemma. In the sequel we shall denote by $\mathrm{sc}(u, \Omega)$ the least constant C such that u is $(-C)$-convex, i.e. $u + C|x|^2/2$ is convex (recall Definition 20.8).

Theorem 21.17. *Let $B \subset \mathbb{R}^n$ a ball of radius R and $u \in C(\overline{B})$ semiconvex, with*

$$\max_{\overline{B}} u > \max_{\partial B} u.$$

(Notice that this implies $\mathrm{sc}(u, B) > 0$)). Then, if we let

$$G^\delta = \Big\{x \in B | \textit{there exists } p \in \overline{B}_\delta \textit{ s.t. } u(y) \leq u(x) + \langle p, x - y\rangle \ \forall y \in B\Big\}$$

it must be

$$\mathscr{L}^n(G^\delta) \geq \frac{\omega_n \delta^n}{[\mathrm{sc}(u, B)]^n}$$

for $0 < \delta < (\max_{\overline{B}} u - \min_{\overline{B}} u)/(2R)$.

Proof. We assume first that u is also in $C^1(B)$. Pick a $\delta > 0$, so small that $2R\delta < \max_{\overline{B}} u - \max_{\partial B} u$ and consider a perturbation $u(y) + \langle p, y\rangle$ with $|p| < \delta$. We claim that such functions necessarily attains its maximum in B. Indeed, this immediately comes from the two inequalities

$$\max_{\partial B}(u + \langle p, y\rangle) \leq \max_{\partial B} u + \delta R$$

and

$$\max_{\overline{B}}(u+\langle p,y\rangle)\geq \max_{\overline{B}} u-\delta R.$$

Consequently, there exists $x\in B$ such that $\nabla u(x)=p$. This shows that $\nabla u(G^\delta)=\overline{B}_\delta$. To go further, we need the area formula. In this case, it gives

$$\int_{G^\delta}|\det D^2u|\,dx=\int_{\overline{B}_\delta}\operatorname{card}\big(\{x|\nabla u(x)=p\}\big)\,dp\geq \omega_n\delta^n$$

by the previous statement. On the other hand

$$\int_{G^\delta}\left|\det D^2u\right|\,dx\leq [\operatorname{sc}(u,B)]^n\,\mathscr{L}^n(G^\delta).$$

Indeed, since the points in G^δ are maxima for the function $u(y)+\langle p,y\rangle$, this implies $D^2u(x)\leq 0$ for any $x\in G_\delta$ and, by semi-convexity, $D^2u(x)\geq -\operatorname{sc}(u,B)I$. If we combine these two inequalities, we get

$$\mathscr{L}^n(G^\delta)\geq \frac{\omega^n\delta^n}{[\operatorname{sc}(u,B)]^n}$$

which is nothing but the thesis.

In the general case we argue by approximation, finding smooth functions u_n such that $u_n\to u$ uniformly and $\operatorname{sc}(u_n,B)\to\operatorname{sc}(u,B)$; to conclude, it suffices to notice that limit of points in $G^\delta(u_n)$ belongs to $G^\delta(u)$, hence $\mathscr{L}^n(G^\delta(u))\geq\limsup_n\mathscr{L}^n(G^\delta(u_n))$. □

We can now prove Jensen's maximum principle.

Proof. Let x_0 be a local maximum of u. We can choose $R>0$ sufficiently small so that $u(x)\leq u(x_0)$ in $\overline{B}_R(x_0)$ and, without loss of generality, we can assume $u(x_0)=0$. This becomes a strict local maximum for the function $\widetilde{u}(x)=u(x)-|x-x_0|^4$. It is also easy to verify that $\widetilde{u}$ is semi-convex in $\overline{B}_R(x_0)$. We now apply the previous lemma to $\widetilde{u}$: for any $\delta=1/k$ we obtain that $\mathscr{L}^n(G^{1/k})>0$ and (thanks to the Alexandrov theorem) this means that there exists a sequence of points (x_k) such that $\widetilde{u}$ is pointwise second-order differentiable at x_k and for appropriate vectors p_k with $|p_k|\leq 1/k$ the function $\widetilde{u}(y)-\langle p_k,y\rangle$ has a local maximum at x_k. Since $|p_k|\to 0$, any limit point of (x_k) for $k\to\infty$ has to be a local maximum for $\widetilde{u}$, but in $\overline{B}_R(x_0)$ this necessarily implies $x_k\to x_0$. Moreover $p_k=D\widetilde{u}(x_k)\to 0$ and $D^2\widetilde{u}(x_k)\leq 0$. As a consequence,

$$Du(x_k)=D\widetilde{u}(x_k)+4|x_k-x_0|^2(x_k-x_0)\to 0$$

and

$$\begin{aligned} D^2u(x_k) &= D^2\widetilde{u}(x_k)+8(x_k-x_0)\otimes(x_k-x_0)+4|x_k-x_0|^2I\\ &\leq \ \leq D^2\widetilde{u}(x_k)+12|x_k-x_0|^2I\ . \end{aligned}$$

Setting $\epsilon_k=12|x_k-x_0|^2$ we get the thesis. □

We now introduce another important tool in the theory of viscosity solutions.

Definition 21.18 (Inf and sup convolutions)**.** *Given $u : A \to \mathbb{R}$ and a parameter $\epsilon > 0$, we can build the regularized functions*

$$u^\epsilon(x) := \sup_{y \in A} \left\{ u(y) - \frac{1}{\epsilon}|x-y|^2 \right\} \tag{21.8}$$

which is called the sup-convolution of u and verifies $u^\epsilon \geq u$, and

$$u_\epsilon(x) := \inf_{y \in A} \left\{ u(y) + \frac{1}{\epsilon}|x-y|^2 \right\}. \tag{21.9}$$

which is called the inf-convolution of u and verifies $u_\epsilon \leq u$.

In the next proposition we summarize the main properties of sup-convolutions; analogous properties hold for inf-convolutions.

Proposition 21.19 (Properties of sup convolutions)**.** *Assume that u is u.s.c. on A and that $u(x) \leq K(1+|x|)$ for some constant $K > 0$.*

(i) u^ϵ is semiconvex and $\mathrm{sc}(u^\epsilon, \mathbb{R}^n) \leq 2/\epsilon$*;*

(ii) $u^\epsilon \geq u$ and $u^\epsilon \downarrow u$ pointwise, locally uniformly if u is continuous;

(iii) If $F(Du, D^2u) \leq 0$ in the sense of viscosity solutions on A, then $F(Du^\epsilon, D^2u^\epsilon) \leq 0$ on A^ϵ, where

$$A^\epsilon := \{x \in \mathbb{R}^n |\ \textit{the supremum in (21.8) is attained}\}.$$

Proof. (i) First of all, notice that by the linear growth assumption, the function u^ϵ is real-valued for any $\epsilon > 0$. Moreover, by its very definition

$$u^\epsilon(x) + \frac{1}{\epsilon}|x|^2 = \sup_{y \in A} u(y) - \frac{1}{\epsilon}|y|^2 + \frac{2}{\epsilon}\langle x, y\rangle$$

and the functions in the right hand side are affine with respect to x. It follows that the left hand side is convex, which means $\mathrm{sc}(u^\epsilon, \mathbb{R}^n) \leq 2/\epsilon$.
(ii) The inequality $u^\epsilon \geq u$ and the monotonicity in ϵ are trivial. In addition, we can take quasi-maxima (y_ϵ) satisfying

$$u^\epsilon(x) \leq u(y_\epsilon) - \frac{\delta_\epsilon^2}{\epsilon} + \epsilon \leq K(1+|y_\epsilon|) - \frac{\delta_\epsilon^2}{\epsilon} + \epsilon \leq K(1+|x|+|\delta_\epsilon|) - \frac{\delta_\epsilon^2}{\epsilon} + \epsilon\ .$$

with $\delta_\epsilon = |y_\epsilon - x|$. Via these two inequalities, one first sees that $y_\epsilon \to x$ so that, exploiting the upper semicontinuity of u and neglecting the quadratic term in the first inequality we get

$$u(x) \geq \limsup_{\epsilon \to 0} u(y_\epsilon) \geq \limsup_{\epsilon \to 0} u^\epsilon(x) \ .$$

If u is continuous, the claim comes form Dini's Lemma concerning monotone convergence. (iii) Let $x_0 \in A^\epsilon$ and let $y_0 \in \mathbb{R}^n$ be the corresponding maximum, so that $u^\epsilon(x_0) = u(y_0) - |x_0 - y_0|^2/\epsilon$. Let then be φ a smooth function such that $u^\epsilon - \varphi$ has a local maximum in x_0 and, without loss of generality, we can take $u^\epsilon(x_0) = \varphi(x_0)$. Define $\psi(x) := \varphi(x - y_0 + x_0)$: we claim that $u - \psi$ has a local maximum at y_0 with value $|x_0 - y_0|^2/\epsilon$. If we prove this claim, then it must be

$$F(D\psi(y_0), D^2\psi(y_0)) \leq 0$$

and, by the definition of ψ, this is equivalent to

$$F(D\varphi(x_0), D^2\varphi(x_0)) \leq 0.$$

Hence, it is enough to prove the claim. On the one hand

$$u(y_0) - \psi(y_0) = u(y_0) - \varphi(x_0) = u(y_0) - u^\epsilon(x_0) = \frac{1}{\epsilon}|x_0 - y_0|^2,$$

while on the other $u^\epsilon(x) \leq \varphi(x)$ in $B_r(x_0)$ gives

$$u(y) - \frac{1}{\epsilon}|x - y|^2 \leq \varphi(x) \quad \forall x \in B_r(x_0), \, \forall y \in \mathbb{R}^n$$

and, letting $y = x - x_0 + y_0$ with $x \in B_r(x_0)$, this implies

$$u(y) \leq \psi(y) + \frac{1}{\epsilon}|x_0 - y_0|^2 \ \forall y \in B_r(y_0) \ .$$

□

21.5 Existence and uniqueness results

In this section we can collect some existence and uniqueness results for second-order equations. The main tool is the comparison principle, stated below. In this section we shall always assume that A is a bounded open set in $\mathbb{R}^n$.

