Scuola Normale Superiore
Université Paris-Saclay
Université Franco Italienne | Università Italo Francese

Classe di Scienze

Corso di perfezionamento in
Matematica

XXXIII ciclo

Tesi in cotutela con
Université Paris-Saclay

Abundance of rational points

MAT/02 (principale), MAT/03 (secondario)

Candidato
dr. Julian Lawrence Demeio

Direttore di tesi Scuola Normale Superiore
prof. Umberto Zannier

Direttore di tesi Université Paris-Saclay
prof. David Harari

Contents

Abstract

For a smooth algebraic variety X defined over a number field K, one could ask several questions about the abundance of its rational points.

This thesis revolves, in particular, around the following three properties: Hilbert Property, weak approximation and strong approximation. The first concerns, more or less, the question of extending the Hilbert Irreducibility Theorem to an arbitrary X (in the sense that the parameters of the Theorem are allowed to vary through rational points of this variety), the interesting case being when X is non-rational, for otherwise one recovers precisely the original theorem of Hilbert. The other two concern the question of density of rational points of X in the adelic ones (possibly with some places removed). The adjective "weak" is more commonly used when talking about proper varieties, and the adjective "strong" is used otherwise.

In the first original work that is part of this thesis, we prove that, under a technical assumption, a proper algebraic surface X, with Zariski-dense rational points, that is endowed with two or more genus 1 fibrations, has the Hilbert Property. This result generalizes an earlier result of Corvaja and Zannier, who proved the Hilbert Property for the Fermat surface $x^4 + y^4 = z^4 + w^4$. The technique used is similar to theirs, the main idea being that of transporting rational points around the surface using the elliptic fibers of the various fibrations.

In the second part of the thesis, we prove that on an arbitrary homogeneous space X, under some technical assumptions, the étale-Brauer–Manin obstruction is the only one to strong approximation. This obstruction is obtained by applying the more classical Brauer–Manin obstruction on all finite étale torsors over X. The proof is basically a reduction to a theorem of Borovoi and Demarche, who proved that (again under technical assumptions) strong approximation up to Brauer–Manin obstruction holds on homogeneous spaces with **connected** stabilizers.

In this part of the thesis we also prove a compatibility result, suggested to be true by work of Cyril Demarche, between Brauer pairing and the so-called abelianization map, for homogeneous spaces of the form G/H, with H connected and linear.

Finally, in the third and last part of the thesis, we explore the problem of "ramified descent", or, in other words, the question of which adelic points of X may be lifted to (a desingularization of a twist of) a fixed geometrically integral and geometrically Galois cover $\varphi : Y \to X$, with commutative geometric Galois group (although in some parts of the work this commutativity assumption is not needed). The case where the cover is unramified is already well-studied, and, therefore, the interest lies in the ramified case (whence the terminology "ramified descent"). We prove that a certain naturally defined "descent set" provides an obstruction to Hasse principle and weak approximation on X (the main difficulty in proving this lies in showing that rational points that lie on the branch locus of φ are unobstructed).

Moreover, in analogy with the classical unramified case, we construct a subgroup B_φ of the Brauer group of X such that the the descent set associated to φ lies in the Brauer–Manin set associated to B_φ. Interestingly enough, the transcendental part of B_φ may provide a non-trivial obstruction, contrary to what happens in the unramified case. It seems reasonable to expect that this B_φ is the only obstruction to the "ramified descent" problem.

Résumé

Pour une variété algébrique lisse X définie sur un corps de nombres K, on peut se poser plusieurs questions sur l'abondance de ses points rationnels.

En particulier, cette thèse s'intéresse aux trois propriétés suivantes : propriété de Hilbert, l'approximation faible et l'approximation forte. La première concerne plus ou moins la question de l'extension duthéorème d'irréductibilité de Hilbert à une X arbitraire (par quoi nous entendons que les paramètres duthéorème peuvent varier parmi les points rationnels de cette variété), le cas intéressant étant lorsque X est non rationnel, car sinon on retrouve précisément le théorème originel de Hilbert. Les deux autres concernent la question de la densité des points rationnels de X dans les points adéliques (possiblement en dehors d'ensemble finis de places). L'adjectif "faible" est normalement utilisé pour parler de variétés propres, et l'adjectif "fort" est utilisé autrement.

Dans le premier travail original qui fait partie de cette thèse, nous montrons que, sous une hypothèse technique, une surface algébrique X propre, avec les points rationnels Zariski-denses, et qui est dotée de deux ou plusieurs fibrations de genre 1, a la propriété de Hilbert. Ce résultat généralise un résultat antérieur de Corvaja et Zannier, qui ont prouvé la propriété de Hilbert pour la surface de Fermat $x^4+y^4=z^4+w^4$. La technique utilisée est similaire à la leur, l'idée principale étant de transporter les points rationnels autour de la surface à l'aide des fibres elliptiques des différentes fibrations.

Dans la deuxième partie de la thèse, nous montrons que pour tous les espaces homogènes X, sous certaines hypothèses techniques, l'obstruction de Brauer–Manin étale est la seule à l'approximation forte. Cette obstruction est notamment obtenue en appliquant l'obstruction de Brauer–Manin sur tous les torseurs étales finis sur X. Notre preuve est essentiellement une réduction à un théorème de Borovoi et Demarche, qui ont montré que (toujours sous des hypothèses techniques) pour les espaces homogènes avec stabilisateurs **connexes** l'obstruction de Brauer–Manin est la seule à l'approximation forte.

Dans cette partie de la thèse, nous prouvons aussi un résultat de compatibilité, suggéré par des travaux de Cyril Demarche, entre l'accouplement de Brauer et l'application dite d'abélianisation, pour des espaces homogènes de la forme G/H, avec H connexe et linéaire.

Enfin, dans la troisième et dernière partie de la thèse, nous explorons le problème de la "descente ramifiée", ou, en d'autres termes, la question de quels points adéliques de X peuvent être relevés à (une désingularisation d'un tordu d') un revêtement (possiblement ramifié) géométriquement intègre et géométriquement de Galois $\varphi : Y \to X$ fixé, avec un groupe de Galois géométrique commutatif (bien que dans certaines parties du travail cette hypothèse de commutativité ne soit pas nécessaire). Le cas où le revêtement est non ramifié étant déjà bien étudié, on est intéressé principalement au cas ramifié (d'où la terminologie "descente ramifiée"). Nous prouvons qu'un certain "ensemble de descente", défini naturellement, fournit une obstruction au principe de Hasse et à l'approximation faible sur X (la difficulté principale pour prouver cela réside dans la démonstration que les points rationnels de X qui se trouvent sur le lieu de ramification de φ ne sont pas obstrués).

De plus, par analogie avec le cas classique non ramifié, on construit un sous-groupe B_φ du groupe de Brauer de X tel que l'ensemble de descente associé à φ se trouve dans l'ensemble de Brauer–Manin associé à B_φ. On prouve aussi, à l'aide d'un exemple explicite, que la partie transcendante de B_φ peut fournir une obstruction non triviale, contrairement à ce qui se passe

dans le cas non ramifié. Il semble raisonnable de s'attendre à ce que le groupe B_φ soit la seule obstruction au problème de la “descente ramifiée”.

Chapter 1

Introduction and Overview

In an algebraic variety X defined over a number field K, one can be interested in the abundance of rational points. In this thesis, this theme, along with questions connected to it, will be the main motivation for the various problems we will be analyzing. Let us start, before delving in the specifics of the work, by briefly motivating such an interest to the reader.

Of course, the simplest example of an abundance property would just be the question of whether $X(K)$ is Zariski-dense in X. Although this question can already be engaging, a positive answer to it is often not enough for applications.

Notoriously, this is the case for applications to the Inverse Galois Problem, i.e. the question “Is every finite group a Galois group over $\mathbb{Q}$?”. This old unsolved problem in mathematics can instead be approached via the “Hilbert Property”, a particular abundance property for rational points which is stronger than Zariski-density and on which we will say more later.

Another issue which might motivate the reader is the question of weak approximation, which we enunciate here in its simplest form, in the case of a projective hypersurface $X : \{f = 0\}$, over $K = \mathbb{Q}$. Let us assume that the homogeneous polynomial f has integral coefficients. We say that X satisfies weak approximation if, for every positive integer n and for every non-zero solution mod n of X and every real point in $X(\mathbb{R})$, there exists an integral solution of $f = 0$ such that it reduces to the given mod n solution, and such that it approximates arbitrarily well the real one. When X is birational to a projective space (these varieties are called *rational*), then one may easily prove that weak approximation holds. However, the general question is well-known to have a negative answer, but one can still be interested in obtaining a positive answer in some specific instances (preferably non-rational). We will later describe in more detail what is known about this question (and related ones, the so-called “Hasse principle” and “strong approximation”).

This approximation property is far from disconnected from the Hilbert Property. In fact, the weak approximation property implies the Hilbert Property for a given projective variety. Unfortunately, the former has the disadvantage of not being very common, even among unirational varieties (although, at least conjecturally, in this case it should hold in a “weaker” form), but especially of being much harder to prove than the latter.

In the rest of this introduction we describe in more detail the context and work of this thesis. Specifically, in Section 1.1 we describe the various abundance properties to which we have hinted at and some basic facts concerning them. In Section 1.2 we describe the first original work that is part of this thesis. In this work the author proves the Hilbert Property of some projective surfaces with multiple elliptic fibrations. In Section 1.3 we describe some well-known obstructions to the approximation properties mentioned above. We also spend here some words on the so-called “descent” technique. In Section 1.4 we describe the second original work that is part of this thesis, which concerns the strong approximation property on homogeneous spaces. In Section 1.5 the last original work that is part of this thesis is described. This concerns the matter of “descent on ramified covers” (“classical” descent is done on unramified covers).

1.1 Various abundance properties

Hilbert Property Let us start by formulating the aforementioned inverse Galois Problem:

Question (Inverse Galois Problem over $\mathbb{Q}$). Is every finite group a Galois group over $\mathbb{Q}$?

Hilbert was the first to purposefully analyze this question [Hil92]. He proved that the answer is positive for A_n and S_n for all n. In order to do so, he proved his famous Irreducibility Theorem (more on this later).

Later, his idea was used again by Emmy Noether [Noe17], who observed that the Hilbert Irreducibility Theorem could be used to deduce a positive answer to the Inverse Galois Problem for all groups, if one was able to obtain a positive answer to the following question (which we formulate here in modern terminology):

Question (Noether's Problem). Is the quotient $\mathbb{A}^n_{\mathbb{Q}}/G$ (where the action of G on $\mathbb{A}^n_{\mathbb{Q}}$ is faithful and linear) a rational variety for all finite (constant) groups G?

Let us recall that a variety X, of dimension n, defined over a field k, is said to be *rational* if there exists a birational map $X \dashrightarrow \mathbb{P}^n_k$.

As is now very well-known, the above problem has a negative answer in general (see [Swa69] for the first counterexample and [Sal84] for the first counterexample over $\mathbb{C}$). Noether's idea was therefore abandoned for some time. It was then around the 1980's that the idea was somehow revived. It was noted that, in fact, even if hoping for the quotients $\mathbb{A}^n_{\mathbb{Q}}$ to be rational would be too much, it might be possible that the variety $\mathbb{A}^n_{\mathbb{Q}}/G$ would still contain "a lot of" rational points. To be precise, one can make sense here of the vague concept of "containing a lot of rational points" by introducing so-called thin sets and the Hilbert Property. The definitions that we are about to give are due to Serre [Ser08] (thin sets) and Colliot-Thélène and Sansuc [CTS87b] (Hilbert Property, which they call "being of Hilbert type").

Definition 1.1.1 (Thin sets)**.** Let X be a geometrically irreducible variety defined over a field K. A subset S of the rational points $X(K)$ is said to be *thin* if it is contained in a union of two types of sets:

(T1) the rational points $Z(K)$ of a proper subvariety of X;

(T2) the rational points coming from a non-trivial cover, i.e. sets of the form $\pi(Y(K))$, where $\pi : Y \to X$ is a finite morphism of degree > 1, and Y is irreducible.

A variety X/K satisfies the *Hilbert Property* (HP) if $X(K)$ is not thin.

The Irreducibility Theorem of Hilbert translates then into this terminology into saying that $\mathbb{A}^n_K$ has the Hilbert Property when K is a number field.

The connection with the Inverse Galois Problem stems from the following proposition (see [Ser08, Cor. 3.3.2], which was already implicit in [CTS87b, Rmk. 7.12]):

Proposition 1.1.2. *Let X be a variety defined over a field K, endowed with a generically free action of a finite group G. If the quotient X/G has the Hilbert Property, then G is a Galois group over K.*

As one may easily see, the Hilbert Property is a birational invariant, hence yet another way of rephrasing Hilbert's Irreducibility Theorem is to say that rational varieties have the Hilbert Property.

Let us also mention a negative example: abelian varieties do not have the Hilbert Property. This is because all rational points on an abelian variety A may be lifted to a cover defined by $A \to A, P \mapsto [2]P + R$, where R is one of finitely many representatives for the quotient $A(K)/[2]A(K)$. Note that the finiteness of this quotient, or, in other words, the weak Mordell-Weil theorem, is used in an essential way here!

The following conjecture, posed by Colliot-Thélène and Sansuc [CTS87b, p. 190], would imply that, for every finite group G and every embedding $G \hookrightarrow SL_{n,K}$, the variety SL_n/G, which is unirational, has the Hilbert Property. By the proposition above, this in turn would imply a positive answer to the Inverse Galois Problem for all finite groups G.[1]

Conjecture 1.1.3. *All unirational varieties over number fields satisfy the Hilbert Property.*

Recall that a variety is *unirational* if it is dominated, as a K-variety, by a rational one.

A more powerful conjectural link between geometry of a variety and HP was suggested by Corvaja and Zannier [CZ17]. Namely, they proved in [CZ17, Theorem 1.6] that every smooth proper variety with the Hilbert Property is geometrically simply connected, and then posed the following concerning the converse:

Question 1.1.4. Does every **geometrically simply connected** variety X/K with Zariski-dense rational points satisfy the Hilbert Property?

Corvaja and Zannier answer the above question affirmatively for the Fermat quartic $x^4+y^4 = z^4+w^4 = 0$. The first original work that is part of this thesis (see Section 1.2 and Chapter 2) contains generalizations of this result.

Hasse principle and weak approximation Weak approximation is an abundance property for rational points, and it is stronger than the Hilbert Property. The Hasse principle is not an abundance property, rather it concerns the existence of rational points, but it goes hand in hand with weak approximation, so we introduce it as well.

Let $X \subseteq \mathbb{P}^n$ be a projective algebraic variety defined over $\mathbb{Q}$, defined by the system of homogeneous polynomial equation:

$$\begin{cases} f_1(x_0,\dots,x_n) = 0 \\ \vdots \\ f_k(x_0,\dots,x_n) = 0 \end{cases},$$

which we may assume to have integer coefficients.

Definition 1.1.5 (Hasse principle, I)**.** We say that X satisfies the *Hasse principle* if the following implication holds: "If there exists a non-zero mod n solution to the system above for all natural numbers $n > 0$, and a non-zero real solution, then there exists a non-zero rational solution to the system". A family of varieties is said to satisfy the *Hasse principle* if all of its members do.

This definition is independent of the choice of equations. In fact, we give below a reformulation that does not depend on this choice.

A classical example where the Hasse principle holds is that of smooth quadrics (in any dimension), i.e., for $X = \{f = 0\}$, with f quadratic and non-degenerate. This is the famous theorem of Hasse-Minkowski.

Another, more abstract, way to define the Hasse principle is the following, which makes sense for all varieties X defined over a number field K.

Definition 1.1.6 (Hasse principle, II)**.** We say that X satisfies the *Hasse principle* (HaP) if the following implication holds: "If $X(K_v) \neq \emptyset$ for all places v of K, then $X(K) \neq \emptyset$". A family of varieties is said to satisfy the *Hasse principle* if all of its members do.

Again, by the Hasse-Minkowski theorem, this property is satisfied by smooth quadrics.

We formulate now the weak approximation property (we just give the abstract definition here, as the "concrete" one was already hinted at at the very beginning of this introduction).

[1] In the article of Colliot-Thélène and Sansuc the conjecture is left more as a curiosity: "It would certainly be of interest to discuss the existence of k-unirational varieties which are not of Hilbert type." However, it is now widely believed that these "k-unirational varieties [...] of interest" should not exist, especially in view of a (stronger) version of Conjecture 1.1.3 that followed it, i.e., Conjecture 1.3.4.

Definition 1.1.7. Assume that $X(K) \neq \emptyset$. We say that X has the *weak approximation* (WA) property if, for all finite subsets $S \subseteq M_K$, the rational points $X(K)$ are dense in $\prod_{v\in S} X(K_v)$. We say that X has the *weak weak approximation* (WWA) property if there exists a finite $S_0 \subseteq M_K$ such that, for all finite subsets $S \subseteq M_K$ disjoint from S_0, the rational points $X(K)$ are dense in $\prod_{v\in S} X(K_v)$.

The WA property is satisfied by projective spaces (with this formulation, this is a consequence of the Chinese Remainder Theorem).

The WWA property is strictly related to the Hilbert Property, by the following result, due to Ekedahl [Eke90] and Colliot-Thélène [Ser08, Section 3.5].

Theorem 1.1.8 (Ekedahl, Colliot-Thélène)**.** *Every variety X/K with WWA satisfies the Hilbert Property.*

Colliot-Thélène in fact conjectured, in an unpublished letter (see [Ser08, Sec. 3.5]), that the WWA would hold on all unirational varieties. Note that, by the result above, this conjecture is stronger than Conjecture 1.1.3. He would then make, with Sansuc, an even stronger and more precise conjecture (more on this in Section 1.3, see also [CTS80, p. 228][CTS21, Sec. 14.1]).

Strong approximation The Hasse principle and weak approximation are properties that concern *rational* points (and that is why we formulated them for projective varieties). There exists an analogue property, called *strong* approximation that concerns, in a certain sense, *integral* points, and this is relevant only for open varieties (for proper varieties it reduces to weak approximation).

We give two definitions, which are actually not equivalent, the second is stronger than the first. The second one, which is more technical, is the one that is usually used in the literature. But the first one, more concrete, is much more intuitive and therefore might spark more interest.

Definition 1.1.9 (Strong Approximation for integral points)**.** An affine variety $X \subseteq \mathbb{A}^n_{\mathbb{Q}}$, defined by a system of polynomial equations:

$$\begin{cases} f_1(x_1,\dots,x_n) = 0 \\ \vdots \\ f_k(x_1,\dots,x_n) = 0 \end{cases},$$

which we assume to have integral coefficients is said to satisfy the strong approximation property for integral points if, for every positive integer n and every solution mod n of the system, there exists an integral solution of the system that lifts the mod n one.

For the second definition, it is convenient to make use of the formalism of adeles and adelic points.

Adeles and adelic points. We recall that the ring of adeles (resp. S-adeles, for some subset $S \subseteq M_K$) $\mathbb{A}_K$ (resp. $\mathbb{A}_K^S$) of K is defined to be the restricted product $\prod'_{v\in M_K} K_v$ (resp. $\prod'_{v\in M_K\setminus S} K_v$), with respect to $O_v \subseteq K_v$. Note that $\mathbb{A}_K$ and $\mathbb{A}_K^S$ are extensions of K. So, it makes sense to talk of $\mathbb{A}_K$-points of X, i.e., points of X with coordinates in $\mathbb{A}_K$. The set of $\mathbb{A}_K$-points of X are called the adelic points of X and are denoted by $X(\mathbb{A}_K)$. Analogously, $X(\mathbb{A}_K^S)$ denotes S-adelic, or $\mathbb{A}_K^S$-points.

An adelic (resp. S-adelic) point of X is, equivalently, given by the following data: a K_v-point P_v of X for each $v \in M_K$ (resp. $M_K \setminus S$) such that P_v is integral* in X for almost all v.

* When X is quasi-projective, by *integral* we mean the following (for the general case one has to resort to scheme-theoretic models). Let X' be the closure of X in projective space, and let $D := X' \setminus X$, then a v-adic point of X is *integral* if it does not reduce to D modulo v. Since

the integrality condition was imposed on almost all v, one can easily prove that this definition depends just on the K-variety X and not the chosen embedding in projective space. Note that, when X is affine, a v-adic point is integral if and only if it has integral coordinates.

Definition 1.1.10 (Strong Approximation)**.** A variety X defined over a number field K, with $X(K) \neq \emptyset$ is said to satisfy strong approximation off a finite subset $S \subseteq M_K$ if $X(K)$ is adelically dense in $X(\mathbb{A}_K^S) = \prod'_{v \notin S} X(K_v)$, where the restricted product is taken with respect to $X(O_v) \subseteq X(K_v)$.

Using the Chinese Remainder Theorem, one proves that the two definitions connect in the following way: if an affine vareity $X/\mathbb{Q}$ has strong approximation (according to the second definition) off ∞, then it has strong approximation according to the first definition as soon as it has solutions in $\mathbb{Z}_p$ for all primes p. This is because, in the adelic terminology, the first definition is asking for the density of $X(\mathbb{Z})$ in $\prod_{p \text{ prime}} X(\mathbb{Z}_p)$, while the second is asking (in this case) for the density of $X(\mathbb{Q})$ in $\prod'_{p \text{ prime}} X(\mathbb{Q}_p)$. Since $X(\mathbb{Q}) \cap \prod_{p \text{ prime}} X(\mathbb{Z}_p) = X(\mathbb{Z})$, the second condition implies the first, i.e., the second definition is stronger.

As mentioned above, when X is proper, Definition 1.1.10 above gives back the definition of weak approximation"off S". Therefore, usually strong approximation is studied in the context of affine varieties, but not exclusively, see e.g. [CLX19].

It is worth mentioning that, when X is affine and of positive dimension, one cannot have "strong approximation off the empty set", or, in other words, $X(K)$ is never dense in $X(\mathbb{A}_K)$. This is because, in the affine case, $X(K)$ is actually discrete in $X(\mathbb{A}_K)$ (because K is discrete in the adeles $\mathbb{A}_K$).

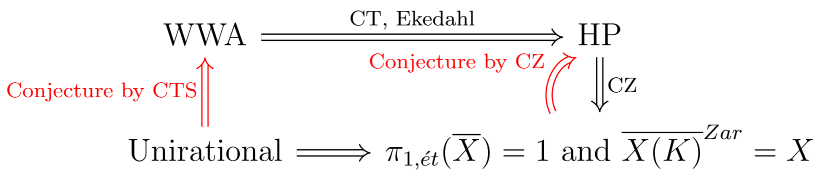
Strong approximation holds on affine spaces $\mathbb{A}_K^n$ by the Chinese Remainder Theorem.

Notoriously, strong approximation has been proven for the semisimple simply connected groups satisfying a certain non-compactness assumption (see [PRR93, Theorem 7.12]). The result is very sophisticated: Platonov's proof makes use of the Kneser-Tits conjecture, which he proved [Pla69].

Advantages and disadvantages of the various properties Of course stronger properties are usually harder to prove. In particular, strong approximation is known for very few varieties, with the most important class being the aforementioned semisimple simply connected linear groups (for instance, SL_n and the spin groups $Sp_n(f)$, for an indefinite quadratic form f). On the other hand weak approximation and the Hasse principle have been proven for slightly more varieties. These include, for instance, compactifications of some principal homogeneous spaces of tori (when there is no Brauer–Manin obstruction, which we will define later).

The Hilbert Property is known for some K3 surfaces (see next section).

Schematic summary of the implications among the various properties For proper geometrically connected smooth X/K:



where: CT=Colliot-Thélène, S=Sansuc; CZ=Corvaja, Zannier.

Any of the conjectures $\Rightarrow$ would imply the HP for the quotients SL_n/G (G finite) for all G, hence a positive answer to the Inverse Galois Problem by the argument presented before Conjecture 1.1.3.

1.2 Elliptic fibrations and the Hilbert Property

We are ready to introduce the first original result that is part of this thesis. This result concerns the search for a sufficient condition for obtaining the Hilbert Property of a surface (defined over a number field). We will be particularly interested in K3 surfaces. To explain why, we start by recalling (in spirit of analogy) the now well-known complete qualitative description of rational points on algebraic curves.

