WELL-POSEDNESS OF THE TWO-DIMENSIONAL NONLINEAR SCHRÖDINGER EQUATION WITH CONCENTRATED NONLINEARITY

RAFFAELE CARLONE, MICHELE CORREGGI, AND LORENZO TENTARELLI

Abstract. We consider a two-dimensional nonlinear Schrödinger equation with concentrated nonlinearity. In both the focusing and defocusing case we prove local well-posedness, i.e., existence and uniqueness of the solution for short times, as well as energy and mass conservation. In addition, we prove that this implies global existence in the defocusing case, irrespective of the power of the nonlinearity, while in the focusing case blowing-up solutions may arise.

Contents

1. Introduction and Main Results 1
1.1. The model 2
1.2. Local well-posedness and conservation laws 5
1.3. Global well-posedness and blow-up alternative 7
2. Proofs 8
2.1. Preliminary results 8
2.2. A derivation of the charge equation 17
2.3. Local well-posedness 19
2.4. Conservation laws 26
2.5. Global well-posedness and blow-up alternative 36
References 37

1. Introduction and Main Results

The nonlinear Schrödinger (NLS) equation plays a relevant role in several sectors of physics, where it appears very often as an effective evolution equation describing the behavior of a microscopic system on a macroscopic or mesoscopic scale. A typical example is provided by the time evolution of Bose-Einstein condensates, which is known to be well approximated by a NLS-type equation going under the name of Gross-Pitaevskii equation [DGPS]. There are however other examples in which the physical meaning of the NLS equation is totally different, as, e.g., the propagation of light in nonlinear optics, the behavior of water or plasma waves, the signal transmission through neurons (FitzHugh-Nagumo model), etc. (see, e.g., [Ma] and references therein).

Thanks to its physical relevance, the NLS equation has attracted a lot interest within the mathematical community as well, and several monographs are devoted to its detailed study (see, e.g., [C]). Here we focus on the simple but nontrivial case of a nonlinearity affecting the evolution only at finitely many points, i.e., a NLS equation with *concentrated nonlinearity*. Roughly speaking

Date: February 14, 2017.

the model we want to investigate is described by the two-dimensional formal equation

$$i\partial_t \psi_t = \Big(-\Delta + \sum_{j=1}^{N} \mu_j \delta(\mathbf{x} - \mathbf{y}_j) \Big) \psi_t, \tag{1.1}$$

where any coupling parameter $\mu_j = \mu_j(\psi_t(\mathbf{y}_j))$ depends itself on the value of the function ψ_t at $\mathbf{y}_j$ (see below).

Such a model has been used in physics to describe very different phenomena, mostly related to solid state physics: the charge accumulation in semiconductor interfaces or heterostructures can be modelled indeed by nonlinear effects concentrated in a small spatial region [BKB, J-LPC, J-LPS, MA, N]. The idea is that the nonlinear term takes into account the many-body interaction effects on the scattering of an electron through a barrier or by an impurity in the medium [MB]. In nonlinear optics similar models arise in the description of the nonlinear propagation in a Kerr-type medium in presence of localized defects [SKB, S *et al*, Y], but several other applications are suggested in acoustic, conventional and high-T_c superconductivity, light propagation in photonic crystals etc. (see [S *et al*] and references therein). More recently the nonlinear propagation in presence of a concentrated defect has been suggested as a dynamic model for the evolution of Bose-Einstein condensates in optical lattices, where the isolated defect is generated by a focused laser beam [DM, LKMF].

From the mathematical point of view, the expression between brackets in the above formula is purely formal, at least in two or more dimensions, and, in order to give it a rigorous meaning, one can follow different paths, as, e.g., classifying the self-adjoint extensions of suitable symmetric operators [AGH-KH] or investigating the properties of the associated quadratic forms [DFT1]. The reason why such models (a.k.a. *solvable models*), involving zero-range or point interactions, have attracted so much interest in the past is that the time evolution described by (1.1) can be simplified and in fact reduced to an ODE-type evolution of finitely many complex numbers named *charges* (see below), which are proportional to the values of ψ_t at the singular points. This was first observed in the corresponding time-dependent linear models [DFT2, SY] (see also [CD, CDFM, CCF, CCNP, NP] for similar results) and later used also in the nonlinear framework.

Analogous 1 and 3D models have indeed already been studied in details in the literature [A, AT, ADFT1, ADFT2]: it has been proven that the weak Cauchy problem associated to (1.1) in 1 or 3D (or rather to its rigorous analogue) admits a unique solution in the proper energy space for short times and that, under additional assumptions of the parameter (e.g., in the defocusing case), such a solution is in fact global in time (thanks to the mass and energy conservation). Further results about the possible emergence of blow-up solutions have also been established; so that the 1 and 3D models are basically completely understood. On the opposite, no results about the well-posedness (neither local nor global) of the 2D equation are available so far, mostly due to hard technical difficulties emerging in 2D (see the discussion at the end of next Sect.). It is also worth mentioning that the 1D and 3D analogues of the model above have been rigorously derived in [CFNT1, CFNT2] from the ordinary NLS equation in a suitable scaling limit of nonlinearity concentration.

1.1. **The model.** We specify now more precisely the model we want to investigate. We are interested in discussing a specific form of 2D NLS equation with concentrated nonlinearities at finitely many points $\mathbf{y}_1, \ldots, \mathbf{y}_N \in \mathbb{R}^2$, with $\mathbf{y}_i \neq \mathbf{y}_j$ for $i \neq j$. The precise definition of the model is similar to the 3D one, with some small but relevant modifications mostly due to the peculiar behavior of the 2D Green function (see below).

We start by recalling the properties of the linear version of the evolution problem (1.1), which has been studied in [CCF]: the idea is to reformulate (1.1) as the Schrödinger equation $i\partial_t \psi_t = H_{\boldsymbol{\alpha}(t)} \psi_t$ associated to a time-dependent Schrödinger operator $H_{\boldsymbol{\alpha}(t)}$ on $L^2(\mathbb{R}^2)$, defined as

$$\left(H_{\boldsymbol{\alpha}(t)} + \lambda\right) \psi = (-\Delta + \lambda)\, \phi_\lambda, \tag{1.2}$$

with domain

$$\mathscr{D}\left(H_{\boldsymbol{\alpha}(t)}\right) = \Big\{ \psi \in L^2(\mathbb{R}^2) \,\big|\, \psi = \phi_\lambda + \frac{1}{2\pi} \sum_{j=1}^N q_j(t) K_0\left(\sqrt{\lambda}|\mathbf{x} - \mathbf{y}_j|\right), \phi_\lambda \in H^2(\mathbb{R}^2),$$
$$\lim_{\mathbf{x} \to \mathbf{y}_j} \phi_\lambda(\mathbf{x}) = \left(\alpha_j(t) + \tfrac{1}{2\pi} \log \tfrac{\sqrt{\lambda}}{2} + \tfrac{\gamma}{2\pi}\right) q_j(t) - \frac{1}{2\pi} \sum_{k \neq j} K_0\left(\sqrt{\lambda}|\mathbf{y}_j - \mathbf{y}_k|\right) q_k(t) \Big\}, \tag{1.3}$$

where $\lambda > 0$, $K_0(\sqrt{\lambda}|\mathbf{x}|)$ denotes the inverse Fourier transform of $(|\mathbf{p}|^2 + \lambda)^{-1}$, i.e., the modified Bessel function of second kind of order 0 (a.k.a. Macdonald function [AS, Sect. 9.6]), γ is the Euler constant and the function $\boldsymbol{\alpha}(t) = (\alpha_1(t), \ldots, \alpha_N(t))$ is assumed to be of class C^1.

Wave functions in the operator domain are thus decomposable into a regular part ϕ_λ, belonging to the domain of the free Laplacian, plus a more singular term proportional to the Green function of $-\Delta + \lambda$, which shows logarithmic singularities at the points $\mathbf{y}_1, \ldots, \mathbf{y}_N$ [CCF]. The interaction is replaced with a boundary condition linking the values of the regular part ϕ_λ at points $\mathbf{y}_1, \ldots, \mathbf{y}_N$ to the coefficients of the singular one.

Remark 1.1 (Domain decomposition).
In the definition of the domain (1.3) a first difference with the 3D case emerges: the operator domain $\mathscr{D}(H_{\boldsymbol{\alpha}(t)})$ is obviously independent of the parameter λ, but, while in 3D one is allowed to take $\lambda = 0$ (with some little care about the large $|\mathbf{x}|$ decay), the same is not possible in 2D. Due to its infrared singularity, the 2D Green function actually diverges when $\lambda \to 0$ and therefore such a choice is forbidden.

The Cauchy problem for the linear evolution equation, i.e.,

$$\begin{cases} i\partial_t \psi_t = H_{\boldsymbol{\alpha}(t)} \psi_t, \\ \psi_{t=0} = \psi_0, \end{cases} \tag{1.4}$$

with $\psi_0 \in \mathscr{D}(H_{\boldsymbol{\alpha}(0)})$, was studied in [CCF], where it was proven that $H_{\boldsymbol{\alpha}(t)}$ generates a two-parameter unitary group $U(t,s)$ and therefore, if $\phi \in \mathscr{D}(H_{\boldsymbol{\alpha}(0)})$, then also $\psi_t \in \mathscr{D}(H_{\boldsymbol{\alpha}(t)})$ for any time $t \in \mathbb{R}$.

Equivalently one can consider the quadratic form $\mathcal{F}_{\boldsymbol{\alpha}(t)}$ associated to the operator $H_{\boldsymbol{\alpha}(t)}$,

$$\mathcal{F}_{\boldsymbol{\alpha}(t)}[\psi] = \int_{\mathbb{R}^2} \mathrm{d}\mathbf{x} \left\{ |\nabla \phi_\lambda|^2 + \lambda\, |\phi_\lambda|^2 - \lambda |\psi|^2 \right\} + \sum_{j=1}^N \left(\alpha_j(t) + \tfrac{1}{2\pi} \log \tfrac{\sqrt{\lambda}}{2} - \tfrac{\gamma}{2\pi} \right) |q_j|^2$$
$$+ \frac{1}{2\pi} \sum_{j \neq k} q_j^* q_k K_0(\sqrt{\lambda}\, |\mathbf{y}_j - \mathbf{y}_k|)$$

with time-independent domain

$$\mathscr{D}[\mathcal{F}] = \Big\{ \psi \in L^2(\mathbb{R}^2) \,\big|\, \psi = \phi_\lambda + \frac{1}{2\pi} \sum_{j=1}^N q_j K_0\left(\sqrt{\lambda}|\mathbf{x} - \mathbf{y}_j|\right), \phi_\lambda \in H^1(\mathbb{R}^2), q_j \in \mathbb{C} \Big\}, \tag{1.5}$$

and the weaker version of the Cauchy problem (1.4):

$$\begin{cases} i\partial_t \langle \chi | \psi_t \rangle = \mathcal{F}_{\boldsymbol{\alpha}(t)}[\chi, \psi_t], & \forall \chi \in \mathscr{D}[\mathcal{F}], \\ \psi_{t=0} = \psi_0, \end{cases} \tag{1.6}$$

where the initial datum ψ_0 also belongs to the form domain $\mathscr{D}[\mathcal{F}]$, $\langle \cdot | \cdot \rangle$ stands for the scalar product in $L^2(\mathbb{R}^2)$ and $\mathcal{F}_{\boldsymbol{\alpha}(t)}[\,\cdot\,,\,\cdot\,]$ is the sesquilinear form associated to $\mathcal{F}_{\boldsymbol{\alpha}(t)}$ defined, e.g., by polarization. The well-posedness of the above Cauchy problem is also proven in [CCF]. Note that, unlike the operator domain, functions in the form domain $\mathscr{D}[\mathcal{F}]$ do not have to satisfy any boundary condition.

A solution to both linear problems (1.4) and (1.6) (see [CCF, Sect. 2.2]) is provided by the following ansatz

$$\boxed{\psi_t(\mathbf{x}) = (U_0(t)\psi_0)(\mathbf{x}) + \frac{i}{2\pi} \sum_{j=1}^{N} \int_0^t \mathrm{d}\tau \, U_0(t-\tau; |\mathbf{x} - \mathbf{y}_j|) \, q_j(\tau),} \tag{1.7}$$

where $U_0(t) = e^{i\Delta t}$ denotes the free propagator, whose integral kernel is given by

$$U_0(t; |\mathbf{x}|) = \frac{e^{-\frac{|\mathbf{x}|^2}{4it}}}{2it}, \qquad t \in \mathbb{R}, \, \mathbf{x} \in \mathbb{R}^2,$$

and the function $\mathbf{q}(t) = (q_1(t), \ldots, q_N(t))$ is the solution of a Volterra-type equation of the form

$$q_j(t) + 4\pi \int_0^t \mathrm{d}\tau \, \mathcal{I}(t-\tau) \, \alpha_j(\tau) q_j(\tau) + \sum_{k=1}^{N} \int_0^t \mathrm{d}\tau \, \mathcal{K}_{jk}(t-\tau) \, q_k(\tau) = f_j(t) \tag{1.8}$$

(see below for more details).

The nonlinear model we plan to investigate in this article is the analogue of (1.4) (resp. (1.6)), where the parameters $\boldsymbol{\alpha}(t)$ depend themselves on the values of the charge $\mathbf{q}(t)$, i.e.,

$$\boxed{\alpha_j(t) = \beta_j \, |q_j(t)|^{2\sigma_j}, \qquad \beta_j \in \mathbb{R}, \sigma_j \in \mathbb{R}^+.} \tag{1.9}$$

Hence for any wave function in the nonlinear operator domain, the above nonlinearity can be translated into N nonlinear boundary conditions, i.e.,

$$\lim_{\mathbf{x} \to \mathbf{y}_j} \phi_\lambda(\mathbf{x}) = \left(\beta_j \, |q_j(t)|^{2\sigma_j} + \tfrac{1}{2\pi} \log \tfrac{\sqrt{\lambda}}{2} + \tfrac{\gamma}{2\pi} \right) q_j(t) - \frac{1}{2\pi} \sum_{k \neq j} K_0(\sqrt{\lambda} |\mathbf{y}_j - \mathbf{y}_k|) q_k(t). \tag{1.10}$$

We will show that a weak solution to the Cauchy problem (1.4) (i.e., a solution to (1.6)) with the *nonlinear* condition (1.9) is provided by the very same ansatz as in (1.7), i.e.,

$$\psi_t(\mathbf{x}) = (U_0(t)\psi_0)(\mathbf{x}) + \frac{i}{2\pi} \sum_{j=1}^{N} \int_0^t \mathrm{d}\tau \, U_0(t-\tau; |\mathbf{x} - \mathbf{y}_j|) \, q_j(\tau),$$

where $\mathbf{q}(t)$ is now the solution of the Volterra-type *nonlinear* equation

$$\boxed{\begin{aligned} q_j(t) &+ 4\pi\beta_j \int_0^t \mathrm{d}\tau \, \mathcal{I}(t-\tau) |q_j(\tau)|^{2\sigma_j} q_j(\tau) \\ &+ \sum_{k=1}^{N} \int_0^t \mathrm{d}\tau \, \mathcal{K}_{jk}(t-\tau) q_k(\tau) = 4\pi \int_0^t \mathrm{d}\tau \, \mathcal{I}(t-\tau) (U_0(\tau)\psi_0)(\mathbf{y}_j), \end{aligned}} \tag{1.11}$$

with $\mathcal{I}$ denoting the Volterra function of order -1

$$\mathcal{I}(t) := \int_0^\infty \mathrm{d}\tau \, \frac{t^{\tau-1}}{\Gamma(\tau)}, \tag{1.12}$$

and where $\mathcal{K}_{jk}$, $j,k = 1,\ldots,N$, is defined by

$$\mathcal{K}_{jk}(t) := \begin{cases} -2\left(\log 2 - \gamma + \frac{i\pi}{4}\right)\mathcal{I}(t), & \text{if } j = k, \\ -\mathcal{I}(t)\mathcal{R}_{jk}(t), & \text{if } j \neq k, \end{cases} \tag{1.13}$$

with

$$\mathcal{R}_{jk}(t) := \frac{t\, e^{\frac{i|\mathbf{y}_j-\mathbf{y}_k|^2}{4t}}}{\mathcal{I}(t)} \int_0^\infty \mathrm{d}v \, \frac{1}{1+vt}\mathcal{I}\left(\tfrac{vt^2}{1+vt}\right) \exp\left\{\tfrac{i|\mathbf{y}_j-\mathbf{y}_k|^2 v}{4}\right\}. \tag{1.14}$$

Notice that the choice of the initial time $t = 0$ is completely arbitrary: everything we prove in this paper holds as well if the initial time $t = 0$ is replaced with any $s \geqslant 0$.

For the sake of completeness we also formulate the weak counterpart of the evolution problem (1.7) and (1.11), which reads as follows: let the initial datum ψ_0 belong to the form domain $\mathscr{D}[\mathcal{F}]$, then

$$\boxed{\begin{cases} i\partial_t \langle \chi | \psi_t \rangle = \mathcal{F}_{\boldsymbol{\alpha}(t)}\left[\chi, \psi_t\right]\big|_{\left\{\alpha_j(t)=\beta_j|q_j(t)|^{2\sigma_j},\, j=1,\ldots,N\right\}}, \\ \psi_{t=0} = \psi_0, \end{cases}} \tag{1.15}$$

for any $\chi \in \mathscr{D}[\mathcal{F}]$.

The form of the Volterra equation (1.11) makes apparent a major difference with the 1 and 3D cases, which is also one of the main reasons why the 2D one called for a more refined analysis: the integral operator with kernel $\mathcal{I}(t-\tau)$ defined in (1.12) is a characteristic feature of the 2D problem and poses hard technical issues (see, e.g., [CFT]). In 3D (in 1D the equation is even simpler) the role of I is played by the Abel-1/2 integral operator with kernel $1/\sqrt{t-\tau}$, which enjoys a lot of useful regularizing properties, making the investigation of (1.11) much easier. In that case, by taking smooth enough initial data, the regularity easily propagates to $\mathbf{q}(t)$, so that the ansatz (1.7) belongs to the operator domain and therefore it provides strong solution to the Cauchy problem. The extension to rougher initial data is then obtained by density. In 2D already the first step, i.e., the regularity of $\mathbf{q}(t)$ for smooth initial data, is challenging and the whole proof strategy has to be dramatically changed (see Sect. 2).

Moreover, the lack of regularity of $\mathbf{q}(t)$ prevents the use of any density argument, which is precisely the route followed in 1 and 3D: indeed, it is impossible in 2D to restrict the set of initial data, prove the well-posedness and then extend the result to all initial data by density. On the opposite, our strategy relies on a contraction argument, which does not allow to propagate any additional regularity from the initial datum to $\mathbf{q}(t)$ (and then to ψ_t). In addition, the appropriate contraction space is $H^{1/2}(0,T)$, which is known to have a sort of pathological behavior, i.e., failure of the Hardy inequality, absence of natural extensions to $H^{1/2}(\mathbb{R})$ etc. (see below), and makes the technical side of the proof really tricky.

1.2. **Local well-posedness and conservation laws.** The first result we prove concerning the evolution problem described above is a local well-posedness for initial data in a suitable subset of the form domain that we define as follows (we set $p = |\mathbf{p}|$ for short)

$$\mathscr{D} := \left\{\psi \in \mathscr{D}[\mathcal{F}] \;\middle|\; (1+p^\epsilon)\ \widehat{\phi_\lambda}(\mathbf{p}) \in L^1(\mathbb{R}^2), \text{ for some } \epsilon > 0\right\}, \tag{1.16}$$

where $\widehat{\phi_\lambda}$ stands for the Fourier transform (see (2.2)) of the regular part ϕ_λ.

Theorem 1.1 (Local well-posedness).
Let $\psi_0 \in \mathscr{D}$ and $\sigma_j \geqslant \frac{1}{2}$, for any $j = 1, \ldots, N$. Then, there exists $T > 0$ such that there is a unique solution to (1.15) *belonging to $\mathscr{D}[\mathcal{F}]$ for any $t \leqslant T$ and it is given by* (1.7)*, with $\mathbf{q}(t)$ the unique solution to* (1.11)*.*

Remark 1.2 (Charge $\mathbf{q}(t)$).
The above Theorem contains in fact two results: the most important one is the local well-posedness of the weak Cauchy problem (1.15), but that result actually follows from the properties of the solution to the Volterra-type equation (1.11). In fact, once established the existence and uniqueness of $\mathbf{q}(t)$ in $C[0,T] \cap H^{1/2}(0,T)$ (see Propositions 2.2, 2.3 and 2.4), one can prove that such a regularity transfers to the wave function defined by (1.7) and then, thanks to the special form of (1.7), that ψ_t solves (1.6). It has to be stressed that the regularity of $\mathbf{q}$ is, in fact, borderline to make this argument work and a very fine analysis has to performed.

Remark 1.3 (Uniqueness of ψ_t).
One could think that the ansatz (1.7) might not be the unique solution of the weak problem (1.15). However, it is easy to see that this in not the case and ψ_t is in fact the unique solution of (1.15). Suppose that, for a given initial datum $\psi_0 \in \mathscr{D}$, there was another solution $\tilde{\psi}_t$. Then, by definition, it should decompose as

$$\tilde{\psi}_t = \tilde{\phi}_{\lambda,t} + \frac{1}{2\pi} \sum_{j=1}^{N} \tilde{q}_j(t) K_0 \left(\sqrt{\lambda} |\mathbf{x} - \mathbf{y}_j| \right),$$

for some bounded charge $\tilde{\mathbf{q}}(t)$ different from $\mathbf{q}(t)$. However, one could as well decompose $\tilde{\psi}_t$ as (see, e.g., Sect. 2.2)

$$\tilde{\psi}_t(\mathbf{x}) = \chi_{\lambda,t}(\mathbf{x}) + \frac{i}{2\pi} \sum_{j=1}^{N} \int_0^t \mathrm{d}\tau \, U_0\left(t - \tau; |\mathbf{x} - \mathbf{y}_j|\right) \tilde{q}_j(\tau),$$

for some function $\chi_{\lambda,t}$. Now, it is not difficult to see (Sect. 2.2) that this function can solve (1.15) if and only if $\chi_{\lambda,t} = U_0(t)\psi_0$ and $\tilde{\mathbf{q}}$ solves the charge equation (1.11). Uniqueness of the solution of (1.11) implies then the result. In fact, in Sect. 2.2 the previous argument is carried out in the case of strong solutions, following the original proof of [A] for the linear problem (and with the extra assumption $\mathbf{q}(0) = \mathbf{0}$). However, it is possible to prove that it can be adapted to weak solutions.

