ON THE LIMITING BEHAVIOR OF PARAMETER-DEPENDENT NETWORK CENTRALITY MEASURES*

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Abstract. We consider a broad class of walk-based, parameterized node centrality measures for network analysis. These measures are expressed in terms of functions of the adjacency matrix and generalize various well-known centrality indices, including Katz and subgraph centralities. We show that the parameter can be "tuned" to interpolate between degree and eigenvector centralities, which appear as limiting cases. Our analysis helps explain certain correlations often observed between the rankings obtained using different centrality measures and provides some guidance for the tuning of parameters. We also highlight the roles played by the spectral gap of the adjacency matrix and by the number of triangles in the network. Our analysis covers both undirected and directed networks, including weighted ones. A brief discussion of PageRank is also given.

Key words. centrality, communicability, adjacency matrix, spectral gap, matrix functions, network analysis, PageRank

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1. Introduction. The mathematical and computational study of complex networks has experienced tremendous growth in recent years. A wide variety of highly interconnected systems, both in nature and in the man-made world of technology, can be modeled in terms of networks. Network models are now commonplace not only in the "hard" sciences but also in economics, finance, anthropology, urban studies, and even in the humanities. As more and more data has become available, the need for tools to analyze these networks has increased and a new field of network science has come of age [1, 17, 24, 47].

Since graphs, which are abstract models of real-world networks, can be described in terms of matrices, it comes as no surprise that linear algebra plays an important role in network analysis. Many problems in this area require the solution of linear systems, the computation of eigenvalues and eigenvectors, and the evaluation of matrix functions. Also, the study of dynamical processes on graphs gives rise to systems of differential and difference equations posed on graphs [2]; the behavior of the solution as a function of time is strongly influenced by the structure (topology) of the underlying graph, which in turn is reflected in the spectral properties of matrices associated with the graph [6].

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One of the most basic questions about network structure is the identification of the "important" nodes in a network. Examples include essential proteins in protein-protein interaction networks, keystone species in ecological networks, authoritative webpages on the World Wide Web, influential authors in scientific collaboration networks, leading actors in the Internet Movie Database, and so forth; see, e.g., [24, 46] for details and many additional examples. When the network being examined is very small (say, on the order of 10 nodes), this determination of importance can often be done visually, but as networks increase in size and complexity, visual analysis becomes impossible. Instead, computational measures of node importance, called *centrality measures*, are used to rank the nodes in a network. There are many different centrality measures in use; see, for example, [13, 24, 40, 47] for extensive treatments of centrality and discussion of different ranking methods. Many authors, however, have noted that different centrality measures often provide rankings that are highly correlated, at least when attention is restricted to the most highly ranked nodes; see, e.g., [21, 32, 41, 45], as well as the results in [5].

In this paper we analyze the relationship between degree centrality, eigenvector centrality, and various centrality measures based on the diagonal entries (for undirected graphs) and row sums of certain (analytic) functions of the adjacency matrix of the graph. These measures contain as special cases the well-known Katz centrality, subgraph centrality, total communicability, and other centrality measures which depend on a tuneable parameter. We also include a brief discussion of PageRank [48]. We point out that Kleinberg's HITS algorithm [37], as a type of eigenvector centrality, is covered by our analysis, as is the extension of subgraph centrality to digraphs given in [4].

As mentioned, there are a number of other ranking methods in use, yet in this paper we limit ourselves to considering centrality measures based on functions of the adjacency matrix, in addition to degree and eigenvector centralities. The choice of which of the many centrality measures to study and why is something that must be considered carefully; see the discussion in [15]. In this paper we focus our attention on centrality measures that have been widely tested and that can be expressed in terms of linear algebra (more specifically, in terms of the adjacency matrix of the network). We additionally restrict our scope to centrality measures that we can demonstrate (mathematically) to be related to one other. Hence, we did not include in our analysis two popular centrality measures, betweenness centrality [30] and closeness centrality [31], which do not appear to admit a simple expression in terms of the adjacency matrix. Our results help explain the correlations often observed between the rankings produced by different centrality measures and may be useful in tuning parameters when performing centrality calculations.

The paper is organized as follows. Sections 2 and 3 contain background information on graphs and centrality measures. In section 4 we describe the general class of functional centrality measures considered in this paper and present some technical lemmas on power series needed for our analysis. In section 5 we state and prove our main results, which show that degree and eigenvector centralities are limiting cases of the parameterized ones. Section 6 contains a brief discussion of the limiting behavior of PageRank and related techniques. In section 7 we provide an interpretation of our results in terms of graph walks and discuss the role played by the spectral gap and by triangles in the network. Related work is briefly reviewed in section 8. A short summary of numerical experiments aimed at illustrating the theory is given in section 9 (the details of the experiments can be found in the supplementary materials accompanying this paper, linked from the main article webpage). Conclusions are given in

section 10.

2. Background and definitions. In this section we recall some basic concepts from graph theory that will be used in the rest of the paper. A more complete overview can be found, e.g., in [20]. For ease of exposition only *unweighted* and *loopless* graphs are considered in this section, but nearly all of our results admit a straightforward generalization to graphs with (positive) edge weights, and several of the results also apply in the presence of loops; see the end of section 5, as well as section 6.

A *directed graph*, or *digraph*, $G = (V, E)$ is defined by a set of n nodes (also referred to as vertices) V and a set of edges $E = \{(i, j) \,|\, i, j \in V\}$. Note that, in general, $(i, j) \in E$ does not imply $(j, i) \in E$. When this happens, G is *undirected* and the edges are formed by unordered pairs of vertices. The *out-degree* of a vertex i, denoted by d_i^{out}, is given by the number of edges with i as the starting node, i.e., the number of edges in E of the form (i, k). Similarly, the *in-degree* of node i is the number d_i^{in} of edges of the form (k, i). If G is undirected, then $d_i^{out} = d_i^{in} = d_i$, the *degree* of node i.

A *walk* of length k in G is a list of nodes $i_1, i_2, \ldots, i_k, i_{k+1}$ such that for all $1 \le l \le k$, there is a (directed) edge between i_l and i_{l+1}. A *closed walk* is a walk where $i_1 = i_{k+1}$. A *path* is a walk with no repeated nodes, and a *cycle* is a closed walk with no repeated nodes except for the first and the last one. A graph is *simple* if it has no *loops* (edges from a node i to itself), no multiple edges, and unweighted edges. An undirected graph is *connected* if there exists a path between every pair of nodes. A directed graph is *strongly connected* if there exists a directed path between every pair of nodes.

Every graph G can be represented as a matrix through the use of an *adjacency matrix* $A = (a_{ij})$ with

$$a_{ij} = \begin{cases} 1 & \text{if } (i, j) \text{ is an edge in } G, \\ 0 & \text{else.} \end{cases}$$

If G is a simple, undirected graph, A is binary and symmetric with zeros along the main diagonal. In this case, the eigenvalues of A will be real. We label the eigenvalues of A in nonincreasing order: $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$. If G is connected, then $\lambda_1 > \lambda_2$ by the Perron–Frobenius theorem [44, page 673]. Since A is a symmetric, real-valued matrix, we can decompose A into $A = Q\Lambda Q^T$, where $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_1 > \lambda_2 \ge \cdots \ge \lambda_n$, $Q = [\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n]$ is orthogonal, and $\mathbf{q}_i$ is the eigenvector associated with λ_i. The *dominant eigenvector*, $\mathbf{q}_1$, can be chosen to have positive entries when G is connected: we write this as $\mathbf{q}_1 > \mathbf{0}$.

If G is a strongly connected digraph, its adjacency matrix A is irreducible, and conversely. Let $\rho(A) = r$ be the spectral radius of A. Then, again by the Perron–Frobenius theorem, $\lambda_1 = r$ is a simple eigenvalue of A and both the left and right eigenvectors of A associated with λ_1 can be chosen to be positive. If G is also diagonalizable, then there exists an invertible matrix X such that $A = X\Lambda X^{-1}$, where $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_1 \ge |\lambda_i|$ for $2 \le i \le n$, $X = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n]$, and $(X^{-1})^T = [\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n]$. The left eigenvector associated with λ_i is $\mathbf{y}_i$, and the right eigenvector associated with λ_i is $\mathbf{x}_i$. In the case where G is not diagonalizable, A can be decomposed using the Jordan canonical form

$$A = XJX^{-1} = X \begin{pmatrix} \lambda_1 & \mathbf{0} \\ \mathbf{0} & \hat{J} \end{pmatrix} X^{-1},$$

where J is the Jordan matrix of A, except that we place the 1×1 block corresponding to λ_1 first for notational convenience. The first column $\mathbf{x}_1$ of X is the dominant right eigenvector of A, and the first column $\mathbf{y}_1$ of X^{-T} is the dominant left eigenvector of A (equivalently, the dominant right eigenvector of A^T).

