

ON SINGULAR LIOUVILLE EQUATIONS AND SYSTEMS

ANDREA MALCHIODI⁽¹⁾

Expository article

In memory of Abbas

ABSTRACT. We consider some singular Liouville equations and systems motivated by uniformization problems in a non-smooth setting, as well as from models in Mathematical Physics. We will study existence of solutions from a variational point of view, using suitable improvements of the Moser-Trudinger inequality. These reduce the problem to a topological one by studying the concentration property of conformal volume, which will be constrained by the functional inequalities of geometric flavour. We will mainly describe some common strategies from the papers [12], [13], [21] in simple situations, to give an idea to the non-expert reader about the general methods we use.

1. INTRODUCTION

One among the most classical problems in Riemannian geometry is to find *canonical metrics* on a given manifold. In two dimensions a natural choice is to *uniformize* a surface looking for metrics of constant Gaussian curvature. One way to achieve this is to choose a *conformal representative*, namely a metric pointwise scaled by a suitable positive function. Given a compact, boundary-less surface (Σ, g) with Gaussian curvature K_g , consider the conformal change $g \mapsto \tilde{g} = e^{2w}g$, where w is a smooth function on Σ . It is known that under conformal changes the Gaussian curvature transforms according to the formula

$$(1) \quad -\Delta_g w + K_g = K_{\tilde{g}} e^{2w}.$$

Hence looking for constant $K_{\tilde{g}}$ amounts to solving the following PDE on Σ

$$(U) \quad -\Delta_g w + K_g = \bar{K} e^{2w},$$

where $\bar{K} \in \mathbb{R}$. By the Gauss-Bonnet formula, the sign of \bar{K} has to be the same as that of the Euler characteristic of Σ .

We introduce next a singular version of (U). Singular objects attracted a lot of attention over the past decades, since they arise in many different situations such as limits of Einstein metrics ([2], [14], [66]), Kähler-Einstein metrics ([28]), as well as in physical applications such as the study of interfaces or in general relativity.

One of the simplest singular objects consist of two-dimensional surfaces with finitely-many conical points. The *model object* is a standard cone, that can be realized with an isometry from a planar circular sector. Isometries preserve the Gaussian curvature and hence a cone is geometrically *flat* on its side surface, but in a weak sense the curvature behaves like a measure at the conical tip. Precisely, if the opening angle θ of the cone is written as $\theta = 2\pi(1 + \alpha)$, $\alpha > -1$, the curvature at the vertex is a Dirac mass with amplitude $-2\pi\alpha$.

With this model in mind, for p_1, \dots, p_m points in Σ and $\alpha_1, \dots, \alpha_m > -1$, we will consider the following problem on a compact, closed surface (Σ, g) of total volume 1

$$(2) \quad -\Delta_g u + K_g = \rho e^{2u} - 2\pi \sum_{j=1}^m \alpha_j \delta_{p_j}; \quad \rho \in \mathbb{R}.$$

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⁽¹⁾ Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa, Italy. email: andrea.malchiodi@sns.it.

Equation (2) is a singular version of (U), and a solution will endow Σ with a constant-curvature metric on $\Sigma \setminus \cup_{i=1}^m \{p_i\}$, and conical angles $\theta_i = 2\pi(1 + \alpha_i)$ at each point p_i . All the singular structure is encoded in the *divisor*, written as a formal sum

$$\underline{\alpha} := \sum_{i=1}^m \alpha_i p_i.$$

Still by the Gauss-Bonnet formula (assuming without loss of generality that $Vol_g(\Sigma) = 1$), that can be obtained rounding-off the conical points and applying the usual Gauss-Bonnet theorem, the constant ρ should satisfy the geometric constraint

$$(3) \quad \rho = 2\pi\chi(\Sigma) + 2\pi \sum_{i=1}^m \alpha_i.$$

As we will see, equations (or systems) like (2) also have applications in physics and for those integer values of the parameters α_i are particularly interesting. Geometrically, these would correspond to *orbifold points* with angle greater than 2π , but for most of this paper we will limit ourselves to describe the case of negative alpha's, which is simpler to analyse. Liouville equations arise in Mathematical Physics as well, to describe mean field vorticity in steady flows (see [18], [24]), Chern-Simons vortices in superconductivity or Electroweak theory (see [64], [70]). For these problems ρ represents a positive physical parameter, and is not assumed to satisfy (3). The points p_j are called *vortices*, and describe either points where vorticity is imposed by external forces ([69]), or vortex points, namely zeroes of the Higgs field with vanishing order α_i .

To study existence for (2), it is useful to *desingularize* the problem, as one could exploit its *variational structure*. Consider the Green's function of $-\Delta_g$ on Σ with pole p , namely the solution to

$$(4) \quad -\Delta_g G_p(x) = \delta_p - 1 \quad \text{on } \Sigma, \quad \text{with} \quad \int_{\Sigma} G_p(x) dV_g = 0.$$

It is a standard fact that G_p has the asymptotic behaviour $G_p \simeq -\frac{1}{2\pi} \log d_g(x, p)$ near the singularity, where $d_g(\cdot, \cdot)$ stands for the distance induced by the background metric g . Consider the change of variables

$$(5) \quad u \mapsto u + 2\pi \sum_{j=1}^m \alpha_j G_{p_j}(x) :$$

after this, (2) becomes

$$(6) \quad -\Delta_g u = \rho \left(\tilde{h}(x) e^{2u} - \tilde{a}(x) \right) \quad \text{on } \Sigma,$$

where $\tilde{h}(x) = e^{-2\pi \sum_{j=1}^m \alpha_j G_{p_j}(x)}$. Here $\tilde{a}(x)$ is a smooth function on Σ such that $\int_{\Sigma} \tilde{a}(x) dV_g = 1$, while by the asymptotics of G_{p_j} \tilde{h} satisfies

$$(7) \quad \tilde{h} > 0 \text{ on } \Sigma \setminus \cup_j \{p_j\}; \quad \tilde{h}(x) \simeq \gamma_j d_g(x, p_j)^{2\alpha_j} \quad \text{near } p_j$$

for some constant $\gamma_j > 0$.

Solutions to (6) can be found as critical points of the Euler-Lagrange energy

$$(8) \quad I_{\rho, \underline{\alpha}}(u) = \int_{\Sigma} |\nabla_g u|^2 dV_g + 2\rho \int_{\Sigma} \tilde{a}(x) u dV_g - \rho \log \int_{\Sigma} \tilde{h}(x) e^{2u} dV_g; \quad u \in H^1(\Sigma).$$

Let us recall that in two dimensions $H^1(\Sigma)$ embeds into $L^p(\Sigma)$ for any $p \in (1, \infty)$: the embedding can be indeed extended up to exponential class. The well-known Moser-Trudinger inequality holds, giving a quantitative estimate on exponentials of Sobolev functions

$$(9) \quad \log \int_{\Sigma} e^{2(u-\bar{u})} dV_g \leq \frac{1}{4\pi} \int_{\Sigma} |\nabla_g u|^2 dV_g + C_{\Sigma, g},$$

where $\bar{u} = \int_{\Sigma} u dV_g$ stands for the average of u on Σ .

In the singular case the Moser-Trudinger inequality on Σ has a different best constant, as was proven by Chen and Troyanov in [25] and [68] (see also [23]).

Proposition 1.1. ([25], [68]) *Let $\alpha_j > -1$ for all j , and let $\tilde{h} : \Sigma \rightarrow \mathbb{R}$ be as in (7). Then one has the inequality*

$$(10) \quad \log \int_{\Sigma} \tilde{h}(x) e^{2(u-\bar{u})} dV_g \leq \frac{1}{4\pi \min\{1, 1 + \min_j \alpha_j\}} \int_{\Sigma} |\nabla u|^2 dV_g + C_{\tilde{h},g}$$

for all $u \in H^1(\Sigma)$.

Notice that the multiplicative constant appearing in the last formula is determined by the most singular behaviour of the function \tilde{h} , see (7), that becomes unbounded at the points p_i with negative α_i 's.

Depending then on the value of ρ , we distinguish three geometric cases: the subcritical, critical and the supercritical ones. In the first one $\rho < 4\pi \min\{1, 1 + \min_j \alpha_j\}$, and the latter term in $I_{\rho,\alpha}$ can be absorbed into the first one, giving *coercivity* of the energy. As a consequence, one always finds solutions using the direct methods of the Calculus of Variations, i.e. taking weak limits of minimizing sequences. See for example [51], [65], [67]. In the *regular case* (U), this situation corresponds to the negative or zero curvature case. In the second case ($\rho = 4\pi \min\{1, 1 + \min_j \alpha_j\}$) the energy $I_{\rho,\alpha}$ is bounded below but coercivity is lost, so it is unclear whether minimizing sequences would converge. If compactness fails, a typical behaviour of solutions (described in more detail later) leads to indefinite concentration of conformal volume at a finite number of points. For example, in the positive curvature case of (U) (i.e. on the sphere), the loss of compactness is caused by the action of the Möbius group, which might cause all conformal volume to concentrate to a single point, but leaving the Euler-Lagrange energy for (U) invariant. A careful blow-up analysis of the minimizing sequences might still lead to existence results: we will not discuss the details here, referring the reader to [34], [59] (and to [33] specifically for the uniformization problem). The third case ($\rho > 4\pi \min\{1, 1 + \min_j \alpha_j\}$) is the most delicate one, and has no regular counterpart in (U). The fact that ρ exceeds the Chen-Troyanov constant causes unboundedness from below of the energy, so it is hopeless to try to find global minima as before. Worse than that, there are situations in which solutions do not exist: one well-known example is the *tear-drop*, namely a spherical surface with only one singularity. It is known that there is indeed no constant curvature metric on such an object (see also [20], [39], [40], [56] for more general results of this type).

The supercritical case will be the one we will mostly be interested in, and we will show that a variational approach might still give conclusions in the search of critical points for $I_{\rho,\alpha}$ of saddle type. In order to find them, as for the Direct minimization methods, one fundamental condition is compactness. Concerning problem (2), an alternative was proved in [9] (after previous results in [16], [48], [47] for the regular case): either a sequence $(u_n)_n$ of solutions to (E_{ρ_n}) (with $\rho_n \rightarrow \bar{\rho} \in \mathbb{R}$) stays uniformly bounded, or it develops a finite number of spheres at regular points and/or *American footballs* at singular points, see Theorem 3.5 for a precise statement. An American football is obtained from a sphere (possibly covered multiple times) by cutting two meridians and by gluing the remaining edges. This results in a constant-curvature singular surface having two equal conical angles $\theta = 2\pi(1 + \alpha)$: by the modified Gauss-Bonnet formula (1) the total curvature of this object must be $4\pi(1 + \alpha)$.

In the blow-up alternative all the curvature is exhausted in this way and therefore ρ_n , the total curvatures of the conformal metric $\tilde{h}e^{2u_n}$, must converge to a number in this discrete set

$$(11) \quad \mathfrak{S} = \left\{ \rho \mid \rho = 4\pi n + 4\pi \sum_{i \in I} (1 + \alpha_i), \quad n \in \mathbb{N}, \quad I \subseteq \{1, \dots, m\} \right\} \subseteq \mathbb{R}_+.$$

On the other hand, if ρ does not belong to this set, solutions have to stay compact and variational methods can be applied. Recall that in the super-critical regime the Euler-Lagrange energy is unbounded from below. However there is a way to describe how the lower bounds fail, in terms of concentration of conformal volume.

It turns out that the multiplicative constant in (10) improves if the conformal volume *spreads* over Σ , see Lemma 2.1. Having a better constant implies more chances to bound the energy from below, and therefore a low energy forbids too much spreading of the volume. Suppose that all the weights α_i are negative: localizing (10) via cut-off functions near a regular or a singular points, in the denominator one finds respectively the value 4π or $4\pi(1 + \alpha_i)$. This suggests to introduce a *weighted cardinality* χ on points of

Σ as follows: set

$$(12) \quad \begin{cases} \chi(q) = 4\pi & \text{if } q \in \Sigma \setminus \{p_1, \dots, p_m\}; \\ \chi(p_i) = 4\pi(1 + \alpha_i) & \text{for all } i = 1, \dots, m. \end{cases}$$

Define also

$$\mathcal{P}(\Sigma) = \{\mu : \mu \text{ is a probability measure on } \Sigma\}.$$

As the total curvature we have at hand is ρ , this *counting* suggest that the limit measures for small energy should be the following

$$(13) \quad \Sigma_{\rho, \alpha} = \{\mu \in \mathcal{P}(\Sigma) : 4\pi\chi(\text{supp}(\sigma)) < \rho\}.$$

Without singularities, such spaces coincide with the measures supported on a given number of points (depending on ρ), and were useful in studying problems in higher dimensions or of higher order, see e.g. [4], [36], [37], [52].

For simplicity, we will assume here the following upper bound on ρ

$$(14) \quad \rho < 4\pi \min\{1, \min_{i \neq j} (2 + \alpha_i + \alpha_j)\}.$$

In this case, setting

$$(15) \quad \mathcal{A} = \{p_i : 4\pi(1 + \alpha_i) < \rho\},$$

one can check that $\Sigma_{\rho, \alpha}$ takes the simple form

$$\Sigma_{\rho, \alpha} = \{\delta_p : p \in \mathcal{A}\}.$$

In this case one has the following result, whose statement becomes rather simple

Theorem 1.2. ([21], *particular case*) *Suppose that $\alpha_i < 0$ for all i and that ρ satisfies (14). Then (6) admits a solution provided $\text{card}(\mathcal{A}) > 1$.*

While the main result in [21] is more general and deals with sets $\Sigma_{\rho, \alpha}$ of arbitrary structure (see also [6], [54]), the proof of existence is rather simple to explain under the assumptions of Theorem 1.2, and we will treat only this case in the present notes. See also [27] for an existence result relying on degree theory.

