

Quantum flags, and new bounds on the quantum capacity of the depolarizing channel

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A new bound for the quantum capacity of the d -dimensional depolarizing channels is presented. Our derivation makes use of a flagged extension of the map where the receiver obtains a copy of a state σ_0 whenever the messages are transmitted without errors, and a copy of a state σ_1 when instead the original state gets fully depolarized. By varying the overlap between the flags states, the resulting transformation nicely interpolates between the depolarizing map (when $\sigma_0 = \sigma_1$), and the d -dimensional erasure channel (when σ_0 and σ_1 have orthogonal support). In our analysis we compute the product-state classical capacity, the entanglement assisted capacity and, under degradability conditions, the quantum capacity of the flagged channel. From this last result we get the upper bound for the depolarizing channel, which by a direct comparison appears to be tighter than previous available results for $d > 2$, and for $d = 2$ it is tighter in an intermediate regime of noise. In particular, in the limit of large d values, our findings presents a previously unnoticed $\mathcal{O}(1)$ correction.

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Quantum Shannon theory [1, 2] provides a characterization of the maximum transmission rates (capacities) achievable in sending classical or quantum data through a quantum channel. The depolarizing channel (DC) [3] is the simplest and most symmetric non-unitary quantum channel but still, despite the considerable efforts that have been spent on such issue [4–16] its so-called quantum capacity [17–19] is not known. DCs have a peculiar position in the theory which make them an important error model for finite dimensional systems, like qubits in a quantum computer. Indeed by pre- and post-processing and classical communication via twirling [20], any other channel can be mapped into a DC whose quantum capacity is lower than or equal to the quantum capacity of the original channel [21]. Accordingly the value of the quantum capacity of DCs can be used to bound the minimum number of physical qubits needed to preserve quantum information in quantum processors and memories. In the view of these facts it is clear that the DC quantum capacity problem is of primary importance in quantum information theory: solving it would likely help in understanding the peculiar difficulties of quantum communication and error correction.

The evaluation of most capacities cannot be performed algorithmically, since it requires in principle an infinite sequence of optimizations, at variance with the classical case [22]. For a particular kind of channels, the degradable channels, the quantum capacity is given by the one-shot quantum capacity, which is a single-letter formula [23]. However, the DC is not degradable and the one-shot quantum capacity is known to be just a lower bound. The main result of this paper is a new analytic upper bound to the quantum capacity of the DC valid for any finite dimension, which outperforms previous results in many different regimes. To achieve this goal we rely on flagged extension of quantum channels, a construction which, in other contexts, proved to be a powerful tool,

see e.g. the result on the superadditivity of coherent information reported in Ref. [24]. In our case we define the flagged depolarizing channel (FDC) assuming that if Alice sends the density matrix ρ , with probability p Bob receives such state together with an ancillary system prepared into the state σ_0 , and with probability $1 - p$ the completely mixed state together with the ancillary system in σ_1 . The density matrices σ_0 and σ_1 behave as flags that encode information about what happened to the input and, at variance with previous approaches [4, 6, 9], are not assumed to be necessarily orthogonal – when this happens Bob can know exactly if he received the original message or an error, and our FDC is equivalent to the erasure channel [25]. By tracing out the flags, Bob effectively receives the output of a DC. This means that FDC is a better communication line than its associated DC, therefore every capacity of the former is larger than or equal to the corresponding value of the latter. Most importantly it is possible to find p, σ_0, σ_1 in such a way that the FDC is degradable obtaining a bound for the quantum capacity of the associated DC. When compared with previous results our findings provide a better estimate of the quantum capacity of the DC for all choices of d and p , except for $d = 2$. In this case the bounds in [7, 8] perform better at low noise, while for higher noise the new bound is better, surpassing also the one in [9] in an intermediate region. Most notably the improvement increases in the large d limit: the gap between the best upper bound and lower bound of the quantum capacity is given by a $\mathcal{O}(1)$ function of p which is differentiable in $p = 0$, in contrast with previous bounds for which the $\mathcal{O}(1)$ term of the gap is the binary entropy $h(p)$.

Preliminaries.– Given a finite dimensional Hilbert space \mathcal{H} , we write the space of linear operators on \mathcal{H} as $\mathcal{L}(\mathcal{H})$ and the set of density operators as $\mathfrak{S}(\mathcal{H})$. The action of a quantum channel $\Lambda : \mathcal{L}(\mathcal{H}_A) \rightarrow \mathcal{L}(\mathcal{H}_B)$ connecting two systems described by the Hilbert spaces \mathcal{H}_A and

\mathcal{H}_B , is a Completely Positive Trace Preserving (CPTP) map [1] on $\mathcal{L}(\mathcal{H}_A)$ which can always cast in the Stinespring representation form,

$$\Lambda(\theta) = \text{tr}_{E'}(U_{AE} \theta_A \otimes |e\rangle \langle e|_E U_{AE}^\dagger), \quad (1)$$

where $|e\rangle_E$ is the state of environment interacting with the system A , and U_{AE} is an unitary interaction acting on $\mathcal{H}_A \otimes \mathcal{H}_E \cong \mathcal{H}_B \otimes \mathcal{H}_{E'}$. In this setting the complementary channel $\tilde{\Lambda} : \mathcal{L}(\mathcal{H}_A) \rightarrow \mathcal{L}(\mathcal{H}_{E'})$ is defined as the CPTP mapping

$$\tilde{\Lambda}(\theta) := \text{tr}_B(U_{AE} \theta_A \otimes |e\rangle \langle e|_E U_{AE}^\dagger). \quad (2)$$

The channel Λ is said to be degradable if there exists a third CPTP channel $W : \mathcal{L}(\mathcal{H}_B) \rightarrow \mathcal{L}(\mathcal{H}_{E'})$ (dubbed degrading channel) such that $W \circ \Lambda = \tilde{\Lambda}$. Similarly, it is said to be anti-degradable if instead there exists a CPTP channel $V : \mathcal{L}(\mathcal{H}_{E'}) \rightarrow \mathcal{L}(\mathcal{H}_B)$ such that $V \circ \tilde{\Lambda} = \Lambda$. Finally we call N a degradable extension of Λ if N is degradable and there is a second channel R such that $R \circ N = \Lambda$.