Remark 1.4 (Condition on the nonlinearity).
Although not so relevant for most physical applications, it is worth discussing briefly the role of the condition $\sigma_j \geqslant \frac{1}{2}$, for any j. There is no analogue of such a condition in the proof of local well-posedness for the 1D and 3D models. We believe it is only a technical assumption needed in a single step of the proof. More precisely it is due to the different strategy we have to follow in the first part of the proof, i.e., the contraction argument used in the analysis of the charge equation, which requires to assume $\sigma_j \geqslant \frac{1}{2}$ (see Lemma 2.1 and Remark 2.1). Obviously, the case $\sigma_j = 0$, $\forall j = 1, \ldots, N$, is also included, corresponding to the linear evolution problem studied in [CCF].

Remark 1.5 (Condition on the initial state).
We point out that the assumption on the initial state $\psi_0 \in \mathscr{D} \subsetneq \mathscr{D}[\mathcal{F}]$ is more restrictive then one would expect, since not only $\phi \in \mathscr{D}[\mathcal{F}]$ but also the Fourier transform of its regular part must be in $L^1(\mathbb{R}^2)$. This, for instance, ensures that the time-evolution of the regular part $U_0(t)\phi_{\lambda,0}$ is a continuous function, in order to be able to evaluate it at the singular points $\mathbf{y}_1, \ldots, \mathbf{y}_N$. The condition is deeply related to the lack of regularizing properties of the operator I and in this respect the choice of $\mathscr{D}$ is the most reasonable. In fact, the extra regularity assumed on the initial

state is also an important ingredient of a technical estimate proven in Lemma 2.8. No analogue of this condition is however present in the 1 and 3D cases and it might as well be that such an extra assumption is not needed for a weak solution.

In addition, we can claim a conservation result, that also plays a crucial role in the proof of the global existence of the solutions mentioned above:

Theorem 1.2 (Conservation laws).
Let $\psi_0 \in \mathscr{D}$, ψ_t be the wave function defined by (1.7) *and* (1.11) *and $T > 0$ the existence time provided by Theorem 1.1. Then, the mass $M(t) = \|\psi_t\|_2$ and the energy*

$$E(t) = \|\phi_{1,t}\|^2_{H^1(\mathbb{R}^2)} + \sum_{j=1}^{N} \left(\frac{\beta_j}{\sigma_j + 1} |q_j(t)|^{2\sigma_j} + \frac{\gamma - \log 2}{2\pi} \right) |q_j(t)|^2 - \frac{1}{\pi} \sum_{k>j=1}^{N} K_0(|\mathbf{y}_j - \mathbf{y}_k|) \operatorname{Re}\left(q_j^*(t) q_k(t)\right) \tag{1.17}$$

are conserved for every $t \in [0, T]$.

Remark 1.6 (Dependence on T).
We stress that, as it is, the conservation of the mass and the energy does not actually depend on T. We claimed that they are conserved quantities only for $t \in [0, T]$, since at this point we know that $\psi_t \in \mathscr{D}[\mathcal{F}]$ only for $t \in [0, T]$. However, it is clear that, as one proves that this is true for every $t \geqslant 0$, then one immediately extends the conservation to any $t \geqslant 0$.

Remark 1.7 (Choice of the spectral parameter λ).
The decomposition of functions in the form domain $\mathscr{D}[\mathcal{F}]$ defined in (1.5) depends on a spectral parameter $\lambda > 0$, although the domain itself is independent of λ. In (1.17) we have made the choice to pick $\lambda = 1$ (as suggested in [A]). It is worth recalling that this is an arbitrary choice and any other choice would imply an equivalent conservation law, but a different decomposition.

Remark 1.8 (Energy form).
Another difference between the 2D case and the 3D one is apparent in the form of the energy (1.17): instead of the L^2 norm of the gradient of the regular part of the wave function, (1.17) contains (first term) the full H^1 norm of $\phi_{1,t}$. This is again a consequence of the impossibility to choose $\lambda = 0$ as a spectral parameter in the form domain decomposition.

1.3. **Global well-posedness and blow-up alternative.** As we are going to see, the energy conservation is the key to prove the global well-posedness of the solution for $\beta_j > 0$, $j = 1, \ldots, N$. On the opposite, in the focusing case, i.e., if $\beta_j < 0$ for some $j = 1, \ldots, N$, the solution might be non-global due to a blow-up at finite time. It is important to remark that, unlike the NLS with concentrated nonlinearity in 3D, one expects that no critical power occurs in the 2D focusing case and hence, as soon as $\beta_j < 0$, a blow-up solution might show up [A]. We plan to deal with this question in a forthcoming paper.

Theorem 1.3 (Global well-posedness).
Let $\sigma_j \geqslant \frac{1}{2}$ and $\beta_j > 0$, $\forall j = 1, \ldots, N$. Then, the solution to (1.15) *provided by ψ_t defined by* (1.7) *and* (1.11) *is global in time, for any initial datum $\psi_0 \in \mathscr{D}$.*

As anticipated, in the focusing case we have a blow-up alternative:

Proposition 1.1 (Blow-up alternative).
Let $\sigma_j \geqslant \frac{1}{2}$, $\beta_j < 0$ for some $j = 1, \ldots, N$, and $\psi_0 \in \mathscr{D}$. Then, the solution to (1.15) *provided by ψ_t defined by* (1.7) *and* (1.11) *is either global in time or it blows-up in a finite time.*

Remark 1.9 (Behavior as $t \to +\infty$).
In fact the proofs of the Theorem 1.3 and Proposition 1.1 provide more information than what is contained in the statements. Indeed, while in the defocusing case $\beta_j > 0$, for any j, the charge $\mathbf{q}(t)$ is uniformly bounded, i.e., $\limsup_{t\to+\infty} |\mathbf{q}(t)| < +\infty$, in the focusing one, i.e., if some $\beta_j < 0$, the global existence of the solution does not imply its boundedness at ∞. More precisely it may happen that the maximal existence time for $\mathbf{q}(t)$ is $+\infty$ but $\limsup_{t\to+\infty} |\mathbf{q}(t)| = +\infty$.

Acknowledgements. The authors acknowledge the support of MIUR through the FIR grant 2013 "Condensed Matter in Mathematical Physics (Cond-Math)" (code RBFR13WAET). The authors also thank A. Fiorenza (Università "Federico II" di Napoli) and A. Teta (Università degli Studi di Roma "La Sapienza") for fruitful discussions about the topic of the paper.

2. Proofs

This Sect. is devoted to the proofs of our main results. We divide this section in five steps:

(i) we point out in Sect. 2.1 some relevant properties of Sobolev spaces of fractional index and of the integral operator I;
(ii) we present in Sect. 2.2 a justification for the ansatz (1.7) and the charge equation (1.11);
(iii) we prove existence, uniqueness and regularity of the solution of (1.11) and show how this allows to prove Theorem 1.1 (Sect. 2.3);
(iv) we prove in Sect. 2.4 mass and energy conservation (Theorem 1.2);
(v) we use the conservation laws of Sect. 2.4 to prove global existence and blow-up alternative (Theorem 1.3 and Proposition 1.1).

We stress that the proof strategy differs very much from the one followed in 1 or 3D. In those cases the core of the argument heavily relies on the regularizing properties of the Abel operator, which is involved in the integral version of the charge equation. Such an operator guarantees the minimal amount of regularity on $\mathbf{q}(t)$ needed to ensure that the ansatz ψ_t solves the weak problem (1.15), at least if the initial datum is regular enough. Unfortunately, the 2D analogue of the Abel operator is the integral operator I, which does not provide any improvement of regularity (see Lemma 2.4). Therefore the strategy itself of the proof needs to be modified: the required regularity of $\mathbf{q}(t)$ is indeed obtained by applying a suitable contraction argument to the charge equation. There are however some drawbacks in this approach, taking the form of additional conditions on the initial state, i.e., $\psi_0 \in \mathscr{D}$, and on the nonlinearity exponent, i.e., $\sigma_j \geqslant 1/2$.

2.1. Preliminary results.

We start by recalling briefly some facts on Sobolev spaces with fractional index. Let $-\infty \leqslant a < b \leqslant +\infty$ and $\nu \in (0,1)$, we denote by $H^\nu(a,b)$ the Sobolev space defined by

$$H^\nu(a,b) = \left\{ f \in L^2(a,b) \mid [f]^2_{\dot{H}^\nu(a,b)} < \infty \right\},$$

where

$$[f]^2_{\dot{H}^\nu(a,b)} := \int_{[a,b]^2} \mathrm{d}t\, \mathrm{d}\tau\, \frac{|f(t) - f(\tau)|^2}{|t-\tau|^{1+2\nu}}.$$

The space $H^\nu(a,b)$ is a Hilbert space with the natural norm

$$\|f\|^2_{H^\nu(a,b)} = \|f\|^2_{L^2(a,b)} + [f]^2_{\dot{H}^\nu(a,b)}. \tag{2.1}$$

When $a = -\infty$ and $b = +\infty$, $H^{\nu}(\mathbb{R})$ can be equivalently defined using the Fourier transform $\widehat{f}$ of f and, for any $f \in L^2(\mathbb{R}^d)$, we will use the following convention

$$\widehat{f}(\mathbf{p}) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \mathrm{d}\mathbf{t}\, e^{-i\mathbf{p}\cdot\mathbf{t}} f(\mathbf{t}). \tag{2.2}$$

Consistently, the convolution of two functions $f, g \in L^2(\mathbb{R}^d)$ is defined as

$$(f * g)(\mathbf{x}) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \mathrm{d}\mathbf{y}\, f(\mathbf{x}-\mathbf{y}) g(\mathbf{y}). \tag{2.3}$$

We start by discussing a technical point concerning the extension of functions in $H^{1/2}(0,T)$: it is known that if $f \in H^{\nu}(0,T)$, for $\nu < 1/2$ (also for $\nu > 1/2$ but with the additional assumption that $f(0) = f(T) = 0$), then $\mathbb{1}_{[0,T]}(t) f(t) \in H^{\nu}(\mathbb{R})$ (see, e.g., [CFNT2, Lemma 2.1]). However, the case $\nu = 1/2$ is very special and not included in the above result because the Hardy inequality, which is a key ingredient of the proof, fails in $H^{1/2}$ [KP]. In the Proposition below we show that if $f \in H^{1/2}(0,T)$ is continuous and satisfies an additional condition, then the extension to an $H^{1/2}$ function of the real line supported on a compact set is possible. We introduce an ad hoc space of continuous functions: for $\beta > 0$ we set

$$C_{\log,\beta}[0,T] := \left\{ f \in C[0,T] \,\middle|\, \exists C < +\infty, \forall t,s \in [0,T], |f(t) - f(s)| \leqslant C \left|\log|t-s|\right|^{-\beta} \right\}. \tag{2.4}$$

Hence, functions in $C_{\log,\beta}$ satisfies a sort of "weak" Hölder continuity condition, which is going to play a very important role in the Proposition below.

Proposition 2.1 (Extension of functions in $C_{\log,\beta}$).
Let $T > 0$ and $\beta > 1/2$, then for any $f(t) \in C_{\log,\beta}[0,T] \cap H^{1/2}(0,T)$ with $f(T) = 0$, the function

$$f_{\mathrm{e}}(t) := \begin{cases} f(t), & \text{if } t \in [0,T], \\ f(-t), & \text{if } t \in [-T,0], \\ 0, & \text{otherwise}, \end{cases} \tag{2.5}$$

belongs to $H^{1/2}(\mathbb{R})$.

Proof. The function f_{e} is obtained from f by reflecting it in an even way, so that $\mathrm{supp}(f_{\mathrm{e}}) = [-T,T]$. Of course $f_{\mathrm{e}}(t) = f(t)$ for $t \in [0,T]$ and

$$\|f_{\mathrm{e}}\|^2_{L^2(-T,T)} = 2\|f\|^2_{L^2(0,T)}, \qquad [f_{\mathrm{e}}]^2_{\dot{H}^{1/2}(-T,T)} \leqslant 4\,[f]^2_{\dot{H}^{1/2}(0,T)},$$

and therefore, if $f \in H^{1/2}(0,T)$, then $f_{\mathrm{e}} \in H^{1/2}(-T,T)$. Also $\|f_{\mathrm{e}}\|_{L^2(\mathbb{R})} = \|f_{\mathrm{e}}\|_{L^2(-T,T)}$. Now a simple computation yields

$$[f_{\mathrm{e}}]^2_{\dot{H}^{1/2}(\mathbb{R})} = [f_{\mathrm{e}}]^2_{\dot{H}^{1/2}(-T,T)} + 2\int_{-T}^{T} \mathrm{d}t \left(\frac{1}{t+T} + \frac{1}{T-t} \right) |f_{\mathrm{e}}(t)|^2, \tag{2.6}$$

and, if we show that the second term on the r.h.s. is finite, we complete the proof. A direct inspection of those integrals reveals that the integrand is an integrable function with possibly some singularity at the boundary of the domain, where we have to verify that it still is integrable. This request can be easily seen to be that

$$\frac{|f_{\mathrm{e}}(t)|^2}{T-t} = \frac{|f(t)|^2}{T-t}$$

is integrable at $t = T$. However, since by assumption $f(T) = 0$, the fact that $f \in C_{\log,\beta}[0,T]$ implies that, for t in a neighborhood of T,

$$|f(t)| \leqslant \frac{C}{|\log(T-t)|^{\beta}},$$

for some $\beta > 1/2$. Hence

$$\frac{|f_{\mathrm{e}}(t)|^2}{T-t} \leqslant \frac{C}{(T-t)|\log(T-t)|^{2\beta}},$$

which is integrable close to $t = T$. □

Another useful result we prove is about the Lipschitz continuity of the map $f \mapsto |f|^{2\sigma} f$ w.r.t. to the H^{ν} and L^{∞} norm. Such a result will play an important role when inspecting the regularity of the solution to the charge equation.

Lemma 2.1.
Let $\sigma \geqslant \frac{1}{2}$, $\nu \in [0,1]$ and T, $M > 0$. Assume also that f and g are functions satisfying

$$\|f\|_{L^{\infty}(0,T)} + \|f\|_{H^{\nu}(0,T)} \leqslant M, \qquad \|g\|_{L^{\infty}(0,T)} + \|g\|_{H^{\nu}(0,T)} \leqslant M. \tag{2.7}$$

Then, there exists a constant $C > 0$ independent of f, g, M and T, such that

$$\left\||f|^{2\sigma} f - |g|^{2\sigma} g\right\|_{L^{\infty}(0,T)} \leqslant C M^{2\sigma} \|f - g\|_{L^{\infty}(0,T)} \tag{2.8}$$

and

$$\left\||f|^{2\sigma} f - |g|^{2\sigma} g\right\|_{H^{\nu}(0,T)} \leqslant C \max\left\{1, \sqrt{T}\right\} M^{2\sigma} \|f - g\|_{H^{\nu}(0,T)}. \tag{2.9}$$

Proof. Let us first focus on (2.8): denote by $\varphi : \mathbb{C} \to \mathbb{C}$ the function $\varphi(z) = |z|^{2\sigma} z$. For $\sigma \geqslant \frac{1}{2}$, $\varphi \in C^2(\mathbb{R}^2; \mathbb{C})$, as a function of the real and imaginary parts of z. Moreover for $z_1, z_2 \in \mathbb{C}$,

$$\varphi(z_1) - \varphi(z_2) = (z_1 - z_2)\, \psi_1(z_1, z_2) + (z_2 - z_1)^* \, \psi_2(z_1, z_2), \tag{2.10}$$

with

$$\psi_1(z_1, z_2) = \int_0^1 \mathrm{d}s \, \partial_z \varphi(z_1 + s(z_2 - z_1)), \qquad \psi_2(z_1, z_2) = \int_0^1 \mathrm{d}s \, \partial_{z^*} \varphi(z_1 + s(z_2 - z_1)).$$

where $\partial_z \varphi = (\sigma + 1)|z|^{2\sigma}$ and $\partial_{z^*} \varphi = \sigma |z|^{2(\sigma - 1)} z^2$. Consequently,

$$|\psi_j(z_1, z_2)| \leqslant C \int_0^1 \mathrm{d}s \, |z_1 + s(z_2 - z_1)|^{2\sigma}, \quad j = 1, 2. \tag{2.11}$$

Thus,

$$|\varphi(z_1) - \varphi(z_2)| \leqslant C \max\{|z_1|, |z_2|\}^{2\sigma} |z_1 - z_2|$$

and, then, setting $z_1 = f(t)$ and $z_2 = g(t)$, (2.7) immediately entails (2.8).

Let us now consider (2.9). Setting again $z_1 = f(t)$ and $z_2 = g(t)$ in (2.10), we have that

$$\varphi(f(t)) - \varphi(g(t)) = (f(t) - g(t))\, \psi_1(f(t), g(t)) + (f(t) - g(t))^* \, \psi_2(f(t), g(t)).$$

For any pair of functions $f_1, f_2 \in H^{\nu}(0,T) \cap L^{\infty}(0,T)$

$$\|f_1 f_2\|_{H^{\nu}(0,T)} \leqslant \sqrt{2\|f_1\|_{L^{\infty}(0,T)} + 2\|f_2\|_{L^{\infty}(0,T)}} \; \|f_1 - f_2\|_{H^{\nu}(0,T)},$$

as it can be easily seen by exploiting (2.1). Hence, since by (2.7) and (2.11),

$$\|\psi_j(g(t), f(t))\|_{L^{\infty}(0,T)} \leqslant C M^{2\sigma}, \quad j = 1, 2,$$

then, denoting $\phi_j(t) := \psi_j(f(t), g(t))$ for short,

$$\begin{aligned}\|\varphi(f(t)) - \varphi(g(t))\|_{H^\nu(0,T)} &\leqslant \|\phi_1 \cdot (f-g)\|_{H^\nu(0,T)} + \|\phi_2 \cdot (f-g)\|_{H^\nu(0,T)} \\ &\leqslant C \max\{M, M^{2\sigma}\} \|f-g\|_{H^\nu(0,T)} \left[\|\phi_1\|_{H^\nu(0,T)} + \|\phi_2\|_{H^\nu(0,T)}\right]. \quad (2.12)\end{aligned}$$

Therefore it remains to verify that $\phi_j \in H^\nu(0,T)$ and estimate its norm: the L^2 norm of ϕ_j can be bounded as

$$\|\phi_j\|_{L^2(0,T)} \leqslant C\sqrt{T}\, M^{2\sigma}, \quad j = 1, 2. \quad (2.13)$$

Hence, it is left to prove that the semi-norms are also bounded. To this aim one notes that, for fixed $z_1, z_2, w_1, w_2 \in \mathbb{C}$, one can write

$$\psi_j(z_2, w_2) - \psi_j(z_1, w_1) = \psi_j(z_2, w_2) - \psi_j(z_2, w_1) + \psi_j(z_2, w_1) - \psi_j(z_1, w_1), \quad (2.14)$$

and, arguing as before,

$$\begin{aligned}\psi_j(z_2, w_2) - \psi_j(z_2, w_1) = (w_2 - w_1) &\int_0^1 \mathrm{d}s\, \partial_z \chi_j(w_1 + s(w_2 - w_1)) \\ &+ (w_2 - w_1)^* \int_0^1 \mathrm{d}s\, \partial_{z^*} \chi_j(w_1 + s(w_2 - w_1)), \quad (2.15)\end{aligned}$$

where we have set $\chi_j(z) := \psi_j(z_2, z)$. Similarly

$$\begin{aligned}\psi_j(z_2, w_1) - \psi_j(z_1, w_1) = (z_2 - z_1) &\int_0^1 \mathrm{d}s\, \partial_z \xi_j(z_1 + s(z_2 - z_1)) \\ &+ (z_2 - z_1)^* \int_0^1 \mathrm{d}s\, \partial_{z^*} \xi_j(z_1 + s(z_2 - z_1)), \quad (2.16)\end{aligned}$$

with $\xi_j(z) := \psi_j(z, w_1)$. Now, since

$$\chi_{1,2}(z) = \int_0^1 \mathrm{d}s\, \partial_{z/z^*}\varphi(z_2 + s(z - z_2)), \qquad \xi_{1,2}(z) = \int_0^1 \mathrm{d}s\, \partial_{z/z^*}\varphi(z + s(z - w_1)),$$

and

$$\begin{aligned}\partial_z^2 \varphi(z) &= \sigma(\sigma+1)\, |z|^{2(\sigma-1)}\, z^*, \\ \partial_z \partial_{z^*} \varphi(z) &= \sigma(\sigma+1)\, |z|^{2(\sigma-1)}\, z, \\ \partial_{z^*}^2 \varphi(z) &= \sigma(\sigma-1)\, |z|^{2(\sigma-2)}\, z^3,\end{aligned}$$

plugging (2.15) and (2.16) into (2.14), one sees that

$$|\psi_j(z_2, w_2) - \psi_j(z_1, w_1)| \leqslant C \max\{|z_1|, |z_2|, |w_1|, |w_2|\}^{2\sigma-1} \left(|z_2 - z_1| + |w_2 - w_1|\right), \quad (2.17)$$

which yields

$$[\phi_j]_{\dot{H}^\nu(0,T)} = [\psi_j(f(t), g(t))]_{\dot{H}^\nu(0,T)} \leqslant C M^{2\sigma-1} \left([f]_{\dot{H}^\nu(0,T)} + [g]_{\dot{H}^\nu(0,T)}\right) \leqslant C M^{2\sigma}.$$

Thus, combining with (2.13),

$$\|\psi_j(f(t), g(t))\|_{H^\nu(0,T)} \leqslant C \max\left\{1, \sqrt{T}\right\} M^{2\sigma},$$

so that (2.9) is proved. □

Remark 2.1. (Condition $\sigma \geqslant \frac{1}{2}$)
We stress that assuming $\sigma \geqslant \frac{1}{2}$ is crucial in the proof (2.9), in particular when assuming that $|z|^{2\sigma} z \in C^2(\mathbb{R}^2; \mathbb{C})$ or, equivalently, in assuring that the exponent $2\sigma - 1$ in (2.17) is positive and therefore the functions $f(t)$ and $g(t)$ can be replaced in the upper bound with their suprema.