On rational curves, rational points are of course very abundant. On curves of genus ≥ 2, there are only finitely many rational points (by Faltings' Theorem). On genus 1 curves, which represent somehow a limit case, rational points are (potentially) Zariski-dense, but they are not that abundant: for instance the Hilbert Property does not hold on elliptic curves (as mentioned before, this is essentially a consequence of the weak Mordell-Weil theorem).

The behaviour of rational points on surfaces (actually, varieties in general), is thought, by analogy, to be similar. In particular, rational points are expected to be abundant on varieties that are geometrically rational (as soon as there is at least one), while, on the other end of the spectrum, they are expected to not be Zariski-dense on surfaces of general type (see the famous Bombieri-Lang conjecture). The limiting case would be surfaces of Kodaira dimension 0. These are divided, by the Enriques–Kodaira classification, into abelian varieties, Enriques surfaces, bielliptic surfaces (and in all three classes the surfaces are not geometrically simply connected, hence the Hilbert Property never holds by a result of Corvaja and Zannier [CZ17, Theorem 1.6]), and K3 surfaces. So, K3 surfaces represent a sort of "limiting case" where the HP could still hold.

The first result in this direction was a recent theorem of Corvaja and Zannier [CZ17, Theorem 1.4], who proved that the Hilbert Property holds for the Fermat surface $x^4 + y^4 = z^4 + w^4$.

Before their result, nothing was known about abundance of rational points (in any sense that is stronger than Zariski or real density) in K3 surfaces (or, in fact, any variety that is not unirational or abelian).

We prove the following theorem, which generalizes their result.

Theorem 1.2.1. *Let E be a projective smooth geometrically connected algebraic surface, defined over a number field K. Suppose that there exist $n \geq 2$ distinct genus one fibrations $\pi_1, \ldots, \pi_n : E \to \mathbb{P}^1$. Let $F \subset E$ denote the union of the divisors of E that are contained, for each $i = 1, \ldots, n$, in a fiber of π_i. If $E \setminus F$ is simply connected and $E(K)$ is Zariski-dense in E, then E has the Hilbert Property.*

The main idea behind the proof of Theorem 1.2.1 is inspired from that of [CZ17, Theorem 1.4]. Namely, one can use the elliptic fibrations to "move points around" the variety X. A sketch of how this is done is the following: given a point $P \in E(K)$, one may construct *degree zero* divisors on the genus one curves $\pi_i^{-1}(P), i = 1, \ldots, n$. One shows, using Merel-Parent's Theorem [Mer96], that, if P is chosen in a sufficiently generic way (hence the assumption that $E(K)$ is Zariski-dense), then these divisors are not torsion, which allows to construct infinitely many K-rational points on the curves $\pi_i^{-1}(P)$. For each of these points, one can restart the process. To show that, after iterating indefinitely, the Hilbert Property actually holds, one uses as fundamental ingredients Faltings' Theorem and the Hilbert Irreducibility Theorem (or, equivalently, the Hilbert Property for $\mathbb{P}^1_K$).

We use Theorem 1.2.1 to give explicit examples of K3 surfaces with the Hilbert Property. Some of the examples are produced starting from a construction presented in [GS19], by Garbagnati and Salgado. Others are Kummer surfaces, for which the Hilbert Property was suggested to be true by Corvaja and Zannier [CZ17]:

Theorem 1.2.2. *Let E_1 and E_2 be elliptic curves over K with infinitely many rational points. The Hilbert Property holds for the quotient $(E_1 \times E_2)/ < \iota >$, where $\iota : E_1 \times E_2 \to E_1 \times E_2$ is the involution $([-1], [-1])$.*

1.3 Obstructions to approximation properties

The other original works in this thesis are more related to obstructions to approximation properties.

Brauer group Let us recall the two possible definitions of Brauer group of a smooth variety X/K. The reason we are interested in this group is that it provides obstructions to all the approximation properties (and the Hasse principle) described in Section 1.1. Sometimes it is even the "only obstruction" (see Conjecture 1.3.4 below and the discussion that follows)!

The first definition we give is more explicit, but it is "the correct one" only in the smooth and proper case. The second definition we give is in fact valid for any scheme X (so, for that definition, there is no need to assume that X is a smooth K-variety).

We recall that the *Brauer group* of a field F is defined as the group of central simple algebras over F, quotiented by the equivalence relation generated by the relations $A \sim M_n(A)$ for all $n \geq 1$ and all central simple algebras A/F, and with group law defined by $(A_1, A_2) \mapsto A_1 \otimes_F A_2$.

Recall that, for a complete DVR R with perfect residue field κ and fraction field k, we have the following short exact sequence [CTS21, Thm. 3.6.2]:

$$1 \to \operatorname{Br} \kappa \to \operatorname{Br} k \xrightarrow{\text{res}} H^1(\kappa, \mathbb{Q}/\mathbb{Z}) \to 1.$$

We refer to [CTS21, Sec. 3.6] for a definition of the various morphisms involved (and especially of the residue map res, which plays an important role in the first definition of $\operatorname{Br} X$).

Definition 1.3.1 (Brauer group, I)**.** Assume X is smooth and proper over a field K of characteristic 0. The Brauer group $\operatorname{Br} X$ of X is the subgroup of the Brauer group of $K(X)$ of elements such that, for all DVRs R containing K whose fraction field is $K(X)$, they lie in the kernel of the following residue map:

$$\text{res}_R : \operatorname{Br}(K(X)) = \operatorname{Br}(\operatorname{Frac}(R)) \to \operatorname{Br}(\operatorname{Frac}(\widehat{R})) \xrightarrow{\text{res}} H^1(\kappa, \mathbb{Q}/\mathbb{Z}),$$

where $\widehat{R}$ is the completion of R at its maximal ideal, κ is its residue field. The elements that lie in $\operatorname{Br} X$ are also called the *unramified* elements of $\operatorname{Br}(K(X))$.

Note that, as is clear from the definition, the Brauer group is a birational invariant among proper smooth varieties.

Definition 1.3.2 (Brauer group, II)**.** The Brauer group of X is the group $H^2_{ét}(X, \mathbb{G}_m)$.

As mentioned before, the "correct" definition (i.e., the one usually presented as definition in the literature, and that we will adopt in the rest of this thesis) is the second one. The equivalence of it to the first definition in the smooth case is a deep result that follows from the purity theorem (see [CTS21, Theorem 3.7.1]). Moreover, another well-known consequence of the purity theorem is that for a central simple algebra $A/K(X)$ (again with $\operatorname{char} K = 0$), A is unramified if and only if $\text{res}_R(A) = 0$ for all DVRs R equal to $\mathcal{O}_{X,\eta} \subseteq K(X)$, for some codimension 1 point $\eta \in X$. In other words, the unramifiedness condition in the first definition may be checked just on the DVRs associated to codimension 1 points on X.

The Brauer–Manin obstruction We recall that, when X is a variety defined over a number field K, and there exists a canonical pairing (called the *Brauer–Manin* pairing):

$$\begin{array}{ccc} \operatorname{Br}(X) \times X(\mathbb{A}_K) & \longrightarrow & \mathbb{Q}/\mathbb{Z}, \\ (b, x) & \longmapsto & \langle b, x \rangle, \end{array}$$

which is defined as follows: if $x = (x_v)_{v \in M_K}$, then $\langle b, x \rangle = \sum_{v \in M_K} \text{inv}_v(x_v^* b)$, where $x_v^* b \in H^2(\Gamma_{K_v}, \overline{K_v}^*)$ denotes the pullback of b along $x_v : \operatorname{Spec} K_v \to X$, and $\text{inv}_v : H^2(\Gamma_{K_v}, \overline{K_v}^*) =$

$\mathrm{Br}(K_v) \to \mathbb{Q}/\mathbb{Z}$ is the usual invariant map (see e.g. [Har20, Thm 8.9] for a definition). The pairing is continuous in x and additive in b. If $x \in X(K) \subset X(\mathbb{A}_K)$ or b comes from $\mathrm{Br}\, K$, then $\langle b, x\rangle = 0$ (see [Sko01, Sec. 5] for a proof, this is essentially a consequence of the classical Albert–Brauer–Hasse–Noether Theorem).

We recall, moreover, that the pairing (being continuous on $X(\mathbb{A}_K)$, and taking values in a discrete group) is constant on the archimedean connected components of X, hence it induces a pairing:

$$\mathrm{Br}(X) \times X(\mathbb{A}_K)_\bullet \longrightarrow \mathbb{Q}/\mathbb{Z},$$

where $X(\mathbb{A}_K)_\bullet$ denotes the topological quotient of $X(\mathbb{A}_K)$ where the archimedean connected components are collapsed to a point.

We will describe in the next paragraph how the pairing above is explicitly computable, using quadratic reciprocity, for quaternion algebras in $\mathrm{Br}\, X$. Apart from that, we refer the reader to [Sko01, Section 5] for more details on the pairing.

We denote by $X(\mathbb{A}_K)^{\mathrm{Br}\, X}$ the following (closed) subset of $X(\mathbb{A}_K)$:

$$\{x \in X(\mathbb{A}_K) \mid \ \langle b, x\rangle = 0 \ \forall b \in \mathrm{Br}(X)\}.$$

We then have that $\overline{X(K)} \subset X(\mathbb{A}_K)^{\mathrm{Br}\, X} \subset X(\mathbb{A}_K)$, i.e. $\mathrm{Br}(X)$ provides an obstruction to the existence and (adelic) density of K-rational points.

Note that this means that, when X is proper, $\mathrm{Br}\, X$ obstructs both the Hasse principle and weak approximation.

Definition 1.3.3. One says that a proper variety X satisfies the Hasse principle and weak approximation up to the Brauer–Manin obstruction if the rational points $X(K)$ are dense in $X(\mathbb{A}_K)^{\mathrm{Br}\, X}$, i.e., $\overline{X(K)} = X(\mathbb{A}_K)^{\mathrm{Br}\, X}$.

Warning. Even when X satisfies the Hasse principle and WA up to the Brauer–Manin obstruction, it does not necessarily mean that it has "a lot of" rational points. This essentially happens because $\mathrm{Br}\, X/\mathrm{Br}\, K$ (the "obstructing set") may well be infinite. For instance, all smooth curves with a rational point are conjectured to satisfy $\overline{X(K)} = X(\mathbb{A}_K)_\bullet^{\mathrm{Br}\, X}$ [Sto07a, Conj. 9.1] (and the conjecture is known when, for instance, the Jacobian of the curve has finitely many rational points and its Tate-Shafarevich group is finite [Sko01, Cor. 6.2.5] [Sch99]). However, for curves of genus $g \geq 1$, $\mathrm{Br}\, X/\mathrm{Br}\, K$ is infinite, and, when $g \geq 2$, the rational points are finitely many by Faltings' Theorem.

On the other hand, when X is rationally connected, then one has that $\mathrm{Br}\, X/\mathrm{Br}\, K$ is finite (this follows from [CTS21, Theorem 5.5.2] since in this case $\mathrm{Pic}\, \overline{X}$ is torsion-free and $H^i(X, \mathcal{O}_X) = 0$ for $i \geq 1$). In this case, $X(\mathbb{A}_K)^{\mathrm{Br}\, X}$ is described, as a subset of $X(\mathbb{A}_K)$, by finitely many conditions (corresponding to the elements of $\mathrm{Br}\, X/\mathrm{Br}\, K$), and these conditions factor through $\prod_{v\in S} X(K_v)$ for some *finite* S. Moreover, in this case, $X(\mathbb{A}_K)^{\mathrm{Br}\, X}$ is adelically open. In particular, if one knows that the Brauer–Manin obstruction is the only one to the Hasse principle and weak approximation on X, then, if $X(K_v) \neq \emptyset$ for all $v \in M_K$, it satisfies weak weak approximation (since it satisfies weak approximation "off S").

In particular, by the Theorem 1.1.8 of Ekedahl-Colliot-Thélène, this implies that the following conjecture, found in [CTS80, p. 228], is stronger than Conjecture 1.1.3:

Conjecture 1.3.4 (Colliot-Thélène and Sansuc, strong version)**.** *All K-unirational varieties satisfy the weak approximation property up to the Brauer–Manin obstruction. I.e., if X/K is unirational, then $\overline{X(K)} = X(\mathbb{A}_K)^{\mathrm{Br}\, X}$.*

The conjecture above is known for some classes of varieties, let us just mention two of these: pencils in conics with at most 5 singular fibers (among which there are various unirational surfaces) [CT90] [SS91], and smooth compactifications of homogeneous spaces under linear connected group with connected stabilizers [Bor95] (see also the discussion in [CTS21, p. 350]). For more examples, we refer to [CTS21, Sec. 14.11].

The Brauer group also provides an obstruction to strong approximation, in the following two senses.

Obstruction to strong approximation off S. Not the whole Brauer group obstructs approximation off S for all S. So, we need to define the subgroup that provides the correct obstruction.

For a K-variety X we define the S-modified Brauer–Manin group of X to be the following:

$$\mathrm{Br}^S X := \mathrm{Ker}\left(\mathrm{Br}\, X \to \prod_{v\in S} \mathrm{Br}\, X_{K_v}\right).$$

We define the Brauer–Manin set *outside* S of X as follows:

$$X(\mathbb{A}_K^S)^{\mathrm{Br}^S X} := \begin{cases} \{x \in X(\mathbb{A}_K^S) :\ \langle x, B\rangle = 0 \ \text{ for all } B \in \mathrm{Br}^S X\} \text{ if } X(K_S) \neq \emptyset, \\ \emptyset \text{ otherwise.} \end{cases} \tag{1.3.1}$$

We have an inclusion $\overline{X(K)}^S \subset X(\mathbb{A}_K^S)^{\mathrm{Br}^S X}$, where $\overline{\star}^S$ denotes the closure in the S-adeles. In other words $\mathrm{Br}^S X$ obstructs strong approximation off S. That was the first sense.

Obstruction to density of $X(K)$ in $X(\mathbb{A}_K)_\bullet$. We already discussed how the Brauer group provides an obstruction to this density. We still call this an obstruction to strong approximation, because adelic density of $X(K)$ in $X(\mathbb{A}_K)_\bullet$ could be thought of as strong approximation off archimedean places "except that we are keeping track of the connected components on the real places".

An explicit example: Brauer Manin obstruction associated to quaternion algebras To the reader that has never encountered the Brauer–Manin obstruction before, we hope that this paragraph can shed some light on its computability.

It is quite easy to compute the Brauer–Manin obstruction associated to unramified quaternion algebras in $\mathrm{Br}(K(X))$. (By a deep theorem of Merkurjev these algebras actually generate $\mathrm{Br}(K(X))[2]$, but this does not really interest us now.)

Recall that a quaternion algebra over a field F of characteristic $\neq 2$ is an algebra defined as:

$$F < u, v > /(u^2 - f, v^2 - g, uv + vu),$$

where f, g are elements of F^*. The algebra above is denoted by $(f, g)_F$, or simply (f, g).

For a quaternion algebra $(f, g) \in \mathrm{Br}(K(X))[2]$ and a DVR R containing K with fraction field $K(X)$, one can compute that its residue $\mathrm{res}_R((f, g))$ (defined as in Definition 1.3.1) is equal to the reduction of

$$(-1)^{ab} f^b g^{-a} \in R^* \tag{1.3.2}$$

in $\kappa^*/(\kappa^*)^2 \cong H^1(\kappa, \frac{1}{2}\mathbb{Z}/\mathbb{Z}) \subseteq H^1(\kappa, \mathbb{Q}/\mathbb{Z})$, where a (resp. b) is the v-valuation of f (resp. g).

This makes it very easy to check if (f, g) is unramified.

For an unramified quaternion algebra (f, g) the associated Brauer–Manin pairing computes:

$$((f, g), -) : X(\mathbb{A}_K)_\bullet \longrightarrow \mathbb{Q}/\mathbb{Z}, \quad (P_v)_{v\in M_K} \mapsto \sum_{v\in M_K} (f(P_v), g(P_v))_v,$$

when this is well-defined (i.e. outside $\mathrm{div}(f) \cup \mathrm{div}(g)$). Here $(-,-)_v : K_v^* \times K_v^* \to \{\pm 1\} \cong \frac{1}{2}\mathbb{Z}/\mathbb{Z}$ denotes the v-adic Hilbert symbol, defined, for odd v, by the formula:

$$(a, b)_v := (-1)^{\alpha\beta\epsilon(v)} \left(\frac{x}{p}\right)^{\beta} \left(\frac{y}{p}\right)^{\alpha}, \text{ with } \epsilon(v) = (Nv-1)/2, Nv = \text{ cardinality of the residue field of } K_v$$

for $a = p^\alpha x$, and $b = p^\beta y$, with x and y v-adic integers coprime to v. We omit the formula for even v, which can be found e.g. in [Ser73, Chapter III]. Since the Brauer–Manin pairing is continuous, its value on adelic points that are supported on $\mathrm{div}(f) \cup \mathrm{div}(g)$ can be inferred from the formula above by continuity.

Example 1.3.5. Let us give a famous example of a curve (due to Reichardt and Lind in the 40's) where the Hasse principle does not hold, and explain how this is explained by the Brauer–Manin obstruction. The curve $X/\mathbb{Q}$ is defined to be the smooth completion of the rational affine curve $X^0 : 2y^2 = x^4 - 17$. Indeed, consider the quaternion algebra $B := (17, y) \in \mathrm{Br}(\mathbb{Q}(X))[2]$. One computes using the formula (1.3.2) that $B \in \mathrm{Br}\, X$. Moreover, one has that:

- $\mathrm{inv}_p((17, y)) = 0$ for all $p \neq 17$ and all $(x, y) \in X^0(\mathbb{Q}_p)$ (we take by definition $\mathbb{Q}_\infty = \mathbb{R}$);
- $\mathrm{inv}_2((17, y)) = \frac{1}{2}$ for all $(x, y) \in X^0(\mathbb{Q}_{17})$.

It follows that $(B, -) : X^0(\mathbb{A}_\mathbb{Q}) \to \mathbb{Q}/\mathbb{Z}$ is constantly equal to $1/2$. By the continuity of the Brauer–Manin pairing, we infer that $(B, -) : X(\mathbb{A}_\mathbb{Q}) \to \mathbb{Q}/\mathbb{Z}$ is the constant function $1/2$ as well. Since $X(\mathbb{Q}) \subseteq X(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}\, X} \subseteq X(\mathbb{A}_\mathbb{Q})^B = \emptyset$, it follows that $X(\mathbb{Q}) = \emptyset$.

We now turn onto describing the so-called "étale-Brauer–Manin" obstruction, which is in general finer than the former. To describe this obstruction, it is necessary to first define so-called "descent sets".

While we are defining the "descent sets", we will also take the opportunity to mention how these sets may be completely characterized, in specific instances, in terms of a Brauer–Manin obstruction. We do so because the third original work of this thesis will look in direction of an "analogue" of this characterization in the context of finite ramified covers.

Descent We are going to assume that the reader is familiar with the notion of algebraic group. We give [Mil17] for a reference on this.

Recall that, whenever we have an étale cover $\varphi : Y \to X$, of proper geometrically integral K-varieties that is geometrically Galois (or, in other words, an étale torsor, which we will define in a second, under a finite group scheme), then (by a standard consequence of the Chevalley-Weil theorem) there are finitely many twists $\varphi_i : Y_i \to X, i = 1, \dots, r$ of φ such that Y_i has points everywhere locally, and these lift all K-rational points of X:

$$X(K) = \cup_i \varphi_i(Y_i(K)).$$

When $Y \to X$ is the isogeny $[2] : E \to E$ of an elliptic curve, the finiteness of the number of twists with all local points is basically (a consequence of the proof of) the classical weak Mordell-Weil theorem.

One deduces the following "restriction" on $X(K)$:

$$X(K) \subseteq X(\mathbb{A}_K)^\varphi := \cup_i \varphi_i(Y_i(\mathbb{A}_K)).$$

In complete analogy, one may define a similar descent set even when X is not proper (although the finiteness of twists with all local points does not hold anymore), and when $Y \to X$ is not finite, but we need to require though that it is a torsor under some algebraic group G/K.

Namely, recall that, for an algebraic group G/K, a *torsor* under G is a morphism $\varphi : Y \to X$ of K-schemes such that Y is endowed with a G-action commuting with projection to X, whose action restricted to the fibers of φ is free and transitive. In other words, there exists an action $m : Y \times_K G \to Y$ that commutes with projection to X and such that the morphism $Y \times_K G \xrightarrow{(id,m)} Y \times_X Y$ is an isomorphism.

Given a cocycle $\sigma \in Z^1(K, G)$, and a torsor $\varphi : Y \to X$, we may twist the K-scheme Y, which is endowed with a G-action, by the cocycle σ and thus obtain a twisted scheme Y^σ. We refer to [Sko01, Sec 2.2] for the definition of this twist. The twisted scheme Y^σ induces a twisted form $\varphi^\sigma : Y^\sigma \to X$ of the torsor. If the images of two cocycles $\sigma, \sigma' \in Z^1(K, G)$ are the same in $H^1(K, G)$, then there is a (non-canonical) isomorphism of torsors $Y^\sigma \cong Y^{\sigma'}$ over X under G.

For any torsor $\varphi : Y \xrightarrow{G} X$ under an algebraic group G/K, one has that:

$$X(K) \subset \bigcup_{[\sigma]\in H^1(K,G)} \varphi^\sigma(Y^\sigma(K)), \tag{1.3.3}$$

where $[\sigma]$ denotes the class of σ in $H^1(K, G)$. For a torsor $\varphi : Y \xrightarrow{G} X$, we define the *descent set* for f as:

$$X(\mathbb{A}_K)^\varphi := \bigcup_{[\sigma]\in H^1(K,G)} \varphi^\sigma(Y^\sigma(\mathbb{A}_K)) \subseteq X(\mathbb{A}_K).$$

When X is proper, the union on the right is still finite even when G is not finite (when G is finite this reduces to the discussion at the beginning of this paragraph). On the other hand, when X is not proper, the union on the right is not necessarily finite, but it is locally finite (i.e., over each locally compact open in $X(\mathbb{A}_K)^\varphi$, we can restrict to a finite number of σ), as proven by Cao, Demarche and Xu in [CDX19, Prop. 6.4]. In particular, this proves that this union is closed.

Historically, there has been an interest in comparing the descent set $X(\mathbb{A}_K)^\varphi$ with the Brauer–Manin obstruction, or in other words, investigating about the existence of a $B_\varphi \subseteq \operatorname{Br} X$ such that $X(\mathbb{A}_K)^\varphi = X(\mathbb{A}_K)^{B_\varphi}$. This was done for two reasons, somehow lying "in opposite directions".

The first was to use this comparison to "produce rational points". Namely, Skorobogatov [Sko01, Theorem 6.1.2(a)], following an earlier version of this theorem by Colliot-Thélène and Sansuc [CTS80] [CTS87a] proved the following (assume X proper):

Theorem 1.3.6. *Assume that G/K is of multiplicative type, i.e., an extension of a finite abelian group scheme by a K-torus. We then have $X(\mathbb{A}_K)^\varphi = X(\mathbb{A}_K)^{B_\varphi}$ with $B_\varphi = \operatorname{Im}(H^1(K, G') \xrightarrow{-\cup[Y]} H^2_{ét}(X, \mathbb{G}_m) = \operatorname{Br} X)$, where G' is the Cartier dual of G and $[Y] \in H^1_{ét}(X, G)$ is the class corresponding to the G-torsor $Y \to X$.*

Using this theorem, which constitutes somehow the bulk of what is usually referred to as "descent theory" or the "descent method", several successes towards Conjecture 1.3.4 were obtained: the examples that we mentioned after stating the conjecture all use somehow the descent technique.

It might be worth sketching here the idea of the "descent method". Applying Theorem 1.3.6, one shows the following: given a torsor $\varphi : Y \to X$ under some multiplicative group G/K, if all twists of Y satisfy the Hasse principle and the weak approximation property when they have a rational point, then Conjecture 1.3.4 holds for X. So, it all boils down to constructing such a Y. Luckily one does not have to guess: there is a so-called "universal" one (which was also the first one for which Theorem 1.3.6 was proven) as soon as the set $X(\mathbb{A}_K)^{\operatorname{Br} X}$ is non-empty, and so this torsor is usually chosen.