Throughout the paper, I denotes the $n \times n$ identity matrix.

3. Node centrality. As we discussed in the introduction, many measures of node centrality have been developed and used over the years. In this section we review and motivate several centrality measures to be analyzed in the rest of the paper.

3.1. Some common centrality measures. Some of the most common measures include degree centrality, eigenvector centrality [9], PageRank [48], betweenness centrality [12, 30], Katz centrality [36], and subgraph centrality [26, 27]. More recently, total communicability has been introduced as a centrality measure [5]. A node is deemed "important" according to a given centrality index if the corresponding value of the index is high relative to that of other nodes.

Most of these measures are applicable to both undirected and directed graphs. In the directed case, however, each node can play two roles: *sink* and *source*, or *receiver* and *broadcaster*, since a node in general can be both a starting point and an arrival point for directed edges. This has led to the notion of *hubs* and *authorities* in a network, with hubs being nodes with a high broadcast centrality index and authorities being nodes with a high receive centrality index. For the types of indices considered in this paper, broadcast centrality measures correspond to quantities computed from the adjacency matrix A, whereas authority centrality measures correspond to the same quantities computed from the transpose A^T of the adjacency matrix. When the graph is undirected, $A = A^T$ and the broadcast and receive centrality scores of each node coincide.

Examples of (broadcast) centrality measures are
- out-degree centrality: $d_i^{out} := [A\mathbf{1}]_i$;
- (right) eigenvector centrality: $C_{ev}(i) := \mathbf{e}_i^T \mathbf{q}_1 = q_1(i)$, where $\mathbf{q}_1$ is the dominant (right) eigenvector of A;
- exponential subgraph centrality: $SC_i(\beta) := [\mathrm{e}^{\beta A}]_{ii}$;
- resolvent subgraph centrality: $RC_i(\alpha) := [(I - \alpha A)^{-1}]_{ii}$;
- total communicability: $TC_i(\beta) := [\mathrm{e}^{\beta A}\mathbf{1}]_i = \mathbf{e}_i^T \mathrm{e}^{\beta A}\mathbf{1}$; and
- Katz centrality: $K_i(\alpha) := [(I - \alpha A)^{-1}\mathbf{1}]_i = \mathbf{e}_i^T(I - \alpha A)^{-1}\mathbf{1}$.

Here $\mathbf{e}_i$ is the ith standard basis vector, $\mathbf{1}$ is the vector of all ones, $0 < \alpha < \frac{1}{\lambda_1}$ (see below), and $\beta > 0$. We note that the vector of all ones is sometimes replaced by a *preference vector* $\mathbf{v}$ with positive entries; for instance, $\mathbf{v} = \mathbf{d} := A\mathbf{1}$ (the vector of node out-degrees).

Replacing A with A^T in the definitions above we obtain the corresponding authority measures. Thus, out-degree centrality becomes in-degree centrality, right eigenvector centrality becomes left eigenvector centrality, and row sums are replaced by column sums when computing the total communicability centrality. Note, however, that the exponential and resolvent subgraph centralities are unchanged when replacing A with A^T, since $f(A^T) = f(A)^T$ for any matrix function [34, Theorem 1.13]. Hence, measures based on the diagonal entries cannot differentiate between the two roles a node can play in a directed network, and for this reason they are mostly used in the undirected case only (but see [4] for an adaptation of subgraph centrality to digraphs).

Often, the value $\beta = 1$ is used in the calculation of exponential subgraph centrality and total communicability. The parameter β can be interpreted as an *inverse temperature* and has been used to model the effects of external disturbances on the network. As $\beta \to 0+$, the "temperature" of the environment surrounding the network increases, corresponding to more intense external disturbances. Conversely, as $\beta \to \infty$, the temperature goes to 0 and the network "freezes." We refer the reader to [25] for an extensive discussion and applications of these physical analogies.

3.2. Justification in terms of graph walks. The justification behind using the (scaled) matrix exponential to compute centrality measures can be seen by considering the power series expansion of $e^{\beta A}$:

$$(3.1) \qquad e^{\beta A} = I + \beta A + \frac{(\beta A)^2}{2!} + \cdots + \frac{(\beta A)^k}{k!} + \cdots = \sum_{k=0}^{\infty} \frac{(\beta A)^k}{k!}.$$

It is well known that given an adjacency matrix A of an unweighted network, $[A^k]_{ij}$ counts the total number of walks of length k between nodes i and j. Thus $SC_i(\beta) = [e^{\beta A}]_{ii}$, the exponential subgraph centrality of node i, counts the total number of closed walks in the network which are centered at node i, weighing walks of length k by a factor of $\frac{\beta^k}{k!}$. Unlike degree, which is a purely local index, subgraph centrality takes into account the short, medium, and long-range influence of all nodes on a given node (assuming G is strongly connected). Assigning decreasing weights to longer walks ensures the convergence of the series (3.1) while guaranteeing that short-range interactions are given more weight than long-range ones [27].

Total communicability is closely related to subgraph centrality. This measure also counts the number of walks starting at node i, scaling walks of length k by $\frac{\beta^k}{k!}$. However, rather than just counting closed walks, total communicability counts all walks between node i and every node in the network. The name stems from the fact that $TC_i(\beta) = \sum_{j=1}^{n} C_{ij}(\beta)$, where $C_{ij}(\beta) := [e^{\beta A}]_{ij}$, the communicability between nodes i and j, is a measure of how "easy" it is to exchange a message between nodes i and j over the network; see [25] for details. Although subgraph centrality and total communicability are clearly related, they do not always provide the same ranking of the nodes. Furthermore, unlike subgraph centrality, total communicability can distinguish between the two roles a node can play in a directed network. More information about the relation between the two measures can be found in [5].

The matrix resolvent $(I - \alpha A)^{-1}$ was first used to rank nodes in a network in the early 1950s, when Katz used the column sums to calculate node importance [36]. Since then, the diagonal values have also been used as a centrality measure; see [26]. The resolvent subgraph centrality score of node i is given by $[(I - \alpha A)^{-1}]_{ii}$, and the Katz centrality score is given by either $[(I - \alpha A)^{-1}\mathbf{1}]_i$ or $[(I - \alpha A^T)^{-1}\mathbf{1}]_i$, depending on whether hub or authority scores are desired. As mentioned, $\mathbf{1}$ may be replaced by an arbitrary (positive) preference vector, $\mathbf{v}$.

As when using the matrix exponential, these resolvent-based centrality measures count the number of walks in the network, penalizing longer walks. This can be seen by considering the Neumann series expansion of $(I - \alpha A)^{-1}$, valid for $0 < \alpha < \frac{1}{\lambda_1}$:

$$(3.2) \qquad (I - \alpha A)^{-1} = I + \alpha A + \alpha^2 A^2 + \cdots + \alpha^k A^k + \cdots = \sum_{k=0}^{\infty} \alpha^k A^k.$$

The resolvent subgraph centrality of node i, $[(I - \alpha A)^{-1}]_{ii}$, counts the total number of closed walks in the network which are centered at node i, weighing walks of length

k by α^k. Similarly, the Katz centrality of node i counts all walks beginning at node i, penalizing the contribution of walks of length k by α^k. The bounds on α ($0 < \alpha < \frac{1}{\lambda_1}$) ensure that the matrix $I - \alpha A$ is invertible and that the power series in (3.2) converges to its inverse. The bounds on α also force $(I - \alpha A)^{-1}$ to be nonnegative, as $I - \alpha A$ is a nonsingular M-matrix. Hence, both the diagonal entries and the row/column sums of $(I - \alpha A)^{-1}$ are positive and can thus be used for ranking purposes.

4. A general class of functional centrality measures. In this section we establish precise conditions that a matrix function $f(A)$, where A is the adjacency matrix of a network, should satisfy in order to be used in the definition of walk-based centrality measures. We consider in particular analytic functions expressed by power series, with a focus on issues like convergence, positivity, and dependence on a tuneable parameter $t > 0$. We also formulate some auxiliary results on power series that will be crucial for the analysis to follow. For an introduction to the properties of analytic functions see, e.g., [42].

4.1. Admissible matrix functions. As discussed in subsection 3.2, walk-based centrality measures (such as Katz or subgraph centrality) lead to power series expansions in the (scaled) adjacency matrix of the network. While exponential- and resolvent-based centrality measures are especially natural (and well studied), there are a priori infinitely many other matrix functions which could be used [49, 26]. Not every function of the adjacency matrix, however, is suitable for the purpose of defining centrality measures, and some restrictions must be imposed.