On the other hand, (2.8) only requires $|z|^{2\sigma} z \in C^1(\mathbb{R}^2; \mathbb{C})$ and hence is valid for $\sigma \geqslant 0$. In fact the estimate (2.12) holds true for $\sigma \geqslant 0$ as well, but the stricter request $\sigma \geqslant \frac{1}{2}$ enters into the derivation of the bounds for ϕ_j, as explained above.

In the second part of the section, we investigate some properties of the integral operator $\mathcal{I}$ associated with the Volterra function of order -1, defined by (1.12), i.e.,

$$(If)(t) := \int_0^t \mathrm{d}\tau \, \mathcal{I}(t-\tau) f(\tau). \tag{2.18}$$

First, we recall some basic properties of $\mathcal{I}(t)$ (for further details we refer to [E1, Sec. 18.3], where $\mathcal{I}(t)$ is denoted by $\nu(t, -1)$). One striking relation involving $\mathcal{I}(t)$ is the inversion formula of the Laplace transform [E2, SKM]. Denoting by

$$(\mathcal{L}f)(p) = \int_0^\infty \mathrm{d}t \, e^{-pt} f(t)$$

the usual action of the Laplace transform, then

$$\mathcal{L}^{-1}\left(\frac{p}{\log(p)}\right)(t) = \mathcal{I}(t).$$

We also stress the asymptotic expansions of $\mathcal{I}(t)$ as $t \to 0$ and $t \to \infty$ (see again [E1]):

$$\mathcal{I}(t) \underset{t\to 0}{=} \frac{1}{t \log^2\left(\frac{1}{t}\right)} \left[1 + \mathcal{O}(|\log t|^{-1})\right], \tag{2.19}$$

$$\mathcal{I}(t) \underset{t\to\infty}{=} e^t + \mathcal{O}(t^{-1}).$$

Since $\mathcal{I}$ is continuous for $t > 0$ the previous expansions entail that

$$\mathcal{I}(t) \in L^1_{\mathrm{loc}}(\mathbb{R}^+) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^+ \backslash \{0\})$$

Furthermore, it is also worth to point out some features of the function $\mathcal{N}$, defined as

$$\mathcal{N}(t) := \int_0^t \mathrm{d}\tau \, \mathcal{I}(\tau). \tag{2.20}$$

Clearly the fact that $\mathcal{I}(t) \in L^1_{\mathrm{loc}}(\mathbb{R}^+)$ implies that the function $\mathcal{N}$ is absolutely continuous on any bounded interval $[0, T]$, $T > 0$, and $\mathcal{N}(0) = 0$. In addition, as $\mathcal{I}$ is strictly positive, $\mathcal{N}$ is strictly increasing on $[0, \infty)$ and the asymptotic expansion as $t \to 0$ is

$$\mathcal{N}(t) \underset{t\to 0}{=} \int_{-\infty}^{\log t} \mathrm{d}x \, \frac{1}{x^2}\left(1 + \mathcal{O}(x^{-1})\right) = \frac{1}{\log\left(\frac{1}{t}\right)} + \mathcal{O}(|\log t|^{-2}). \tag{2.21}$$

Another important property of $\mathcal{N}(t)$ is stated in next

Lemma 2.2.
Let $\mathcal{N}(t)$ be defined in (2.20)*, then, for any $T > 0$, $\mathcal{N}(t) \in H^\nu(0, T)$, $\forall \nu \in \left[0, \frac{1}{2}\right]$, and*

$$\lim_{T\to 0} \|\mathcal{N}\|_{H^\nu(0,T)} = 0. \tag{2.22}$$

Proof. The absolute continuity of $\mathcal{N}(t)$ in the interval $[0,T]$ implies that $\mathcal{N} \in L^2(0,T)$, for any finite T. Consequently it is left to prove that the seminorm $[\mathcal{N}]_{\dot{H}^{1/2}(0,T)}$ in bounded. An easy computation shows that

$$[\mathcal{N}]^2_{\dot{H}^{1/2}(0,T)} = 2\int_0^T \mathrm{d}t \int_0^{\frac{t}{2}} \mathrm{d}s \left|\frac{\mathcal{N}(t)-\mathcal{N}(s)}{t-s}\right|^2 + 2\int_0^T \mathrm{d}t \int_{\frac{t}{2}}^t \mathrm{d}s \left|\frac{\mathcal{N}(t)-\mathcal{N}(s)}{t-s}\right|^2.$$

Looking at the first integral and recalling that $\mathcal{N}$ is increasing, we find

$$\left|\frac{\mathcal{N}(t)-\mathcal{N}(s)}{t-s}\right|^2 \leqslant 4\,\frac{\mathcal{N}^2(t)}{t^2}, \quad \forall s \in \left(0, \tfrac{t}{2}\right).$$

Hence,

$$\int_0^T \mathrm{d}t \int_0^{\frac{t}{2}} \mathrm{d}s \left|\frac{\mathcal{N}(t)-\mathcal{N}(s)}{t-s}\right|^2 \leqslant 2\int_0^T \mathrm{d}t\, \frac{\mathcal{N}^2(t)}{t} < \infty,$$

since, by (2.21), $\frac{\mathcal{N}^2(t)}{t} \sim \mathcal{I}(t)$, when $t \to 0$, and thus is integrable over $[0,T]$, for any T finite.

Applying Cauchy inequality to the second integral, we get

$$\int_0^T \mathrm{d}t \int_{\frac{t}{2}}^t \mathrm{d}s \left|\frac{\mathcal{N}(t)-\mathcal{N}(s)}{t-s}\right|^2 = \int_0^T \mathrm{d}t \int_{\frac{t}{2}}^t \mathrm{d}s \left|\frac{1}{t-s}\int_s^t \mathrm{d}\tau\, \mathcal{I}(\tau)\right|^2$$

$$\leqslant \int_0^T \mathrm{d}t \int_{\frac{t}{2}}^t \mathrm{d}s\, \frac{1}{t-s}\int_s^t \mathrm{d}\tau\, \mathcal{I}^2(\tau).$$

Furthermore since $\mathcal{I}$ is positive and convex (see [CFT]), it is $\mathcal{I}^2(\tau) \leqslant \mathcal{I}^2(t) + \mathcal{I}^2(s)$ for every $\tau \in [s,t]$, so that

$$\int_0^T \mathrm{d}t \int_{\frac{t}{2}}^t \mathrm{d}s\, \frac{1}{t-s}\int_s^t \mathrm{d}\tau\, \mathcal{I}^2(\tau) \leqslant \int_0^T \mathrm{d}t \int_{\frac{t}{2}}^t \mathrm{d}s\, (\mathcal{I}^2(t) + \mathcal{I}^2(s)).$$

Now, noting that $\log^{-4}(1/s) \leqslant \log^{-4}(1/t)$ for all $s \in (t/2, t)$ and using again (2.19),

$$\int_0^T \mathrm{d}t \int_{\frac{t}{2}}^t \mathrm{d}s\, \mathcal{I}^2(s) \sim \int_0^T \mathrm{d}t \int_{\frac{t}{2}}^t \mathrm{d}s\, \frac{1}{s^2 \log^4(\frac{1}{s})} \leqslant C\int_0^T \mathrm{d}t\, \frac{1}{t\log^4(\frac{1}{t})} < \infty,$$

whereas, on the other hand,

$$\int_0^T \mathrm{d}t \int_{\frac{t}{2}}^t \mathrm{d}s\, \mathcal{I}^2(t) \leqslant C\int_0^T \mathrm{d}t\, t\mathcal{I}^2(t) \sim C\int_0^T \mathrm{d}t\, \frac{1}{t\log^4(\frac{1}{t})} < \infty.$$

Thus

$$\int_0^T \mathrm{d}t \int_{\frac{t}{2}}^t \mathrm{d}s \left|\frac{\mathcal{N}(t)-\mathcal{N}(s)}{t-s}\right|^2 < \infty.$$

In conclusion we proved that $[\mathcal{N}]_{\dot{H}^{1/2}(0,T)} < \infty$. The same inequalities also imply (2.22). □

Corollary 2.1.
Let $\mathcal{R}_{jk}$ be defined in (1.14) *and set*

$$\widetilde{\mathcal{N}}_{jk}(t) := \int_0^t \mathrm{d}\tau\, \mathcal{I}(\tau)\mathcal{R}_{jk}(\tau), \tag{2.23}$$

then $\widetilde{\mathcal{N}}_{jk}(t) \in H^\nu(0,T)$, $\forall \nu \in \left[0, \frac{1}{2}\right]$ and $j,k = 1,\dots,N$, $j \neq k$.

Proof. The proof is analogue to the proof of Lemma 2.2 and we omit for the sake of brevity. We point out that boundedness and continuity of $\mathcal{R}_{jk}(t)$ as a function of $t \geqslant 0$ play an important role as well as the fact that $\mathcal{R}_{jk}(t) \to 0$, as $t \to 0$. All these properties can however be proven by direct inspection of the explicit expression (1.14). □

In [CFT] the operator I is investigated in details and several useful properties are established. Here, we only show the most relevant ones for our application (we also mention some proofs for the sake of completeness).

Lemma 2.3.
Let $T > 0$ and $f \in L^\infty(0,T)$, then $If \in C[0,T]$ and

$$\|If\|_{L^\infty(0,T)} \leqslant C_T \|f\|_{L^\infty(0,T)}, \tag{2.24}$$

with $C_T > 0$ independent of f and such that $C_T \xrightarrow[T\to 0]{} 0$.

Proof. Recalling (2.18) and (2.21), (2.24) is immediate. Then, it is left to prove that If is continuous. To this aim, fix $t_0 \in [0,T)$ and $t \in (t_0, T]$. Easy computations yield

$$If(t) - If(t_0) = \int_0^T \mathrm{d}\tau\, \mathcal{I}(t-\tau)\mathbb{1}_{[t_0,t]}(\tau) f(\tau) - \int_0^T \mathrm{d}\tau\, \left(\mathcal{I}(t_0-\tau) - \mathcal{I}(t-\tau)\right)\mathbb{1}_{[0,t_0]}(\tau) f(\tau)$$

and therefore

$$\begin{aligned} |If(t) - If(t_0)| &\leqslant \int_0^T \mathrm{d}\tau\, \mathcal{I}(t-\tau)\mathbb{1}_{[t_0,t]}(\tau)|f(\tau)| + \int_0^T \mathrm{d}\tau\, \left|\mathcal{I}(t_0-\tau) - \mathcal{I}(t-\tau)\right|\mathbb{1}_{[0,t_0]}(\tau)|f(\tau)| \\ &\leqslant \mathcal{N}(t-t_0)\|f\|_{L^\infty(0,T)} + \|f\|_{L^\infty(0,T)} \int_0^{t_0} \mathrm{d}\tau\, \left|\mathcal{I}(\tau) - \mathcal{I}(t-t_0+\tau)\right|, \end{aligned} \tag{2.25}$$

Therefore, the first term converges to zero by the continuity of $\mathcal{N}$, while the second one tends to zero by dominated convergence. Indeed, it suffices to bound from above the integrand in the second term by an integrable function independent of t. Since t varies in a bounded set and $\mathcal{I}(t)$ is bounded for $t > 0$ finite, we have

$$|\mathcal{I}(\tau) - \mathcal{I}(t-t_0+\tau)| \leqslant \mathcal{I}(\tau) + \sup_{\delta\in[0,T]} \mathcal{I}(\tau+\delta),$$

and the r.h.s. is integrable for $\tau \in [0, t_0]$.

Since the same holds if $t < t_0$, one has that $If(t) \to If(t_0)$ as $t \to t_0$, which concludes the proof. □

Lemma 2.4.
Let $f \in H^{1/2}(0,T) \cap L^\infty(0,T)$, $T > 0$, then $If \in H^{1/2}(0,T)$ and, in particular, there exists $C_T > 0$ independent of f and satisfying $C_T \xrightarrow[T\to 0]{} 0$, such that

$$\|If\|_{H^{1/2}(0,T)} \leqslant C_T \left(\|f\|_{L^\infty(0,T)} + \|f\|_{H^{1/2}(0,T)} \right). \tag{2.26}$$

Proof. Let us divide the proof in two parts: we first estimate the L^2 norm of If and then the semi-norm $[If]_{H^{1/2}(0,T)}$.

Let $T > 0$ be finite and $f \in H^{1/2}(0,T) \cap L^\infty(0,T)$. In order to extend the operator I to an operator on the line, we set $f_e(t) := \mathbb{1}_{[0,T]}(t) f(t)$ and define

$$(I_e f)(t) := \int_0^t \mathrm{d}\tau\, \mathcal{I}_e(t-\tau) f_e(\tau), \qquad t \in \mathbb{R},$$

where

$$\mathcal{I}_e(t) := \mathbb{1}_{[0,T]}(t)\mathcal{I}(t). \tag{2.27}$$

Since $(I_e f)(t) = (If)(t)$ for all $t \in [0,T]$,

$$\|If\|_{L^2(0,T)} = \|I_e f\|_{L^2(0,T)} \leqslant \|I_e f\|_{L^2(\mathbb{R})}. \tag{2.28}$$

Now, applying the Fourier transform on $\mathbb{R}$ to $I_e f$ and using the identity

$$\mathbb{1}_{[0,t]}(\tau) = \mathbb{1}_{\mathbb{R}^+}(\tau) - \mathbb{1}_{\mathbb{R}^+}(\tau - t),$$

one gets

$$\widehat{I_e f} = \widehat{\mathcal{I}_e * (\mathbb{1}_{\mathbb{R}^+} f_e)} - \widehat{(\mathbb{1}_{\mathbb{R}^-}\mathcal{I}_e) * f_e} = \widehat{\mathcal{I}}_e \widehat{\mathbb{1}_{\mathbb{R}^+} f_e} - \widehat{\mathbb{1}_{\mathbb{R}^-}\mathcal{I}_e}\widehat{f}_e = \widehat{\mathcal{I}}_e \widehat{f}_e,$$

since by construction $\mathbb{1}_{\mathbb{R}^+}(t) f_e(t) = f_e(t)$ and $\mathbb{1}_{\mathbb{R}^-}(t)\mathcal{I}_e(t) = 0$. Hence by (2.28) and the above identity

$$\|If\|^2_{L^2(0,T)} \leqslant \int_{\mathbb{R}} \mathrm{d}k \, \big|\widehat{\mathcal{I}}_e(k)\big|^2 \big|\widehat{f}_e(k)\big|^2, \tag{2.29}$$

but $|\widehat{\mathcal{I}}_e(k)| \leqslant C\mathcal{N}(T)$ and therefore

$$\|If\|^2_{L^2(0,T)} \leqslant C\mathcal{N}^2(T) \, \|f_e\|^2_{L^2(\mathbb{R})} = C\mathcal{N}^2(T) \, \|f\|^2_{L^2(0,T)}, \tag{2.30}$$

which implies the result via Lemma 2.2.

We now focus on the seminorm $[If]_{\dot{H}^{1/2}(0,T)}$. First, we note that, for every $0 < s < t < T$,

$$(If)(t) - (If)(s) = \int_s^t \mathrm{d}\tau \, \mathcal{I}(\tau) f(t-\tau) + \int_0^s \mathrm{d}\tau \, \mathcal{I}(\tau)(f(t-\tau) - f(s-\tau)),$$

so that

$$[If]^2_{\dot{H}^{1/2}(0,T)} \leqslant 4\int_0^T \mathrm{d}t \int_0^t \mathrm{d}s \left|\frac{1}{t-s}\int_s^t \mathrm{d}\tau \, \mathcal{I}(\tau) f(t-\tau)\right|^2 + 4\int_0^T \mathrm{d}t \int_0^t \mathrm{d}s \left|\int_0^s \mathrm{d}\tau \, \mathcal{I}(\tau)\frac{f(t-\tau) - f(s-\tau)}{t-s}\right|^2. \tag{2.31}$$

Now, one can easily see that, since $f \in L^\infty(0,T)$,

$$4\int_0^T \mathrm{d}t \int_0^t \mathrm{d}s \left|\frac{1}{t-s}\int_s^t \mathrm{d}\tau \, \mathcal{I}(\tau) f(t-\tau)\right|^2 \leqslant 4\|f\|^2_{L^\infty(0,T)} \int_0^T \mathrm{d}t \int_0^t \mathrm{d}s \left|\frac{\mathcal{N}(t) - \mathcal{N}(s)}{t-s}\right|^2 = 2\|f\|^2_{L^\infty(0,T)}[\mathcal{N}]^2_{\dot{H}^{1/2}(0,T)} \leqslant 2\|f\|^2_{L^\infty(0,T)}\|\mathcal{N}\|^2_{H^{1/2}(0,T)} \tag{2.32}$$

where the last factor $\|\mathcal{N}\|_{H^{1/2}(0,T)}$ is finite by Lemma 2.2. On the other hand by Cauchy-Schwarz inequality, monotonicity of $\mathcal{N}$ and positivity of $\mathcal{I}$, we have

$$\begin{aligned} 4\int_0^T \mathrm{d}t \int_0^t \mathrm{d}s &\left|\int_0^s \mathrm{d}\tau \, \mathcal{I}(\tau)\frac{f(t-\tau) - f(s-\tau)}{t-s}\right|^2 \\ &\leqslant 4\mathcal{N}(T) \int_0^T \mathrm{d}t \int_0^t \mathrm{d}s \int_0^s \mathrm{d}\tau \, \mathcal{I}(\tau) \left|\frac{f(t-\tau) - f(s-\tau)}{t-s}\right|^2 \\ &\leqslant 4\mathcal{N}(T) \int_0^T \mathrm{d}\tau \, \mathcal{I}(\tau) \int_0^{T-\tau} \mathrm{d}t \int_0^{t-\tau} \mathrm{d}s \left|\frac{f(t) - f(s)}{t-s}\right|^2 \leqslant 2\mathcal{N}^2(T) \, [f]^2_{\dot{H}^{1/2}(0,T)} \end{aligned}$$

and plugging the above inequality and (2.32) into (2.31),

$$[If]^2_{\dot{H}^{1/2}(0,T)} \leqslant C \max\left\{\|\mathcal{N}\|^2_{\dot{H}^{1/2}(0,T)}, \mathcal{N}^2(T)\right\} \left(\|f\|^2_{L^\infty(0,T)} + \|f\|^2_{\dot{H}^{1/2}(0,T)}\right).$$

Finally, the above estimate in combination with (2.30) yields

$$\|If\|_{\dot{H}^{1/2}(0,T)} \leqslant C \max\left\{ \|\mathcal{N}\|_{\dot{H}^{1/2}(0,T)}, \mathcal{N}(T) \right\} \left(\|f\|_{L^\infty(0,T)} + \|f\|_{\dot{H}^{1/2}(0,T)} \right)$$

and, since both $\mathcal{N}(T)$ and $\|\mathcal{N}\|_{\dot{H}^{1/2}(0,T)}$ converges to zero as $T \to 0$ by Lemma 2.2, the proof is complete. □

In view of Corollary 2.1, the results about the operator I proven above can be easily extended to the operator

$$\left(\tilde{I}_{jk} f \right)(t) := \int_0^t \mathrm{d}\tau \, I(t-\tau) \mathcal{R}_{jk}(t-\tau) \, f(\tau). \tag{2.33}$$

We state such a result in next

Corollary 2.2.
Let $f \in H^{1/2}(0,T) \cap L^\infty(0,T)$, $T > 0$, then, $\tilde{I}_{jk} f \in H^{1/2}(0,T)$ for any $j \neq k$, and, in particular, there exists $C_T > 0$ independent of f and satisfying $C_T \xrightarrow[T \to 0]{} 0$, such that

$$\left\| \tilde{I}_{jk} f \right\|_{H^{1/2}(0,T)} \leqslant C_T \left(\|f\|_{L^\infty(0,T)} + \|f\|_{H^{1/2}(0,T)} \right). \tag{2.34}$$

Finally, we point out some relevant properties of the integral operator J, defined by

$$(Jf)(t) := \int_0^t \mathrm{d}\tau \, \mathcal{J}(t-\tau) f(\tau), \qquad \mathcal{J}(t-\tau) := -\gamma - \log(t-\tau). \tag{2.35}$$

Lemma 2.5.
For any $t \in \mathbb{R}^+$ and $f \in L^1(0,t)$,

$$(JIf)(t) = (IJf)(t) = \int_0^t \mathrm{d}\tau \, f(\tau). \tag{2.36}$$

Proof. We first observe that one has the identity

$$\int_0^t \mathrm{d}\tau \, \mathcal{I}(\tau) \left(-\gamma - \log(t-\tau) \right) = \int_0^t \mathrm{d}\tau \, \mathcal{I}(t-\tau) \left(-\gamma - \log \tau \right) = 1. \tag{2.37}$$

In [SKM, Lemma 32.1] it is indeed proven that (in the formula stated in the cited Lemma one has to take $\alpha = 1, h = 0$)

$$\int_0^t \mathrm{d}\tau \, \left(\log \tau - \psi(1) \right) \partial_t \nu(t-\tau) = -1,$$

where ν here denotes the Volterra function of order 0. However, using [E1, Eq. (12), Sect. 18.3], one can recognize that $\partial_t \nu(t) = \mathcal{I}(t)$ (and that $\psi(1) = -\gamma$).

Let us then prove the identity involving IJ. The proof of the other one is perfectly analogous and we omit it for the sake of brevity. First of all, in the expression

$$(IJf)(t) = \int_0^t \mathrm{d}\tau \int_0^{t-\tau} \mathrm{d}\sigma \, \mathcal{I}(\tau) \mathcal{J}(t-\tau-\sigma) f(\sigma),$$

one can exchange the order of the integration, since

$$\int_0^t \mathrm{d}\tau \int_0^{t-\tau} \mathrm{d}\sigma \, \mathcal{I}(\tau) \mathcal{J}(t-\sigma-\tau) f(\sigma) = \int_0^t \mathrm{d}\sigma \int_0^{t-\sigma} \mathrm{d}\tau \, \mathcal{I}(\tau) \mathcal{J}(t-\sigma-\tau) f(\sigma).$$

Using (2.37), we conclude that

$$(IJf)(t) = \int_0^t \mathrm{d}\sigma \int_0^{t-\sigma} \mathrm{d}\tau \, \mathcal{I}(\tau) \mathcal{J}(t-\sigma-\tau) f(\sigma) = \int_0^t \mathrm{d}\sigma \, f(\sigma).$$

□

2.2. **A derivation of the charge equation.** Before starting to discuss the charge equation, it is worth making a brief excursus on a heuristic computation, which motivates ansatz (1.7) and equation (1.11). In the following, we assume for the sake of simplicity that $q_j(0) = 0$, for every $j = 1, \dots, N$. However, one can prove that such an assumption is not restrictive.