We next discuss the *singular Toda system* arising in Chern-Simons theory, which represents a non-abelian counterpart of (2). Specifically, we consider the following system

$$(16) \quad \begin{cases} -\Delta u_1 = 2\rho_1 \left(\frac{h_1 e^{u_1}}{\int_{\Sigma} h_1 e^{u_1} dV_g} - 1 \right) - \rho_2 \left(\frac{h_2 e^{u_2}}{\int_{\Sigma} h_2 e^{u_2} dV_g} - 1 \right) - 4\pi \sum_{j=1}^m \alpha_{1,j} (\delta_{p_j} - 1), \\ -\Delta u_2 = 2\rho_2 \left(\frac{h_2 e^{u_2}}{\int_{\Sigma} h_2 e^{u_2} dV_g} - 1 \right) - \rho_1 \left(\frac{h_1 e^{u_1}}{\int_{\Sigma} h_1 e^{u_1} dV_g} - 1 \right) - 4\pi \sum_{j=1}^m \alpha_{2,j} (\delta_{p_j} - 1), \end{cases}$$

where h_1, h_2 are smooth positive functions on Σ , and the coefficients $\alpha_{i,j}$ are again larger than -1 . In geometry, (16) describes Frenet frames of holomorphic curves in $\mathbb{C}\mathbb{P}^n$, see [15, 19, 29], with the p_i 's stand for *ramification points* of the curves. From the physical point of view, abelian Chern-Simons vortices have been quite well studied for some time, see e.g. [17], [22], [58], [63], while the treatment of the non-abelian case is more recent, see e.g. [38], [45], [46], [59].

With a change of variable similar to (5) the latter problem transforms into

$$(17) \quad \begin{cases} -\Delta u_1 = 2\rho_1 \left(\frac{\tilde{h}_1 e^{u_1}}{\int_{\Sigma} \tilde{h}_1 e^{u_1} dV_g} - 1 \right) - \rho_2 \left(\frac{\tilde{h}_2 e^{u_2}}{\int_{\Sigma} \tilde{h}_2 e^{u_2} dV_g} - 1 \right), \\ -\Delta u_2 = 2\rho_2 \left(\frac{\tilde{h}_2 e^{u_2}}{\int_{\Sigma} \tilde{h}_2 e^{u_2} dV_g} - 1 \right) - \rho_1 \left(\frac{\tilde{h}_1 e^{u_1}}{\int_{\Sigma} \tilde{h}_1 e^{u_1} dV_g} - 1 \right), \end{cases}$$

where the functions \tilde{h}_i satisfy

$$(18) \quad \tilde{h}_i > 0 \quad \text{on} \quad \Sigma \setminus \{p_1, \dots, p_m\}; \quad \tilde{h}_i(x) \simeq d(x, p_j)^{2\alpha_{i,j}} \quad \text{near } p_j, \quad i = 1, 2.$$

As for the scalar case one gains the variational structure, with Euler-Lagrange functional

$$(19) \quad J_{\rho, \alpha}(u_1, u_2) = \int_{\Sigma} Q(u_1, u_2) dV_g + \sum_{i=1}^2 \rho_i \left(\int_{\Sigma} u_i dV_g - \log \int_{\Sigma} \tilde{h}_i e^{u_i} dV_g \right),$$

where $Q(u_1, u_2)$ is defined as:

$$(20) \quad Q(u_1, u_2) = \frac{1}{3} (|\nabla u_1|^2 + |\nabla u_2|^2 + \nabla u_1 \cdot \nabla u_2).$$

For the regular Toda system a sharp Moser-Trudinger inequality was found in [44] (see also [30] and [71] for other Liouville systems), where it was shown that

$$(21) \quad 4\pi \sum_{i=1}^2 \log \int_{\Sigma} e^{u_i - \bar{u}_i} dV_g \leq \int_{\Sigma} Q(u_1, u_2) dV_g + C; \quad u \in H^1(\Sigma).$$

Notice that one always has the inequality $Q(u_1, u_2) \geq \frac{1}{4} |\nabla u_1|^2$, and hence (21) can be thought of as an extension of (9). Our next goal is to introduce the following theorem, which extends both (10) and (21).

Theorem 1.3. ([12]) *Suppose $p_1, \dots, p_m \in \Sigma$ and $\alpha_{i,j}$, $i = 1, 2$, $j = 1, \dots, m$, satisfy $\alpha_{i,j} > -1$ for all i, j . Then, if \tilde{h}_i satisfy (18), the following inequality holds*

$$(22) \quad 4\pi \sum_{i=1}^2 \min \left\{ 1, 1 + \min_j \alpha_{i,j} \right\} \log \int_{\Sigma} \tilde{h}_i e^{u_i - \bar{u}_i} dV_g \leq \int_{\Sigma} Q(u_1, u_2) dV_g + C \quad u_1, u_2 \in H^1(\Sigma).$$

The constants in the above inequality are sharp.

The above result is a first step for a variational attack for the study of (2). In the recent paper [11] the case of non-negative coefficients and positive genus has been treated using simply inequality (21), as the corresponding functions \tilde{h}_i are uniformly bounded (see also [53], [60] and [55] for the regular case and [5] for the scalar singular case). Inequality (22) is indeed needed in a general situation.

Using blow-up analysis it is possible to show that inequality (22) holds for any smaller couple of coefficients on the left-hand side, and moreover that there exist extremal functions for the corresponding Euler functionals (19). This is what we will present in these notes. One can then pass to the limit for these extremals when the parameters approach the critical ones. The presence of singularities causes a variety of blow-up behaviours (different blow-up rates for the two components, and blow-up at regular or singular points): using a Pohozaev identity from the recent paper [49] these can be reduced to two cases only.

The above reasoning in terms of volume concentration for the scalar singular equation (see the comments before Theorem 1.2) allows to prove a related alternative for the two components of the system. As a counterpart of (14), we define

$$(23) \quad \bar{\rho}_1 := 4\pi \min \left\{ 1, \min_{m \neq m'} (2 + \alpha_{1m} + \alpha_{1m'}) \right\} \quad \bar{\rho}_2 := 4\pi \min \left\{ 1, \min_{m \neq m'} (2 + \alpha_{2m} + \alpha_{2m'}) \right\},$$

and suppose that $\rho_i < \bar{\rho}_i$. Define also

$$(24) \quad \mathcal{A}_i = \{p_j : 4\pi(1 + \alpha_{i,j}) < \rho_i\}, \quad i = 1, 2.$$

Under the above condition on the ρ_i 's it turns out that for $J_{\rho, \underline{\alpha}}(u)$ low either $\tilde{h}_1 e^{u_1}$ concentrates near a singular point in \mathcal{A}_1 or $\tilde{h}_2 e^{u_2}$ concentrates near a singular point in \mathcal{A}_2 . To express this (non-exclusive) alternative, it is natural to introduce the *join* of two topological spaces X and Y (see for instance [41]):

$$(25) \quad X \star Y := \frac{X \times Y \times [0, 1]}{\sim},$$

where \sim is the equivalence relation among triples (x, y, t) given by

$$(x, y, 0) \sim (x, y', 0) \quad \forall x \in X, \forall y, y' \in Y \quad (x, y, 1) \sim (x', y, 1) \quad \forall x, x' \in X, \forall y \in Y.$$

The join of the sets \mathcal{A}_1 and \mathcal{A}_2 could then be used to characterize low-energy levels of $J_{\rho, \underline{\alpha}}$, with the join parameter $s \in [0, 1]$ expressing whether $\tilde{h}_1 e^{u_1}$ is distributionally closer to a Dirac mass or whether $\tilde{h}_2 e^{u_2}$ is closer to a Dirac mass (for example $s = \frac{1}{2}$ would describe couples with the same scale of concentration). This description is however not optimal in general, as it does not take into account the interaction between two components u_1 and u_2 via the mixed term in the quadratic form Q , which penalizes aligned gradients. For the regular case of (16), in [55] it was shown that the *relative rate of concentration* of the two components plays a role in this matter.

It turns out that if u_1, u_2 concentrate near the same point and with the same scale (a more precise definition is given below), then the Moser-Trudinger constants for the system double. As a consequence of this fact it follows that, when $\rho_1, \rho_2 \in (4\pi, 8\pi)$ and no singularities occur, then join elements of the form $(x, x, \frac{1}{2})$, $x \in \Sigma$ have to be excluded (see [42] for higher values of ρ_1). We will present a new improved inequality from [13] for the singular system (16), in order to understand at the same time the effect of the interaction of the two components between themselves and with the singularities. As a consequence of this improved inequality we deduce the next result, which to be stated needs the definition of the counterpart to (11). Let $\Gamma'_{i, \mathcal{M}} \subset \mathbb{R}_+$ be defined, for $i = 1, 2$ and $\mathcal{M} \subset \{1, \dots, m\}$, by

$$\Gamma'_{i, \mathcal{M}} := 4\pi \left\{ n + \sum_{j' \in \mathcal{M}'} (1 + \alpha_{ij'}) + \sum_{j \in \mathcal{M}} (2 + \alpha_{1j} + \alpha_{2j}) : n \in \mathbb{N}, \mathcal{M}' \subset \{1, \dots, m\} \setminus \mathcal{M} \right\}.$$

and define also

$$(26) \quad \Gamma_{\alpha_1, \alpha_2} = \cup_{\mathcal{M} \subset \{1, \dots, m\}} \left(\Gamma'_{1, \mathcal{M}} \times \left[\sum_{j \in \mathcal{M}} 4\pi(1 + \alpha_{2j}), +\infty \right) \cup \left[\sum_{j \in \mathcal{M}} 4\pi(1 + \alpha_{1j}), +\infty \right) \times \Gamma'_{2, \mathcal{M}} \right).$$

We then set

$$(27) \quad \Gamma = \Gamma_{\alpha_1, \alpha_2}.$$

Theorem 1.4. ([13]) *Let Γ as in (26), $(\bar{\rho}_1, \bar{\rho}_2)$ be as in (23), and let $\rho \in \mathbb{R}_+^2 \setminus \Gamma$ satisfy $\rho_i < \bar{\rho}_i$ for both $i = 1, 2$. Define integer numbers M_1, M_2, M_3 by:*

$$(28) \quad \begin{aligned} M_1 &:= \#\{j : 4\pi(1 + \alpha_{1j}) < \rho_1\} & M_2 &:= \#\{j : 4\pi(1 + \alpha_{2j}) < \rho_2\} \\ M_3 &:= \#\{j : 4\pi(1 + \alpha_{ij}) < \rho_i \text{ and } \rho_i < 4\pi(2 + \alpha_{1j} + \alpha_{2j}) \text{ for both } i = 1, 2\}. \end{aligned}$$

Then system (16) admits solutions provided the following condition holds

$$(M_1, M_2, M_3) \notin \{(1, j, 0), (j, 1, 0), (2, 2, 1), (2, 3, 2), (3, 2, 2), j \in \mathbb{N}\}.$$

By the previous description low sub-levels of $J_{\rho, \alpha}$ can be identified with the topological join of \mathcal{A}_1 and of \mathcal{A}_2 , with some points removed. Under the assumptions on the ρ_i 's, this join consists of a graph \mathcal{X} made of segments whose end-points belong to $\{p_1, \dots, p_m\}$. The conditions on (M_1, M_2, M_3) in the previous theorem ensure that this graph is non contractible. It turns out that the above assumptions on the M_i 's are necessary: in fact in [13] a non-existence result for every case not covered by the theorem is proved.

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2. VARIATIONAL ASPECTS OF SINGULAR LIOUVILLE EQUATIONS

In this section we treat problem (2) via variational methods. We first show that, in a regime where coercivity fails, low energy implies volume concentration at suitable points. We will then this characterization to build min-max schemes leading to existence of solutions.

2.1. Improved Moser-Trudinger inequalities. In this subsection we describe how improved Moser-Trudinger inequalities can be employed to deduce information on functions whose Euler-Lagrange energy is small enough. We would like to give some conditions on a function u in order to obtain lower bounds on the energy even when we are beyond the coercivity threshold. Indeed, the *spreading* of the function e^{2u} over the surface gives sufficient conditions to obtain this lower bound, deduced via some *improvement* of the Moser-Trudinger inequality. Two well-known examples were due to J.Moser and T.Aubin, see [3], [57]. Moser proved that one can replace $\frac{1}{4\pi}$ by $\frac{1}{8\pi}$ on the standard sphere (S^2, g_{S^2}) provided u is

antipodally symmetric. Aubin showed instead that on (S^2, g_{S^2}) one can take any constant larger than $\frac{1}{8\pi}$ provided u is *balanced*, which means that

$$\int_{S^2} x_i e^{2u} dV_{S^2} = 0; \quad i = 1, 2, 3.$$

Here x_i stand for the Euclidean i -th coordinate function, so the balancing condition means having zero center of mass in \mathbb{R}^3 for the conformal volume.

W.Chen and C.Li, [26], extended this argument to arbitrary surfaces, showing that if e^{2u} has integral bounded from below into two separate subsets of Σ , then the constant $\frac{1}{4\pi}$ in (9) can be basically divided by two. The result was then extended in [37], [36] for an arbitrary number of spreading regions.