The second reason for investigating the existence of B_φ was to find an obstruction "finer than Brauer–Manin". This is explained in more detail in the next paragraph.

Descent and étale-Brauer–Manin obstructions Because of (1.3.3), one has the following two inclusions:

$$X(K) \subset X(\mathbb{A}_K)^{ét,\mathrm{Br}} := \bigcap_{\substack{f:Y\xrightarrow{G}X \\ G \text{ finite} \\ \text{group scheme}}} \bigcup_{[\sigma]\in H^1(K,G)} f^\sigma(Y^\sigma(\mathbb{A}_K)^{\operatorname{Br} Y^\sigma}),$$

and
$$X(K) \subset X(\mathbb{A}_K)^{desc} := \bigcap_{\substack{f:Y \xrightarrow{G} X \\ G \text{ linear}}} X(\mathbb{A}_K)^f.$$

Hence, both $X(\mathbb{A}_K)^{ét,\mathrm{Br}}$ and $X(\mathbb{A}_K)^{desc}$ provide obstructions to the existence of K-rational points. Moreover, as remarked above, it follows by [CDX19, Prop. 6.4] that the sets on the right are closed in $X(\mathbb{A}_K)$. Hence both $X(\mathbb{A}_K)^{ét,\mathrm{Br}}$ and $X(\mathbb{A}_K)^{desc}$ provide obstructions to (adelic) density of K-rational points as well. Moreover, in [CDX19] the authors prove that the two obstructions are in fact equal, i.e. $X(\mathbb{A}_K)^{ét,\mathrm{Br}} = X(\mathbb{A}_K)^{desc}$.

The étale-Brauer–Manin obstruction was first investigated (not with this name) in [Sko99] where Skorobogatov used it to find $X/\mathbb{Q}$ a (proper, smooth) surface such that
$$\emptyset = X(\mathbb{Q}) = X(\mathbb{A}_\mathbb{Q})^{ét,\mathrm{Br}} \subsetneq X(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}\, X}.$$

As for the Brauer–Manin obstruction, the étale-Brauer–Manin obstruction provides an obstruction to strong approximation in two senses.

Obstruction to strong approximation off S.

As with the Brauer–Manin obstruction, it is necessary here to give a new definition:
$$X(\mathbb{A}_K^S)^{ét,\mathrm{Br}^S} = \bigcap_{\substack{f:Y \xrightarrow{F} X \\ F \text{ finite} \\ \text{group scheme}}} \bigcup_{[\sigma]\in H^1(K,F)} f^\sigma(Y^\sigma(\mathbb{A}_K^S)^{\mathrm{Br}^S Y^\sigma}). \tag{1.3.4}$$

Using [CDX19, Prop. 6.4], one again proves that $X(\mathbb{A}_K^S)^{ét,\mathrm{Br}^S}$ is closed, hence $\overline{X(K)}^S \subseteq X(\mathbb{A}_K^S)^{ét,\mathrm{Br}^S}$. In other words, the set just defined provides an obstruction to strong approximation off S.

Obstruction to density of $X(K)$ in $X(\mathbb{A}_K)_\bullet$. We have:
$$\overline{X(K)} \subseteq X(\mathbb{A}_K)_\bullet^{ét,\mathrm{Br}},$$
where the latter is defined in a completely analogous way as $X(\mathbb{A}_K)^{ét,\mathrm{Br}}$. As explained above, we still think of density in $X(\mathbb{A}_K)_\bullet$ as a form of strong approximation "except that we are keeping track of connected components at real places".

The étale-Brauer–Manin obstruction is known to not be the only obstruction to the Hasse principle [Poo10].

In the case of strong approximation for homogeneous spaces, we will, however, prove that this is the only obstruction. This is the second original work that is part of this thesis, which we describe in the next section.

1.4 Approximation properties on Homogeneous spaces

This section presents the second original part of this thesis, which constitutes Chapter 3. We start by giving a bit of historical context to the results of this paper.

Previously known results on homogeneous spaces The first known results for approximation properties on homogeneous spaces concern simply connected semisimple algebraic groups. We state them here in their completeness, as they are particularly simple to state, and constitute the "building blocks" of all later results.

Theorem 1.4.1 (Chernousov-Harder-Kneser-Platonov)**.** *Let G/K be a semisimple simply connected algebraic group. The following hold:*

(HaP) Principal homogeneous spaces of G satisfy the Hasse principle.

(WA) Any smooth compactification of G/K satisfies weak approximation.

(SA) G/K satisfies strong approximation off a finite nonempty S if and only if, for every simple factor G_i of G, $\prod_{v\in S} G_i(K_v)$ is not compact.

Since, for almost all $v \in M_K$, $G(K_v)$ is not compact, the last statement of the theorem is non-empty. Moreover, using this non-compactness fact, one can actually easily prove that the second statement of the theorem is implied by the last.

The Hasse principle is due to Kneser (classical groups) [Kne66a], Harder [Har65][Har66] (exceptional groups other than E_8), and Chernousov [Che89] (E_8) , and the strong approximation is due to Kneser [Kne66b] and Platonov [Pla69], who provided an independent proof, based on his proof of the Kneser-Tits conjecture. All of these results can be found in the book [PRR93]. A more uniform proof of weak approximation has also been given by Harari [Har94, Theorem 5.3.1].

In addition to the above theorem, one also has the important result of Kneser [Kne65] [Kne69], who proved that all principal homogeneous spaces under G over a non-archimedean local completion K_v of K are actually trivial, i.e., are K_v-isomorphic to G_{K_v}. In particular, combining this with the first statement of the theorem above, one has that the following morphism:

$$H^1(K,G) \to \prod_{v\in M_K^\infty} H^1(K_v,G)$$

is injective. [This is actually even bijective, see [PRR93, Theorem 6.6].]

The strong approximation theorem of Kneser and Platonov has then been generalized several times, for example by Harari [Har08], Colliot-Thélène and Xu [CTX09], and, most recently, by Borovoi and Demarche [BD13], whose results encompass all the previous ones. In [BD13], the authors study the case of a homogeneous space under a connected (not necessarily linear) group G with connected geometric stabilizers, and prove that, under some technical assumptions, the Brauer–Manin obstruction to strong approximation is the only one. A precise statement follows:

Theorem 1.4.2 (Borovoi-Demarche-'13)**.** *Let G be a connected algebraic group over a number field K. We assume that the Tate-Shafarevich group $Ш(K, G^{ab})$ is finite, where G^{ab} denotes the maximal abelian quotient of G. Let X be a left homogeneous space under G. Let $S \supset M_K^\infty$ be a finite set of places of K. We assume that $G^{sc}(K)$ is dense in $G^{sc}(\mathbb{A}_K^S)$. Set $S_f := S \cap M_K^{fin} = S \setminus M_K^\infty$. Then the set $X(\mathbb{A}_K)_\bullet^{\text{ét},\mathrm{Br}}$ is equal to the closure of the set $G^{scu}(K_{S_f}) \cdot X(K) \subset X(\mathbb{A}_K)_\bullet$ for the adelic topology.*

For what exactly G^{sc} and G^{scu} are, we defer to Chapter 3.

The finiteness of the Tate-Shafarevich group of G^{ab} is of course, just conjectured to be true (and known for some specific cases of modular abelian varieties). It is, however, by now standard in the field to assume this conjecture to be true.

Étale-Brauer–Manin obstruction and homogeneous spaces Now we can finally talk about the second original work that is part of this thesis. The main purpose of this work is to prove that the étale-Brauer–Manin obstruction is "the only one" on homogeneous spaces, although other results which might be of independent interest are obtained.

Indeed, as already proved in [Dem17], the Brauer–Manin obstruction to strong approximation is in general not the only one when the geometric stabilizers of X are not connected. In particular, Demarche showed that there is a non-trivial étale-Brauer–Manin obstruction to strong approximation for some homogeneous spaces with finite stabilizers. Hence, the "connectedeness of stabilizers" hypothesis in the theorem of Borovoi and Demarche 1.4.2 is necessary. On the other hand, in this work, we build on [BD13, Thm 1.4] to prove the following:

Theorem 1.4.3. *Let G be a connected algebraic group over a number field K. We assume that the Tate-Shafarevich group $Ш(K, G^{ab})$ is finite. Let X be a homogeneous space under G. Let $S \supset M_K^\infty$ be a finite set of places of K. We assume that $G^{sc}(K)$ is dense in $G^{sc}(\mathbb{A}_K^S)$. Set $S_f := S \cap M_K^{fin} = S \setminus M_K^\infty$. Then the set $X(\mathbb{A}_K)_\bullet^{\text{ét,Br}}$ is equal to the closure of the set $G^{scu}(K_{S_f}) \cdot X(K) \subset X(\mathbb{A}_K)_\bullet$ for the adelic topology.*

The condition that $G^{sc}(K)$ is dense in $G^{sc}(\mathbb{A}_K^S)$ is equivalent, by the strong approximation Theorem 1.4.1 (SA) of Kneser and Platonov, to the condition that, for each simple component G_i of G^{sc}, there exists a $v \in S$ such that $G_i(K_v)$ is not compact.

Note that the above theorem says that the étale-Brauer–Manin obstruction is the only obstruction (at least, when each simple component of G^{sc} is non-compact at at least one archimedean component) to strong approximation "in the second sense", as we previously defined it. We will then also prove another theorem, namely Theorem 3.6.5 where the analogue for strong approximation "in the first sense", i.e. off a finite S_0 (possibly different from S) is proven.

The main idea of the proof of Theorem 3.1.1 is to show that, if there is no étale-Brauer–Manin obstruction to the existence of K-rational points on X, then there is a homogeneous space Z under G, with connected geometric stabilizers, and a finite G-equivariant morphism $Z \to X$ that makes Z a torsor over X (under some finite group scheme). This allows one to apply the aforementioned result of Borovoi and Demarche to Z. To obtain the torsor $Z \to X$ we crucially rely on [CDX19, Lemma 7.1], a result that first appeared (in the proper case) in [Sto07b].

As we mentioned, in this work we also prove a variant (see Theorem 3.6.5) of the Theorem above where one removes an additional finite set of places. In the course of proving Theorem 3.6.5 we will also obtain Theorem 3.6.1, which is an analogue of the strong approximation result by Borovoi and Demarche [BD13] (so, in the context of connected stabilizers) with a finite set of places removed. This appears to be new, and, as it will be remarked, does not seem to follow directly by projection from [BD13, Thm 1.4], as one may think at first sight.

In order to prove Theorem 3.6.5, we will need a detailed description of the values of elements of $\mathrm{Br}(X_{K_v})$ on $X(K_v)$, which we will obtain by using the abelianization construction of Demarche [Dem13] (that in turn builds on earlier work of Borovoi [Bor96]). The explicit description that we will need is given in Theorem 3.5.1, which appears to be new, and possibly also of independent interest. Standard dévissage methods seem not to yield a proof of Theorem 3.5.1, so our approach relies on an explicit (and not very exciting) Galois cocycle computation instead.

Part of the results of this paper were also obtained independently and almost simultaneously by Francesca Balestrieri [BalMS]. Namely, she proves Theorem 3.1.1 in the case that X has a rational point and is a homogeneous space under a linear group. The results of this work that are not covered by hers are Theorem 3.1.1 in the case that $X(K) = \emptyset$ (in particular, handling this case is what forces us to use some Weil restriction arguments, that appear in Section 3.4), Theorems 3.6.1 and 3.6.5 (which deal with the question of strong approximation after removing some non-archimedean places), and the compatibility Theorem 3.5.1, which aims to connect [Dem13, Cor. 6.3] with [BD13, Thm 1.4].

1.5 Ramified descent

We talk now about the last work that is part of this thesis.

The purpose of this paper is to study whether the link between the Brauer–Manin obstruction and descent set (for groups of multiplicative type) described in Theorem 1.3.6 can be extended to the setting of covers. I.e., restrict to the case where $Y \to X$ is finite (and hence, so is the group scheme G), but, importantly, allow for ramification (of course, $Y \to X$ will no longer be a torsor, it will be so only generically), and ask whether the "descent set" of $Y \to X$, which we will define in a second, can be described by a Brauer–Manin condition. The long-term interest for doing so would be that some varieties have a particularly easy-to-describe finite abelian ramified cover.

For instance, this is the case with Kummer surfaces, that have a $2:1$ cover that is a principal homogeneous space under an abelian variety. If one is able to obtain a good description of the descent set (for instance, in terms of a Brauer–Manin condition), one might hope to use these covers to deduce information on the rational points on X.

Let X/K be a smooth geometrically connected variety and let $\psi : Y \to X$ be an integral cover (generally speaking, we use the letter φ for torsors and ψ or π for possibly ramified covers), i.e., a finite surjective morphism with Y normal and integral. Assume that ψ is generically a torsor under a finite group scheme G/K, and let $U \subseteq X$ be an open subvariety over which ψ is étale (and hence a G-torsor), and let $V := \psi^{-1}(V)$. The *descent set* for the ramified cover $\psi : Y \to X$ is defined to be the set:

$$X(\mathbb{A}_K)^{\psi} := \overline{\bigcup_{\xi \in H^1(K,G)} \psi_\xi(V_\xi(\mathbb{A}_K))} \subseteq X(\mathbb{A}_K).$$

Even though we give the definition in general, we will mainly be interested in the case where X is proper, where one has that $X(\mathbb{A}_K) = X(K_\Omega)$.

We prove the following theorem, which, in the case where X is proper, proves that this descent set provides an obstruction to the Hasse principle and weak approximation.

Theorem 1.5.1. *The inclusion* $\overline{X(K)} \subseteq X(\mathbb{A}_K)^{\psi}$ *holds.*

Actually, what we prove is even more precise: namely that all rational points of X may be lifted to rational points of a twist of a smooth compactification Y^{sm} of V. Note that, for K-rational points in U, this is immediate (and well-known). However, it is less so for points lying in the branch locus of $Y \to X$.

The proof of this theorem is quite quick. The author's first proof relied on a reduction to the case of curves, where it reduces to the well-known fact that the absolute decomposition groups of the DVRs $R \subseteq K(X)$ are semi-direct products of their inertia subgroup and their unramified quotient, which holds in general for all DVRs of residual characteristic 0. Very kindly, Olivier Wittenberg has suggested an alternative and much cleaner proof, that is presented in this thesis.

The author's motivation towards the matter has been sparked by the following side question, posed by David Harari in a workshop:

Question 1.5.2. Could the descent set for ramified covers be linked to a non-algebraic Brauer–Manin obstruction?

The curiosity behind the question lies in the fact that in "classical" (non-ramified) contexts, the Brauer group B_ψ is always algebraic (recall Theorem 1.3.6).

We answer the question of Harari affirmatively. More specifically, we construct a (non-algebraic) subgroup $\mathrm{Br}_\psi X \subseteq \mathrm{Br}\, X$ (although this is the main example to keep in mind, the definition does not require that G is abelian) such that:

Theorem 1.5.3. *The inclusion* $X(\mathbb{A}_K)^{\psi} \subseteq X(\mathbb{A}_K)^{\mathrm{Br}_\psi X}$ *holds. Moreover, even when G is abelian, the group* $\mathrm{Br}_\psi X$ *is not necessarily algebraic, and the transcendental part may provide non-trivial obstruction.*

The subgroup $\mathrm{Br}_\psi X$ is defined as

$$\mathrm{Br}\, X \cap \mathrm{Im}\, H^2(\Gamma, \mu_\infty) \subseteq H^2(\Gamma_{K(X)}, \overline{K(X)}^*) = \mathrm{Br}(K(X)),$$

where Γ denotes the (profinite) Galois group of the extension $\overline{K}(Y)/K(X)$, and "Im" refers to the image under the natural morphism $H^2(\Gamma, \mu_\infty) \to H^2(\Gamma_{K(X)}, \overline{K(X)}^*)$. Once the definition has been given, the proof of the first part of the theorem is mostly just a verification.

We then provide an example where the group $\mathrm{Br}_\psi X$ is entirely transcendental (i.e., $\mathrm{Br}_\psi X \cap \mathrm{Br}_1 X = \mathrm{Br}_0 X$, the image of $\mathrm{Br}\, K$, which notoriously does not give any obstruction) and provides a non-trivial obstruction. Here X is a compactification of a quotient SL_n/G, where G is a nilpotent group of class 2, and ψ is $SL_n/G' \to SL_n/G$.

Moreover, this appears to be only the second known example of transcendental obstruction to weak approximation for quotients SL_n/G, or, in other words (see [Har07, Section 1.2]), to the so-called *Grunwald Problem* for a finite group G. The first such example is Theorem 1.2 of [DAN17]. Our example seems to be much more explicit than that of [DAN17]. Indeed, in the latter, the authors provide an example of a quotient SL_n/G where weak approximation holds up to Brauer-Manin obstruction but such that the algebraic obstruction is trivial, which indirectly proves that the transcendental obstruction is non-trivial. By contrast, we prove that some explicit classes of the transcendental Brauer group of the varieties in question provide an obstruction to weak approximation. As in [DAN17], also in our example the algebraic obstruction vanishes.

Since Theorem 4.1.1 proves that $X(\mathbb{A}_K)^\psi \subseteq X(\mathbb{A}_K)^{\mathrm{Br}_\psi X}$, and $\overline{X(K)}$ is contained in both of them, it might be interesting to compare these two obstruction sets, which might lead one to ask the following (assume X/K is proper, smooth, geometrically connected):

Question 1.5.4. Let $\psi : Y \to X$ be, as above, an integral cover that is generically a torsor under a finite **abelian** group scheme G/K. Does one have that $X(\mathbb{A}_K)^\psi = X(\mathbb{A}_K)^{\mathrm{Br}_\psi X}$? What about the analogue question when G is solvable?

The question above will be the object of study of future work of the author, where partial answers should be obtained.

Note that a positive answer to the question above would, for instance, guarantee that, if Y is a variety all of whose G-twists satisfy the Hasse principle, then X satisfies the Hasse principle as well.

Let us mention that, when Y is an equivariant compactification of SL_n and G acts as a subgroup, the question above is already known to have a positive answer, as in this case it follows from weak approximation (up to the Brauer–Manin obstruction) on the variety SL_n/G and the triviality of $H^1(K_v, SL_n)$ for all local completions K_v of K. That SL_n/G satisfies the weak approximation property up to the Brauer–Manin obstruction when G is supersolvable is proven in the recent work of Harpaz and Wittenberg [HW20, Théorème B].

Another case where the question is already answered is when ψ is unramified and G is abelian, as in this case, as we will prove in Section 4.4.4, $\mathrm{Br}_\psi X \supseteq B_\psi = \mathrm{Im}(H^1(K, G') \xrightarrow{-\cup[Y]} H^2_{ét}(X, \mathbb{G}_m) = \mathrm{Br}\, X)$, and this implies that the sequence of inclusions $X(\mathbb{A}_K)^\psi \subseteq X(\mathbb{A}_K)^{\mathrm{Br}_\psi X} \subseteq X(\mathbb{A}_K)^{B_\psi}$ must be an equality by Theorem 1.3.6.

Chapter 2

Elliptic Fibrations and the Hilbert Property

2.1 Introduction

Recall that, motivated by the search for varieties with the Hilbert Property, in this chapter we prove the following:

Theorem 2.1.1. *Let E be a smooth projective geometrically connected algebraic surface, defined over a number field K. Suppose that there exist $n \geq 2$ elliptic fibrations $\pi_1, \ldots, \pi_n : E \to \mathbb{P}_1$. Let $F \subset E$ be the union of the divisors of E that are contained, for each $i = 1, \ldots, n$, in a fiber of π_i. If $E \setminus F \neq \emptyset$ is simply connected and $E(K)$ is Zariski-dense in E, then E has the Hilbert Property.*

The first to have used multiple elliptic fibrations to prove the Hilbert Property are Corvaja and Zannier, who proved it for the Fermat surface $x^4 + y^4 = z^4 + w^4$ [CZ17, Theorem 1.4]. Our proof basically develops from their ideas.

We then use this theorem to give explicit examples of K3 surfaces with the Hilbert Property.

2.2 Background

This section contains some preliminaries, including reminders on the Hilbert Property. Moreover, we shall take care here of most of the notation that will be used in the chapter.

Notation Throughout this chapter, except when stated otherwise, k denotes a perfect field and K a number field. A (k-)*variety* is an algebraic quasi-projective variety (defined over a field k), not necessarily irreducible or reduced. Unless specified otherwise, we will always work with the Zariski topology.

Given a morphism $f : X \to Y$ between k-varieties, and a point $s : \mathrm{Spec}(k(s)) \to Y$, we denote by $f^{-1}(s)$ the scheme-theoretic fibered product $\mathrm{Spec}(k(s)) \times_Y X$, and call it the *fiber* of f in s. Hence, with our notation, this is not necessarily reduced.

A geometrically integral k-variety X is a k-variety such that $X_{\bar{k}}$ is integral.

A morphism $f : Y \to X$ between normal k-varieties is a *cover* if it is finite.

When $k \subset \mathbb{C}$, a smooth k-variety X is *simply connected* if $X_{\mathbb{C}}$, with its euclidean topology, is a simply connected topological space.

Given a morphism $f : Y \to X$ between integral k-varieties, with Y normal, we will make use of the notion of *relative normalization* of X in Y. We refer the reader to [Sta20, Tag 0BAK] or [Liu02, Definition 4.1.24] for its definition, and recall here the properties that are needed in this thesis. Namely, we will need that the *relative normalization* of X in Y is a finite morphism

$n : \hat{X} \to X$ such that $\hat{X}$ is normal, and such that there exists a factorization $f = \varphi \circ n$, where $\varphi : Y \to \hat{X}$ has a geometrically integral generic fiber. The *normalization* of an integral variety X is the usual normalization $\hat{X}$ of X [Liu02, Sec. 4.1.2].

The domain $\mathbf{Dom}(f)$ of a rational map $f : Y \dashrightarrow X$ between integral k-varieties is the maximal open Zariski subset $U \subset Y$ such that $f|_U$ extends to a morphism on U.

Given a rational map $f : Y \dashrightarrow X$ between integral k-varieties, and a birational transformation $b : Y' \dashrightarrow Y$, we will denote with abuse of notation, when there is no risk of confusion, the map $f \circ b$ by f. We say that f is *well-defined* on Y' if $\mathbf{Dom}(f \circ b) = Y'$.

Ramification We say that a morphism $f : Y \to X$ between k-varieties is *unramified* at a closed point $y \in Y$ if its differential $df_y : T_yY \to T_{f(y)}X \otimes_{k(f(y))} k(y)$ is injective. Otherwise we say that f is *ramified* at y.

Since k is perfect, [Vak, Exercise 21.6.I] implies that our definition of unramifiedness at a closed point coincides with the more standard one saying that $f : Y \to X$ is unramified at $y \in Y$ if $y \notin \operatorname{Supp} \Omega^1_{Y/X}$.

The set of closed points where f is ramified are the closed points of a (reduced) closed subscheme of Y, and we will refer to this closed subscheme as the *ramification locus*. The image of the ramification locus under f is the *branch locus*. We recall that, by Zariski's Purity Theorem (see [Zar58] or [Sta20, Tag 0BMB]), when f is finite, Y is normal and X is smooth, the branch locus of a finite morphism $f : Y \to X$ is a divisor. Hence, in this case, we will also refer to the branch locus as the *branch divisor*. We remark that the ramification locus of $f : Y \to X$ is also the locus cut out by the 0-th Fitting ideal of $\Omega_{Y/X}$, and, as such, is invariant under base change (see [Tei77, Ch.1] or [Sta20, Tag07Z6,Tag 0C3H] for more details on Fitting ideals). A simply connected variety X does not have any geometrically integral cover of degree > 1 which is unramified in codimension 1.

Cubic hypersurfaces Cubic hypersurfaces are hypersurfaces in $\mathbb{P}_n$ defined by a cubic homogeneous polynomial. We recall the following:

Theorem 2.2.1 (Segre)**.** *Let X be a smooth cubic projective hypersurface, defined over a field k of characteristic 0, with a k-rational point. Then X is unirational.*

Hilbert Property For a more detailed exposition of the basic theory of the Hilbert Property and thin sets we refer the interested reader to [Ser08, Ch. 3]. We limit ourselves here to recalling the most common definition (which is not the one given in the introduction), and some recent results.

