



A maximal restriction theorem and Lebesgue points of functions in $\mathcal{F}(L^p)$

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Abstract. Fourier restriction theorems, whose study had been initiated by E.M. Stein, usually describe a family of a priori estimates of the L^q -norm of the restriction of the Fourier transform of a function f in $L^p(\mathbb{R}^n)$ to a given subvariety S , endowed with a suitable measure. Such estimates allow to define the restriction $\mathcal{R}f$ of the Fourier transform of an L^p -function to S in an operator theoretic sense. In this article, we begin to investigate the question what is the “intrinsic” pointwise relation between $\mathcal{R}f$ and the Fourier transform of f , by looking at curves in the plane, for instance with non-vanishing curvature. To this end, we bound suitable maximal operators, including the Hardy–Littlewood maximal function of the Fourier transform of f restricted to S .

1. Introduction

The restriction problem for the Fourier transform in \mathbb{R}^n was introduced by Elias M. Stein, who proved the first result in any dimension [2], p. 28, later improved by the sharper Stein–Tomas method [5]. Since then more and more sophisticated techniques have been introduced to attack the still open problems in this area, concerning the maximal range of exponents for which the restriction inequality holds.

In two-dimensions, the restriction estimate for the circle had been proved already, in an almost optimal range of exponents, by Fefferman and Stein [2], p. 33. Shortly later, sharp estimates were obtained by Zygmund [8] for the circle and by Carleson and Sjölin [1] and Sjölin [3] for a class of curves including strictly convex C^2 curves.

The present paper does not mean to proceed along these lines, but rather to propose a reflection on the measure-theoretic meaning of the restriction phenomenon and possibly suggest some related problems.

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A restriction theorem is usually meant as a family of *a priori* inequalities

$$(1.1) \quad \|\widehat{f}|_S\|_{L^q(S,\mu)} \leq C\|f\|_{L^p(\mathbb{R}^n)},$$

where $f \in \mathcal{S}(\mathbb{R}^n)$, S is a surface with appropriate curvature properties, and μ a suitably weighted finite surface measure on S . The validity of such an inequality implies the existence of a bounded *restriction operator* $\mathcal{R}: L^p(\mathbb{R}^n) \rightarrow L^q(S, \mu)$ such that $\mathcal{R}f = \widehat{f}|_S$ when f is a Schwartz function.

In general terms our question is: assuming that (1.1) holds, what is the “intrinsic” pointwise relation between $\mathcal{R}f$ and \widehat{f} for a general L^p -function f ?

A partial answer follows directly from the restriction inequality. Assume that (1.1) holds for given p, q . This forces the condition $p < 2$, so that $\widehat{f} \in L^{p'}$. Fix an approximate identity $\chi_\varepsilon(x) = \varepsilon^{-n}\chi(x/\varepsilon)$ with $\chi \in \mathcal{S}(\mathbb{R}^n)$, $\int \chi = 1$. Then, with $\psi = \mathcal{F}^{-1}\chi$,

$$\widehat{f} * \chi_\varepsilon = \widehat{f\psi(\varepsilon \cdot)}$$

is well defined on S and coincides with $\mathcal{R}(f\psi(\varepsilon \cdot))$. Moreover, $f\psi(\varepsilon \cdot) \rightarrow f$ in $L^p(\mathbb{R}^n)$, so that $(\widehat{f} * \chi_\varepsilon)|_S \rightarrow \mathcal{R}f$ in $L^q(S, \mu)$. Hence, for a subsequence $\varepsilon_k \rightarrow 0$, the χ_{ε_k} -averages of \widehat{f} converge pointwise to $\mathcal{R}f$ μ -a.e.

It is natural to ask if the limit over all ε exists μ -a.e. We give positive answers in two dimensions to this and related questions.

We recall that, for a curve S in the plane, necessary conditions on p, q for having (1.1) are $p < 4/3$ and $p' \geq 3q$ and that they are also sufficient when S is C^2 with nonvanishing curvature and μ is the arclength measure, or, more generally, when S is just C^2 and convex, and μ is the *affine arclength measure* [3]. Notice that the two measures differ by a factor comparable to the $1/3$ power of the curvature, so that the affine arclength is concentrated on the set of points with nonvanishing curvature and ordinary arclength is damped near these points.