The classical capacity $C(\Lambda)$ of Λ is the highest achievable rate at which classical data can be faithfully transmitted through such channel. Following [27, 28] it can be computed as $C(\Lambda) = \lim_{n \rightarrow \infty} C_n(\Lambda) = \lim_{n \rightarrow \infty} \frac{1}{n} \chi(\Lambda^{\otimes n})$, with $\chi(\Lambda) = \max_{\{p_i; \rho_i\}} \chi(\{p_i; \Lambda(\rho_i)\})$ where the Holevo quantity of an ensemble is defined as $\chi(\{p_i; \rho_i\}) := S(\sum_i p_i \rho_i) - \sum_i p_i S(\rho_i)$, $S(\rho)$ being the Von Neumann entropy. Similarly the entanglement assisted classical capacity $C_E(\Lambda)$ measures the highest rate at which the classical information can be transmitted through Λ when Alice and Bob share unlimited resource of entanglement. From [29] it follows that $C_E(\Lambda) = \max_{\rho} I(\rho, \Lambda)$, where the mutual information is defined as $I(\rho, \Lambda) := S(\rho) + S(\Lambda(\rho)) - S(\tilde{\Lambda}(\rho))$. Finally the quantum capacity $Q(\Lambda)$ gives the highest rate at which quantum information can be transmitted over many uses of Λ . In this case from [19, 30] we get $Q(\Lambda) = \lim_{n \rightarrow \infty} Q_n(\Lambda) = \lim_{n \rightarrow \infty} \max_{\rho \in \mathfrak{S}(\mathcal{H}_A^{\otimes n})} \frac{1}{n} J(\rho, \Lambda^{\otimes n})$, with $J(\rho, \Lambda^{\otimes n}) := S(\Lambda^{\otimes n}(\rho)) - S(\tilde{\Lambda}^{\otimes n}(\rho))$. For a degradable channel the regularization limit on n is not needed and the expression for $Q(\Lambda)$ reduces to single-letter formula

$$Q(\Lambda) = Q_1(\Lambda) := \max_{\rho \in \mathfrak{S}(\mathcal{H}_A)} J(\rho, \Lambda). \quad (3)$$

The FDC model.— In a standard approach to quantum communication the interaction between the quantum carriers of the information and their environment, the associated interaction time, as well as the state of environment are assumed to be known. However, it is possible to think about scenarios where the state of environment is changing in time and it can be monitored with quantum measurements. In this setting, suppose that with probability p_i the state of environment is the state σ_i , and that when this happens information carrier gets transformed by a a given CPT transformation

Λ_i . If there was no other information except the probability distribution of environment, the complete channel would be just the weighted sum of each individual map, i.e. $\Lambda := \sum_i p_i \Lambda_i$. Instead, we assume that in our case Bob collects a copy of the environment: in this case the complete channel can be written as

$$\mathbb{A}[\dots] := \sum_i p_i \Lambda_i[\dots] \otimes \sigma_i, \quad (4)$$

where now the σ_i s live on an ancillary space \mathcal{H}_1 on which Bob has complete access. More abstractly, this model can be also seen as a quantum channel with *quantum flags*, where with probability p_i the channel acts as Λ_i and Bob receives a quantum flag σ_i which encodes in a quantum state the information about which channel is acting. As Λ can be obtained from \mathbb{A} by simply tracing away the flags, it turns out that the capacities of the latter provide natural upper bounds for the corresponding ones of the former, i.e.

$$Q(\Lambda) \leq Q(\mathbb{A}), \quad (5)$$

where we specified this property in the case of the quantum capacity. A special example of a channel of the form (4) was considered in [4, 6] where the σ_i were assumed to be orthogonal pure states. Here, on the contrary we allow the σ_i 's to be mixed and not necessarily orthogonal and focus on the case where the resulting mapping has the form

$$\mathbb{A}_p^d[\dots] = (1-p)[\dots] \otimes \sigma_0 + p \text{Tr}[\dots] \frac{I^d}{d} \otimes \sigma_1. \quad (6)$$

This channel acts on a d dimensional Hilbert space and it can be expressed as in (4) with two components, the first associated with the identity channel and the second associated with a completely depolarizing transformation that replaces every input with the completely mixed state I^d/d . Notice however that Eq. (6) describes a proper CPTP mapping also for values of p larger than 1 – indeed its Choi state [1, 2] can be easily shown to be positive for any $p > 0$ such that $\frac{p}{d} \sigma_0 + (1-p) \sigma_1 \geq 0$. Most importantly, irrespectively from the value of σ_0 and σ_1 , by removing the flag states from (6) via partial trace reduces to a standard DC,

$$\Lambda_p^d[\dots] := (1-p)[\dots] + p \text{Tr}[\dots] \frac{I^d}{d}. \quad (7)$$

Therefore, invoking the monotonicity (5) we can upper bound the rather elusive quantum capacity of Λ_p^d , with the quantum capacity of \mathbb{A}_p^d which, as we shall see in the following section, that it is relatively easy to characterize.