Proposition 21.20 (Comparison principle)**.** *Let $F : A \times \mathbf{Sym}^{n\times n} \to \mathbb{R}$ be continuous and satisfying, for some $\lambda > 0$, the strict monotonicity condition*

$$F(x, S + tI) \geq F(x, S) + \lambda t \qquad \forall t \geq 0$$

and the uniform continuity assumption

$$\lim_{x\to\bar{x}} F(x, S) = F(\bar{x}, S) \qquad \textit{uniformly in } S \in \mathrm{Sym}^{n\times n}, \textit{ for all } \bar{x} \in A.$$

Let $\underline{u}$, $\overline{u} : A \to \mathbb{R}$ be respectively a bounded u.s.c. subsolution and a bounded l.s.c. supersolution to $-F(x, D^2u) = 0$ in A, with $(\underline{u})^* \leq (\overline{u})_*$ on ∂A. Then $\underline{u} \leq \overline{u}$ on A.

Notice that the uniform continuity assumption, though restrictive, covers the equations of the form $G(D^2u) + f(x)$ with f continuous in A. Also, the proof we shall give shows that in the case when F is independent of x the strict monotonicity assumption is not needed.

A direct consequence of the comparison principle (take $\underline{u} = \overline{u} = u$) is the following uniqueness result:

Theorem 21.21 (Uniqueness of continuous solutions)**.** *Let F be as in the comparison principle and $h \in C(\partial A)$. Then the problem*

$$\begin{cases} -F(x, D^2u(x)) = 0 & \textit{in } A; \\ \\ u = h & \textit{on } \partial A \end{cases} \tag{21.10}$$

admits at most one viscosity solution $u \in C(\overline{A})$.

At the level of existence, we can exploit Theorem 21.12 to obtain the following result:

Theorem 21.22 (Existence of continuous solutions)**.** *Let F be as in the comparison principle and let f and g be respectively a subsolution and a supersolution of $-F(x, D^2u) = 0$, such that $f_* > -\infty$, $g^* < +\infty$ and $f \leq g$ on A. If $g^* \leq f_*$ on ∂A, there exists a solution to* (21.10) *with $h = g^* = f_*$.*

In order to prove this last result, it suffices to take any solution u given by Perron's method, so that $f \leq u \leq g$ in A. It follows that $u^* \leq g^* \leq f_* \leq u_*$ on ∂A and the comparison principle (with $\underline{u} = u^*$, $\overline{u} = u_*$) gives $u^* \leq u_*$ on A, i.e. u is continuous.

The rest of the section will be devoted to the proof of the comparison principle, which uses besides doubling of variables, inf-sup convolutions and Jensen's maximum principle.

First of all, for $\gamma > 0$ we set

$$F_\gamma(x, S) := F(x, S - \gamma I) \le F(x, S) - \gamma\lambda$$

and we see that we can assume with no loss of generality that $\underline{u}$ satisfies the stronger property $-F_\gamma(x, D^2 u) \le 0$ in the viscosity sense. Indeed, once we are able to show that this assumption implies the comparison property, we get

$$\underline{u} - \delta + \frac{\gamma(\delta)}{2}|x|^2 \le \overline{u} \qquad \text{in } A$$

for some $\gamma < \delta$ depending only on δ and A. Indeed, for $\delta > 0$ fixed still $\underline{u} - \delta + \frac{\gamma}{2}|x|^2 \le \overline{u}$ in ∂A for γ small enough, and satisfies $-F_\gamma(x, D^2 u) \le 0$. To conclude the proof in the general case, it suffices to pass to the limit as $\delta \downarrow 0$.

Assume by contradiction that $d_0 := \underline{u}(x_0) - \overline{u}(x_0) > 0$ for some $x_0 \in A$, and let us consider the sup convolution

$$u^\epsilon(x) := \sup_{x' \in A} \underline{u}(x') - \frac{1}{\epsilon}|x - x'|^2 \tag{21.11}$$

of $\underline{u}$ and the inf convolution

$$u_\epsilon(y) := \sup_{y' \in A} \overline{u}(y') - \frac{1}{\epsilon}|y - y'|^2 \tag{21.12}$$

of $\overline{u}$; since $u^\epsilon \ge \underline{u}$ and $u_\epsilon \le \overline{u}$ we have

$$\max_{\overline{A} \times \overline{A}} u^\epsilon(x) - u_\epsilon(y) - \frac{1}{4\epsilon}|x - y|^4 \ge u^\epsilon(x_0) - u_\epsilon(x_0) \ge \underline{u}(x_0) - \overline{u}(x_0) = d_0$$

and we shall denote by $(x_\epsilon, y_\epsilon) \in \overline{A} \times \overline{A}$ a maximizing pair, so that

$$d_0 + \frac{1}{\epsilon}|x_\epsilon - y_\epsilon|^4 \le u^\epsilon(x_\epsilon) - u_\epsilon(y_\epsilon) \le \sup \underline{u} - \inf \overline{u}\ . \tag{21.13}$$

Also, we notice that

$$u^\epsilon(x_\epsilon) := \max_{x' \in \overline{A}} (\underline{u})^*(x') - \frac{1}{\epsilon}|x_\epsilon - x'|^2, \quad u_\epsilon(y_\epsilon) = \min_{y' \in \overline{A}} (\overline{u})_*(y') - \frac{1}{\epsilon}|y_\epsilon - y'|^2,$$

and we shall denote by $x'_\epsilon \in \overline{A}$ and $y'_\epsilon \in \overline{A}$ maximizers and minimizers respectively.

Now we claim that:

(a) $\liminf_{\epsilon\downarrow 0} \operatorname{dist}(x_\epsilon, \partial A) > 0$ and $\liminf_{\epsilon\downarrow 0} \operatorname{dist}(y_\epsilon, \partial A) > 0$;

(b) for ϵ small enough the supremum in (21.11) with $x \in B_\epsilon(x_\epsilon)$ is attained and the infimum in (21.12) with $y \in B_\epsilon(y_\epsilon)$ is attained.

To prove (a), notice that if $(\bar{x}, \bar{y})$ is any limit point of (x_ϵ, y_ϵ) as $\epsilon \downarrow 0$, then (21.13) gives $\bar{x} = \bar{y}$ and

$$d_0 \le \limsup_{\epsilon\downarrow 0} (\underline{u})^*(x'_\epsilon) - (\overline{u})_*(y'_\epsilon) - \frac{|x_\epsilon - x'_\epsilon|^2 + |y_\epsilon - y'_\epsilon|^2}{\epsilon} .$$

Since the supremum of $(\underline{u})^* - (\overline{u})_*$ is finite, this implies that $|x_\epsilon - x'_\epsilon| \to 0$, $|y_\epsilon - y'_\epsilon| \to 0$, hence $(x'_\epsilon, y'_\epsilon) \to (\bar{x}, \bar{x})$ as well and semicontinuity gives $d_0 \le (\underline{u})^*(\bar{x}) - (\overline{u})_*(\bar{x})$. By assumption $(\underline{u})^* \le (\overline{u})_*$ on ∂A, therefore $\bar{x} \in A$ and this proves (a).

To prove (b), it suffices to choose, thanks to (a), $\epsilon_0 > 0$ and $\delta_0 > 0$ small enough, so that $\operatorname{dist}(x_\epsilon, \partial A) \ge \delta_0$ for $\epsilon \in (0, \epsilon_0)$. Now, for all x we have

$$\underline{u}(x') - \frac{1}{\epsilon}|x' - x|^2 \le \max \underline{u} - \frac{1}{\epsilon}|x' - x|^2 < \underline{u}(x) \le u^\epsilon(x)$$

as soon as $|x - x'|^2 > \epsilon \operatorname{osc}(\underline{u})$. Hence, any maximizer x' belongs to the closed ball centered at x with radius $\sqrt{\epsilon \operatorname{osc}(\underline{u})}$. If $|x - x_\epsilon| < \epsilon$ and $\epsilon < \epsilon_0$, since $\operatorname{dist}(x_\epsilon, \partial A) \ge \delta_0$ this implies that any maximizer x' is in A for $\epsilon < \epsilon_0$ small enough. The argument for y_ϵ is similar.

Let us fix ϵ small enough so that (b) holds and both x_ϵ and y_ϵ belong to A, and let us apply Jensen's maximum principle to the (locally) semiconvex function

$$w(x, y) := u^\epsilon(x) - u_\epsilon(y) - \frac{1}{4\epsilon}|x - y|^4$$

to find $z_n := (x_{\epsilon,n}, y_{\epsilon,n}) \to (x_\epsilon, y_\epsilon)$ and $\delta_n \downarrow 0$ such that w is pointwise second-order differentiable at z_n, $Dw(z_n) \to 0$ and $D^2 w(z_n) \le \delta_n I$. By statement (b) and Proposition 21.19(iii), for n large enough (such that $|z_n - (x_\epsilon, y_\epsilon)| < \epsilon$) we have

$$-F_\gamma(x_{\epsilon,n}, D^2 u^\epsilon(x_{\epsilon,n})) \le 0, \qquad -F(y_{\epsilon,n}, D^2 u_\epsilon(y_{\epsilon,n})) \ge 0. \tag{21.14}$$

On the other hand, the upper bound on $D^2 w(z_n)$ gives

$$\begin{cases} D^2 u^\epsilon(x_{\epsilon,n}) - \frac{2}{\epsilon}(x_{\epsilon,n} - y_{\epsilon,n}) \otimes (x_{\epsilon,n} - y_{\epsilon,n}) - \frac{1}{\epsilon}|x_{\epsilon,n} - y_{\epsilon,n}|^2 I \le \delta_n I \\ -D^2 u_\epsilon(y_{\epsilon,n}) - \frac{2}{\epsilon}(x_{\epsilon,n} - y_{\epsilon,n}) \otimes (x_{\epsilon,n} - y_{\epsilon,n}) - \frac{1}{\epsilon}|x_{\epsilon,n} - y_{\epsilon,n}|^2 I \le \delta_n I . \end{cases} \tag{21.15}$$

By (21.15) we obtain that $D^2 u^\epsilon(x_{\epsilon,n})$ are uniformly bounded above, and they are also uniformly bounded below, since u^ϵ is semiconvex. Since similar remarks apply to $D^2 u_\epsilon(y_{\epsilon,n})$, we can assume with no loss of generality that $D^2 u^\epsilon(x_{\epsilon,n}) \to X_\epsilon$ and $D^2 u_\epsilon(y_{\epsilon,n}) \to Y_\epsilon$, and

(21.15) gives $X_\epsilon \leq Y_\epsilon$. On the other hand, from (21.14) we get $-F_\gamma(x_\epsilon, X_\epsilon) \leq 0$ and from the strict monotonicity property of $F(x_\epsilon, \cdot)$ we get

$$F(x_\epsilon, Y_\epsilon) \geq F(x_\epsilon, X_\epsilon) \geq \lambda\gamma, \qquad -F(y_\epsilon, Y_\epsilon) \geq 0 \ .$$

Hence $F(x_\epsilon, Y_\epsilon) - F(y_\epsilon, Y_\epsilon) \geq \lambda\gamma$. Finding a (common) limit point $\bar{x} \in A$ of x_ϵ and y_ϵ as $\epsilon \downarrow 0$ we contradict the uniform continuity assumption on $F(\cdot, S)$ at $x = \bar{x}$.