Lemma 2.1. ([26]) *Let Ω_1, Ω_2 be subsets of Σ satisfying $d_g(\Omega_1, \Omega_2) \geq \delta_0$, where δ_0 is a positive real number, and let $\gamma_0 \in (0, \frac{1}{2})$. Then, for any $\varepsilon > 0$ there exists a constant $C = C(\varepsilon, \delta_0, \gamma_0)$ such that*

$$\log \int_{\Sigma} e^{2(u-\bar{u})} dV_g \leq C + \frac{1}{8\pi - \varepsilon} \int_{\Sigma} |\nabla_g u|^2 dV_g$$

for all the functions $u \in H^1(\Sigma)$ satisfying

$$(29) \quad \frac{\int_{\Omega_i} e^{2u} dV_g}{\int_{\Sigma} e^{2u} dV_g} \geq \gamma_0, \quad \forall i = 1, 2.$$

PROOF. Assume without loss of generality that $\bar{u} = 0$: one can find two functions g_1, g_2 such that

$$(30) \quad \begin{cases} g_i(x) \in [0, 1] & \text{for every } x \in \Sigma; \\ g_i(x) = 1, & \text{for every } x \in \Omega_i, i = 1, 2; \\ g_i(x) = 0, & \text{if } d_g(x, \Omega_i) \geq \frac{\delta_0}{4}; \\ \|g_i\|_{C^4(\Sigma)} \leq C_{\delta_0}, \end{cases}$$

where C_{δ_0} is a positive constant depending only on δ_0 . By interpolation, for any $\varepsilon > 0$ there exists $C_{\varepsilon, \delta_0}$ (depending only on ε and δ_0) such that, for any $v \in H^1(\Sigma)$ and for any $i = 1, 2$ we have

$$(31) \quad \int_{\Sigma} |\nabla_g(g_i v)|^2 dV_g \leq \int_{\Sigma} g_i^2 |\nabla_g v|^2 dV_g + \varepsilon \int_{\Sigma} |\nabla_g v|^2 dV_g + C_{\varepsilon, \delta_0} \int_{\Sigma} v^2 dV_g.$$

We next notice that

$$\int_{\Sigma} e^{2u} dV_g \leq \frac{1}{\gamma_0} \int_{\Omega_i} e^{2u} dV_g \leq \int_{\Sigma} e^{2g_i u} dV_g.$$

Using the standard Moser-Trudinger inequality one finds

$$\log \int_{\Sigma} e^{2u} dV_g \leq \log \frac{1}{\gamma_0} + \frac{1}{4\pi} \sum_{i=1}^2 \int_{\Sigma} |\nabla_g(g_i v)|^2 dV_g + \sum_{i=1}^2 \bar{g}_i \bar{u} + C_{l, \Sigma, g}.$$

By (31) we then deduce

$$\log \int_{\Sigma} e^{2u} dV_g \leq \log \frac{1}{\gamma_0} + \frac{1+\varepsilon}{4\pi} \sum_{i=1}^2 \int_{\Sigma} |\nabla_g u|^2 dV_g + \sum_{i=1}^2 \bar{g}_i \bar{u} + C_{l, \Sigma, g} C_{\varepsilon, \delta_0} \int_{\Sigma} v^2 dV_g.$$

Since we are assuming the average of u to be zero, the average terms in the last formula are bounded by a constant times the Dirichlet norm of u by Poincaré's inequality. Therefore, using the elementary inequality $t \leq \varepsilon t^2 + \frac{1}{4\varepsilon}$ we find that

$$\log \int_{\Sigma} e^{2u} dV_g \leq \log \frac{1}{\gamma_0} + \frac{1+\varepsilon}{4\pi} \sum_{i=1}^2 \int_{\Sigma} |\nabla_g u|^2 dV_g + C_{l, \Sigma, g, \varepsilon} + C_{\varepsilon, \delta_0} \int_{\Sigma} v^2 dV_g.$$

It can be shown, for example using truncations (in height or in Fourier modes) that the last term is lower order and it can be absorbed into the Dirichlet energy multiplied by an arbitrarily small constant, concluding the proof. ■

We discuss next the counterpart of the previous result in presence of singularities, for which we recall the inequality in Proposition 1.1. Again, we wish to derive some improved inequalities in terms of the spreading of the function

$$\tilde{f}_u := \frac{\tilde{h}(x)e^{2u}}{\int_{\Sigma} \tilde{h}(x)e^{2u} dV},$$

appearing in the singular Euler-Lagrange energy.

Similarly to Lemma 2.1, (10) can also be localized. If some portion of \tilde{f}_u is localized near a regular point, the corresponding gain in the constant will still be 4π . If instead \tilde{f}_u is localized near a singular point p_i with negative weight α_i , we will gain locally a quantity of size $4\pi(1 + \alpha_i)$. One gets therefore the following result.

Lemma 2.2. ([21]) *Let $n \in \mathbb{N}$, let $I \subseteq \{1, \dots, m\}$ with $n + \text{card}(I) > 0$, and let $\alpha_i < 0$ for all $i \in I$. Assume there exist $r > 0$, $\delta_0 > 0$ and pairwise distinct points $\{q_1, \dots, q_n\} \subseteq \Sigma \setminus \{p_1, \dots, p_m\}$ such that:*

- *for any couple $\{a, b\} \subseteq \{q_1, \dots, q_n \cup (\cup_{i \in I} p_i)\}$ with $a \neq b$ one has $\text{dist}_g(B_r(a), B_r(b)) \geq 4\delta_0$;*
- *for any $a \in \{q_1, \dots, q_n\}$ one has $d_g(p_i, B_r(a)) \geq 4\delta_0$ for any $i \in \{1, \dots, m\} \setminus I$;*

and consider any $\gamma_0 \in (0, \frac{1}{n + \text{card}(I)})$.

Then, for any $\tilde{\epsilon} > 0$ there exists a constant $C := C(\Sigma, g, n, I, r, \delta_0, \gamma_0, \tilde{\epsilon})$ such that

$$(32) \quad \log \int_{\Sigma} \tilde{h}e^{2(u-\bar{u})} dV_g \leq \frac{1}{4\pi(n + \sum_{i \in I}(1 + \alpha_i) - \tilde{\epsilon})} \int_{\Sigma} |\nabla_g u|^2 dV_g + C$$

for all functions $u \in H^1(\Sigma)$ satisfying

$$\frac{\int_{B_r(a)} \tilde{h}e^{2u} dV_g}{\int_{\Sigma} \tilde{h}e^{2u} dV_g} \geq \gamma_0, \quad \forall a \in \{q_1, \dots, q_n \cup (\cup_{i \in I} p_i)\}.$$

The above lemma states that the more the *conformal volume* is spread over the surface, the more one gains in the Moser-Trudinger inequality, especially when some volume accumulates near regular points. In this situation, one then get lower bounds on the energy even in supercritical regimes. Therefore, if the energy gets low enough, one should expect concentration of volume. We state next two lemmas making this reasoning rigorous via a covering procedure.

Lemma 2.3. *Let $\ell \in \mathbb{N}$, and fix two positive number ε and r . Suppose for a non-negative function $f \in L^1(\Sigma)$ with $\|f\|_{L^1(\Sigma)} = 1$ the following condition holds*

$$\int_{\cup_{i=1}^{\ell} B_r(q_i)} f dV_g < 1 - \varepsilon \quad \text{for every } \ell\text{-tuple } q_1, \dots, q_{\ell} \in \Sigma.$$

Then there exist $\bar{\varepsilon} > 0$ and $\bar{r} > 0$, depending only on ε, r, ℓ and Σ (but not on f), and $\ell + 1$ points $\bar{q}_1, \dots, \bar{q}_{\ell+1} \in \Sigma$ (which depend on f) satisfying

$$\int_{B_{\bar{r}}(\bar{q}_1)} f dV_g \geq \bar{\varepsilon}, \dots, \int_{B_{\bar{r}}(\bar{q}_{\ell+1})} f dV_g \geq \bar{\varepsilon}; \quad B_{2\bar{r}}(\bar{q}_i) \cap B_{2\bar{r}}(\bar{q}_j) = \emptyset \text{ for } i \neq j.$$

PROOF. Arguing by contradiction, assume that for every $\bar{\varepsilon}, \bar{r} > 0$ there exists f as in the statement and such that for every $(\ell + 1)$ -tuple of points $q_1, \dots, q_{\ell+1}$ in Σ we have

$$(33) \quad \int_{B_{\bar{r}}(q_j)} f dV_g \geq \bar{\varepsilon} \quad \forall j = 1, \dots, \ell + 1 \quad \Rightarrow \quad B_{2\bar{r}}(q_i) \cap B_{2\bar{r}}(q_j) \neq \emptyset \text{ for some } i \neq j.$$

Let $\bar{r} = \frac{r}{8}$, where r is given in the statement. We can find $h \in \mathbb{N}$ and h points $x_1, \dots, x_h \in \Sigma$ such that $\cup_{i=1}^h B_{\bar{r}}(x_i)$ covers Σ . For ε as in the statement of the Lemma, we also define $\bar{\varepsilon} = \frac{\varepsilon}{2h}$. We remark that the choice of \bar{r} and $\bar{\varepsilon}$ depends on r, ε, ℓ and Σ only, as required.

Let $\{\tilde{x}_1, \dots, \tilde{x}_j\} \subseteq \{x_1, \dots, x_h\}$ denote the points for which $\int_{B_{\bar{r}}(\tilde{x}_i)} f dV_g \geq \bar{\varepsilon}$. Define $\tilde{x}_{j_1} = \tilde{x}_1$, and let A_1 denote the set

$$A_1 = \{\cup_i B_{\bar{r}}(\tilde{x}_i) : B_{2\bar{r}}(\tilde{x}_i) \cap B_{2\bar{r}}(\tilde{x}_{j_1}) \neq \emptyset\} \subseteq B_{4\bar{r}}(\tilde{x}_{j_1}).$$

If there exists \tilde{x}_{j_2} with $B_{2\bar{r}}(\tilde{x}_{j_2}) \cap B_{2\bar{r}}(\tilde{x}_{j_1}) = \emptyset$, we set

$$A_2 = \{\cup_i B_{\bar{r}}(\tilde{x}_i) : B_{2\bar{r}}(\tilde{x}_i) \cap B_{2\bar{r}}(\tilde{x}_{j_2}) \neq \emptyset\} \subseteq B_{4\bar{r}}(\tilde{x}_{j_2}).$$

Proceeding in this way, we choose recursively points $\tilde{x}_{j_3}, \tilde{x}_{j_4}, \dots, \tilde{x}_{j_s}$ such that

$$B_{2\bar{r}}(\tilde{x}_{j_s}) \cap B_{2\bar{r}}(\tilde{x}_{j_a}) = \emptyset \quad \forall 1 \leq a < s,$$

and introduce sets A_3, \dots, A_s by

$$A_s = \{\cup_i B_{\bar{r}}(\tilde{x}_i) : B_{2\bar{r}}(\tilde{x}_i) \cap B_{2\bar{r}}(\tilde{x}_{j_s}) \neq \emptyset\} \subseteq B_{4\bar{r}}(\tilde{x}_{j_s}).$$

Because of (33), the process cannot go further than \tilde{x}_{j_ℓ} , and hence $s \leq \ell$. Using the definition of \bar{r} we obtain

$$(34) \quad \cup_{i=1}^j B_{\bar{r}}(\tilde{x}_i) \subseteq \cup_{i=1}^s A_i \subseteq \cup_{i=1}^s B_{4\bar{r}}(\tilde{x}_{j_i}) \subseteq \cup_{i=1}^s B_r(\tilde{x}_{j_i}).$$

Then by our choice of $h, \bar{\varepsilon}, \{\tilde{x}_1, \dots, \tilde{x}_j\}$ and by (34) one has

$$\begin{aligned} \int_{\Sigma \setminus \cup_{i=1}^s B_r(\tilde{x}_{j_i})} f dV_g &\leq \int_{\Sigma \setminus \cup_{i=1}^j B_{\bar{r}}(\tilde{x}_i)} f dV_g \leq \int_{(\cup_{i=1}^h B_{\bar{r}}(\tilde{x}_i)) \setminus (\cup_{i=1}^j B_{\bar{r}}(\tilde{x}_i))} f dV_g \\ &< (h-j)\bar{\varepsilon} \leq \frac{\bar{\varepsilon}}{2}. \end{aligned}$$

Finally, if we chose $q_i = \tilde{x}_{j_i}$ for $i = 1, \dots, s$ and $q_i = \tilde{x}_{j_s}$ for $i = s+1, \dots, \ell$, we get a contradiction to the assumptions of the lemma. ■

Using Lemmas 2.1 and 2.3 it is possible to analyse the volume concentration for function with large negative energy, showing that it has to concentrate near at most one singular point in \mathcal{A} , see (15).

Lemma 2.4. *Suppose the assumptions of Theorem 1.2 hold true. Then for any $\varepsilon > 0$ and any $r > 0$ there exists a large $L = L(\varepsilon, r)$ such that for every $u \in H^1(\Sigma)$ with $I_{\rho, \underline{\alpha}}(u) \leq -L$ there exist $p_i \in \mathcal{A}$ such that*

$$(35) \quad \frac{1}{\int_{\Sigma} \tilde{h}(x) e^{2u} dV_g} \int_{\Sigma \setminus B_r(p_i)} \tilde{h}(x) e^{2u} dV_g < \varepsilon.$$

PROOF. We first claim that the conformal volume of functions with low energy must concentrate near a single point. Indeed, suppose by contradiction that there exist $\varepsilon, r > 0$ and $(u_n)_n \subseteq H^1(\Sigma)$ with $I_{\rho, \underline{\alpha}}(u_n) \rightarrow -\infty$ and such that for every point $q \in \Sigma$ one has $\int_{\cup_{i=1}^k B_r(q_i)} e^{u_n} dV_g < 1 - \varepsilon$. Noting that $I_{\rho, \underline{\alpha}}$ is invariant under adding constants, we can assume that for every n we have the normalization $\int_{\Sigma} \tilde{h}(x) e^{2u_n} dV_g = 1$. Then we can apply Lemma 2.3 with $\ell = 1$, $f = \tilde{h}(x) e^{2u_n}$, and afterwards Lemma 2.2 with $n + \text{card}(I) = 2$ and δ_0, γ_0 sufficiently small (depending on r and ε) to obtain

$$\begin{aligned} I_{\rho, \underline{\alpha}}(u_n) &\geq \int_{\Sigma} |\nabla_g u_n|^2 dV_g + 2\rho \int_{\Sigma} \tilde{a} u_n dV_g - C\rho \\ &\quad - \frac{\rho}{4\pi \min_{i \neq j} (2 + \alpha_i + \alpha_j) - \tilde{\varepsilon}} \int_{\Sigma} |\nabla_g u_n|^2 dV_g - \rho \bar{u}_n, \end{aligned}$$

with C and $\tilde{\varepsilon}$ independent of n ($\tilde{\varepsilon}$ arbitrarily small). Since we are assuming $\rho < 4\pi \min_{i \neq j} (2 + \alpha_i + \alpha_j)$, we can choose $\tilde{\varepsilon} > 0$ so small that $1 - \frac{\rho}{4\pi \min_{i \neq j} (2 + \alpha_i + \alpha_j) - \tilde{\varepsilon}} := \delta > 0$. Hence using also the Poincaré inequality we find

$$\begin{aligned} (36) \quad I_{\rho, \underline{\alpha}}(u_n) &\geq \delta \int_{\Sigma} |\nabla_g u_n|^2 dV_g + 2\rho \int_{\Sigma} \tilde{a}(u_n - \bar{u}_n) dV_g - C\rho \\ &\geq \delta \int_{\Sigma} |\nabla_g u_n|^2 dV_g - C \left(\int_{\Sigma} |\nabla_g u_n|^2 dV_g \right)^{\frac{1}{2}} - C\rho \geq -C. \end{aligned}$$

This lower bound contradicts the fact that $I_{\rho, \underline{\alpha}}(u_n) \rightarrow -\infty$, and proves our claim.

To conclude the proof, we must show that the volume concentrates near a singular point in \mathcal{A} . In order to show this, it is sufficient to argue as before and still apply Lemma 2.2 with $I = \emptyset$ and $n = 1$, using the fact that $\rho < 4\pi$ and that the local Moser-Trudinger constant is bigger than ρ if the singular concentration point does not belong to \mathcal{A} . ■

By the previous lemma it follows that if the Euler-Lagrange energy is low enough then the function $\tilde{h}(x) e^{2u}$, normalized in L^1 , is localized near at most one singular point of Σ . Choosing ε and r sufficiently small, one can easily see that the point p_i in the statement of Lemma 2.4 must be unique, and therefore we obtain a canonical map from low energy levels into the set of singular points.