FDC capacities.— A fundamental ingredient in studying the capacities of \mathbb{A}_p^d is that such channel is covariant under the action of arbitrary unitary transformations U of $SU(d)$, i.e. $\mathbb{A}_p^d[U \dots U^\dagger] = (U \otimes I) \mathbb{A}_p^d[\dots] (U^\dagger \otimes I)$, the operators I being the identity on the flags. This implies that the output von Neumann entropy associated

with a generic pure input state is a constant quantity $t(p, d, \sigma_0, \sigma_1)$ which does not explicitly depend upon the specific value of $|\psi\rangle$, but only upon the parameters that characterize the map i.e. $S(\Lambda_p^d[|\psi\rangle\langle\psi|]) = t(p, d, \sigma_0, \sigma_1)$. In the Supplemental Material (SM), using the concavity properties of $\chi(\{p_i, \rho_i\})$ and $I(\rho, \Lambda)$ [1], the product state classical capacity of the channel and the entanglement assisted capacity are shown to correspond to

$$\begin{aligned} C_1(\Lambda_p^d) &= \log d + S((1-p)\sigma_0 + p\sigma_1) - t(p, d, \sigma_0, \sigma_1), \\ C_E(\Lambda_p^d) &= 2 \log d + S((1-p)\sigma_0 + p\sigma_1) \\ &\quad - t(p, d^2, \sigma_0, \sigma_1). \end{aligned} \quad (8)$$

For finding the quantum capacity we restrict the problem to the case where $\sigma_1 = |e_1\rangle\langle e_1|$ is a pure state, and σ_0 is diagonalizable in that basis, i.e. $\sigma_0 = c^2 |e_1\rangle\langle e_1| + (1-c^2) |e_1^\perp\rangle\langle e_1^\perp|$. For this case both Λ_p^d and its complementary counterpart can be parametrised by the fidelity between σ_0 and σ_1 , i.e. via the parameter c (in particular we can write $\Lambda_{p,c}^d(\rho) := (1-p)\rho \otimes (c^2 |e_1\rangle\langle e_1| + (1-c^2) |e_1^\perp\rangle\langle e_1^\perp|) + p \frac{I^d}{d} \otimes |e_1\rangle\langle e_1|$). In the SM, using a simple measurement and action channel as a candidate for the degrading channel, we showed that $\Lambda_{p,c}^d$ is degradable for c fulfilling the inequality

$$c \leq c(p) := \sqrt{(1-2p)/(2-2p)}. \quad (9)$$

In this regime, the quantum capacity of $\Lambda_{p,c}^d$ is equal to the product state quantum capacity Q_1 i.e. $Q(\Lambda_{p,c}^d) = Q_1(\Lambda_{p,c}^d)$, and we should maximize the coherent information J to compute $Q_1(\Lambda_{p,c}^d)$. In general, maximizing the coherent information J is not an easy task: in our case however the problem however gets simplified again thanks to the degradability condition of the channel. When this property holds, in fact $J(\rho, \Lambda_{p,c}^d)$ is concave in the input state ρ [31], which by covariance of $\Lambda_{p,c}^d$ implies that the coherent information is maximized on the maximally mixed state

$$\begin{aligned} Q(\Lambda_{p,c}^d) &= Q_1(\Lambda_{p,c}^d) = \max_\rho J(\rho, \Lambda_{p,c}^d) = J\left(\frac{I^d}{d}, \Lambda_{p,c}^d\right) \\ &= \log d + S((1-p)\sigma_0 + p\sigma_1) - t(p, d^2, \sigma_0, \sigma_1). \end{aligned} \quad (10)$$

Upper bounds for the DC quantum capacity.— According to Eq. (5), the quantum capacity of the DC Λ_p^d can be upper bounded by the capacity of $\Lambda_{p,c}^d$, irrespectively from the choice we make on the parameter c , as long as the degradability constraint (9) holds true. Intuitively however, as c gets larger, the bound gets better, because channel (6) gets closer to Λ_p^d . To get the best upper bound for the quantum capacity of Λ_p^d we hence set $c = c(p)$. Accordingly, using the expression for $t(p, d^2, \sigma_0, \sigma_1)$ computed in the SM, our best way to upper bound $Q(\Lambda_p^d)$ is provided by

$$\begin{aligned} Q(\Lambda_p^d) &\leq Q(\Lambda_{p,c(p)}^d) = \log d + \eta\left(\frac{1}{2}\right) \\ &\quad - \eta\left(\frac{1}{2} - \frac{(d^2-1)p}{d^2}\right) - (d^2-1)\eta\left(\frac{p}{d^2}\right), \end{aligned} \quad (11)$$

where $\eta(z) := -z \log(z)$ (as discussed in the SM, an alternative bound can be obtained by choosing the flag states to be pure. The resulting expression is however much more involved than (11) and a numerical check reveals that it is less performing than the latter).

In order to test the quality of our finding we now proceed with a comparison with the limits previously proposed in the literature. We start considering first the low noise regime ($p \ll 1$) where (11) gives

$$Q(\Lambda_{p,c(p)}^d) = \log d + \frac{d^2-1}{d^2} (\log\left(\frac{p}{d^2}\right) - \log e + 1) p + O(p^2). \quad (12)$$

For $d = 2$, the above expression is less tight if compared with the bound of Ref. [8] which for this special regime implies

$$Q(\Lambda_p^2) \leq Q(\Lambda_{p,c(p)}^d) - \frac{3}{4}p + O(p^2 \log p). \quad (13)$$

Things however change when we move out from the $d = 2$, low noise regime. To our knowledge, there are two bounds obtained from the degradable extension of the d dimensional depolarizing channel. The first one is given in Ref. [6] and consists in the following expression

$$\begin{aligned} Q(\Lambda_p^d) &\leq f_{1,d}(p) := \eta\left(\frac{1+(d-1)\gamma}{d}\right) + (d-1)\eta\left(\frac{1-\gamma}{d}\right) \\ &\quad - \eta\left(1 - \frac{(d-1)\gamma}{d}\right) - (d-1)\eta\left(\frac{\gamma}{d}\right), \end{aligned} \quad (14)$$

with $\gamma = \frac{2d}{d^2-1} \left(\sqrt{1 - p \frac{d^2-1}{d^2}} - \left(1 - p \frac{d^2-1}{2}\right) \right)$. The second one was instead obtained by using the fact that Λ_p^d is degradable and anti-degradable when $p = \frac{d}{2(d+1)}$, see [6, 32]. Using this fact, [4, 6] showed we have

$$Q(\Lambda_p^d) \leq f_{2,d}(p) := \left(1 - \frac{2p(d+1)}{d}\right) \log d. \quad (15)$$