21.6 Hölder regularity

Definition 21.23 (Tangent paraboloids)**.** *Consider a paraboloid P (generally, $P(x) = c + \langle p, x\rangle + 1/2\langle Sx, x\rangle$), we say that P is a paraboloid centered in x_0 with opening M if*

$$P(x) = c + \frac{M}{2}|x - x_0|^2 \ .$$

Given a function $u : \Omega \to \mathbb{R}$ and a subset $A \subset \Omega \subset \mathbb{R}^n$, we denote

$$\overline{\theta}(x_0, A, u) := \inf\left\{M \;\middle|\; \exists\, P(x) = c + M/2|x - x_0|^2 \text{ with } u(x_0) = P(x_0) \text{ and } u \leq P \text{ on } A\right\}$$

Moreover, we put

$$\underline{\theta}(x_0, A, u) := \overline{\theta}(x_0, A, u)$$

and finally

$$\theta(x_0, A, u) := \max\left\{\underline{\theta}(x_0, A, u), \overline{\theta}(x_0, A, u)\right\} \ .$$

Given a function $u : \Omega \to \mathbb{R}$, let us consider the symmetric difference quotient in the direction ξ

$$\Delta^2_{h,\xi}u(x_0) := \Delta_{h,\xi}(\Delta_{h,\xi}u)(x_0) = \frac{u(x_0 + h\xi) - u(x_0 - h\xi) - 2u(x_0)}{h^2} \sim \frac{\partial^2 u}{\partial \xi^2}(x_0) \ .$$

Notice that the symmetric difference quotient satisfies, by applying twice the integration by parts formula for $\Delta_{h,\xi}$,

$$\int_\Omega u\Delta^2_{h,\xi}\phi\, dx = \int_\Omega \phi\Delta^2_{h,\xi}u\, dx \tag{21.16}$$

whenever $u \in L^1_{\text{loc}}(\Omega)$, $\phi \in L^\infty(\Omega)$ has compact support, $|\xi| = 1$ and the h-neighbourhood of suppϕ is contained in Ω.

Remark 21.24. If a paraboloid P with aperture M “touches” u from above (i.e. $P(x_0) = u(x_0)$ and $P(x) \geq u(x)$ in some ball $B_r(x_0)$), then

$$\Delta^2_{h,\xi}u(x_0) \leq \Delta^2_{h,\xi}P(x_0) = M \qquad \text{with } |\xi| = 1 \text{ and } |h| \leq r \ ,$$

and a similar property holds for paraboloids touching from below. Thus we deduce the inequalities

$$-\underline{\theta}(x_0, B_r(x_0), u) \leq \Delta^2_{h,\xi}u(x_0) \leq \overline{\theta}(x_0, B_r(x_0), u) \ .$$

Proposition 21.25. *If $u : \Omega \to \mathbb{R}$ satisfies*

$$\theta_\epsilon := \theta(\cdot, B_\epsilon(\cdot) \cap \Omega, u) \in L^p(\Omega)$$

for some $\epsilon > 0$ and $1 < p \leq \infty$, then u belongs to $W^{2,p}(\Omega)$ and, more precisely,

$$\|D_{\xi\xi} u\|_{L^p(|Omega)} \leq \|\theta_\epsilon\|_{L^p(\Omega)} \qquad \forall \xi \in \mathbf{S}^{n-1}. \tag{21.17}$$

Remark 21.26. By bilinearity it is possible to obtain estimates from (21.17) an estimate on mixed second derivatives:

$$\|D_{\xi\eta} u\|_{L^p(\Omega)} \leq |\xi||\eta|\|\theta_\epsilon\|_{L^p(\Omega)} \qquad \forall \xi,\, \eta \in \mathbb{R}^n,\, \xi \perp \eta\ .$$

Proof. Fix a smooth function $\varphi \in C^\infty_c(\Omega)$, then

$$\left| \int_\Omega u(x) \frac{\partial^2 \varphi}{\partial \xi^2}(x)\, dx \right| = \left| \lim_{h \to 0} \int_\Omega u(x) \Delta^2_{h,\xi} \varphi(x)\, dx \right|$$
$$= \quad \left| \lim_{h \to 0} \int_\Omega (\Delta^2_{h,\xi} u(x)) \varphi(x)\, dx \right| \leq \|\theta_\epsilon\|_{L^p(\Omega)} \|\varphi\|_{L^{p'}(\Omega)}\ ,$$

while we pass from the first to the second line with (21.16). Thanks to Riesz Representation Theorem, we know that the map $\varphi \mapsto \int_\Omega u(x) \frac{\partial^2 \varphi}{\partial \xi^2}(x)\, dx$ admits a representation with an element of $L^p(\Omega)$ which represents the derivative $D_\xi \xi u$ in the sense of distributions and which satisfies (21.17). □

Corollary 21.27. *If $\Omega \subset \mathbb{R}^n$ is convex and $\theta_\epsilon \in L^\infty(\Omega)$, then*

$$\mathrm{Lip}(\nabla u, \Omega) \leq \|\theta_\epsilon\|_{L^\infty(\Omega)}\ .$$

Proof. We recall that since Ω is convex and v is scalar we have $\|Dv\|_{L^\infty(\Omega)} = \mathrm{Lip}(v, \Omega)$ (while, in general, $\|Dv\|_{L^\infty(\Omega)} \leq \mathrm{Lip}(v, \Omega)$). If v takes values in $\mathbb{R}^n$ (in our case $v = \nabla u : \Omega \to \mathbb{R}^n$), then by the same smoothing argument used in the scalar case we can always show that

$$\||Dv|_{\mathcal{L}}\|_{L^\infty(\Omega)} = \mathrm{Lip}(v, \Omega)$$

(here $|A|_{\mathcal{L}}$ denoted the operator norm of A) because, when v is continuously differentiable

$$v(x) - v(y) = \int_0^1 Dv((1-t)x + ty)(y - x)\, dt \leq |x - y| \int_0^1 |Dv|_{\mathcal{L}}((1-t)x + ty)\, dt\ .$$

In our case Dv is a symmetric square matrix and its $\mathcal{L}$-norm coincides with its largest eigenvalue, that is

$$|Dv|_{\mathcal{L}} = \sup_{|\xi|=1} |\langle Dv\xi, \xi \rangle|\ .$$

□

At this point our aim is the study of nonlinear PDE as

$$-F(x, D^2u(x)) + f(x) = 0 \tag{21.18}$$

with F nondecreasing on $\mathrm{Sym}^{n\times n}$ (the laplacian, for example).

Definition 21.28 (Ellipticity)**.** *In the problem* (21.18) *we have ellipticity with constants* $\Lambda \geq \lambda > 0$ *if*

$$\lambda\|N\| \leq F(M+N) - F(M) \leq \Lambda\|N\| \qquad \forall\, N \geq 0 \tag{21.19}$$

where $\|N\|$ *means the largest eigenvalue of the symmetric matrix* N*.*

Remark 21.29. Every symmetric matrix N admits a unique decomposition as a sum

$$N = N^+ - N^-$$

with N^+, $N^- \geq 0$ and $N^+N^- = 0$. It can be obtained simply diagonalizing $N = \sum_{i=1}^n \rho_i e_i \otimes e_i$ and then choosing $N^+ := \sum_{\rho>0} \rho_i e_i \otimes e_i$ and $\sum_{\rho_i\leq 0} \rho_i e_i \otimes e_i$. Observing this, we are able to improve the definition of elliptic problem replacing (21.19) with

$$F(M+N) - F(M) \leq \Lambda\|N^+\| - \lambda\|N^-\| \tag{21.20}$$

for every symmetric matrix N.

Example 21.30. Consider the case

$$F(M) = \mathrm{tr}(BM)$$

where $B = (b_{ij})_{i,j=1,\dots,n}$ belongs to the set

$$\mathcal{A}_{\lambda,\Lambda} := \left\{B \in \mathrm{Sym}^{n\times n} | \lambda I \leq B \leq \Lambda I\right\} .$$

Fix the symmetric matrix N. In order to verify (21.19) we choose the coordinate system in which $N = \mathrm{diag}(\rho_1, \dots, \rho_n)$, thus

$$F(M+N) - F(M) = \mathrm{tr}(BN) = \sum_{i=1}^n b_{ii}\rho_i \geq \lambda \sum_{i=1}^n \rho_i \geq \lambda\rho_{\max}$$

and analogously

$$F(M+N) - F(M) = \mathrm{tr}(BN) = \sum_{i=1}^n b_{ii}\rho_i \leq \Lambda \sum_{i=1}^n \rho_i \leq n\Lambda\rho_{\max} .$$

After this introductory part about definitions and notation, we enter in the core of the matter of the Hölder regularity for viscosity solutions: as in De Giorgi's work on the XIX Hilbert problem, the regularity will be deduced only from inequalities derived from ellipticity, without a specific attention to the original equation. This idea is due to Pucci, who the main tools we are going to introduce are named after.

