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On the Regularity of Alexandrov Surfaces with Curvature Bounded Below

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Abstract: In this note, we prove that on a surface with Alexandrov's curvature bounded below, the distance derives from a Riemannian metric whose components, for any $p \in [1,2)$, locally belong to $W^{1,p}$ out of a discrete singular set. This result is based on Reshetnyak's work on the more general class of surfaces with bounded integral curvature.

Keywords: Alexandrov spaces; surfaces with bounded integral curvature; potential theory on surfaces

MSC: 53B21, 51F99, 31A05

1 Introduction

It is known that a finite dimensional Alexandrov space (X, d) can be equipped with a Riemannian metric defined almost everywhere that induces the original distance d. On the manifold part of the Alexandrov space, the Riemannian metric components are known to be BV_{loc} functions, as proved by Perelman in [12]. For geometric purposes it would be convenient to find an atlas where there is no Cantor part in the derivatives of the metric. Indeed, it is well-known that a continuous function f of a real variable and with bounded variation does not satisfy, in general, the fundamental theorem of calculus:

$$f(b) - f(a) = \int_a^b f'(s)\, ds,$$

while a Sobolev function does.

In this note, we show that an Alexandrov surface with a lower curvature bound admits a weak Riemannian metric with components in the Sobolev space.

Theorem 1.1. *Let (S, d) be a closed surface with curvature bounded below. Then S is a topological (smooth) surface and the distance d derives from a Riemannian metric g defined $\mathcal{H}^2$-almost everywhere. Furthermore, for all $p \in [1,2)$ there exists a discrete set $\mathfrak{S}_p \subset S$ such that the components of g, read in a local chart, belong to $W^{1,p}_{\mathrm{loc}}(S \setminus \mathfrak{S}_p, \mathcal{H}^2)$. Moreover,*

$$Reg(S) := \{x \in S;\, T_xS = \mathbb{R}^2\} \subset S \setminus \mathfrak{S}_p.$$

Last, there exists a well-defined Levi-Civita connection (acting on smooth vector fields) with components locally in L^p for any $p \in [1,2)$.

Remark 1.2. $\mathcal{H}^2$ stands for the 2-Hausdorff measure relative to d. In [11, 12], the weak Riemannian metric is defined at (and only at) each point of $Reg(S)$, S being a finite dimensional Alexandrov space.

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Remark 1.3. This result applies to convex surfaces in Euclidean space. Using the differential structure induced by the ambient space, it can be proved that the metric components are functions of locally bounded variation (indeed, the charts as well as their inverse functions are convex, their first derivatives are then in BV_{loc}), this estimate is sharp. Therefore, even in this simple case the above result leads to something new. By analogy one can think to the graph of the Cantor-Vitali function that, when viewed in a tilted system of coordinates, is the graph of a Lipschitz function; more generally this is true for the graph of any (multivalued) monotone operator in n variables, see [1].

Our proof strongly relies on Reshetnyak's work on the more general class of surfaces with *bounded integral curvature* [14]. Reshetnyak's work is recalled in the next section where we present all the necessary material to prove our result. The proof itself is given in the last part of this note.

2 Surfaces with bounded integral curvature

2.1 Definition and properties

Definition 2.1 (Upper angle and excess)**.** Let (S, d) be a complete geodesic space and let bac be a geodesic triangle. The upper angle $\overline{\angle} bac$ at a is defined as the upper limit of the angle $\angle_0 bac$ of a (Euclidean) *comparison triangle* (namely a Euclidean triangle whose sidelengths are the same as the ones of the geodesic triangle bac in S) when b, c go to a along the given geodesics. The excess of bac is then defined as

$$\delta(bac) = \overline{\angle} bac + \overline{\angle} abc + \overline{\angle} bca - \pi.$$

Definition 2.2 (Surface with bounded integral curvature)**.** Let (S, d) be a complete godesic space such that S is a closed topological surface. Then (S, d) is said to be a surface of bounded integral curvature if for any point in S there exists a neighborhood homeomorphic to a disc such that the sum of the excesses of finitely many non overlapping simple triangles is bounded from above by a constant depending only on the chosen neighborhood. A triangle is said to be simple if its boundary is made of three geodesic segments, it is convex with respect to its boundary and homeomorphic to a disc.

Remark 2.3. This definition is taken from [14].