Theorem 1.1. *Let S be a C^2 regular curve in \mathbb{R}^2 and $f \in L^p(\mathbb{R}^2)$.*

- (i) *Assume that $1 \leq p < 4/3$ and let $\chi \in \mathcal{S}(\mathbb{R}^2)$ with $\int \chi = 1$. Then, with respect to arclength measure, for almost every $x \in S$ at which the curvature does not vanish, $\lim_{\varepsilon \rightarrow 0} \widehat{f} * \chi_\varepsilon(x) = \mathcal{R}f(x)$.*
- (ii) *Assume that $1 \leq p < 8/7$. Then, with respect to arclength measure, almost every $x \in S$ at which the curvature does not vanish is a Lebesgue point for \widehat{f} and the regularized value of \widehat{f} at x coincides with $\mathcal{R}f(x)$.*

To be more explicit, the latter statement says that at μ -almost every $x \in S$,

$$\lim_{r \rightarrow 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |\widehat{f}(y) - \mathcal{R}f(x)| dy = 0,$$

where $B_r(x)$ denotes the Euclidean ball in \mathbb{R}^2 of radius $r > 0$ centered at x .

Several questions remain open, regarding extensions to less regular curves, to other values of p in the range $8/7 \leq p < 4/3$, or to higher dimensions. We just mention here that, in dimension $d \geq 3$, our method gives results for a class of curves including $\Gamma(t) = (t, t^2, \dots, t^d)$.

Theorem 1.1 is a direct consequence of certain “maximal restriction theorems” concerning restrictions to S of truncated maximal functions of the Fourier transform. Since maximal restriction inequalities may also have an intrinsic interest, we go beyond what is strictly needed to deduce Theorem 1.1 and consider (truncated) two-parameter maximal functions, such as the strong maximal function, relative to any coordinate system in \mathbb{R}^2 .

In Theorem 2.1 below we prove that, for a convex C^2 curve, the two-parameter maximal operator defined in (2.1), is L^p - L^q bounded for p, q in the full range of validity of the restriction theorem, with the L^q -norm on S relative to affine arclength measure.

In Corollary 2.3 we deduce the same L^p - L^q estimates, but in the smaller range $p < 8/7$, for the truncated strong maximal function, which does not only control averages of \hat{f} , but also those of $|\hat{f}|$.

Notice that the convexity condition imposed in Theorem 2.1 can be removed in passing to the more qualitative Theorem 1.1 since, for a general C^2 regular curve, the subset of points of nonvanishing curvature is a countable union of disjoint convex sub-arcs.

The proof of Theorem 2.1 is based on the Kolmogorov–Seliverstov–Plessner linearization method [7], Ch. XIII. This leads to proving uniform estimates for a family of linear operators to which a modification of the basic approach of [1], [8] for curves in \mathbb{R}^2 can be applied. For this reason our method is limited to the two-dimensional context. Unfortunately, the usual TT^* method of Stein–Tomas does not seem to be applicable, even for the Hardy–Littlewood maximal function.

We would like to point the reader to a recent paper of M. Vitturi [6], where a maximal restriction theorem is proved for the sphere in dimension $n \geq 3$, with $p \leq 4/3$ (resp. $p \leq 8/7$).

2. The strong maximal function of \hat{f} along a curve

Let $S = \{\Gamma(t) : t \in I\}$, where Γ is a C^2 curve in \mathbb{R}^2 with nonnegative signed curvature, i.e., with $\kappa(t) = \det(\Gamma', \Gamma'')(t) \geq 0$. Denote by $d\mu(t) = \kappa^{1/3}(t) dt$ the pull-back to I of the affine arclength measure on S .

We assume for simplicity that S is parametrized by the x -coordinate as $\Gamma(x) = (x, \varphi(x))$, i.e., that S is the graph of a convex C^2 function φ on a bounded interval I . Notice that the measure μ is concentrated on the set where $\kappa = \varphi'' > 0$.

We consider the two-parameter maximal function¹

$$(2.1) \quad \mathcal{M}f(x) = \sup_{0 < \varepsilon', \varepsilon'' < 1} \left| \int \widehat{f}(x + s, \varphi(x) + t) \chi_{\varepsilon'}(s) \chi_{\varepsilon''}(t) ds dt \right|,$$

where $\chi_\varepsilon(\cdot) = \varepsilon^{-1} \chi(\cdot/\varepsilon)$, with $\chi \in \mathcal{S}(\mathbb{R})$, even, with $\int \chi = 1$.

¹Theorem 2.1 also holds if $\chi \otimes \chi$ is replaced by a general $\chi \in \mathcal{S}(\mathbb{R}^2)$, because this can be expanded into a rapidly decreasing series $\sum_j \chi_j' \otimes \chi_j''$.