Definition 21.31 (Pucci's extremal operators)**.** *Given ellipticity constants $\Lambda \geq \lambda > 0$ and a symmetric matrix M with eigenvalues $\{\rho_1, \dots, \rho_n\}$, Pucci's extremal operators are defined by*

$$\begin{aligned} \mathcal{M}^-(M) &:= \lambda \sum_{\rho_i > 0} \rho_i + \Lambda \sum_{\rho_i < 0} \rho_i \\ \mathcal{M}^+(M) &:= \Lambda \sum_{\rho_i > 0} \rho_i + \lambda \sum_{\rho_i < 0} \rho_i \ . \end{aligned}$$

Remark 21.32. Example 21.30 shows that

$$\mathcal{M}^-(M) = \inf_{B \in \mathcal{A}_{\lambda,\Lambda}} \mathrm{tr}(BM) \tag{21.21}$$

$$\mathcal{M}^+(M) = \sup_{B \in \mathcal{A}_{\lambda,\Lambda}} \mathrm{tr}(BM) \ . \tag{21.22}$$

As a matter of fact, denoting with (b_{ij}) the coefficients of the matrix $B \in \mathcal{A}_{\lambda,\Lambda}$,

$$\mathrm{tr}(BM) = \sum_{i=1}^n b_{ii}\rho_i \geq \lambda \sum_{\rho_i > 0} \rho_i + \Lambda \sum_{\rho_i < 0} \rho_i \tag{21.23}$$

and the equality in (21.23) holds if

$$B = \sum_{\rho_i > 0} \lambda e_i \otimes e_i + \sum_{\rho_i < 0} \Lambda e_i \otimes e_i \ .$$

Remark 21.33. Pucci's extremal operators defined in Definition 21.31 satisfy the following properties:

(a) trivially $\mathcal{M}^- \leq \mathcal{M}^+$ and $\mathcal{M}^-(-M) = -\mathcal{M}^+(M)$ for every symmetric matrix M, moreover $\mathcal{M}^\pm$ are positively 1-homogeneous;

(b) for every M, N it is simple to obtain from (21.21) and (21.22) that

$$\mathcal{M}^+(M) + \mathcal{M}^-(N) \leq \mathcal{M}^+(M+N) \leq \mathcal{M}^+(M) + \mathcal{M}^+(N)$$

and similarly

$$\mathcal{M}^-(M) + \mathcal{M}^-(N) \leq \mathcal{M}^-(M+N) \leq \mathcal{M}^-(N) + \mathcal{M}^+(M) \ ;$$

(c) they are elliptic (i.e., they satisfy (21.19)) with constants $\lambda, n\Lambda$, because of Example 21.30.

Definition 21.34. *With the previous notations, we will denote*

$$\begin{aligned} \mathrm{Sub}(f) &:= \left\{u : \Omega \to \mathbb{R} \,\middle|\, -\mathcal{M}^+(u) + f \leq 0\right\} \\ \mathrm{Sup}(f) &:= \left\{u : \Omega \to \mathbb{R} \,\middle|\, -\mathcal{M}^-(u) + f \geq 0\right\} \ , \end{aligned}$$

finally

$$\mathrm{Sol}(f) := \mathrm{Sub}(-|f|) \cap \mathrm{Sup}(|f|) \ . \tag{21.24}$$

Remark 21.35. Roughly speaking, the classes defined above correspond to De Giorgi's class DG, since u solution to (21.18) imply $u \in \mathrm{Sol}(f)$ (where we use the ellipticity constants of F to define Pucci's operators); thus, if we are able to infer obtain regularity of functions in $\mathrm{Sol}(f)$ we can "forget" the specific equation.

22 Regularity theory for viscosity solutions

22.1 The Alexandrov-Bakelman-Pucci estimate

Let us introduce the following notation:

$$\mathrm{Sup}(f) := \left\{ u | -\mathcal{M}^-(u) + f \geq 0 \right\}$$

and

$$\mathrm{Sub}(f) := \left\{ u | -\mathcal{M}^+(u) + f \leq 0 \right\}$$

where $\mathcal{M}^\pm$ are Pucci's extremal operators (depending only on the ellipticity coefficients λ and Λ). Notice that since $\mathcal{M}^+ \geq \mathcal{M}^-$ the intersection of the two sets can be nonempty.

In the sequel, a constant will be called *universal* if it depends only on the space dimension n and the ellipticity constants λ, Λ.

The estimate we want to prove is named after Aleksandrov, Bakelman and Pucci and is therefore called *ABP weak maximum principle.*

Theorem 22.1. *Let $u \in \mathrm{Sup}(f) \cap C(\overline{B}_r)$ with $u \geq 0$ on ∂B_r. and $f \in C(\overline{B}_r)$. Then*

$$\max_{\overline{B}_r} u^- \leq cr \left(\int_{\{u=\Gamma_u\}} \left(f^+\right)^n \, dx \right)^{1/n}$$

with c is universal and Γ_u is defined below.

Since f^+ measures in some sense how far u is from being concave, the estimate above can be seen as a quantitative extension of the fact that a concave function in a ball attains its minimum on the boundary of the ball.

Definition 22.2 (Definition of Γ_u)**.** *Assume the function u^- is extended to all $\overline{B}_{2r} \setminus \overline{B}_r$ as the null function (this extension is continuous, since u^- is null on ∂B_r). We then define*

$$\Gamma_u = \sup \left\{ L | L \text{ affine}, \; L \leq -u^- \text{ on } \overline{B}_{2r} \right\}.$$

In order to prove the ABP estimate we set $M := \max_{\overline{B}_r} u^-$ and assume with no loss of generality that $M > 0$.

The following facts are either trivial consequences of the definitions or easy applications of the tools introduced in the Convex Analysis part: first, $-M \leq \Gamma_u \leq 0$; as a consequence $\Gamma_u \in W^{1,\infty}_{\mathrm{loc}}(B_{2r})$; finally since u is differentiable a.e. by Rademacher's theorem and the graph of the subdifferential is closed, we get $\partial\Gamma_u(x) \neq \emptyset$ for all $x \in B_{2r}$. We will use this last property to provide a supporting hyperplane to Γ_u at any point in $\overline{B}_r$.

We need some preliminary results, here is the first one.

Theorem 22.3. *Assume $u \in C(\overline{B}_r)$, $u \geq 0$ on ∂B_r and $\Gamma_u \in C^{1,1}(B_r)$. Then*

$$\max_{\overline{B}_r} u^- \leq cr \left(\int_{B_r} \det \nabla^2 \Gamma_u \, dx \right)^{1/n} .$$

Remark 22.4. *The previous theorem implies the ABP estimate, provided we show that*

- $\Gamma_u \in C^{1,1}(B_r)$, *as a consequence of* $u \in \mathrm{Sup}(f)$;
- *on the set* $\{u > \Gamma_u\}$ *(which is the so-called* non-contact region*) one has* $\det \nabla^2 \Gamma_u = 0$ *a.e.*
- *on the set* $\{u = \Gamma_u\}$ *(which is the so-called* contact region*) one has* $\det \nabla^2 \Gamma_u \leq f$ *a.e.*

We can now prove this first result.

Proof. Let $x_1 \in B_r$ be such that $u(x_1) = M$. Fix ξ with $|\xi| < M/3r$ and denote by L_α the affine function $L(x) = -\alpha + \langle x, \xi \rangle$. It is obvious that if $\alpha \gg 1$, then the corresponding hyperplane lies below the graph of (the extended version of) $-u^-$ and there is a minimum value of α such that this happens, that is $-u^- \leq L_\alpha$ on $\overline{B}_{2r}$. The graph will then meet the corresponding hyperplane at some point, say $x_0 \in \overline{B}_{2r}$. If it were $|x_0| > r$, then $L_\alpha(x_0) = 0$, but on the other hand $|L_\alpha(x_1)| \geq M$ and since $|x_0 - x_1| \leq 3r$ L_α would have slope $|\xi| \geq M/3r$, which is a contradiction. Hence the contact point must lie inside the ball B_r and therefore $B_{M/3r}(0) \subset \nabla \Gamma_u(B_r)$. If we measure the corresponding volumes and use the area formula, we get

$$\omega_n (M/3r)^n \leq \int_{Br} \left(\det \nabla^2 \Gamma_u \, dx \right)$$

or equivalently

$$M \leq 3 \omega_n^{-1/n} r \left(\int_{B_r} \det \nabla^2 \Gamma_u \, dx \right)^{1/n} .$$

This proves the claim with $c = 3\omega_n^{-1/n}$. □

Let us now come to the next steps. The next theorem shows that regularity, measured in terms of opening of paraboloids touching Γ_u from above, propagates from the contact set to the non-contact set. It turns out that the regularity in the contact set is a direct consequence of the subsolution property.

Theorem 22.5 (Propagation of regularity)**.** *Let $u \in C(\overline{B}_r)$ and suppose there exist $0 < \epsilon \leq r$, $M \geq 0$ such that for all $x_0 \in \overline{B}_r \cap \{u = \Gamma_u\}$ there exists a paraboloid with opening coefficient less than M which has a contact point from above with the graph of Γ_u in $B_\epsilon(x_0)$. Then $\Gamma_u \in C^{1,1}(\overline{B}_r)$ and $\det \nabla^2 u = 0$ a.e. on $\{u > \Gamma_u\}$.*

With the notation introduced before, the assumption of Theorem 22.5 means $\overline{\theta}(\Gamma_u, B_\epsilon(x_0))(x_0) \leq M$. Since Γ_u is convex, the corresponding quantity $\underline{\theta}$ is null. Recall also that we have already proved that $\overline{\theta}$, $\underline{\theta} \in L^\infty$ implies $u \in C^{1,1}$.

Theorem 22.6. *Let $v \in \mathrm{Sup}(f)$ in B_δ, φ convex in B_δ with $0 \leq \varphi \leq v$ and $v(0) = \varphi(0) = 0$. Then $\varphi(x) \leq c\left(\sup_{B_\delta} f^+\right)|x|^2$ in $B_{\nu\delta}$, where ν and C are universal constants.*

We can get a naive interpretation of this lemma (or, better, of its infinitesimal version as $\delta \downarrow 0$) by this formal argument: $v - \varphi$ has a local minimum at 0 implies $\nabla^2 v(0) \geq \nabla^2 \varphi(0)$ and by the assumption $v \in \mathrm{Sup}(f)$ this gives $\mathcal{M}^-(\varphi(0)) \leq f$.