A first obvious condition is that the function should be defined by a power series with *real* coefficients. This guarantees that $f(z)$ takes real values when the argument is real and that $f(A)$ has real entries for any real A. In [49] (see also [26]), the authors proposed considering only analytic functions admitting a Maclaurin series expansion of the form

$$(4.1) \qquad f(z) = \sum_{k=0}^{\infty} c_k z^k, \quad c_k \geq 0 \quad \text{for} \quad k \geq 0.$$

This ensures that $f(A)$ will be nonnegative for any adjacency matrix A. In [26] it is further required that $c_k > 0$ for all $k = 1, 2, \ldots, n - 1$, so as to guarantee that $[f(A)]_{ij} > 0$ for all $i \neq j$ whenever the network is (strongly) connected.[1] Although not explicitly stated in [26], it is clear that if one wants *all* the walks (of any length) in G to make a positive contribution to a centrality measure based on f, then one should impose the more restrictive condition $c_k > 0$ for all $k \geq 0$. Note that c_0 plays no significant role, since it is just a constant value added to all the diagonal entries of $f(A)$ and therefore does not affect the rankings. However, imposing $c_0 > 0$ guarantees that all entries of $f(A)$ are positive and leads to simpler formulas. Another tacit assumption in [26] is that only power series with a positive radius of convergence should be considered.

In the following, we will denote by $\mathcal{P}$ the class of analytic functions that can be expressed as sums of power series with strictly positive coefficients on some open neighborhood of 0. We note in passing that $\mathcal{P}$ forms a *positive cone* in function space, i.e., $\mathcal{P}$ is closed under linear combinations with positive coefficients.

Clearly, given an arbitrary adjacency matrix A, the matrix function $f(A)$, with $f \in \mathcal{P}$, need not be defined; indeed, f must be defined on the spectrum of A [34]. If f

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[1] We recall that a nonnegative $n \times n$ matrix A is irreducible if and only if $(I + A)^{n-1} > 0$. See, e.g., [35, Theorem 6.2.24].

is entire (i.e., analytic in the whole complex plane, like the exponential function), then $f(A)$ will always be defined, but this is not the case of functions with singularities, such as the resolvent. However, this difficulty can be easily circumvented by introducing a (scaling) parameter t and by considering for a given A the parameterized matrix function $g(t, A) := f(tA)$ only for values of t such that the power series

$$f(tA) = c_0 I + c_1 tA + c_2 t^2 A^2 + \cdots = \sum_{k=0}^{\infty} c_k t^k A^k$$

is convergent, that is, such that $|t\lambda_1(A)| < R_f$, where R_f denotes the radius of convergence of the power series representing f. In practice, for the purposes of this paper, we will limit ourselves to *positive* values of t in order to guarantee that $f(tA)$ is entrywise positive, as required by the definition of a centrality index. We summarize our discussion so far in the following lemma.

LEMMA 1. *Let $\mathcal{P}$ be the class of all analytic functions that can be expressed by a Maclaurin series with strictly positive coefficients in an open disk centered at 0. Given an irreducible adjacency matrix A and a function $f \in \mathcal{P}$ with radius of convergence $R_f > 0$, let $t^* = R_f/\lambda_1(A)$. Then $f(tA)$ is defined and strictly positive for all $t \in (0, t^*)$. If f is entire, then one can take $t^* = \infty$.*

Restriction of f to the class $\mathcal{P}$ and use of a positive parameter t, which will depend on A and f in case f is not entire, allows one to define the notion of f-centrality (as well as f-communicability, f-betweenness, and so forth; see [26]). Exponential subgraph centrality (with $t = \beta$) is an example of an entire function (hence all positive values of β are feasible), while resolvent subgraph centrality (with $t = \alpha$) exemplifies the situation where the parameter must be restricted to a finite interval, in this case $(0, \frac{1}{\lambda_1(A)})$ (since the geometric series $1 + x + x^2 + \cdots$ has radius of convergence $R_f = 1$).

We consider now two subclasses of the class $\mathcal{P}$ previously introduced. We let $\mathcal{P}_\infty$ denote the set of all power series in $\mathcal{P}$ with radius of convergence $R_f = \infty$, and we let $\mathcal{P}^\infty$ denote the set of all power series with finite radius of convergence R_f such that

$$(4.2) \qquad \sum_{k=0}^{\infty} c_k R_f^k = \lim_{t \to 1^-} \sum_{k=0}^{\infty} c_k t^k R_f^k = \infty$$

(we note that the first equality above follows from Abel's theorem [42, page 229]). The exponential and the resolvent are representative of functions in $\mathcal{P}_\infty$ and $\mathcal{P}^\infty$, respectively. It is worth emphasizing that, together, $\mathcal{P}_\infty$ and $\mathcal{P}^\infty$ do not exhaust the class $\mathcal{P}$. For example, the function $f(z) = \sum_{k=0}^{\infty} \frac{z^k}{k^2}$ is in $\mathcal{P}$, but it is not in $\mathcal{P}_\infty$ (since its radius of convergence is $R_f = 1$) or in $\mathcal{P}^\infty$, since

$$\lim_{t \to 1^-} \sum_{k=0}^{\infty} \frac{t^k R_f^k}{k^2} = \sum_{k=0}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} < \infty.$$

In section 5 we will analyze centrality measures based on functions f in $\mathcal{P}$ and its subclasses, $\mathcal{P}_\infty$ and $\mathcal{P}^\infty$.

4.2. Asymptotic behavior of the ratio of two power series. In our study of the limiting behavior of parameter-dependent functional centrality measures we will need to investigate the asymptotic behavior of the ratio of two power series with

positive coefficients. The following technical lemmas will be crucial for our analysis.

LEMMA 2. *Let the power series $\sum_{k=0}^{\infty} a_k t^k$, $\sum_{k=0}^{\infty} b_k t^k$ have positive real coefficients and be convergent for all $t \geq 0$. If $\lim_{k \to \infty} \frac{a_k}{b_k} = 0$, then*

$$\lim_{t \to \infty} \frac{\sum_{k=0}^{\infty} a_k t^k}{\sum_{k=0}^{\infty} b_k t^k} = 0.$$

Proof. Let $\varepsilon > 0$ be arbitrary, and let $N > 0$ be such that $\frac{a_k}{b_k} < \varepsilon$ for all $k > N$. Also, let $g(t) = \sum_{k=0}^{\infty} a_k t^k$ and $h(t) = \sum_{k=0}^{\infty} b_k t^k$. We have

$$(4.3) \qquad \frac{g(t)}{h(t)} = \frac{T_N^g(t) + g_1(t)}{T_N^h(t) + h_1(t)} = \frac{T_N^g(t)}{T_N^h(t) + h_1(t)} + \frac{g_1(t)}{T_N^h(t) + h_1(t)},$$

with $T_N^g(t) = \sum_{k=0}^{N} a_k t^k$, $T_N^h(t) = \sum_{k=0}^{N} b_k t^k$ and $g_1(t)$, $h_1(t)$ being the tails of the corresponding series. The first term on the right-hand side of (4.3) manifestly tends to zero as $t \to \infty$. The second term is clearly bounded above by $g_1(t)/h_1(t)$. The result then follows from

$$g_1(t) = \sum_{k=N+1}^{\infty} a_k t^k = \sum_{k=N+1}^{\infty} \left(\frac{a_k}{b_k} \right) b_k t^k < \varepsilon \, h_1(t)$$

and the fact that ε is arbitrary. □

LEMMA 3. *Let $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$ be given, with $\lambda_1 > |\lambda_2| \geq \cdots \geq |\lambda_n|$, and let $f \in \mathcal{P}_\infty \cup \mathcal{P}^\infty$ be defined at these points. Then*

$$(4.4) \qquad \lim_{t \to t^* -} \frac{t^j f^{(j)}(t\lambda_i)}{f(t\lambda_1)} = 0 \quad for \quad j = 0, 1, \ldots, \quad i = 2, \ldots, n,$$

where $t^* = R_f/\lambda_1$ and R_f is the radius of convergence of the series defining f around 0 (finite or infinite according to whether $f \in \mathcal{P}^\infty$ or $f \in \mathcal{P}_\infty$, respectively).

Proof. Consider first the case $t^* < \infty$. In this case the assumption that $f \in \mathcal{P}^\infty$ guarantees (cf. (4.2)) that the denominator of (4.4) tends to infinity, whereas the numerator remains finite for all $i \neq 1$ and all j. Indeed, each derivative $f^{(j)}(z)$ of $f(z)$ can be expressed by a power series having the same radius of convergence as the power series expressing $f(z)$. Since each $t^*\lambda_i$ (with $i \neq 1$) falls inside the circle of convergence, we have $|f^{(j)}(t^*\lambda_i)| < \infty$ for each $j \geq 0$ and hence (4.4).