Definition 2.4 (Curvature measure)**.** Let (S, d) a surface of bounded integral curvature. Then its curvature measure ω is defined as $\omega = \omega^+ - \omega^-$ where for any open set O,

$$\omega^+(O) = \sup_{T_i} \sum \max\{\delta(T_i), 0\}$$

where the supremum is taken over finite sums of simple triangles contained in O. Analogously

$$\omega^-(O) = \sup_{T_i} \sum -\min\{\delta(T_i), 0\}.$$

Then, for an arbitrary Borel set E,

$$\omega^\pm(E) := \inf_{O \supset E} \omega^\pm(O)$$

where the infimum is taken over open sets.

The above definition gives rise to a signed measure, as proved by Alexandrov and Zalgaller:

Theorem 2.5 ([2, Chapter 5])**.** *Let (S, d) be a surface with bounded integral curvature. Then the curvature measure ω defined in Definition 2.4 is the restriction to open sets of a signed Borel measure with locally finite total variation on S.*

2.2 Reshetnyak's subharmonic metric

Reshetnyak proved the existence of non-trivial coordinates on a surface with bounded integral curvature.

Theorem 2.6 ([14, Theorem 7.1.2]). *Let (S, d) be a closed surface of bounded integral curvature ω and let*

$$\Omega_1 = \left\{z \in S : \ \omega^+(\{z\}) < 2\pi\right\}. \tag{2.1}$$

Then, for all $z \in \Omega_1$ there exist a chart (U, ϕ) with $z \in U$ and a Riemannian metric g defined on $V = \phi(U) \subset \mathbb{R}^2$ by the formula

$$g(x_1, x_2) = \exp(-2u(x_1, x_2))\,(dx_1^2 + dx_2^2)$$

where $u = u_+ - u_-$ with $u_\pm \in L^1_{\rm loc}(V)$ pointwise defined and satisfying

$$\Delta u_\pm = \omega^\pm \qquad \text{in } V \tag{2.2}$$

in the weak sense.

Remark 2.7. In the above theorem, we adopt the convention $\Delta = \partial^2_{x_1x_1} + \partial^2_{x_2x_2}$. In particular, the functions $u_\pm$ are subharmonic. Let us recall that when $g_{ij} = \exp(-2u)\delta_{ij}$ with u smooth, then $\Delta u = K_g$, where K_g is the sectional curvature of g (see for instance [3]).

Proposition 2.8. *With the same hypotheses as in the above theorem, setting for $q \geq 4$*

$$\Omega_q = \left\{z \in S : \ \omega^+(\{z\}) < \frac{2\pi}{q}\right\},$$

and given $z \in \Omega_q$, we can choose (U, ϕ) in such a way that:

(a) the metric components g_{ij} and the volume form $\sqrt{\det(g)}$ belong to $L^q(V, dx_1dx_2)$;
(b) the distributional derivatives $\frac{\partial g_{ij}}{\partial x_k}$ of g belong to $L^p(V, \sqrt{\det(g)}\,dx_1dx_2)$ where $p = 2 - 6/(q+2)$.
(c) the Christoffel symbols Γ^k_{ij} belong to $L^p(V, \sqrt{\det(g)}\,dx_1dx_2)$ where $p = 2 - 2/q$.

The proof of the above proposition follows from some classical facts in potential theory that we briefly recall. First, the *logarithmic potential* v of a signed Borel measure μ with finite total variation in $\mathbb{R}^2$ is defined by the formula

$$v(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \ln|x - \xi|\,d\mu(\xi) \qquad x \in \mathbb{R}^2. \tag{2.3}$$

The logarithmic potential satisfies the following properties (see, for instance, [7, 8, Chap 3,4, and 16] for proofs).

Theorem 2.9. *Let μ be a signed Borel measure with finite total variation in $\mathbb{R}^2$ and let v be defined as in* (2.3). *Then*

(a) v is smooth in the complement of the support of μ;
(b) $v \in W^{1,p}_{\rm loc}(\mathbb{R}^2)$ for any $p \in [1, 2)$ if μ has compact support;
(c) one has

$$\Delta v = \mu \qquad \text{in the weak sense in } \mathbb{R}^2. \tag{2.4}$$

Notice that if μ is nonnegative and supported in $\overline{D}$ then v is pointwise defined in D, possibly equal to $-\infty$, with

$$v(x) \leq \frac{1}{2\pi}\ln(1 + |x|)\mu(\overline{D}) \qquad \forall x \in D. \tag{2.5}$$

The following more refined estimate is taken from [16, Corollary 4.3] (note that in this paper, the Laplacian operator as well as the logarithmic potential differ from ours by a sign).