Now it is possible to see how these tools allow to prove the ABP estimate.

Proof. Pick a point $x_0 \in B_r \cap \{u = \Gamma_u\}$ and let L be a supporting hyperplane for Γ_u at x_0, so that $\Gamma_u \geq L$ and $\Gamma_u(x_0) = L_{x_0}$. With respect to the statement of the Theorem 22.6, define $\varphi := \Gamma_u - L$, $v := -u^- - L = u \wedge 0 - L$ (and notice that v is a supersolution because $v \in \mathrm{Sup}(f\chi_{B_r})$.) Now, $\varphi(x_0) = v(x_0)$ implies, by means of Theorem 22.6,

$$\overline{\theta}(\varphi, B_{\nu\delta}(x_0))(x_0) \leq c \sup_{B_\delta(x_0)} f^+ \qquad \forall x_0 \in \overline{B}_r \tag{22.1}$$

with ν and c universal, for all $\delta \in (0, r)$. Hence $\overline{\theta}(\Gamma_u, B_{\nu\delta}(x_0))(x_0) \leq c \sup_{B_\delta} f^+$. By Theorem 22.5 we get $\Gamma_u \in C^{1,1}$ and $\det \nabla^2 \Gamma_u = 0$ a.e. in the non-contact region. Finally, to get the desired estimate, we have to show that a.e. in the contact region one has $\det \nabla^2 \Gamma_u \leq c(f^+)^n$. But this comes at once by passing to the limit as $\delta \to 0$ in (22.1) at any differentiability point x_0 of Γ_u. In fact, all the eigenvalues of $\nabla^2 \Gamma_u(x_0)$ do not exceed $cf^+(x_0)$ and the conclusion follows. □

Now we prove Theorem 22.5.

Proof. Let $r \in (0, \delta/4)$ and call $\overline{c} := \left(\sup_{\overline{B}_r} \varphi\right)/r^2$. Let then $x_0 \in \partial B_r$ be a maximum point. By means of a rotation, we can write $x = (x', x_n)$, $x' \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}$, and assume $x_0 = (0, r)$. Consider the intersection A of the closed strip defined by the hyperplanes $x_n = r$ and $x_n = -r$ with the ball $\overline{B}_\delta$. We clearly have that $\partial A = A_1 \cup A_2 \cup A_3$, where $A_1 = \overline{B}_{\delta/2} \cap \{x_n = r\}$, $A_2 = \overline{B}_{\delta/2} \cap \{x_n = -r\}$ and $A_3 = \partial B_{\delta/2} \cap \{|x_n| < r\}$.

We claim that $\varphi \geq \overline{c}r^2$ on A_1. To this aim, we first prove that $\varphi(y) \leq \varphi(x_0) + o(|y - x_0|)$ for $y \to x_0$, $y \in H := \{x_n = r\}$. In fact, this just comes from $\varphi(ry/|y|) \leq \varphi(x_0)$ and observing that $\varphi(y) - \varphi(ry/|y|) = o(|y - x_0|)$, because φ is Lipschitz continuous. On the other hand, we have that $\xi \in \partial\varphi_{|H}(x_0)$ implies $\varphi(y) \geq \varphi(x_0) + \langle \xi, y - x_0 \rangle$ for all $y \in H$. Hence, by comparison, it must be $\xi = 0$ and so $\varphi(y) \geq \overline{c}r^2$ on A_1 (this should be seen as a nonsmooth version of the Lagrange multiplier theorem).

As a second step, set $p(x) = \overline{c}/8(x_n + r)^2 - 4\overline{c}r^2/\delta^2 \, |x'|^2$ and notice that the following properties hold:

(a) on A_1, $p(x) \le \overline{c}/2r^2 \le \varphi(x)$;

(b) on A_2, $p(x) \le 0 \le \varphi(x)$ (and in particular $p(x) \le v(x)$);

(c) on A_3, $\delta^2/4 = |x'|^2 + x_n^2 \le |x'|^2 + r^2 \le |x'|^2 + \delta^2/16$, which implies $|x'|^2 \ge 3/16\delta^2$.

By means of the last estimate we get $p(x) \le (\overline{c}/2)r^2 - (3/4)\overline{c}r^2 \le 0 \le \varphi$ and obviously $p(0) = \overline{c}r^2/8 > 0$. Now, we can rigidly move this paraboloid (in partial analogy with the strategy described above) until we get a limit paraboloid $p' = p - \alpha$ (for some translation parameter $\alpha > 0$) lying below the graph of v and touching it at some point, say y. Since $p \le v$ on ∂A, the point y is internal to A.

By the supersolution property $\mathcal{M}^-(\nabla^2 p) \le f(y) \le \sup_{B_\delta} f$ we get (since we have an explicit expression for p)

$$\lambda\frac{\overline{c}}{4} - 8(n-1)\Lambda\overline{c}\frac{r^2}{\delta^2} \le \sup_{B_\delta} f.$$

But now we can fix r such that $8(n-1)\Lambda\overline{c}r^2/\delta^2 \le \lambda\overline{c}/8$ (it is done by taking r so that $8r \le \delta\sqrt{\lambda/((n-1)\Lambda)}$): we have therefore $\overline{c} \le \frac{8}{\lambda}\sup_{B_\delta} f$ with $\nu := \frac{1}{8}\sqrt{\lambda/((n-1)\Lambda)}$. $\square$

It remains to prove Theorem 22.6.

Proof. Recall first that we are assuming the existence of an uniform estimate

$$\overline{\theta}(\Gamma_u, B_\epsilon(x))(x) \le M \quad \forall x \in \overline{B}_r \cap \{u = \Gamma_u\}.$$

Consider now any point $x_0 \in \overline{B}_r \cap \{u > \Gamma_u\}$ and call L a supporting hyperplane for Γ_u at x_0. We claim that:

(a) There exist $n+1$ points $x_1, \dots, x_{n+1}$ such that $x_0 \in S := \mathrm{co}(x_1, \dots, x_{n+1})$ (here and in the sequel co stands for convex hull) and, moreover, all such points belong to $\overline{B}_r \cap \{u = \Gamma_u\}$ with at most one exception lying on ∂B_{2r}. In addition $\Gamma_u \equiv L$ on S;

(b) $x_0 = \sum_{i=1}^{n+1} \lambda_i x_i$ with at least one index i verifying both $x_i \in B_r \cap \{u = \Gamma_u\}$ and $\lambda_i \ge 1/(3n)$.

To show the utility of this claim, just consider how these two facts imply the thesis: on the one hand, if Γ_u is differentiable at x_0, we get $\det \nabla^2\Gamma_u(x_0) = 0$ because $\Gamma_u = L$ on S. On the other hand we may assume, without loss of generality that $x_1 \in \{u = \Gamma_u\} \cap \overline{B_r}$ and $\lambda_1 \ge (1/3n)$ so that since

$$x_0 + h = \lambda_1\left(x_1 + \frac{h}{\lambda_1}\right) + \lambda_2 x_2 + \cdots + \lambda_{n+1}x_{n+1},$$

one has

$$\begin{aligned}
\Gamma_u(x_0+h) &\le \lambda_1\Gamma_u\left(x_1+h/\lambda_1\right)+\lambda_2\Gamma_u(x_2)+\cdots+\lambda_{n+1}\Gamma_u(x_{n+1})\\
&\le \lambda_1\left[L(x_1)+k\left|\frac{h}{\lambda_1}\right|^2\right]+\lambda_2L(x_2)+\cdots+\lambda_{n+1}L(x_{n+1})=L(x_0)+k\left|h\right|^2/\lambda_1\\
&\le \Gamma_u(x_0)+3nk\left|h\right|^2
\end{aligned}$$

and this estimate is clearly uniform since we only require $|h/\lambda_1|\le\epsilon$, which is implied by $|h|\le\epsilon/(3n)$.

Hence, the problem is reduced to proving the two claims above. This is primarily based on a standard result in Convex Analysis, which is recalled here for completeness.

Theorem 22.7 (Carathéodory)**.** *Let V be a finite-dimensional real vector space and let $n:=\dim(V)$. If $C\subset V$ is a closed set, then for all $x\in\mathrm{co}(C)$ (the convex hull of C) there exist $x_1,\dots,x_{n+1}\in C$, $\lambda_1,\dots,\lambda_{n+1}\ge 0$ such that*

$$x=\sum_{i=1}^{n+1}\lambda_i x_i,\qquad \sum_{i=1}^{n+1}\lambda_i=1.$$

Set then $C':=\left\{x\in\overline{B}_{2r}|L(x)=-u^-(x)\right\}\ne\emptyset$ and $C=\mathrm{co}(C')$; we claim that $x_0\in C$. In fact, if this weren't the case, there would exist an hyperplane L' such that $L'(x_0)>0$ and $L'_{|C}<0$ and hence, taking $\delta>0$ sufficiently small we would have $L+\delta L'(x_0)>L(x_0)$ and also $L+\delta L'\le -u^-$, which contradicts the maximality of L. Now, we can write $x_0=\sum_{i=1}^{n+1}\lambda_i x_i$ with $x_i\in\{-u^-=L\}\subset\{-u^-=\Gamma_u\}$. In case there were two distinct points x_i, x_j with $|x_i|>r$ and $|x_j|>r$ (and so $L(x_i)=0$, $L(x_j)=0$) then the function Γ_u would achieve its maximum, equal to 0, in the interior of B_{2r} and so (by the convexity of Γ_u) it must be identically $\Gamma_u=0$ on B_{2r}, in contrast with the assumption $M=\max u^->0$. Let us now prove that $\Gamma_u=L$ on S. The implication $\ge$ is trivial, the other one is clear for each $x=x_i$ and is obtained by means of the convexity of Γ_u at all points in S. We also need to show that any exceptional point, if any, must lie on ∂B_{2r}. Call such a point x_i with $|x_i|>r$ and assume by contradiction $|x_i|<2r$: then $L(x_i)=0$ and $\Gamma_u(x_i)=0$, again contradicting the fact that Γ_u can't have interior maximum points.