Given that all of these bounds, including our bound, are obtained from degradable extensions of DCs and the convexity of upper bounds obtained from degradable extensions [4], we can obtain the following upper bound (see the SM for the detailed proof)

$$Q(\Lambda_p^d) \leq \text{conv}\{Q(\Lambda_{p,c(p)}^d), f_{1,d}(p), f_{2,d}(p)\}, \quad (16)$$

where the convex hull $\text{conv}\{g_1(p), g_2(p), \dots\}$ is defined as the maximal convex function that is less than or equal to all the $g_i(p)$ s. Figure 2 compares the new bound with previous benchmarks for $d = 2$, [7, 8] for low noise and [9] for high noise, showing that the new bound is better in an intermediate regime. Figure 1 represents $Q(\Lambda_{p,c(p)}^d)$, $f_{1,d}(p)$, $f_{2,d}(p)$, and the convex hull for $d = 4$ and $d = 10$.

To be more quantitative, we can study the asymptotic expansion of the capacities of the various extensions for large d . Defining $\delta(p) := \eta\left(\frac{1}{2}\right) - \eta\left(\frac{1}{2} - p\right) + \eta(1-p)$ one

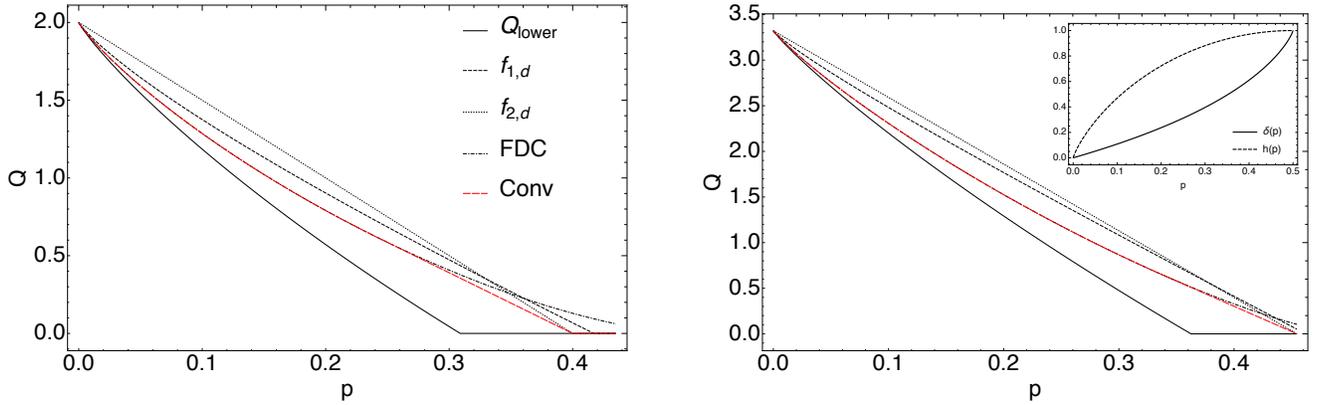


FIG. 1: $d = 4$ (left) and $d = 10$ (right): Q_{lower} is the lower bound from Eq. (18), $f_{1,d}$ and $f_{2,d}$ are previous bounds from degradable extensions (see Eq. (14) and (15)), FDC is our bound presented in Eq. (11), while finally Conv is the convex hull of all the bounds defined in Eq. 16. Inset: comparison for the $\mathcal{O}(1)$ gaps for large d between upper bounds and the hashing lower bound, as a function of p . For previous bounds the gap is $h(p)$, for ours it is given by the function (18).

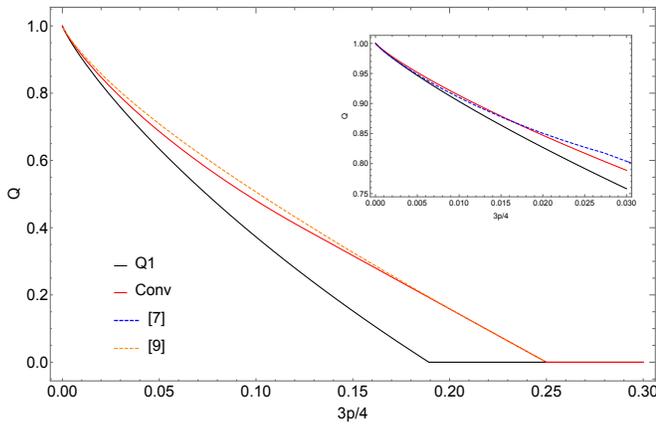


FIG. 2: $d = 2$ case: Q_1 is the lower bound from Eq. (18), Conv is the convex hull of all the bounds defined in Eq. 16. In the main plot, we compare our bound with [9], in the inset with [7] at low noise.

can show that

$$\begin{aligned} Q(\Lambda_{p,c(p)}^d) &= (1 - 2p) \log d - h(p) + \delta(p) + \mathcal{O}\left(\frac{1}{\log d}\right), \\ f_{1,d}(p) &= (1 - 2p) \log d + \mathcal{O}\left(\frac{\log d}{d}\right), \\ f_{2,d}(p) &= (1 - 2p) \log d + \mathcal{O}\left(\frac{\log d}{d}\right), \end{aligned}$$

which should be compared with the lower bound of $Q(\Lambda_p^d)$ one get by taking the value of the single shot coherent information evaluated on the completely mixed state, i.e.

$$\begin{aligned} Q(\Lambda_p^d) &\geq Q_{lower}(\Lambda_p^d) := J\left(\frac{I^d}{d}, \Lambda_p^d\right) \\ &= \log d - \eta\left(1 - p + \frac{p}{d^2}\right) - (d^2 - 1)\eta\left(\frac{p}{d^2}\right) \\ &= (1 - 2p) \log d - h(p) + \mathcal{O}\left(\frac{1}{\log d}\right). \end{aligned} \quad (17)$$

As we can see, our bound is the only one that shows an $\mathcal{O}(1)$ term which is not zero (and negative). Furthermore,

the gap between our bound and the lower bound scales as

$$Q(\Lambda_{p,c(p)}^d) - Q_{lower}(\Lambda_p^d) = \delta(p) + \mathcal{O}\left(\frac{1}{\log d}\right). \quad (18)$$

On the contrary the differences between the other upper bounds and the lower bound exhibit a $\mathcal{O}(1)$ gap equal to $h(p)$ which, as shown in Fig. 1 is larger than (18) for $p < \frac{1}{2}$ (where the quantum capacity is not zero). In particular, it appears that our inequality gives a much better bound for low p , since $h(p)$ has derivative that diverges as $-\log p$ when $p \rightarrow 0$, while $\delta(p)$ scales linearly in p .