Definition 2.2.2. Let X be a geometrically integral variety, defined over a field k. A *thin* subset $S \subset X(k)$ is any set contained in a union $D(k) \cup \bigcup_{i=1,\dots,n} \pi_i(E_i(k))$, where $D \subsetneq X$ is a closed subvariety, and $\pi_i : E_i \to X$ are generically finite morphisms of degree > 1, and the E_i's are irreducible.

Remark 2.2.3. When X is normal, one can substitute "generically finite morphisms" with "covers" in Definition 2.2.2 to get an equivalent definition. In fact let, for each $i = 1, \dots, n$, $\pi'_i : E'_i \to X$ be the relative normalization of X in $\pi_i : E_i \dashrightarrow X$, and let $b_i : E'_i \dashrightarrow E_i$ be a birational map such that $\pi'_i = \pi_i \circ b_i$. We have that $D(k) \cup \bigcup_i (\pi_i(E_i(k))) \subset (D(k) \cup \bigcup_i \pi_i(Y_i(k))) \cup \bigcup_i (\pi'_i(E'_i(k)))$, where $Y_i \subset E_i$ denotes the (closure of the) locus where b_i^{-1} is not an isomorphism. Moreover, since any closed subvariety $D \subsetneq X$ is contained in a divisor, one can substitute in Definition 2.2.2 "closed subvariety" with "divisor".

Recall that a geometrically integral k-variety X has the Hilbert Property (HP) if and only if $X(k)$ is not thin.

Theorem 2.2.4 (Bary-Soroker, Fehm, Petersen). *Let $f : X \to S$ be a morphism of geometrically integral K-varieties. Suppose that there exists a non-thin subset $A \subset S(K)$ such that, for each $s \in A$, $f^{-1}(s)$ has the HP. Then X/K has the HP.*

Proof. See [BSFP14, Theorem 1.1]. □

Definition 2.2.5. Let $\mathcal{E}$ be a normal projective algebraic K-surface. We say that a morphism $\pi : \mathcal{E} \to \mathbb{P}_1$ is an *elliptic fibration* if its generic fiber is a smooth, geometrically connected, genus 1 curve.

The following theorem is included just for completeness. In fact, Theorem 2.1.1, which we are going to prove in Section 2.4, is a stronger version of it.

Theorem 2.2.6. *Let K be a number field, and E be a projective smooth simply connected algebraic K-surface, endowed with two elliptic fibrations $\pi_i : E \to \mathbb{P}_1/K$, $i = 1, 2$, such that $\pi_1 \times \pi_2 : E \to \mathbb{P}_1 \times \mathbb{P}_1$ is a finite morphism. Suppose moreover that the following hold:*

(a) The K-rational points $E(K)$ are Zariski-dense in E;

(b) Let $\eta_1 \cong \operatorname{Spec} K(\lambda)$ be the generic point of the codomain of π_1. All the branch points (i.e. the images of the ramification points) of the morphism $\pi_2|_{\pi_1^{-1}(\eta_1)}$ are non-constant in λ, and the same holds upon inverting π_1 and π_2.

Then the surface E/K has the Hilbert Property.

Proof. See [Dem20, Theorem 1.4]. □

2.3 Hilbert Property for cubic hypersurfaces

Theorem 2.3.1. *Let $X \subset \mathbb{P}_n/K$, $n \geq 3$ be a smooth cubic projective hypersurface, with a K-rational point. Then X has the Hilbert Property.*

In this section our base field will always be a number field K.

We need the following lemma, which is implicit in [Lui12].

Lemma 2.3.2. *Let $\pi : \mathcal{E} \to \mathbb{P}_1$ be an elliptic fibration, defined over a number field K. Then, there exists a non-empty open Zariski subset $U_\pi \subset \mathcal{E}$ such that, for any $P \in U_\pi(K)$, $\pi^{-1}(\pi(P))$ is smooth and $\#\pi^{-1}(\pi(P))(K) = \infty$.*

Proof. Let p_λ be the generic point of $\mathbb{P}_1$ and $U \ni p_\lambda$ be a Zariski neighborhood of p_λ such that $\pi_U := \pi|_{\pi^{-1}(U)}$ is a smooth morphism, and such that there exists a divisor H of $\pi^{-1}(U)$ which is flat and relatively ample over U. We notice that such a couple (U, H) always exists: in fact it suffices to choose an ample divisor H_λ on the generic fiber $\mathcal{E}_\lambda := \pi^{-1}(p_\lambda)$, and then define H to be the Zariski closure of H_λ in $\pi^{-1}(U)$, where U is defined to be a sufficiently small Zariski neighborhood of p_λ.

Let $J : \mathcal{E}^0 \to U \subset \mathbb{P}_1$ be the Jacobian, or "Pic^0"-, fibration corresponding to π_U (see [SI13, II.10.3] or [Kle05, Definition 4.1, Theorem 4.8, Sec. 5] for a definition and basic functoriality properties, which we are implicitly going to use). For $p \in U$, we denote by J_p the $\operatorname{Spec} k(p)$-scheme $J^{-1}(p)$. We note that, by functoriality [Kle05, Definition 4.1], $J^{-1}(p)$ may be canonically identified with the Jacobian variety of $\pi^{-1}(p)$.

Let now

$$\psi_\lambda : \mathcal{E}_\lambda \to J_\lambda$$

be the morphism defined, pointwise, by $\psi_\lambda(Q) := [(\deg H_\lambda) \cdot Q - H_\lambda]$, and

$$\Psi : \pi^{-1}(U') \to J^{-1}(U')$$

be an extension of ψ_λ to a Zariski neighbourhood $U' \subset U$ of p_λ. By functoriality [Kle05, Definition 4.1], we know that, for each $p \in U'$, and each $Q \in \pi^{-1}(p)$, $\Psi(Q) = [(\deg H) \cdot Q - H_p]$, where $H_p = H|_{\pi^{-1}(p)}$, and $[(\deg H) \cdot Q - H_p] \in J_p$.

Let N be the least common divisor of the orders of torsion of the groups $\tilde{E}(K)$, where $\tilde{E}$ varies in the set of elliptic curves defined over the number field K. This is a finite number by the Mazur-Merel-Parent Theorem (see e.g. the article by Parent [Par99]). The claim of the lemma is then satisfied with $U_\pi := \pi^{-1}(U) \setminus ([N] \circ \Psi)^{-1}(O)$, where O denotes, with abuse of notation, the image of the zero section of J. □

Proof of Theorem 2.3.1. We note that X is K-unirational by Theorem 2.2.1, in particular it has Zariski-dense K-rational points.

We prove the result by induction on n.

Case $n = 3$.

We assume by contradiction that X does not have the HP. By Remark 2.2.3, there exist then irreducible covers $\varphi_i : Y_i \to X$, $i = 1, \dots, m$ of degree $\deg \varphi_i > 1$ and a divisor $D \subset X$ such that $X(K) \subset \bigcup_i \varphi_i(Y_i(K)) \cup D(K)$. We may assume, without loss of generality, that the Y_i's are geometrically integral (see the *Remark on irreducible varieties* in [Ser08, p. 20]), and that the φ_i's are finite morphisms (see Remark 2.2.3). Let us denote now by R_i the branch divisor of φ_i. Since X is a smooth cubic, it is isomorphic to a (smooth) hyperplane section of the image of the (cubic) Veronese embedding $\mathbb{P}_n \hookrightarrow \mathbb{P}_{\binom{n+3}{3}-1}$. Hence, by Lefschetz' hyperplane Theorem [Mil63, Theorem 7.4], it is simply connected. It follows that the R_i's are non-empty for each $i = 1, \dots, m$.

Let us denote by $\mathbb{A}_4^*$ the dual affine space of $\mathbb{A}_4$, minus the origin. To each element $h \in \mathbb{A}_4^*$ corresponds a hyperplane $H(h) := \{\mathbf{x} \in \mathbb{P}_3 \mid h(\mathbf{x}) = 0\} \subset \mathbb{P}_3$. Let $(h_1, h_2) \in \mathbb{A}_4^*(K) \times \mathbb{A}_4^*(K)$ be such that:

i. $H(h_1) \cap H(h_2) \cap X$ has three distinct $\bar{K}$-points (and hence, as a direct consequence, $H(h_1) \cap H(h_2)$ is a line), and it is disjoint from the union of the R_i's;

ii. $H(h_1) \cap X$ and $H(h_2) \cap X$ are smooth curves.

We note that all conditions contain an open and non-empty subvariety of $\mathbb{A}_4^* \times \mathbb{A}_4^*$ as a consequence of Bertini's Theorem [Har77, Corollary III.10.9] applied to the linear system of hyperplanes of $\mathbb{P}_3$. Hence such a couple (h_1, h_2) always exists.

We notice that the morphism $[h_1 : h_2] : X \setminus H(h_1) \cap H(h_2) \to \mathbb{P}_1$ is non-constant on each of the irreducible components of the R_i's. In fact, if it were constant on an irreducible component $R_i^j \subset R_i$, we would have that there exist $\alpha, \beta \in K$ such that $\alpha h_1 + \beta h_2 = 0$ on R_i^j, with at least one between α and β being non-zero. We assume, without loss of generality, that $\alpha \neq 0$. We would then have that the intersection $H(h_1) \cap H(h_2) \cap R_i^j = H(h_2) \cap H(\alpha h_1 + \beta h_2) \cap R_i^j = H(h_2) \cap R_i^j \neq \emptyset$, which gives a contradiction.

Let $\{P_1, P_2, P_3\}$ be the intersection $H(h_1) \cap H(h_2) \cap X$, and let $\pi : X \setminus \{P_1, P_2, P_3\} \to \mathbb{P}_1$ be the following morphism:

$$P \longmapsto [h_1(P) : h_2(P)].$$

The map π extends naturally to a morphism $\hat{\pi} : \hat{X} \to \mathbb{P}_1$, where $\hat{X} = \mathrm{Bl}_{P_1 + P_2 + P_3} X$ denotes the blowup of X in the (smooth) subscheme $P_1 + P_2 + P_3 \subset X$. We note that, since X is a cubic surface, the morphism $\hat{\pi}$ is an elliptic fibration.

We claim now that the morphisms $\pi \circ \varphi_i : Y_i \setminus \varphi_i^{-1}(\{P_1, P_2, P_3\}) \to \mathbb{P}_1$ have geometrically integral generic fiber for each $i = 1, \dots, m$. In fact, let us assume by contradiction that there existed an $i \in \{1, \dots, m\}$ such that this is not true. Let

$$\pi \circ \varphi_i : Y_i \setminus \varphi_i^{-1}(\{P_1, P_2, P_3\}) \xrightarrow{\pi'} C \xrightarrow{r} \mathbb{P}_1 \tag{2.3.1}$$

be the relative normalization factorization of $\pi \circ \varphi_i|_{Y_i \setminus \varphi_i^{-1}(\{P_1,P_2,P_3\})}$. We would have that $\deg r > 1$. The factorization 2.3.1 yields a morphism $\varphi_i' : Y_i \setminus \varphi_i^{-1}(\{P_1, P_2, P_3\}) \to X \setminus \{P_1, P_2, P_3\} \times_{\mathbb{P}_1} C$, and a factorization:

$$\varphi_i : Y_i \setminus \varphi_i^{-1}(\{P_1, P_2, P_3\}) \xrightarrow{\varphi_i'} \widehat{X \setminus \{P_1, P_2, P_3\} \times_{\mathbb{P}_1} C} \xrightarrow{\alpha} X \setminus \{P_1, P_2, P_3\},$$

where $\widehat{X \setminus \{P_1, P_2, P_3\} \times_{\mathbb{P}_1} C}$ denotes the normalization of $X \setminus \{P_1, P_2, P_3\} \times_{\mathbb{P}_1} C$.

Hence the branch locus of φ_i would contain the branch locus of α, which would be non-empty if $\deg r > 1$ (since $X \setminus \{P_1, P_2, P_3\}$ is simply connected) and contained in a finite union of fibers of π. This contradicts our choice of (h_1, h_2).

Let us denote now by $\hat{Y}_i$ the desingularization of $Y_i' = Y_i \times_X \hat{X}$, and by $\psi_i : \hat{Y}_i \to \mathbb{P}_1$ the composition of the desingularization morphism $\hat{Y}_i \to Y_i'$, the projection $Y_i' \to \hat{X}$ and the map $\hat{\pi} : \hat{X} \to \mathbb{P}_1$. By the Theorem of generic smoothness [Har77, Corollary III.10.7] we know that there exists a non-empty open subset $V_i \subset \mathbb{P}_1$ such that, for each $t \in V_i(\overline{K})$, $\psi_i^{-1}(t)$ is smooth (and we may assume, by further restricting V_i, geometrically connected as well, because ψ_i has a geometrically integral generic fiber). Let us denote now by $D' \subset \hat{X}$ the proper subvariety, which is the union of all the following:

i. the fibers $\hat{\pi}^{-1}(x)$, for each $x \notin V_i$, for each $i = 1, \ldots, m$;

ii. the proper transform of $D \subset X$, and the exceptional locus of $\hat{X} \to X$;

iii. the proper transform of $X \setminus U$, where U is defined as in Lemma 2.3.2 for $(\mathcal{E}, \pi) = (\hat{X}, \hat{\pi})$.

Let us choose now a K-rational point $P \in (\hat{X} \setminus D')(K)$, and let us denote by E_P the fiber $\hat{\pi}^{-1}(\hat{\pi}(P))$. We know, by Lemma 2.3.2, that E_P has infinitely many K-rational points. We have assumed, however, that $X(K) \subset \bigcup_i \varphi_i(Y_i(K)) \cup D(K)$, and hence

$$E_P(K) \subset \bigcup_i \varphi_i(\psi_i^{-1}(\hat{\pi}(P))(K)) \cup (E_P \cap D')(K). \tag{2.3.2}$$

We claim that the right hand side of 2.3.2 is finite. In fact, for each $i = 1, \ldots, m$, the morphism $\psi_i^{-1}(\hat{\pi}(P)) \to E_P$ is ramified by the invariance of the ramification locus under base change, and, since the curve $\psi_i^{-1}(\hat{\pi}(P))$ is a smooth geometrically connected complete curve, by Riemann-Hurwitz theorem, it is of genus > 1. As a consequence, by Faltings' theorem, $\psi_i^{-1}(\hat{\pi}(P))(K)$ is finite for each $i = 1, \ldots, m$. Moreover, $(E_P \cap D')$ is obviously finite, hence we have proved that the right hand side of 2.3.2 is finite. As we noted before, however, $E_P(K)$ is infinite. We have obtained a contradiction, proving the theorem in the case $n = 3$.

Case $n \geq 4$.

First of all we note that, for each $P \in X(\bar{K})$, the generic hyperplane of $\mathbb{P}_n$ passing through P cuts X in a smooth irreducible cubic of dimension $n - 2$. In fact, the smoothness outside of P is a direct consequence of Bertini's theorem [Har77, Remark III.10.9.2], while the smoothness at P follows from the fact that the generic hyperplane at P is transversal to the tangent plane $T_P X$ of X at P.

We choose now a K-rational point $P \in X(K)$, and two K-rational (distinct) hyperplanes $H_0 := \{h_0 = 0\}, H_\infty := \{h_\infty = 0\}$ passing through P, such that $H_0 \cap X$ is smooth. Let $L := H_0 \cap H_\infty$.

Let us consider the following morphism:

$$\varphi : X \setminus L \cap X \longrightarrow \mathbb{P}_1, \quad p \longmapsto [h_0(p) : h_\infty(p)],$$

which extends naturally to a morphism $\hat{\varphi} : \mathrm{Bl}_{L \cap X} X \to \mathbb{P}_1$. For $t = [t_1 : t_2] \in \mathbb{P}_1$, the scheme-theoretic fiber $\hat{\varphi}^{-1}(t)$ is isomorphic to the intersection $H_t \cap X$, where H_t denotes the hyperplane

$t_1h_0+t_2h_\infty = 0$ in $\mathbb{P}_n$. Since $H_0 \cap X$ is smooth, the intersection $H_t \cap X$ is smooth for t in a Zariski open subset $V \subset \mathbb{P}_1(\bar{K})$, containing 0. For $x \in V(K)$, the fiber $\hat{\varphi}^{-1}(x)$ is a smooth cubic in an $(n-1)$-dimensional projective space, with a K-rational point in it (namely, P). Hence, by the induction hypothesis, this fiber has the HP, and hence, since $\mathbb{P}_1/K$ has the HP, $\mathrm{Bl}_{L\cap X} X$ (and, therefore, X) has the HP as well by Theorem 2.2.4. □

Remark 2.3.3. As Colliot-Thélène has pointed out to the author, Theorem 2.3.1, in the dim $X = 2$ case, follows from [SD01], where Swinnerton-Dyer proves that a cubic surface X/K with a K-rational point has the weak weak approximation property (see [Ser08, Definition 3.5.6]), which is stronger than the Hilbert Property *loc. cit.*. One can then apply the induction argument used in the proof of 2.3.1 to prove that the Hilbert Property holds in general for smooth cubic hypersurfaces X/K with a K-rational point.

2.4 Surfaces with the Hilbert Property

The proof of Theorem 2.1.1 uses the following lemma, which is Lemma 3.2 of [CZ17].

Lemma 2.4.1. *Let G be a finitely generated abelian group of positive rank. Let $n \in \mathbb{N}$ and $\{h_u+H_u\}_{u=1,\dots,n}$ be a collection of finite index cosets in G, i.e. $h_u \in G$, $H_u < G$ and $[G : H_u] < \infty$ for each $u = 1,\dots,n$. If $G \setminus \bigcup_{u=1,\dots,n}(h_u + H_u)$ is finite, then $\bigcup_{u=1,\dots,n}(h_u + H_u) = G$.*

Notation 2.4.2. Let E be a smooth projective geometrically connected k-surface, endowed with fibrations $\pi_1,\dots,\pi_n : E \to \mathbb{P}_1,\ n \geq 2$. We call the *fixed locus of* $\pi_1,\dots,\pi_n$ the following reduced subvariety of E:

$$\mathfrak{Fix}(\pi_1,\dots,\pi_n) = \bigcup\{D : D \text{ is a divisor in } E \text{ and } \pi_i|_D \text{ is constant } \forall i = 1,\dots,n\}$$

Remark 2.4.3. The subvariety $F \subset E$ described in Theorem 2.1.1 is exactly $\mathfrak{Fix}(\pi_1,\dots,\pi_n)$.

Proof of Theorem 2.1.1. Suppose by contradiction that there exist $m \in \mathbb{N}$, irreducible covers $\varphi_i : Y_i \to E,\ i = 1,\dots,m$ and a proper subvariety $D \subsetneq E$, such that $E(K) \subset D(K) \cup \bigcup_i \varphi_i(Y_i(K))$, and $\deg \varphi_i \geq 2$. We may assume, without loss of generality, that the Y_i's are smooth and geometrically connected.

We say that a cover Y_i, and the corresponding i, is $\{j_1,\dots,j_k\}$-*unramified*, where $\{j_1,\dots,j_k\} \subset \{1,\dots,m\}$, if, for each $j \in \{j_1,\dots,j_k\}$, the branch divisor of φ_i is contained in a finite union of fibers of π_j. We say that it is $\{j_1,\dots,j_k\}$-*ramified* otherwise. By the simply connectedness hypothesis, no cover is $\{1,\dots,m\}$-unramified.

We say that a point $P \in E(K)$ is j-*good* if the fiber $\pi_j^{-1}(\pi_j(P))$ is smooth and geometrically connected of genus 1, and $\#\pi_j^{-1}(\pi_j(P))(K) = \infty$. By Lemma 2.3.2, there exists a non-empty open subset $U \subset E$ such that, for each $P \in U(K)$, P is j-good for each $j = 1,\dots,m$. We assume moreover, without loss of generality, that $U \cap D = \emptyset$.

For each $1 \leq i \leq m,\ 1 \leq j \leq n$, let

$$\pi_j \circ \varphi_i : Y_i \xrightarrow{r_{ij}} C_{ij} \to \mathbb{P}_1$$

be the relative normalization factorization of $\pi_j \circ \varphi_i$. We have that the geometric generic fiber of r_{ij} is irreducible, C_{ij} is a smooth complete geometrically connected curve and $C_{ij} \to \mathbb{P}_1$ is a finite morphism. Let $\widehat{E \times_{\mathbb{P}_1} C_{ij}} \to E \times_{\mathbb{P}_1} C_{ij}$ be a desingularization of $E \times_{\mathbb{P}_1} C_{ij}$. Then there exists a commutative diagram as follows:

$$\begin{array}{ccccc}
\hat{Y}_i & \xrightarrow{\psi_{ij}} & \widehat{E \times_{\mathbb{P}_1} C_{ij}} & \longrightarrow & E \\
\downarrow{\scriptstyle b_i} & & & \nearrow_{\varphi_i} & \\
Y_i & & & &
\end{array}$$

where $b_i : \hat{Y}_i \to Y_i$ is the composition of a finite sequence of blowups.

When i is j-ramified, since the branch locus of φ_i contains at least one component transverse to the fibration π_j, and the morphism $\widehat{E \times_{\mathbb{P}_1} C_{ij}} \to E$ is j-unramified, ψ_{ij} must have at least one irreducible component of the branch locus which is transverse to the fibration $\widehat{E \times_{\mathbb{P}_1} C_{ij}} \to C_{ij}$. Hence, by the invariance of the ramification locus under base change, when i is not j-unramified, the geometric generic fiber of $\hat{Y}_i \to C_{ij}$, which is isomorphic to the geometric generic fiber of r_{ij}, is ramified over the geometric generic fiber of $\widehat{E \times_{\mathbb{P}_1} C_{ij}} \to C_{ij}$, which has genus 1. Therefore, it is a curve of genus > 1. Hence, when i is j-ramified, the geometric generic fiber of $\pi_j \circ \varphi_i$ is a finite union of curves of genus > 1. When i is j-unramified one shows analogously that the geometric generic fiber of $\pi_j \circ \varphi_i$ is a finite union of curves of genus 1.

We have that, for each $P \in U(K)$:

$$P \in \pi_n^{-1}(\pi_n(P))(K) \cap U \subset$$
$$\bigcup_{i\ n-\text{unramified}} \varphi_i(Y_i(K)) \cap \pi_n^{-1}(\pi_n(P)) \cup \bigcup_{i\ n-\text{ramified}} \varphi_i(Y_i(K)) \cap \pi_n^{-1}(\pi_n(P)) =$$
$$\bigcup_{i\ n-\text{unramified}} \varphi_i((\pi_n \circ \varphi_i)^{-1}(\pi_n(P)(K))) \cup \bigcup_{i\ n-\text{ramified}} \varphi_i((\pi_n \circ \varphi_i)^{-1}(\pi_n(P)(K))).$$

Hence, as noted before, when i is n-ramified, after restricting (without loss of generality) U to a smaller (non-empty) Zariski open subset, $(\pi_n \circ \varphi_i)^{-1}(\pi_n(P))$ is a curve of genus > 1. Therefore, by Falting's Theorem, we deduce that:

$$\pi_n^{-1}(\pi_n(P))(K) \subset \bigcup_{i\ n-\text{unramified}} \varphi_i((\pi_n \circ \varphi_i)^{-1}(\pi_n(P)(K))) \cup A_0(P),$$

where $A_0(P)$ is a finite set. Moreover, when i is n-unramified, after restricting (without loss of generality) U to a smaller (non-empty) Zariski open subset, $(\pi_n \circ \varphi_i)^{-1}(\pi_n(P))$ is a curve of genus 1. Therefore, by the weak Mordell-Weil theorem, we have that, for each $i = 1, \dots, m$, $(\pi_n \circ \varphi_i)^{-1}(\pi_n(P))(K) \subset \pi_n^{-1}(\pi_n(P))(K)$ is either empty or a finite index coset.