Theorem 2.1. *The inequality*

$$(2.2) \quad \|\mathcal{M}f\|_{L^q(I,\mu)} \leq C_p \|f\|_{L^p(\mathbb{R}^2)},$$

holds for $1 \leq p < 4/3$ and $p' \geq 3q$.

Proof. We may and shall assume $f \in \mathcal{S}(\mathbb{R}^2)$ and, since μ is finite, $p' = 3q$ by Hölder's inequality. We linearize \mathcal{M} by defining, for fixed measurable functions $\varepsilon'(x), \varepsilon''(x)$ on I with values in $(0, 1)$,

$$(2.3) \quad \begin{aligned} \mathcal{R}_{\varepsilon', \varepsilon''} f(x) &= \int \widehat{f}(x + s, \varphi(x) + t) \chi_{\varepsilon'}(s) \chi_{\varepsilon''}(t) ds dt \\ &= \int f(\xi, \eta) \int e^{-i(\xi(x+s)+\eta(\varphi(x)+t))} \chi_{\varepsilon'}(s) \chi_{\varepsilon''}(t) ds dt d\xi d\eta \\ &= \int \widehat{\chi}(\varepsilon'(x)\xi) \widehat{\chi}(\varepsilon''(x)\eta) e^{-i(\xi x + \eta \varphi(x))} f(\xi, \eta) d\xi d\eta. \end{aligned}$$

The formal adjoint of $\mathcal{R}_{\varepsilon', \varepsilon''}$ is

$$(2.4) \quad \begin{aligned} \mathcal{E}_{\varepsilon', \varepsilon''} g(\xi, \eta) &= \mathcal{R}_{\varepsilon', \varepsilon''}^* g(\xi, \eta) \\ &= \int_I \widehat{\chi}(\varepsilon'(x)\xi) \widehat{\chi}(\varepsilon''(x)\eta) e^{i(\xi x + \eta \varphi(x))} g(x) \kappa^{1/3}(x) dx. \end{aligned}$$

It suffices to prove the inequality

$$(2.5) \quad \|\mathcal{E}_{\varepsilon', \varepsilon''} g\|_{L^{p'}(\mathbb{R}^2)} \leq C_p \|g\|_{L^q(I,\mu)}, \quad g \in C_c^\infty(I),$$

uniformly in the functions $\varepsilon'(x), \varepsilon''(x)$. We introduce a truncation in ξ and η , in order to gain decay at infinity for $\mathcal{E}_{\varepsilon', \varepsilon''} g$. Fixing another function χ_0 smooth on \mathbb{R} , supported in $[-2, 2]$ and equal to 1 on $[-1, 1]$, we define, for $\lambda \gg 1$,

$$(2.6) \quad \mathcal{E}_{\varepsilon', \varepsilon''}^\lambda g(\xi, \eta) = \chi_0\left(\frac{\xi}{\lambda}\right) \chi_0\left(\frac{\eta}{\lambda}\right) \int_I \widehat{\chi}(\varepsilon'(x)\xi) \widehat{\chi}(\varepsilon''(x)\eta) e^{i(\xi x + \eta \varphi(x))} g(x) \kappa^{1/3}(x) dx.$$

It will then suffice to prove (2.5) with $\mathcal{E}_{\varepsilon', \varepsilon''}$ replaced by $\mathcal{E}_{\varepsilon', \varepsilon''}^\lambda$, uniformly in $\varepsilon'(x), \varepsilon''(x)$ and λ .

We start from the identity

$$(2.7) \quad \|\mathcal{E}_{\varepsilon', \varepsilon''}^\lambda g\|_{p'} = \|(\mathcal{E}_{\varepsilon', \varepsilon''}^\lambda g)^2\|_{p'/2}^{1/2}.$$

If U is the open subset of I where $\kappa(x) > 0$, the measure μ is concentrated on U , so we have

$$\begin{aligned} (\mathcal{E}_{\varepsilon', \varepsilon''}^\lambda g)^2(\xi, \eta) &= \chi_0^2\left(\frac{\xi}{\lambda}\right) \chi_0^2\left(\frac{\eta}{\lambda}\right) \int_{U^2} \widehat{\chi}(\varepsilon'(x)\xi) \widehat{\chi}(\varepsilon''(x)\eta) \widehat{\chi}(\varepsilon'(y)\xi) \widehat{\chi}(\varepsilon''(y)\eta) \\ &\quad \times e^{i(\xi(x+y)+\eta(\varphi(x)+\varphi(y)))} g(x) \kappa^{1/3}(x) g(y) \kappa^{1/3}(y) dx dy \\ &= \chi_0^2\left(\frac{\xi}{\lambda}\right) \chi_0^2\left(\frac{\eta}{\lambda}\right) \int_{U^2} \widehat{\chi}(\varepsilon'(x)\xi) \widehat{\chi}(\varepsilon''(x)\eta) \widehat{\chi}(\varepsilon'(y)\xi) \widehat{\chi}(\varepsilon''(y)\eta) \\ &\quad \times e^{i(\xi(x+y)+\eta(\varphi(x)+\varphi(y)))} G_0(x, y) dx dy, \end{aligned}$$

with $G_0 = (g\kappa^{1/3}) \otimes (g\kappa^{1/3})$.