Proposition 2.5. *Under the assumptions of Theorem 1.2 there exist a large constant L and a continuous map $\Psi : \{I_{\rho, \alpha} \leq -L\} \rightarrow \mathcal{A}$ such that, if $I_{\rho, \alpha}(u_n) \rightarrow -\infty$, then $\frac{1}{\int_{\Sigma} \tilde{h}(x) e^{2u_n} dV_g} \tilde{h}(x) e^{2u} \rightarrow \delta_{\Psi(u_n)}$ as $n \rightarrow \infty$.*

Remark 2.6. *If $\text{card}(\mathcal{A}) > 1$, since \mathcal{A} is discrete, from the continuity of Ψ it follows that $\{I_{\rho, \alpha} \leq -L\}$ is disconnected for L sufficiently large.*

We will see in the next section how to construct a sort of *inverse map* to Ψ , which will allow to prove existence of solutions via suitable min-max schemes.

3. PROOF OF THEOREM 1.2

In this section we prove our first existence result, concerning singular Liouville equations.

3.1. Test function estimates. For each singular point p_i and $\lambda > 0$, define the function

$$(37) \quad \varphi_{i, \lambda}(x) = \log \frac{\lambda}{(1 + \lambda^2 d(x, x_i)^{2(1+\alpha_i)})}.$$

This function satisfies the following properties.

Proposition 3.1. *Let p_i be a singular point in \mathcal{A} . Then, as $\lambda \rightarrow +\infty$ the following properties hold true*

$$(i) \quad \frac{1}{\int_{\Sigma} \tilde{h}(x) e^{2\varphi_{i, \lambda}} dV_g} \tilde{h}(x) e^{2\varphi_{i, \lambda}} \rightarrow \delta_{p_i} \text{ weakly in the sense of distributions;}$$

$$(ii) \quad I_{\rho, \alpha}(\varphi_{i, \lambda}) \rightarrow -\infty.$$

PROOF. To prove (i) we first notice that, since $\tilde{h}(x) \simeq d(x, p_i)^{2\alpha_i}$ one has

$$\tilde{h}(x) e^{2\varphi_{i, \lambda}(x)} \geq \frac{\lambda^{-2\alpha_i} \lambda^2}{(1 + \lambda^2 d(x, x_i)^{2(1+\alpha_i)})^2} \geq C^{-1} \frac{\lambda^{-2\alpha_i} \lambda^2}{(\lambda^2 \lambda^{-2(1+\alpha_i)})^2} \geq C^{-1} \lambda^2 \quad \text{in } B_{2\lambda^{-1}(p_i)} \setminus B_{\lambda^{-1}(p_i)}.$$

Integrating it follows that

$$(38) \quad \int_{\Sigma} \tilde{h}(x) e^{2\varphi_{i, \lambda}} dV_g \geq \int_{B_{2\lambda^{-1}(p_i)} \setminus B_{\lambda^{-1}(p_i)}} \tilde{h}(x) e^{2\varphi_{i, \lambda}} dV_g \geq C^{-1}.$$

On the other hand, one has that

$$\int_{\Sigma \setminus B_{2\lambda^{-1}(p_i)}} \tilde{h}(x) e^{2\varphi_{i, \lambda}} dV_g \leq C \int_{2\lambda^{-1}}^{\infty} \frac{r^{2\alpha_i} \lambda^2}{(1 + \lambda^2 r^{2(1+\alpha_i)})^2} r dr.$$

By the change of variables $\lambda r^{1+\alpha_i} = s^{1+\alpha_i}$ we then obtain

$$\int_{\Sigma \setminus B_{2\lambda^{-1}(p_i)}} \tilde{h}(x) e^{2\varphi_{i, \lambda}} dV_g \leq C \int_{2\lambda^{-\frac{\alpha_i}{1+\alpha_i}}}^{\infty} \frac{s^{2\alpha_i+1}}{(1 + s^{2(1+\alpha_i)})^2} ds \rightarrow 0$$

as $\lambda \rightarrow +\infty$. From the latter formula and (38) then (i) follows.

To show (ii), we prove the following estimates

$$(39) \quad \rho \int_{\Sigma} \varphi_{i, \lambda} dV_g = -(\rho + o_{\lambda}(1)) \log \lambda \quad (o_{\lambda}(1) \rightarrow 0 \text{ as } \lambda \rightarrow +\infty);$$

$$(40) \quad \int_{\Sigma} |\nabla_g \varphi_{i, \lambda}|^2 dV_g \leq 8(1 + \alpha_i) \pi (1 + o_{\lambda}(1)) \log \lambda \quad \text{as } \lambda \rightarrow +\infty.$$

Once we have these, (ii) follows immediately.

Proof of (39). Fixing any $\delta > 0$ small we have that

$$\log \frac{\lambda}{1 + \lambda^2 \text{diam}(\Sigma)^{2(1+\alpha_i)}} \leq \varphi_{i, \lambda}(y) \leq \log \frac{\lambda}{1 + \lambda^2 \delta^{2(1+\alpha_i)}}, \quad y \in \Sigma \setminus B_{\delta}(p_i),$$

and

$$(41) \quad \log \frac{\lambda}{1 + \lambda^2 \delta^{2(1+\alpha_i)}} \leq \varphi_{i, \lambda}(y) \leq \log \lambda, \quad \text{for } y \in B_{2\delta}(p_i).$$

From these two estimates and some elementary computations we deduce that

$$\rho \int_{\Sigma} \varphi_{i,\lambda} dV_g = \rho [(-\log \lambda)(1 + O(\delta^2)) + O(1) + O(\delta^2)(|\log \lambda| + |\log \delta|)],$$

as $\lambda \rightarrow +\infty$. By the arbitrariness of δ , (39) follows.

Proof of (40). We will show the following two pointwise estimates on the gradient of $\varphi_{i,\lambda}$

$$(42) \quad |\nabla \varphi_{i,\lambda}(y)| \leq C\lambda; \quad \text{for every } y \in \Sigma,$$

where C is a constant independent of σ and λ , and

$$(43) \quad |\nabla \varphi_{i,\lambda}(y)| \leq \frac{2(1 + \alpha_i)}{d(y, p_i)}.$$

To check (42) we notice that

$$(44) \quad \frac{\lambda^2 d(y, p_i)}{1 + \lambda^2 d(y, p_i)^{2(1+\alpha_i)}} \leq C\lambda^{\frac{1}{1+\alpha_i}},$$

where C is a fixed constant (independent of λ). Moreover

$$(45) \quad \nabla \varphi_{i,\lambda}(y) = -2(1 + \alpha_i)\lambda^2 \frac{d(y, p_i)^{1+2\alpha_i} \nabla_y(d_i(y))}{(1 + \lambda^2 d(y, p_i)^{2(1+\alpha_i)})}.$$

Using $|\nabla_y d_i(y)| \leq 1$ and inserting (44) into (45) we obtain immediately (42). Similarly, erasing the term 1 from the denominator we deduce (43).

The estimate (42) then implies

$$(46) \quad \int_{B_{\frac{1}{\lambda^{1+\alpha_i}}}(p_i)} |\nabla_g \varphi_{i,\lambda}|^2 dV_g \leq C$$

for some fixed C depending only on Σ and α_i . On the other hand, using polar coordinates centred at p_i and using (43) one finds that

$$\int_{\Sigma \setminus B_{\frac{1}{\lambda^{1+\alpha_i}}}(p_i)} |\nabla_g \varphi_{i,\lambda}|^2 dV_g \leq 4 \int_{\Sigma \setminus B_{\frac{1}{\lambda^{1+\alpha_i}}}(p_i)} \frac{(1 + \alpha_i)^2}{d_g(y, x_i)^2} dV_g \leq 8\pi(1 + o_\lambda(1))(1 + \alpha_i) \log \lambda$$

as $\lambda \rightarrow +\infty$. From (46) and the last formula we finally deduce (40). ■

3.2. Min-max scheme and existence. We next introduce a variational scheme for obtaining existence of solutions for (6). Let first L be so large that Proposition 2.5 applies with $\frac{L}{4}$, and then choose two distinct points p_{i_1}, p_{i_2} such that $4\pi(1 + \alpha_i) < \rho$.

Choose next $\bar{\lambda}$ be so large that $I_{\rho,\alpha}(\varphi_{i_1,\bar{\lambda}}) \leq -L$ and $I_{\rho,\alpha}(\varphi_{i_2,\bar{\lambda}}) \leq -L$ (see Proposition 3.1 (ii)). Fixing this value of $\bar{\lambda}$, we define the family of maps

$$(47) \quad \Pi_{\bar{\lambda}} = \left\{ \varpi : [0, 1] \rightarrow H^1(\Sigma) : \varpi \text{ is continuous and } \varpi(0) = \varphi_{i_1,\bar{\lambda}}, \varpi(1) = \varphi_{i_2,\bar{\lambda}} \right\}.$$

Lemma 3.2. $\Pi_{\bar{\lambda}}$ is non-empty and moreover, letting

$$\bar{\Pi}_{\bar{\lambda}} = \inf_{\varpi \in \Pi_{\bar{\lambda}}} \sup_{t \in [0,1]} I_{\rho,\alpha}(\varpi(t)), \quad \text{one has} \quad \bar{\Pi}_{\bar{\lambda}} > -\frac{L}{2}.$$

PROOF. To show that $\Pi_{\bar{\lambda}} \neq \emptyset$, it suffices to consider the map

$$(48) \quad \bar{\varpi}(t) = (1-t)\varphi_{i_1,\bar{\lambda}} + t\varphi_{i_2,\bar{\lambda}}.$$

Arguing by contradiction, suppose that $\bar{\Pi}_{\bar{\lambda}} \leq -\frac{L}{2}$. Then there would exist a map $\varpi \in \Pi_{\bar{\lambda}}$ with $\sup_{t \in [0,1]} I_{\rho,\alpha}(\varpi(t)) \leq -\frac{3}{8}L$. Since by our choice of L Proposition 2.5 applies with $\frac{L}{4}$, the composition

$$t \mapsto \Psi \circ \varpi(t)$$

is well defined and continuous. However, by Proposition 3.1 (i) and Proposition 2.5 one has that

$$\Psi \circ \varpi(0) = p_1; \quad \Psi \circ \varpi(1) = p_2,$$

which contradicts the continuity of this map. ■

By the statement of Lemma 3.2 and standard variational arguments, one can find a *Palais-Smale sequence* $(u_n)_n$ for $I_{\rho,\alpha}$ at level $\bar{\Pi}_{\bar{\lambda}}$, namely a sequence for which

$$I_{\rho,\alpha}(u_n) \rightarrow \bar{\Pi}_{\bar{\lambda}}; \quad I'_{\rho}(u_n) \rightarrow 0.$$

Unfortunately it is not known whether Palais-Smale sequences admit converging subsequences. To show this property, recall first that $u \mapsto e^{2u}$ is compact from $H^1(\Sigma)$ to $L^p(\Sigma)$, which by Hölder's inequality implies the compactness of $u \mapsto \tilde{h}(x)e^{2u}$ from $H^1(\Sigma)$ into $L^1(\Sigma)$. Therefore it would be sufficient to show that a Palais-Smale sequence is bounded.

This indeed can be proved indirectly, following an argument in [61], by slightly modifying the value of the parameter ρ . We choose a small $\rho_0 > 0$, and allow ρ to vary in the interval $[1 - \rho_0, 1 + \rho_0]$. We consider then the functional $I_{\rho,\alpha}$ for these values of ρ . If ρ_0 is sufficiently small, the interval $[1 - \rho_0, 1 + \rho_0]$ will be compactly contained in the complement of the set \mathfrak{S} , see (11). Following the previous estimates with minor changes, one easily checks that the min-max scheme applies uniformly for $\rho \in [1 - \rho_0, 1 + \rho_0]$ and for $\bar{\lambda}$ sufficiently large. Precisely, given any large $L > 0$, there exist ρ_0 sufficiently small and $\bar{\lambda}$ so large that for $\rho \in [1 - \rho_0, 1 + \rho_0]$

$$(49) \quad \sup_{t \in \{0,1\}} I_{\rho,\alpha}(\varpi(t)) < -2L; \quad \bar{\Pi}_{\rho} := \inf_{\varpi \in \bar{\Pi}_{\bar{\lambda}}} \sup_{t \in [0,1]} I_{\rho,\alpha}(\varpi(t)) > -\frac{L}{2},$$

where $\bar{\Pi}_{\bar{\lambda}}$ is defined in (47). Moreover, using for example the test map (48), one shows that for ρ_0 sufficiently small there exists a large constant \bar{L} such that

$$(50) \quad \bar{\Pi}_{\rho} \leq \bar{L} \quad \text{for every } \rho \in [1 - \rho_0, 1 + \rho_0].$$

We have the following result, regarding the dependence in ρ of the min-max value $\bar{\Pi}_{\rho}$, see [35].

Lemma 3.3. *Let $\bar{\lambda}$ be so large and ρ_0 be so small that (49) holds. Then the function*

$$\rho \mapsto \frac{\bar{\Pi}_{\rho}}{\rho} \quad \text{is non-increasing in } [1 - \rho_0, 1 + \rho_0].$$

PROOF. For $\rho' \geq \rho$, we have

$$\frac{I_{\rho,\alpha}(u)}{\rho} - \frac{I_{\rho'}(u)}{\rho'} = \frac{1}{2} \left(\frac{1}{\rho} - \frac{1}{\rho'} \right) \int_{\Sigma} |\nabla_g u|^2 dV_g \geq 0,$$

which clearly implies $\frac{\bar{\Pi}_{\rho}}{\rho} \geq \frac{\bar{\Pi}_{\rho'}}{\rho'}$. This concludes the proof. ■

From Lemma 3.3 we deduce that the function $\rho \mapsto \frac{\bar{\Pi}_{\rho}}{\rho}$ is differentiable almost everywhere, and we obtain the following corollary.

Corollary 3.4. *Let $\bar{\lambda}$ and ρ_0 be as in Lemma 3.3, and let $\Lambda \subset [1 - \rho_0, 1 + \rho_0]$ be the (dense) set of ρ for which the function $\frac{\bar{\Pi}_{\rho}}{\rho}$ is differentiable. Then for $\rho \in \Lambda$ the functional $I_{\rho,\alpha}$ possesses a bounded Palais-Smale sequence $(u_l)_l$ at level $\bar{\Pi}_{\rho}$, weakly converging to a solution of (6).*

PROOF. The existence of a Palais-Smale sequence $(u_l)_l$ follows from Lemma 3.2, and the boundedness is proved exactly as in [35], Lemma 3.2. ■

From the above result we obtained a sequence $\rho_k \rightarrow \rho$ such that $I_{\rho_k,\alpha}$ has a critical point. We have next the following result.