Lemma 2.10. *Let μ be a nonnegative and finite Borel measure concentrated on D with $\mu(D) < 2\pi/q$ for some $q > 1$. Then the logarithmic potential v of μ satisfies*

$$\| \exp(-2v)\|_{L^q(D)} \le \left(\frac{2\pi}{\delta+2}2^{\delta+2}\right)^{1/q},$$

where $\delta = -q\mu(D)/\pi > -2$.

Building on these results, we can now prove the estimates on the metric components.

2.3 Proof of Proposition 2.8

We prove statements (b) and (c) for $q > 4$ and with the strict inequalities $p < 2 - 6/(q+2)$, $p < 2 - 2/q$ in (b) and (c) respectively. Then the case $q = 4$ and the cases when p reachs the extremal value can be proved using $\Omega_q = \cup_{q'>q}\Omega_{q'}$, hence any compact subset of Ω_q is contained in $\Omega_{q'}$ for some $q' > q$.

According to Theorem 2.6 we can work in local coordinates, assuming that $z = 0$ and that $g_{ij} = \exp(-2u)\delta_{ij}$ in the open unit Euclidean disc D. With the aim of proving (a) and (b), let us also note that $g_{ii} = \sqrt{\det(g)} = \exp(-2u)$.

Defining

$$h_\pm(x) := u_\pm(x) - \frac{1}{2\pi}\int_D \ln|x-\xi|\, d\omega^\pm(\xi),$$

it follows that $h_\pm$ are harmonic in the weak sense in D, thus smooth in (the open set) D. Therefore, in order to obtain local estimates in D on u, we need only to estimate the logarithmic potentials $v_\pm$ relative to $\omega^\pm$.

Under the further assumption

$$\omega^+(D) < \frac{2\pi}{q} \tag{2.6}$$

we get that $v_\pm$ are uniformly bounded from above in D, more precisely

$$v_\pm \le \frac{\ln 2}{2\pi}|\omega|(D). \tag{2.7}$$

Moreover, since we have assumed (2.6), Lemma 2.10 applies to v_+ yields $\exp(-2v_+) \in L^q(D)$. By combining the two estimates, we obtain that

$$\exp(-2u) \in L^q_{\mathrm{loc}}(D)$$

and (a) is proved.

To prove (b), we fix $p \in [1,2)$ and any $r > 1$ such that $pr < 2$. Then, we choose $q = (p+1)r/(r-1)$ and assume that (2.6) holds. By definition of the metric g, we have that $g_{ij} = e^{-2u}\delta_{ij}$ thus

$$\frac{\partial g_{ii}}{\partial x_k} = -2\frac{\partial u}{\partial x_k}e^{-2u}.$$

By the above discussion and Theorem 2.9 (b), we get the $W^{1,\alpha}_{\mathrm{loc}}(D)$ regularity of u for any $\alpha \in [1,2)$. Now, by integrating against the volume form (here sD denotes the concentric disk scaled by a factor $s > 0$), we get

$$\begin{aligned}\int_{sD}\left|\frac{\partial g_{ii}}{\partial x_k}\right|^p\sqrt{\det(g_{i,j})}\,dx &= 2^p\int_{sD}\left|\frac{\partial u}{\partial x_k}\right|^p e^{-2(p+1)u}\,dx\\ &\le 4\left\|\frac{\partial u}{\partial x_k}\right\|^p_{L^{rp}(sD)}\|\exp(-2(p+1)u)\|_{L^{r/(r-1)}(sD)}\\ &= 4\left\|\frac{\partial u}{\partial x_k}\right\|^p_{L^{rp}(sD)}\|\exp(-2u)\|^{p+1}_{L^q(sD)} < +\infty\end{aligned}$$

for all $s \in (0,1)$. To conclude the proof, notice that $r < 2/p$ yields $p < 2 - 6/(q+2)$.

Similarly,

$$\int_{sD}\left|\frac{\partial u}{\partial x_k}\right|^p\sqrt{\det(g_{i,j})}\,dx < +\infty \tag{2.8}$$

whenever $p < 2 - 2/q$.