Next, we consider the case where $t^* = \infty$. Let $i \neq 1$, and assume $\lambda_i \neq 0$ (the result is trivial for $\lambda_i = 0$). Since f is entire, so are all its derivatives and, moreover,

$$(4.5) \qquad |f^{(j)}(t\lambda_i)| = \left| \sum_{k=0}^{\infty} (k+j)^{\underline{j}} c_{k+j} t^k \lambda_i^k \right| \leq \sum_{k=0}^{\infty} (k+j)^{\underline{j}} c_{k+j} t^k |\lambda_i|^k < \infty,$$

where we have used the (standard) notation $(k+j)^{\underline{j}} = (k+j)(k+j-1)\cdots(k+1)$ (with the convention $k^{\underline{0}} = 1$). Now let $a_k = (k+j)^{\underline{j}} c_{k+j} \lambda_i^k$ be the coefficient of t^k in the power series expansion of $t^j f^{(j)}(t\lambda_i)$, and let $b_k = c_k \lambda_i^k$ be the coefficient of t^k in the power series expansion of $f(t\lambda_1)$; then

$$(4.6) \qquad \frac{a_k}{b_k} = (k+j)^{\underline{j}} \left(\frac{|\lambda_i|}{\lambda_1} \right)^k \quad for \ all \quad k \geq j.$$

Since exponential decay trumps polynomial growth, we conclude that the expression in (4.6) tends to zero as $k \to \infty$. Using Lemma 2 we get the desired conclusion. □

As we will see in the next section, the limit (4.4) with $j = 0$ will be instrumental in our analysis of undirected networks, while the general case is needed for the analysis of directed networks.

5. Limiting behavior of parameterized centrality measures. One difficulty in measuring the "importance" of a node in a network is that it is not always clear which of the many centrality measures should be used. Additionally, it is not clear a priori when two centrality measures will give similar node rankings on a given network. When using parameter-dependent indices, such as Katz, exponential, or resolvent-based subgraph centrality, the necessity of choosing the value of the parameter adds another layer of difficulty. For instance, it is well known that using different choices of α and β in Katz and subgraph centralities will generally produce different centrality scores and can lead to different node rankings. However, experimentally, it has been observed that different centrality measures often provide rankings that are highly correlated [5, 21, 32, 41, 45]. Moreover, in most cases, the *rankings* are quite stable, in the sense that they do not appear to change much for different choices of α and β, even if the actual *scores* may vary by orders of magnitude [38]. With Katz and subgraph centralities this happens in particular when the parameters α and β approach their limits:

$$\alpha \to 0+, \quad \alpha \to \frac{1}{\lambda_1}-, \quad \beta \to 0+, \quad \beta \to \infty.$$

Noting that the first derivatives of the node centrality measures grow unboundedly as $\alpha \to \frac{1}{\lambda_1}-$ and as $\beta \to \infty$, the centrality scores are extremely sensitive to (vary extremely rapidly with) small changes in α when α is close to $\frac{1}{\lambda_1}-$ and in β when β is even moderately large. Yet, the rankings produced stabilize quickly and do not change much (if at all) when α and β approach these limits. The same is observed as $\alpha, \beta \to 0+$.

The remainder of this section is devoted to proving that the same behavior can be expected, more generally, when using parameterized centrality measures based on analytic functions $f \in \mathcal{P}$. The observed behavior for Katz and subgraph centrality measures is thus explained and generalized.

It is worth noting that while all the parameterized centrality measures considered here depend continuously on $t \in [0, t^*)$, the rankings do not: hence, the limiting behavior of the ranking as the parameter tends to zero cannot be obtained by simply setting the parameter to zero.

5.1. Undirected networks. We begin with the undirected case. The following theorem is our main result. It completely describes the limiting behavior, for "small" and "large" values of the parameter, of parameterized functional centrality measures based on either the diagonal entries or the row sums. Recall that a nonnegative matrix A is *primitive* if $\lambda_1 > |\lambda_i|$ for $i = 2, \ldots, n$; see, e.g., [44, page 674].

THEOREM 5.1. *Let $G = (V, E)$ be a connected, undirected, unweighted network with adjacency matrix A, assumed to be primitive, and let $f \in \mathcal{P}$ be defined on the spectrum of A. Let $SC_i(t) = [f(tA)]_{ii}$ be the f-subgraph centrality of node i, and let $\mathbf{SC}(t)$ be the corresponding vector of f-subgraph centralities. Also, let $TC_i(t) = [f(tA)\mathbf{1}]_i$ be the total f-communicability of node i, and let $\mathbf{TC}(t)$ be the corresponding vector. Then, the following hold:*

  (i) *as $t \to 0+$, the rankings produced by both $\mathbf{SC}(t)$ and $\mathbf{TC}(t)$ converge to those produced by $\mathbf{d} = (d_i)$, the vector of degree centralities;*

 (ii) *if, in addition, $f \in \mathcal{P}_\infty \cup \mathcal{P}^\infty$, then for $t \to t^*-$ the rankings produced by both $\mathbf{SC}(t)$ and $\mathbf{TC}(t)$ converge to those produced by eigenvector centrality, i.e., by the entries of $\mathbf{q}_1$, the dominant eigenvector of A;*

(iii) *the conclusion in* (ii) *still holds if the vector of all ones* $\mathbf{1}$ *is replaced by any preference vector* $\mathbf{v} > \mathbf{0}$ *in the definition of* $\mathbf{TC}(t)$.

Proof. To prove (i), consider first the Maclaurin expansion of $SC_i(t)$:

$$SC_i(t) = c_0 + c_1 t[A]_{ii} + c_2 t^2 [A^2]_{ii} + c_3 t^3 [A^3]_{ii} + \cdots = c_0 + 0 + c_2 t^2 d_i + c_3 t^3 [A^3]_{ii} + \cdots .$$

Let $\phi(t) := \frac{1}{c_2 t^2}[\mathbf{SC}(t) - c_0 \mathbf{1}]$. The rankings produced by $\phi(t)$ will be the same as those produced by $\mathbf{SC}(t)$, as the scores for each node have all been shifted and scaled in the same way. Now, the ith entry of $\phi(t)$ is given by

$$(5.1) \qquad \phi_i(t) = \frac{1}{c_2 t^2}[SC_i(t) - c_0] = d_i + \frac{c_3}{c_2} t[A^3]_{ii} + \frac{c_4}{c_2} t^2 [A^4]_{ii} + \cdots ,$$

which tends to d_i as $t \to 0+$. Thus, as $t \to 0+$, the rankings produced by the f-subgraph centrality scores reduce to those produced by the degrees.

Similarly, we have

$$(5.2) \quad TC_i(t) = [f(tA)\mathbf{1}]_i = [c_0 \mathbf{1} + c_1 tA\mathbf{1} + c_2 t^2 A^2 \mathbf{1} + \cdots]_i = c_0 + c_1 t d_i + c_2 t^2 [A\mathbf{d}]_i + \cdots .$$

Subtracting c_0 from $[f(tA)\mathbf{1}]_i$ and dividing the result by $c_1 t$ leaves the quantity $d_i + O(t)$; hence for $t \to 0+$ we again obtain degree centrality.

To prove (ii), consider first the expansion of $SC_i(t)$ in terms of the eigenvalues and eigenvectors of A:

$$SC_i(t) = \sum_{k=1}^{n} f(t\lambda_k) q_k(i)^2 = f(t\lambda_1) q_1(i)^2 + \sum_{k=2}^{n} f(t\lambda_k) q_k(i)^2,$$

where $q_k(i)$ is the ith entry of the (normalized) eigenvector $\mathbf{q}_k$ of A associated with λ_k. Let $\psi(t) := \frac{1}{f(t\lambda_1)}\mathbf{SC}(t)$. As in the proof of (i), the rankings produced by $\psi(t)$ are the same as those produced by $\mathbf{SC}(t)$, since the scores for each node have all been rescaled by the same amount. Next, the ith entry of $\psi(t)$ is

$$(5.3) \qquad\qquad \psi_i(t) = q_1(i)^2 + \sum_{k=2}^{n} \frac{f(t\lambda_k)}{f(t\lambda_1)} q_k(i)^2.$$

Since A is primitive, we have $\lambda_1 > \lambda_k$ for $2 \le k \le n$. Hence, applying Lemma 3 with $j = 0$ we conclude that $\psi_i(t) \to q_1(i)^2$ as $t \to t^*-$. By the Perron–Frobenius theorem we can choose $\mathbf{q}_1 > \mathbf{0}$; hence the rankings produced by $q_1(i)^2$ are the same as those produced by $q_1(i)$. Thus, as $t \to t^*-$, the rankings produced by the f-subgraph centrality scores reduce to those obtained with eigenvector centrality.