Theorem 3.5. ([9], [7], [8]) *Let Σ be a compact surface, and let u_i solve (6) with \tilde{h} as in (7), $\rho = \rho_i$, $\rho_i \rightarrow \bar{\rho}$. Suppose that $\int_{\Sigma} f_{u_i} dV_g \leq \bar{C}$ for some fixed $\bar{C} > 0$. Then along a subsequence u_{i_k} one of the following alternatives holds:*

- (i): u_{i_k} is uniformly bounded from above on Σ ;

(ii): $\max_{\Sigma} \left(2u_{i_k} - \log \int_{\Sigma} f_{u_{i_k}} dV_g \right) \rightarrow +\infty$ and there exists a finite blow-up set $S = \{q_1, \dots, q_l\} \in \Sigma$ such that

(a) for any $s \in \{1, \dots, l\}$ there exist $x_k^s \rightarrow q_s$ such that $u_{i_k}(x_k^s) \rightarrow +\infty$ and $u_{i_k} \rightarrow -\infty$ uniformly on the compact sets of $\Sigma \setminus S$,

(b) $\rho_{i_k} \tilde{f}_{u_{i_k}} \rightarrow \sum_{s=1}^l \beta_s \delta_{q_s}$ in the sense of measures, with $\beta_s = 4\pi$ for $q_s \neq \{p_1, \dots, p_m\}$, or $\beta_s = 4\pi(1 + \alpha_j)$ if $q_s = p_j$ for some $j = \{1, \dots, m\}$. In particular one has that

$$\bar{\rho} = 4\pi n + 4\pi \sum_{j \in J} (1 + \alpha_j),$$

for some $n \in \mathbb{N} \cup 0$ and $J \subseteq \{1, \dots, m\}$ (possibly empty) satisfying $n + \text{card}(J) > 0$.

PROOF OF THEOREM 1.2 By Corollary 3.4 there exists a sequence $\rho_k \rightarrow \rho$ such that $I_{\rho_k, \alpha}$ has a critical point u_k . By Theorem 3.5, since $\rho \notin \mathfrak{S}$, u_k must then converge to a solution of (2). ■

We also refer to [35], [62] for previous results on surfaces with positive genus concerning the regular case of (2). The above method can actually be used to find multiplicity results as well for generic data, see [31], [32].

4. A MOSER-TRUDINGER INEQUALITY FOR SINGULAR SYSTEMS

In this section we are going to prove the following Moser-Trudinger type inequality. It is a weaker version of Theorem 1.3, but anyway sufficient to prove the existence result Theorem 1.4. We denote as x^- the negative part of a real number x , that is $x^- := \begin{cases} 0 & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}$, and we set for $i \in \{1, 2\}$

$$(51) \quad \tilde{\alpha}_i = - \max_{j \in \{1, \dots, m\}} \alpha_{i,j}^-.$$

Proposition 4.1. *Let Σ be a closed surface with area $|\Sigma| = 1$, \tilde{h}_i be as in (18), and $\tilde{\alpha}_i$ be as in (51). Then, for any $\rho = (\rho_1, \rho_2) \in \mathbb{R}_+^2$ satisfying $\rho_i < 4\pi(1 + \tilde{\alpha}_i)$ for both $i \in \{1, 2\}$ there exists $C(\rho) > 0$ such that the Euler-Lagrange functional (19) verifies*

$$J_{\rho, \alpha}(u) > -C(\rho) \quad \forall u \in H^1(\Sigma)^2.$$

Definition 4.2. *As in [44], we define the set of admissible parameters Λ as*

$$\Lambda := \{ \rho \in \mathbb{R}_+^2 : J_{\rho, \alpha} \text{ is bounded from below} \}.$$

Clearly, Λ preserves the partial order of \mathbb{R}_+^2 , that is if $\rho \in \Lambda$ then $\tilde{\rho} \in \Lambda$ until $\tilde{\rho}_i \leq \rho_i$ for both $i \in \{1, 2\}$; in these terms, Proposition 4.1 is equivalent to saying

$$(0, 4\pi(1 + \tilde{\alpha}_1)) \times (0, 4\pi(1 + \tilde{\alpha}_2)) \subset \Lambda.$$

Remark 4.3. *One can easily see that Λ is not empty: since it holds*

$$\frac{|\nabla u_1|^2 + |\nabla u_2|^2}{6} \leq Q(u_1, u_2)$$

one can apply the scalar Moser-Trudinger inequality (10) to both components to get

$$\left(0, \frac{8}{3}\pi(1 + \tilde{\alpha}_1) \right) \times \left(0, \frac{8}{3}\pi(1 + \tilde{\alpha}_2) \right) \subset \Lambda.$$

To prove Proposition 4.1, some lemmas will be needed. First of all, we notice that when the parameter ρ is in the interior of the set Λ , then the energy functional is not only bounded from below, but even coercive and it has a minimizer; on the other hand, if ρ is on the boundary of Λ , then $J_{\rho, \alpha}$ cannot be coercive.

Lemma 4.4. *For any $\rho \in \overset{\circ}{\Lambda}$ there exists a constant C such that*

$$J_{\rho, \alpha}(u) \geq \frac{\int_{\Sigma} (|\nabla u_1|^2 + |\nabla u_2|^2) dV_g}{C} - C$$

Moreover, $J_{\rho, \alpha}$ admits a minimizer $u = (u_1, u_2)$ that solves (17).

PROOF. Taking $\delta \in \left(0, \frac{d(\rho, \partial\Lambda)}{\sqrt{2}}\right)$, we have $(1 + \delta)\rho \in \Lambda$ so $J_{(1+\delta)\rho, \underline{\alpha}}(u) \geq -C$; therefore, we can write

$$\begin{aligned} J_{\rho, \underline{\alpha}}(u) &= \frac{\delta}{1 + \delta} \int_{\Sigma} Q(u_1, u_2) dV_g + \frac{J_{(1+\delta)\rho, \underline{\alpha}}(u)}{1 + \delta} \geq \\ &\geq \frac{\delta}{6(1 + \delta)} \int_{\Sigma} (|\nabla u_1|^2 + |\nabla u_2|^2) dV_g - C \end{aligned}$$

and the first claim follows.

To prove the rest we notice that, if we restrict ourselves to the subset of $H^1(\Sigma)^2$ consisting of all functions satisfying $\int_{\Sigma} \tilde{h}_i e^{u_i} dV_g = 1$, the energy is coercive since, from Poincaré's inequality and (10)

$$\begin{aligned} \int_{\Sigma} u_i^2 dV_g &= \int_{\Sigma} (u_i - \bar{u}_i)^2 dV_g + (\bar{u}_i)^2 \leq \\ &\leq C \int_{\Sigma} |\nabla u_i|^2 dV_g + \left(C + \frac{1}{16\pi(1 + \tilde{\alpha}_i)} \int_{\Sigma} |\nabla u_i|^2 dV_g \right)^2 \leq \\ &\leq C \left(1 + \int_{\Sigma} |\nabla u_i|^2 dV_g \right)^2. \end{aligned}$$

Being $J_{\rho, \underline{\alpha}}$ weakly lower-semicontinuous as well, the existence of a minimizer follows from the direct methods of calculus of variations. ■

Lemma 4.5. *For any $\rho \in \partial\Lambda$ there exists a sequence $\{\tilde{u}_k\}_{k \in \mathbb{N}} \subset H^1(\Sigma)^2$ verifying*

$$\int_{\Sigma} (|\nabla \tilde{u}_{1,k}|^2 + |\nabla \tilde{u}_{2,k}|^2) dV_g \xrightarrow{k \rightarrow +\infty} +\infty \quad \lim_{k \rightarrow +\infty} \frac{J_{\rho, \underline{\alpha}}(\tilde{u}_k)}{\int_{\Sigma} (|\nabla \tilde{u}_{1,k}|^2 + |\nabla \tilde{u}_{2,k}|^2) dV_g} \leq 0.$$

PROOF. Suppose by contradiction that

$$\int_{\Sigma} (|\nabla u_{1,k}|^2 + |\nabla u_{2,k}|^2) dV_g \xrightarrow{k \rightarrow +\infty} +\infty \quad \Rightarrow \quad \frac{J_{\rho, \underline{\alpha}}(u_k)}{\int_{\Sigma} (|\nabla u_{1,k}|^2 + |\nabla u_{2,k}|^2) dV_g} \geq \theta > 0$$

for any choice of $\{u_k\}$. This would mean that $J_{\rho, \underline{\alpha}}(u) \geq \frac{\theta}{2} \int_{\Sigma} (|\nabla u_1|^2 + |\nabla u_2|^2) dV_g - C$, hence for any small δ we would get

$$\begin{aligned} J_{(1+\delta)\rho, \underline{\alpha}}(u) &= (1 + \delta)J_{\rho, \underline{\alpha}}(u) - \delta \int_{\Sigma} Q(u_1, u_2) dV_g \geq \\ &\geq \left((1 + \delta)\frac{\theta}{2} - \frac{\delta}{2} \right) \int_{\Sigma} (|\nabla u_1|^2 + |\nabla u_2|^2) dV_g - C \\ &\geq -C \end{aligned}$$

hence $(1 + \delta)\rho \in \Lambda$, whereas one clearly has $(1 - \delta)\rho \in \Lambda$; this is in contradiction to $\rho \in \partial\Lambda$. ■

We then need a basic calculus lemma. Its proof will be omitted, as it can be found in [44] (following an idea of W. Ding).

Lemma 4.6 ([44], Lemma 4.4). *Let $\{a_k\}_{k \in \mathbb{N}}$ and $\{b_k\}_{k \in \mathbb{N}}$ be two sequences of real numbers satisfying*

$$a_k \xrightarrow{k \rightarrow +\infty} +\infty \quad \text{and} \quad \lim_{k \rightarrow +\infty} \frac{b_k}{a_k} \leq 0.$$

Then there exists a smooth function $F : [0, +\infty) \rightarrow \mathbb{R}$ satisfying, up to subsequences,

$$0 < F'(t) < 1 \quad \text{for any } t \geq 0 \quad F'(t) \xrightarrow{t \rightarrow +\infty} 0 \quad F(a_k) - b_k \xrightarrow{k \rightarrow +\infty} +\infty.$$

The latter lemma will be applied to the sequences

$$a_k = \int_{\Sigma} Q(\tilde{u}_{1,k}, \tilde{u}_{2,k}) dV_g, \quad b_k = J_{\rho, \underline{\alpha}}(\tilde{u}_k)$$

where \tilde{u}_k is as in Lemma 4.5, and we will consider the auxiliary functional

$$\tilde{J}_{\rho, \underline{\alpha}}(u) := J_{\rho, \underline{\alpha}}(u) - F \left(\int_{\Sigma} Q(u_1, u_2) dV_g \right),$$

whose behaviour is described by the following lemma.

Lemma 4.7. *For any $\rho \in \overset{\circ}{\Lambda}$ the functional $\tilde{J}_{\rho, \underline{\alpha}}$ is bounded from below on $H^1(\Sigma)^2$ and its infimum is achieved by a function satisfying*

$$\begin{cases} -\left(1 - \frac{2}{3}\mathfrak{g}(u)\right) \Delta u_i + \frac{\mathfrak{g}(u)}{3} \Delta u_{3-i} = 2\rho_i \left(\tilde{h}_i e^{u_i} - 1\right) - \rho_{3-i} \left(\tilde{h}_{3-i} e^{u_{3-i}} - 1\right); \\ \int_{\Sigma} \tilde{h}_i e^{u_i} dV_g = 1, \end{cases}$$

where $\mathfrak{g}(u) = F' \left(\int_{\Sigma} Q(u_1, u_2) dV_g \right)$. On the other hand, if $\rho \in \partial\Lambda$ then $\inf_{H^1(\Sigma)^2} \tilde{J}_{\rho, \underline{\alpha}} = -\infty$.

PROOF. For $\rho \in \overset{\circ}{\Lambda}$ one can argue as in Lemma 4.4, yielding lower semi-continuity from the regularity of F and coercivity from the behaviour of F' at infinity.

For $\rho \in \partial\Lambda$, taking \tilde{u}_k as in Lemma 4.5 and applying Lemma 4.6 one gets

$$\tilde{J}_{\rho, \underline{\alpha}}(\tilde{u}_k) = b_k - F(a_k) \xrightarrow{k \rightarrow +\infty} -\infty.$$

This concludes the proof. ■

We will need next the following two results. The first one is from [12], and its proof is rather similar to from Theorem 3.1 in [44].

Theorem 4.8. ([12]) *Let \tilde{h}_i as in (18), let $u_k = (u_{1,k}, u_{2,k}) \in H^1(\Sigma)^2$ be solutions of*

$$\begin{cases} -\Delta u_{i,k} = 2V_{i,k} \tilde{h}_i e^{u_{i,k}} - V_{3-i,k} \tilde{h}_{3-i} e^{u_{3-i,k}} + \psi_{i,k} \\ \int_{\Sigma} \tilde{h}_i e^{u_{i,k}} dV_g \leq C \\ \|\psi_{i,k}\|_{L^p(\Sigma)} \leq C \\ V_{i,k} \xrightarrow{k \rightarrow +\infty} 1 \text{ in } L^\infty(\Sigma) \end{cases} \quad i \in \{1, 2\},$$

for some $p > 1$, $C > 0$ and define the sets S_i as

$$S_i := \left\{ p \in \Sigma : \exists x_k \xrightarrow{k \rightarrow +\infty} p \text{ such that } u_{i,k}(x_k) \xrightarrow{k \rightarrow +\infty} +\infty \right\}.$$

Then, after taking subsequences, one of the following alternatives happens.

- (1) For each $i \in \{1, 2\}$, either $u_{i,k}$ is bounded in $L^\infty(\Sigma)$ or it tends uniformly to $-\infty$.
- (2) $S_i \neq \emptyset$ for some $i \in \{1, 2\}$; in this case, S_i is finite and either $u_{j,k}$ is bounded in $L_{loc}^\infty(\Sigma \setminus (S_1 \cup S_2))$ or it converges to $-\infty$ in $L_{loc}^\infty(\Sigma \setminus (S_1 \cup S_2))$ for each $j \in \{1, 2\}$; moreover, if $S_i \setminus S_{3-i} \neq \emptyset$, then the latter alternative occurs for $u_{i,k}$.