Hence, by Lemma 2.4.1, we deduce that:

$$P \in \pi_n^{-1}(\pi_n(P))(K) \subset \bigcup_{i\ n-\text{unramified}} \varphi_i(Y_i(K)).$$

Therefore we have that

$$E(K) \subset \bigcup_{i\ n-\text{unramified}} \varphi_i(Y_i(K)) \cup (E \setminus U).$$

We have now reduced to the case where all the covers Y_i are n-unramified. Proceeding with an easy induction on n, we may reduce to the case where all the covers Y_i are $\{1, \dots, n\}$-unramified. But there are no such covers by hypothesis, whence we deduce that $E(K)$ is not Zariski-dense in E, leading to the desired contradiction. □

2.4.1 A family of K3 surfaces with the Hilbert Property

As an application of Theorem 2.1.1 we describe now a family of K3 surfaces with the Hilbert Property.[1]

[1] K3 surfaces (and, in general, Calabi-Yau varieties) represent a "limiting case" for the study of rational points, at least conjecturally. In fact, the conjectures of Vojta suggest that on algebraic varieties there should be "less" rational points as the canonical bundle gets "bigger". Hence, since for K3 surfaces the canonical bundle is trivial by definition, we expect the rational points here not to be "too many", yet their existence (and Zariski-density) is not precluded. In fact, proving the HP, we are providing some examples of abundance of rational points in such surfaces.

For $\lambda \in K^*$, and $c_1, c_2 \in K[x, y, z]$ cubic homogeneous polynomials, let

$$X'_\lambda(c_1, c_2) := \{([w_0 : w_1], [x : y : z]) \in \mathbb{P}_1 \times \mathbb{P}_2 \mid w_0^2 c_1(x, y, z) = \lambda w_1^2 c_2(x, y, z)\}. \tag{2.4.1}$$

The surfaces $X'_\lambda(c_1, c_2)$ are, up to a birational transformation, endowed with multiple elliptic fibrations, usually defined over $\bar{K}$, whose construction we recall in the next paragraphs. In some particular cases, when enough of these fibrations are defined over K, this allows us to use Theorem 2.1.1 to prove the Hilbert Property of these varieties.

Remark 2.4.4. When $c_1(x, y, z) = f_1(x, z)$ does not depend on y, $c_2(x, y, z) = f_2(y, z)$ does not depend on x, and both f_1 and f_2 do not have multiple roots, equation (2.4.1) describes a Kummer surface (i.e. a quotient of an abelian surface by the group of isomorphisms $\{\pm 1\}$).

In fact, in this case, equation (2.4.1) describes, up to a birational transformation, the quotient of $E_1 \times E_2$ by the group $\{\pm 1\}$, where E_1 and E_2 are the elliptic curves defined by the following Weierstrass equations:

$$E_1 : \ w^2 = f_1(x, z), \quad E_2 : \ w^2 = f_2(y, z). \tag{2.4.2}$$

Construction of the Elliptic Fibrations We give now an explicit construction of a smooth model of of $X'_\lambda(c_1, c_2)$ and of the elliptic fibrations it is endowed with, under a genericity assumption on c_1 and c_2. We avoid going into detail, as these constructions are described thoroughly by Garbagnati and Salgado in [GS19].

Let $P_1, \ldots, P_9$ be 9 (distinct) points in $\mathbb{P}_2(\bar{K})$ such that:

i. $P_1, \ldots, P_4$ are the four points of intersection of two smooth conics $Q_1^1 := \{q_1^1 = 0\}, Q_2^1 := \{q_2^1 = 0\}$ in $\mathbb{P}_2$, defined over K;

ii. $P_5, \ldots, P_8$ are the four points of intersection of two smooth conics $Q_1^2 := \{q_1^2 = 0\}, Q_2^2 := \{q_2^2 = 0\}$ in $\mathbb{P}_2$, defined over K;

iii. The eight points $P_1, \ldots, P_8$ are in generic position, in the sense that no three of these points lie on a line, and no six of these points lie on a conic;

iv. $P_1, \ldots, P_9$ are the nine points of intersection of two smooth cubics $C_1 := \{c_1 = 0\}, C_2 := \{c_2 = 0\}$ in $\mathbb{P}_2$, defined over K.

Definition 2.4.5. We say that a 9-tuple $(P_1, \ldots, P_9) \in P_2(\bar{K})^9$ is *good* if it satisfies the four conditions above.

We note that, by choosing two sufficiently Zariski generic 4-tuples $p_1, \ldots, p_4$ and $p_5, \ldots, p_8$ such that both $p_1 + \cdots + p_4$ and $p_5 + \cdots + p_8$ are defined over K, and letting p_9 be the unique ninth intersection of the pencil of cubics through $p_1, \ldots, p_8$, the 9-tuple $p_1, \ldots, p_9$ is good.

We assume hereafter that a choice of a good 9-tuple of points $P_1, \ldots, P_9$, of the two cubics C_1, C_2 and of the conics Q_j^i, $i, j = 1, 2$ has been made.

Let $R := \mathrm{Bl}_{P_1 + \cdots + P_9} \mathbb{P}_2$ be the blowup of $\mathbb{P}_2$ in the nine points $P_1, \ldots, P_9$. The two cubics C_1, C_2 define an elliptic fibration on R, which we denote by $\mathcal{C}$, defined as $\mathcal{C}(p) = [c_1(p) : c_2(p)]$.

The fibers of $\mathcal{C}$ are by construction the proper transforms of the elements of the pencil generated by C_1, C_2.

For $\lambda \in K^*$, let $f_\lambda : \mathbb{P}_1 \to \mathbb{P}_1$ be the morphism defined by $f_\lambda([w_0 : w_1]) = [w_0^2 : \lambda w_1^2]$. Let also $X_\lambda(c_1, c_2)$ be the smooth surface defined as the fibered product $R \times_{\mathcal{C}, f_\lambda} \mathbb{P}_1$, $\alpha_\lambda : X_\lambda(c_1, c_2) \to R$ be the projection on the first factor, and $\varphi_\lambda : X_\lambda \to \mathbb{P}_1$ be the projection on the second factor. The surface $X_\lambda(c_1, c_2)$ is a K3 surface.

Note 1. We observe that, by construction, $X_\lambda(c_1, c_2)$ is birational to $X'_\lambda(c_1, c_2)$.

The surface $X_\lambda(c_1, c_2)$ is endowed with at least three elliptic fibrations. The first one is φ_λ. The second and third one, which we denote by $\widetilde{\mathcal{Q}}^1$ and $\widetilde{\mathcal{Q}}^2$, are the proper transforms of the two pencils of conics generated, respectively, by $\{Q_1^1, Q_2^1\}$ and by $\{Q_1^2, Q_2^2\}$. I.e., $\widetilde{\mathcal{Q}}^i = \mathcal{Q}^i \circ \alpha_\lambda$, where the maps $\mathcal{Q}^i : R \to \mathbb{P}_1$ are defined as $\mathcal{Q}^i(p) = [q_1^i(p) : q_2^i(p)],\ i = 1, 2$.

Proposition 2.4.6. *Let $P_1, \ldots, P_9 \in \mathbb{P}_2(\bar{K})$ be a good 9-tuple of points, and $C_1 := \{c_1 = 0\}, C_2 := \{c_2 = 0\}$ be two smooth cubics such that $C_1 \cap C_2 = \{P_1, \ldots, P_9\}$. If $X_\lambda(c_1, c_2)$ has Zariski-dense K-rational points, then it has the Hilbert Property.*

Proof. Since $\mathfrak{Fix}(\varphi_\lambda, \widetilde{\mathcal{Q}}^1, \widetilde{\mathcal{Q}}^2) = \emptyset$, this is an immediate consequence of Theorem 2.1.1 applied to $X_\lambda(c_1, c_2)$ (which, being a K3 surface, is simply connected), with fibrations $\varphi_\lambda, \widetilde{\mathcal{Q}}^1$ and $\widetilde{\mathcal{Q}}^2$. □

Remark 2.4.7. One can show that, given cubic polynomials c_1, c_2 satisfying the hypothesis of Proposition 2.4.6, there exist always infinitely many $\lambda \in K^*$ such that $X_\lambda(c_1, c_2)$ has Zariski-dense K-rational points. To show this one can use a result of Van Lujik, namely [Lui12, Theorem 2.2], where he shows that under some mild hypothesis a surface V/K endowed with two elliptic fibrations has Zariski-dense K-rational points as soon as it posses one K-rational point outside a specific Zariski-closed subvariety (for $V = X_\lambda(c_1, c_2)$ we will denote the latter with $D_\lambda \subsetneq X_\lambda(c_1, c_2)$). Moreover, the resulting Zariski-closed subvariety is invariant by "twists" of the variety that preserve, in a specific sense, the fibrations. We notice that the variety $X_\lambda(c_1, c_2)$ is a twist of the variety $X_1(c_1, c_2)$, in fact the morphism

$$\begin{array}{rccc} \varphi_\lambda : & X_\lambda(c_1, c_2)_{\bar{K}} & \longrightarrow & X_1(c_1, c_2)_{\bar{K}} \\ & ((x:y:z), w) & \longmapsto & ((x:y:z), \sqrt{\lambda}w) \end{array}, \tag{2.4.3}$$

(we are using the notation of 2.4.1, the reader may easily verify that the rational map defined through 2.4.3 is indeed a morphism) is a $\bar{K}$-isomorphism. The interested reader may now verify that the hypothesis of [Lui12, Theorem 2.2] are satisfied with $(V, f_1, f_2) = (X_1(c_1, c_2), \tilde{\mathcal{Q}}^1, \tilde{\mathcal{Q}}^2)$, $(W, g_1, g_2) = (X_\lambda(c_1, c_2), \tilde{\mathcal{Q}}^1, \tilde{\mathcal{Q}}^2)$, $\varphi = \varphi_\lambda$, and with the rational maps $\alpha_1, \alpha_2, \beta_1, \beta_2$ intended to be the ones constructed in [Lui12, Remark 2.4]. One may easily check now that there exists a Zariski-closed subvariety Z of $\mathbb{P}_2$ such that, if $P = (x : y : z) \notin Z$, the point $((x : y : z), 1) \in X_\lambda(c_1, c_2) \setminus D_\lambda$, where $\lambda = c_1(x, y, z)/c_2(x, y, z)$. Since, for $(x : y : z) \in \mathbb{P}_2(K) \setminus Z$, the rational function $c_1(x, y, z)/c_2(x, y, z)$ assumes infinitely many values, one has, as a direct consequence of [Lui12, Theorem 2.2], that $X_\lambda(c_1, c_2)$ has Zariski-dense K-rational points for infinitely many $\lambda \in K^*$.

2.4.2 Kummer surfaces

The following proposition is another application of Theorem 2.1.1.

Proposition 2.4.8. *Let E_1 and E_2 be two elliptic curves defined over a number field K, with positive Mordell-Weil rank. The Kummer surface $S := E_1 \times E_2/\{\pm 1\}$ has the Hilbert Property.*

Proof. A desingularization of S, which we denote by $\tilde{S}$, may be obtained as the quotient by $\{\pm 1\}$ of the blow up $\widehat{E_1 \times E_2}$ of $E_1 \times E_2$ in the 16 2-torsion points. We denote the set of the images of these points in S with $\mathcal{T}$, and the corresponding exceptional lines in $\tilde{S}$ with $\mathcal{L}$. Moreover, we denote by $b : \tilde{S} \to S$ the just described desingularization morphism, and by $q : E_1 \times E_2 \to S$ the quotient map.

The surface $\tilde{S}$ has at least three elliptic fibrations, defined over K. Two of these, which we denote by $\pi_i,\ i = 1, 2$ are the following compositions:

$$\tilde{S} \to S = E_1 \times E_2/\{\pm 1\} \to E_i/\{\pm 1\} \cong \mathbb{P}_1,\ i = 1, 2.$$

If $E_1 \coloneqq \{y_1^2 z_1 = f_1(x_1, z_1)\}$ and $E_2 \coloneqq \{y_2^2 z_2 = f_2(x_2, z_2)\}$, then, as noted in Remark 2.4.4, S is birational to the surface defined by the following equation for $([x : y : z], [w_1 : x_2]) \in \mathbb{P}_2 \times \mathbb{P}_1$:

$$w_1^2 f_2(x, z) = w_2^2 f_1(y, z).$$

We have that:

$$[w_1 : w_2] \circ q = [y_1 z_2 : y_2 z_1]. \tag{2.4.4}$$

We then define the third fibration, π_3, to be the extension (as a rational map) to $\tilde{S}$ of the map $([x : y : z], [w_1 : w_2]) \to [w_1 : w_2]$. Let us check that $\mathbf{Dom}(\pi_3) = \tilde{S}$. The map $[y_1 z_2 : y_2 z_1]$ is well-defined on $\widehat{E_1 \times E_2}$. Moreover, since $[y_1 z_2 : y_2 z_1] : \widehat{E_1 \times E_2} \to \mathbb{P}_1$ is invariant by the action of $\{\pm 1\}$, it induces indeed a well-defined morphism on the quotient $\tilde{S} = \widehat{E_1 \times E_2}/\{\pm 1\}$.

We have that, since y_i is a local parameter at points of order 2 in E_i, and z_i is a local parameter at $O \in E_i$, the morphism $[y_1 z_2 : y_2 z_1] : \widehat{E_1 \times E_2} \to \mathbb{P}_1$ is non-constant on the exceptional lines lying over the points (T_1, T_2), when both T_1 and T_2 have order 2 or both have order 0, and it is constant on the other exceptional lines.

It follows that $\mathfrak{Fix}(\pi_1, \pi_2, \pi_3)$ is the union of the 6 exceptional lines in $\tilde{S}$ lying over the points (T_1, T_2), when exactly one of T_1, T_2 has order 2 and the other is O. We denote the union of these 6 points in S with $\mathcal{T}^b$, and the corresponding lines in $\tilde{S}$ with $\mathcal{L}^b \coloneqq b^{-1}(\mathcal{T}^b)$. Since, by hypothesis, $S(K) \supset q(E_1(K) \times E_2(K))$ is Zariski-dense in S, the proposition follows from Theorem 2.1.1 and the following lemma. □

Remark 2.4.9. In Proposition 2.4.8 the hypothesis that the two curves E_1, E_2 have positive Mordell-Weil rank is just used to guarantee that K-rational points are Zariski-dense in S. However, one can remove this hypothesis in the case that the j-invariants of E_1 and E_2 are not both equal to 0 or 1728. In fact, under these assumptions, Kuwata and Wang showed in [WK93] that K-rational points are always Zariski-dense. (They work in the specific case $K = \mathbb{Q}$, but the part of their paper where they prove Zariski-density of rational points can be rephrased ad litteram over any number field.)

Lemma 2.4.10. *The surface $(\tilde{S} \setminus \mathcal{L}^b)/\mathbb{C}$ is (topologically) simply connected.*

Proof. Let $\Lambda_1 =< e_1, e_2 >$ and $\Lambda_2 =< e_3, e_4 >$ be lattices in $\mathbb{C}$ such that $E_1 \cong \mathbb{C}/\Lambda_1$, and $E_2 \cong \mathbb{C}/\Lambda_2$ as analytic spaces. We observe that the universal cover of $\tilde{S} \setminus \mathcal{L} = S \setminus \mathcal{T}$ is the following composition

$$\mathbb{C}^2 \setminus \frac{1}{2}(\Lambda_1 \times \Lambda_2) \to \mathbb{C}/\Lambda_1 \times \mathbb{C}/\Lambda_2 \setminus q^{-1}(\mathcal{T}) \cong E_1 \times E_2 \setminus q^{-1}(\mathcal{T}) \xrightarrow{q} S \setminus \mathcal{T}.$$

Therefore

$$\pi_1(S \setminus \mathcal{T}, p_0) \cong (\Lambda_1 \times \Lambda_2) \rtimes < \iota >,$$

where p_0 denotes a point infinitesimally near to $q((O, O)) \in S$, the action of ι on $\Lambda_1 \times \Lambda_2$ is given by $(a, b) \to (-a, -b)$, and the element ι corresponds to a (single) loop around $q((O, O)) \in S$ (and hence $\iota^2 = 1$). For any 2-torsion point T in $E_1 \times E_2$, let $\iota_T \in \pi_1(S \setminus \mathcal{T}, p_0)$ denote the element corresponding to a (single) loop around the point $q(T) \in S$. A priori this element is well-defined only after a choice of a path between $q(p_0)$ and a point infinitesimally near to $q(T)$ has been made. This choice can be done arbitrarily and it is in fact irrelevant for our purposes. We will assume anyway that the path chosen is the geodetic, using the distance induced by the universal cover

$$\mathbb{R}^4 \xrightarrow{(e_1|e_2|e_3|e_4)} \mathbb{C} \times \mathbb{C} \to \mathbb{C}/\Lambda_1 \times \mathbb{C}/\Lambda_2 \cong E_1 \times E_2 \xrightarrow{q} S,$$

where the $\mathbb{R}^4$ on the left is endowed with the Euclidean metric.

We have that, if $T = \sum_{i=1}^{4} \frac{\epsilon_i}{2} e_i$, where $\epsilon_i \in \{0,1\}$, then

$$\iota_T = \left(\sum_{i=1}^{4} \epsilon_i e_i \right) \iota.$$

Let $H \subset (\Lambda_1 \times \Lambda_2) \rtimes \mathbb{Z}/2\mathbb{Z}$ denote the minimal normal subgroup containing the ι_T's, for $T \in \mathcal{T}^g := \mathcal{T} \setminus \mathcal{T}^b$. We note that $(e_1+e_4)/2$, $(e_1+e_3)/2$, $(e_1+e_3+e_4)/2 \in \mathcal{T}^g$, and hence $e_1 = (e_1+e_4)+(e_1+e_3)-(e_1+e_3+e_4) \in H$. Analogously one has that $e_i \in H$ for every $1 \leq i \leq 4$. Therefore, $H = (\Lambda_1 \times \Lambda_2) \rtimes < \iota >$.

We now observe that, in the blown-up surface $\tilde{S} \setminus \mathcal{L}^b$, the loop ι_T becomes trivial for any point $T \in \mathcal{T}^g$. In fact, a small topological neighborhood L_T^ϵ of the exceptional line $L_T := b^{-1}(T) \subset \tilde{S}$ is retractible on L_T itself, which is simply connected.

Therefore, by Van Kampen's Theorem applied to $\tilde{S} \setminus \mathcal{L}^b = (\tilde{S} \setminus \mathcal{L}) \cup \bigcup_{T \in \mathcal{T}^g} L_T^\epsilon$, we have that

$$\pi_1(\tilde{S} \setminus \mathcal{L}^b) \cong {}^{(\Lambda_1 \times \Lambda_2) \rtimes < \iota >}\!\big/_{H} \cong \{1\},$$

as we wanted to prove. □

Remark 2.4.11. Corvaja and Zannier proved in [CZ17] that there are Zariski-dense K-rational points in $E_1 \times E_1/\{\pm 1\}$ that are not image of K-rational points in $E_1 \times E_2$. It is possible that combining the technique presented in [CZ17] with other results, for instance the ones contained in [Zan10], one may obtain a different proof of Proposition 2.4.8, at least in the case where $E_1 = E_2$.

Chapter 3

The étale Brauer–Manin obstruction to strong approximation on homogeneous spaces

3.1 Introduction

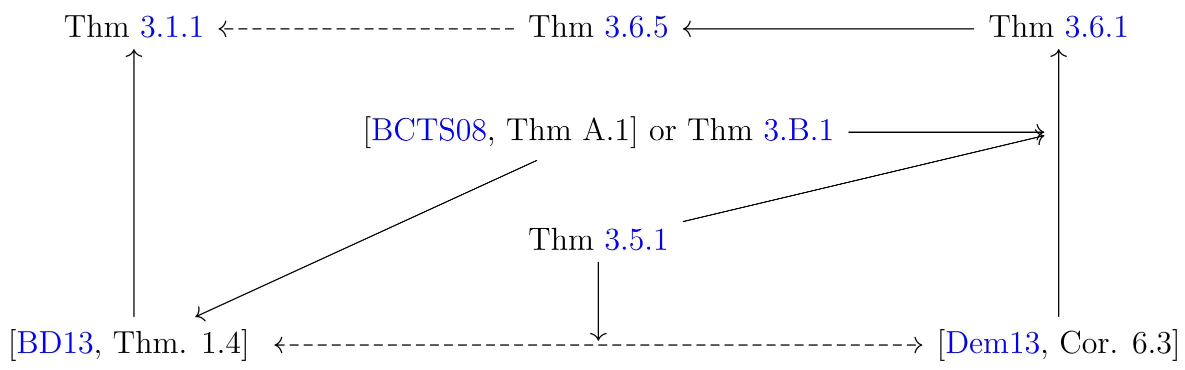
Recall that in this chapter, we want to prove that the étale-Brauer–Manin obstruction is the only one on homogeneous spaces. More precisely:

Theorem 3.1.1. *Let G be a connected algebraic group over a number field K. We assume that the Tate-Shafarevich group $Ш(K, G^{ab})$ is finite. Let X be a left homogeneous space under G. Let $S \supset M_K^\infty$ be a finite set of places of K. We assume that $G^{sc}(K)$ is dense in $G^{sc}(\mathbb{A}_K^S)$. Set $S_f := S \cap M_K^{fin} = S \setminus M_K^\infty$. Then the set $X(\mathbb{A}_K)_\bullet^{\text{ét,Br}}$ is equal to the closure of the set $G^{scu}(K_{S_f}) \cdot X(K) \subset X(\mathbb{A}_K)_\bullet$ for the adelic topology.*

We will then also prove Theorem 3.6.5, which is an analogue of the Theorem above with finitely many places removed. In the course of doing so we will also prove Theorem 3.6.1, which is an analogue of the result of Borovoi and Demarche [BD13, Thm. 1.4] with finitely many places removed. This appears to be new, and, as it will be remarked, does not seem to follow directly by projection from [BD13, Thm 1.4], as one may think at first sight. So, to prove this, we will need a "Brauer pairing compatibility" result, which is Theorem 3.5.1.

3.1.1 Structure of the chapter

In Section 3.2 and Section 3.3 we set up the notation and recall some of the preliminaries for the results that are proved in this chapter. In Section 3.4 we prove Theorem 3.1.1. In Section 3.5, we prove Theorem 3.5.1, which we use to prove the equivalence of [Dem13, Cor. 6.3] with [BD13, Thm. 1.4], so that we can then use this equivalence in Section 3.6 to prove Theorem 3.6.1, and then use this last to prove Theorem 3.6.5. We also remark that the proof of Theorem 3.6.5 is logically independent from the proof of Theorem 3.1.1, and reproves it completely. However, since, to prove Theorem 3.6.5, we use Theorem 3.5.1 with its long calculation, we preferred keeping the two theorems separate in the exposition. Finally, the appendices contain no new results, but just facts that are recalled for convenience, or because they did not appear explicitly in the literature. We include a diagram of the logical implications of this chapter and a couple of results in the literature that play an essential role:



In the diagram above, a non-dotted arrow between X and Y indicates that X is used (either in the current mathematical literature or in this thesis) to prove Y. An arrow from X pointing to another arrow indicates that X gets used (again, either in the current mathematical literature or in this thesis) to prove the implication it points to. A dotted arrow indicates that it is possible to prove Y using X (without using any of the other results appearing in the diagram, except at most the ones pointing to the dotted arrow from X to Y).

3.2 Notation

Unless specified otherwise, k will always denote a field of characteristic 0 and K a number field. For a field k, $\overline{k}$ will always denote a (fixed) algebraic closure of k, and Γ_k the profinite group $\mathrm{Gal}(\overline{k}/k)$. For a number field K, M_K will denote the set of places of K, M_K^{fin} (resp. M_K^∞) the finite (resp. archimedean) places. For a place $v \in M_K$, K_v will denote the v-adic completion of K, and, for $v \in M_K^{fin}$, $O_v \subset K_v$ will denote the v-adic integers. The topological ring of adeles of K, i.e. the ring $\prod'_{v\in M_K} K_v$ (the restricted product being on $O_v \subset K_v$) is denoted by $\mathbb{A}_K$. For a finite subset $S \subset M_K$, $\mathbb{A}_K^S$ denotes the topological ring of S-adeles, i.e. the ring $\prod'_{v\in M_K\setminus S} K_v$, K_S denotes the product $\prod_{v\in S} K_v$, and S_f will denote the intersection $S \cap M_K^{fin}$.