We want to make the change of variables $z_1 = x + y, z_2 = \varphi(x) + \varphi(y)$. It follows from the convexity of φ that the map $\Phi(x, y) = (x + y, \varphi(x) + \varphi(y))$ is injective on each of the subsets $U_{\pm}^2 = \{(x, y) \in U^2 : x \preceq y\}$ and that $\det \Phi'(x, y) = \varphi'(y) - \varphi'(x) \neq 0$ on U^2 .

With $A = \Phi(U_+) = \Phi(U_-)$, we set, for $z = (z_1, z_2) \in A$,

$$\begin{aligned}
 (x_{\pm}(z), y_{\pm}(z)) &= (\Phi|_{U_{\pm}^2})^{-1}(z) \\
 \varepsilon_1^{\pm}(z) &= \varepsilon'(x_{\pm}(z)), \quad \varepsilon_2^{\pm}(z) = \varepsilon'(y_{\pm}(z)) \\
 \varepsilon_1''^{\pm}(z) &= \varepsilon''(x_{\pm}(z)), \quad \varepsilon_2''^{\pm}(z) = \varepsilon''(y_{\pm}(z)) \\
 G_{\pm}(z) &= \frac{G_0(x_{\pm}(z), y_{\pm}(z))}{|\varphi'(x_{\pm}(z)) - \varphi'(y_{\pm}(z))|}.
 \end{aligned}
 \tag{2.8}$$

Then

$$\begin{aligned}
 \mathcal{E}_{\varepsilon', \varepsilon''}^{\lambda} g(\xi, \eta)^2 &= \chi_0^2\left(\frac{\xi}{\lambda}\right) \chi_0^2\left(\frac{\eta}{\lambda}\right) \sum_{\pm} \int_A \widehat{\chi}(\varepsilon_1^{\pm}(z)\xi) \\
 &\quad \times \widehat{\chi}(\varepsilon_1''^{\pm}(z)\eta) \widehat{\chi}(\varepsilon_2^{\pm}(z)\xi) \widehat{\chi}(\varepsilon_2''^{\pm}(z)\eta) e^{i(\xi z_1 + \eta z_2)} G_{\pm}(z) dz.
 \end{aligned}
 \tag{2.9}$$

We are so led to consider the operator

$$\begin{aligned}
 T_{\bar{\varepsilon}}^{\lambda} G(\xi, \eta) &= \chi_0^2\left(\frac{\xi}{\lambda}\right) \chi_0^2\left(\frac{\eta}{\lambda}\right) \int_A \widehat{\chi}(\varepsilon_1'(z)\xi) \widehat{\chi}(\varepsilon_1''(z)\eta) \widehat{\chi}(\varepsilon_2'(z)\xi) \widehat{\chi}(\varepsilon_2''(z)\eta) e^{i(\xi z_1 + \eta z_2)} G(z) dz,
 \end{aligned}$$

for arbitrary measurable functions $\bar{\varepsilon} = (\varepsilon_1', \varepsilon_1'', \varepsilon_2', \varepsilon_2'')$ on A with values in $(0, 1)^4$ and arbitrary continuous functions G on A .

Lemma 2.2. *For $1 \leq p \leq 2$, $T_{\bar{\varepsilon}}^{\lambda}$ is bounded from $L^p(A)$ to $L^{p'}(\mathbb{R}^2)$, uniformly in $\bar{\varepsilon}$ and λ .*

Proof. The statement is trivial for $p = 1$.