Similarly, we have

$$(5.4) \quad TC_i(t) = \sum_{k=1}^{n} f(t\lambda_k)(\mathbf{q}_k^T \mathbf{1}) q_k(i) = f(t\lambda_1)(\mathbf{q}_1^T \mathbf{1}) q_1(i) + \sum_{k=2}^{n} f(t\lambda_k)(\mathbf{q}_k^T \mathbf{1}) q_k(i).$$

Note that $\mathbf{q}_1^T \mathbf{1} > 0$, since $\mathbf{q}_1 > \mathbf{0}$. Dividing both sides by $f(t\lambda_1)\mathbf{q}_1^T \mathbf{1}$ and taking the limit as $t \to t^*-$ we obtain the desired result.

Finally, (iii) follows by just replacing $\mathbf{1}$ with $\mathbf{v}$ in the foregoing argument.   ☐

By specializing the choice of f to the matrix exponential and resolvent, we immediately obtain the following corollary of Theorem 5.1.

COROLLARY 1. *Let* $G = (V, E)$ *be a connected, undirected, unweighted network with adjacency matrix* A, *assumed to be primitive. Let* $EC_i(\beta) = [\mathrm{e}^{\beta A}]_{ii}$ *and*

	Limiting ranking scheme	
Method	Degree	Eigenvector
$RC(\alpha)$, $K(\alpha)$	$\alpha \to 0+$	$\alpha \to \frac{1}{\lambda_1}-$
$EC(\beta)$, $TC(\beta)$	$\beta \to 0+$	$\beta \to \infty$

$RC_i(\alpha) = [(I - \alpha A)^{-1}]_{ii}$ *be the exponential and resolvent subgraph centralities of node i. Also, let* $TC_i(\beta) = [e^{\beta A}\mathbf{1}]_i$ *and* $K_i(\alpha) = [(I - \alpha A)^{-1}\mathbf{1}]_i$ *be the total communicability and Katz centrality of node i, respectively. Then, the limits in Table 1 hold. Moreover, the limits for* $TC_i(\beta)$ *and* $K_i(\alpha)$ *remain the same if the vector* $\mathbf{1}$ *is replaced by an arbitrary preference vector* $\mathbf{v} > \mathbf{0}$.

REMARK 1. The restriction to primitive matrices is required in order to have $\lambda_1 > \lambda_k$ for $k \neq 1$, so that Lemma 3 can be used in the proof of Theorem 5.1. At first sight, this assumption may seem somewhat restrictive; for instance, bipartite graphs would be excluded, since they have $\lambda_n = -\lambda_1$. In practice, however, there is no loss of generality. Indeed, if A is imprimitive, we can replace A with the (always primitive) matrix $A_\varepsilon = (1 - \varepsilon)A + \varepsilon I$ with $0 < \varepsilon < 1$, compute the quantities of interest using $f(tA_\varepsilon)$, and then let $\varepsilon \to 0$. Note that $\rho(A_\varepsilon) = \rho(A)$; hence the radius of convergence is unchanged. Also note that for some centrality measures, such as those based on the matrix exponential, it is not even necessary to take the limit for $\varepsilon \to 0$. Indeed, we have $e^{\beta A_\varepsilon} = e^{\beta \varepsilon}e^{\beta(1-\varepsilon)A}$. The prefactor $e^{\beta \varepsilon}$ is just a scaling that does not affect the rankings, and $e^{\beta(1-\varepsilon)A}$ and $e^{\beta A}$ have identical limiting behavior for $\beta \to 0$ or $\beta \to \infty$.

5.2. Directed networks. Here we extend our analysis to directed networks. The discussion is similar to that for the undirected case, except that now we need to distinguish between receive and broadcast centralities. Also, the Jordan canonical form must replace the spectral decomposition in the proofs.

THEOREM 5.2. *Let* $G = (V, E)$ *be a strongly connected, directed, unweighted network with adjacency matrix A, and let* $f \in \mathcal{P}$ *be defined on the spectrum of A. Let* $TC_i^b(t) = [f(tA)\mathbf{1}]_i$ *be the broadcast total f-communicability of node i and* $\mathbf{TC}^b(t)$ *be the corresponding vector of broadcast total f-communicabilities. Furthermore, let* $TC_i^r(t) = [f(tA^T)\mathbf{1}]_i = [f(tA)^T\mathbf{1}]_i$ *be the receive total f-communicability of node i and* $\mathbf{TC}^r(t)$ *be the corresponding vector of receive total f-communicabilities. Then, the following hold:*

  (i) *as* $t \to 0+$, *the rankings produced by* $\mathbf{TC}^b(t)$ *converge to those produced by the out-degrees of the nodes in the network;*

 (ii) *as* $t \to 0+$, *the rankings produced by* $\mathbf{TC}^r(t)$ *converge to those produced by the in-degrees of the nodes in the network;*

(iii) *if* $f \in \mathcal{P}_\infty \cup \mathcal{P}^\infty$, *then as* $t \to t^*-$, *the rankings produced by* $\mathbf{TC}^b(t)$ *converge to those produced by* $\mathbf{x}_1$, *where* $\mathbf{x}_1$ *is the dominant right eigenvector of A;*

 (iv) *if* $f \in \mathcal{P}_\infty \cup \mathcal{P}^\infty$, *then as* $t \to t^*-$, *the rankings produced by* $\mathbf{TC}^r(t)$ *converge to those produced by* $\mathbf{y}_1$, *where* $\mathbf{y}_1$ *is the dominant left eigenvector of A;*

  (v) *results* (iii) *and* (iv) *still hold if* $\mathbf{1}$ *is replaced by an arbitrary preference vector* $\mathbf{v} > \mathbf{0}$ *in the definitions of* $\mathbf{TC}^b(t)$ *and* $\mathbf{TC}^r(t)$.

Proof. The proofs of (i) and (ii) are analogous to that for $TC_i(t)$ in part (i) of Theorem 5.1, keeping in mind that the entries of $A\mathbf{1}$ are the out-degrees and those of $A^T\mathbf{1}$ are the in-degrees of the nodes of G.

TABLE 2
Limiting behavior of different ranking schemes in the directed case.

Method	Limiting ranking scheme			
	Out-degree	In-degree	Right eigenvector	Left eigenvector
$K^b(\alpha)$	$\alpha \to 0+$		$\alpha \to \frac{1}{\lambda_1}-$	
$K^r(\alpha)$		$\alpha \to 0+$		$\alpha \to \frac{1}{\lambda_1}-$
$EC^b(\beta)$	$\beta \to 0+$		$\beta \to \infty$	
$EC^r(\beta)$		$\beta \to 0+$		$\beta \to \infty$

To prove (iii), observe that if f is defined on the spectrum of A, then

$$(5.5) \qquad f(A) = \sum_{k=1}^{s} \sum_{j=0}^{n_k-1} \frac{f^{(j)}(\lambda_k)}{j!} (A - \lambda_k I)^j G_k,$$

where s is the number of distinct eigenvalues of A, n_k is the index of the eigenvalue λ_k (that is, the order of the largest Jordan block associated with λ_k in the Jordan canonical form of A), and G_k is the oblique projector with range $\mathcal{R}(G_k) = \mathcal{N}((A - \lambda_k I)^{n_k})$ and null space $\mathcal{N}(G_k) = \mathcal{R}((A - \lambda_k I)^{n_k})$; see, e.g., [34, section 1.2.2] or [44, section 7.9]. Using (5.5) and the fact that λ_1 is simple by the Perron–Frobenius theorem, we find that

$$TC_i^b(t) = f(t\lambda_1)(\mathbf{y}_1^T \mathbf{1})x_1(i) + \sum_{k=2}^{s} \sum_{j=0}^{n_k-1} \frac{t^j f^{(j)}(t\lambda_k)}{j!}[(A - \lambda_k I)^j G_k \mathbf{1}]_i.$$

Noting that $\mathbf{y}_1^T \mathbf{1} > 0$, let $\boldsymbol{\psi}^b(t) := \frac{1}{f(t\lambda_1)(\mathbf{y}_1^T\mathbf{1})}\mathbf{TC}^b(t)$. The rankings produced by $\boldsymbol{\psi}^b(t)$ will be the same as those produced by $\mathbf{TC}^b(t)$. Now, the ith entry of $\boldsymbol{\psi}^b(t)$ is

$$(5.6) \qquad \psi_i^b(t) = x_1(i) + \sum_{k=2}^{s} \sum_{j=0}^{n_k-1} \frac{t^j f^{(j)}(t\lambda_k)}{j!f(t\lambda_1)(\mathbf{y}_1^T\mathbf{1})}[(A - \lambda_k I)^j G_k \mathbf{1}]_i.$$

Without loss of generality, we can assume that $\lambda_1 > |\lambda_k|$ for $k \neq 1$ (see Remark 1). By Lemma 3 the second term on the right-hand side of (5.6) vanishes as $t \to t^*-$, and therefore $\psi_i^b(t) \to x_1(i)$; that is, the rankings given by $\boldsymbol{\psi}^b(t)$ reduce to those given by the right dominant eigenvector $\mathbf{x}_1$ of A in the limit $t \to t^*-$.