Neglecting any regularity issue, we can compute the time derivative of (1.7) and obtain that, at least formally,

$$\begin{aligned} i\partial_t \psi_t(\mathbf{x}) &= (-\Delta U_0(t)\psi_0)(\mathbf{x}) - \frac{1}{2\pi}\sum_{j=1}^N q_j(t) + \frac{1}{2\pi}\sum_{j=1}^N \int_0^t \mathrm{d}\tau\, \partial_\tau U_0(t-\tau; |\mathbf{x}-\mathbf{y}_j|)\, q_j(\tau) \\ &= (-\Delta U_0(t)\psi_0)(\mathbf{x}) - \frac{1}{2\pi}\sum_{j=1}^N \int_0^t \mathrm{d}\tau\, U_0(t-\tau; |\mathbf{x}-\mathbf{y}_j|)\, \dot{q}_j(\tau), \quad (2.38) \end{aligned}$$

where we used the fact that (as $q_j(0) = 0$) $\psi_0 \in H^2(\mathbb{R}^2)$ and that (by definition) $i\partial_t U_0(t)\psi_0 = -\Delta U_0(t)\psi_0$. Hence, applying the Fourier transform on $\mathbb{R}^2$, the above expression becomes (we set $p = |\mathbf{p}|$)

$$i\widehat{\partial_t \psi_t}(\mathbf{p}) = p^2 e^{-ip^2 t}\widehat{\psi_0}(\mathbf{p}) - \frac{1}{2\pi}\sum_{j=1}^N \int_0^t \mathrm{d}\tau\, e^{-i\mathbf{p}\cdot\mathbf{y}_j} e^{-ip^2(t-\tau)}\, \dot{q}_j(\tau). \quad (2.39)$$

The l.h.s. of (2.38) equals (compare with (1.15)), at least in a weak sense, the action of H_0 on the regular part of the wave function ψ_t (see (1.2) and (1.3)), i.e.,

$$\begin{aligned} & p^2\left(\widehat{\psi_t}(\mathbf{p}) - \frac{1}{2\pi}\sum_{j=1}^N \frac{q_j(t)e^{-i\mathbf{p}\cdot\mathbf{y}_j}}{p^2+\lambda}\right) - \frac{\lambda}{2\pi}\sum_{j=1}^N \frac{q_j(t)e^{-i\mathbf{p}\cdot\mathbf{y}_j}}{p^2+\lambda} \\ &\qquad = p^2 e^{-ip^2 t}\widehat{\psi_0}(\mathbf{p}) + \frac{1}{2\pi}\sum_{j=1}^N \int_0^t \mathrm{d}\tau\, e^{-i\mathbf{p}\cdot\mathbf{y}_j}\partial_\tau\left(e^{-ip^2(t-\tau)}\right) q_j(\tau) - \frac{1}{2\pi}\sum_{j=1}^N q_j e^{-i\mathbf{p}\cdot\mathbf{y}_j} \\ &\qquad = p^2 e^{-ip^2 t}\widehat{\psi_0}(\mathbf{p}) - \frac{1}{2\pi}\sum_{j=1}^N \int_0^t \mathrm{d}\tau\, e^{-i\mathbf{p}\cdot\mathbf{y}_j} e^{-ip^2(t-\tau)}\, \dot{q}_j(\tau), \quad (2.40) \end{aligned}$$

which is equal to (2.39). Therefore, for any $\mathbf{q}(t)$ and ψ_0 such that the r.h.s. of (2.40) makes sense, the ansatz (1.7) does solve the time-dependent Schrödinger equation, at least in a weak sense.

Under restrictive assumptions on ψ_0, however, the ansatz ψ_t must belong to the (nonlinear) operator domain $\mathscr{D}(H_{\boldsymbol{\alpha}(t)})$, with $\alpha_j = \beta_j |q_j(t)|^{2\sigma_j}$, $j = 1, \dots, N$, i.e., it must satisfy the boundary conditions (1.10), which can be cast in the form

$$\begin{aligned} \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{d}\mathbf{p}\, e^{i\mathbf{p}\cdot\mathbf{y}_j}\widehat{\phi_{\lambda,t}}(\mathbf{p}) &= \left(\beta_j |q_j(t)|^{2\sigma_j} + \tfrac{1}{2\pi}\log\tfrac{\sqrt{\lambda}}{2} - \tfrac{\gamma}{2\pi}\right) q_j(t) \\ &\quad - \frac{1}{2\pi}\sum_{k\neq j} q_k(t) K_0\left(\sqrt{\lambda}|\mathbf{y}_j - \mathbf{y}_k|\right), \quad (2.41) \end{aligned}$$

In fact, as we are going to see, the above condition will force $\mathbf{q}(t)$ to be a solution to the charge equation (1.11). Indeed, since

$$\phi_{\lambda,t} = \psi_t - \frac{1}{2\pi}\sum_{k=1}^N q_k(t) K_0\left(\sqrt{\lambda}|\cdot - \mathbf{y}_k|\right),$$

by (1.7),

$$\frac{1}{2\pi}\int_{\mathbb{R}^2}\mathrm{d}\mathbf{p}\, e^{i\mathbf{p}\cdot\mathbf{y}_j}\left\{e^{-ip^2t}\widehat{\psi_0}(\mathbf{p})+\frac{i}{2\pi}\sum_{k=1}^{N}\int_0^t \mathrm{d}\tau\, e^{-i\mathbf{p}\cdot\mathbf{y}_k}e^{-ip^2(t-\tau)}\,q_k(\tau)-\frac{1}{2\pi}\sum_{k=1}^{N}\frac{q_k(t)e^{-i\mathbf{p}\cdot\mathbf{y}_k}}{p^2+\lambda}\right\}$$
$$=\left(\beta_j|q_j(t)|^{2\sigma_j}+\tfrac{1}{2\pi}\log\tfrac{\sqrt{\lambda}}{2}-\tfrac{\gamma}{2\pi}\right)q_j(t)-\frac{1}{2\pi}\sum_{k\neq j}q_k(t)K_0\left(\sqrt{\lambda}|\mathbf{y}_j-\mathbf{y}_k|\right).$$

The last off-diagonal term cancels exactly and thus the identity becomes

$$\frac{1}{2\pi}\int_{\mathbb{R}^2}\mathrm{d}\mathbf{p}\, e^{i\mathbf{p}\cdot\mathbf{y}_j}\left\{e^{-ip^2t}\widehat{\psi_0}(\mathbf{p})+\frac{i}{2\pi}\sum_{k=1}^{N}\int_0^t \mathrm{d}\tau\, e^{-i\mathbf{p}\cdot\mathbf{y}_k}e^{-ip^2(t-\tau)}\,q_k(\tau)-\frac{1}{2\pi}\frac{q_j(t)e^{-i\mathbf{p}\cdot\mathbf{y}_j}}{p^2+\lambda}\right\}$$
$$=\left(\beta_j|q_j(t)|^{2\sigma_j}+\tfrac{1}{2\pi}\log\tfrac{\sqrt{\lambda}}{2}+\tfrac{\gamma}{2\pi}\right)q_j(t).$$

Combining the last diverging term on the l.h.s. with the second one via an integration by parts (here we implicitly assume that the charge is regular enough), we get

$$\frac{1}{2\pi}\int_{\mathbb{R}^2}\mathrm{d}\mathbf{p}\left\{e^{i\mathbf{p}\cdot\mathbf{y}_j}e^{-ip^2t}\widehat{\psi_0}(\mathbf{p})-\frac{1}{2\pi(p^2+\lambda)}\int_0^t\mathrm{d}\tau\, e^{-ip^2(t-\tau)}\,\left[\dot{q}_j(\tau)-i\lambda q_j(\tau)\right]\right.$$
$$\left.+\frac{i}{2\pi}\sum_{k\neq j}\int_0^t\mathrm{d}\tau\, e^{i\mathbf{p}\cdot(\mathbf{y}_j-\mathbf{y}_k)}e^{-ip^2(t-\tau)}\,q_k(\tau)\right\}=\left(\beta_j|q_j(t)|^{2\sigma_j}+\tfrac{1}{2\pi}\log\tfrac{\sqrt{\lambda}}{2}+\tfrac{\gamma}{2\pi}\right)q_j(t),$$

The $\mathbf{p}$ integral of the second term on the l.h.s. contains an infrared singularity for $t=\tau$ which is proportional to $\log(t-\tau)$: in fact by [GR, Eqs. 3.722.1 & 3.722.3]

$$\left(U_0(t)K_0(\sqrt{\lambda}\,\cdot\,)\right)(\mathbf{0})=\int_{\mathbb{R}^2}\mathrm{d}\mathbf{p}\,\frac{e^{-ip^2(t-\tau)}}{p^2+\lambda}=-\pi e^{i\lambda(t-\tau)}\left[\mathrm{ci}(\lambda(t-\tau))-i\,\mathrm{si}(\lambda(t-\tau))\right]$$
$$=-\pi e^{i\lambda(t-\tau)}\left(\gamma+\log\lambda+\log(t-\tau)\right)+e^{i\lambda(t-\tau)}Q(\lambda;t-\tau),\quad (2.42)$$

where $\mathrm{si}(\,\cdot\,)$ and $\mathrm{ci}(\,\cdot\,)$ stand for the sine and cosine integral functions [AS, Eqs. 5.2.1 & 5.2.2] and (see, e.g., [AS, Eq. 5.2.16])

$$Q(\lambda;t-\tau):=-\pi\bigg(\sum_{n=1}^{\infty}\frac{((t-\tau)^2\lambda^2)^n}{2n(2n)!}-i\,\mathrm{si}((t-\tau)\lambda)\bigg)\quad (2.43)$$

(note that $Q(0;t-\tau)=-\frac{i\pi^2}{2}$). Hence, we obtain

$$\left(U_0(t)\psi_0\right)(\mathbf{y}_j)+\frac{i}{2\pi}\sum_{k\neq j}\int_0^t\mathrm{d}\tau\, U_0(t-\tau;|\mathbf{y}_j-\mathbf{y}_k|)\,q_k(\tau)-\left(\beta_j|q_j(t)|^{2\sigma_j}+\tfrac{1}{2\pi}\log\tfrac{\sqrt{\lambda}}{2}+\tfrac{\gamma}{2\pi}\right)q_j(t)$$
$$=-\frac{1}{4\pi}\int_0^t\mathrm{d}\tau\,\left(\gamma+\log(t-\tau)+\log\lambda-\tfrac{1}{\pi}Q(\lambda;t-\tau)\right)\partial_\tau\left(e^{i\lambda(t-\tau)}q_j(\tau)\right)$$

and taking the limit $\lambda\to 0$ (notice the exact cancellation of the diverging $\log\lambda$ terms)

$$(U_0(t)\psi_0)(\mathbf{y}_j)+\frac{i}{2\pi}\sum_{k\neq j}\int_0^t\mathrm{d}\tau\, U_0(t-\tau;|\mathbf{y}_j-\mathbf{y}_k|)\,q_k(\tau)$$
$$-\left(\beta_j|q_j(t)|^{2\sigma_j}-\tfrac{1}{2\pi}\log 2+\tfrac{\gamma}{2\pi}-\tfrac{i}{8}\right)q_j(t)=-\frac{1}{4\pi}\int_0^t\mathrm{d}\tau\,\left(\gamma+\log(t-\tau)\right)\dot{q}_j(\tau).$$

If we now apply to both sides the integral operator I defined in (2.18) and exploit the property proven in Lemma 2.5, we find

$$\int_0^t d\tau\,\mathcal{I}(t-\tau)(U_0(\tau)\psi_0)(\mathbf{y}_j) + \frac{i}{2\pi}\sum_{k\neq j}\int_0^t d\tau\,\mathcal{I}(t-\tau)\int_0^\tau d\mu\,U_0(\tau-\mu;|\mathbf{y}_j-\mathbf{y}_k|)q_k(\mu)+$$

$$-\int_0^t d\tau\,\mathcal{I}(t-\tau)\beta_j|q_j(\tau)|^{2\sigma_j}q_j(\tau) + \frac{1}{2\pi}\left(\log 2-\gamma+\tfrac{i\pi}{4}\right)\int_0^t d\tau\,\mathcal{I}(t-\tau)q_j(\tau) = \frac{q_j(t)}{4\pi}.$$

Now, with the change of variable $\frac{1}{\tau-\mu} = \frac{1}{t-\mu}+v$, one can see that

$$\int_0^t d\tau\,\mathcal{I}(t-\tau)\int_0^\tau d\mu\,U_0(\tau-\mu;|\mathbf{y}_j-\mathbf{y}_k|)q_k(\mu) = \frac{1}{2i}\int_0^t d\tau\,\mathcal{I}(t-\tau)\mathcal{R}_{jk}(t-\tau)q_k(\tau),$$

with $\mathcal{R}_{jk}$ defined by (1.14). Then, multiplying each term by 4π and suitably rearranging terms, one obtains (1.11).

2.3. **Local well-posedness.** In order to prove Theorem 1.1, the first step is to discuss existence, uniqueness and Sobolev regularity of any solution of (1.11). We split the results into two separate Propositions to make the proof strategy clearer: by general arguments about Volterra-type integral equations and the properties of (1.11), we obtain existence and uniqueness of a continuous solution $\mathbf{q}(t)$ up to some maximal existence time T_*, which might as well be $+\infty$. Then, in order to derive the Sobolev regularity of $\mathbf{q}(t)$, we use the aforementioned contraction, which works on some a priori shorter interval $[0,T]$, $T<T_*$. In Proposition 2.4 we will however show how one can extend such a regularity to the whole existence interval, provided a property of the source term holds true (it will be proven in Lemma 2.10).

Preliminarily, note that (1.11) can be written in a compact form as

$$\mathbf{q}(t) + \int_0^t d\tau\left(\mathbf{g}(t,\tau,\mathbf{q}(\tau)) + \mathcal{K}(t,\tau)\,\mathbf{q}(\tau)\right) = \mathbf{f}(t), \tag{2.44}$$

where $\mathcal{K}$ is the matrix of components $\mathcal{K}_{jk}$ defined in (1.13) and $\mathbf{g}=(g_1,\dots,g_N)$, $\mathbf{f}=(f_1,\dots,f_N)$ are defined respectively by

$$g_j(t,\tau,\mathbf{q}(\tau)) = 4\pi\beta_j\mathcal{I}(t-\tau)|q_j(\tau)|^{2\sigma_j}q_j(\tau), \qquad j=1,\dots,N, \tag{2.45}$$

$$f_j(t) = 4\pi\int_0^t d\tau\,\mathcal{I}(t-\tau)(U_0(\tau)\psi_0)(\mathbf{y}_j), \qquad j=1,\dots,N. \tag{2.46}$$

Proposition 2.2 (Continuity of $\mathbf{q}(t)$)**.**
Let $\sigma_j \geqslant \frac{1}{2}$ for every $j=1,\dots,N$, and $\psi_0\in\mathscr{D}$. Then, there exists $T_*>0$ such that (2.44) admits a unique solution $\mathbf{q}(t)\in C[0,T_*)$. Moreover either $T_*=+\infty$, i.e., the solution is global in time, or $T_*<+\infty$ and $\lim_{t\to T_*}|\mathbf{q}(t)|=+\infty$.

Proof. The result is obtained by directly applying [Mi, Theorem 1.2]: it claims that there exists $T_*>0$ for which (2.44) admits a unique solution $\mathbf{q}\in C[0,T^*)$, with the claimed properties, provided

(i) $\mathbf{f}$ is continuous on $\mathbb{R}^+$;
(ii) for every $t'>0$ and every bounded set $\mathcal{B}\subset\mathbb{C}^N$, there exists a measurable function $m(t,\tau)$ such that

$$|\mathbf{g}(t-\tau,\mathbf{q}) + \mathcal{K}(t,\tau)\mathbf{q}| \leqslant m(t,\tau), \quad \forall\,0\leqslant\tau\leqslant t\leqslant t', \quad \forall\mathbf{q}\in\mathcal{B},$$

with

$$\sup_{t\in[0,t']}\int_0^t \mathrm{d}\tau\, m(t,\tau)<\infty, \qquad \int_0^t \mathrm{d}\tau\, m(t,\tau)\xrightarrow[t\to 0]{} 0;$$

(iii) for every compact interval $I\subset\mathbb{R}^+$, every continuous function $\boldsymbol{\varphi}: I\to\mathbb{C}^N$ and every $t_0\in\mathbb{R}^+$,

$$\lim_{t\to t_0}\int_I \mathrm{d}\tau\ \left[\mathbf{g}(t-\tau,\boldsymbol{\varphi}(\tau))-\mathbf{g}(t_0-\tau,\boldsymbol{\varphi}(\tau))+(\mathcal{K}(t,\tau)-\mathcal{K}(t_0,\tau))\boldsymbol{\varphi}(\tau)\right]=0; \tag{2.47}$$

(iv) for every $t'>0$ and every bounded $\mathcal{B}\subset\mathbb{C}^N$, there exists a measurable function $h(t,\tau)$ such that

$$\left|\mathbf{g}(t-\tau,\mathbf{q}_1)-\mathbf{g}(t-\tau,\mathbf{q}_2)+\mathcal{K}(t,\tau)(\mathbf{q}_1-\mathbf{q}_2)\right|\leqslant h(t,\tau)\left|\mathbf{q}_1-\mathbf{q}_2\right|,$$

for all $0\leqslant\tau\leqslant t\leqslant t'$ and all $\mathbf{q}_1$, $\mathbf{q}_2\in\mathcal{B}$, with $h(t,\cdot\,)\in L^1(0,t)$ for all $t\in[0,t']$ and

$$\int_t^{t+\varepsilon}\mathrm{d}\tau\, h(t+\varepsilon,\tau)\xrightarrow[\varepsilon\to 0]{} 0.$$

Let us now verify all the hypothesis. First, consider point (i): since $\psi_0\in\mathscr{D}[\mathcal{F}]$,

$$4\pi(U_0(\tau)\psi_0)(\mathbf{y}_j)=\underbrace{4\pi\left(U_0(\tau)\phi_{\lambda,0}\right)(\mathbf{y}_j)}_{A_1(\tau)}+\underbrace{2q_j(0)\left(U_0(\tau)K_0\left(\sqrt{\lambda}|\cdot-\mathbf{y}_j|\right)\right)(\mathbf{y}_j)}_{A_2(\tau)}$$
$$+\underbrace{2\sum_{k\neq j}q_k(0)\left(U_0(\tau)K_0\left(\sqrt{\lambda}|\cdot-\mathbf{y}_k|\right)\right)(\mathbf{y}_j)}_{A_3(\tau)}. \tag{2.48}$$

Observing that

$$A_1(\tau)=2\int_{\mathbb{R}^2}\mathrm{d}\mathbf{p}\, e^{i\mathbf{p}\cdot\mathbf{y}_j}e^{-ip^2\tau}\widehat{\phi_{\lambda,0}}(\mathbf{p})$$

and recalling that $\widehat{\phi_{\lambda,0}}\in L^1(\mathbb{R}^2)$ by assumption, one sees that A_1 is bounded and therefore IA_1 is continuous as well by Lemma 2.3. On the other hand, by the definition of Bessel functions, one has

$$A_3(\tau)=\frac{1}{\pi}\sum_{k\neq j}q_k(0)\int_{\mathbb{R}^2}\mathrm{d}\mathbf{p}\, e^{i\mathbf{p}\cdot(\mathbf{y}_j-\mathbf{y}_k)}\frac{e^{-ip^2\tau}}{p^2+\lambda}=2\sum_{k\neq j}q_k(0)\int_0^\infty \mathrm{d}p\, p\frac{e^{-ip^2\tau}}{p^2+\lambda}J_0(p|\mathbf{y}_j-\mathbf{y}_k|)$$
$$=\sum_{k\neq j}q_k(0)\int_0^\infty \mathrm{d}\varrho\,\frac{e^{-i\varrho\tau}}{\varrho+\lambda}J_0(\sqrt{\varrho}|\mathbf{y}_j-\mathbf{y}_k|)=\sqrt{2\pi}\,\widecheck{G_3}(-\tau),$$

where $G_3(\varrho):=\mathbb{1}_{[0,+\infty)}(\varrho)\sum_{k\neq j}q_k(0)\frac{J_0(\sqrt{\varrho}|\mathbf{y}_j-\mathbf{y}_k|)}{\varrho+\lambda}$. Consequently,

$$\|A_3\|^2_{H^\nu(\mathbb{R})}=2\pi\int_{\mathbb{R}}\mathrm{d}\varrho\,(1+\varrho^2)^\nu\,|G_3(-\rho)|^2\leqslant C\sum_{i=1}^N|q_i(0)|^2\sum_{k\neq j}\int_0^\infty\mathrm{d}p\,(1+p^4)^\nu p\,\frac{J_0^2(p|\mathbf{y}_j-\mathbf{y}_k|)}{(p^2+\lambda)^2}$$
$$\leqslant C\sum_{k\neq j}\int_0^\infty\mathrm{d}p\,\frac{(1+p^4)^\nu p}{(p+1)(p^2+\lambda)^2},$$

which entails that $A_3\in H^\nu(0,T)$ for every $\nu<\frac{3}{4}$. Since by Sobolev embeddings this implies that A_3 is continuous too, then, again by Lemma 2.3, one finds that IA_3 is continuous. Finally, exploiting (2.35), (2.42) and (2.43),

$$A_2(\tau) = \frac{1}{\pi} q_j(0) \int_{\mathbb{R}^2} \mathrm{d}\mathbf{p}\, \frac{e^{-ip^2\tau}}{p^2+\lambda} = q_j(0) \left[-e^{i\lambda\tau}(\gamma + \log\lambda + \log\tau) + \frac{e^{i\lambda\tau}}{\pi} Q(\lambda;\tau) \right]$$
$$= q_j(0) \underbrace{e^{i\lambda\tau}\mathcal{J}(\tau)}_{A_{2,1}(\tau)} + q_j(0) \underbrace{\tfrac{e^{i\lambda\tau}}{\pi}\left(-\pi\log\lambda + Q(\lambda;\tau)\right)}_{A_{2,2}(\tau)}. \quad (2.49)$$

Now, it is clear that $A_{2,2}(\tau)$ is smooth, so that $IA_{2,2}$ is continuous. Furthermore, by (2.37),

$$(IA_{2,1})(t) = 1 + \int_0^t \mathrm{d}\tau\, \mathcal{I}(t-\tau) a_{2,1}(\tau), \qquad a_{2,1}(\tau) := \left(e^{i\lambda\tau} - 1\right)\mathcal{J}(\tau).$$

Since $a_{2,1}$ is continuous (actually belongs to $H^1(0,T)$), then $IA_{2,1}$ is continuous too. Summing up, we have thus shown that f_j (defined by (2.46)) is continuous, and so is $\mathbf{f}$.