When $\varphi : X \to Y$ is a morphism defined over K, we will denote by $\varphi_v : X_{K_v} \to Y_{K_v}$ the induced morphism among the base-changed varieties $X_{K_v} = X \times_K K_v$ and $Y_{K_v} = Y \times_K K_v$.

All schemes appearing are separated, therefore, we tacitly always assume this hypothesis throughout the chapter.

A variety over a field k is an integral scheme of finite type over k.

When X is a variety defined over K, the notation $X(\mathbb{A}_K)_\bullet$ will denote the adelic points where each archimedean component is collapsed to the (discrete) topological space of its connected components.

An *adelic-like object* is a product $\prod'_{v\in M_K} P_v$, where P_v (parametrized by the places $v \in M_K$) are sets, such that almost all of them have an integral version P_{O_v} (endowed with a natural morphism $P_{O_v} \to P_v$), and the restricted product is taken with respect to these integral versions.

For a subset $Y \subset \prod'_{v\in M_K} P_v$ of an adelic-like object and a set $S \subset M_K$, we denote by Y_S the set $(\pi^S)^{-1}(\pi^S(Y))$, where $\pi^S : \prod'_{v\in M_K} P_v \to \prod'_{v\in M_K\setminus S} P_v$ is the standard projection.

Group actions (and, correspondingly, homogeneous spaces) will be assumed to be left actions unless specified otherwise. In particular, most of the torsors appearing will, instead, be right torsors. This will be specified each time.

Let G be a connected algebraic group (not necessarily linear) over k. Then, according to Chevalley's Theorem (see [Con02] for a proof), G fits into a (canonical) short exact sequence:

$$1 \to G^{lin} \to G \to G^{ab} \to 1,$$

where G^{lin} is a connected linear k-group, and G^{ab} is a k-abelian variety.

For a connected algebraic group G over k, we will use the following notation (borrowing it from [BD13]):

G^u is the unipotent radical of G^{lin};

$G^{red} = G^{lin}/G^u$ is the reductive k-group associated with G;

G^{ant} the maximal anti-affine subvariety of G (we refer to [Mil17, Ch. 10] for the definition and main properties);

$G^{ss} = [G^{red}, G^{red}] \subset G^{red}$ is the commutator subgroup of G^{red} (it is a semisimple k-group);

G^{sc} is the universal cover of G^{ss}, a simply connected semisimple k-group (we say that a k-group is *simply connected* if it is geometrically simply connected);

G^{ssu} is the inverse image of G^{ss} under the projection $G^{lin} \to G^{red}$, which fits into an exact sequence:

$$1 \to G^u \to G^{ssu} \to G^{ss} \to 1;$$

G^{scu} is the fibered product $G^{sc} \times_{G^{red}} G^{lin}$ (with its canonical group structure), which fits into an exact sequence:

$$1 \to G^u \to G^{scu} \to G^{sc} \to 1.$$

We have a canonical homomorphism $G^{scu} = G^{sc} \times_{G^{red}} G^{lin} \to G^{lin} \hookrightarrow G$.

For a field k, Sch_k will denote the category of quasi-projective $\mathrm{Spec}\, k$-schemes.

When X is defined over L, a finite extension of k, we denote by $R_{L/k}X$ the Weil restriction to k of X (see [Sch94, Ch. 4] for the definition and basic properties of the functor $R_{L/k}$). When L is Galois over k, with Galois group Γ (which we consider naturally endowed with its *left* action on L/k), there is an action of Γ on Sch_L, which may be described as follows. If $(X, p_X) \in \mathrm{Sch}_L$, where X is a scheme and $p_X : X \to \mathrm{Spec}\, L$ is the structural morphism, then $(X, p_X)^\gamma := (X, \gamma \circ P_X)$, where $\gamma : \mathrm{Spec}\, L \to \mathrm{Spec}\, L$ denotes (the map induced by) conjugation by γ (we refer to [Sch94, 4.11.1] for more details).

Whenever we have two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$, and a collection of morphisms $\mathcal{F}^Y : F(Y) \to G(Y)$, indexed by the objects $Y \in \mathcal{C}$, we say that such a collection is a natural transformation if, for every morphism $f : Y_1 \to Y_2$ in $\mathcal{C}$, we have that $G(f) \circ \mathcal{F}^{Y_1} = \mathcal{F}^{Y_2} \circ F(f)$. We also say, with a slight abuse of notation, that the morphisms are a natural transformation when their collection is.

For a product $\prod_{i \in I} X_i$ and a subset $J \subset I$ we denote (when there is no risk of confusion) by π_J the projection $\prod_{i \in I} X_i \to \prod_{i \in J} X_i$.

If A is a topological abelian group, $A^\wedge$ will denote its completion with respect to open finite index subgroups. When A is discrete, $A^\wedge$ is the profinite completion.

For a torsion abelian group A, A^D will denote the dual $\mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$ endowed with the compact-open topology. This is a profinite group. If A is a profinite abelian group, A^D will denote the torsion group $\mathrm{Hom}_{cont}(A, \mathbb{Q}/\mathbb{Z})$, where $\mathbb{Q}/\mathbb{Z}$ is endowed with its discrete topology.

If G is a group, and M is an abelian group, we will say that a set-morphism $G \to \mathrm{Aut}(M)$ is a group *pseudo-action* of G on M. We will keep using the term "group action" when $G \to \mathrm{Aut}(M)$ is a group-homomorphism.

When $[H \to G]$ is a crossed module, and there is no risk of confusion, we will use the notation ${}^g h$ to denote the left action of $g \in G$ on $h \in H$. We remind the reader that this action is compatible with the left action of H by conjugation on itself.

If G/k is an algebraic group, and $k \subset F$ is a field extension, we will use the notation $H^i(F, G)$ (with $i \in \mathbb{N}$ and $i = 0, 1$ if G is not abelian) to denote the cohomology group/pointed set $H^i(\Gamma_F, G(\overline{F}))$. If G is not abelian we assume that the cohomological set $H^1(\Gamma_F, G(\overline{F}))$ is the one of right cocycles (i.e. those that correspond to left F-torsors under G through [Sko01, p.18, 2.10]). We will use the notation $H^1_{lt}(\Gamma_F, G(\overline{F}))$ to denote left cocycles instead.

If $\eta \in H^1(K, G)$, we use the notation G^η to denote the inner twist of G by η, and G_η (resp. $G_{\eta'}$) to denote the left (resp. right) principal homogeneous space of G obtained by twisting G by

η (resp. $\eta' = \eta^{-1}$, which is a right cocycle). This twist is naturally endowed with a right (resp. left) action of G^η. See [Sko01, p. 12-13] for more details on these constructions.

If Z is a k-variety endowed with a right G-action, and $\eta \in H^1(K,G)$, we use the notation Z_η to denote the twisted k-variety $(Z \times^G G_\eta)$ [Sko01, p. 20]. This is naturally endowed with a right G^η-action. Analogously, one can twist with respect to a left action and a left cocycle $\eta' \in H^1_{lt}(K,G)$.

3.3 Reminders

Even though we already reminded in the introduction what the Brauer–Manin obstruction and the étale-Brauer–Manin obstruction are, for convenience, let us recall their definitions here as well.

We recall that, when X is a variety defined over a number field K, the Brauer group $\operatorname{Br}(X)$ is defined as $H^2(X_{ét}, \mathbb{G}_m)$, and there exists a canonical pairing (called the *Brauer Manin* pairing):

$$\begin{array}{ccc} \operatorname{Br}(X) \times X(\mathbb{A}_K) & \longrightarrow & \mathbb{Q}/\mathbb{Z}, \\ (b,x) & \longmapsto & \langle b,x\rangle, \end{array}$$

which is defined as follows: if $x = (x_v)_{v\in M_K}$, then $\langle b,x\rangle = \sum_{v\in M_K} \operatorname{inv}_v(x_v^*b)$, where $x_v^*b \in H^2(\Gamma_{K_v}, \overline{K_v}^*)$ denotes the pullback of b along $x_v : \operatorname{Spec} K_v \to X$, and $\operatorname{inv}_v : H^2(\Gamma_{K_v}, \overline{K_v}^*) = \operatorname{Br}(K_v) \to \mathbb{Q}/\mathbb{Z}$ is the usual invariant map (see e.g. [Har20, Thm 8.9] for a definition). The pairing is continuous in x and additive in b. If $x \in X(K) \subset X(\mathbb{A}_K)$ or b comes from $\operatorname{Br} K$, then $\langle b,x\rangle = 0$ (see [Sko01, Sec. 5] for a proof, this is essentially a consequence of the classical Albert–Brauer–Hasse–Noether Theorem).

We recall, moreover, that the pairing (being continuous on $X(\mathbb{A}_K)$, and taking values in a discrete group) is constant on the archimedean connected components of X, hence it induces a pairing:

$$\operatorname{Br}(X) \times X(\mathbb{A}_K)_\bullet \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

We refer the reader to [Sko01, Section 5] for more details on the Brauer–Manin pairing.

We denote by $X(\mathbb{A}_K)^{\operatorname{Br} X}$ the following (closed) subset of $X(\mathbb{A}_K)$:

$$\{x \in X(\mathbb{A}_K) \mid \langle b,x\rangle = 0 \ \forall b \in \operatorname{Br}(X)\}.$$

We then have that $\overline{X(K)} \subset X(\mathbb{A}_K)^{\operatorname{Br} X} \subset X(\mathbb{A}_K)$, i.e. $\operatorname{Br}(X)$ provides an obstruction to the existence and (adelic) density of K-rational points.

We also recall that, for an algebraic group G/K, and for all (right) torsors $f : Y \xrightarrow{G} X$ under G, one has that:

$$X(K) \subset \bigcup_{[\sigma]\in H^1_{lt}(K,G)} f^\sigma(Y^\sigma(K)), \tag{3.3.1}$$

where, for any cocycle $\sigma \in Z^1(K,G)$, $f^\sigma : Y^\sigma \xrightarrow{G^\sigma} X$ denotes the torsor f twisted by σ (see [Sko01, Sec 2.2]), and $[\sigma]$ denotes the class of σ in $H^1_{lt}(K,G)$. For a right torsor $f : Y \xrightarrow{G} X$, we define:

$$X(\mathbb{A}_K)^f := \bigcup_{[\sigma]\in H^1_{lt}(K,G)} f^\sigma(Y^\sigma(\mathbb{A}_K)).$$

Because of (3.3.1), one has the following two inclusions:

$$X(K) \subset X(\mathbb{A}_K)^{ét,\operatorname{Br}} := \bigcap_{\substack{f:Y \xrightarrow{G} X \\ G \text{ finite} \\ \text{group scheme}}} \bigcup_{[\sigma]\in H^1_{lt}(K,G)} f^\sigma(Y^\sigma(\mathbb{A}_K)^{\operatorname{Br} Y^\sigma}),$$

and

$$X(K) \subset X(\mathbb{A}_K)^{desc} := \bigcap_{\substack{f:Y \xrightarrow{G} X \\ G \text{ linear}}} X(\mathbb{A}_K)^f.$$

Hence, both $X(\mathbb{A}_K)^{\acute{e}t,\mathrm{Br}}$ and $X(\mathbb{A}_K)^{desc}$ provide obstructions to the existence of K-rational points. Less obviously, Cao, Demarche and Xu [CDX19, Prop. 6.4] prove, through a Chevalley-Weil-like argument, that both $X(\mathbb{A}_K)^{\acute{e}t,\mathrm{Br}}$ and $X(\mathbb{A}_K)^{desc}$ are closed in $X(\mathbb{A}_K)$, hence they provide an obstruction to (adelic) density of K-rational points as well. Moreover, they prove that the two obstructions are in fact equal, i.e. $X(\mathbb{A}_K)^{\acute{e}t,\mathrm{Br}} = X(\mathbb{A}_K)^{desc}$.

The question of strong approximation asks whether, for a K-variety X (where X is not necessarily proper), and a finite subset $S \subset M_K$, $X(K)$ is dense in $X(\mathbb{A}_K^S)$ (where it is embedded through its diagonal image). Unfortunately, when $S = \emptyset$ one does not expect in general to have strong approximation for affine varieties. This is mainly because of a compactness issue. Namely, in the affine case, $X(K)$ is closed in $X(\mathbb{A}_K)$ (since $\mathbb{A}^n(K)$ is closed in $\mathbb{A}^n(\mathbb{A}_K)$), so there is no chance for $X(K)$ to be dense in it, unless $X(K) = X(\mathbb{A}_K)$, which can never happen if $\dim X \geq 1$ and $X(\mathbb{A}_K) \neq \emptyset$. Using the same argument, one sees that, in order for an affine variety X with $X(\mathbb{A}_K^S) \neq \emptyset$ to satisfy strong approximation outside S, there must exist at least one $v \in S$ for which $X(K_v)$ is not compact.

3.3.1 Reminders on Galois cocycles

Let Γ be a profinite group, and $D^\bullet = [\cdots \to 0 \to M_{-n} \xrightarrow{f_{-n}} \ldots \xrightarrow{f_{n-1}} M_n \to 0 \to \cdots]$ be a complex of discrete (left) Γ-modules, where M_i is in degree i.

For a (left) Γ-module M, we employ the following notation for its cohomology, which we compute through the standard resolution (see, for instance, [NSW08, Sec. 1.2]):

i. for $j \geq 0$, $C^j := C^j(\Gamma, M)$ denotes the abelian group $\mathrm{Fun}(\Gamma^j, M)$ (and we will denote functions in $\mathrm{Fun}(\Gamma^j, M)$ with the notation $\alpha_{\sigma_1,\ldots,\sigma_j}$), for $j < 0$ we set $C^j := 0$;

ii. $X^j := X^j(\Gamma, M)$ denotes the (acyclic) Γ-module $\mathrm{Fun}(\Gamma^{j+1}, M)$ (and we will denote functions in $X^j(\Gamma, M) = \mathrm{Fun}(\Gamma^{j+1}, M)$ with the notation $\alpha(\sigma_0, \ldots, \sigma_j)$), for $j < 0$ we set $X^j := 0$;

iii. $\partial^j : C^j \to C^{j+1}$ denotes the morphism

$$\left(\partial^j \alpha\right)_{\sigma_1,\ldots,\sigma_{j+1}} := {}^{\sigma_1}\alpha_{\sigma_2,\ldots,\sigma_{j+1}} + \sum_{i=1}^{j} (-1)^i \alpha_{\sigma_1,\ldots,\sigma_{i-1},\sigma_i\sigma_{i+1},\ldots,\sigma_{j+1}} + (-1)^{j+1}\alpha_{\sigma_1,\ldots,\sigma_j},$$

and $\partial_X^j : X^j \to X^{j+1}$ denotes the morphism

$$\left(\partial^j \alpha\right)(\sigma_0, \ldots, \sigma_{j+1}) := \sum_{i=0}^{j+1} (-1)^i \alpha(\sigma_0, \ldots, \hat{\sigma}_i, \ldots, \sigma_{j+1});$$

iv. $Z^j := Z^j(\Gamma, M) := \mathrm{Ker}(\partial^j)$, and $B^j := \mathrm{Im}(\partial^{j-1})$ (and $B^0 := 0$), and $H^j(\Gamma, M) := {}^{Z^j}\!/_{B^j}$;

v. for an element $\beta \in Z^j$ we denote by $[\beta]$ its image in H^j;

vi. For a pairing $\star \cdot \star : M \times M' \to M''$, $\alpha \in C^j(\Gamma, M)$ and $\beta \in C^k(\Gamma, M')$, $\alpha \cup \beta \in C^{j+k}(\Gamma, M'')$ denotes the cocycle $(\alpha \cup \beta)_{\sigma_1,\ldots,\sigma_{j+k}} = \alpha_{\sigma_1,\ldots,\sigma_j} \cdot {}^{\sigma_1\cdots\sigma_j}\beta_{\sigma_{j+1},\ldots,\sigma_{j+k}}$.

We recall that, for each $j \geq 0$, there is an isomorphism $(X^j(\Gamma, M))^\Gamma \cong C^j(\Gamma, M)$, to be found, for instance in [NSW08, p. 14]. These isomorphisms commute with the differentials ∂^j/∂_X^j.

Remark 3.3.1. When Γ pseudo-acts on M (instead of acting, see Section 3.2), we still use the above notation, which keeps making sense (even though in this case it is not directly related to any cohomology construction, at least to the author's knowledge).

For the complex $D^\bullet$, we may compute its hypercohomology using the acyclic resolution $D^\bullet \to \mathrm{Tot}^\bullet(X^\bullet(\Gamma, D^\bullet))$, where the latter denotes the totalizing complex of the double complex $X^\bullet(\Gamma, D^\bullet)$. We recall that, for a bicomplex $\mathcal{C} = C^{\bullet,\bullet}$ (with differentials $\partial_1 : C^{\bullet,\bullet} \to C^{\bullet+1,\bullet}, \partial_2 : C^{\bullet,\bullet} \to C^{\bullet,\bullet+1}$), the totalizing complex (see e.g. [Wei94, p. 8]) $\mathrm{Tot}^\bullet(\mathcal{C})$ is defined through $\mathrm{Tot}^n(\mathcal{C}) := \oplus_{i+j=n} C^{i,j}$, with differential $(-1)^j\partial_1 + \partial_2$. This resolution realizes the hypercohomology $H^j(\Gamma, C)$ as the cohomology of a complex $(C^j(\Gamma, C), \partial^j)_{j\in\mathbb{Z}}$, defined as follows.

i. For $j \in \mathbb{Z}$, $C^j := C^j(\Gamma, D^\bullet) := C^{j-n}(\Gamma, M_{-n}) \bigoplus \cdots \bigoplus C^{j+n}(\Gamma, M_n)$; we indicate an element $\underline{\alpha} \in C^j$ with the notation $\underline{\alpha} = (\alpha_{-n}; \alpha_{-n+1}; \cdots; \alpha_n)$,

ii. $\partial^j : C^j \to C^{j+1}$ denotes the morphism

$$\partial^j(\alpha_{-n}; \alpha_{-n+1}; \cdots; \alpha_n) = (\partial^j\alpha_n; (-1)^{j-n} f_{-n}(\alpha_{-n}) + \partial^{j+n-1}\alpha_{-n+1}; \cdots; (-1)^{j+n-1} f_{n-1}(\alpha_{n-1}) + \partial^{j+n}\alpha_n)$$

We also employ the following notation:

iii. the j-cocyles are $Z^j := Z^j(\Gamma, D^\bullet) := \mathrm{Ker}(\partial^j)$, the j-coboundaries are $B^j := \mathrm{Im}(\partial^{j-1})$, the j-hypercohomology is $\mathbb{H}^j(\Gamma, D^\bullet) := Z^j / B^j$;

iv. for an element $\beta \in Z^j$ we denote by $[\beta]$ its image in $\mathbb{H}^j$.

Notation 3.3.2. When $\alpha, \beta \in C^j(\Gamma, D^\bullet)$ we use the notation $\alpha \triangleq \beta$ to mean that $\alpha - \beta \in B^j(\Gamma, D^\bullet)$.

Notation 3.3.3. To avoid having too many subscripts, in the course of the proof of Theorem 3.5.1 we will use σ, η instead of σ_1, σ_2.

Remark 3.3.4. Let $M[2] \xrightarrow{i} C := [M_{-2} \xrightarrow{f_2} M_{-1} \xrightarrow{f_1} M_0]$ be a morphism of complexes of Γ-modules that is a quasi-isomorphism, and $\alpha = (a_{\sigma_1,\sigma_2}, b_\sigma, c) \in Z^0(\Gamma, C)$. Let $b'_\sigma \in \mathrm{Fun}(\Gamma, M_{-2})$ be such that $f_2(b'_\sigma) = b_\sigma - (\partial c')$, where $c' \in M_{-1}$ is such that $f_1(c') = c$. Let $\beta \in Z^2(\Gamma, M) = Z^0(\Gamma, M[2])$ be such that $i_*(\beta) = a_{\sigma_1,\sigma_2} + (\partial b'_\sigma) \in Z^0(\Gamma, C)$, then it satisfies that $i_*[\beta] = [\alpha] \in \mathbb{H}^0(\Gamma, C)$.

The following lemma is well-known (since it basically just unravels the definition of derived cup product in a special case), however the author was not able to find a reference with the explicit signs, which will be needed for the computation in the proof of Theorem 3.5.1.

Lemma 3.3.5. *Let* $\mathcal{C} := [\cdots \to C_n \xrightarrow{d} C_{n+1} \to \ldots]$ *and* $\mathcal{C}' := [\ldots \xrightarrow{d} C'_n \to C'_{n+1} \to \ldots]$ *be two bounded complexes of* Γ_k*-modules, and let*

$$C_i \otimes C'_{-i} \xrightarrow{(.)} \bar{k}^*, \ i \in \mathbb{Z}, \tag{3.3.2}$$

be compatible pairings of Γ_k*-modules, i.e. such that the following pairing diagrams are commutative for all* $i \in \mathbb{Z}$*:*

$$\begin{array}{ccccc} C_i & \times & C'_{-i} & \longrightarrow & \bar{k}^* \\ \downarrow d & & \uparrow d & & \| \\ C_{i+1} & \times & C'_{-i-1} & \longrightarrow & \bar{k}^* \end{array} .$$

Then, we have a (canonical) pairing:

$$\mathbb{H}^i(k, \mathcal{C}) \otimes \mathbb{H}^j(k, \mathcal{C}') \xrightarrow{\cup} \mathbb{H}^{i+j}(k, \bar{k}^*),$$

that is induced from the following map on the level of cochains:

$$\begin{array}{ccccc} C^i(k,\mathcal{C}) & \otimes & C^j(k,\mathcal{C}') & \overset{\cup}{\to} & C^{i+j}(k,\bar{k}^*), \\ (\alpha_h)_{h\in\mathbb{Z}} & \otimes & (\alpha'_h)_{h\in\mathbb{Z}} & \mapsto & \sum_{h\in\mathbb{Z}}(-1)^{jh+\binom{h}{2}}\alpha_h\cup\alpha_{-h}. \end{array} \tag{3.3.3}$$

Moreover, the pairing induced by (3.3.3) coincides with the derived cup-product associated with the derived pairing $\mathcal{C}\otimes^L\mathcal{C}'\to \mathrm{Tot}^\bullet(\mathcal{C}^\bullet\otimes\mathcal{C}'^\bullet)\to\overline{k}^*[0]$ *induced by the pairings (3.3.2).*

Proof. This is basically just unraveling the definitions of derived cup-product. We recall that, if we have a morphism $\mathrm{Tot}^\bullet(\mathcal{C}^\bullet\otimes\mathcal{C}'^\bullet)\to K^\bullet$, this induces a morphism $\mathcal{C}^\bullet\otimes^L\mathcal{C}'^\bullet\to\mathrm{Tot}^\bullet(\mathcal{C}^\bullet\otimes\mathcal{C}'^\bullet)\to K^\bullet$. We denote by $R^\bullet\Gamma : D(\mathrm{Ab}_{\Gamma_k})\to D(\mathrm{Ab})$ the functor that computes group cohomology of Γ_K-modules through the standard resolution (so, as recalled in Subsection 3.3.1). The derived cup-product is then defined (in this case) as the following composition:

$$\mathrm{Tot}^\bullet(H^\bullet(R^\bullet\Gamma\mathcal{C})\otimes H^\bullet(R^\bullet\Gamma\mathcal{C}'))\to H^\bullet(\mathrm{Tot}^\bullet(R^\bullet\Gamma\mathcal{C}\otimes R^\bullet\Gamma\mathcal{C}'))\to H^\bullet(R^\bullet\Gamma(\mathrm{Tot}^\bullet(\mathcal{C}\otimes\mathcal{C}')))\to H^\bullet(R^\bullet\Gamma(K^\bullet)).$$

Unraveling the morphisms above with the correct signs, reveals that the pairing is exactly the one in (3.3.3) (the correct signs can also be found by checking that (3.3.3) is the only choice of signs compatible with the Leibniz rule for cup products, and with trivial signs at the degree 0 level). □

3.3.2 Reminders on abelian(ized) cohomology

We briefly recall here a construction of Demarche [Dem13], as well as his main result in *loc. cit.*. We refer the reader to his paper for a more detailed exposition and for proofs of the results that follow.