For $p = 2$ we prove the equivalent statement that $(T_{\bar{\varepsilon}}^{\lambda})^* T_{\bar{\varepsilon}}^{\lambda} : L^2(A) \rightarrow L^2(A)$. We have

$$(T_{\bar{\varepsilon}}^{\lambda})^* T_{\bar{\varepsilon}}^{\lambda} G(z) = \int_A K_{\bar{\varepsilon}}^{\lambda}(z, w) G(w) dw,$$

where, for $(z, w) \in A^2$,

$$\begin{aligned}
 K_{\bar{\varepsilon}}^{\lambda}(z, w) &= \int_{\mathbb{R}^2} e^{-i(\xi, \eta) \cdot (z-w)} \\
 &\quad \times \chi_0^4\left(\frac{\xi}{\lambda}\right) \chi_0^4\left(\frac{\eta}{\lambda}\right) \widehat{\chi}(\varepsilon_1'(z)\xi) \widehat{\chi}(\varepsilon_2'(z)\xi) \widehat{\chi}(\varepsilon_1''(z)\eta) \widehat{\chi}(\varepsilon_2''(z)\eta) \\
 &\quad \times \widehat{\chi}(\varepsilon_1'(w)\xi) \widehat{\chi}(\varepsilon_2'(w)\xi) \widehat{\chi}(\varepsilon_1''(w)\eta) \widehat{\chi}(\varepsilon_2''(w)\eta) d\xi d\eta.
 \end{aligned}
 \tag{2.10}$$

Let

$$\begin{aligned}
 \varepsilon'(z, w, \lambda) &= \max \{ \varepsilon_1'(z), \varepsilon_2'(z), \varepsilon_1'(w), \varepsilon_2'(w), \lambda^{-1} \}, \\
 \varepsilon''(z, w, \lambda) &= \max \{ \varepsilon_1''(z), \varepsilon_2''(z), \varepsilon_1''(w), \varepsilon_2''(w), \lambda^{-1} \}.
 \end{aligned}
 \tag{2.11}$$

Using iteratively the property that, given two Schwartz functions f, g on \mathbb{R} , the product $f(at)g(bt)$ can be expressed as $h((a \vee b)t)$ with each Schwartz norm $\|h\|_{(N)} = \max_{|\alpha| \leq N} \sup_x (1 + |x|)^N |\partial^\alpha h(x)|$ controlled by the same norm of f and g , we can write

$$\begin{aligned} \chi_0^4\left(\frac{\xi}{\lambda}\right) \widehat{\chi}(\varepsilon'_1(z)\xi) \widehat{\chi}(\varepsilon'_2(z)\xi) \widehat{\chi}(\varepsilon'_1(w)\xi) \widehat{\chi}(\varepsilon'_2(w)\xi) &= \psi'_{z,w,\lambda}(\varepsilon'(z, w, \lambda)\xi) \\ \chi_0^4\left(\frac{\eta}{\lambda}\right) \widehat{\chi}(\varepsilon''_1(z)\eta) \widehat{\chi}(\varepsilon''_2(z)\eta) \widehat{\chi}(\varepsilon''_1(w)\eta) \widehat{\chi}(\varepsilon''_2(w)\eta) &= \psi''_{z,w,\lambda}(\varepsilon''(z, w, \lambda)\eta), \end{aligned}$$

with $\psi'_{z,w,\lambda}, \psi''_{z,w,\lambda} \in \mathcal{S}(\mathbb{R})$ uniformly bounded in each Schwartz norm.

Then

$$(2.12) \quad K_{\tilde{\varepsilon}}^\lambda(z, w) = \frac{1}{\varepsilon'(z, w, \lambda)\varepsilon''(z, w, \lambda)} \widehat{\psi}'_{z,w,\lambda}\left(\frac{z_1 - w_1}{\varepsilon'(z, w, \lambda)}\right) \widehat{\psi}''_{z,w,\lambda}\left(\frac{z_2 - w_2}{\varepsilon''(z, w, \lambda)}\right),$$

so that, for every N , we have the uniform bound

$$|K_{\tilde{\varepsilon}}^\lambda(z, w)| \leq C_N \frac{1}{\varepsilon'(z, w, \lambda)\varepsilon''(z, w, \lambda)} \left(1 + \frac{|z_1 - w_1|}{\varepsilon'(z, w, \lambda)}\right)^{-N} \left(1 + \frac{|z_2 - w_2|}{\varepsilon''(z, w, \lambda)}\right)^{-N}.$$

We now make a double partition of A^2 , depending on which of the three parameters z, w, λ determines the value of ε' and ε'' respectively:

$$A^2 = E'_1 \cup E'_2, \quad A^2 = E''_1 \cup E''_2,$$

such that

$$\begin{aligned} \varepsilon'(z, w, \lambda) &= \begin{cases} \varepsilon'_1(z) \text{ or } \varepsilon'_2(z) \text{ or } \lambda^{-1} & \text{on } E'_1, \\ \varepsilon'_1(w) \text{ or } \varepsilon'_2(w) & \text{on } E'_2, \end{cases} \\ \varepsilon''(z, w, \lambda) &= \begin{cases} \varepsilon''_1(z) \text{ or } \varepsilon''_2(z) \text{ or } \lambda^{-1} & \text{on } E''_1, \\ \varepsilon''_1(w) \text{ or } \varepsilon''_2(w) & \text{on } E''_2. \end{cases} \end{aligned}$$