Now we prove part (b) of the claim. If all points x_j verify $|x_j|\le r$, then $\max\lambda_i\ge\frac{1}{n+1}>\frac{1}{3n}$, otherwise if one point, say x_1, satisfies $|x_1|=2r$, then $\lambda_i<1/(3n)$ for all $i\ge 2$ implies $\lambda_1\ge 2/3$ and therefore

$$r\ge|x_0|\ge 2\lambda_1 r-\sum_{i=2}^{n+1}\lambda_i|x_i|>\frac{4}{3}r-\frac{n}{3n}r=r.$$

□

22.2 The Harnack Inequality

In this section we shall prove Harnack's inequality for functions in the class $\mathrm{Sol}(f) := \mathrm{Sub}(f) \cap \mathrm{Sup}(f)$ where, according to Definition 21.34, the sets $\mathrm{Sup}(|f|)$ and $\mathrm{Sub}(|f|)$ are defined through Pucci's extremal operators (with fixed ellipticity constants $0 < \lambda \leq \Lambda$) so that, in the sense of viscosity solutions,

$$\begin{aligned} u \in \mathrm{Sup}(|f|) \qquad &\Longleftrightarrow \qquad -\mathcal{M}^-(u) + |f| \geq 0 \ ; \\ u \in \mathrm{Sub}(|f|) \qquad &\Longleftrightarrow \qquad -\mathcal{M}^+(u) + |f| \leq 0 \ . \end{aligned}$$

We shall use the standard notation $Q_r(x)$ for the closed n-cube in $\mathbb{R}^n$ with side length r, $Q_r = Q_r(0)$ and always assume that f is continuous. In the proof of Lemma 22.13 below, however, we shall apply the ABP estimate to a function $w \in \mathrm{Sup}(g)$ with g upper semicontinuous. Since there exists g_n continuous with $g_n \downarrow g$ and $w \in \mathrm{Sup}(g_n)$, the ABP estimate holds even in this case.

Theorem 22.8. *Consider a function $u : Q_1 \to \mathbb{R}$ with $u \geq 0$ and $u \in \mathrm{Sol}(f) \cap C(Q_1)$. There exists a universal[7] constant C_H such that*

$$\sup_{x \in Q_{1/2}} u(x) \leq C_H \left(\inf_{x \in Q_{1/2}} u(x) + \|f\|_{L^n(Q_1)} \right) \ . \tag{22.2}$$

Let us show how (22.2) leads to the Hölder regularity result for viscosity solutions of a fully non linear elliptic PDE

$$-F(D^2 u(x)) + f(x) = 0 \ .$$

1. As usual, we need to control the oscillation (now on cubes), defined by

   $$\omega_r := M_r - m_r$$

   with $M_r := \sup_{x \in Q_r} u(x)$ and $m_r := \inf_{x \in Q_r} u(x)$.

   In the same context of Theorem 22.8, there exists a universal constant $\mu \in (0,1)$ such that

   $$\omega_{1/2} \leq \mu \omega_1 + 2\|f\|_{L^n(Q_1)} \ . \tag{22.3}$$

   Indeed, we apply Harnack's inequality (22.2)

   - to the function $u - m_1$, so that

     $$M_{1/2} - m_1 \leq C_H \left(m_{1/2} - m_1 + \|f\|_{L^n(Q_1)} \right) \ ; \tag{22.4}$$

[7] As we said in the previous section, "universal" means that the constant depends only on the space dimension n and the ellipticity constants λ, Λ.

- to the function $M_1 - u$, so that

$$M_1 - m_{1/2} \leq C_H \left(M_1 - M_{1/2} + \|f\|_{L^n(Q_1)} \right) \ . \tag{22.5}$$

Adding (22.4) and (22.5) we get

$$\omega_1 + \omega_{1/2} \leq C_H \left(\omega_1 - \omega_{1/2} + 2\|f\|_{L^n(Q_1)} \right) \ ,$$

which proves (22.3) because

$$\omega_{1/2} \leq \frac{C_H - 1}{C_H + 1}\omega_1 + 2\frac{C_H}{C_H + 1} \ |f\|_{L^n(Q_1)} \leq \frac{C_H - 1}{C_H + 1}\omega_1 + 2\,\|f\|_{L^n(Q_1)} \ .$$

We spend a line to remark that clearly $\mu = (C_H - 1)/(C_H + 1)$, C_H being the universal constant in (22.2). It is crucial for the decay of the oscillation that $\mu < 1$.

2. Thanks to a rescaling argument (which we will be hugely used also in the proof of Harnack's inequality), we can generalize (22.3). Fix a radius $0 < r \leq 1$ and put

$$u_r(y) := \frac{u(ry)}{r^2} \qquad f_r(y) = f(ry) \qquad \text{when } y \in Q_1 \ .$$

Notice that (22.3) holds also for u_r (with the corresponding source f_r) because Pucci's operators are homogeneous. Moreover, passing to a smaller scale, the L^n-norm improves.

For simplicity we keep the notation ω_r for the oscillation of the function u, we use $\mathrm{osc}(\cdot, Q_r)$ otherwise. We can estimate

$$\begin{aligned} \omega_{r/2} &= r^2\mathrm{osc}(u_r, Q_{1/2}) \leq \mu r^2\mathrm{osc}(u, Q_1) + 2r^2\|f_r\|_{L^n(Q_1)} \\ &= \mu\omega_r + 2r\|f\|_{L^n(Q_r)} \leq \mu\omega_r + 2r\|f\|_{L^n(Q_1)} \ . \end{aligned}$$

3. By the iteration lemmas we used so frequently in the elliptic regularity chapters, we are immediately able to conclude that

$$\omega_r \leq Cr^\alpha \quad \text{with } \left(\frac{1}{2}\right)^\alpha = \mu \ ,$$

for all $r \in (0, 1]$, with C dependent only on μ and $\|f\|_{L^n(Q_1)}$, thus we have Hölder regularity.

In order to prove Harnack's inequality, we will pass through the following reformulation of Theorem 22.8.

Theorem 22.9. *There exist universal constants $\epsilon_0(n,\lambda,\Lambda)$, $C(n,\lambda,\Lambda)\in\mathbb{R}$ such that if $u: Q_1\to\mathbb{R}$ is a nonnegative function, belonging to* $\mathrm{Sol}(f)\cap C(Q_{4\sqrt{n}})$ *on $Q_{4\sqrt{n}}$, with $\inf_{x\in Q_{1/4}} u(x)\le 1$ and*

$$\|f\|_{L^n(Q_{4\sqrt{n}})}\le\epsilon_0\ ,$$

then

$$\sup_{x\in Q_{1/4}} u(x)\le C\ . \tag{22.6}$$

Remark 22.10. Theorem 22.8 and Theorem 22.9 are easily seen to be equivalent: since we will prove the second one, it is more important for us to check that Theorem 22.8 follows from Theorem 22.9.
For some positive $\delta>0$ (needed to avoid a potential division by 0) consider the function

$$v:=\frac{u}{\inf_{Q_{1/4}}u+\delta+\|f\|_{L^n(Q_{4\sqrt{n}})}/\epsilon_0}\ .$$

Since $\inf_{Q_{1/4}} v\le 1$ and, denoting by f_v the source term associated with v, $\|f_v\|_{L^n(Q_{4\sqrt{n}})}\le\epsilon_0$, we have $\sup_{x\in Q_{1/4}}v(x)\le C$, hence

$$\sup_{x\in Q_{1/4}}u(x)\le C\left(\inf_{x\in Q_{1/4}}u(x)+\delta+\|f\|_{L^n(Q_{4\sqrt{n}})}/\epsilon_0\right)\ .$$

We let $\delta\to 0$ and we obtain Harnack's inequality with the cubes $Q_{1/4}$, $Q_{4\sqrt{n}}$; by a scaling argument, this means

$$\sup_{x\in Q_r(x_0)}u(x)\le C\left(\inf_{x\in Q_r(x_0)}u(x)+r\|f\|_{L^n(Q_{16r\sqrt{n}}(x_0))}\right)\ . \tag{22.7}$$

Now, we pass to the cubes $Q_{1/2}$, Q_1 with a simple covering argument: there exists an integer $N=N(n)$ such that for all $x\in Q_{1/2}$, $y\in Q_1$ we can find points x_i, $1\le i\le N$, with $x_i=x$, $x_N=y$ and $x_{i+1}\in Q_r(x_i)$ for $1\le i<N$, with $r=r(n)$ so small that all cubes $Q_{16r\sqrt{n}}(x_i)$ are contained in Q_1. By applying repeatedly (22.7) we get (22.2) with $C_H\sim C^N$.

We describe the strategy of the proof of Theorem 22.9, even if the full proof will be completed at the end of this section.
We will study the map

$$t\mapsto\mathscr{L}^n\left(\{u>t\}\cap Q_1\right)$$

in order to prove:

- a decay estimate of the form $\mathscr{L}^n\left(\{u>t\}\cap Q_1\right)\le dt^{-\epsilon}$, thanks to the fact that $u\in\mathrm{Sup}(|f|)$ (see Lemma 22.13),

- the full thesis of Theorem 22.9 using the fact that also $u \in \mathrm{Sub}(-|f|)$.

The first goal will be achieved using the Alexandrov-Bakelman-Pucci inequality of the previous section. The structure of the proof remembers that of De Giorgi's regularity theorem, as we said, and we will complete it through the following lemmas and remarks.

The first lemma is a particular case of Calderón-Zygmund decomposition.

Lemma 22.11 (Dyadic Lemma)**.** *Consider Borel sets $A \subset B \subset Q_1$ with $\mathscr{L}^n(A) \le \delta < 1$. If the implication*

$$\mathscr{L}^n(A \cap Q) < \delta \mathscr{L}^n(Q) \implies \tilde{Q} \subset B \ , \tag{22.8}$$

holds for any dyadic cube $Q \subset Q_1$, with $\tilde{Q}$ being the predecessor of Q, then

$$\mathscr{L}^n(A) \le \delta \mathscr{L}^n(B) \ .$$

Proof. We apply the construction of Calderón-Zygmund (seen in Theorem 14.1) to $f = \chi_A$: there exists a countable family of cubes $\{Q_i\}_{i \in I}$, pairwise disjoint, such that

$$\chi_A \le \delta \qquad \text{a.e. on } Q_1 \setminus \bigcup_{i \in I} Q_i \tag{22.9}$$

and

$$\mathscr{L}^n(A \cap \tilde{Q}_i) \le \delta \mathscr{L}^n(\tilde{Q}_i) \ .$$

Since $\delta < 1$ and χ_A is a characteristic function, (22.9) means that $A \subset \bigcup_{i \in I} Q_i$ up to negligible sets, thus

$$\mathscr{L}^n(A) \le \sum_{i \in I} \mathscr{L}^n(A \cap \tilde{Q}_i) \le \sum_{i \in I} \delta \mathscr{L}^n(\tilde{Q}_i) \le \delta \mathscr{L}^n(B) \ .$$

□

It is bothering, but necessary to go on with the proof, to deal at the same time with balls and cubes: balls emerge from the radial construction in the next lemma and cubes are needed in Calderón-Zygmund theorem, needed elsewhere.