For every $\mathbf{q} \in \mathcal{B}$, with $\mathcal{B}$ bounded,

$$|\mathbf{g}(t,\tau,\mathbf{q}) + \mathcal{K}(t,\tau)\mathbf{q}| \leqslant C\,\mathcal{I}(t-\tau)$$

and, since $\mathcal{I} \in L^1_{\mathrm{loc}}(\mathbb{R}^+)$, (ii) is satisfied.

In addition, let $I = [a,b]$ be an interval, $\boldsymbol{\varphi} : I \to \mathbb{C}^N$ a continuous function and $t_0 \in \mathbb{R}^+$. The integral in (2.47) consists, up to some constants, of terms like

$$\int_a^b \mathrm{d}\tau\ \left[\mathcal{I}(t-\tau) - \mathcal{I}(t_0-\tau)\right] \left[\beta_j|\varphi_j(\tau)|^{2\sigma_j}\varphi_j(\tau) - 2(\log 2 - \gamma + \tfrac{i\pi}{4})\varphi_j(\tau)\right]$$

or

$$\int_a^b \mathrm{d}\tau\, \varphi_k(\tau)\left[\mathcal{I}(t-\tau)\mathcal{R}_{jk}(t-\tau) - \mathcal{I}(t_0-\tau)\mathcal{R}_{jk}(t_0-\tau)\right].$$

Hence (iii) is satisfied by dominated convergence (see, e.g., the discussion of (2.25)).

Finally, we see that, as $\mathbf{q}_1$, $\mathbf{q}_2 \in \mathcal{B}$,

$$|\mathbf{g}(t,\tau,\mathbf{q}_1) - \mathbf{g}(t,\tau,\mathbf{q}_2) + \mathcal{K}(t,\tau)(\mathbf{q}_1 - \mathbf{q}_2)|$$
$$\leqslant C\mathcal{I}(t-\tau)\sum_{j=1}^N \left||q_{1,j}|^{2\sigma_j}q_{1,j} - |q_{2,j}|^{2\sigma_j}q_{2,j}\right| + |\mathcal{K}(t,\tau)||\mathbf{q}_1 - \mathbf{q}_2| \leqslant C\,\mathcal{I}(t-\tau)|\mathbf{q}_1 - \mathbf{q}_2|.$$

Consequently, setting $h(t,\tau) = C\,\mathcal{I}(t-\tau)$, (iv) is satisfied. □

Proposition 2.3 (Sobolev regularity of $\mathbf{q}(t)$)**.**
Let $\sigma_j \geqslant \frac{1}{2}$ for every $j = 1,\dots,N$, and $\psi_0 \in \mathscr{D}$. Then, there exists $0 < T < T_*$, such that $\mathbf{q}(t) \in H^{1/2}(0,T)$.

Proof. The key observation is that, if one proves that the map

$$\mathcal{G}(\mathbf{q})[t] = \mathbf{f}(t) - \int_0^t \mathrm{d}\tau\, \Big(\mathbf{g}(t,\tau,\mathbf{q}(\tau)) + \mathcal{K}(t,\tau)\mathbf{q}(\tau)\Big)$$

is a contraction in a suitable subset of $C[0,T] \cap H^{1/2}(0,T)$, for a sufficiently small $T \in (0,T^*)$, then (2.44) has a unique solution in this subset. Hence such a solution must coincide with the unique continuous solution provided by Proposition 2.2, which thus belongs to $H^{1/2}(0,T)$.

For fixed $0 < T < T^*$, the first point is to investigate the Sobolev regularity of the forcing term $\mathbf{f}$. We know that $4\pi(U_0(\tau)\psi_0)(\mathbf{y}_j) = A_1(\tau) + A_2(\tau) + A_3(\tau)$, with A_i defined in (2.48). Concerning A_1, we write

$$A_1(\tau) = 2\int_{\mathbb{R}^2} \mathrm{d}\mathbf{p}\, e^{i\mathbf{p}\cdot\mathbf{y}_j} e^{-ip^2\tau}\widehat{\phi_{\lambda,0}}(\mathbf{p}) = 2\int_{\mathbb{R}^2} \mathrm{d}\mathbf{p}\, e^{-ip^2\tau}\left(\widehat{T_{-\mathbf{y}_j}\phi_{\lambda,0}}\right)(\mathbf{p})$$
$$= 2\pi\int_0^\infty \mathrm{d}\varrho\, e^{-i\varrho\tau}\left\langle \widehat{T_{-\mathbf{y}_j}\phi_{\lambda,0}}\right\rangle(\sqrt{\varrho}) = (2\pi)^{3/2}\widecheck{G_1}(-\tau)$$

where $T_{\mathbf{y}}$ is the translation operator, i.e., $(T_{\mathbf{y}}\psi)(\mathbf{x}) := \psi(\mathbf{x}-\mathbf{y})$,

$$G_1(\varrho) := \mathbb{1}_{[0,+\infty)}(\varrho)\left\langle \widehat{T_{-\mathbf{y}_j}\phi_{\lambda,0}}\right\rangle(\sqrt{\varrho}),$$

and $\langle f\rangle$ denotes the angular average of a function on $\mathbb{R}^2$, i.e.,

$$\langle f\rangle(\varrho) = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\vartheta\, f(\varrho\cos\vartheta, \varrho\sin\vartheta).$$

Consequently, one finds that

$$\|A_1\|^2_{H^\nu(\mathbb{R})} = (2\pi)^3\int_{\mathbb{R}}\mathrm{d}\varrho\,\left(1+\varrho^2\right)^\nu|G_1(-\varrho)|^2 = 16\pi^3\int_0^\infty \mathrm{d}p\,\left(1+p^4\right)^\nu p\left|\left\langle\widehat{T_{-\mathbf{y}_j}\phi_{\lambda,0}}\right\rangle(p)\right|^2$$
$$\leqslant C\int_{\mathbb{R}^2}\mathrm{d}\mathbf{p}\,\left(1+p^4\right)^\nu\left|\left(\widehat{T_{-\mathbf{y}_j}\phi_{\lambda,0}}\right)(\mathbf{p})\right|^2$$

so that $A_1 \in H^{1/2}(0,T)$, since $\phi_{\lambda,0} \in H^1(\mathbb{R}^2)$ by assumption. As A_1 is bounded too we have, by Lemma 2.4, that $IA_1 \in H^{1/2}(0,T)$. On the other hand, we recall that $A_2 \in H^{1/2}(0,T) \cap C[0,T]$ and thus $IA_2 \in H^{1/2}(0,T)$ by Lemma 2.4. Finally, since $A_{2,2}$ is smooth, $IA_{2,2}$ is smooth as well and, as $IA_{2,1} = 1 + Ia_{2,1}$ with $a_{2,1} \in H^1(0,T)$, we have that $IA_{2,1} \in H^{1/2}(0,T)$. Summing up, recalling (2.46) and (2.48), we proved that $\mathbf{f} \in H^{1/2}(0,T)$.

We introduce now the contraction space: let

$$\mathcal{A}_T = \left\{\mathbf{q} \in C[0,T] \cap H^{1/2}(0,T)\;\middle|\; \|\mathbf{q}\|_{L^\infty(0,T)} + \|\mathbf{q}\|_{H^{1/2}(0,T)} \leqslant \mathrm{b}_T\right\},$$

with $\mathrm{b}_T = 2\max\{\|\mathbf{f}\|_{L^\infty(0,T)} + \|\mathbf{f}\|_{H^{1/2}(0,T)}, 1\}$. The set $\mathcal{A}_T$ is a complete metric space with the norm induced by $C[0,T] \cap H^{1/2}(0,T)$, i.e.,

$$\|\cdot\|_{\mathcal{A}_T} = \|\cdot\|_{L^\infty(0,T)} + \|\cdot\|_{H^{1/2}(0,T)}.$$

In order to prove that $\mathcal{G}$ defines a contraction on $\mathcal{A}_T$, we need to show that $\mathcal{G}$ maps $\mathcal{A}_T$ into itself and the contraction condition on the norms is satisfied.

We start by proving that $\mathcal{G}(\mathcal{A}_T) \subset \mathcal{A}_T$. Letting $\mathbf{q} \in \mathcal{A}_T$, one immediately sees (recalling *Step 1*) that $\mathcal{G}(\mathbf{q})[t]$ is continuous. Then, we split $\mathcal{G}(\mathbf{q})[t]$ into two terms:

$$\mathcal{G}_1(\mathbf{q})[t] = \int_0^t \mathrm{d}\tau\,\mathbf{g}(t,\tau,\mathbf{q}(\tau)), \qquad \mathcal{G}_2(\mathbf{q})[t] = \int_0^t \mathrm{d}\tau\,\mathcal{K}(t,\tau)\mathbf{q}(\tau).$$

From (2.8) and (2.9), (2.18), (2.26), one obtains

$$\|\mathcal{G}_1(\mathbf{q})\|_{H^{1/2}(0,T)} \leqslant \sum_{j=1}^N \left\|I\,|q_j|^{2\sigma_j}q_j\right\|_{\mathcal{A}_T} \leqslant C_T\sum_{j=1}^N\left\||q_j|^{2\sigma_j}q_j\right\|_{\mathcal{A}_T} \leqslant C_T\sum_{j=1}^N \mathrm{b}_T^{2\sigma_j}\|q_j\|_{\mathcal{A}_T} \leqslant C_T\mathrm{b}_T\sum_{j=1}^N\mathrm{b}_T^{2\sigma_j}$$

where, from now on, C_T stands for a generic positive constant such that $C_T \to 0$, as $T \to 0$, and which may vary from line to line. In addition, using (2.8) and (2.24), one sees that

$$\|\mathcal{G}_1(\mathbf{q})\|_{L^\infty(0,T)} \leqslant \sum_{j=1}^N \left\|I\,|q_j|^{2\sigma_j} q_j\right\|_{L^\infty(0,T)} \leqslant C_T \sum_{j=1}^N \left\||q_j|^{2\sigma_j} q_j\right\|_{L^\infty(0,T)} \leqslant C_T \mathrm{b}_T \sum_{j=1}^N \mathrm{b}_T^{2\sigma_j},$$

so that

$$\|\mathcal{G}_1(\mathbf{q})\|_{\mathcal{A}_T} \leqslant C_T \mathrm{b}_T \sum_{j=1}^N \mathrm{b}_T^{2\sigma_j}. \tag{2.50}$$

On the other hand, by Corollary 2.1, we find that $\|\mathcal{G}_2(\mathbf{q})\|_{H^{1/2}(0,T)} \leqslant C_T \|\mathbf{q}\|_{\mathcal{A}_T} \leqslant C_T \mathrm{b}_T$, while, from (2.24), $\|\mathcal{G}_2(\mathbf{q})\|_{L^\infty(0,T)} \leqslant C_T \|\mathbf{q}\|_{L^\infty(0,T)} \leqslant C_T \mathrm{b}_T$. Thus, we have

$$\|\mathcal{G}_2(\mathbf{q})\|_{\mathcal{A}_T} \leqslant C_T \|\mathbf{q}\|_{\mathcal{A}_T} \leqslant C_T \mathrm{b}_T.$$

Putting it together with (2.50), we finally get

$$\|\mathcal{G}(\mathbf{q})\|_{\mathcal{A}_T} \leqslant \mathrm{b}_T \left[\frac{1}{2} + C_T \left(1 + \sum_{j=1}^N \mathrm{b}_T^{2\sigma_j}\right)\right].$$

Consequently, as the term in brackets is equal to $\frac{1}{2} + o(1)$ as $T \to 0$, for T sufficiently small $\mathcal{G}(\mathbf{q}) \in \mathcal{A}_T$.

Therefore, it is left to prove that $\mathcal{G}$ is actually a norm contraction. Given two functions $\mathbf{q}_1$, $\mathbf{q}_2 \in \mathcal{A}_T$, we have

$$\mathcal{G}(\mathbf{q}_1) - \mathcal{G}(\mathbf{q}_2) = \mathcal{G}_1(\mathbf{q}_1) - \mathcal{G}_1(\mathbf{q}_2) + \mathcal{G}_2(\mathbf{q}_1 - \mathbf{q}_2).$$

Arguing as before, one sees that $\|\mathcal{G}_2(\mathbf{q}_1 - \mathbf{q}_2)\|_{\mathcal{A}_T} \leqslant C_T \|\mathbf{q}_1 - \mathbf{q}_2\|_{\mathcal{A}_T}$. On the other hand, using again (2.24) and Lemma 2.1 and 2.4,

$$\left\|I\left(|q_{j,1}|^{2\sigma_j} q_{j,1} - |q_{j,2}|^{2\sigma_j} q_{j,2}\right)\right\|_{\mathcal{A}_T} \leqslant C_T \left\||q_{j,1}|^{2\sigma_j} q_{j,1} - |q_{j,2}|^{2\sigma_j} q_{j,2}\right\|_{\mathcal{A}_T} \leqslant C_T \mathrm{b}_T^{2\sigma_j} \|q_{j,1} - q_{j,2}\|_{\mathcal{A}_T}.$$

Then, as b_T is bounded, $\|\mathcal{G}_1(\mathbf{q}_1) - \mathcal{G}_1(\mathbf{q}_2)\|_{\mathcal{A}_T} \leqslant C_T \|\mathbf{q}_1 - \mathbf{q}_2\|_{\mathcal{A}_T}$, so that

$$\|\mathcal{G}(\mathbf{q}_1) - \mathcal{G}(\mathbf{q}_2)\|_{\mathcal{A}_T} \leqslant C_T \|\mathbf{q}_1 - \mathbf{q}_2\|_{\mathcal{A}_T}.$$

Hence, since $C_T \to 0$ as $T \to 0$, $\mathcal{G}$ is a contraction on $\mathcal{A}_T$, provided that T is small enough. ☐

Remark 2.2 (Contraction time).
We stress that the value of T depends on the convolution kernel $\mathcal{I}(t - \tau)$ and its asymptotic behavior close to 0. In particular, we can apply the contraction argument to the evolution from $s > 0$ to $t > s$ through (1.11) of an initial datum ψ_s such that $I((U_0(\cdot)\psi_s)(\mathbf{y}_j))(\cdot)$ belongs to $C[s,t] \cap H^{1/2}(s,t)$. A priori the contraction time might be different in this case (e.g., shorter), but it can not collapse to 0, since it depends only on the properties of I and the regularity of ψ_s which can be proved by a contraction argument on the preceding interval.

The contraction time T provided by Proposition 2.3 is a priori shorter than the maximal existence time of a continuous solution T_* given by Proposition 2.2. However, in view of Remark 2.2, we can aim at extending the Sobolev regularity of $\mathbf{q}(t)$ up to T_*, as discussed in next

Proposition 2.4 (Regularity extension of $\mathbf{q}(t)$)**.**
Let $\mathbf{q}(t)$ *be the solution of* (1.11) *provided by Proposition 2.3,* T_* *the maximal existence time given*

in Proposition 2.2 and T the contraction time given by Proposition 2.3. Assume also that for any $T_1 > 0$ such that $T + T_1 < T_*$ and, for any $j = 1, \dots, N$,

$$\int_0^t \mathrm{d}\tau\, \mathcal{I}(t-\tau)\, (U_0(\tau)\psi_T)\, (\mathbf{y}_j) \in C[0,T_1] \cap H^{1/2}(0,T_1), \tag{2.51}$$

then $\mathbf{q}(t) \in H^{1/2}(0,T)$ for any $T < T_*$.

Proof. The key observation is that, thanks to the assumption (2.51), the contraction argument can be repeated starting from $t = T$. Indeed, we can consider the charge equation with initial time $t = T$ and initial datum ψ_T. The analogue of Proposition 2.2 ensures then the existence of a unique continuous solution $\tilde{\mathbf{q}}(t)$ of such an equation for $t < T + T_*'$, with $T_*' > 0$ its maximal existence time. Obviously, by taking the limit $t \to 0$ of the charge equation, one gets $\tilde{\mathbf{q}}(0) = \mathbf{q}(T)$ and therefore $\mathbf{q}$ can be extended continuously to $[T, T + T_*')$ by setting

$$\mathbf{q}_{\mathrm{e}}(t) = \begin{cases} \mathbf{q}(t), & \text{if } t \in [0,T], \\ \tilde{\mathbf{q}}(t-T), & \text{if } t \in [T, T_*'). \end{cases}$$

Indeed $\mathbf{q}_{\mathrm{e}}(t)$ solves the same equation as $\mathbf{q}$ and therefore the uniqueness of the continuous solution in $[0, T_*)$ implies that $\mathbf{q}_{\mathrm{e}} = \mathbf{q}$ and $T + T_*' = T_*$. Now, in view of the Remark 2.2 above, $\tilde{\mathbf{q}}(t) \in C[0,T_1] \cap H^{1/2}(0,T_1)$, for some $T_1 > 0$, by a direct repetition of the contraction argument used in the proof of Proposition 2.3. By a simple bootstrap the result is thus proven. □

At this stage it is useful to sum up the results we have proven on the solution of the charge equation (1.11). Indeed, there are two positive times $T < T_*$ so that there is a unique continuous solution to (1.11) in $[0,T]$, for any $T < T_*$, and such a solution also belongs to $H^{1/2}(0,T)$, provided the property (2.51) holds true (it will actually be proven in Lemma 2.8 to prove the global well-posedness). Note that T_* might as well be $+\infty$ and the solution be global in time, in which case it is also $H^{1/2}$ on any bounded subset of $\mathbb{R}^+$.

We can now prove Theorem 1.1, since the existence and uniqueness of the charge $\mathbf{q}(t)$ will imply that the ansatz (1.7) is a solution to the weak Cauchy problem (1.15). In order to see that and make the derivation of the charge equation discussed in Sect. 2.2 correct, we need to handle integral expressions involving the derivative of $\mathbf{q}(t)$. This will be done as explained in the following Remark.

Remark 2.3 (Integration of $\dot{\mathbf{q}}$ – part I).
In the following of the paper we will often manage integrals involving the distributional derivative of the charge $\mathbf{q}(t)$. Clearly, such a notation is purely formal since we do not actually know whether $\mathbf{q}(t)$ is an absolutely continuous function. Hence, its derivates might not be integrable in Lebesgue sense. However, $q_j(t)\mathbb{1}_{[0,T]}$ is a compactly supported distribution belonging to $\mathcal{E}'$, the dual of $\mathcal{E} = C^\infty(\mathbb{R})$. Hence, its distributional derivative is well defined and it still belongs to $\mathcal{E}'$. On the other hand, for any continuous function f, one obviously has

$$\dot{f}(t)\mathbb{1}_{[0,T]} = \frac{\mathrm{d}}{\mathrm{d}t}\left(f(t)\mathbb{1}_{[0,T]}\right) - f(T)\delta(t-T) + f(0)\delta(t), \tag{2.52}$$

and since the r.h.s. is in $\mathcal{E}'$, the same holds for the l.h.s.. Hence we can give a meaning to the expression

$$\int_0^t \mathrm{d}\tau\, g(\tau)\dot{q}_j(\tau), \tag{2.53}$$

whenever $g \in C^\infty(\mathbb{R})$, as the distributional pairing between $\mathcal{E}'$ and $\mathcal{E}$. Of course the above is not the Lebesgue integral and we should have used a different symbol. However, in order to avoid a too heavy notation, we make a little abuse and keep the same integral symbol. Note that with such a convention we actually have

$$\int_0^t \mathrm{d}\tau\, \dot{\mathbf{q}}(\tau) = \mathbf{q}(t) - \mathbf{q}(0), \tag{2.54}$$

since the function 1 is smooth.
Of course if we knew a priori that $\mathbf{q} \in W^{1,1}(0,T)$, then there would be no problem in the definition of any integral involving $\dot{\mathbf{q}}$ against a continuous function.

Proof of Theorem 1.1. Let ψ_t be the function defined by (1.7) and (1.11). For the sake of simplicity we split the proof in two steps. In the former we show that $\psi_t \in \mathscr{D}[\mathcal{F}]$, in the latter, we prove that ψ_t is a solution of the weak problem (1.15).

In order to prove that $\psi_t \in \mathscr{D}[\mathcal{F}]$, it is sufficient to show that

$$\psi_t(\mathbf{x}) - \frac{1}{2\pi}\sum_{j=1}^N q_j(t) K_0\left(\sqrt{\lambda}|\mathbf{x}-\mathbf{y}_j|\right) \in H^1(\mathbb{R}^2). \tag{2.55}$$

Exploiting (1.7) and the Fourier transform, we can see that the previous expression reads

$$e^{-ip^2t}\widehat{\psi_0}(\mathbf{p}) + \frac{i}{2\pi}\sum_{j=1}^N \int_0^t \mathrm{d}\tau\, e^{-i\mathbf{p}\cdot\mathbf{y}_j} e^{-ip^2(t-\tau)} q_j(\tau) - \frac{1}{2\pi}\frac{q_j(t)e^{-i\mathbf{p}\cdot\mathbf{y}_j}}{p^2+\lambda}.$$

Hence, integrating by parts (in view of Remark 2.3), one finds

$$e^{-ip^2t}\left(\widehat{\psi_0}(\mathbf{p}) - \sum_{j=1}^N \frac{q_j(0)e^{-i\mathbf{p}\cdot\mathbf{y}_j}}{2\pi(p^2+\lambda)}\right) - \frac{e^{-i\mathbf{p}\cdot\mathbf{y}_j}}{2\pi(p^2+\lambda)}\sum_{j=1}^N\int_0^t \mathrm{d}\tau\, e^{-ip^2(t-\tau)}(\dot{q}_j(\tau) - i\lambda q_j(\tau)). \tag{2.56}$$

Note that the integral of $\dot{q}_j$ on the r.h.s. has to be understood as explained in Remark 2.3, which can be done since $e^{-ip^2(t-\tau)}$ is a smooth function of τ.