On any intersection $E'_j \cap E''_k = E_{jk}$, each of ε' and ε'' depends on only one of the variables z, w . We decompose

$$|(T_{\tilde{\varepsilon}}^\lambda)^* T_{\tilde{\varepsilon}}^\lambda G(z)| \leq \sum_{j,k=1}^2 \int_A \mathbf{1}_{E_{jk}}(z, w) |K_{\tilde{\varepsilon}}^\lambda(z, w)| |G(w)| dw = \sum_{j,k=1}^2 U_{jk} |G|(z),$$

In the case $j = k = 1$ we have

$$\begin{aligned} U_{11} |G|(z) &\leq C \int_A \frac{1}{\tilde{\varepsilon}'(z)\tilde{\varepsilon}''(z)} \left(1 + \frac{|z_1 - w_1|}{\tilde{\varepsilon}'(z)}\right)^{-2} \left(1 + \frac{|z_2 - w_2|}{\tilde{\varepsilon}''(z)}\right)^{-2} |G(w)| dw \\ &\leq C M_s G(z), \end{aligned}$$

where M_s denotes the strong maximal function in \mathbb{R}^2 . Hence U_{11} is bounded on L^2 .

In the case $j = k = 2$, it is sufficient to observe that U_{22}^* has the same form as U_{11} to obtain the same conclusion.

Suppose now that $j \neq k$, say $j = 1, k = 2$, i.e., with ε' depending on z and ε'' on w . Then, extending G to be 0 on $\mathbb{R}^2 \setminus A$,

$$\begin{aligned} U_{12}|G|(z) &\leq C \int_A \frac{1}{\tilde{\varepsilon}'(z)\tilde{\varepsilon}''(w)} \left(1 + \frac{|z_1 - w_1|}{\tilde{\varepsilon}'(z)}\right)^{-2} \left(1 + \frac{|z_2 - w_2|}{\tilde{\varepsilon}''(w)}\right)^{-2} |G(w)| dw \\ &= C \int_{\mathbb{R}} \frac{1}{\tilde{\varepsilon}'(z)} \left(1 + \frac{|z_1 - w_1|}{\tilde{\varepsilon}'(z)}\right)^{-2} \\ &\quad \times \left(\int_{\mathbb{R}} \frac{1}{\tilde{\varepsilon}''(w)} \left(1 + \frac{|z_2 - w_2|}{\tilde{\varepsilon}''(w)}\right)^{-2} |G(w_1, w_2)| dw_2 \right) dw_1 \\ &= C \int_{\mathbb{R}} \frac{1}{\tilde{\varepsilon}'(z)} \left(1 + \frac{|z_1 - w_1|}{\tilde{\varepsilon}'(z)}\right)^{-2} (T|G|)(w_1, z_2) dw_1 \\ &\leq C M_1(T|G|)(z_1, z_2), \end{aligned}$$

where $M_1 f(z_1, z_2)$ denotes the one-dimensional Hardy–Littlewood maximal function of $f(\cdot, z_2)$ evaluated at z_1 and

$$Tf(w_1, z_2) = \int_{\mathbb{R}} \frac{1}{\tilde{\varepsilon}''(w)} \left(1 + \frac{|z_2 - w_2|}{\tilde{\varepsilon}''(w)}\right)^{-2} f(w_1, w_2) dw_2.$$

In analogy with the previous case, the operator T^* ,

$$T^*h(w_1, w_2) = \int_{\mathbb{R}} \frac{1}{\tilde{\varepsilon}''(w)} \left(1 + \frac{|z_2 - w_2|}{\tilde{\varepsilon}''(w)}\right)^{-2} h(w_1, z_2) dz_2,$$

is dominated by

$$\sup_{0 < \varepsilon < 1} \int_{\mathbb{R}} \frac{1}{\varepsilon} \left(1 + \frac{|z_2 - w_2|}{\varepsilon}\right)^{-2} |h(w_1, z_2)| dz_2 = M_2 h(w_1, w_2),$$

M_2 being now the Hardy–Littlewood maximal operator in the second variable. It follows that T , and hence U_{12} , is bounded on L^2 and this proves the statement for $p = 2$.