Lemma 22.12 (Truncation Lemma)**.** *If we fix the dimension of the space and the ellipticity constants $0 < \lambda \le \Lambda$, there exists a universal function $\varphi \in C^\infty(\mathbb{R}^n)$ such that*

(i) $\varphi \ge 0$ *on* $\mathbb{R}^n \setminus B_{2\sqrt{n}}(0)$*;*

(ii) $\varphi \le -2$ *on the cube* Q_3*;*

(iii) finally $\mathcal{M}^+(D^2\varphi) \le C_\varphi \chi_{\overline{Q}_1}$ *on* $\mathbb{R}^n$*.*

Proof. We recall some useful inclusions:

$$B_{1/4} \subset Q_1 \subset Q_3 \subset B_{3\sqrt{n}/2} \subset B_{2\sqrt{n}} \ .$$

We first define

$$\varphi(x) = M_1 - M_2|x|^{-\alpha} \quad \text{when } |x| \geq 1/4 \ .$$

When α is fixed, we can find $M_1 = M_1(\alpha) \geq 0$ and $M_2 = M_2(\alpha) \geq 0$ such that

(i) $\varphi\big|_{\partial B_{2\sqrt{n}}} \equiv 0$, so that $\varphi \geq 0$ on $\mathbb{R}^n \setminus B_{2\sqrt{n}}$;

(ii) $\varphi\big|_{\partial B_{3\sqrt{n}/2}} \equiv 0$, so that $\varphi \leq -2$ on Q_3.

After choosing a smooth extension for φ on $B_{1/4}$, we conclude checking that there exists an exponent α that is suitable to verify the third property of the statement. We compute

$$D^2\left(|x|^{-\alpha}\right) = -\frac{\alpha}{|x|^{\alpha+2}} I + \alpha(\alpha+2)\frac{x \otimes x}{|x|^{\alpha+4}} \ ,$$

thus the eigenvalues of $D^2\varphi$ when $|x| \geq 1/4$ are $M_2\alpha|x|^{-(\alpha+2)}$ with multiplicity $n-1$ and $-M_2\alpha(\alpha+1)|x|^{-(\alpha+2)}$ with multiplicity 1 (this is the eigenvalue due to the radial direction). Hence, when $|x| \geq 1/4$ we have

$$\mathcal{M}^+(D^2\varphi) = \frac{M_2}{|x|^{\alpha+2}}(\Lambda\alpha - \lambda(n-1)\alpha(\alpha+1))$$

so that $\mathcal{M}^+(D^2\varphi) \leq 0$ on $\mathbb{R}^n \setminus \overline{Q}_1$ if we choose $\alpha = \alpha(n, \lambda, \Lambda) \gg 1$. □

Lemma 22.13 (Decay Lemma). *There exist universal constants $\epsilon_0 > 0$, $M > 1$ and $\mu \in (0,1)$ such that if $u \in \mathrm{Sup}(|f|)$, $u \geq 0$ on $Q_{4\sqrt{n}}$, $\inf_{Q_3} u \leq 1$ and $\|f\|_{L^n(Q_{4\sqrt{n}})} \leq \epsilon_0$, then for every integer $k \geq 1$*

$$\mathscr{L}^n\left(\{u > M^k\} \cap Q_1\right) \leq (1-\mu)^k \ . \tag{22.10}$$

Proof. We prove the first step, that is

$$\mathscr{L}^n(\{u > M\} \cap Q_1) \leq (1-\mu) \ . \tag{22.11}$$

We use the Alexandrov-Bakelman-Pucci estimate of Theorem 22.1 for the function w, defined as the function u additively perturbed with the truncation function φ of Lemma 22.12. If $w := u + \varphi$, then

(i)

$$w \geq 0 \quad \text{on } \partial B_{2\sqrt{n}} \tag{22.12}$$

because $u \geq 0$ on $Q_{4\sqrt{n}}$ and $\varphi \geq 0$ on $\mathbb{R}^n \setminus B_{2\sqrt{n}}$;

(ii)
$$\inf_{B_{2\sqrt{n}}} w \le \inf_{Q_3} w \le -1 \tag{22.13}$$
because $Q_3 \subset B_{2\sqrt{n}}$ and $\varphi \le -2$ on $B_{2\sqrt{n}}$, and at the same time we are assuming that $\inf_{Q_3} u \le 1$;

(iii) directly from the definition of $\mathrm{Sup}(|f|)$ we get $-\mathcal{M}^-(D^2u) + |f| \ge 0$, moreover $\mathcal{M}^+(D^2\varphi) \le C_\varphi \chi_{\overline{Q}_1}$. Since in general $\mathcal{M}^-(A+B) \le \mathcal{M}^-(A) + \mathcal{M}^+(B)$ (see Remark 21.33), then
$$-\mathcal{M}^-(D^2w) + (|f| + C_\varphi\chi_{\overline{Q}_1}) \ge (-\mathcal{M}^-(D^2u) + |f|) + (-\mathcal{M}^+(D^2\varphi) + C_\varphi\chi_{\overline{Q}_1}) \ge 0 \ . \tag{22.14}$$
The inequality (22.14) means that $w \in \mathrm{Sup}(|f| + C_\varphi\chi_{\overline{Q}_1})$.

Thanks to the ABP estimate, which we can apply to w thanks to (22.12) and (22.14), we get
$$\max_{x\in\overline{B}_{2\sqrt{n}}} w^-(x) \le C_{ABP}\left(\int_{\{w=\Gamma_w\}} \left(|f(y)| + C_\varphi\chi_{\overline{Q}_1}(y)\right)^n \, dy\right)^{1/n} \ . \tag{22.15}$$

Now, remembering that (22.13) holds and that, by definition, $\{w = \Gamma_w\} \subset \{w \le 0\}$, we can expand (22.15) with
$$\begin{aligned}
1 \ &\le \ \max_{x\in\overline{B}_{2\sqrt{n}}} w^-(x) \le C_{ABP}\left(\int_{\{w=\Gamma_w\}} \left(|f(y)| + C_\varphi\chi_{\overline{Q}_1}(y)\right)^n \, dy\right)^{1/n} \\
&\le \ C_{ABP}\left(\int_{\{w\le 0\}} \left(|f(y)| + C_\varphi\chi_{\overline{Q}_1}(y)\right)^n \, dy\right)^{1/n} && (22.16)\\
&\le \ C_{ABP}\|f\|_{L^n(Q_{4\sqrt{n}})} + C_{ABP}C_\varphi\mathscr{L}^n\left(Q_1 \cap \{w \le 0\}\right)^{1/n} && (22.17)\\
&\le \ C_{ABP}\|f\|_{L^n(Q_{4\sqrt{n}})} + C_{ABP}C_\varphi\mathscr{L}^n\left(Q_1 \cap \{u < M\}\right)^{1/n} \ , && (22.18)
\end{aligned}$$
where we pass from line (22.16) to line (22.17) by Minkowski inequality and from line (22.17) to line (22.18) because, if $w(x) \le 0$, then $u(x) \le -\varphi(x)$ and then $u(x) \le M$ with $M = \max\varphi^-$.
Choosing a universal ϵ_0 such that $C_{ABP}\epsilon_0 \le 1/2$ we can resume (22.18) with
$$\mathscr{L}^n\left(Q_1 \cap \{u \le M\}\right)^{1/n} \ge \frac{1}{2C_{ABP}C_\varphi} \ , \tag{22.19}$$
thus, if $\mu := (2C_{ABP}C_\varphi)^{-n}$, we obtain (22.11).

We prove the inductive step: suppose that (22.10) holds for every $j \leq k-1$. We exploit the Dyadic Lemma 22.11 with $A = \{u > M^k\} \cap Q_1$, $B = \{u > M^{k-1}\} \cap Q_1$ and $\delta = 1-\mu$. Naturally $A \subset B \subset Q_1$ and $\mathscr{L}^n(A) \leq \delta$, so, if we are able to check that (22.8) holds, then

$$\mathscr{L}^n\left(Q_1 \cap \{u > M^k\}\right) \leq (1-\mu)\mathscr{L}^n\left(Q_1 \cap \{u > M^{k-1}\}\right) \leq (1-\mu)^k \ .$$

Concerning (22.8), suppose by contradiction that for some dyadic cube $Q \subset Q_1$ we have that

$$\mathscr{L}^n(A \cap Q) < \delta \mathscr{L}^n(Q) \tag{22.20}$$

but $\tilde{Q} \not\subset B$, $\tilde{Q}$ being the predecessor of Q, as usual: there exists $z \in \tilde{Q}$ such that $u(z) \leq M^{k-1}$. Let us rescale and translate the problem, putting $\tilde{u}(y) := u(x)M^{-(k-1)}$ with $x = z + 2^{-i}y$ if Q has edge length 2^{-i}. Because of the rescaling technique, we need to adapt f, that is define a new datum

$$\tilde{f}(y) := \frac{f(x)}{2^{2i}M^{k-1}} \ .$$

The intention of this definition of $\tilde{f}$ is to ensure that $\tilde{u} \in \mathrm{Sup}(|\tilde{f}|)$, in fact

$$-\mathcal{M}^-(D^2\tilde{u}) + |\tilde{f}| = \frac{1}{2^{2i}M^{k-1}}\left(-\mathcal{M}^-(D^2u) + |f|\right) \geq 0 \ .$$