Discussion.—We introduced a specific flagged version of DC which for a certain values of the parameter is degradable allowing us to compute analytic bound for the quantum capacity of the original map. Our result works in any dimension, and it is the tightest available analytical upper bound. Unlike other degradable extensions of depolarizing channel [4, 6], the introduced flags are not orthogonal. The idea we used is of general applicability and could give new good bounds for many other channels.

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Explicit value of $t(p, d, \sigma_0, \sigma_1)$

The fact that \mathbb{A}_p^d is covariant under $SU(d)$ implies that the output von Neumann entropy associated with a generic input state is a constant $t(p, d, \sigma_0, \sigma_1)$ that explicitly does not depend upon the specific value of $|\psi\rangle$ but only upon the parameters that characterize the map, i.e. p, σ_0, σ_1 and d . A simple algebra permits us to explicit determine the value of $t(p, d, \sigma_0, \sigma_1)$ obtaining

$$\begin{aligned} t(p, d, \sigma_0, \sigma_1) &:= S(\mathbb{A}_p^d[|\psi\rangle\langle\psi|]) = S\left((1-p)|\psi\rangle\langle\psi| \otimes \sigma_0 + p\frac{I^d}{d} \otimes \sigma_1\right) \\ &= h\left(\frac{d(1-p)+p}{d}\right) + \frac{p(d-1)}{d} \log(d-1) + \frac{d(1-p)+p}{d} S\left(\frac{(1-p)\sigma_0 + \frac{p}{d}\sigma_1}{\frac{d(1-p)+p}{d}}\right) + \frac{p(d-1)}{d} S(\sigma_1), \end{aligned} \quad (19)$$

where $h(x) := -x \log x - (1-x) \log(1-x)$ is the binary entropy.

Classical capacities of the FDC

By convexity of the von Neumann entropy, it follows that

$$\min_{\rho} S(\mathbb{A}_p^d[\rho]) = t(p, d, \sigma_0, \sigma_1). \quad (20)$$

Using above observation we compute the Holevo capacity of the map $C_1(\mathbb{A}_p^d)$. Notice that for any ensemble $\{p_i; \rho_i\}$, one can create a larger ensemble $\{p_i, dU; U\rho_i U^\dagger\}$, where the state $U\rho_i U^\dagger$ is extracted with probability density $p_i dU$, where dU is the Haar measure of $SU(d)$. By the concavity of the Holevo quantity it follows

$$\chi(\{p_i; \mathbb{A}_p^d[\rho_i]\}) \leq \chi(\{p_i, dU; \mathbb{A}_p^d[U\rho_i U^\dagger]\}) = \log d + S((1-p)\sigma_0 + p\sigma_1) - \sum_i p_i S(\mathbb{A}_p^d[\rho_i]), \quad (21)$$

where we used the depolarizing identity $\int dU U\rho U^\dagger = \frac{I^d}{d}$.

We can now invoke (20) to put an upper bound on $\chi(\{p_i, dU; \mathbb{A}_p^d[U\rho_i U^\dagger]\})$ by replacing all the $S(\mathbb{A}_p^d[\rho_i])$ terms with the constant $t(p, d, \sigma_0, \sigma_1)$. The resulting quantity no longer depends on the input of the channel and provide an

achievable maximum for the Holevo information of the channel yielding the identity

$$C_1(\mathbb{A}_p^d) = \log d + S((1-p)\sigma_0 + p\sigma_1) - t(p, d, \sigma_0, \sigma_1), \quad (22)$$

(the achievability being granted e.g. by ensembles of the form $\{dU; U|\psi\rangle\langle\psi|U^\dagger\}$, with $|\psi\rangle$ arbitrarily chosen).

To compute the entanglement assisted capacity of \mathbb{A}_p^d , we use the fact that the quantum mutual information of a channel is concave in ρ [1]. Exploiting this and the covariance of \mathbb{A}_p^d under $SU(d)$ we can then write

$$I\left(\frac{I^d}{d}, \mathbb{A}_p^d\right) = I\left(\int U\rho U^\dagger dU, \mathbb{A}_p^d\right) \geq \int I(U\rho U^\dagger, \mathbb{A}_p^d) dU = I(\rho, \mathbb{A}_p^d). \quad (23)$$

Therefore, we can conclude that the state that maximizes the quantum mutual information is $\frac{I}{d}$ and after some algebra we get

$$C_E(\mathbb{A}_p^d) = I\left(\frac{I^d}{d}, \mathbb{A}_p^d\right) = 2 \log d + S((1-p)\sigma_0 + p\sigma_1) - t(p, d^2, \sigma_0, \sigma_1), \quad (24)$$