The conclusion for $1 < p < 2$ follows by Riesz–Thorin interpolation. □

We go back to the proof of Theorem 2.1, recalling that we are assuming $p' = 3q$. Observing that $p'/2 > 2$ and combining together (2.7), (2.9) and Lemma 2.2, we have

$$\|\mathcal{E}_{\varepsilon', \varepsilon''}^\lambda g\|_{L^{p'}(\mathbb{R}^2)} \leq C (\|G_+\|_{L^r(A)} + \|G_-\|_{L^r(A)})^{1/2},$$

with G_\pm as in (2.8) and $r = (p'/2) = \frac{p}{2-p}$. To express the right-hand side in terms of the original function g , we find that

$$\begin{aligned} \|G_+\|_{L^r(A)}^r &= \int_A \left| \frac{G_0(x_+(z), y_+(z))}{\varphi'(x_+(z)) - \varphi'(y_+(z))} \right|^r dz \\ &= \int_{U_+} \frac{|G_0(x, y)|^r}{|\varphi'(x) - \varphi'(y)|^{r-1}} dx dy \\ &= \int_{U_+} \frac{|g(x)|^r |g(y)|^r}{|\varphi'(x) - \varphi'(y)|^{r-1}} \kappa(x)^{r/3} \kappa(y)^{r/3} dx dy. \end{aligned}$$

Making the change of variables

$$u = \varphi'(x), \quad v = \varphi'(y),$$

and setting $x(u) = (\varphi')^{-1}(u)$, $y(v) = (\varphi')^{-1}(v)$, we obtain that

$$\|G_+\|_{L^r(A)}^r = \int_{\varphi'(U_+)} \frac{|g(x(u))|^r |g(y(v))|^r}{|u-v|^{r-1}} \kappa(x(u))^{r/3-1} \kappa(y(v))^{r/3-1} du dv.$$

Notice that $1 \leq r < 2$, so that we can interpret, up to a constant factor, the integral as the pairing $\langle I^{2-r} f, f \rangle$, where I^α denotes fractional integration of order α and

$$f(u) = |g(x(u))|^r \kappa(x(u))^{r/3-1}.$$

By the Hardy–Littlewood–Sobolev inequality,

$$\|G_+\|_{L^r(A)}^r \leq C_r \|f\|_{L^s(\varphi'(U_+))}^2,$$

with $s = 2/(3-r)$. The same estimate holds for G_- , so that, for this value of s ,

$$\begin{aligned} \|\mathcal{E}_{\varepsilon'}^\lambda g\|_{L^{p'}(\mathbb{R}^2)} &\leq C_p \|f\|_{L^s(\varphi'(U))}^{1/r} = C_p \left(\int_{\varphi'(U)} |g(x(u))|^{\frac{2r}{3-r}} \kappa(x(u))^{-2/3} du \right)^{\frac{3-r}{2r}} \\ &= C_p \left(\int_U |g(x)|^{\frac{2r}{3-r}} \kappa(x)^{1/3} du \right)^{\frac{3-r}{2r}} = \|g\|_{L^{\frac{2r}{3-r}}(I,\mu)}. \end{aligned}$$

But $2r/(3-r) = (p'/3)' = q'$, with q as in the statement of the theorem. □

Consider now the truncated strong maximal function of \widehat{f} ,

$$(2.13) \quad \mathcal{M}^+ f(x) = \sup_{0 < \varepsilon', \varepsilon'' < 1/4} \frac{1}{4\varepsilon' \varepsilon''} \int_{|s| < \varepsilon', |t| < \varepsilon''} |\widehat{f}(x+s, \varphi(x)+t)| ds dt, \quad x \in I.$$

From Theorem 2.1 we obtain the following inequality for \mathcal{M}^+ for a more restricted range of p .

Corollary 2.3. *The inequality*

$$(2.14) \quad \|\mathcal{M}^+ f\|_{L^q(I,\mu)} \leq C_p \|f\|_{L^p(\mathbb{R}^2)}, \quad f \in \mathcal{S}(\mathbb{R}^2),$$

holds for $1 \leq p < 8/7$ and $p' \geq 3q$.