Theorem 4.9. ([49], Proposition 3.1) *Let $u_k = (u_{1,k}, u_{2,k}) \in H^1(\Sigma)^2$ be solutions of (17), let $\hat{\alpha}_i(p)$ stand for 0 if p is regular and for $\alpha_{i,j}$ if $p = p_j$. Define also*

$$(52) \quad \sigma_i(p) := \lim_{r \rightarrow 0} \lim_{k \rightarrow +\infty} \int_{B_r(p)} \tilde{h}_i e^{u_{i,k}} dV_g.$$

Then, one has

$$(53) \quad \sigma_1(p)^2 - \sigma_1(p)\sigma_2(p) + \sigma_2(p)^2 = 4\pi(1 + \hat{\alpha}_1(p))\sigma_1(p) + 4\pi(1 + \hat{\alpha}_2(p))\sigma_2(p).$$

We are now in position to prove the main result of this section.

PROOF OF PROPOSITION 4.1 Suppose by contradiction that

$$(0, 4\pi(1 + \tilde{\alpha}_1)) \times (0, 4\pi(1 + \tilde{\alpha}_2)) \not\subset \Lambda;$$

then there is some $\bar{\rho} \in \partial\Lambda$ with $\bar{\rho}_i < 4\pi(1 + \tilde{\alpha}_i)$ for both $i \in \{1, 2\}$.

Consider a sequence $\{\rho_k\}_{k \in \mathbb{N}} \in \overset{\circ}{\Lambda}$ with $\rho_k \xrightarrow{k \rightarrow +\infty} \bar{\rho}$ and a minimizer u_k for $\tilde{J}_{\rho_k, \underline{\alpha}}$, as in Lemma 4.7; then, $v_k := u_k + \log \rho_k$ solves

$$\begin{cases} -\Delta v_{i,k} = 2 \frac{6-5\mathfrak{g}(v_k)}{6-8\mathfrak{g}(v_k)+2\mathfrak{g}(v_k)^2} \left(\tilde{h}_i e^{v_{i,k}} - \rho_{i,k}\right) - \frac{3-4\mathfrak{g}(v_k)}{3-4\mathfrak{g}(v_k)+\mathfrak{g}(v_k)^2} \left(\tilde{h}_{3-i} e^{v_{3-i,k}} - \rho_{3-i,k}\right); \\ \int_{\Sigma} \tilde{h}_i e^{v_{i,k}} dV_g = \rho_{i,k}, \end{cases}$$

with $\frac{6 - 5\mathfrak{g}(v_k)}{6 - 8\mathfrak{g}(v_k) + 2\mathfrak{g}(v_k)^2}$ and $\frac{3 - 4\mathfrak{g}(v_k)}{3 - 4\mathfrak{g}(v_k) + \mathfrak{g}(v_k)^2}$ both uniformly converging to 1, so Theorem 4.8 can be applied to this sequence. The normalization on the integral implies that $u_{i,k}$ cannot tend to $-\infty$ for any $i \in \{1, 2\}$; moreover, we can also exclude boundedness in $L^\infty(\Sigma)$ because this would imply convergence to a minimizer \bar{u} of $\tilde{J}_{\bar{\rho}, \alpha}$, contradicting Lemma 4.7.

The only case left is the blow-up around at least one point p : Pohozaev's identity (53) implies that if there is a singularity of mass $\alpha_{i,j}$ on p then $\sigma_i \geq 4\pi(1 + \alpha_{i,j})$ for some $i \in \{1, 2\}$, whereas if p is a regular point then there is a component with a mass of at least 4π around it; in both cases, for such an i we obtain:

$$4\pi(1 + \tilde{\alpha}_i) \leq \lim_{r \rightarrow 0} \lim_{k \rightarrow +\infty} \int_{B_r(p)} \tilde{h}_i e^{v_{i,k}} dV_g \leq \lim_{k \rightarrow +\infty} \int_{\Sigma} \tilde{h}_i e^{v_{i,k}} dV_g = \bar{\rho}_i < 4\pi(1 + \tilde{\alpha}_i),$$

that is a contradiction. ■

We conclude the section by showing a partial converse of Theorem 4.1, namely that for higher values of the parameter ρ the functional $J_{\rho, \alpha}$ is unbounded from below. Estimates of this type will be needed in the next Section and are in the spirit of Proposition 3.1.

Proposition 4.10. *If $\rho_i > 4\pi(1 + \tilde{\alpha}_i)$ for some $i \in \{1, 2\}$, then $\inf_{H^1(\Sigma)^2} J_{\rho, \alpha} = -\infty$ that is*

$$\Lambda \subset (0, 4\pi(1 + \tilde{\alpha}_1)] \times (0, 4\pi(1 + \tilde{\alpha}_2)].$$

PROOF. We will give the proof for $i = 1$, being nearly identical for $i = 2$. Choosing a point p_1 such that $\hat{\alpha}_1(p_1) = \tilde{\alpha}_1$, we define for large λ

$$\varphi_{1,\lambda}(x) = \log \left(\frac{\lambda^{1+\tilde{\alpha}_1}}{1 + (\lambda d(x, p_1))^{2(1+\tilde{\alpha}_1)}} \right)^2; \quad \varphi_{2,\lambda}(x) = -\frac{1}{2} \log \left(\frac{\lambda^{1+\tilde{\alpha}_1}}{1 + (\lambda d(x, p_1))^{2(1+\tilde{\alpha}_1)}} \right)^2.$$

Using the fact that $|\nabla (d(x, p_1)^{2(1+\tilde{\alpha}_1)})| \leq 2(1 + \tilde{\alpha}_1)d(x, p_1)^{1+2\tilde{\alpha}_1}$, we obtain

$$\begin{aligned} |\nabla \varphi_{1,\lambda}(x)| &= \left| \frac{-2\lambda^{2(1+\tilde{\alpha}_1)} |\nabla (d(x, p_1)^{2(1+\tilde{\alpha}_1)})|}{1 + (\lambda d(x, p_1))^{2(1+\tilde{\alpha}_1)}} \right| \leq \\ &\leq \frac{4(1 + \tilde{\alpha}_1)\lambda^{2(1+\tilde{\alpha}_1)} d(x, p_1)^{1+2\tilde{\alpha}_1}}{1 + (\lambda d(x, p_1))^{2(1+\tilde{\alpha}_1)}} \leq \\ &\leq \min \left\{ C\lambda, \frac{4(1 + \tilde{\alpha}_1)}{d(x, p_1)} \right\}, \end{aligned}$$

and therefore

$$\begin{aligned} \int_{\Sigma} Q(\varphi_{1,\lambda}, \varphi_{2,\lambda}) dV_g &= \frac{1}{4} \int_{\Sigma} |\nabla \varphi_{1,\lambda}|^2 dV_g \leq \\ (54) \quad &\leq C\lambda^2 \int_{B_{\frac{1}{\lambda}}(p_1)} dV_g + 4(1 + \tilde{\alpha}_1)^2 \int_{\Sigma \setminus B_{\frac{1}{\lambda}}(p_1)} \frac{dV_g}{d(\cdot, p_1)^2} \leq \\ &\leq C + 8\pi(1 + \tilde{\alpha}_1)^2 \log \lambda. \end{aligned}$$

Moreover, being

$$(55) \quad \max\{1, (\lambda d(x, p_1))^{2(1+\tilde{\alpha}_1)}\} \leq 1 + (\lambda d(x, p_1))^{2(1+\tilde{\alpha}_1)} \leq C \max\{1, (\lambda d(x, p_1))^{2(1+\tilde{\alpha}_1)}\},$$

one gets the following estimate on the average of $\varphi_{1,\lambda}$

$$\overline{\varphi_{1,\lambda}} = \int_{\Sigma} (\max\{2(1 + \tilde{\alpha}_1) \log \lambda, -2(1 + \tilde{\alpha}_1)(\log \lambda + 2 \log d(\cdot, p_1))\} + O(1)) dV_g.$$

Dividing Σ into the two regions where the above maximum is attained and using the integrability of $\log d(\cdot, p_1)$ in two dimensions one gets

$$\begin{aligned}
\overline{\varphi_{1,\lambda}} &= 2(1 + \tilde{\alpha}_1) \log \lambda \int_{B_{\frac{1}{\lambda}}(p_1)} dV_g - 2(1 + \tilde{\alpha}_1) \log \lambda \int_{\Sigma \setminus B_{\frac{1}{\lambda}}(p_1)} dV_g - \\
(56) \quad &- 4(1 + \tilde{\alpha}_1) \int_{\Sigma \setminus B_{\frac{1}{\lambda}}(p_1)} \log d(\cdot, p_1) dV_g + O(1) \\
&= -2(1 + \tilde{\alpha}_1) \log \lambda + O(1),
\end{aligned}$$

and clearly also $\overline{\varphi_{2,\lambda}} = (1 + \tilde{\alpha}_1) \log \lambda + O(1)$.

For a small but fixed $\delta > 0$ we have, again by (55),

$$\begin{aligned}
(57) \quad \int_{\Sigma} \tilde{h}_1 e^{\varphi_{1,\lambda}} dV_g &\geq C \int_{B_{\delta}(p_1) \setminus B_{\frac{1}{\lambda}}(p_1)} d(\cdot, p_1)^{2\tilde{\alpha}_1} e^{\varphi_{1,\lambda}} dV_g \geq \\
&\geq \frac{C}{\lambda^{2(1+\tilde{\alpha}_1)}} \int_{B_{\delta}(p_1) \setminus B_{\frac{1}{\lambda}}(p_1)} \frac{dV_g}{d(\cdot, p_1)^{4+2\tilde{\alpha}_1}} \geq C;
\end{aligned}$$

on the other hand, we can write that

$$(58) \quad \int_{\Sigma} \tilde{h}_2 e^{\varphi_{2,\lambda}} dV_g \geq C \lambda^{1+\tilde{\alpha}_1} \int_{\Sigma \setminus B_{\frac{1}{\lambda}}(p_1)} \tilde{h}_2 d(\cdot, p_1)^{2(1+\tilde{\alpha}_1)} dV_g \geq C \lambda^{1+\tilde{\alpha}_1}.$$

Therefore, from (45), (56), (57), (58) we conclude that

$$J_{\rho, \underline{\alpha}}(u) \leq 2(1 + \tilde{\alpha}_1)(4\pi(1 + \tilde{\alpha}_1) - \rho_1) \log \lambda + O(1) \xrightarrow{\lambda \rightarrow \infty} -\infty,$$

as desired. ■

5. IMPROVED VECTORIAL INEQUALITIES

First of all, we recall the following result from [10], which extends Lemma 2.2 to the vectorial case.

Lemma 5.1. ([10], Lemma 4.3) *Let $\delta > 0$, $J_1, K_1, J_2, K_2 \in \mathbb{N}$ be given, let $\{m_{11}, \dots, m_{1J_1}, m_{21}, \dots, m_{2J_2}\} \subset \{1, \dots, m\}$, $\{\Omega_{ij}\}_{i=1,2}^{j=1, \dots, J_i+K_i}$ be open subsets of Σ such that*

$$\begin{aligned}
\alpha_{im_{ij}} &\leq 0 \quad \forall i = 1, 2, j = 1, \dots, J_i \\
d(\Omega_{ij}, \Omega_{ij'}) &\geq \delta \quad \forall i = 1, 2, \forall j, j' = 1, \dots, J_i + K_i, j \neq j' \\
d(p_j, \Omega_{ij}) &\geq \delta \quad \forall i = 1, 2, \forall j = 1, \dots, K_i + M_i, \forall j = 1, \dots, m, j \neq m_{ij};
\end{aligned}$$

and $u \in H^1(\Sigma)^2$ satisfy

$$\int_{\Omega_{ij}} f_{i,u} dV_g \geq \delta \quad \forall i = 1, 2, \forall j = 1, \dots, J_i + K_i.$$

Then, for any $\varepsilon > 0$ there exists $C = C_{\Sigma, \delta, J_1, K_1, J_2, K_2, \varepsilon} > 0$ such that

$$4\pi \sum_{i=1}^2 \left(K_i + \sum_{j=1}^{J_i} (1 + \alpha_{im_{ij}}) \right) \left(\log \int_{\Sigma} \tilde{h}_i e^{u_i} dV_g - \int_{\Sigma} u_i dV_g \right) \leq (1 + \varepsilon) \int_{\Sigma} Q(u) dV_g + C.$$

We will consider next some improved functional inequalities that take into account the relative *speeds of concentration* of the two components of the system. Let us first set

$$(59) \quad \mathcal{L} := \left\{ f \in L^1(\Sigma) : f > 0 \text{ a.e. in } \Sigma, \int_{\Sigma} f dV_g = 1 \right\}.$$

We will define, for each $f \in \mathcal{L}$, a *center of mass* and a *scale of concentration*, inspired by [55] (Proposition 3.1) but such that the center of mass belongs to a given finite set $\mathcal{F} \subset \Sigma$ (which will be, in our applications, a subset of the singular points). As in [55], we will map \mathcal{L} onto the topological cone over \mathcal{F} of height δ , which is defined by

$$(60) \quad \mathcal{C}_{\delta} \mathcal{F} := \frac{\mathcal{F} \times [0, \delta]}{\sim},$$

where the equivalence relation \sim is given by $(x, \delta) \sim (x', \delta)$ for any $x \in \Sigma$. The meaning of such an identification is the following: if a function $f \in \mathcal{L}$ does not concentrate around any point $x \in \mathcal{F}$, then we may not be able to define a center of mass. In this case, the equivalence relation in the definition of the cone leaves it undetermined.

Lemma 5.2. *Let $\mathcal{F} := \{x_1, \dots, x_K\} \subset \Sigma$ be a given finite set and $\mathcal{L}, \mathcal{C}_\delta$ be defined by (59) and (60). Then, for $\delta > 0$ small enough there exists a map $\psi = (\beta, \varsigma) = (\beta_{\mathcal{F}}, \varsigma_{\mathcal{F}}) : \mathcal{L} \rightarrow \mathcal{C}_\delta \mathcal{F}$ such that:*

- If $\varsigma(f) = \delta$, then either $\int_{\Sigma \setminus \cup_{x \in \mathcal{F}} B_\delta(x)} f dV_g \geq \delta$ or there exists $x', x'' \in \mathcal{F}$ with $x' \neq x''$ and

$$\int_{B_\delta(x')} f dV_g \geq \delta \quad \int_{B_\delta(x'')} f dV_g \geq \delta$$

- If $\varsigma(f) < \delta$, then

$$\int_{B_{\varsigma(f)}(\beta(f))} f dV_g \geq \delta \quad \int_{\Sigma \setminus B_{\varsigma(f)}(\beta(f))} f dV_g \geq \delta.$$

Moreover, if $f^n \xrightarrow{n \rightarrow +\infty} \delta_x$ for some $x \in \mathcal{F}$, then $(\beta(f^n), \varsigma(f^n)) \xrightarrow{n \rightarrow +\infty} (x, 0)$.