Now, if this functions belongs to $L^2(\mathbb{R}^2, (p^2+1)\,\mathrm{d}\mathbf{p})$, then (2.55) is fulfilled. For the first term this is immediate since it represents the Fourier transform of $U_0(t)\phi_{\lambda,0}$, which is in $H^1(\mathbb{R}^2)$, since $\phi_{\lambda,0}$ does. Concerning the second term, we first set $\lambda = 1$ for the sake of simplicity, and change variables to get

$$\int_{\mathbb{R}^2} \mathrm{d}\mathbf{p}\,(1+p^2)\left|\frac{e^{-i\mathbf{p}\cdot\mathbf{y}_j}}{2\pi(p^2+1)}\sum_{j=1}^N\int_0^t \mathrm{d}\tau\, e^{-ip^2(t-\tau)}(\dot{q}_j(\tau) - iq_j(\tau))\right|^2$$
$$\leqslant C\sum_{j=1}^N\int_0^\infty \mathrm{d}\varrho\,\frac{1}{1+\varrho}\left[\left|\int_0^t \mathrm{d}\tau\, e^{i\varrho\tau}\dot{q}_j(\tau)\right|^2 + \left|\int_0^t \mathrm{d}\tau\, e^{i\varrho\tau}q_j(\tau)\right|^2\right].$$

Now, one can check that

$$\int_0^t \mathrm{d}\tau\, e^{i\varrho\tau}\dot{q}_j(\tau) = \sqrt{2\pi}\,\hat{\dot{\xi}}_j(-\varrho), \qquad \int_0^t \mathrm{d}\tau\, e^{i\varrho\tau}q_j(\tau) = \sqrt{2\pi}\,\widehat{\mathbb{1}_{[0,t]}q_j}(-\varrho),$$

where

$$\xi_j(\tau) := \begin{cases} q_j(0), & \text{if } \tau \leqslant 0, \\ q_j(\tau), & \text{if } 0 < \tau < t, \\ q_j(t), & \text{if } \tau \geqslant t. \end{cases} \tag{2.57}$$

Note that $\dot{\xi}_j$ is a distribution belonging to $\mathcal{E}'$, as $\dot{q}_j \mathbb{1}_{[0,T]}$ does, therefore we can define its Fourier transform, which is in fact a (sufficiently) smooth function. Consequently,

$$\sum_{j=1}^{N} \int_0^\infty \mathrm{d}\varrho \, \frac{1}{1+\varrho} \left[\left| \int_0^t \mathrm{d}\tau \, e^{i\varrho\tau} \dot{q}_j(\tau) \right|^2 + \left| \int_0^t \mathrm{d}\tau \, e^{i\varrho\tau} q_j(\tau) \right|^2 \right]$$
$$\leqslant 2\pi \sum_{j=1}^{N} \int_{\mathbb{R}} \mathrm{d}\varrho \, \frac{\left| \hat{\dot{\xi}}_j(\varrho) \right|^2}{1+|\varrho|} + \sum_{j=1}^{N} \int_{\mathbb{R}} \mathrm{d}\varrho \, \frac{\left| \widehat{\mathbb{1}_{[0,t]} q_j}(\varrho) \right|^2}{1+|\varrho|}.$$

Now, since $q_j \in H^{1/2}(0,T)$, $\dot{\xi}_j \in H^{-1/2}(\mathbb{R})$ (see [CCF]) and thus the right hand side of the previous inequality is finite. Summing up, (2.56) belongs to $L^2(\mathbb{R}^2, (p^2+1)\,\mathrm{d}\mathbf{p})$ and then (2.55) is satisfied.

Once we know that $\psi_t \in \mathscr{D}[\mathcal{F}]$, it just remains to show that it solves the weak Cauchy problem (1.15). However, once that the function $\mathbf{q}$ is fixed by the charge equation (thanks to Propositions 2.2, 2.3 and 2.4) the required computations are completely analogous analogous to those of the linear non-autonomous problem, provided that one sets $\alpha(t) = |q_j(t)|^{2\sigma_j}$, and in that case this has been already proved in [CCF]. □

2.4. **Conservation laws.** In this section we prove the conservation of mass and energy claimed in Theorem 1.2, which in turn will be the key to prove the global existence stated in Theorem 1.3. We recall the results proven in Propositions 2.2 and 2.3: there exists some $T_* > 0$ such that there is a unique continuous solution of (1.11) in $[0, T_*)$, which also belongs to $H^{1/2}(0,T)$ for some $0 < T < T_*$.

Before proceeding further, however, another Remark is in order about the use we will make of the derivative of $\mathbf{q}$. In view of Remark 2.3 it can be "integrated" against smooth functions by exploiting the distributional pairing. Here we aim at giving a meaning to some more singular expressions:

Remark 2.4 (Integration of $\dot{\mathbf{q}}(t)$ – part II).
Thanks to Proposition 2.3, we know that for some $T < T_*$, $\mathbf{q} \in H^{1/2}(0,T)$. We claim that this is sufficient to give a rigorous meaning to the expression

$$\int_0^T \mathrm{d}t \, f(t) \dot{q}_j(t),$$

for any function $f \in C_{\log,\beta}[0,T] \cap H^{1/2}(0,T)$, $\beta > 1/2$. The idea is to use the pairing provided by the duality between $H^{1/2}(\mathbb{R})$ and $H^{-1/2}(\mathbb{R})$, which allows to interpret the integral of $f^* g$, with $f \in H^{1/2}(\mathbb{R})$ and $g \in H^{-1/2}(\mathbb{R})$, as

$$\int_{\mathbb{R}} \mathrm{d}t \, f^*(t) g(t) = \int_{\mathbb{R}} \mathrm{d}p \, \left(\sqrt{p^2+1} \widehat{f}^*(p) \right) \left(\tfrac{1}{\sqrt{p^2+1}} \widehat{g}(p) \right), \tag{2.58}$$

where the symbol on the l.h.s. is not the Lebesgue integral, while on the r.h.s. we are integrating the product of two L^2 functions. Note that such a duality fails in general on a compact subset of the real line.

So, if $f \in C_{\log,\beta}[0,T] \cap H^{1/2}(0,T)$, we can rewrite

$$\int_0^T \mathrm{d}t \, f(t) \dot{q}_j(t) = f(T) \left(q_j(T) - q_j(0) \right) + \int_0^T \mathrm{d}t \, \left(f(t) - f(T) \right) \dot{q}_j(\tau)$$

and, since both f and q are continuous, $f(T)$ is well defined as well as $q_j(T)$ and $q_j(0)$. Next we observe that $f(t) - f(T)$ satisfies the hypothesis of Proposition 2.1 with $\beta > 1/2$ and therefore there exists an extension $f_{\rm e} \in H^{1/2}(\mathbb{R})$ of $f(t) - f(T)$, such that

$$\int_0^T \mathrm{d}t\ (f(t) - f(T))\,\dot{q}_j(\tau) = \int_{\mathbb{R}} \mathrm{d}t\, f_{\rm e}(t)\dot{\xi}_j(\tau),$$

where ξ_j is defined in (2.57). Here we are using that $\mathrm{supp}(\dot{\xi}_j) \subset [0,T]$. Now, since $f_{\rm e} \in H^{1/2}(\mathbb{R})$ and $\dot{\xi}_j \in H^{-1/2}(\mathbb{R})$ (cfr. [CCF]), then the last integral is meant as in (2.58).

Before attacking the proof, we state a technical Lemma, which is a consequence of the charge equation (1.11) and which will be used in the derivation of the mass and energy conservation.

Lemma 2.6.
Let $\mathbf{q}(t)$ be the solution of (1.11) *provided by Proposition 2.3 and T_* the maximal existence time given in Proposition 2.2, then for every $j = 1, \dots, N$*

$$\begin{aligned}(U_0(t)\psi_0)(\mathbf{y}_j) = -\frac{i}{2\pi}\sum_{k\neq j}\int_0^t \mathrm{d}\tau\, U_0(t-\tau;|\mathbf{y}_j-\mathbf{y}_k|)q_k(\tau) + \left(\beta_j|q_j(t)|^{2\sigma_j} + \frac{\gamma-\log 2}{2\pi}\right)q_j(t)\\ - \frac{iq_j(t)}{8} + \frac{1}{4\pi}\frac{d}{dt}\int_0^t \mathrm{d}\tau\,(-\gamma-\log(t-\tau))q_j(\tau), \quad (2.59)\end{aligned}$$

for a.e. $t \in [0,T]$ with $T < T_*$.

Proof. Dividing the charge equation (1.11) by 4π and recalling (1.13) and (1.14), we get

$$\begin{aligned}&\frac{1}{4\pi}q_j(t) + \int_0^t \mathrm{d}\tau\,\mathcal{I}(t-\tau)\big(\beta_j|q_j(\tau)|^{2\sigma_j} + \tfrac{\gamma-\log 2}{2\pi} - \tfrac{i}{8}\big)q_j(\tau)\\ &\quad - \frac{i}{2\pi}\sum_{k\neq j}\int_0^t \mathrm{d}\tau\,\mathcal{I}(t-\tau)\int_0^\tau \mathrm{d}s\,U_0\left(\tau-s;|\mathbf{y}_j-\mathbf{y}_k|\right)q_k(s) = \int_0^t \mathrm{d}\tau\,\mathcal{I}(t-\tau)(U_0(\tau)\psi_0)(\mathbf{y}_j)\end{aligned}$$

where we used backwards the same change of variable exploited at the end of Sect. 2.2. Hence, applying to both sides the operator J defined by (2.35) and recalling Lemma 2.5, we obtain

$$\begin{aligned}&\frac{1}{4\pi}(Jq_j)(t) + \int_0^t \mathrm{d}\tau\,\left(\beta_j|q_j(\tau)|^{2\sigma_j} + \frac{\gamma-\log 2}{2\pi} - \frac{i}{8}\right)q_j(\tau)\\ &\quad - \frac{i}{2\pi}\sum_{k\neq j}\int_0^t \mathrm{d}\tau\int_0^\tau \mathrm{d}s\,U_0\left(\tau-s;|\mathbf{y}_j-\mathbf{y}_k|\right)q_k(s) = \int_0^t \mathrm{d}\tau\,(U_0(\tau)\psi_0)(\mathbf{y}_j)\end{aligned}$$

and, in particular, that Jq_j is absolutely continuous. Then, differentiating in t and rearranging terms, we obtain (2.59). □

In view of Remark 2.4 the following technical results will prove to be very useful.

Lemma 2.7.
Let $\mathbf{q}(t)$ be the solution of (1.11) *provided by Proposition 2.3 and T_* the maximal existence time given in Proposition 2.2, then*

$$\mathbf{q}(t) \in C_{\log,\beta}[0,T], \qquad \forall \beta \leqslant 1, \quad (2.60)$$

for any $T < T_*$.

Proof. Fix $T < T_*$. We first remark that, since we are always considering continuous functions on compact sets, proving that $f \in C_{\log,\beta}[0,T]$ is equivalent to show that

$$\lim_{\delta\to 0} |\log\delta|^\beta\,|f(s+\delta) - f(s)| \leqslant C < +\infty, \quad (2.61)$$

for any $s \in [0,T]$ (where at the extreme points the limit has to be suitably adjusted).

From the charge equation (1.11), we get

$$q_j(t) = -4\pi\beta_j \int_0^t \mathrm{d}\tau\, \mathcal{I}(t-\tau)|q_j(\tau)|^{2\sigma_j} q_j(\tau) \\ - \sum_{k=1}^N \int_0^t \mathrm{d}\tau\, \mathcal{K}_{jk}(t-\tau) q_k(\tau) + 4\pi \int_0^t \mathrm{d}\tau\, \mathcal{I}(t-\tau)(U_0(\tau)\psi_0)(\mathbf{y}_j), \quad (2.62)$$

i.e., $q_j(t) = I_1(t) + I_2(t) + I_3(t)$ (with obvious meaning of the three terms).

Let us consider first $I_1(t)$. The case $t = 0$, $\delta > 0$ is easier to deal with: since $\mathbf{q}(t)$ is bounded on $[0,T]$,

$$|I_1(\delta) - I_1(0)| \leqslant C \|q_j\|_{L^\infty(0,T)}^{2\sigma_j+1} \int_0^\delta \mathrm{d}\tau\, \mathcal{I}(\delta - \tau) \leqslant C\mathcal{N}(\delta) \underset{\delta\to 0}{\sim} \frac{C}{|\log\delta|}$$

where we recall the definition of $\mathcal{N}$ given by (2.20) and its asymptotic behavior in (2.21). On the other hand, if we consider the case $t \in (0,T)$, $\delta > 0$ ($\delta < 0$ is analogous), then we see that

$$I_1(t+\delta) - I_1(t) = \int_t^{t+\delta} \mathrm{d}\tau\, \mathcal{I}(t+\delta-\tau)|q_j(\tau)|^{2\sigma_j} q_j(\tau) \\ + \int_0^t \mathrm{d}\tau\, \left[\mathcal{I}(t+\delta-\tau) - \mathcal{I}(t-\tau)\right] |q_j(\tau)|^{2\sigma_j} q_j(\tau) := I_{1,1}(\delta,t) + I_{1,2}(\delta,t).$$

Now, arguing as before, one easily finds that $I_{1,1}(\delta,t) \sim \frac{1}{|\log\delta|}$ as $\delta \to 0$. Furthermore, again by the boundedness of $\mathbf{q}$, one has

$$|I_{2,1}(\delta,t)| \leqslant C \int_0^t \mathrm{d}\tau\, |\mathcal{I}(\tau+\delta) - \mathcal{I}(\tau)|\,. \quad (2.63)$$

Since $\mathcal{I}$ is continuous, coercive and strictly convex [CFT, H], it has a unique minimum point $t_{\min} > 0$. If $t + \delta \leqslant t_{\min}$, then

$$\int_0^t \mathrm{d}\tau\, |\mathcal{I}(\tau+\delta) - \mathcal{I}(\tau)| = \mathcal{N}(\delta) + \mathcal{N}(t) - \mathcal{N}(t+\delta) \leqslant \mathcal{N}(\delta) + \delta \sup_{\tau\in[t,t+\delta]} \mathcal{I}(\tau) \underset{\delta\to 0}{\sim} \frac{1}{|\log\delta|}.$$

Thus, combining with (2.63), one has $I_{1,2}(\delta,t) \sim \frac{1}{|\log\delta|}$ as $\delta \to 0$. If, on the opposite, $t \geqslant t_{\min}$ (the case $t < t_{\min} < t+\delta$ can be excluded for δ small enough), then

$$\int_0^t \mathrm{d}\tau\, |\mathcal{I}(\tau+\delta) - \mathcal{I}(\tau)| = \int_0^{t_{\min}-\delta} \mathrm{d}\tau\, \left(\mathcal{I}(\tau) - \mathcal{I}(\tau+\delta)\right) + \int_{t_{\min}-\delta}^{t_{\min}} \mathrm{d}\tau\, |\mathcal{I}(\tau+\delta) - \mathcal{I}(\tau)| \\ + \int_{t_{\min}}^t \mathrm{d}\tau\, \left(\mathcal{I}(\tau+\delta) - \mathcal{I}(\tau)\right) \leqslant \mathcal{N}(t_{\min}-\delta) - \mathcal{N}(t_{\min}) + \mathcal{N}(\delta) \\ + \mathcal{N}(t+\delta) - \mathcal{N}(t) - \mathcal{N}(t_{\min}+\delta) + \mathcal{N}(t_{\min}) + C\delta \leqslant \mathcal{N}(t+\delta) - \mathcal{N}(t) + \mathcal{N}(\delta) + C\delta$$

and, arguing as before, we obtain $I_{1,2}(\delta,t) \sim \frac{1}{|\log\delta|}$, as $\delta \to 0$.

Therefore, it is left to investigate the behavior of $I_2(t)$ and $I_3(t)$. Exploiting the properties of $\mathcal{R}_{jk}$ (recall that it a bounded and continuous function), one can easily see that $I_2(t)$ can be studied in the same way as $I_1(t)$. On the contrary, $I_3(t)$ requires some further remark, since $(U_0(\tau)\psi_0)(\mathbf{y}_j)$ is not bounded on $[0,T]$. However, from (2.48) it can be split into the sum of three terms A_1, A_2 and A_3. The first and the last ones are bounded and hence it is possible to use the previous strategy to prove that IA_1 and IA_3 have the needed property. Concerning A_2, arguing as in the proof of Proposition 2.2, one sees that it can be split, in turn, in two terms $A_{2,1}$ and $A_{2,2}$, where the second one is bounded and the first one satisfies the following property

$$\int_0^t \mathrm{d}\tau\, \mathcal{I}(t-\tau) A_{2,1}(\tau) = 1 + \int_0^t \mathrm{d}\tau\, \mathcal{I}(t-\tau) a_{2,1}(\tau) \tag{2.64}$$

with $a_{2,1}(\tau)$ bounded. Consequently, IA_2 can be bounded exactly as above. □

Lemma 2.8.
Let $\phi \in \mathscr{D}$ with $\mathscr{D}$ defined in (1.16)*, then*

$$(U_0(t)\phi_{1,0})(\mathbf{x}) \in C_{\log,\beta}[0,T] \cap H^{1/2}(0,T), \qquad \forall \beta \in \mathbb{R}^+, \tag{2.65}$$

$$(U_0(t)K_0(\cdot\, - \mathbf{y})(\mathbf{x}) \in C_{\log,\beta}(0,T] \cap H^{1/2}(0,T), \qquad \forall \beta \in \mathbb{R}^+, \tag{2.66}$$

for any T finite and for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$.

Proof. The fact that $(U_0(t)\phi_{1,0})(\mathbf{x}) \in H^{1/2}(0,T)$, for any finite $T > 0$, was already discussed in the proof of Proposition 2.3. Hence we have only to verify the other property. By expressing the quantity using the Fourier transform, we have

$$|(U_0(t+\delta)\phi_{1,0})(\mathbf{x}) - (U_0(t)\phi_{1,0})(\mathbf{x})| \leqslant \int_{\mathbb{R}^2} \mathrm{d}\mathbf{p} \left| e^{-ip^2\delta} - 1 \right| \left| \widehat{\phi_{1,0}}(\mathbf{p}) \right|.$$

Again to show that $(U_0(t)\phi_{1,0})(\mathbf{x}) \in C_{\log,\beta}[0,T]$, it suffices to prove that the analogue of (2.61) holds true, but, for any $\epsilon > 0$,

$$|\log\delta|^\beta \left| e^{-ip^2\delta} - 1 \right| \leqslant 2^{1-\epsilon/2} p^\epsilon \delta^{\epsilon/2} |\log\delta|^\beta \xrightarrow[\delta\to 0]{} 0,$$

for any β finite, and the result is thus a direct consequence of the properties of ψ_0 (recall the definition of $\mathscr{D}$ in (1.16)) and dominated convergence.

From (2.49) and smoothness of $A_{2,2}$, we see that the properties of $(U_0(t)K_0(\cdot\, - \mathbf{y})(\mathbf{x})$, $t > 0$, are the same as $e^{it}\log t$ and $\log t$ belongs to $H^{1/2}(0,T)$, for any finite $T > 0$, while e^{it} is a smooth function and can be dropped. Therefore it suffices to show that $\log\tau \in C_{\log,\beta}(0,T]$, but for any $t > 0$,

$$|\log\delta|^\beta \,|\log(t+\delta) - \log t| \leqslant \frac{1}{t}\delta|\log\delta|^\beta \xrightarrow[\delta\to 0]{} 0,$$

for any $\beta > 0$. □

Lemma 2.9.
Let $\mathbf{q}(t)$ be the solution of (1.11) *provided by Proposition 2.3 and T_* the maximal existence time given in Proposition 2.2, then for every $j = 1, \ldots, N$,*

$$\int_0^t \mathrm{d}\tau\ (-\gamma - \log(t-\tau))\, \dot{\mathbf{q}}(\tau) \in C_{\log,\beta}(0,T] \cap H^{1/2}(0,T), \qquad \forall \beta \leqslant 1, \tag{2.67}$$

for any $T < T_*$.

Proof. First of all we notice that, in light of Remark 2.4, the above expression can be given a meaning using the pairing between $H^{1/2}(\mathbb{R})$ and $H^{-1/2}(\mathbb{R})$: by replacing $\dot{q}_j$ with $\dot{\xi}_j$ defined in (2.57), one can extend the integral to the whole real line. A change of variable then yields

$$\int_0^t \mathrm{d}\tau\ (-\gamma - \log(t-\tau))\, \dot{q}_j(\tau) = \int_{\mathbb{R}} \mathrm{d}\tau\, \mathbb{1}_{[0,t]}(\tau)\, (-\gamma - \log(\tau))\, \dot{\xi}_j(t-\tau),$$

where the characteristic function of $[0,t]$ can be removed thanks to the definition of ξ_j. Since $\log \tau \in H^{1/2}(0,T) \cap C_{\log,\beta}(0,T]$, for any finite T and $\beta > 0$ (see the proof of Lemma 2.8), we can extend it to a function in $H^{1/2}(\mathbb{R})$.