Proof. As before, we assume $p' = 3q$. Let $h = f * f^*$, where

$$f^*(x, y) = \overline{f(-x, -y)}.$$

Then $\widehat{h} = |\widehat{f}|^2$, so that $\|h\|_r \leq \|f\|_p^2$, with $r = p/(2 - p) < 4/3$. Then, for s such that $r' = 3s$, $\|\mathcal{M}h\|_s \leq C_r \|f\|_p^2$. But, for $\varepsilon', \varepsilon'' < 1/4$ and χ as in (2.1),

$$\begin{aligned} & \frac{1}{4\varepsilon'\varepsilon''} \int_{|s|<\varepsilon', |t|<\varepsilon''} |\widehat{f}(x + s, \varphi(x) + t)| \, ds \, dt \\ & \leq \left(\frac{1}{4\varepsilon'\varepsilon''} \int_{|s|<\varepsilon', |t|<\varepsilon''} |\widehat{f}(x + s, \varphi(x) + t)|^2 \, ds \, dt \right)^{1/2} \\ & = \left(\frac{1}{4\varepsilon'\varepsilon''} \int_{|s|<\varepsilon', |t|<\varepsilon''} \widehat{h}(x + s, \varphi(x) + t) \, ds \, dt \right)^{1/2} \\ & \leq \left(\int \widehat{h}(x + s, \varphi(x) + t) \chi_{4\varepsilon'}(s) \chi_{4\varepsilon''}(t) \, ds \, dt \right)^{1/2} \leq (\mathcal{M}h(x))^{1/2}. \end{aligned}$$

Hence $\|\mathcal{M}^+ f\|_{L^q(I, \mu)} \leq \|\mathcal{M}h\|_{q/2}^{1/2}$ and it can be easily checked that $q/2 = s$. □

3. Lebesgue points of \widehat{f} along a curve

Adapting standard arguments, cf. [4], we obtain the following reformulation of Theorem 1.1 (ii), where B_ε denotes the disk of radius ε centered at 0.

Corollary 3.1. *Let $1 \leq p < 8/7$ and let S be a C^2 curve in the plane. Given $f \in L^p(\mathbb{R}^2)$, for almost every $x \in I$ relative to affine arclength,*

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{|B_\varepsilon|} \int_{B_\varepsilon} |\widehat{f}(x + x', \varphi(x) + y') - \mathcal{R}f(x, \varphi(x))| \, dx' \, dy' = 0.$$

Proof. We may restrict ourselves to a subset of S which is the graph of a C^2 function φ on an interval I with $\varphi'' \neq 0$. Let μ be as in Section 2.

Given $\tau > 0$, let $g \in \mathcal{S}(\mathbb{R}^2)$ such that $\|f - g\|_p < \tau$. Since $\mathcal{R}g = \widehat{g}|_S$,

$$\begin{aligned} F(x) &= \limsup_{\varepsilon \rightarrow 0} \frac{1}{|B_\varepsilon|} \int_{B_\varepsilon} |\widehat{f}(x + x', \varphi(x) + y') - \mathcal{R}f(x, \varphi(x))| \, dx' \, dy' \\ &\leq \limsup_{\varepsilon \rightarrow 0} \frac{1}{|B_\varepsilon|} \int_{B_\varepsilon} |(\widehat{f - g})(x + x', \varphi(x) + y')| \, dx' \, dy' + |\mathcal{R}(f - g)(x, \varphi(x))| \\ &\leq \mathcal{M}^+(f - g)(x) + |\mathcal{R}(f - g)(x, \varphi(x))|. \end{aligned}$$

Hence, if $q = p'/3$, $\|F\|_{L^q(I, \mu)} \leq C\tau$ for every $\tau > 0$, i.e., $F = 0$ μ -a.e. □

References

- [1] CARLESON, L. AND SJÖLIN, P.: Oscillatory integrals and a multiplier problem for the disc. *Studia Math.* **44** (1972), 287–299.
- [2] FEFFERMAN, C.: Inequalities for strongly singular convolution operators. *Acta Math.* **124** (1970), 9–36.

- [3] SJÖLIN, P.: Fourier multipliers and estimates of the Fourier transforms of measures carried by smooth curves in \mathbb{R}^2 . *Studia Math.* **51** (1974), 169–182.
- [4] STEIN, E. M.: *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series 30, Princeton University Press, Princeton, NJ, 1970.
- [5] TOMAS, P. A.: A restriction theorem for the Fourier transform. *Bull. Amer. Math. Soc.* **81** (1975), 477–478.
- [6] VITTURI, M.: A note on maximal Fourier restriction for spheres in all dimensions. Preprint, arXiv:1703.09495, 2017.
- [7] ZYGMUND, A.: *Trigonometric series. Vols. I, II*. Cambridge University Press, New York, 1959.
- [8] ZYGMUND, A.: On Fourier coefficients and transforms of functions of two variables. *Studia Math.* **50** (1974), 189–201.

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