In what follows X will always denote a quotient G/H, where G is a connected k-group, and H is a connected linear k-closed subgroup.

Construction of abelianized cohomology We assume first that G^{lin} is reductive. Let $H^{sc}\xrightarrow{\rho'_H}H^{red}$ and $G^{sc}\xrightarrow{\rho_G}G$ be as in Section 3.2, let $s : H^{red}\to H$ be a section of the projection $H\to H^{red}$, and let ρ_H be the composition $H^{sc}\xrightarrow{\rho'_H}H^{red}\xrightarrow{s}H$. Let $T_H\subset T_G$ be two maximal tori in, respectively, $s(H^{red})$ and G. Let SA_G be a maximal semi-abelian subvariety of G containing T_G, and let T_H^{sc} and T_G^{sc} be, respectively, the tori $T_H\times_H H^{sc}$ and $SA_G\times_G G^{sc}$.

Let C_X be the cone of the following morphism of complexes:

$$[T_H^{sc}\xrightarrow{\rho_H}T_H]\xrightarrow{[\iota^{sc},\iota]}[T_G^{sc}\xrightarrow{\rho_G}SA_G],$$

where both complexes start in degree -1, $\iota : H\to G$ denotes the closed embedding and $\iota^{sc} : H^{sc}\to G^{sc}$ denotes the unique morphism of groups such that $\rho_G\circ\iota^{sc}=\iota\circ\rho_H$. We call C_X the *abelianized complex* of X. Moreover, we set the notation $C_H:=[T_H^{sc}\xrightarrow{\rho_H}T_H]$ and $C_G:=[T_G^{sc}\xrightarrow{\rho_G}SA_G]$.

For a field $F\supset k$, we denote the (Galois) hypercohomology of C_X by $\mathbb{H}^i_{ab}(F,X):=\mathbb{H}^i(F,C_X)$, and we refer to it as the *abelianized cohomology* of X.

When G^{lin} is not reductive, let $G':=G/G^u$, where $G^u\subset G$ denotes the unipotent radical of G, let $H_1:=\iota(H)/(\iota(H)\cap G^u)$, and let $X':=G'/H_1$. There exists a natural surjection $X\to X'$. One can repeat the above "abelianization" construction for X', and defines $\mathbb{H}^i_{ab}(F,X):=\mathbb{H}^i(F,C_{X'})$. We refer to [Dem13] for more details on this.

Abelianization morphism For each field $F\supset k$, there exists an *abelianization* map [Dem13]:

$$\mathrm{ab}^0_F : X(F)\to\mathbb{H}^0(F,C_X). \tag{3.3.4}$$

The map (3.3.4) factors (by construction) through $X'(F)$ as follows: $X(F) \to X'(F) \to \mathbb{H}^0(F, C_{X'})$. When F is a local field there is a natural topology on $\mathbb{H}^0(F, C_X)$, see [Dem13, p. 20].

With the above notation, let $k = K$ be a number field. We choose smooth group-scheme models, over $\operatorname{Spec} O_{K,S}$, where S is some finite set of primes of K, $\tilde{\iota} : \mathcal{H} \to \mathcal{G}$, for H, G and the closed embedding ι. We define $\mathcal{X} := \mathcal{G}/\mathcal{H}$, and define $C_{\mathcal{X}}$ to be the abelianized complex of $\mathcal{X}$. This is a complex of fppf sheaves on $\operatorname{Spec} O_{K,S}$, see [Dem13, Sec. 2.4.1] for the construction of $C_{\mathcal{X}}$ and of the abelianization map $\mathcal{X}(Z) \xrightarrow{\mathrm{ab}^0_{\mathcal{Z}}} \mathbb{H}^0_{fppf}(Z, C_{\mathcal{X}}) := \mathbb{H}^0_{fppf}(Z, p_Z^* C_{\mathcal{X}})$ for an $O_{K,S}$-scheme $p_Z : Z \to \operatorname{Spec} O_{K,S}$. Importantly, $\mathrm{ab}^0_{\mathcal{Z}}$ is functorial in $\mathcal{Z}$. Note that $\mathbb{H}^0_{fppf}(Z, C_{\mathcal{X}}) = \mathbb{H}^0_{\acute{e}t}(Z, C_{\mathcal{X}})$, since $C_{\mathcal{X}}$ is a complex of smooth group schemes.

We use the notation $\mathbb{P}^i(K, C) := \prod'_{v \in M_F} \mathbb{H}^i(F_v, C)_\bullet$, where the restricted product is taken over $\mathbb{H}^i(O_v, C) \to \mathbb{H}^i(F_v, C)$ (after an implied choice of an integral model for C has been made) , and $\mathbb{H}^i(F_v, C)_\bullet$ denotes the usual hypercohomology for $v \in M_K^{fin}$ and hypercohomology modified à la Tate (as defined in [HS05, p. 103]) for $v \in M_K^\infty$.

The following lemma is implicit in [Dem13]:

Lemma 3.3.6. *The following hold:*

i. *For a non-archimedean $v \in M_K$, and $F = K_v$, the local abelianization map (3.3.4) $ab^0_{K_v}$ is surjective;*

ii. *For all $v \in M_K$, $F = K_v$, $ab^0_{K_v}$ is continuous and open, and for $v \notin S \cup M_K^\infty$, $(ab^0_{K_v})(\mathcal{X}(O_v)) = \operatorname{Im} \mathbb{H}^0(O_v, C_{\mathcal{X}}) \subset \mathbb{H}^0(K_v, C_X)$;*

Moreover we have that the restricted product morphism $ab^0 : X(\mathbb{A}_K)_\bullet \to \mathbb{P}^0(K, C_X)$ is continuous and open.

Proof. The last statement is an immediate consequence of the points above. For the first point and the first part of the second point above see [Dem13, Cor. 2.21] and [Dem13, p. 21]. We have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{X}(O_v) & \xrightarrow{\mathrm{ab}^0_{O_v}} & \mathbb{H}^0(O_v, C_{\mathcal{X}}) \\ \downarrow & & \downarrow \\ X(K_v) & \xrightarrow{\mathrm{ab}^0_{K_v}} & H^0(K_v, C_X) \end{array} .$$

The inclusion $\mathrm{ab}^0_{O_v}(\mathcal{X}(O_v)) \subset \operatorname{Im} \mathbb{H}^0(O_v, C_{\mathcal{X}})$ follows. For the identity, see the following diagram (which is commutative by [Dem13, Thm. 2.9]) with exact (as pointed-sets) rows:

$$\begin{array}{ccccccc} \mathcal{H}(O_v) & \longrightarrow & \mathcal{G}(O_v) & \longrightarrow & \mathcal{X}(O_v) & \longrightarrow & H^1(O_v, \mathcal{H}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathbb{H}^0(O_v, C_{\mathcal{H}}) & \longrightarrow & \mathbb{H}^0(O_v, C_{\mathcal{G}}) & \longrightarrow & \mathbb{H}^0(O_v, C_{\mathcal{X}}) & \longrightarrow & \mathbb{H}^1(O_v, C_{\mathcal{H}}) \end{array} ,$$

and the following short exact sequence of pointed sets:

$$\mathcal{G}(O_v) \to \mathbb{H}^0(O_v, \mathcal{C}_{\mathcal{G}}) \to H^1(O_v, \mathcal{G}^{sc})$$

and notice that $H^1(O_v, \mathcal{H}) = 0$, $H^1(O_v, \mathcal{G}^{sc}) = 0$ by [PRR93, Thm 6.1]. Moreover, by [PRR93, Thm 6.1] again, $H^1(O_v, T_{\mathcal{H}}) = 0$ and $H^1(O_v, T_{\mathcal{H}^{sc}}) = 0$. We also have $H^2(O_v, T_{\mathcal{H}^{sc}}) = 0$ because $H^2(O_v, (T_{\mathcal{H}^{sc}})_{tor}) \twoheadrightarrow H^2(O_v, T_{\mathcal{H}^{sc}})$, where, for a commutative algebraic group scheme $G \to S$ the subscript *tor* defines the increasing limit $\lim_{n\to\infty} G[n]$ (the limit here is taken with respect to the partial order $n \prec m$ if $n|m$), and $H^2(O_v, (T_{\mathcal{H}^{sc}})_{tor}) = 0$ because the cohomological dimension of $\operatorname{Spec} \mathbb{F}_v$ is 1. Hence $\mathbb{H}^1(O_v, \mathcal{C}_{\mathcal{H}}) = \mathbb{H}^1(O_v, [T_{\mathcal{H}^{sc}} \to T_{\mathcal{H}}]) = 0$. Hence, the surjectivity of $X(O_v) \to \mathbb{H}^0(O_v, \mathcal{C}_{\mathcal{X}})$ follows. □

Relation to the Brauer–Manin obstruction Let C_X^d be the dual complex of C_X (this complex is described explicitly in [Dem13, Sec. 5], and its definition will be recalled in Subsection 3.5.1, diagram (3.5.6)). We have, for every field $F \supset k$, a pairing (see Lemma 3.3.5):

$$\mathbb{H}^0(F, C_X) \times \mathbb{H}^2(F, C_X^d) \to \mathbb{H}^2(F, \overline{F}^*) = \mathrm{Br}(F). \tag{3.3.5}$$

When F is a local field, (3.3.5) induces, through the local invariant $\mathrm{inv}_v : \mathrm{Br}(F) \to \mathbb{Q}/\mathbb{Z}$, a pairing:

$$\mathbb{H}^0(F, C_X)^\wedge \times \mathbb{H}^2(F, C_X^d) \to \mathbb{Q}/\mathbb{Z}, \tag{3.3.6}$$

which is perfect when $v \in M_K^{fin}$ (since C_X is a cone of a complex of 1-motives, this basically follows by devissage from [HS05, Thm 0.1], and is proved explicitly in Lemma 3.6.9). For a number field K, the local pairing (3.3.6) on its completions, induces a morphism (see [Dem13, p.21]):

$$\mathbb{P}^0(K, C_X)^\wedge \xrightarrow{\vartheta} \mathbb{H}^2(K, C_X^d)^D. \tag{3.3.7}$$

Demarche [Dem11] defined a morphism

$$\alpha : \mathbb{H}^2(k, C_X^d) \to \mathrm{Br}_a(X_k, G), \tag{3.3.8}$$

where $\mathrm{Br}_1(X,G) \coloneqq \mathrm{Ker}(\mathrm{Br}(X) \to \mathrm{Br}(\bar{G})) \subset \mathrm{Br}(X)$ and $\mathrm{Br}_a(X,G) \coloneqq \mathrm{Br}_1(X,G)/\mathrm{Br}(k)$ (we refer to *loc. cit.* or to Subsection 3.5.1 for more details). We will often tacitly identify $\mathrm{Br}_a(X,G)$ with $\mathrm{Ker}(e^* : \mathrm{Br}_1(X,G) \to \mathrm{Br}(k))$, where $e : \mathrm{Spec}\, k \to G/H$ is the trivial element.

Moreover, Demarche proved that α sits in the following exact sequence (which will not be needed in this thesis, but is included for completeness):

$$\mathrm{NS}(\bar{G}^{\mathrm{ab}})^{\Gamma_k} \to \mathbb{H}^2(k, C_X^d) \xrightarrow{\alpha} \mathrm{Br}_a(X,G) \to H^1(k, \mathrm{NS}(\bar{G}^{\mathrm{ab}})).$$

In Section 3.5 we will prove Theorem 3.5.1, which proves that α is compatible with the Brauer–Manin and local pairings.

The following is the main theorem of [Dem13].

Theorem 3.3.7 (Demarche)**.** *Let K be a number field, G a connected K-group, S a finite set of places of K. Let H be a connected linear K-subgroup of G, and let $X \coloneqq G/H$. We assume that the group G^{sc} satisfies strong approximation outside S, and that $\text{Ш}(K, G^{ab})$ is finite. Let C_X^d and $\vartheta : X(\mathbb{A}_K)_\bullet \to \mathbb{H}^2(K, C_X^d)^D$ be defined as above. Then the kernel of ϑ (i.e. $\vartheta^{-1}(\{0\})$) is the closure of $G^{scu}(K_{S_f}) \cdot X(K)$ in $X(\mathbb{A}_K)_\bullet$.*

Remark 3.3.8. As remarked by Demarche in [Dem13, Rmq 6.4], Theorem 3.3.7 implies the main theorem of [BD13], once a suitable compatibility of Brauer and local duality pairings is proven. The compatibility needed is exactly the one proven in Theorem 3.5.1. In fact, in Section 3.6, this connection will be made explicit in Theorem 3.6.1 and Remark 3.6.4.

3.3.3 Reminders on the morphism α

In this subsection we recall part of the construction of α (see (3.3.8)), given in [Dem11], of which we borrow the notation. Throughout H will be a linear connected subgroup of a connected algebraic group G with G^{lin} reductive, both defined over a field k.

We fix a Levi decomposition of $H = H^u \cdot H^{red}$, such that $Z_{H^{\mathrm{red}}}$ is contained in the maximal torus T_G of G^{lin} (keeping the notation of Subsection 3.3.2). We put $Z' \coloneqq T_G/Z_{H^{\mathrm{red}}}$, $Z_1 \coloneqq G/Z_{H^{\mathrm{red}}}$, $H' \coloneqq H^{\mathrm{red}}/Z_{H^{\mathrm{red}}}$, $\widetilde{H} \coloneqq \mathrm{Ker}(H \to H') = H^u \cdot Z_{H^{\mathrm{red}}}$, and $Z \coloneqq G/\widetilde{H}$. We have a natural morphism $Z_1 \to Z$ that gives Z_1 the structure of a H^u-torsor over Z. We notice that $X = Z/H'$, and that we have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccc} H^u & \hookrightarrow & \tilde{H} & \twoheadrightarrow & Z_{H^{red}} \\ \| & & \downarrow & & \downarrow \\ H^u & \hookrightarrow & H & \twoheadrightarrow & H^{red} \\ & & \downarrow & & \downarrow \\ & & H' & = & H' \end{array}$$

We use the following notation

$$Q_{geom} := [\bar{k}(Z)^* \to \mathrm{Div}(\bar{Z}) \to \mathrm{Pic}'(\bar{Z}/\bar{X})],$$

where the complex ends in degree 2. We refer to [Dem11, p. 4] for the definition of $\mathrm{Pic}'(\bar{Z}/\bar{X})$, which we remind being isomorphic to $\mathrm{Pic}(\overline{H'})$.

Remark 3.3.9. We recall from [Dem11, p. 5, Rmk. 1.3] that we may define the complex Q_{geom} (in the same way as above) letting Z be, instead of the quotient $G/\tilde{H}$, *any* form of the (right) $\overline{H}'$-torsor $\overline{G/\tilde{H}} \to \overline{X}$.

The following proposition is the first step in Demarche's construction of the aforementioned morphism α. We refer to [Wei94, p. 9] for a definition of the truncation operator. The following is consequence of [Dem11, Thm. 2.1, Prop. 2.2]:

Proposition 3.3.10 (Demarche)**.** *We keep the above notation. Let $p : Z \to X$ denote the natural projection, and $p_X : X \to \mathrm{Spec}\, k$ the structural morphism. There exists a natural morphism*

$$\tau_{\leq 2}\mathbf{R}p_{X_*}\mathbf{G}_{mX} \leftarrow Q_{geom}, \tag{3.3.9}$$

which induces an isomorphism:

$$\mathbb{H}^2(k, Q_{geom}) \xrightarrow{\sim} \mathrm{Br}_1(X, G). \tag{3.3.10}$$

The second step is a morphism $\mathbb{H}^2(k, C_X^d) \to \mathbb{H}^2(k, Q_{geom})/\operatorname{Br} k \cong \mathrm{Br}_a(X, G)$, but this will be recalled in detail in the Section 3.5, so we skip its construction for now.

3.4 Sufficiency of the étale-Brauer–Manin obstruction

In this section, we prove Theorem 3.1.1.

Remark 3.4.1. In the proof of Theorem 3.1.1 we are going to use the aforementioned result of Borovoi and Demarche. Their main theorem in [BD13] can be stated exactly as Theorem 3.1.1, but restricting to the case when X has connected geometric stabilizers, and substituting $X(\mathbb{A}_K)_\bullet^{\text{ét,Br}}$ with $X(\mathbb{A}_K)_\bullet^{\mathrm{Br}\, X}$.

Example 3.4.2. This example is borrowed from [Dem17, Thm 2.1]. Let p be a prime, and H a finite constant non-commutative group of order p^n, such that the p^{n+1}-roots of unity are contained in a number field K. Let $X := G/H$, where G/K is any semisimple simply connected algebraic group, and $H \hookrightarrow G$ is any embedding. Then, for any S, one has that $\pi^S(X(\mathbb{A}_K)^{\mathrm{Br}\, X}) \neq \overline{X(K)}^S$ (where $\overline{X(K)}^S$ denotes the closure of $X(K)$ in $X(\mathbb{A}_K^S)$, and $\pi^S : X(\mathbb{A}_K) \to X(\mathbb{A}_K^S)$ denotes the projection). In particular, in general, one could not hope for the statement of Theorem 3.1.1 to be true with $X(\mathbb{A}_K)_\bullet^{\text{ét,Br}}$ replaced by $X(\mathbb{A}_K)_\bullet^{\mathrm{Br}\, X}$, i.e. the Brauer–Manin obstruction is *not* the only one to strong approximation for homogeneous space.

Remark 3.4.3. In view of the example above, it would be interesting to know if, for a general homogeneous space X, there are any intermediate obstructions $X(\mathbb{A}_K)_\bullet^{\text{ét,Br}} \subset X(\mathbb{A}_K)_\bullet^{?} \subset X(\mathbb{A}_K)_\bullet^{\mathrm{Br}\, X}$ such that $G^{scu}(K_{S_f}) \cdot X(K)$ is dense in $X(\mathbb{A}_K)_\bullet^{?}$.

Remark 3.4.4. The group $Ш(K, G^{ab})$ is defined as the kernel of the map

$$H^1(K, G^{ab}) \to \prod_{v \in M_K} H^1(K_v, G^{ab}).$$

It is strongly conjectured to be always finite, and it is known to be in some specific cases, for instance, when G^{ab} is an elliptic curve of analytic rank 0 or 1 defined and $K = \mathbb{Q}$ (see [Kol88]).

Remark 3.4.5. We remind the reader of the aforementioned Theorem of Platonov [PRR93, Theorem 7.12], which states that $G^{sc}(K)$ is dense in $G^{sc}(\mathbb{A}_K^S)$ if and only if G^{sc} has no K-simple component $G_i \subset G^{sc}$ with $G_i(K_S)$ compact. This makes the hypothesis "$G^{sc}(K)$ is dense in $G^{sc}(\mathbb{A}_K^S)$" of Theorem 3.1.1 easily verifiable.

Remark 3.4.6. We note that, if $x \in X(\mathbb{A}_K)_\bullet$ is in $X(\mathbb{A}_K)_\bullet^{\text{ét,Br}}$, then Theorem 3.1.1 tells us that it lies in the closure of $G^{scu}(K_{S_f}) \cdot X(K)$. Hence, its projection x^{S_f} to $X(\mathbb{A}_K^{S_f})_\bullet$ lies in the closure of $X(K)$ in $X(\mathbb{A}_K^{S_f})_\bullet$ (since the projection of $G^{scu}(K_{S_f})$ is trivial). Therefore, Theorem 3.1.1 may be seen as a theorem saying that, under its assumptions, the étale-Brauer–Manin obstruction to strong approximation is the only one for homogeneous spaces.

Remark 3.4.7. We notice that, although, by the previous remark, we have that, for a finite $S \subset M_K^{fin}$ large enough, $\overline{X(K)}^S \supset \pi^S(X(\mathbb{A}_K)_\bullet^{\text{ét,Br}})$ (where $\pi^S : X(\mathbb{A}_K)_\bullet \to X(\mathbb{A}_K^S)_\bullet$ denotes the standard projection and $\overline{\star}^S$ denotes the closure in the S-adeles), this is not necessarily an equality, as the example presented in Proposition 3.4.8, which follows, clearly points out. However, one can still aim to describe the set $\overline{X(K)}^S \subset X(\mathbb{A}_K^S)_\bullet$, although it becomes a less trivial consequence of Theorem 3.1.1. This is done in Section 3.6.

Proposition 3.4.8. *Keeping the notation of the above remark, let $K = \mathbb{Q}$, $X = \mathbb{G}_m/\mathbb{Q}$ and $S = \{2\}$. We have that $X(\mathbb{A}_K)_\bullet^{\text{ét,Br}} = X(\mathbb{A}_\mathbb{Q})_\bullet^{\text{Br}\,X} = \overline{X(\mathbb{Q})} = X(\mathbb{Q}) = \mathbb{Q}^*$, while $\overline{X(\mathbb{Q})}^S \subset X(\mathbb{A}_\mathbb{Q}^S)_\bullet$ is not countable.*

Proof. We have the following inclusions:

$$X(\mathbb{A}_\mathbb{Q})_\bullet^{\text{Br}\,X} \supset X(\mathbb{A}_\mathbb{Q})_\bullet^{\text{ét,Br}\,X} \supset X(\mathbb{Q}) = \mathbb{Q}^*.$$

Moreover, by [Har08, Thm. 4] (applied to the case $X = \mathbb{G}_m$), we have that $X(\mathbb{A}_\mathbb{Q})_\bullet^{\text{Br}\,X} = \mathbb{G}_m(\mathbb{A}_\mathbb{Q})_\bullet^{\text{Br}\,\mathbb{G}_m} \subset \overline{\mathbb{Q}^*} = \mathbb{Q}^*$ (the closure being in the idelic topology of $(\mathbb{I}_\mathbb{Q})_\bullet = (\mathbb{G}_m)(\mathbb{A}_\mathbb{Q})_\bullet)$. Hence we have that all the inclusions above are equalities.

On the other hand, we have that $\overline{X(\mathbb{Q})}^S$, being the closure of $\mathbb{Q}^*$ in $(\mathbb{I}_\mathbb{Q}^S)_\bullet$, is equal to $\mathbb{Q}^* \cdot 2^{\hat{\mathbb{Z}}}$, where the embedding $2^{\hat{\mathbb{Z}}} \hookrightarrow (\mathbb{I}_\mathbb{Q}^S)_\bullet$ is defined as described in the next paragraph.

The morphism

$$2^\star : \mathbb{Z} \to (\mathbb{I}_\mathbb{Q}^S)_\bullet, \quad n \mapsto 2^n,$$

is continuous if we endow $\mathbb{Z}$ with the profinite topology (i.e. the one induced by the embedding $\mathbb{Z} \hookrightarrow \widehat{\mathbb{Z}}$, where $\widehat{\mathbb{Z}}$ is endowed with its profinite topology) and $(\mathbb{I}_\mathbb{Q}^S)_\bullet$ with its natural topology. Therefore, since $(\mathbb{I}_\mathbb{Q}^S)_\bullet$ is complete, there is a unique continuous extension of $2^\star$ to $\hat{\mathbb{Z}}$, which defines an embedding $2^{\hat{\mathbb{Z}}} \hookrightarrow (\mathbb{I}_\mathbb{Q}^S)_\bullet$.

□

3.4.1 Preliminaries for the proof of Theorem 3.1.1

We shall need the following more or less standard facts about Weil restriction (we recall that, according to our notation, Sch_k denotes quasi-projective $\mathrm{Spec}\, k$-schemes):

Proposition 3.4.9. *Let L/k be a finite Galois extension with Galois group Γ.*

(i) The functor $R_{L/k} : \mathrm{Sch}_L \to \mathrm{Sch}_k$ is right-adjoint to the base-change functor $\mathrm{Sch}_k \to \mathrm{Sch}_L$.

(ii) For every k-variety $X \in \mathrm{Sch}_k$ there exists a closed embedding $\iota_X : X \hookrightarrow R_{L/k}X_L$. Moreover, the collection $\iota_\star, \star \in \mathrm{Sch}_k$ is a natural transformation between the identity functor on Sch_k and $R_{L/k} \circ (\star \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, L)$.