It remains to prove the last item concerning the Christoffel symbols. Recall that for a smooth metric g such that $g_{ij} = \exp(-2u)\delta_{ij}$, it is a classical result that

$$D_{\frac{\partial}{\partial x_1}} \frac{\partial}{\partial x_2} = -\partial_{x_1} u \frac{\partial}{\partial x_2} - \partial_{x_2} u \frac{\partial}{\partial x_1} ,$$
$$D_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_i} = -\partial_{x_i} u \frac{\partial}{\partial x_i} + \partial_{x_{i+1}} u \frac{\partial}{\partial x_{i+1}} , \tag{2.9}$$

see for instance [15, p 399] or [3] for a general formula relating the Levi-Civita connections of two conformal metrics. This formula yields $|\Gamma_{ij}^k| = \left|\frac{\partial u}{\partial x_s}\right|$ with $s = k$ or $s = k+1$ depending on whether $i = j$ or not. In any case, if we *define* the Christoffel symbols using (2.9), the estimate (2.8) applies and completes the proof of (c).

We conclude this part by explaining why the Levi-Civita connection is well-defined as a tensor on S. Using (2.9) as a definition, it is simple to check that given two smooth vectors fields X, Y, the vector field $D_X Y$ satisfies the torsion free and the compatibility with the metric conditions *locally* when read in a chart. A result due to Huber (see [9] or [14, Theorem 7.1.3]) guarantees that any transition map relative to the subharmonic metric is a conformal map hence smooth. Therefore, the Levi-Civita connection is well-defined *globally* in Ω_q with components locally in L^p for $p = 2 - 2/q$. Note that the same global well-posedness holds for the subharmonic metric.

3 Surfaces with curvature bounded from below

In this part, we prove Theorem 1.1. Prior to that, we report Alexandrov's result concerning the curvature measure of a surface with curvature bounded below in Alexandrov's sense (see [4] or [10]). There are several equivalent definitions of Alexandrov space with curvature bounded below. We exhibit one such definition and refer to [5] for more on this subject.

Definition 3.1 (Surface with curvature bounded below)**.** Let (S, d) be a complete geodesic space whose Hausdorff dimension equals 2. Then (S, d) is a space with curvature bounded below by a number k if every point has a neighborhood U such that for any $a, b, c, d \in U$ the following inequality is satisfied

$$\angle_k bac + \angle_k cad + \angle_k dab \leq 2\pi,$$

where $\angle_k bac$ stands for the angle at a of a comparison triangle in the 2-dimensional space form of curvature k.

Theorem 3.2 ([4, 10])**.** *Let (S, d) be a surface with curvature bounded from below by $k \in \mathbb{R}$. Then, (S, d) has bounded integral curvature. Moreover, its curvature measure ω satisfies*

$$\omega = f d\mathcal{H}^2 + \omega^s$$

where ω^s, the singular part of ω w.r.t. $\mathcal{H}^2$, is a nonnegative and locally finite Borel measure and the density f of the absolutely continuous part of ω satisfies $f \geq k$ $\mathcal{H}^2$-almost everywhere in S.

The proof that a 2-dimensional Alexandrov space with curvature bounded below is a topological surface is usually attributed to Alexandrov (and it is not true in general in higher dimension). This property can also be deduced from results in [6] or in [13] where Perelman proves that any point x in an Alexandrov space admits a neighborhood which is homeomorphic to a neighborhood of the apex of the cone over the space of directions at x (indeeed, a closed 1-dimensional Alexandrov space is homeomorphic to a circle). The precise statement on the curvature measure given above is [10, Proposition 4.5].

Proof of Theorem 1.1. Thanks to Theorem 3.2 and the results in Section 2, there exists a subharmonic metric around any point of S such that $\omega(\{z\}) = \omega^+(\{z\}) < 2\pi$. Moreover, it is a general property of surfaces with bounded integral curvature that around such a point, the distance induced by the subharmonic metric locally coincides with the distance d, see [14, Paragraph 7] for more details.

Now, for any point z in a surface with curvature bounded below, $\omega(\{z\}) = 2\pi - L(\Sigma_z)$ (see for instance [10]) where $L(\Sigma_z)$ stands for the (positive) length of the space of directions at z which is a 1–dimensional space of curvature bounded below by 1. In particular, $\omega(\{z\}) = \omega^+(\{z\}) < 2\pi$ and therefore $\Omega_1 = S$. Moreover, the distance induced by the subharmonic metric *globally* coincides with d.

In this setting, the measure induced by the metric g coincides with the Hausdorff measure $\mathcal{H}^2$ (see [11] for a proof). Proposition 2.8 then yields the result with $\mathfrak{S}_p = S \setminus \Omega_q$, with q related to p by $p = 2 - 6/(q + 2)$.

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