The proof of (iv) is completely analogous to that of (iii).

Finally, the proof of (v) is obtained by replacing $\mathbf{1}$ with $\mathbf{v}$ and observing that the argument used to prove (iii) (and thus (iv)) remains valid. $\quad\square$

By specializing the choice of f to the matrix exponential and resolvent, we immediately obtain the following corollary of Theorem 5.2.

COROLLARY 2. *Let $G = (V, E)$ be a strongly connected, directed, unweighted network with adjacency matrix A. Let $EC_i^b(\beta) = [e^{\beta A}\mathbf{v}]_i$ and $K_i^b(\alpha) = [(I - \alpha A)^{-1}\mathbf{v}]_i$ be the total communicability and Katz broadcast centrality of node i, respectively. Similarly, let $EC_i^r(\beta) = [e^{\beta A^T}\mathbf{v}]_i$ and $K_i^r(\alpha) = [(I - \alpha A^T)^{-1}\mathbf{v}]_i$ be the total communicability and Katz receive centrality of node i. Then, the limits in Table 2 hold. Moreover, all these limits remain the same if the vector $\mathbf{1}$ is replaced by an arbitrary preference vector $\mathbf{v} > \mathbf{0}$.*

This concludes our analysis in the case of simple, strongly connected (di)graphs.

5.3. Extensions to more general graphs. So far we have restricted our discussion to unweighted, loopless graphs. This was done in part for ease of exposition. Indeed, it is easy to see that all of the results in Theorem 5.1 and Corollary 1 remain valid in the case of *weighted* undirected networks if all the weights a_{ij} (with $(i, j) \in E$) are positive and if we interpret the degree of node i to be the *weighted degree*, i.e., the ith row sum $A\mathbf{1}$. The only case that cannot be generalized is that relative to $\mathbf{SC}(t)$ as $t \to 0+$ in Theorem 5.1 and, as a consequence, those relative to $EC_i(\beta)$ and $K_i(\alpha)$ as $\beta, \alpha \to 0+$ in Corollary 1. The reason for this is that in general it is no longer true that $[A^2]_{ii} = d_i$, i.e., the diagonal entries of A^2 are not generally equal to the weighted degrees.

Furthermore, all of the results in Theorem 5.2 and Corollary 2 remain valid in the case of strongly connected, *weighted* directed networks if we interpret the out-degree and in-degree of node i as weighted out- and in-degree, given by the ith row and column sum of A, respectively.

Finally, all the results relative to the limit $t \to t^*-$ in Theorems 5.1 and 5.2 remain valid in the presence of loops (i.e., if $a_{ii} \neq 0$ for some i). Hence, in particular, all the results in Corollaries 1 and 2 concerning the behavior of the various exponential and resolvent-based centrality measures for $\beta \to \infty$ and $\alpha \to \frac{1}{\lambda_1}-$ remain valid in this case.

6. The case of PageRank. In this section we discuss the limiting behavior of the PageRank algorithm [48], which has a well-known interpretation in terms of random walks on a digraph (see, e.g., [39]). Because of the special structure possessed by the matrices arising in this method, a treatment somewhat different from that developed in the previous section is required.

Let $G = (V, E)$ be an arbitrary digraph with $|V| = n$ nodes, and let A be the corresponding adjacency matrix. From A we construct an irreducible, column-stochastic matrix P as follows. Let D be the diagonal matrix with entries

$$
d_{ij} = \begin{cases} d_i^{out} & \text{if } i = j \quad \text{and} \quad d_i^{out} > 0, \\ 1 & \text{if } i = j \quad \text{and} \quad d_i^{out} = 0, \\ 0 & \text{else.} \end{cases}
$$

Now, let

$$(6.1) \qquad\qquad H = A^T D^{-1}.$$

This matrix may have zero columns, corresponding to those indices i for which $d_i^{out} = 0$; the corresponding nodes of G are known as *dangling nodes*. Let I denote the set of such indices, and define the vector $\mathbf{a} = (a_i)$ by

$$
a_i = \begin{cases} 1 & \text{if } i \in I, \\ 0 & \text{else.} \end{cases}
$$

Next, we define the matrix S by

$$(6.2) \qquad\qquad S = H + \frac{1}{n}\mathbf{1}\,\mathbf{a}^T.$$

Thus, S is obtained from H by replacing each zero column of H (if present) by the column vector $\frac{1}{n}\mathbf{1}$. Note that S is column-stochastic but could still be (and very often

is) reducible. To obtain an irreducible matrix, we take $\alpha \in (0, 1)$ and construct the "Google matrix"

$$(6.3) \qquad P = \alpha S + (1 - \alpha)\mathbf{v}\mathbf{1}^T,$$

where $\mathbf{v}$ is an arbitrary probability distribution vector (i.e., a column vector with nonnegative entries summing up to 1). The simplest choice for $\mathbf{v}$ is the uniform distribution, $\mathbf{v} = \frac{1}{n}\mathbf{1}$, but other choices are possible. Thus, P is a convex combination of the modified scaled adjacency matrix S and a rank-one matrix and is column-stochastic. If every entry in $\mathbf{v}$ is strictly positive ($\mathbf{v} > \mathbf{0}$), P is also positive and therefore acyclic and irreducible. The Markov chain associated with P is *ergodic*: it has a unique steady-state probability distribution vector $\mathbf{p} = \mathbf{p}(\alpha) > \mathbf{0}$, given by the dominant eigenvector of P, normalized so that $\mathbf{p}^T\mathbf{1} = 1$: thus, $\mathbf{p}$ satisfies $\mathbf{p} = P\mathbf{p}$, or $(I - P)\mathbf{p} = \mathbf{0}$. The vector $\mathbf{p}$ is known as the *PageRank vector*, and it can be used to rank the nodes in the original digraph G. The success of this method in ranking web pages is universally recognized. It has also been used successfully in many other settings.

The role of the parameter α is to balance the structure of the underlying digraph with the probability of choosing a node at random (according to the probability distribution $\mathbf{v}$) in the course of a random walk on the graph. Another important consideration is the rate of convergence to steady state of the Markov chain: the smaller the value of α, the faster the convergence. In practice, the choice $\alpha = 0.85$ is often recommended.

It was recognized early on that the PageRank vector can also be obtained by solving a nonhomogeneous linear system of equations. In fact, there is more than one such linear system; see, e.g., [39, Chapter 7] and the references therein. One possible reformulation of the problem is given by the linear system

$$(6.4) \qquad (I - \alpha H)\mathbf{x} = \mathbf{v}, \quad \mathbf{p} = \mathbf{x}/(\mathbf{x}^T\mathbf{1}).$$

For each $\alpha \in (0, 1)$, the coefficient matrix in (6.4) is a nonsingular M-matrix; hence it is invertible with a nonnegative inverse $(I - \alpha H)^{-1}$. Note the similarity of this linear system with that corresponding to Katz centrality. Using this equivalence, we can easily describe the limiting behavior of PageRank for $\alpha \to 0+$.

THEOREM 6.1. *Let H be the matrix defined in (6.1), and let $\mathbf{p}(\alpha)$ be the PageRank vector corresponding to a given $\alpha \in (0, 1)$. Assume $\mathbf{v} = \frac{1}{n}\mathbf{1}$ in the definition (6.3) of the Google matrix P. Then, for $\alpha \to 0+$, the rankings given by $\mathbf{p}(\alpha)$ converge to those given by the vector $H\mathbf{1}$, the row sums of H, or, equivalently, by the vector $S\mathbf{1}$, the row sums of S.*

Proof. Note that for each $\alpha \in (0, 1)$ the inverse matrix $(I - \alpha H)^{-1}$ can be expanded in a Neumann series; hence the unique solution of (6.4) can be expressed as

$$(6.5) \qquad \mathbf{x} = \mathbf{v} + \alpha H\mathbf{v} + \alpha^2 H^2\mathbf{v} + \cdots.$$

When $\mathbf{v} = \frac{1}{n}\mathbf{1}$, the rankings given by the entries of $\mathbf{x}$ coincide with those given by the entries of $(\mathbf{x} - \mathbf{v})/\alpha$. But (6.5) implies that

$$\lim_{\alpha \to 0+} \frac{\mathbf{x} - \mathbf{v}}{\alpha} = H\mathbf{v} = \frac{1}{n}H\mathbf{1},$$

showing that the rankings from $\mathbf{p}(\alpha)$ coincide with those from the row sums of H in the limit $\alpha \to 0+$. Finally, $S\mathbf{1} = H\mathbf{1} + (\mathbf{a}^T\mathbf{1}/n)\mathbf{1}$; hence the row sums of S result in the same limit rankings. $\square$

We emphasize that the above result holds only for the case of a uniform personalization vector $\mathbf{v}$.