Detailed analysis of degradability

To find complementary channel $\tilde{\mathbb{A}}_{p,c}^d$ we should first write the Stinespring [1, 2] form of $\mathbb{A}_{p,c}^d$ as it is discussed in Eq. (1). For this purpose we add extra degree of freedom extending the environment Hilbert space to $\mathcal{H}_E = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \otimes \mathcal{H}_4 \otimes \mathcal{H}_5$ where \mathcal{H}_1 , \mathcal{H}_2 and \mathcal{H}_5 are two dimensional, and \mathcal{H}_3 and \mathcal{H}_4 are d dimensional Hilbert spaces. Simple algebra can hence be used to verify that the Stinespring representation of the channel can be obtained through the following unitary interaction

$$U_{AE} |\psi\rangle_A |0\rangle_1 |0\rangle_2 |\Phi^d\rangle_{3,4} |0\rangle_5 = \sqrt{1-p} |\psi\rangle_A |\sigma_0\rangle_{1,2} |\Phi^d\rangle_{3,4} |0\rangle_5 + \sqrt{p} |\Phi^d\rangle_{A,4} |e_1\rangle_1 |e_1\rangle_2 |\psi\rangle_3 |1\rangle_5, \quad (25)$$

where $|0\rangle, |1\rangle$ are two orthogonal states, $|\Phi^d\rangle$ is a maximally entangled state in dimension d , and $|\sigma_0\rangle_{1,2}$ is a purification of σ_0 , and the trace in Eq. (1) is on labels 2,3,4,5 (see the next section in the SM for the details). To find the complementary channel instead of taking trace over states 2,3,4,5 we should take trace over states A,1. Carrying out the calculation we get

$$\begin{aligned} \tilde{\mathbb{A}}_{p,c}^d(|\psi\rangle\langle\psi|) = & (1-p)\sigma_{02} \otimes |\Phi^d\rangle\langle\Phi^d|_{3,4} \otimes |0\rangle\langle 0|_5 + p|e_1\rangle\langle e_1|_2 \otimes |\psi\rangle\langle\psi|_3 \otimes \frac{I^d}{d} \otimes |1\rangle\langle 1|_5 \\ & + \sqrt{p(1-p)} [c \text{tr}_A(|\psi\rangle\langle\psi|_A |\Phi^d\rangle_{3,4} |0\rangle_5 \langle\Phi^d|_{A,4} \langle\psi|_3 \langle 1|_5) + h.c.] \otimes |e_1\rangle\langle e_1|_2. \end{aligned} \quad (26)$$

We now look for the existence of a degrading CPTP channel $W_{p,c}$ connecting $\mathbb{A}_{p,c}^d$ and $\tilde{\mathbb{A}}_{p,c}^d$, i.e. satisfying the condition $W_{p,c} \circ \mathbb{A}_{p,c}^d = \tilde{\mathbb{A}}_{p,c}^d$ or explicitly

$$(1-p)W_{p,c}(\rho \otimes \sigma_{01}) + \frac{p}{d}W_{p,c}(I^d \otimes |e_1\rangle\langle e_1|_1) = \tilde{\mathbb{A}}_{p,c}^d(\rho). \quad (27)$$

As a suitable candidate for $W_{p,c}$ we consider a two-step process which first performs a measurement on system 1 that then triggers an action on A . Specifically for the measurement we assume an orthogonal projection in the basis $|e_1\rangle$ and $|e_1^\perp\rangle$. For the action on A instead we assume that if the measurement outcome is $|e_1\rangle$ we will prepare whatever state was left on A into the fixed state $|e_1^\perp\rangle\langle e_1^\perp|_2 \otimes |\Phi^d\rangle\langle\Phi^d|_{3,4} \otimes |0\rangle\langle 0|_5$; on the contrary, if the result is $|e_1^\perp\rangle$ we operate on A with a channel of the form $\tilde{\mathbb{A}}_{q,c'}^d$ with properly selected parameters q, c' . With this choice, the resulting mapping $W_{p,c}$ on $\rho_{A,1}$ is hence given by

$$W_{p,c}(\rho_{A,1}) := \langle e_1 | \text{tr}_A(\rho_{A,1}) | e_1 \rangle | e_1^\perp \rangle \langle e_1^\perp |_2 \otimes |\Phi^d\rangle\langle\Phi^d|_{3,4} \otimes |0\rangle\langle 0|_5 + \langle e_1^\perp | \text{tr}_A(\rho_{A,1}) | e_1^\perp \rangle \tilde{\mathbb{A}}_{q,c'}^d(\text{tr}_1(\rho_{A,1})). \quad (28)$$

With this choice the condition (27) becomes

$$(1-p)[c^2|e_1^\perp\rangle\langle e_1^\perp|_2 \otimes |\Phi^d\rangle\langle\Phi^d|_{3,4} \otimes |0\rangle\langle 0|_5 + (1-c^2)\tilde{\mathbb{A}}_{q,c'}^d(\rho_A)] + p|e_1^\perp\rangle\langle e_1^\perp|_2 \otimes |\Phi^d\rangle\langle\Phi^d|_{3,4} \otimes |0\rangle\langle 0|_5 = \tilde{\mathbb{A}}_{p,c}^d(\rho_A). \quad (29)$$

which can be satisfied if it is possible to find $q, c' \in [0, 1]$ such that

$$q = \frac{p}{(1-p)(1-c^2)}, \quad c'^2 = \frac{c^2(1-p)}{1-2p-c^2+pc^2}. \quad (30)$$

Doing simple algebra reveals that this is the case for all those cases where the following inequality holds,

$$c \leq \sqrt{\frac{1-2p}{2-2p}}. \quad (31)$$

Under this condition the channel $\mathbb{A}_{p,c}^d$ is degradable

Stinespring representation and complementary channel

Here we show that the mapping

$$U_{AE} |\psi\rangle_A |0\rangle_1 |0\rangle_2 |\Phi^d\rangle_{3,4} |0\rangle_5 = \sqrt{1-p} |\psi\rangle_A |\sigma_0\rangle_{1,2} |\Phi^d\rangle_{3,4} |0\rangle_5 + \sqrt{p} |\Phi^d\rangle_{A,4} |\sigma_1\rangle_{1,2} |\psi\rangle_3 |1\rangle_5, \quad (32)$$

provides a Stinespring representation of the channel $\Lambda_{p,\sigma_0,\sigma_1}^d$. For this purpose we first notice that (32) identifies a unitary transformation because in the domain where we have defined it does preserve the scalar product: indeed introducing the compact notation $|\psi, e\rangle_{AE} := |\psi\rangle_A |0\rangle_1 |0\rangle_2 |\Phi^d\rangle_{3,4} |0\rangle_5$ we have

$${}_{AE} \langle \phi, e | U_{AE}^\dagger U_{AE} |\psi, e\rangle_{AE} = (1-p) {}_A \langle \phi | \psi \rangle_A + p {}_3 \langle \phi | \psi \rangle_3 = {}_A \langle \phi | \psi \rangle_A = {}_{AE} \langle \phi, e | \psi, e \rangle_{AE}. \quad (33)$$