PROOF. Fix $\tau \in (\frac{1}{2}, 1)$, take $\delta \leq \frac{\min_{x, x' \in \mathcal{F}, x \neq x'} d(x, x')}{2}$ and define, for $k = 1, \dots, K$,

$$I_k(f) := \int_{B_\delta(x_k)} f dV_g; \quad I_0(f) := \int_{\Sigma \setminus \cup_{x \in \mathcal{F}} B_\delta(x)} f dV_g = 1 - \sum_{k=1}^K I_k(f),$$

Choose now indices \tilde{k}, \hat{k} such that

$$I_{\tilde{k}}(f) := \max_{k \in \{0, \dots, K\}} I_k(f) \quad I_{\hat{k}}(f) := \max_{k \neq \tilde{k}} I_k(f).$$

We will define the map ψ depending on \tilde{k} and $I_{\tilde{k}}(f)$:

- $\tilde{k} = 0$. Since f has little mass around each of the points x_k , we set $\varsigma(f) = \delta$ and do not define $\beta(f)$, as it would be irrelevant by the equivalence relation in (60). The assertion of the Lemma is verified, up to taking a smaller δ , because

$$\int_{\Sigma \setminus \cup_{x \in \mathcal{F}} B_\delta(x)} f dV_g = I_0(f) \geq \frac{1}{K+1} \geq \delta$$

- $\tilde{k} \geq 1$, $I_{\tilde{k}}(f) \leq \frac{K\tau}{1-\tau} I_{\hat{k}}(f)$. Here, f has still little mass around the point $x_{\tilde{k}}$ (which could not be uniquely defined), so again we set $\varsigma(f) := \delta$. It is easy to see that $I_{\tilde{k}}(f) \geq \frac{1-\tau}{K}$, so

$$\int_{B_\delta(x_{\tilde{k}})} f dV_g \geq \int_{B_\delta(x_{\hat{k}})} f dV_g \geq \frac{1-\tau}{K}$$

- $\tilde{k} \geq 1$, $I_{\tilde{k}}(f) \geq \frac{K\tau}{1-\tau} I_{\hat{k}}(f)$. Now, $I_{\tilde{k}}(f) > \tau$, so one can define a scale of concentration $s(x_{\tilde{k}}, f) \in (0, \delta)$ of f around $x_{\tilde{k}} \in \mathcal{F}$, uniquely determined by

$$\int_{B_{s(x_{\tilde{k}}, f)}(x_{\tilde{k}})} f dV_g = \tau.$$

We can also define a center of mass $\beta(f) = x_{\tilde{k}}$ but we have to interpolate for the scale:

- Case $I_{\tilde{k}}(f) \leq \frac{2K\tau}{1-\tau} I_{\hat{k}}(f)$: setting

$$\varsigma(f) = s(x_{\tilde{k}}, f) + \frac{I_{\tilde{k}}(f)}{\frac{K\tau}{1-\tau} I_{\hat{k}}(f)} (\delta - s(x_{\tilde{k}}, f)),$$

we get $s(x_{\tilde{k}}, f) < \varsigma(f) < \delta$; moreover, $I_{\hat{k}}(f) \geq \frac{1-\tau}{K(1+\tau)}$, hence

$$\int_{B_{\varsigma(f)}(\beta(f))} f dV_g \geq \int_{B_{s(x_{\tilde{k}}, f)}(x_{\tilde{k}})} f dV_g = \tau \geq \delta$$

$$\int_{\Sigma \setminus B_{\varsigma(f)}(\beta(f))} f dV_g \geq \int_{\Sigma \setminus B_\delta(x_{\tilde{k}})} f dV_g \geq \frac{1-\tau}{K(1+\tau)} \geq \delta$$

– Case $I_{\tilde{k}}(f) \geq \frac{2K\tau}{1-\tau} I_{\tilde{k}}(f)$: we just set $\varsigma(f) : s(x_{\tilde{k}}, f)$ and we get

$$\int_{B_{\varsigma(f)}(\beta(f))} f dV_g = \tau \geq \delta \quad \int_{\Sigma \setminus B_{\varsigma(f)}(\beta(f))} f dV_g = 1 - \tau \geq \delta.$$

To prove the final assertion, write (up to sub-sequences), $(\beta_\infty, \varsigma_\infty) = \lim_{n \rightarrow +\infty} (\beta(f^n), \varsigma(f^n))$. For large n we will have

$$\int_{\Sigma \setminus \cup_{x' \in \mathcal{F}} B_\delta(x')} f^n dV_g \leq \frac{\delta}{2} \quad \int_{B_\delta(x'')} f^n dV_g \leq \frac{\delta}{2} \quad \text{for any } x'' \in \mathcal{F} \setminus \{x\},$$

which excludes $\varsigma_\infty = \delta$. We also exclude $\varsigma_\infty \in (0, \delta)$ as it would give

$$\int_{B_{\frac{3}{2}\varsigma_\infty}(\beta_\infty)} f^n dV_g \geq \delta \quad \int_{\Sigma \setminus B_{\frac{3}{2}\varsigma_\infty}(\beta_\infty)} f^n dV_g \geq \delta.$$

which is a contradiction since $\mathcal{F} \cap \left(\overline{A_{\frac{3}{2}\varsigma_\infty, \frac{3}{2}\varsigma_\infty}(\beta_\infty)} \right) = \emptyset$.

Finally, we exclude $\beta_\infty \neq x$ because we would get the following contradiction:

$$\int_{B_\delta(\beta_\infty)} f^n dV_g \geq \delta.$$

This concludes the proof. ■

Define

$$(61) \quad (u_1, u_2) \mapsto \left(\frac{\tilde{h}_1 e^{u_1}}{\int_\Sigma \tilde{h}_1 e^{u_1} dV_g}, \frac{\tilde{h}_2 e^{u_2}}{\int_\Sigma \tilde{h}_2 e^{u_2} dV_g} \right) =: (f_{1,u}, f_{2,u}) :$$

combining such a map ψ with Lemma 5.1 we deduce some extra information on low sub-levels of $J_{\rho, \alpha}$. Recall first the definition of the sets \mathcal{A}_i from (24).

Corollary 5.3. *Let δ, ψ be as in Lemma 5.2 and define, for $u \in H^1(\Sigma)^2$,*

$$\beta_1(u) = \beta_{\mathcal{A}_1}(f_{1,u}), \quad \varsigma_1(u) = \varsigma_{\mathcal{A}_1}(f_{2,u}) \quad \beta_2(u) = \beta_{\mathcal{A}_2}(f_{2,u}), \quad \varsigma_2(u) = \varsigma_{\mathcal{A}_2}(f_{2,u}).$$

Then for any $\delta' > 0$ there exists $L_{\delta'}$ such that if $\varsigma_i(u) \geq \delta'$ for both $i = 1, 2$, then $J_{\rho, \alpha}(u) \geq -L_{\delta'}$.

PROOF. Assume first $\varsigma_1(u) = \delta$: from the statement of Lemma 5.3, we get one of the following:

- $\int_{\Sigma \setminus \cup_{j=1}^m B_\delta(p_j)} f_{1,u} dV_g \geq \frac{\delta}{2}$,
- $\int_{B_\delta(p_j)} f_{1,u} dV_g \geq \frac{\delta}{2M}$ for some $p_j \notin \mathcal{A}_1$,
- $\int_{B_\delta(p'_j)} f_{1,u} dV_g \geq \delta$, $\int_{B_\delta(p''_j)} f_{1,u} dV_g \geq \delta$ for some $j' \neq j''$.

Depending on which possibility occurs, define respectively

- $\Omega_{11} := \Sigma \setminus \cup_{j=1}^m B_\delta(p_j)$,
- $\Omega_{11} := B_\delta(p_j)$,
- $\Omega_{11} := B_\delta(p_{j'})$, $\Omega_{12} := B_\delta(p_{j''})$.

It is easy to verify that such sets satisfy the hypotheses of Lemma 5.1, up to eventually redefining the map ψ with a smaller $\delta \leq \frac{\min_{j \neq j'} d(p_j, p_{j'})}{4}$: in the first case, we have $J_1 = 0, K_1 = 1$, in the second case either $J_1 = 0, K_1 = 1$ or $J_1 = 1, K_1 = 0$ but $\rho < 4\pi(1 + \alpha_{1j})$, and in the third case we have $J_1 = 2, K_1 = 0$.

If $\delta' \leq \varsigma_1(u) < \delta$, then $\int_{\Sigma \setminus B_{\delta'}(\beta_1(u))} f_{1,u} dV_g \geq \delta$, so we have one among the following:

- $\int_{\Sigma \setminus \cup_{j=1}^m B_\delta(x)} f_{1,u} dV_g \geq \frac{\delta}{2}$
- $\int_{B_\delta(\beta_1(u))} f_{1,u} dV_g \geq \delta$, $\int_{B_\delta(p_j)} f_{1,u} dV_g \geq \frac{\delta}{2M}$ for some $p_j \neq \beta_1(u)$.
- $\int_{A_{\delta', \delta}(\beta_1(u))} f_{1,u} dV_g$.

Depending on which is the case, define:

- $\Omega_{11} := \Sigma \setminus \cup_{j=1}^m B_\delta(p_j)$.
- $\Omega_{11} := B_\delta(u)(\beta_1(u))$, $\Omega_{12} := B_\delta(p_j)$.
- $\Omega_{11} := A_{\delta', \delta}(\beta_1(u))$

Repeat the same argument for u_2 to get similarly Ω_{21} , and possibly Ω_{22} . Applying Lemma 5.1 one gets $J_{\rho, \underline{\alpha}}(u) \geq -L_{\delta'}$. ■

With some extra work (see [13] for the details) it can be shown that the vectorial Moser-Trudinger inequality improves each time the two scales (in the sense defined by Lemma 5.2) coincide, no matter how small they are.

Proposition 5.4. ([13]) *Let $\beta_i(u), \varsigma_i(u)$ be as in Corollary 5.3. There exists $L \gg 0$ such that if*

$$\begin{cases} \beta_1(u) = \beta_2(u) = p_m & \text{with } \rho_1, \rho_2 < 4\pi(2 + \alpha_{1m} + \alpha_{2m}) \\ \varsigma_1(u) = \varsigma_2(u) \end{cases},$$

then $J_{\rho, \underline{\alpha}}(u) \geq -L$.

6. PROOF OF THEOREM 1.2

Let us introduce the space \mathcal{X} , which is simply a graph and will be used in our min-max scheme. It is obtained removing some points from the join of the weighted barycenters $\mathcal{A}_1 \star \mathcal{A}_2$ defined by (25). The points to exclude correspond to improved inequalities for functions centered around the same point and at the same rate of concentration (see the previous section for more details).

Precisely, we set

$$(62) \quad \mathcal{X} := \mathcal{A}_1 \star \mathcal{A}_2 \setminus \left\{ \left(p_j, p_j, \frac{1}{2} \right) : \rho_1, \rho_2 < 4\pi(2 + \alpha_{1j} + \alpha_{2j}) \right\}.$$

We will prove that, under the assumptions of Theorem 1.4, the space \mathcal{X} is not contractible, showing that it has a non-trivial homology group. In order to do this, we will recall how to compute the homology groups of the join of two known spaces. Since the join is homotopically equivalent to a smash product of X, Y and S^1 (see [41] for details), its homology groups only depend on the homology of X and Y .

Theorem 6.1. ([41], Theorem 3.21) *Let X and Y be two topological spaces. Then*

$$\tilde{H}_q(X \star Y) = \sum_{q'=0}^q \tilde{H}_{q'}(X) \oplus \tilde{H}_{q-q'-1}(Y).$$

In particular, if $X = (S^{D_1})^{\vee N_1}$ and $Y = (S^{D_2})^{\vee N_2}$ are wedge sum of spheres, then $X \star Y$ has the same homology of $(S^{D_1+D_2+1})^{\vee N_1 N_2}$.

Actually, in the same book [41] it is shown that the following homotopical equivalence holds: $(S^{D_1})^{\vee N_1} \star (S^{D_2})^{\vee N_2} \simeq (S^{D_1+D_2+1})^{\vee N_1 N_2}$. We have then the following result.

Proposition 6.2. *Let \mathcal{X} be as in (62) and suppose we are under the assumptions of Theorem 1.4. Then, the space \mathcal{X} has non-trivial homology groups and it is not contractible.*

PROOF. The spaces \mathcal{A}_i are discrete sets of M_i points, for $i = 1, 2$, that is a wedge sum of $M_i - 1$ copies of S^0 . Therefore, by Theorem 6.1, $\mathcal{A}_1 \star \mathcal{A}_2$ has the same homology as $(S^1)^{\vee (M_1-1)(M_2-1)}$. The set we have to remove from the join consists of M_3 singular points $\{p_{m_1}, \dots, p_{m_{M_3}}\}$ for some $\{m_1, \dots, m_{M_3}\} \subset \{1, \dots, M\}$.