REMARK 2. Since H is a scaled adjacency matrix, each entry of $H\mathbf{1}$ is essentially a weighted in-degree; see also the discussion in section 5.3. Hence, we conclude that the structure of the graph G retains considerable influence on the rankings obtained by PageRank even for very small α, as long as it is nonzero.[2]

REMARK 3. The behavior of the PageRank vector for $\alpha \to 1$ (from the left) has received a great deal of attention in the literature; see, e.g., [7, 8, 39, 52]. Assume that $\lambda = 1$ is the only eigenvalue of S on the unit circle. Then it can be shown (see [43, 52]) that for $\alpha \to 1-$, the rankings obtained with PageRank with initial vector $\mathbf{v}$ converge to those given by the vector

$$(6.6) \qquad \mathbf{x}^* = (I - (I - S)(I - S)^{\sharp})\mathbf{v},$$

where $(I - S)^{\sharp}$ denotes the group generalized inverse of $I - S$ (see [14]). In the case of a uniform personalization vector $\mathbf{v} = \frac{1}{n}\mathbf{1}$, (6.6) is equivalent to using the row sums of the matrix $(I - S)(I - S)^{\sharp}$. As discussed in detail in [8], however, using this vector may lead to rankings that are not very meaningful, since when G is not strongly connected (which is usually the case in practice), it tends to give zero scores to nodes that are arguably the most important. For this reason, values of α too close to 1 are not recommended.

We conclude this section with a few remarks about another technique, known as *DiffusionRank* [53] or *Heat Kernel PageRank* [16]. This method is based on the matrix exponential e^{tP}, where P is column-stochastic, acyclic, and irreducible. For example, P could be the "Google matrix" constructed from a digraph G in the manner described above. It is immediate to see that for all $t > 0$ the column sums of e^{tP} are all equal to e^{t}; hence the scaled matrix $\mathrm{e}^{-t}\mathrm{e}^{tP}$ is column-stochastic. Moreover, its dominant eigenvector is the same as the dominant eigenvector of P, namely, the PageRank vector. It follows from the results found in section 5.2, and can easily be shown directly (see, e.g., [53]), that the node rankings obtained using the row sums of e^{tP} tend, for $t \to \infty$, to those given by PageRank. Hence, the PageRank vector can be regarded as the equilibrium distribution of a continuous-time diffusion process on the underlying digraph.

7. Discussion. The centrality measures considered in this paper are all based on walks on the network. The degree centrality of a node i counts the number of walks of length one starting at i (the degree of i). In contrast, the eigenvector centrality of node i gives the limit as $k \to \infty$ of the percentage of walks of length k which start at node i among all walks of length k (see [19, Theorem 2.2.4] and [24, page 127]). Thus, the degree centrality of node i measures the *local* influence of i and the eigenvector centrality measures the *global* influence of i.

When a centrality measure associated with an analytic function $f \in \mathcal{P}$ is used, walks of all lengths are included in the calculation of centrality scores, and a weight c_k is assigned to the walks of length k, where $c_k \to 0$ as $k \to \infty$. Hence, both local and global influence are now taken into account but with longer walks being penalized more heavily than shorter ones. The parameter t permits further tuning of the weights; as t is decreased, the weights corresponding to larger k decay faster and shorter walks become more important. In the limit as $t \to 0+$, walks of length one (i.e., edges) dominate the centrality scores and the rankings converge to the degree

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[2]See this paper's supplementary materials for a numerical illustration of this statement.

centrality rankings. As t is increased, given a fixed walk length k, the corresponding weight increases more rapidly than those of shorter walks. In the limit as $t \to t^*-$, walks of "infinite" length dominate and the centrality rankings converge to those of eigenvector centrality.

Hence, when using parameterized centrality measures, the parameter t can be regarded as a "knob" that can be used for interpolating, or tuning, between rankings based on local influence (short walks) and those based on global influence (long walks). In applications where local influence is most important, degree centrality will often be difficult to distinguish from any of the parameterized centrality measures with t small. Similarly, when global influence is the only important factor, parameterized centrality measures with $t \approx t^*$ will often be virtually indistinguishable from eigenvector centrality.

Parameterized centrality measures are likely to be most useful when both local and global influence need to be considered in the ranking of nodes in a network. In order to achieve this, "moderate" values of t (not too small and not too close to t^*) should be used.

To make this notion more quantitative, however, we need some way to estimate how fast the limiting rankings given by degree and eigenvector centralities are approached for $t \to 0+$ and $t \to t^*-$, respectively. We start by considering the undirected case (weights and loops are allowed). The approach to the eigenvector centrality limit as $t \to t^*-$ depends on the *spectral gap* $\lambda_1 - \lambda_2$ of the adjacency matrix of the network. This is clearly seen from the fact that the difference between the various parameterized centrality measures (suitably scaled) depends on the ratios $\frac{f(t\lambda_k)}{f(t\lambda_1)}$ for $2 \le k \le n$; see (5.3) and (5.4). Since a function $f \in \mathcal{P}$ is strictly increasing with t (when $t > 0$), a relatively large spectral gap implies that each term containing $\frac{f(t\lambda_k)}{f(t\lambda_1)}$ (with $k \ne 1$) will tend rapidly to zero as $t \to t^*-$, since $f(t\lambda_1)$ will grow much faster than $f(t\lambda_k)$. For example, in the case of exponential subgraph centrality the $k = 2$ term in the sum contains the factor $e^{\beta\lambda_2}/e^{\beta\lambda_1} = e^{\beta(\lambda_2 - \lambda_1)}$, which decays to zero extremely fast for $\beta \to \infty$ if $\lambda_1 - \lambda_2$ is "large," with every other term with $k > 2$ going to zero at least as fast.

More generally, when the spectral gap is large, the rankings obtained using parameterized centrality will converge to those given by eigenvector centrality more quickly as t increases than in the case when the spectral gap is small. Thus, in networks with a large enough spectral gap, eigenvector centrality may as well be used instead of a measure based on the exponential or resolvent of the adjacency matrix. However, it is not always easy to tell a priori when $\lambda_1 - \lambda_2$ is "large enough"; some guidelines can be found in [23]. We also note that the tuning parameter t can be interpreted as a way to artificially widen or shrink the (absolute) gap, thus giving more or less weight to the dominant eigenvector.

The situation is rather more involved in the case of directed networks. Equation (5.6) shows that the difference between the (scaled) parameterized centrality scores and the corresponding eigenvector centrality scores contains terms of the form $\frac{t^j f^{(j)}(t\lambda_k)}{j! f(t\lambda_1)}$ (with $0 \le j \le n_k - 1$, where n_k is the index of λ_k), as well as additional quantities involving powers of $A - \lambda_k I$ and the oblique projectors G_k. Although these terms vanish as $t \to t^*-$, the spectral gap in this case can provide only an *asymptotic* measure of how rapidly the eigenvector centrality scores are approached, unless A is nearly normal.

Next, we turn to the limits as $t \to 0+$. For brevity, we limit our discussion to the undirected case. From (5.1) we see that for small t, the difference between the (scaled

and shifted) f-subgraph centrality $\phi_i(t)$ of node i and the degree d_i is dominated by the term $\frac{c_3}{c_2}t[A^3]_{ii}$. Now, it is well known that the number of triangles (cycles of length 3) that node i participates in is equal to $\Delta_i = \frac{1}{2}[A^3]_{ii}$. It follows that if a node i participates in a large number of triangles, then the corresponding centrality score $\phi_i(t)$ can be expected to approach the degree centrality score d_i more slowly, for $t \to 0+$, than a node j that participates in no (or few) such triangles.

To understand this intuitively, consider two nodes, i and j, both of which have degree k. Suppose node i participates in no triangles and node j participates in $\binom{k}{2}$ triangles. That is, $N(i)$, the set of nodes adjacent to node i, is an independent set of k nodes (*independent* means that no edges are present between the nodes in $N(i)$), while $N(j)$ is a clique (complete subgraph) of size k. In terms of local communities, node i is isolated (does not participate in a local community), while node j sits at the center of a dense local community (a clique of size $k+1$) and participates in links to other nodes only within this small, dense subgraph. Due to this, whenever j communicates with any of its neighbors, this information can quickly be passed among all its neighbors. This allows the clique of size $k + 1$ to act as a sort of "supernode" where j's local influence depends greatly on the local influence of this supernode. That is, even on a local level, it is difficult to separate the influence of j from that of its neighbors. In contrast, node i does not participate in a dense local community and, thus, its local influence depends more on its immediate neighbors than on the neighbors of those neighbors. Therefore, local (i.e., small t) centrality measures on node i will be more similar to degree centrality than those on node j.