Next we notice that by tracing over 2, 3, 4, 5 we get

$$\text{tr}^{(A1)} \left[U_{AE} |\psi, e\rangle_{AE} \langle \psi, e | U_{AE}^\dagger \right] = (1-p) |\psi\rangle_A \langle \psi | \otimes \sigma_0 + p \frac{I^d}{d} \otimes \sigma_1 = \Lambda_{p,\sigma_0,\sigma_1}^d (|\psi\rangle_A \langle \psi |), \quad (34)$$

for all possible input state $|\psi\rangle_A$ (here $\text{tr}^{(A1)}$ indicates that we are taking the partial trace with respect to all degree of freedom of the system but $A, 1$).

From the above definition we now show that Eq. (26) is the complementary channel (2) of $\Lambda_{p,c}^d$, i.e. that the following identity holds true

$$\tilde{\Lambda}_{p,c}^d[\rho] = \text{tr}_{A1} \left[U_{AE} (\rho \otimes |e\rangle \langle e|_E) U_{AE}^\dagger \right], \quad (35)$$

for all input states ρ (here tr_{A1} indicates that the trace is taken on A and 1). Without loss of generality we can always focus on pure input states. Under this condition the right side of the previous expression yields

$$U_{AE} (|\psi\rangle_A \langle \psi | \otimes |e\rangle \langle e|_E) U_{AE}^\dagger = (1-p) |\psi\rangle \langle \psi|_A \otimes |\sigma_0\rangle \langle \sigma_0|_{1,2} \otimes |\Phi^d\rangle \langle \Phi^d|_{3,4} \otimes |0\rangle \langle 0|_5 \quad (36)$$

$$+ p |\Phi^d\rangle \langle \Phi^d|_{A,4} \otimes |e_1\rangle \langle e_1|_1 \otimes |e_1\rangle \langle e_1|_2 \otimes |\psi\rangle \langle \psi|_3 \otimes |1\rangle \langle 1|_5 \quad (37)$$

$$+ \sqrt{p(1-p)} |\psi\rangle_A |\Phi^d\rangle_{3,4} \langle \Phi^d|_{A,4} \langle \psi|_3 \otimes |\sigma_0\rangle_{1,2} \langle e_1|_1 \langle e_1|_2 \otimes |0\rangle \langle 0|_5 + h.c. \quad (38)$$

Given that $|\sigma_0\rangle_{1,2} = c |e_1\rangle |e_1\rangle_{1,2} + \sqrt{1-c^2} |e_1^\perp\rangle |e_1^\perp\rangle_{1,2}$ we can take trace over 2, 3, 4, 5 and get

$$\tilde{\Lambda}_{p,c}^d (|\psi\rangle_A \langle \psi |) = (1-p) \sigma_{02} \otimes |\Phi^d\rangle \langle \Phi^d|_{3,4} \otimes |0\rangle \langle 0|_5 + p |e_1\rangle \langle e_1|_2 \otimes |\psi\rangle \langle \psi|_3 \otimes \frac{I^d}{2} \otimes |1\rangle \langle 1|_5 \quad (39)$$

$$+ \sqrt{p(1-p)} [c \text{tr}_A (|\psi\rangle_A |\Phi^d\rangle_{3,4} |0\rangle_5 \langle \Phi^d|_{A,4} \langle \psi|_3 \langle 1|_5) + h.c.] \otimes |e_1\rangle \langle e_1|_2. \quad (40)$$

Pure flags expansion

The condition for degradability for the pure flags are similar to the case where the flags are mixed but diagonal. In this scenario the channel explicitly writes as

$$\Lambda_{p,c}^d[\dots] = (1-p) [\dots] \otimes |e_0\rangle \langle e_0| + p \frac{I^d}{d} \otimes |e_1\rangle \langle e_1|, \quad (41)$$

where the parameter c refers now to the overlap $c := \langle e_1 | e_0 \rangle$. Notice that the phase in c is not important in studying the degradability of $\Lambda_{p,c}^d$ since the phase in c can be set to zero by acting with a unitary transformation after the action of the channel (41): accordingly in the following we shall assume c to be real without loss of generality.

To find complementary channel $\Lambda_{p,c}^d$ we should first write the Stinespring form of this transformation. The Hilbert space of the environment is decomposed as $\mathcal{H}_E = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \otimes \mathcal{H}_4$ where \mathcal{H}_1 and \mathcal{H}_4 are two dimensional, and \mathcal{H}_2 and \mathcal{H}_3 are d dimensional Hilbert spaces. The unitary interaction between system and environment acts as following

$$U'_{AE} |\psi\rangle_A |0\rangle_1 |\Phi^d\rangle_{2,3} |0\rangle_4 = \sqrt{1-p} |\psi\rangle_A |e_0\rangle_1 |\Phi^d\rangle_{2,3} |0\rangle_4 + \sqrt{p} |\Phi^d\rangle_{A,3} |e_1\rangle_1 |\psi\rangle_2 |1\rangle_4, \quad (42)$$

where $|0\rangle, |1\rangle$ are two orthogonal states, $|\Phi^d\rangle$ is a maximally entangled states in dimension d , $|e\rangle_E = |0\rangle_1 |\Phi^d\rangle_{2,3} |0\rangle_4$, and the trace in Eq. (1) here is on states 2,3,4. Doing simple calculation we can show that this is a Stinespring