By applying the operator J to the charge equation (1.11) and using (2.36), we get

$$\int_0^t \mathrm{d}\tau \; (-\gamma - \log(t-\tau))\, q(\tau) = -4\pi \int_0^t \mathrm{d}\tau \, |q_j(\tau)|^2 \, q_j(\tau) + 4\pi\beta_j \sum_{k=1}^N \int_0^t \mathrm{d}\tau \, \mathcal{Y}_{jk}(t-\tau) q_k(\tau) + 4\pi \int_0^t \mathrm{d}\tau \; (U_0(\tau)\psi_0)(\mathbf{y}_j),$$

where we have set

$$\mathcal{Y}_{jk}(t) := \begin{cases} 2\left(\log 2 - \gamma + \frac{i\pi}{4}\right), & \text{if } j = k, \\ \mathcal{R}_{jk}(t), & \text{if } j \neq k. \end{cases} \tag{2.68}$$

By taking the (weak) derivative of the above identity and observing that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \mathrm{d}\tau \; (-\gamma - \log(t-\tau))\, q(\tau) = -(\gamma + \log t) q(0) + \int_0^t \mathrm{d}\tau \; (-\gamma - \log(t-\tau))\, \dot{q}(\tau),$$

we get

$$\int_0^t \mathrm{d}\tau \; (-\gamma - \log(t-\tau))\, \dot{q}(\tau) = (\gamma + \log t) q(0) + 4\pi \left(U_0(t)\psi_0\right)(\mathbf{y}_j) - 4\pi \, |q_j(t)|^2 \, q_j(t) + 4\pi\beta_j \sum_{k=1}^N \mathcal{Y}_{jk}(0) q_k(t). \tag{2.69}$$

Now the claim follows by simply observing that all the terms on r.h.s. enjoy the required properties. Indeed we have seen that $\log t \in C_{\log,\beta}(0,T] \cap H^{1/2}(0,T)$ for any T finite and $\beta > 0$. Moreover, thanks to Proposition 2.3 and Lemma 2.8, q_j has the needed regularity for any $T < T_*$ and $\beta \leqslant 1$ and this allows us to deal with the last two terms (recall the boundedness of $\mathcal{R}_{jk}$). Finally Lemma 2.8 yields the result once applied to the second term. □

Proof of Theorem 1.2. The proof is divided into two parts, where we prove mass and energy conservation separately.

Part 1. Let us first consider the mass conservation. Using the Fourier transform, (1.7) reads

$$\widehat{\psi}_t(\mathbf{p}) = e^{-ip^2 t} \widehat{\psi_0}(\mathbf{p}) + \frac{i}{2\pi} \sum_{j=1}^N e^{-i\mathbf{p}\cdot\mathbf{y}_j} \int_0^t \mathrm{d}\tau \, e^{-ip^2(t-\tau)} q_j(\tau).$$

Hence,

$$\left|\widehat{\psi}_t(\mathbf{p})\right|^2 = \left|\widehat{\psi_0}(\mathbf{p})\right|^2 + \frac{1}{\pi} \sum_{j=1}^N \mathrm{Im}\left\{ e^{i\mathbf{p}\cdot\mathbf{y}_j} \widehat{\psi_0}(\mathbf{p}) \int_0^t \mathrm{d}\tau \, e^{-ip^2\tau} q_j^*(\tau) \right\} + \frac{1}{4\pi^2} \sum_{j,k=1}^N e^{i\mathbf{p}\cdot(\mathbf{y}_k - \mathbf{y}_j)} \int_0^t \mathrm{d}\tau \int_0^t \mathrm{d}s \, e^{-ip^2(s-\tau)} q_j(\tau) q_k^*(s),$$

so that, denoting by $\mathscr{F}^{-1}$ the anti-Fourier transform on $\mathbb{R}^2$,

$$\|\psi_t\|^2_{L^2(\mathbb{R}^2)} = \|\psi_0\|^2_{L^2(\mathbb{R}^2)} + 2\sum_{j=1}^N \mathscr{F}^{-1}\left\{\mathrm{Im}\left\{\widehat{\psi_0}(\mathbf{p})\int_0^t \mathrm{d}\tau\, e^{-ip^2\tau}q_j^*(\tau)\right\}\right\}(\mathbf{y}_j)$$
$$+\frac{1}{2\pi}\sum_{j,k=1}^N \mathscr{F}^{-1}\left\{\int_0^t \mathrm{d}\tau\int_0^t \mathrm{d}s\, e^{-ip^2(s-\tau)}q_j(\tau)q_k^*(s)\right\}(\mathbf{y}_k-\mathbf{y}_j) =: \|\psi_0\|^2_{L^2(\mathbb{R}^2)} + \Psi + \Phi.$$

Now, by the properties of the Fourier transform and the definition of U_0,

$$\Psi = 2\sum_{j=1}^N \mathrm{Im}\left\{\int_0^t \mathrm{d}\tau\, q_j^*(\tau)\mathscr{F}^{-1}\{e^{-ip^2\tau}\widehat{\psi_0}(\mathbf{p})\}(\mathbf{y}_j)\right\} = 2\sum_{j=1}^N \mathrm{Im}\left\{\int_0^t \mathrm{d}\tau\, q_j^*(\tau)(U_0(\tau)\psi_0)(\mathbf{y}_j)\right\},$$

so that by (2.59) proven in Lemma 2.6,

$$\Psi = \Psi_1+\Psi_2+\Psi_3 := -\frac{1}{\pi}\sum_{j=1}^N\sum_{k\neq j}\mathrm{Re}\left\{\int_0^t \mathrm{d}\tau\int_0^\tau \mathrm{d}s\, q_j^*(\tau)q_k(s)U_0\left(\tau-s;|\mathbf{y}_j-\mathbf{y}_k|\right)\right\}$$
$$-\frac{1}{4}\sum_{j=1}^N\int_0^t \mathrm{d}\tau\,|q_j(\tau)|^2 + \frac{1}{2\pi}\sum_{j=1}^N \mathrm{Im}\left\{\int_0^t \mathrm{d}\tau\, q_j^*(\tau)\frac{d}{d\tau}\int_0^\tau \mathrm{d}s\,(-\gamma-\log(\tau-s)q_j(s))\right\}.$$

On the other hand, Φ can be split into two terms as well:

$$\Phi = \Phi_1+\Phi_2 := \frac{1}{2\pi}\sum_{j=1}^N \mathscr{F}^{-1}\left\{\int_0^t \mathrm{d}\tau\int_0^t \mathrm{d}s\, e^{-ip^2(s-\tau)}q_j(\tau)q_j^*(s)\right\}(\mathbf{0})$$
$$+\frac{1}{2\pi}\sum_{j=1}^N\sum_{k\neq j}\mathscr{F}^{-1}\left\{\int_0^t \mathrm{d}\tau\int_0^t \mathrm{d}s\, e^{-ip^2(s-\tau)}q_j(\tau)q_k^*(s)\right\}(\mathbf{y}_k-\mathbf{y}_j).$$

One can easily see that Φ_2 cancels with Ψ_1, since

$$\Phi_2 = \frac{1}{\pi}\sum_{j=1}^N\sum_{k\neq j}\mathrm{Re}\left\{\int_0^t \mathrm{d}\tau\int_0^\tau \mathrm{d}s\, q_j^*(\tau)q_k(s)\mathscr{F}^{-1}\{e^{-ip^2(\tau-s)}\}(\mathbf{y}_j-\mathbf{y}_k)\right\}$$
$$= \frac{1}{\pi}\sum_{j=1}^N\sum_{k\neq j}\mathrm{Re}\left\{\int_0^t \mathrm{d}\tau\int_0^\tau \mathrm{d}s\, q_j^*(\tau)q_k(s)U_0\left(\tau-s;|\mathbf{y}_j-\mathbf{y}_k|\right)\right\} = -\Psi_1.$$

Then, it is left to prove that $\Psi_2+\Psi_3+\Phi_1=0$. First, one sees that

$$\Phi_1 = \frac{1}{\pi}\sum_{j=1}^N \mathscr{F}^{-1}\left\{\mathrm{Re}\left\{\int_0^t \mathrm{d}\tau\, q_j^*(\tau)\int_0^\tau \mathrm{d}s\, q_j(s)e^{-ip^2(\tau-s)}\right\}\right\}(\mathbf{0}),$$

then we compute

$$\int_0^\tau \mathrm{d}s\, q_j(s)e^{-ip^2(\tau-s)} = i\frac{d}{d\tau}\int_0^\tau \mathrm{d}s\,\frac{e^{-ip^2(\tau-s)}}{p^2+1}q_j(s) - \frac{iq_j(\tau)}{p^2+1} + \int_0^\tau \mathrm{d}s\,\frac{e^{-ip^2(\tau-s)}}{p^2+1}q_j(s),$$

thus obtaining

$$\Phi_1 = -\frac{1}{\pi}\sum_{j=1}^{N}\mathscr{F}^{-1}\left\{\mathrm{Im}\bigg\{\int_0^t \mathrm{d}\tau\, q_j^*(\tau)\frac{d}{d\tau}\int_0^\tau \mathrm{d}s\,\frac{e^{-ip^2(\tau-s)}}{p^2+1}q_j(s)\bigg\}\right\}(\mathbf{0})$$
$$+\frac{1}{\pi}\sum_{j=1}^{N}\mathscr{F}^{-1}\left\{\mathrm{Re}\bigg\{\int_0^t \mathrm{d}\tau\, q_j^*(\tau)\int_0^\tau \mathrm{d}s\,\frac{e^{-ip^2(\tau-s)}}{p^2+1}q_j(s)\bigg\}\right\}(\mathbf{0}).$$

Hence, using again the properties of the Fourier transform, the above expression can be rewritten as

$$\Phi_1 = -\frac{1}{\pi}\sum_{j=1}^{N}\mathrm{Im}\left\{\int_0^t \mathrm{d}\tau\, q_j^*(\tau)e^{i\tau}\frac{d}{d\tau}\left\{e^{-i\tau}\int_0^\tau \mathrm{d}s\, q_j(s)\mathscr{F}^{-1}\left[\frac{e^{-ip^2(\tau-s)}}{p^2+1}\right]\right\}(\mathbf{0})\right\}$$
$$= -\frac{1}{2\pi}\sum_{j=1}^{N}\mathrm{Im}\left\{\int_0^t \mathrm{d}\tau\, q_j^*(\tau)e^{i\tau}\frac{d}{d\tau}\int_0^\tau \mathrm{d}s\, q_j(s)e^{-is}(-\gamma-\log(\tau-s))\right\}$$
$$-\frac{1}{2\pi}\sum_{j=1}^{N}\mathrm{Im}\left\{\int_0^t \mathrm{d}\tau\, q_j^*(\tau)e^{i\tau}\frac{d}{d\tau}\int_0^\tau \mathrm{d}s\, q_j(s)e^{-is}Q(1;\tau-s)\right\} =: \Phi_{1,1}+\Phi_{1,2},$$

where we have made use of (2.42) and (2.43). Now, with some computations, one finds that

$$\Phi_{1,2}+\Psi_2 = -\frac{1}{2\pi^2}\sum_{j=1}^{N}\mathrm{Im}\bigg\{\int_0^t \mathrm{d}\tau\, q_j^*(\tau)\int_0^\tau \mathrm{d}s\, q_j(s)e^{i(\tau-s)}\dot{Q}(1;\tau-s)\bigg\}$$
$$= -\frac{1}{2\pi^2}\sum_{j=1}^{N}\mathrm{Im}\bigg\{\int_0^t \mathrm{d}\tau\, q_j^*(\tau)\int_0^\tau \mathrm{d}s\, q_j(s)\frac{e^{i(\tau-s)}-1}{\tau-s}\bigg\},$$

since $\dot{Q}(1;\tau-s) = \frac{1-e^{-i(\tau-s)}}{\tau-s}$, as it follows from (2.42) and the definition of the sine and cosine integral functions [AS, Eqs. 5.2.1 & 5.2.2]. Similarly,

$$e^{i\tau}\frac{d}{d\tau}\int_0^\tau \mathrm{d}s\, q_j(s)e^{-is}(-\gamma-\log(\tau-s)) = \frac{d}{d\tau}\int_0^\tau \mathrm{d}s\, q_j(s)(-\gamma-\log(\tau-s))$$
$$-\int_0^\tau \mathrm{d}s\, q_j(s)\frac{e^{i(\tau-s)}-1}{\tau-s}$$

leading to $\Phi_{1,2}+\Psi_2+\Phi_{1,1} = -\Psi_3$ and thus completing the proof of the mass conservation.

Part 2. Let us turn now our attention to energy conservation. Since $\psi_0 \in \mathscr{D}[\mathcal{F}]$, taking $\lambda = 1$, (1.7) yields

$$\psi_t(\mathbf{x}) = (U_0(t)\phi_{1,0})(\mathbf{x}) + \frac{1}{2\pi}\sum_{j=1}^{N} q_j(t)K_0(|\mathbf{x}-\mathbf{y}_j|)$$
$$-\frac{1}{2\pi}\sum_{j=1}^{N}\int_0^t \mathrm{d}\tau\,(\dot{q}_j(\tau)-iq_j(\tau))\,(U_0(t-\tau)K_0(|\cdot-\mathbf{y}_j|))(\mathbf{x}), \quad (2.70)$$

where we have integrated by parts and used the simple formula

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[e^{-i(t-\tau)}(U_0(t-\tau)K_0)(\mathbf{x})\right] = ie^{-i(t-\tau)}U_0(t-\tau;\mathbf{x}), \quad (2.71)$$

which can be easily verified by rewriting the quantities via Fourier transform. In light of Remark 2.4 and Lemma 2.8, the term involving $\dot{q}_j$ has to be understood as discussed in Remark 2.4, i.e., as the pairing between a function in $H^{-1/2}(\mathbb{R})$ and another in $H^{1/2}(\mathbb{R})$. In the very same way we get

$$\phi_{1,t}(\mathbf{x}) = (U_0(t)\phi_{1,0})(\mathbf{x}) - \frac{1}{2\pi}\sum_{j=1}^{N}\int_0^t \mathrm{d}\tau\ (\dot{q}_j(\tau) - iq_j(\tau))\,(U_0(t-\tau)K_0(|\cdot - \mathbf{y}_j|))(\mathbf{x}).$$

Then, we can compute the H^1 norm of $\phi_{1,t}$ as

$$\|\phi_{1,t}\|^2_{H^1(\mathbb{R}^2)} = \|\phi_{1,0}\|^2_{H^1(\mathbb{R}^2)} + \Psi_t + \Phi_t, \tag{2.72}$$

where

$$\Psi_t = -2\sum_{j=1}^{N}\mathrm{Re}\bigg(\int_0^t \mathrm{d}\tau\ (\dot{q}_j(\tau) - iq_j(\tau))^*\,(U_0(\tau)\phi_{1,0})(\mathbf{y}_j)\bigg) \tag{2.73}$$

and Φ_t can be split in two parts, i.e., $\Phi_t = \Phi_\mathrm{d} + \Phi_\mathrm{od}$, with

$$\Phi_\mathrm{d} = \frac{1}{4\pi^2}\sum_{j=1}^{N}\int_0^t \mathrm{d}\tau\int_0^t \mathrm{d}s\ (\dot{q}_j(\tau) - iq_j(\tau))\,(U_0(\tau-s)K_0(|\cdot - \mathbf{y}_j|))(\mathbf{y}_j)\,(\dot{q}_j(s) - iq_j(s))^*, \tag{2.74}$$

$$\Phi_\mathrm{of} = \frac{1}{4\pi^2}\sum_{j=1}^{N}\sum_{k\neq j}\int_0^t \mathrm{d}\tau\int_0^t \mathrm{d}s\ (\dot{q}_j(\tau) - iq_j(\tau))$$
$$\times (U_0(\tau-s)K_0(|\cdot - \mathbf{y}_k|))(\mathbf{y}_j)\,(\dot{q}_j(s) - iq_j(s))^*. \tag{2.75}$$

Lemma 2.8 guarantees that the r.h.s. of (2.73)–(2.75) are well defined expressions, which should be understood as explained in Remark 2.4. On the other hand, using (2.42), we can immediately rewrite (2.74) and (2.75) as (recall the definition of $Q(\cdot\,;\cdot)$ in (2.43))

$$\Phi_\mathrm{d} = \frac{1}{2\pi^2}\sum_{j=1}^{N}\mathrm{Re}\bigg\{\int_0^t \mathrm{d}\tau\int_0^\tau \mathrm{d}s\ (\dot{q}_j(\tau) - iq_j(\tau))^*$$
$$\times e^{i(\tau-s)}\left[Q(1;\tau-s) - \pi(\gamma + \log(\tau-s))\right](\dot{q}_j(s) - iq_j(s))\bigg\},$$

$$\Phi_\mathrm{of} = \frac{1}{\pi}\sum_{j=1}^{N}\sum_{k\neq j}\mathrm{Re}\bigg\{\int_0^t \mathrm{d}\tau\int_0^\tau \mathrm{d}s\ (\dot{q}_j(\tau) - iq_j(\tau))^*\,(U_0(\tau-s)K_0(|\cdot - \mathbf{y}_k|))(\mathbf{y}_j)\,(\dot{q}_k(s) - iq_k(s))\bigg\}.$$

Now, using again that $\psi_0 \in \mathscr{D}[\mathcal{F}]$ and (2.59), we obtain

$$(U_0(\tau)\phi_{1,0})(\mathbf{y}_j) = -\frac{1}{2\pi}\sum_{k=1}^{N} q_k(0)\,(U_0(\tau)K_0(|\cdot - \mathbf{y}_k|))(\mathbf{y}_j)$$
$$-\frac{i}{2\pi}\sum_{k\neq j}\int_0^\tau \mathrm{d}s\, U_0(\tau-s;|\mathbf{y}_j - \mathbf{y}_k|)q_k(s) + \left(\beta_j|q_j(\tau)|^{2\sigma_j} + \frac{\gamma - \log 2}{2\pi}\right)q_j(\tau) - \frac{iq_j(\tau)}{8}$$
$$+\frac{q_j(0)}{4\pi}(-\gamma - \log\tau) + \frac{1}{4\pi}\int_0^\tau \mathrm{d}s\,(-\gamma - \log(\tau-s))\dot{q}_j(s), \tag{2.76}$$

where Lemma 2.9 guarantees the well-posedness of last term. Plugging it into the definition of Ψ, we can split Ψ_t into five terms as $\Psi_t = \Psi_1 + \Psi_2 + \Psi_3 + \Psi_4 + \Psi_5$, where $\Psi_1 = \Psi_{1,\mathrm{d}} + \Psi_{1,\mathrm{od}}$, with

$$\Psi_{1,\mathrm{d}} = \frac{1}{\pi}\sum_{j=1}^{N} \mathrm{Re}\bigg(q_j(0) \int_0^t \mathrm{d}\tau \; (\dot{q}_j(\tau) - i q_j(\tau))^* \left(U_0(\tau) K_0(|\cdot - \mathbf{y}_j|)\right)(\mathbf{y}_j) \bigg), \tag{2.77}$$

$$\Psi_{1,\mathrm{od}} = \frac{1}{\pi}\sum_{j=1}^{N}\sum_{k\neq j} \mathrm{Re}\bigg(q_k(0) \int_0^t \mathrm{d}\tau \; (\dot{q}_j(\tau) - i q_j(\tau))^* \left(U_0(\tau) K_0(|\cdot - \mathbf{y}_k|)\right)(\mathbf{y}_j) \bigg). \tag{2.78}$$

Moreover,

$$\Psi_2 = -\frac{1}{\pi}\sum_{j=1}^{N}\sum_{k\neq j} \mathrm{Im}\bigg(\int_0^t \mathrm{d}\tau \int_0^\tau \mathrm{d}s \; (\dot{q}_j(\tau) - i q_j(\tau))^* \, U_0\left(\tau - s; |\mathbf{y}_j - \mathbf{y}_k|\right) q_k(s) \bigg), \tag{2.79}$$

$$\Psi_3 = -\sum_{j=1}^{N} \left(\frac{\beta_j |q_j(t)|^{2\sigma_j}}{\sigma_j + 1} + \frac{\gamma - \log 2}{2\pi} \right) |q_j(t)|^2 + \sum_{j=1}^{N} \left(\frac{\beta_j |q_j(0)|^{2\sigma_j}}{\sigma_j + 1} + \frac{\gamma - \log 2}{2\pi} \right) |q_j(0)|^2, \tag{2.80}$$

where we have used that

$$2\mathrm{Re}\left[(\dot{q}_j(\tau) - i q_j(\tau))^* |q_j(\tau)|^{2\sigma_j} q_j(\tau) \right] = \frac{1}{\sigma_j + 1} \frac{d}{d\tau} \left(|q_j(\tau)|^{2\sigma_j + 2} \right),$$

$$2\mathrm{Re}\left[(\dot{q}_j(\tau) - i q_j(\tau))^* q_j(\tau) \right] = \frac{d}{d\tau} |q_j(\tau)|^2;$$

$\Psi_4 = \Psi_{4,\mathrm{r}} + \Psi_{4,\mathrm{i}}$, with

$$\Psi_{4,\mathrm{r}} = -\frac{1}{2\pi}\sum_{j=1}^{N} \mathrm{Re}\bigg(q_j(0) \int_0^t \mathrm{d}\tau \; (\dot{q}_j(\tau) - i q_j(\tau))^* \left(-\gamma - \log \tau\right) \bigg), \tag{2.81}$$

$$\Psi_{4,\mathrm{i}} = -\frac{1}{4}\sum_{j=1}^{N} \mathrm{Im}\bigg(\int_0^t \mathrm{d}\tau \; (\dot{q}_j(\tau) - i q_j(\tau))^* \, q_j(\tau) \bigg); \tag{2.82}$$

and finally

$$\Psi_5 = -\frac{1}{2\pi}\sum_{j=1}^{N} \mathrm{Re}\bigg(\int_0^t \mathrm{d}\tau \int_0^\tau \mathrm{d}s \, (\dot{q}_j(\tau) - i q_j(\tau))^* (-\gamma - \log(\tau - s)) \dot{q}_j(s) \bigg). \tag{2.83}$$

We stress that all the expressions above (2.77)–(2.83) are well defined thanks to Lemma 2.8 and Remark 2.4 and therefore the decomposition of Ψ_t is meaningful as well.