From a more global perspective, we can expect the degree centrality limit to be attained more rapidly, for $t \to 0+$, for networks with low clustering coefficient[3] than for networks with high clustering coefficient (such as social networks).

For the total communicability centrality, on the other hand, (5.2) suggests that the rate at which degree centrality is approached be dictated, for small t, by the vector $A\mathbf{d} = A^2\mathbf{1}$. Hence, if node i has a large number of next-to-nearest neighbors (i.e., there are many nodes at distance 2 from i), then the degree centrality will be approached more slowly, for $t \to 0+$, than for a node that has no (or few) such next-to-nearest neighbors.

8. Related work. As mentioned in the introduction, correlations between the rankings obtained with different centrality measures, such as degree and eigenvector centralities, have frequently been observed in the literature. A few authors have gone beyond this empirical observation and have proved rigorous mathematical statements explaining some of these correlations in special cases. Here we briefly review these previous contributions and how they relate to our own.

Bonacich and Lloyd showed in [10] that eigenvector centrality is a limiting case of Katz centrality when $\alpha \to \frac{1}{\lambda}-$, but their proof assumes that A is diagonalizable.

A centrality measure closely related to Katz centrality, known as (normalized) α-centrality, was thoroughly studied in [33]. This measure actually depends on *two* parameters α and β and reduces to Katz centrality when $\alpha = \beta$. The authors of [33] show that α-centrality reduces to degree centrality as $\alpha \to 0+$; they also show, but only for *symmetric* adjacency matrices, that α-centrality reduces to eigenvector centrality for $\alpha \to \frac{1}{\lambda}-$ (a result less general than that about Katz centrality in [10]).

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[3]Recall that the *clustering coefficient* of an undirected network $G = (V, E)$ is defined as the average of the *node clustering coefficients* $CC(i) := \frac{2\Delta_i}{d_i(d_i-1)}$ over all nodes $i \in V$ of degree $d_i \geq 2$. See, e.g., [13, page 303].

A proof that Katz centrality (with an arbitrary preference vector $\mathbf{v}$) reduces to eigenvector centrality as $\alpha \to \frac{1}{\lambda}-$ for a general A (that is, without requiring that A be diagonalizable) can be found in [52]. This proof avoids use of the Jordan canonical form but makes use of the Drazin inverse, following [43]. Unfortunately this technique is not easily generalized to centrality measures based on other matrix functions.

Related results can also be found in [51]. In this paper, the authors consider parameter-dependent matrices ("kernels") of the form

$$N_\gamma(B) = B\sum_{k=0}^{\infty}(\gamma B)^k = B(I - \gamma B)^{-1} \quad \text{and} \quad E_\gamma(B) = \mathrm{e}^{\gamma B} = \sum_{k=0}^{\infty}\frac{(\gamma B)^k}{k!},$$

where B is taken to be either AA^T or $A^T A$, with A the adjacency matrix of a (directed) citation network. The authors show that

$$\lim_{\gamma \to \gamma^*-}\left(\frac{1}{\rho(B)} - \gamma\right)N_\gamma(B) = \mathbf{v}\mathbf{v}^T, \quad \lim_{\gamma \to \infty}\mathrm{e}^{-\gamma\rho(B)}E_\gamma(B) = \mathbf{v}\mathbf{v}^T,$$

where $\gamma^* = 1/\rho(B)$ and $\mathbf{v}$ is the dominant eigenvector of B. Noting that $\mathbf{v}$ is the hub vector when $B = AA^T$ and the authority vector when $B = A^T A$, the authors observe that the HITS algorithm [37] is a limiting case of the kernel-based algorithms.

Finally, we mention the work by Romance [50], which introduces a general family of centrality measures which includes as special cases degree centrality, eigenvector centrality, PageRank, α-centrality (including Katz centrality), and many others. Among other results, this general framework allows the author to explain the strong correlation between degree and eigenvector centralities observed in certain networks, such as Erdös–Renyi graphs. We emphasize that the unifying framework presented in [50] is quite different from ours.

In conclusion, our analysis allows us to unify, extend, and complete some partial results that can be found scattered in the literature concerning the relationship among different centrality measures. In particular, our treatment covers a broader class of centrality measures and networks than those considered by earlier authors. In addition, we provide some rules of thumb for the choice of parameters when using measures such as Katz and subgraph centralities (see section 9).

9. Summary of numerical experiments. In this section we briefly summarize the results of numerical experiments aimed at illustrating our theoretical results. A complete description of the tests performed, inclusive of plots and tables, can be found in this paper's supplementary materials.

We examined various parameterized centrality measures based on the matrix exponential and resolvent, including subgraph and total communicability measures. Numerical tests were performed on a set of networks from different application areas (social networks, protein-protein interaction networks, computer networks, collaboration networks, a road network, etc.). Both directed and undirected networks were considered. The tests were primarily aimed at monitoring the limiting behavior of the various centrality measures for $\beta \to 0+$, $\beta \to \infty$ for exponential-type measures and for $\alpha \to 0+$, $\alpha \to \frac{1}{\lambda_1}-$ for resolvent-type measures.

Our experiments confirm that the rankings obtained with exponential-type centrality measures approach quickly those obtained from degree centrality as β gets smaller, with the measure based on the diagonal entries $[\mathrm{e}^{\beta A}]_{ii}$ approaching degree centrality faster, in general, than the measure based on $[\mathrm{e}^{\beta A}\mathbf{1}]_i$. The tests also confirm that for networks with a large spectral gap, the rankings obtained by both of

these measures approach those from eigenvector centrality much more quickly, as β increases, than for the networks with a small spectral gap. These remarks are especially true when only the top ranked nodes are considered.

Similar considerations apply to resolvent-type centrality measures and to directed networks.

Based on our tests, we propose the following rules of thumb when using exponential and resolvent-type centrality measures. For the matrix exponential the parameter β should be chosen in the range $[0.5, 2]$, with smaller values used for networks with a relatively large spectral gap. Using values of β smaller than 0.5 results in rankings very close to those obtained using degree centrality, and using $\beta > 2$ leads to rankings very close to those obtained using eigenvector centrality. Since both degree and eigenvector centralities are cheaper than exponential-based centrality measures, it would make little sense to use the matrix exponential with values of β outside the interval $[0.5, 2]$. As a default value, $\beta = 1$ (as originally proposed in [27]) is a very reasonable choice.

Similarly, resolvent-based centrality measures are most informative when the parameter α is of the form τ/λ_1 with τ chosen in the interval $[0.5, 0.9]$. Outside of this interval, the rankings obtained are very close to the degree (for $\tau < 0.5$) and eigenvector (for $\tau > 0.9$) rankings, especially when attention is restricted to the top ranked nodes. Again, the smaller values should be used when the network has a large spectral gap.

Similar conclusions hold for the choice of the damping parameter α used in the PageRank algorithm, in broad agreement with the results of [7, 8].

10. Conclusions. We have studied a broad family of parameterized network centrality measures that includes subgraph, total communicability, and Katz centralities, as well as degree and eigenvector centralities (which appear as limiting cases of the others as the parameter approaches certain values). Our analysis applies (for the most part) to rather general types of networks, including directed and weighted networks; some of our results also hold in the presence of loops. A discussion of the limiting behavior of PageRank was also given, particularly for small values of the parameter α.

Our results help explain the frequently observed correlations between the degree and eigenvector centrality rankings on many real-world complex networks, particularly those exhibiting a large spectral gap, and why the rankings tend to be most stable precisely near the extreme values of the parameters. This is at first sight surprising, given that as the parameters approach their upper bounds, the centrality scores and their derivatives diverge, indicating extreme sensitivity.

We have discussed the role of network properties, such as the spectral gap and the clustering coefficient, on the rate at which the rankings obtained by a parameterized centrality measure approach those obtained by the degree and eigenvector centralities in the limit. We have further shown that the parameter plays the role of a "knob" that can be used to give more or less weight to walks of different lengths on the graph.

In the case of resolvent- and exponential-type centrality measures, we have provided rules of thumb for the choice of the parameters α and β. In particular, we provide guidelines for the choice of the parameters that produce rankings that are the most different from the degree and eigenvector centrality rankings and, therefore, most useful in terms of providing additional information in the analysis of a given network. Of course, the larger the spectral gap, the smaller the range of parameter values leading to rankings exhibiting a noticeable difference from those obtained from

degree and/or eigenvector centralities. Since degree and eigenvector centralities are considerably less expensive to compute compared to subgraph centrality, for networks with a large spectral gap it may be difficult to justify the use of the more expensive centrality measures discussed in this paper.

Finally, in this paper we have mostly avoided discussing computational aspects of the ranking methods under consideration, focusing instead on the theoretical understanding of the relationship among the various centrality measures. For recent progress on walk-based centrality computations see, e.g., [3, 5, 11, 28, 29].

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