Defining then, for some fixed $\delta < \frac{1}{2}$, $\mathcal{Y} := \cup_{j=1}^{M_3} B_\delta(p_{m_j}, p_{m_j}, \frac{1}{2})$, \mathcal{Y} retracts on $\{p_{m_1}, \dots, p_{m_{M_3}}\}$. On the other hand, $\mathcal{X} \cap \mathcal{Y}$ is a disjoint union of M_3 punctured intervals, that is a discrete set of $2M_3$ points, and $\mathcal{X} \cup \mathcal{Y}$ is the whole join. Therefore, the Mayer-Vietoris sequence yields

$$\underbrace{H_1(\mathcal{X} \cap \mathcal{Y})}_0 \rightarrow H_1(\mathcal{X}) \oplus \underbrace{H_1(\mathcal{Y})}_0 \rightarrow \underbrace{H_1(\mathcal{X} \cup \mathcal{Y})}_{\mathbb{Z}^{(M_1-1)(M_2-1)}} \rightarrow \underbrace{\tilde{H}_0(\mathcal{X} \cap \mathcal{Y})}_{\mathbb{Z}^{2M_3-1}} \rightarrow \tilde{H}_0(\mathcal{X}) \oplus \underbrace{\tilde{H}_0(\mathcal{Y})}_{\mathbb{Z}^{M_3-1}} \rightarrow \underbrace{\tilde{H}_0(\mathcal{X} \cup \mathcal{Y})}_0.$$

The exactness of the sequence implies that $b_1(\mathcal{X}) - \tilde{b}_0(\mathcal{X}) = (M_1 - 1)(M_2 - 1) - M_3$, so if the latter number is not zero we get at least a non-trivial homology group. Algebraic computations show that, under the assumption $M_1, M_2 \geq M_3$, $(M_1 - 1)(M_2 - 1) \neq M_3$ is equivalent to the assumption of Theorem 1.4, and therefore the proof is complete. ■

We will now introduce some test functions from the space \mathcal{X} to arbitrarily low sub-levels of $J_{\rho,\alpha}$. Such test functions will have a profile which resembles the entire solutions of the Liouville equation and of the Toda system. We will use suitable interpolation between each of the above three profiles depending on whether the points in \mathcal{A}_i coincide or not and depending on which of the parameters ρ_i . The map Φ^λ will therefore be defined case by case.

Let us start by setting $\Phi^\lambda(\zeta) = \left(\phi_1 - \frac{\phi_2}{2}, \phi_2 - \frac{\phi_1}{2}\right)$ when $\zeta = (p_j, p_j m, t)$ for some j . The functions ϕ_1, ϕ_2 will be defined in different ways, depending on the relative positions of $\rho_1, \rho_2, \alpha_{1j}, \alpha_{2j}$ in \mathbb{R} . When dealing with the same singular point we set

(\ll) $\rho_1, \rho_2 < 4\pi(2 + \alpha_{1j} + \alpha_{2j})$:

$$\phi_1 := \begin{cases} -2 \log \max \{1, (\lambda d(\cdot, p_j))^{2(1+\alpha_{1j})}\} & \text{if } t < \frac{1}{2} \\ 0 & \text{if } t > \frac{1}{2} \end{cases}$$

$$\phi_2 := \begin{cases} 0 & \text{if } t < \frac{1}{2} \\ -2 \log \max \{1, (\lambda d(\cdot, p_j))^{2(1+\alpha_{2j})}\} & \text{if } t > \frac{1}{2}. \end{cases}$$

(\ll) $\rho_1 < 4\pi(2 + \alpha_{1j} + \alpha_{2j}) < \rho_2$:

$$\phi_1 := -2 \log \max \{1, \max \{1, (\lambda t)^{2(1+\alpha_{2j})}\} (\lambda d(\cdot, p_j))^{2(1+\alpha_{1j})}\}$$

$$\phi_2 := -2 \log \max \{1, (\lambda t d(\cdot, p_j))^{2(2+\alpha_{1j}+\alpha_{2j})}\}.$$

($>$) $\rho_2 < 4\pi(2 + \alpha_{1j} + \alpha_{2j}) < \rho_1$:

$$\phi_1 := -2 \log \max \{1, (\lambda(1-t)d(\cdot, p_j))^{2(2+\alpha_{1j}+\alpha_{2j})}\}$$

$$\phi_2 := -2 \log \max \{1, \max \{1, (\lambda(1-t))^{2(1+\alpha_{1j})}\} (\lambda d(\cdot, p_j))^{2(1+\alpha_{2j})}\}.$$

($>>$) $\rho_1, \rho_2 > 4\pi(2 + \alpha_{1j} + \alpha_{2j})$:

$$\phi_1 := -2 \log \max \left\{ 1, \left(\lambda \frac{\max\{1, \lambda t\}}{\max\{1, \lambda(1-t)\}} \right)^{2+\alpha_{1j}+\alpha_{2j}} d(\cdot, p_m)^{2(1+\alpha_{1j})}, (\lambda d(\cdot, p_j))^{2(2+\alpha_{1j}+\alpha_{2j})} \right\}$$

$$\phi_2 := -2 \log \max \left\{ 1, \left(\lambda \frac{\max\{1, \lambda(1-t)\}}{\max\{1, \lambda t\}} \right)^{2+\alpha_{1j}+\alpha_{2j}} d(\cdot, p_j)^{2(1+\alpha_{2j})}, (\lambda d(\cdot, p_j))^{2(2+\alpha_{1j}+\alpha_{2j})} \right\}.$$

Let us now consider the case $x_1 \neq x_2, x_i \in \mathcal{A}_i$. Here, we define Φ^λ just by linearly interpolating between the test functions defined before:

$$\Phi^\lambda(x_1, x_2, t) = \Phi^{\lambda(1-t)}(x_1, x_1, 0) + \Phi^{\lambda t}(x_2, x_2, 1).$$

We have then the following result.

Proposition 6.3. *The above test functions $\{\Phi^\lambda\}_\lambda : \mathcal{X} \rightarrow H^1(\Sigma)^2$ satisfy*

$$J_{\rho,\alpha}(\Phi^\lambda(\zeta)) \xrightarrow{\lambda \rightarrow +\infty} -\infty \quad \text{uniformly for } \zeta \in \mathcal{X}.$$

We are finally in position to prove Theorem 1.4. The proof will follow by showing that low sub-levels are *dominated* by the space \mathcal{X} (see [41], page 528), which is not contractible by Proposition 6.2.

Lemma 6.4. *For $L \gg 0$ large enough there exist maps $\Phi : \mathcal{X} \rightarrow J_{\rho,\alpha}^{-L}$ and $\Psi : J_{\rho,\alpha}^{-L} \rightarrow \mathcal{X}$ such that $\Psi \circ \Phi$ is homotopically equivalent to $\text{Id}_\mathcal{X}$.*

To prove Lemma 6.4 we need the following estimate. Notice that the choice of τ (see the proof of Lemma 5.2), which was not relevant in all the rest of this paper, will be made in the proof of this lemma to let the following result hold true: for the proof we refer to [12].

Lemma 6.5. *Let δ be as in Lemma 5.2, $\beta_i(u), \sigma_i(u)$ be as in Corollary 5.3 and Φ^λ as in Theorem 6.3. Then, for a suitable choice of τ , there exists $C_0 > 0, \delta' \in (0, \delta)$ such that:*

- If either $t \geq 1 - \frac{C_0}{\lambda}$ or $\begin{cases} t > \frac{1}{2} \\ x_1 = x_2 =: p_j \\ \rho_1, \rho_2 < 4\pi(2 + \alpha_{1j} + \alpha_{2j}) \end{cases}$, then $\sigma_1(\Phi^\lambda(\zeta)) \geq \delta'$;
- otherwise, $\sigma_1(\Phi^\lambda(\zeta)) < \delta$ and $\beta_1(\Phi^\lambda(\zeta)) = x_1$.

- If either $t \leq \frac{C_0}{\lambda}$ or $\begin{cases} t < \frac{1}{2} \\ x_1 = x_2 =: p_j \\ \rho_1, \rho_2 < 4\pi(2 + \alpha_{1j} + \alpha_{2j}) \end{cases}$, then $\sigma_2(\Phi^\lambda(\zeta)) \geq \delta'$;
otherwise, $\sigma_2(\Phi^\lambda(\zeta)) < \delta$ and $\beta_2(\Phi^\lambda(\zeta)) = x_2$.

PROOF OF LEMMA 6.4. Let δ be as in Lemma 5.2, $\beta_i(u), \varsigma_i(u)$ be as in Corollary 5.3 and δ' be as in Lemma 6.5. Take now L so large that Corollary 5.3 and Theorem 5.4 apply.

We define $\Phi = \Phi^{\lambda_0}$ as in Theorem 6.3, with λ_0 such that $\Phi^\lambda(\mathcal{X}) \subset J_{\rho, \underline{\alpha}}^{-2L}$ for any $\lambda \geq \lambda_0$. As for $\Psi : J_{\rho, \underline{\alpha}}^{-2L} \rightarrow \mathcal{X}$, we write

$$\Psi(u) = (\beta_1(u), \beta_2(u), t'(\varsigma_1(u), \varsigma_2(u))) \quad \text{with } t'(\varsigma_1(u), \varsigma_2(u)) = \begin{cases} 0 & \text{if } \varsigma_2(u) \geq \delta' \\ \frac{\delta' - \varsigma_2(u)}{2\delta' - \varsigma_1(u) - \varsigma_2(u)} & \text{if } \varsigma_1(u), \varsigma_2(u) \leq \delta' \\ 1 & \text{if } \varsigma_1(u) \geq \delta' \end{cases}.$$

Let us verify the well-posedness of Ψ . The definition of t' makes sense because, from Corollary 5.3, $J_{\rho, \underline{\alpha}}(u) < -L$ implies $\min\{\varsigma_1(u), \varsigma_2(u)\} \leq \delta'$. Moreover, if $t' > 0$ (resp. $t' < 1$), then $\varsigma_1 < \delta$ is well-defined (respectively, $\varsigma_2 < \delta$ is well-defined), hence β_1 (resp. β_2) is also defined. Finally, Ψ is mapped on \mathcal{X} because, from Theorem 5.4, when $J_{\rho, \underline{\alpha}}(u) < -L$ we cannot have $(\beta_1(u), \beta_2(u), t'(\varsigma_1(u), \varsigma_2(u))) = (p_j, p_j, \frac{1}{2})$ with $\rho_1, \rho_2 < 4\pi(2 + \alpha_{1j} + \alpha_{2j})$.

To get a homotopy between the two maps, we first let λ tend to $+\infty$, in order to get x_1 and x_2 , then we apply a linear interpolation for the parameter t . Writing $\Psi(\Phi^\lambda(\zeta)) = (\beta_1^\lambda(\zeta), \beta_2^\lambda(\zeta), t^{\lambda_0}(\zeta))$, we have $F = F_2 * F_1$, with

$$F_1 : (\zeta, s) = ((x_1, x_2, t), s) \mapsto \left(\beta_1^{\frac{\lambda_0}{1-s}}(\zeta), \beta_2^{\frac{\lambda_0}{1-s}}(\zeta), t^{\lambda_0}(\zeta) \right)$$

$$F_2 : (x_1, x_2, t^{\lambda_0}(\zeta)) \mapsto (x_1, x_2, (1-s)t^{\lambda_0}(\zeta) + st).$$

We have to verify that all is well-defined.

If we cannot define $\beta_1^{\frac{\lambda_0}{1-s}}(\zeta)$, then by Lemma 6.5 we either have $t \geq 1 - \frac{C_0(1-s)}{\lambda_0} \geq 1 - \frac{C_0}{\lambda_0}$ or we are on the first half of the punctured segment. By the same Lemma, we get $\varsigma_1(\Phi^{\lambda_0}(\zeta)) \geq \delta'$, that is $t^{\lambda_0}(\zeta) = 1$.

For the same reason, if $\beta_2^{\frac{\lambda_0}{1-s}}(\zeta)$ is not defined, then $t^{\lambda_0}(\zeta) = 0$, so $F_1 : \mathcal{X} \times [0, 1] \rightarrow \mathcal{A}_1 * \mathcal{A}_2$ makes sense. Its image is actually contained in \mathcal{X} because, from Lemma 6.5, if $x_1 = x_2$ and $\rho < 4\pi(\chi_1(x) + \chi_2(x))$, where we have set

$$\chi_i(\{x\}) := \begin{cases} 1 + \alpha_{i,j} & \text{if } x = p_j \\ 1 & \text{if } x \notin \{p_1, \dots, p_m\}. \end{cases}$$

Then either $t^{\lambda_0}(\zeta) \in \{0, 1\}$, hence in particular it does not equal $\frac{1}{2}$.

Concerning F_2 , the previous Lemma implies $\beta_1^{\frac{\lambda_0}{1-s}}(\zeta) = x_1$ if $t \leq 1 - \frac{C_0}{\lambda}(1-s)$, hence in particular passing to the limit as $s \rightarrow 1$, if $t < 1$. A similar condition holds for β_2 , which gives $F_2(\cdot, 0) = F_1(\cdot, 1)$. If x_1 is not defined then $t^{\lambda_0}(\zeta) = 1$, hence $(1-s)t^{\lambda_0}(\zeta) + st = 1$, and similarly there are no issues when x_2 cannot be defined. Finally, by the argument used before, if $x_1 = x_2 = p_j$ and $\rho_1, \rho_2 < 4\pi(2 + \alpha_{1j} + \alpha_{2j})$, then $(1-s)t^{\lambda_0}(\zeta) + st \neq \frac{1}{2}$. ■

Concerning compactness, we have a useful result which can be deduced from minor modifications of the argument in [50]. It basically states the existence of bounded Palais-Smale sequences for ρ belonging to a dense set of $\mathbb{R}_+^2 \setminus \Gamma$. Putting together with the compactness result stated before, we get:

Lemma 6.6. *Let $\rho \notin \Gamma$ be given and let $a < b$ be such that (16) has no solutions in $\{J_{\rho, \underline{\alpha}} \in [a, b]\}$. Then, $\{J_{\rho, \underline{\alpha}} \leq a\}$ is a deformation retract of $\{J_{\rho, \underline{\alpha}} \leq b\}$.*

We also deduce that $J_{\rho, \underline{\alpha}}$ is uniformly bounded from above on solutions, hence we have:

Corollary 6.7. *Let $\rho \notin \Gamma$ be given. Then, there exists $L > 0$ such that $\{J_{\rho, \underline{\alpha}} \leq L\}$ is a deformation retract of $H^1(\Sigma)^2$; in particular, it is contractible.*

PROOF OF THEOREM 1.4 Suppose by contradiction that the system (16) has no solutions. By Lemma 6.6, $\{J_{\rho,\alpha} \leq -L\}$ is a deformation retract of $\{J_{\rho,\alpha} \leq L\}$, hence by Corollary 6.7 it is contractible. Let $H(\zeta, s) : \mathcal{X} \times [0, 1] \rightarrow \mathcal{X}$ the homotopy equivalence defined in Lemma 6.4 and let H' be a homotopy equivalence between a constant map and $\text{Id}_{\{J_{\rho,\alpha} \leq -L\}}$.

Then $H''(\zeta, s) = \Psi(H'(\Phi(\zeta), s)) : \mathcal{X} \times [0, 1] \rightarrow \mathcal{X}$ is an equivalence between the maps $\Psi \circ \Phi$ and a constant and $H'' * H$ is an equivalence between $\text{Id}_{\mathcal{X}}$ and a constant map. This means that \mathcal{X} is contractible, contradicting Theorem 6.2. ■

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