Using the definition of the free propagator $U_0(\tau)$ and (2.42), one sees that

$$\Psi_{1,\mathrm{d}} = \frac{1}{2\pi}\sum_{j=1}^{N} \mathrm{Re}\bigg\{ q_j(0) \int_0^t \mathrm{d}\tau \; (\dot{q}_j(\tau) - i q_j(\tau))^* \, e^{i\tau} \left[-\gamma - \log \tau - \tfrac{1}{\pi} Q(1;\tau) \right] \bigg\},$$

with Q defined by (2.43). Then, summing and subtracting $(\dot{q}_j(\tau) - i q_j(\tau))^*(-\gamma - \log \tau)$ in the integrand function and defining $\ell(t) := (e^{it} - 1)(-\gamma - \log t)$, we have

$$\Psi_{1,\mathrm{d}} + \Psi_{4,\mathrm{r}} = \frac{1}{2\pi}\sum_{j=1}^{N} \mathrm{Re}\bigg\{ q_j(0) \int_0^t \mathrm{d}\tau \; (\dot{q}_j(\tau) - i q_j(\tau))^* \left[\ell(\tau) + \tfrac{e^{i\tau}}{\pi} Q(1;\tau) \right] \bigg\} =: R_1.$$

On the other hand, we observe that $\Phi_{\mathrm{d}} + \Psi_5 = R_2 + R_3$, where

$$R_2 = \frac{1}{2\pi}\sum_{j=1}^{N} \mathrm{Re}\bigg\{ \int_0^t \mathrm{d}\tau \int_0^\tau \mathrm{d}s \, \left(\dot{q}_j(\tau) - iq_j(\tau)\right)^* \dot{q}_j(s) \left[\ell(\tau - s) + \tfrac{e^{i(\tau-s)}}{\pi} Q(1;\tau - s)\right] \bigg\},$$

$$R_3 = \frac{1}{2\pi}\sum_{j=1}^{N} \mathrm{Im}\bigg\{ \int_0^t \mathrm{d}\tau \int_0^\tau \mathrm{d}s \, (\dot{q}_j(\tau) - iq_j(\tau))^* q_j(s) e^{i(\tau-s)} \left[-\gamma - \log(\tau - s) + \tfrac{1}{\pi} Q(1;\tau - s)\right] \bigg\}.$$

As a consequence, we see that $\Psi_{1,\mathrm{d}} + \Psi_4 + \Psi_5 + \Phi_{\mathrm{d}} = R_1 + R_2 + R_3 + \Psi_{4,\mathrm{i}} =: \Gamma$. Now, an integration by parts shows that

$$\int_0^\tau \mathrm{d}s \, \dot{q}_j(s) \left[\ell(\tau - s) + \tfrac{e^{i(\tau-s)}}{\pi} Q(1;\tau - s)\right] = -\tfrac{i\pi}{2} q_j(\tau) - q_j(0) \left(\ell(\tau) + \tfrac{e^{i\tau}}{\pi} Q(1;\tau)\right) \\ + \tfrac{i}{\pi} \int_0^\tau \mathrm{d}s \, q_j(s) e^{i(\tau-s)} Q(1;\tau - s) + \int_0^\tau \mathrm{d}s \, q_j(s) \left[\dot{\ell}(\tau - s) + \tfrac{e^{i(\tau-s)}}{\pi} \dot{Q}(1;\tau - s)\right]$$

and then, plugging into the definition of R_2, there results that

$$\Gamma = \frac{1}{2\pi}\sum_{j=1}^{N} \mathrm{Im}\bigg\{ \int_0^t \mathrm{d}\tau \int_0^\tau \mathrm{d}s \, \left(\dot{q}_j(\tau) - iq_j(\tau)\right)^* q_j(s) e^{i(\tau-s)} (-\gamma - \log(\tau - s)) \bigg\} \\ + \frac{1}{2\pi}\sum_{j=1}^{N} \mathrm{Re}\bigg\{ \int_0^t \mathrm{d}\tau \int_0^\tau \mathrm{d}s \, \left(\dot{q}_j(\tau) - iq_j(\tau)\right)^* q_j(s) \left[\dot{\ell}(\tau - s) + \tfrac{e^{i(\tau-s)}}{\pi} \dot{Q}(1;\tau - s)\right] \bigg\}$$

However, easy computations (see (2.42)) exploiting the definition of the trigonometric integral functions (see, e.g., [AS, GR]) yield

$$\dot{\ell}(t) + \frac{e^{it}}{\pi} \dot{Q}(1;t) = ie^{it}(-\gamma - \log t)$$

and therefore $\Gamma = 0$.

It remains then to compute the off-diagonal terms $\Theta := \Psi_{1,\mathrm{od}} + \Psi_2 + \Phi_{\mathrm{od}}$. Another integration by parts yields

$$\int_0^\tau \mathrm{d}s \, \left(\dot{q}_k(s) - iq_k(s)\right) \left(U_0(\tau - s) K_0\left(|\cdot - \mathbf{y}_k|\right)\right)(\mathbf{y}_j) = q_k(\tau) K_0\left(|\mathbf{y}_j - \mathbf{y}_k|\right) \\ - q_k(0) \left(U_0(\tau) K_0\left(|\cdot - \mathbf{y}_k|\right)\right)(\mathbf{y}_j) - i \int_0^\tau \mathrm{d}s \, q_k(s) U_0(\tau - s; |\mathbf{y}_j - \mathbf{y}_k|)$$

and consequently, plugging into the definition of Φ_{od}, we find that

$$\Theta = \frac{1}{\pi}\sum_{j=1}^{N}\sum_{k\neq j} K_0(|\mathbf{y}_k - \mathbf{y}_j|) \, \mathrm{Re}\bigg\{ \int_0^t \mathrm{d}\tau \, \left(\dot{q}_j - iq_j(\tau)\right)^* q_k(\tau) \bigg\}$$

due to the definition of the Macdonald function K_0. Moreover, since (see again Remarks 2.3 and 2.4)

$$\int_0^t \mathrm{d}\tau \, \left(\dot{q}_j^*(\tau) q_k(\tau) + \dot{q}_j(\tau) q_k^*(\tau)\right) = \mathrm{Re}\left[q_j^*(\tau) q_k(\tau) - q_j^*(0) q_k(0)\right]$$

and

$$\mathrm{Re}\bigg\{ \int_0^t \mathrm{d}\tau \, \left[(-iq_j(\tau))^* q_k(\tau) + (-iq_k(\tau))^* q_j(\tau)\right] \bigg\} = -\mathrm{Im}\bigg\{ \int_0^t \mathrm{d}\tau \, 2\mathrm{Re}\left[q_j^*(\tau) q_k(\tau)\right] \bigg\} = 0,$$

we have

$$\Theta = \frac{1}{\pi} \sum_{k>j} K_0(|\mathbf{y}_k - \mathbf{y}_j|) \operatorname{Re}\left[q_j^*(\tau) q_k(\tau) - q_j^*(0) q_k(0)\right].$$

Summing up

$$\|\phi_{\lambda,t}\|^2_{H^1(\mathbb{R}^2)} = \|\phi_{\lambda,t}\|^2_{H^1(\mathbb{R}^2)} - \sum_{j=1}^N \left(\tfrac{\beta_j}{\sigma_j+1} |q_j(t)|^{2\sigma_j} + \tfrac{\gamma - \log 2}{2\pi} \right) |q_j(t)|^2$$

$$+ \sum_{j=1}^N \left(\tfrac{\beta_j}{\sigma_j+1} |q_j(0)|^{2\sigma_j} + \tfrac{\gamma - \log 2}{2\pi} \right) |q_j(0)|^2 + \frac{1}{\pi} \sum_{k>j} K_0(|\mathbf{y}_k - \mathbf{y}_j|) \operatorname{Re}\left[q_j^*(\tau) q_k(\tau) - q_j^*(0) q_k(0)\right],$$

so that, in view of (1.17), $E(t) = E(0)$, for any $t \leqslant T < T_*$. □

2.5. **Global well-posedness and blow-up alternative.** Before proving Theorem 1.3 and Proposition 1.1, we need one final Lemma, which allows us to extended the Sobolev regularity of $\mathbf{q}(t)$ to any interval $(0, T)$, with $T < T_*$. We aim at applying Proposition 2.4 and therefore we need to prove (2.51):

Lemma 2.10.
Let $\mathbf{q}(t)$ be the solution of (1.11) *provided by Proposition 2.3 and T_* the maximal existence time given in Proposition 2.2, then for any $T < T_*$ and $0 < T_1 < T_* - T$*

$$\int_0^t \mathrm{d}\tau\, \mathcal{I}(t-\tau) \left(U_0(\tau)\psi_T\right)(\mathbf{y}_j) \in C[0, T_1] \cap H^{1/2}(0, T_1), \tag{2.84}$$

$\forall j = 1, \ldots, N$.

Proof. Using the ansatz (1.7), we see that the above expression splits into two terms:

$$(U_0(\tau)\psi_T)(\mathbf{y}_j) = \left(U_0(\tau + T)\psi_0\right)(\mathbf{y}_j) + \frac{i}{2\pi} \sum_{k=1}^N \int_0^T \mathrm{d}s\, U_0\left(\tau + T - s; |\mathbf{y}_j - \mathbf{y}_k|\right)\, q_k(s)$$

and, using the Fourier transform as well as the properties of the unitary group $U_0(\tau)$, the r.h.s. becomes

$$\int_{\mathbb{R}^2} \mathrm{d}\mathbf{p}\, e^{-i\mathbf{p}\cdot\mathbf{y}_j} e^{-ip^2(\tau+T)} \widehat{\psi_0}(\mathbf{p}) + \frac{i}{2\pi} \sum_{k=1}^N \int_0^T \mathrm{d}s \int_{\mathbb{R}^2} \mathrm{d}\mathbf{p}\, e^{-i\mathbf{p}\cdot(\mathbf{y}_j - \mathbf{y}_k)} e^{-ip^2(\tau+T-s)}\, q_k(s)$$

$$= \int_{\mathbb{R}^2} \mathrm{d}\mathbf{p} e^{-i\mathbf{p}\cdot\mathbf{y}_j} e^{-ip^2(\tau+T)} \widehat{\phi_{\lambda,0}}(\mathbf{p}) + \frac{i}{2\pi} \sum_{k=1}^N \int_{\mathbb{R}^2} \mathrm{d}\mathbf{p} e^{-i\mathbf{p}\cdot(\mathbf{y}_j - \mathbf{y}_k)} \left[\int_0^T \mathrm{d}s\, e^{-ip^2(\tau+T-s)}\, q_k(s) - \frac{i q_k(0)}{p^2 + \lambda} \right].$$

The first term is bounded thanks to the hypothesis on ψ_0 and therefore its integral with $\mathcal{I}(t-\tau)$ is continuous. The second term can be rewritten as (recall Remark 2.3)

$$\frac{1}{2\pi} \sum_{k=1}^N \left[\int_0^T \mathrm{d}s\, \left(\dot{q}_k(s) + i\lambda q_k(s)\right) \left(U_0(\tau + T - s) K_0(\cdot - \mathbf{y}_k)\right)(\mathbf{y}_j) - q_k(T) \left(U_0(\tau) K_0(\cdot - \mathbf{y}_k)\right)(\mathbf{y}_j) \right].$$

The first quantity between brackets is bounded thanks to the properties of $\mathbf{q}(t)$ (see again Remark 2.4 and Lemma 2.8). The remaining part of the expression is not bounded but, using the properties of the integral kernel $\mathcal{I}$ (see (2.64) and the proof of Proposition 2.2), one can show that, when integrated against $\mathcal{I}(t-\tau)$, it provides a continuous function.

In order to prove that the expression belongs to $H^{1/2}(0, T_1)$, one can proceed exactly as in the proof of Proposition 2.3. We omit the details for the sake of brevity. □

Proof of Theorem 1.3. As a preliminary step, we combine Lemma 2.10 with Proposition 2.4, which implies that $\mathbf{q}(t) \in H^{1/2}(0,T)$ for any $T < T_*$ (recall the maximal existence time T_* provided by Proposition 2.2). Hence the energy conservation proven in Theorem 1.2 holds true up to any $T < T_*$. Moreover, it yields that, if $\beta_j > 0$, $\forall j = 1, \ldots, N$,

$$|\mathbf{q}(t)| \leqslant C < +\infty, \qquad \forall t \in [0,T], \tag{2.85}$$

and any $T < T_*$: indeed, by (1.17) we have

$$\sum_{j=1}^{N} \left(\tfrac{\beta_j}{\sigma_j+1} |q_j(t)|^{2\sigma_j} + \tfrac{\gamma - \log 2}{2\pi} \right) |q_j(t)|^2 \leqslant C < +\infty,$$

but $\gamma - \log 2 < 0$. However, if the quantity between brackets is bounded from below by $c > 0$ for any $j = 1, \ldots, N$, then the result easily follows. On the other hand is for some j_* the lower bound on the quantity fails, it means that

$$\tfrac{\beta_j}{\sigma_j+1} |q_{j*}(t)|^{2\sigma_j} \leqslant \tfrac{\log 2 - \gamma}{2\pi} + c,$$

which also implies that q_{j*} is bounded, since $\beta_{j*} > 0$. We can then remove q_{j*} from the upper bound and repeat the argument, so obtaining the result.

Hence, since $\mathbf{q}$ remains bounded as $t \to T$ by a quantity which is independent of T, it must be (recall that T_* is by definition the maximal existence time of $\mathbf{q}(t)$)

$$\limsup_{t \to T^*} |\mathbf{q}(t)| \leqslant C < +\infty, \tag{2.86}$$

which implies that $\mathbf{q}$ can be extended to the whole positive half-line and that $\mathbf{q}$ is the unique solution of (1.11) in $C[0,\infty)$, i.e., it is global in time (see [Mi, Theorem 2.3]). In addition, Proposition 2.4 in combination with Lemma 2.10 implies that $\mathbf{q} \in H^{1/2}(0,T)$, for every finite $T > 0$.

Consequently, arguing as before, one can prove that the function ψ_t defined by (1.7) and (1.11) is in $\mathscr{D}[\mathcal{F}]$ and solves (1.15) for every $t \geqslant 0$, thus proving Theorem 1.3. □

Proof of Proposition 1.1. If $\beta_j < 0$ for some $j \in \{1, \ldots, N\}$, then we have the following alternative: either $\limsup_{t \to T_*} |\mathbf{q}(t)| < +\infty$, which implies that $T_* = +\infty$ and the solution is global in time, or

$$\limsup_{t \to T_*} |\mathbf{q}(t)| = +\infty.$$

In this second case we can still have two opposite alternatives: either $T_* = +\infty$ and, in spite of not being bounded, the solution is nevertheless global in time, or $T_* < +\infty$ and the blow-up occurs. Indeed, by the energy conservation and the diverging limit of $\mathbf{q}$, we obtain

$$\limsup_{t \to T_*} \|\phi_{\lambda,t}\|_{H^1(\mathbb{R}^2)} = +\infty,$$

i.e., ψ_t blows-up at a finite time. □

References

[A] Adami R., A Class of Schrödinger Equations with Concentrated Nonlinearity, Ph.D. Thesis, Università degli Studi di Roma "La Sapienza", 2000.

[AGH-KH] Albeverio S., Gesztesy F., Høegh-Krohn R., Holden H., *Solvable Models in Quantum Mechanics*, Spinger, Berlin, 1988.

[AS] Abramovitz M., Stegun I.A., *Handbook of Mathematical Functions: with Formulas, Graphs, and Mathematical Tables*, Dover, New York, 1965.

[AT] Adami R., Teta A., A Class of Nonlinear Schrödinger Equations with Concentrated Nonlinearity, *J. Funct. Anal.* **180** (2001), 148-175.

[ADFT1] Adami R., Dell'Antonio G., Figari R., Teta A., The Cauchy problem for the Schrödinger equation in dimension three with concentrated nonlinearity, *Ann. I. H. Poincaré – AN* **20** (2003), 477-500.

[ADFT2] Adami R., Dell'Antonio G., Figari R., Teta A., Blow-up solutions for the Schrödinger equation in dimension three with a concentrated nonlinearity, *Ann. I. H. Poincaré – AN* **21** (2004), 121-137.

[BKB] Bulashenko O.M., Kochelap V.A., Bonilla L.L., Coherent patterns and self-induced diffraction of electrons on a thin nonlinear layer, *Phys. Rev. B* **54** (1996), 1537–1540.

[C] Cazenave T., *An introduction to nonlinear Schrödinger equations*, Textos de Métodos Matemáticos **22**, I.M.U.F.R.J., Rio de Janeiro, 1989.

[CCF] Carlone R., Correggi M., Figari R., Two-dimensional Time-dependent Point Interactions, preprint *arXiv:1601.02390 [math-ph]* (2016).

[CCNP] Carlone R., Cacciapuoti C., Noja D., Posilicano A., The 1-D Dirac equation with concentrated nonlinearity, preprint *arXiv:1607.00665 [math-ph]* (2016).

[CD] Correggi M., Dell'Antonio G., Decay of a Bound State under a Time-Periodic Perturbation: a Toy Case, *J. Phys. A: Math. Gen.* **38** (2005), 4769–4781.

[CDFM] Correggi M., Dell'Antonio G., Figari R., Mantile A., Ionization for Three Dimensional Time-dependent Point Interactions, *Commun. Math. Phys.* **257** (2005), 169–192.

[CFNT1] Cacciapuoti C., Finco D., Noja D., Teta A., The NLS Equation in Dimension One with Spatially Concentrated Nonlinearities: the Pointlike Limit, *Lett. Math. Phys.*, **104** (2014), 1557-1570.

[CFNT2] Cacciapuoti C., Finco D., Noja D., Teta A., The point-like limit for a NLS equation with concentrated nonlinearity in dimension three, preprint *arXiv:1511.06731 [math-ph]* (2015).

[CFT] Carlone R., Fiorenza A., Tentarelli L., The action of Volterra integral operators with highly singular kernels on Hölder continuous, Lebesgue and Sobolev functions, preprint *arXiv:1611.08503 [math.AP]* (2016).

[DFT1] Dell'Antonio G., Figari R., Teta A., Hamiltonians for Systems of N Particles Interacting through Point Interactions, *Ann. I. H. Poincaré – PHY* **60** (1994), 253–290.

[DFT2] Dell'Antonio G., Figari R., Teta A., The Schrödinger equation with moving point interactions in three dimensions, in *Stochastic Processes, Physics and Geometry: New Interplays. I: A Volume in Honor of Sergio Albeverio*, Conference Proceedings, Canadian Mathematical Society **28**, 99–113, AMS, 2000.

[DGPS] Dalfovo F., Giorgini S., Pitaevskii L.P., Stringari S., Theory of Bose-Einstein condensation in trapped gases, *Rev. Mod. Phys.* **71** (1999), 463–512.

[DM] Dror N., Malomed B.A., Solitons supported by localized nonlinearities in periodic media, *Phys. Rev. A* **83** (2011), 033828.

[E1] Erdélyi A., *Higher Transcendental Functions, vol. III*, Krieger Publishing, 1981.

[E2] Erdélyi A., *Tables of Integral Transforms, vol. I*, McGraw-Hill, 1954.

[GR] Gradshteyn I.S., Ryzhik I.M., *Tables of Integrals, Series and Products*, Academic Press, San Diego, 2007.

[H] Hardy G.H., *Ramanujan. Twelve Lectures on Subjects Suggested by His Life and Work*, Cambridge University Press, Cambridge, 1940.

[J-LPC] Presilla C., Jona-Lasinio G., Capasso F., Nonlinear feedback oscillations in resonant tunneling through double barriers, *Phys. Rev. B* **43** (1991), 5200–5203.

[J-LPS] Jona Lasinio G., Presilla C., Sjostrand J., On Schrödinger Equations with Concentrated Nonlinearities, *Ann. Phys.* **240** (1995), 1–21.

[KP] Kufner A., Persson L.E., *Weighted inequalities of Hardy type*, World Scientific, Singapore, 2003.

[LKMF] Li K., Kevrekidis P.G., Malomed B.A., Frantzeskakis D.J., Transfer and scattering of wave packets by a nonlinear trap, *Phys. Rev. E* **84** (2011), 056609.

[Ma] Malomed B.A., Nonlinear Schrödinger Equations, in *Encyclopedia of Nonlinear Science* (Scott A. edt.), Routledge, New York, 639–643, 2005.

[Mi] Miller R.K., *Nonlinear Volterra Integral Equations*, W.A. Benjamin Inc., 1971.

[MA] Malomed B.A., Azbel M.Y., Modulational instability of a wave scattered by a nonlinear center, *Phys. Rev. B* **47** (1993), 10402–10406.

[MB] Molina M.I., Bustamante C.A., The attractive nonlinear delta-function potential, *Amer. J. Phys.* **70** (2002), 67–70.

[N] Nier F., The dynamics of some quantum open systems with short-range nonlinearities, *Nonlinearity* **11** (1998), 1127–1172.

[NP] Noja D., Posilicano A., Wave equations with concentrated nonlinearities, *J. Phys. A: Math. Gen.* **38** (2005), 5011.

[S *et al*] Sukhorukov A.A., Kivshar Y.S., Bang O. , Rasmussen J.J., Christiansen P.L., Nonlinearity and disorder: Classification and stability of nonlinear impurity modes, *Phys. Rev. E* **63** (2001), 036601.

[SKB] Sukhorukov A.A., Kivshar Y.S., Bang O., Two-color nonlinear localized photonic modes, *Phys. Rev. E* **60** (1999), R41–R44.

[SKM] Samko S.G., Kilbas A.A., Marichev O.I., *Fractional Integrals and Derivatives*, Gordon and Breach Science Publishers, Philadelphia, Pa., USA, 1993.

[SY] Sayapova M.R., Yafaev D.R., The evolution operator for time-dependent potentials of zero radius, *Proc. Steklov Inst. Math.* **159** (1984), 173–180.

[Y] Yeh P., *Optical Waves in Layered Media*, Wiley, New York, 2005.

Università "Federico II" di Napoli, Dipartimento di Matematica e Applicazioni "R. Caccioppoli", MSA, via Cinthia, I-80126, Napoli, Italy.
E-mail address: raffaele.carlone@unina.it

Università degli Studi di Roma "La Sapienza", Dipartimento di Matematica "G. Castelnuovo", P.le Aldo Moro, 5, 00185, Roma, Italy.
E-mail address: michele.correggi@gmail.com

Università "Federico II" di Napoli, Dipartimento di Matematica e Applicazioni "R. Caccioppoli", MSA, via Cinthia, I-80126, Napoli, Italy.
E-mail address: lorenzo.tentarelli@unina.it