Clearly if $x \in \tilde{Q}$ then $y \in Q_3$, so

$$\inf_{y \in Q_3} \tilde{u}(y) \leq \frac{u(z)}{M^{k-1}} \leq 1 \ .$$

If $\|\tilde{f}\|_{L^n(Q_{4\sqrt{n}})} \leq \epsilon_0$, then, applying what we already saw in (22.19) to $\tilde{u}$ instead of u,

$$\mu \leq \mathscr{L}^n\left(\{\tilde{u} \leq M\} \cap Q_1\right) = 2^{ni}\mathscr{L}^n\left(\{u \leq M^k\} \cap Q\right) \ ,$$

this means that $\mu\mathscr{L}^n(Q) \leq \mathscr{L}^n\left(\{u \leq M^k\} \cap Q\right)$ and, passing to the complement,

$$\mathscr{L}^n\left(\{u > M^k\} \cap Q\right) \leq (1-\mu)\mathscr{L}^n(Q) \ ,$$

which contradicts (22.20).
In order to complete our proof, we show that effectively $\|\tilde{f}\|_{L^n(Q_{4\sqrt{n}})} \leq \epsilon_0$. In general, let us remark that the rescaling technique does not cause any problem at the level of the source term f. Indeed

$$\|\tilde{f}\|_{L^n(Q_{4\sqrt{n}})} = \frac{1}{M^{k-1}2^i}\|f\|_{L^n(Q_{4\sqrt{n}/2^i}(z))} \leq \epsilon_0 \ .$$

$\square$

Corollary 22.14. *There exist universal constants $\epsilon > 0$ and $d \geq 0$ such that if $u \in \mathrm{Sup}(|f|)$, $u \geq 0$ on $Q_{4\sqrt{n}}$, $\inf_{Q_3} u \leq 1$ and $\|f\|_{L^n(Q_{4\sqrt{n}})} \leq \epsilon_0$, then*

$$\mathscr{L}^n\left(\{u > t\} \cap Q_1\right) \leq dt^{-\epsilon} \qquad \forall\, t > 0\ . \tag{22.21}$$

Proof. This corollary is obtained by Lemma 22.13 choosing ϵ such that $(1-\mu) = M^{-\epsilon}$ and $d' = M^{\epsilon} = (1-\mu)^{-1}$: interpolating, for every $t \geq 1/M$ there exists $k \in \mathbb{N}$ such that $M^{k-1} \leq t \leq M^k$, so

$$\mathscr{L}^n\left(\{u > t\} \cap Q_1\right) \leq \mathscr{L}^n\left(\{u > M^{k-1}\} \cap Q_1\right) \leq M^{-\epsilon(k-1)} \leq d'(M^k)^{-\epsilon} \leq d't^{-\epsilon}\ .$$

Choosing $d \geq d'$ such that $1 \leq dt^{-\epsilon}$ for all $t \in (0, 1/M)$ we conclude. □

In the next lemma we use both the subsolution and the supersolution property to improve the decay estimate on $\mathscr{L}^n(\{u > t\})$. The statement is a little technical and the reader might wonder about the choice of the scale l_j as given in the statement of the lemma; it turns out, see (22.26), that this is the largest scale r on which we are able to say that $\mathscr{L}^n\left(\{u \geq \nu^j\} \cap Q_r\right) \ll r^n$, knowing that the global volume $\mathscr{L}^n\left(\{u \geq \nu^j\} \cap Q_1\right)$ is bounded by $d(\nu^j)^{-\epsilon}$.

Lemma 22.15. *Suppose that $u \in \mathrm{Sub}(-|f|)$ on $Q_{4\sqrt{n}}$ and $\|f\|_{L^n(Q_{4\sqrt{n}})} \leq \epsilon_0$ with ϵ_0 given by Lemma 22.13. Assume that (22.21) holds. Then here exist universal constants $M_0 > 1$ and $\sigma > 0$ such that if*

$$x_0 \in Q_{1/2} \quad \text{and} \quad u(x_0) \geq M_0 \nu^{j-1} \text{ for some } j \geq 1\ ,$$

then

$$\exists\, x_1 \in \overline{Q}_{l_j}(x_0) \quad \text{such that} \quad u(x_1) \geq M_0 \nu^j\ ,$$

where $\nu := M_0/(M_0 - 1/2) > 1$ and $l_j := \sigma M_0^{-\epsilon/n}(\nu^{-\epsilon/n})^j$.

Proof. First of all, we fix a large universal constant $\sigma > 0$ such that

$$\frac{1}{2}\sigma^n > d2^{\epsilon}(4\sqrt{n})^n \tag{22.22}$$

and then we choose another universal constant M_0 so large that

$$dM_0^{-\epsilon} < \frac{1}{2} \tag{22.23}$$

and

$$\sigma M_0^{-\epsilon/n} < 2\sqrt{n}\ . \tag{22.24}$$

By contradiction, assume that for some $j \geq 1$ we have

$$\sup_{x \in \overline{Q}_{l_j}(x_0)} u(x) < \nu^j M_0\ . \tag{22.25}$$

We first estimate the superlevels

$$\mathscr{L}^n\left(\{u \geq \nu^j M_0/2\} \cap Q_{l_j/(4\sqrt{n})}(x_0)\right) \leq \mathscr{L}^n\left(\{u \geq \nu^j M_0/2\} \cap Q_1\right)$$
$$\leq d\left(\nu^j M_0/2\right)^{-\epsilon} = d 2^{\epsilon} \nu^{-j\epsilon} M_0^{-\epsilon} < \frac{1}{2}\left(\frac{\sigma}{4\sqrt{n}}\right)^n \nu^{-j\epsilon} M_0^{-\epsilon} = \frac{1}{2}\left(\frac{l_j}{4\sqrt{n}}\right)^n , \quad (22.26)$$

where we used condition (22.22) on σ and the definition of l_j, as given in the statement of the lemma.

We claim that the superlevel can be estimated as follows:

$$\mathscr{L}^n\left(\{u < \nu^j M_0/2\} \cap Q_{l_j/(4\sqrt{n})}(x_0)\right) < \frac{1}{2}\mathscr{L}^n\left(Q_{l_j/(4\sqrt{n})}\right) . \quad (22.27)$$

Obviously the validity of (22.26) and (22.27) is the contradiction that will conclude the proof, so we need only to show (22.27).

Define the auxiliary function

$$v(y) := \frac{\nu M_0 - u(x)\nu^{-(j-1)}}{(\nu-1)M_0} ,$$

where $x = x_0 + \frac{l_j}{4\sqrt{n}} y$. Since $y \in Q_{4\sqrt{n}} \iff x \in Q_{l_j}(x_0)$, by (22.25) the function v is defined and non-negative on $Q_{4\sqrt{n}}$. In addition, $u(x_0) \geq M_0 \nu^{j-1}$ implies that $\inf_{Q_{4\sqrt{n}}} v \leq 1$. Notice that our preliminary choice of ν ($\nu = M_0/(M_0 - 1/2)$) means that $M_0 = \nu/[2(\nu-1)]$, so (modulo the change of variables)

$$\{v > M_0\} = \{u < \nu^j M_0/2\} .$$

Moreover, if we compute the datum f_v which corresponds to v, since the rescaling radius is $l_j/4\sqrt{n}$, we get

$$f_v(y) = \frac{(l_j/4\sqrt{n})^2}{M_0(\nu-1)\nu^{j-1}} f(x)$$

so

$$\|f_v\|_{L^n(Q_{4\sqrt{n}})} = \frac{1}{M_0(\nu-1)\nu^{j-1}} \frac{l_j}{4\sqrt{n}} \|f\|_{L^n(Q_{l_j}(x_0))} \leq \epsilon_0 \quad (22.28)$$

because

$$\frac{1}{M_0(\nu-1)\nu^{j-1}} \frac{l_j}{4\sqrt{n}} = \frac{\sigma M_0^{-\epsilon/n}}{2\sqrt{n}} \nu^{-\epsilon j/n - j} \leq 1$$

thanks to (22.24). The estimate in (22.28) allows us to use Corollary 22.14 for v, that is

$$\mathscr{L}^n\left(\{v > M_0\} \cap Q_1\right) \leq d M_0^{-\epsilon} ,$$

and we can use this in conjunction with (22.23) to obtain that (22.27) holds:

$$\mathscr{L}^n\left(\{u < \nu^j M_0/2\} \cap Q_{l_j/4\sqrt{n}}(x_0)\right) \leq d M_0^{-\epsilon} \mathscr{L}^n\left(Q_{l_j/4\sqrt{n}}\right) < \frac{1}{2}\mathscr{L}^n\left(Q_{l_j/4\sqrt{n}}\right) .$$

□

We can now complete the proof of Theorem 22.9, using Lemma 22.15. Notice that in Theorem 22.9 we made all assumptions needed to apply Lemma 22.15, taking also Corollary 22.14 into account, which ensures the validity of (22.21).
Roughly speaking, if we assume, by (a sort of) contradiction, that u is not bounded above on $Q_{1/2}$, then, thanks to Lemma 22.15, we should be able to find recursively a sequence (x_j) with the property that

$$u(x_j) \geq M_0 \nu^j \quad \text{and} \quad x_{j+1} \in Q_{l_j}(x_j);$$

since $\sum_j l_j < \infty$, the sequence (x_j) admits a converging subsequence, and in the limit point we find a contradiction. However, in order to iterate Lemma 22.15 we have to confine the sequence in the cube $Q_{1/2}$.
To achieve this, we fix a universal positive integer j_0 such that $\sum_{j \geq j_0} l_j < 1/4$ and we assume, by contradiction, that there exists a point $x_0 \in Q_{1/4}$ with $u(x_0) \geq M_0 \nu^{j_0 - 1}$. This time, the sequence (x_k) we generate iterating Lemma 22.15 is contained in $Q_{1/2}$ and

$$u(x_k) \geq M_0 \nu^{j_0 + k - 1} \ . \tag{22.29}$$

When $k \to \infty$ in (22.29) we obtain the contradiction. This way, we obtained also an "explicit" expression of the universal constant in (22.6), in fact we proved in that

$$\sup_{x \in Q_{1/4}} u(x) < M_0 \nu^{j_0 - 1} \ .$$

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