representation of 41. To find the complementary channel instead of taking trace over states 2,3,4 we should take trace over states A,1, carrying out the calculation we get

$$\tilde{\Lambda}_{p,c}^{\prime d}(|\psi\rangle\langle\psi|) = (1-p) |\Phi^d\rangle\langle\Phi^d|_{2,3} \otimes |0\rangle\langle 0|_4 + p |\psi\rangle\langle\psi|_2 \otimes \frac{I_3^d}{2} \otimes |1\rangle\langle 1|_4 \quad (43)$$

$$+ \sqrt{p(1-p)} [c \operatorname{tr}_A(|\psi\rangle\langle\psi|_A |\Phi^d\rangle\langle\Phi^d|_{2,3} |0\rangle\langle 0|_4 \langle\Phi^d|_{A,3} \langle\psi|_2 \langle 1|_4) + h.c.] . \quad (44)$$

As the form of $\tilde{\Lambda}_{p,c}^{\prime d}$ is exactly the same as $\tilde{\Lambda}_{p,c}^d$, the regime where $\Lambda_{p,c}^d$ is degradable is the same as before, i.e.

$$c^2 \leq \frac{1-2p}{2-2p} . \quad (45)$$

In this regime the quantum capacity of $\Lambda_{p,c}^d$ can be computed as in Eq. (10), i.e.

$$Q(\Lambda_{p,c}^d) = \log d + S((1-p) |e_0\rangle\langle e_0| + p |e_1\rangle\langle e_1|) - t(p, d^2, |e_0\rangle, |e_1\rangle) , \quad (46)$$

which after some algebra can be casted into the expression

$$\begin{aligned} Q(\Lambda_{p,c}^d) &= \log d + \eta\left[\frac{1}{2} \left(1 - \sqrt{-2(p-1)p \cos(\theta) + 2(p-1)p + 1}\right)\right] + \eta\left[\frac{1}{2} \left(1 + \sqrt{-2(p-1)p \cos(\theta) + 2(p-1)p + 1}\right)\right] \\ &\quad - \eta\left[\frac{d^2(-p)+d^2-\sqrt{d^4p^2-2d^4p+d^4-2d^2p^2 \cos(\theta)+2d^2p \cos(\theta)+p^2+p}}{2d^2}\right] \\ &\quad - \eta\left[\frac{d^2(-p)+d^2+\sqrt{d^4p^2-2d^4p+d^4-2d^2p^2 \cos(\theta)+2d^2p \cos(\theta)+p^2+p}}{2d^2}\right] + \frac{p(d^2-1)}{d^2} \log\left(\frac{p}{d^2}\right) , \end{aligned} \quad (47)$$

where $\cos(\theta) = 2c^2 - 1$ and $\eta(z) := -z \log(z)$.

Combination of different bounds from degradable extensions

In this section we present one of the results in Ref. [4]. We call N a degradable extension of Λ if N is degradable and there is a second channel R such that $R \circ N = \Lambda$. In Ref. [4] it has been shown that if N_0 is a degradable extension of Λ_0 and N_1 is a degradable extension of Λ_1 then $N = \lambda N_0 \otimes |0\rangle\langle 0| + (1-\lambda)N_1 \otimes |1\rangle\langle 1|$ is a degradable extension of $\Lambda = \lambda\Lambda_0 + (1-\lambda)\Lambda_1$ for every $0 \leq \lambda \leq 1$, and the quantum capacities satisfy the following relation

$$Q(\Lambda) \leq Q_1(N) \leq \lambda Q_1(N_0) + (1-\lambda)Q_1(N_1) . \quad (48)$$

This theorem can be used to show if we have upper bounds for the quantum capacity of two channels, all obtained from degradable extensions, the convex combination of the bounds is also an upper bound for the respective convex combination of the channels. We clarify this with an example: Consider the depolarizing channel i.e. $\Lambda_p^d[\dots] = (1-p)[\dots] + p \operatorname{Tr}[\dots] \frac{I^d}{d}$. The set of all values of p for which Λ_p^d is a CPTP is P , and N_p is a degradable extension of Λ_p^d for all $p \in P$. If $p_0, p_1 \in P$, then N_{p_0}, N_{p_1} are degradable extensions of $\Lambda_{p_0}^d, \Lambda_{p_1}^d$ respectively, then

$$Q(\Lambda_{p_0}^d) \leq g(p_0) := Q_1(N_{p_0}), \quad Q(\Lambda_{p_1}^d) \leq g(p_1) := Q_1(N_{p_1}) . \quad (49)$$

Therefore

$$N = \lambda N_{p_0} \otimes |0\rangle\langle 0| + (1-\lambda)N_{p_1} \otimes |1\rangle\langle 1| , \quad (50)$$

is a degradable extension of $\Lambda_{\lambda p_0 + (1-\lambda)p_1}^d$, then using 48 we get $Q(\Lambda_{\lambda p_0 + (1-\lambda)p_1}^d) \leq \lambda g(p_0) + (1-\lambda)g(p_1)$. As this holds for all $p_0, p_1 \in P$, therefore $\operatorname{conv}\{g(p)\}$ is also an upper bound for the quantum capacity of Λ_p^d , where

$$\operatorname{conv}\{g(p)\} := \inf_{\substack{p_0, p_1 \in P, \\ 0 \leq \lambda \leq 1}} \{\lambda g(p_0) + (1-\lambda)g(p_1) : p = \lambda p_0 + (1-\lambda)p_1\} .$$

In particular, given $g_1(p), \dots, g_n(p)$, all upper bounds for the quantum capacity of depolarizing channel all derived from degradable extensions, then $g_{\min}(p) := \min\{g_1(p), \dots, g_n(p)\}$ is also an upper bound and therefore $\operatorname{conv}\{g_1(p), \dots, g_n(p)\} := \operatorname{conv}\{g_{\min}(p)\}$, is also an upper bound too.