(iii) For every L-variety $Y \in \mathrm{Sch}_L$ there is an isomorphism:

$$\psi^Y = (\psi^Y_\gamma) : (R_{L/k}Y)_L \longrightarrow \prod_{\gamma \in \Gamma} Y^\gamma.$$

Moreover, these morphisms form a natural transformation between the functors $Y \mapsto (R_{L/k}Y)_L$ and $Y \mapsto \prod_{\gamma\in\Gamma} Y^\gamma$.

(iv) For every $X \in \mathrm{Sch}_k$, one has that $\psi^{X_L} \circ (\iota_X)_L = \Delta_{X_L}$, where $\Delta_{X_L} : X_L \to \prod_{\gamma\in\Gamma}(X_L)^\gamma$ denotes the diagonal embedding.

In point (iv) we are implicitly using the fact that, since X is defined over k, then, for every $\gamma \in \Gamma$, there is a natural identification between ${X_L}^\gamma$ and X_L.

Proof.

(i) This is the definition of Weil restriction, which exists by [Sch94, Cor. 4.8.1].

(ii) See [Sch94][4.2.5].

(iii) See [Sch94][4.11.3].

(iv) The morphism $\pi : X_L \to X$ induces a base changed morphism $\pi_L : (X_L)_L \to X_L$. Since L/k is Galois, one may identify $(X_L)_L$ with $\coprod_{\gamma\in\Gamma}(X_L)_\gamma$, which may again be naturally identified with $\coprod_{\gamma\in\Gamma}(X_L)$. Using this identification, the morphism π_L corresponds to the codiagonal morphism.
This identification induces the following commutative diagram:

$$\begin{array}{ccccccc}
\mathrm{Hom}_k(X, R_{L/k}X_L) & \xrightarrow{\pi^*} & \mathrm{Hom}_k(X_L, R_{L/k}X_L) & = & \mathrm{Hom}_L(X_L, (R_{L/k}X_L)_L) & \xrightarrow{(\psi^{X_L})_*} & \mathrm{Hom}_L(X_L, \prod_{\gamma\in\Gamma}(X_L)^\gamma) \\
\uparrow & & \uparrow & & & & \| \\
\mathrm{Hom}_L(X_L, X_L) & \xrightarrow{\pi_L^*} & \mathrm{Hom}_L((X_L)_L, X_L) & & \xrightarrow{\sim} & & \mathrm{Hom}_L(\coprod_{\gamma\in\Gamma}(X_L)^{\gamma^{-1}}, X_L),
\end{array} \tag{3.4.1}$$

where the first two vertical morphisms are the ones induced from the definition of $R_{L/k}$. The commutativity of the first square follows from the definition of $R_{L/k}$, while the commutativity of the second square is the definition of ψ^{X_L} (see [Sch94][4.11.3]).

Point (iv) now follows from considering the identity morphism in the bottom left corner of (3.4.1), and looking at its image in the top right corner of the same diagram following the two distinct paths up-right-right and right-right-up. □

Remark 3.4.10. By Proposition 3.4.9(i) the functor $R_{L/k}$ preserves (fibered) products. Hence, for every couple (Y_1, Y_2) of L-varieties, the morphism $R_{L/k}(Y_1 \times_L Y_2) \xrightarrow{R_{L/k}\pi_1 \times R_{L/k}\pi_2} R_{L/k}Y_1 \times_k R_{L/k}Y_2$ is an isomorphism.

Remark 3.4.11. We observe that, if $m : G_L \times_L Y \to Y$ is an action of G_L on Y, there is a natural action of $R_{L/k}G_L$ on $R_{L/k}Y$ defined by the following composition:

$$R_{L/k}G_L \times_k R_{L/k}Y \xrightarrow{(R_{L/k}\pi_1 \times R_{L/k}\pi_2)^{-1}} R_{L/k}(G_L \times_L Y) \xrightarrow{R_{L/k}m} R_{L/k}Y.$$

Moreover, by functoriality of $R_{L/k}$, this induced action has the property that, for each G_L-equivariant morphism $f : Y_1 \to Y_2$ between G_L-varieties, the morphism $R_{L/k}f$ is a $R_{L/k}G_L$-equivariant morphism.

Let G be a k-algebraic group, L/k be a finite Galois extension, with Galois group Γ, and let Y/L be an L-variety endowed with a (left) G_L-action. We observe that Proposition 3.4.9(ii) (applied on G) gives a natural embedding $G \hookrightarrow R_{L/k}G_L$, which one can easily verify (using the fact that the collection of morphisms $\iota_\star, \star \in \mathrm{Sch}_k$ is a natural transformation) to be a group homomorphism (and, hence, embedding).

Proposition 3.4.12. *With the above notation, the following hold:*

(i) If $Y = X_L$, with X defined over k, and the action of G_L is induced by base change from one of G on X, then the natural embedding:

$$\iota_X : X \hookrightarrow R_{L/k}X_L$$

is G-equivariant (where the G-action on $R_{L/k}X_L$ is the one induced from the action of $R_{L/k}G_L$ on $R_{L/k}X_L$, defined as in Remark 3.4.11, restricted to G through the embedding $\iota_G : G \hookrightarrow R_{L/k}G_L$).

(ii) There exists a natural G_L-equivariant isomorphism:

$$\psi^Y := \prod_{\gamma\in\Gamma} \psi^Y_\gamma : (R_{L/k}Y)_L \longrightarrow \prod_{\gamma\in\Gamma} Y^\gamma,$$

where the action of G_L on $(R_{L/k}Y)_L$ is the one induced from the action of $R_{L/k}G_L$ on $R_{L/k}Y$, defined as in Remark 3.4.11, restricted to G through the embedding $\iota_G : G \hookrightarrow R_{L/k}G_L$, and the action on $\prod_{\gamma\in\Gamma} Y^\gamma$ is induced from the diagonal embedding $\Delta_{G_L} : G_L \to \prod_{\gamma\in\Gamma} G_L^\gamma \cong \prod_{\gamma\in\Gamma} G_L$.

Proof.

(i) Let $m_X : G \times_k X \to X$ be the morphism defining the action of G on X. Then, (i) follows from the commutativity of the following diagram:

$$\begin{array}{ccccc}
G \times_k X & = & G \times_k X & \xrightarrow{\;m_X\;} & X \\
\downarrow{\scriptstyle \iota_X \times \iota_G} & & \downarrow{\scriptstyle \iota_{X\times G}} & & \downarrow{\scriptstyle \iota_X} \\
R_{L/k}G_L \times_k R_{L/k}X_L & \xrightarrow{(R_{L/k}\pi_1 \times R_{L/k}\pi_2)^{-1}} & R_{L/k}(G_L \times_L X_L) & \xrightarrow{\;R_{L/k}m_X\;} & R_{L/k}X_L,
\end{array}$$

which, in turn, is a consequence of the fact that, for $\mathcal{X} \in \mathrm{Sch}_k$, the morphisms $\iota_{\mathcal{X}} : \mathcal{X} \to R_{L/k}\mathcal{X}_L$ introduced in Proposition 3.4.9[(i)] are a natural transformation.

(ii) Let $m_Y : G_L \times_L Y \to Y$ be the morphism defining the action of G_L on Y. Then, (ii) follows from the commutativity of the following diagram:

$$\begin{array}{ccccccc}
G_L \times_L (R_{L/k}Y)_L & \xrightarrow{(\iota_G)_L \times id} & (R_{L/k}G_L)_L \times_L (R_{L/k}Y)_L & \xrightarrow{(R_{L/k}\pi_1 \times R_{L/k}\pi_2)^{-1}} & (R_{L/k}(G_L \times_L Y))_L & \xrightarrow{(R_{L/k}m_Y)_L} & (R_{L/k}Y)_L \\
\downarrow{\scriptstyle \iota \times \psi^Y} & & \downarrow{\scriptstyle \psi^{G_L} \times \psi^Y} & & \downarrow{\scriptstyle \psi^{(G_L \times_L Y)}} & & \downarrow{\scriptstyle \psi^Y} \\
\prod_{\gamma\in\Gamma} G_L \times_L Y^\gamma & \xrightarrow{(\Delta_{G_L}) \times id} & \prod_{\gamma\in\Gamma} (G_L)^\gamma \times_L \prod_{\gamma\in\Gamma} Y^\gamma & \xrightarrow{\;\sim\;} & \prod_{\gamma\in\Gamma} (G_L \times_L Y)^\gamma & \xrightarrow{\prod_{\gamma\in\Gamma} m^\gamma} & \prod_{\gamma\in\Gamma} Y^\gamma.
\end{array}$$

The commutativity of the first square follows from the equality $\Delta_{G_L} = \psi^{G_L} \circ (\iota_G)_L$, which was proven in Proposition 3.4.9[(iv)]. The commutativity of the central and last square is a consequence of the fact that, for $\mathcal{Y} \in \mathrm{Sch}_L$, the morphisms $\psi^{\mathcal{Y}}$ introduced in Proposition 3.4.9[(ii)] are a natural transformation. $\square$

Notation 3.4.13. For a group G/k acting on a variety Y, and a point $y \in Y(\bar{k})$, we denote by $\mathfrak{Stab}_{\bar{G}}\, y$ the stabilizer of y in $\bar{G} \coloneqq G_{\bar{k}}$.

Let G be an algebraic k-group and let Y be a G_L-variety (i.e. an L-variety endowed with a left G_L-action). We know by Remark 3.4.11 and the discussion following it that G acts on $(R_{L/k}Y)$ through the diagonal embedding.

Corollary 3.4.14. *Keeping the notation of Proposition 3.4.12(ii), we have that, for each $\bar{x} \in (R_{L/k}Y)(\bar{k})$:*

$$\mathfrak{Stab}_{\bar{G}}(\bar{x}) = \cap_{\gamma\in\Gamma} \mathfrak{Stab}_{\bar{G}}(\psi_\gamma^Y(x)).$$

Proof. This immediately follows from Proposition 3.4.12(ii). □

Proposition 3.4.15. *Let $\pi : Z \to X$ be a finite surjective G-equivariant morphism of k-varieties endowed with a (left) G-action. If X is a homogeneous space under G and Z is geometrically integral, then Z is a homogeneous space as well.*

Proof. Let $\bar{z} \in \bar{Z}(\bar{k})$ be any geometric point, and let $Y = \bar{G}\cdot\bar{z}$ be its $\bar{G}$-orbit. We assume that Y is endowed with the $\bar{k}$-variety structure that comes from the natural isomorphism $\bar{G}/\,\mathfrak{Stab}_{\bar{G}}\,\bar{z} \cong Y$. In particular, we have by [Mil17, Lemma 9.30] that Y is locally closed in $\bar{Z}$.

Since X is a homogeneous space, we have that $\pi(Y(\bar{k})) = \pi(\bar{G}\cdot\bar{z}(\bar{k})) = \pi(\bar{z})\cdot\bar{G}(\bar{k}) = X(\bar{k})$. Hence, the morphism $\pi|_Y : Y \to X$ is surjective on $\bar{k}$-points, hence dominant (by Nullstellensatz). Therefore, if $Y' \subset \bar{Z}$ denotes the Zariski-closure of Y in $\bar{Z}$ (which coincides with the Zariski-closure of $Y(\bar{k})$ in $\bar{Z}$ by Nullstellensatz), we have that $\dim X = \dim\bar{Z} \geq \dim Y' \geq \dim Y \geq \dim X$, and, hence, since $\bar{Z}$ is irreducible, $Y' = \bar{Z}$.

We want to show that actually $Y = \bar{Z}$. Since Y is locally closed in $\bar{Z}$ and $\bar{Z}$ is reduced, it is enough, by the Nullstellensatz, to show that $\bar{Z}(\bar{k}) = Y(\bar{k})$. We assume, by contradiction, that there exists a $\bar{s} \in \bar{Z}(\bar{k}) \setminus Y$. We have, as before, that $\bar{G}\cdot\bar{s}$ is dense in $\bar{Z}$. Therefore, since both $\bar{G}\cdot\bar{s}$ (which we give again a $\bar{k}$-variety structure as before) and Y are constructible and dense in $\bar{Z}$, they both contain some non-empty Zariski open subset of $\bar{Z}$, and their intersection is non-empty. This is a contradiction because we assumed $\bar{s} \notin Y = \bar{G}\cdot\bar{z}$, hence $\bar{G}\cdot\bar{s}\cap\bar{G}\cdot\bar{z} = \emptyset$. □

3.4.2 Proof of Theorem 3.1.1

The lemmas that follow play a major role (especially Lemma 3.4.17) in the proof of Theorem 3.1.1.

Lemma 3.4.16. *Let G be a connected algebraic group over k, X be a k-scheme of finite type endowed with a G-action, and $X^0 \subset X$ be a connected component of X. Then the G-action on X induces one on X^0 (i.e. there exists a unique action on X^0 that makes the embedding $X^0 \hookrightarrow X$ G-equivariant).*

Proof. The proof is straightforward. □

We recall that, if F is a group acting on the right on a k-variety Z, and G acts on the left with an action that commutes with the one of F, and $\eta \in H^1_{lt}(K, F)$, there is a natural left action of G on Z_η, commuting with the right F^η-action.

Lemma 3.4.17. *Let G be a connected algebraic group over K and X a left G-homogeneous space, and assume that there exists a finite group scheme $F/\operatorname{Spec} K$, and a right F-torsor $\varphi : Z \to X$ such that Z is endowed with a left G-action, commuting with the F-action, with connected geometric stabilizers and such that φ is G-equivariant. Suppose that $X(\mathbb{A}_K)^{ét} \neq \emptyset$. Then, there exists a $\eta \in H^1_{lt}(K, F)$ and a connected component Z' of Z_η such that Z', endowed with the G-action of Lemma 3.4.16, is a G-homogeneous space with connected stabilizers. Moreover, there is a finite subgroup $F' \subset F^\eta$ such that $Z' \to X$ is a right F'-torsor.*

Proof. By [CDX19, Lemma 7.1] there exists an element $\eta \in H^1_{lt}(K, F)$ and a connected component Z' of Z_η such that Z' is geometrically connected. Since G is connected, we have by Lemma 3.4.16 that there is a left G-action on Z' that makes the embedding $Z' \hookrightarrow Z_\eta$ G-equivariant.

Let us now prove that Z' is a homogeneous space. We know that X is a homogeneous space, and that Z' is smooth (because $Z' \to X$ is étale and X is smooth) and geometrically connected. Hence, since $Z' \to X$ is finite and G-equivariant, Z' is a homogeneous space by Proposition 3.4.15. Moreover, by our assumption, the geometric stabilizers of the G-action are connected on Z, so, in particular, they are on Z'.

Letting F' be the stabilizer of Z' under the F^η-action, the last part is straightforward. □

Lemma 3.4.18. *Let X be a (left) homogeneous space under a connected K-group G. There exists a finite group scheme $F/\operatorname{Spec} K$, and a right F-torsor $\varphi : Z \to X$ such that Z is endowed with a left G-action with connected geometric stabilizers and such that φ and the F-action are G-equivariant.*

Proof. Let L/K be a Galois extension such that there exists a point $\bar{x} \in X(L)$. Let $H = \mathfrak{Stab}_{G_L}(\bar{x})$ and let $H^0 \leq H$ be the connected component of H in which lies the identity. We have, by [Mil17, Proposition 1.39], that H^0 is a normal subgroup of H. We denote by H_f the (finite) quotient H/H^0. Let $Y := G_L/H^0$. We have a G_L-equivariant morphism:

$$\psi : Y = G_L/H^0 \to G_L/H \cong X_L, \tag{3.4.2}$$

where the last isomorphism is induced from the map $G_L \to X_L$, $g \mapsto g \cdot \bar{x}$. The identifications of (3.4.2) make Y a right H_f-torsor over X_L (see [Sko01, Section 3.2]), and the right H_f-action commutes with the right G_L action. Hence the induced morphism

$$R_\psi : R_{L/K}Y \to R_{L/K}X_L,$$

makes $R_{L/K}Y$ a right $F := R_{L/K}H_f$-torsor over $R_{L/K}X_L$. The left F-action commutes with the left $R_{L/K}G_L$-action on $R_{L/K}Y$ (defined as in Remark 3.4.11). We endow $R_{L/K}Y$ with the left G-action given by restricting the $R_{L/K}G_L$-action to a G-action through the embedding $\iota_G : G \hookrightarrow R_{L/K}G_L$.

Let $\iota_X : X \to R_{L/K}X_L$ be the morphism of Proposition 3.4.9(iii). Let Z be the fibered product $X \times_{R_{L/K}X_L} R_{L/K}Y$. We notice that, since, by functoriality of $R_{L/K}$ and Proposition 3.4.12(i), R_ψ and ι_X are both G-equivariant, the k-variety Z is equipped with a natural left G-action. Moreover, the projection $Z \to X$ can be endowed with the structure of a right $R_{L/K}H_f$-torsor over X (since $Z \to X$ is just a base change of the right $R_{L/K}H_f$-torsor $R_\varphi : R_{L/K}Y \to R_{L/K}X_L$) .

Lastly, we prove that the geometric stabilizers of Z are connected. Let $\bar{z} \in \bar{Z}$ be a geometric point. Since $\bar{Z} \hookrightarrow \overline{R_{L/K}Y_K}$ (where the morphism is $\bar{G}$-equivariant), we have that, by Corollary 3.4.14, there exists a $g \in G(\bar{K})$ such that $\bar{S} := \mathfrak{Stab}_{\bar{G}}(\bar{z}) \subset g\bar{H}^0g^{-1}$, where $\bar{H}^0 = H^0_{\bar{K}}$. Moreover, since $\dim \bar{S} = \dim \bar{G} - \dim \bar{Z} = \dim \bar{G} - \dim \bar{X} = \dim \bar{H}^0$, and $\bar{H}^0$ is integral and algebraic subgroups are always closed, we actually have that $\bar{S} = g\bar{H}^0g^{-1}$, which is connected. □

Lemma 3.4.19. *Let X be a (left) homogeneous space under a connected K-group G, with linear stabilizers. Suppose there is no étale Brauer–Manin obstruction for the variety X, i.e. that there exists*

$$(P_v)_{v \in M_K} \in X(\mathbb{A}_K)^{\text{ét,Br}}$$

Then, there exists a homogeneous space Z under G with geometrically connected stabilizers, an adelic point $(Q_v) \in Z(\mathbb{A}_K)^{\operatorname{Br} Z}$, and a G-equivariant morphism $\psi : Z \to X$ such that $(\psi_v(Q_v)) = (P_v)$. Moreover, Z is a (right) torsor over X under a finite group scheme.

Proof. We know by Lemma 3.4.17 (whose hypothesis hold by Lemma 3.4.18) that there exists a finite group scheme F and a right F-torsor $\psi' : Z' \to X$, where Z' is a homogeneous space with geometrically connected stabilizers and ψ' is G-equivariant. Since $(P_v)_{v \in M_K} \in X(\mathbb{A}_K)^{\text{ét,Br}}$, we know that there exists an element $\eta \in H^1_{lt}(K,F)$ and an element $(Q_v)_{v\in M_K} \in Z(\mathbb{A}_K)^{\mathrm{Br}\,Z}$, where $Z := Z'_\eta$, such that $(\psi_v(Q_v))_{v\in M_K} = (P_v)_{v\in M_K}$, where $\psi := (\psi')^\eta : Z \to X$. We observe that $Z = Z'_\eta$ is still a G-homogeneous space (since it is a twist of a G-homogeneous space, with respect to an action that commutes with the G one) and $Z \to X$ is a right torsor under F^η. □

Lemma 3.4.20. *Let X and Y be connected $\bar{k}$-varieties, with Y simply connected, and let $y_0 \in Y(\bar{k})$. Let $\varphi : \mathcal{Y} \to X \times_{\bar{k}} Y$ be an étale cover such that there exists a section $\sigma_{y_0} : X \times_{\bar{k}} \{y_0\} \to \mathcal{Y}|_{X\times_{\bar{k}}\{y_0\}}$ to the restricted cover $\varphi|_{X\times_{\bar{k}}\{y_0\}}$. There exists then a unique section $\sigma : X \times_{\bar{k}} Y \to \mathcal{Y}$ to φ extending σ_{y_0}.*

Proof. We can assume, without loss of generality, that $\mathcal{Y}$ is connected (otherwise we can restrict φ to the connected component containing the image of σ_{y_0}).

Let $x_0 \in X(\bar{k})$ be any point, which we are going to use as a "basepoint". We have a canonical embedding $\iota : \pi_1(Y, y_0) \hookrightarrow \pi_1(X \times Y, (x_0, y_0))$. Since Y is simply connected and we are in characteristic 0, we have that $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0) \cong \pi_1(X, x_0)$ through natural isomorphisms (this follows from GAGA-like theorems, see [Gro71, XIII 4.6], whose hypothesis hold by [EH02]). Hence, the natural embedding ι is an isomorphism.

Let now $P := \sigma_{y_0}((x_0, y_0)) \in \mathcal{Y}$. By construction P is $\pi_1(X, x_0)$-, and, hence, $\pi_1(X \times Y, (x_0, y_0))$-invariant. By the standard theory of étale covers, this means that the connected étale cover $\mathcal{Y} \to X \times Y$ has degree 1, and, hence, is an isomorphism. In particular, the cover $\mathcal{Y} \to X \times Y$ has a unique section. □

The following lemma is a slightly more general case of [HW20, Prop. 5.1].

Lemma 3.4.21. *Let G be a simply connected (recall from Section 3.2 that this means that G is geometrically simply connected) linear k-group, X be a k-variety endowed with a G-action, and $\varphi : Z \to X$ be an étale cover. There exists then a unique G-action on Z such that φ is G-equivariant.*

Proof. It is sufficient, by Galois descent, to prove the existence and uniqueness over $\bar{k}$. So we can assume without loss of generality that $k = \bar{k}$. Let $m_X : G \times_k X \to X$ be the G-action on X. We consider the following diagram:

$$\begin{array}{ccc} G \times_{\bar{k}} Z & \overset{m_Z}{\dashrightarrow} & Z \\ \downarrow & & \downarrow \\ G \times_{\bar{k}} X & \xrightarrow{m_X} & X \end{array}, \tag{3.4.3}$$

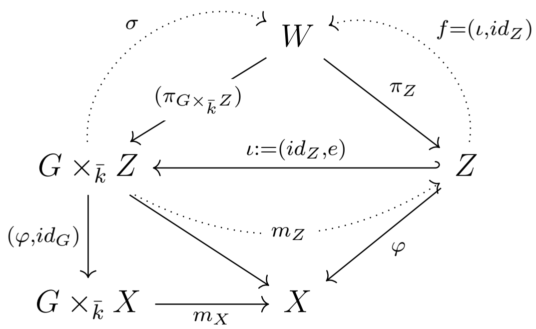
which we would like to complete with a (unique) group action m_Z on the first row that makes it commute.

Let us consider the following commutative diagram:

$$\begin{tikzcd} G \times_{\bar{k}} Z \arrow[d, "(\varphi, id_G)"'] \arrow[dr, dotted, "m_Z"] & & Z \arrow[ll, hook', "\iota := (e, id_Z)"'] \arrow[dl, "\varphi"] \\ G \times_{\bar{k}} X \arrow[r, "m_X"'] & X & \end{tikzcd}, \tag{3.4.4}$$

We claim that there is a unique m_Z that makes diagram (3.4.4) above commute with all but ι, and such that ι is a section of it. From this, and the fact that m_X is a group action, it is a straightforward verification to see that m_Z is a group action itself.

We enlarge the commutative diagram above to the following:



, (3.4.5)

where $W := (G\times_{\bar k} Z)\times_X Z$, and hence the square $[Z, X, G\times_{\bar k} Z, W]$ is cartesian by definition. The existence and uniqueness of the sought morphism m_Z (such that the lower trapezoid commutes and ι is a section of it) is equivalent to the existence and uniqueness of a morphism $\sigma : G\times_{\bar k} Z \to W$ such that it is a section of $(id_{G\times_{\bar k}Z}, \varphi)$ and such that $\pi_Z\circ\sigma\circ\iota = id_Z$. Lemma 3.4.20 implies that the existence and uniqueness of such a section is equivalent to the existence and uniqueness of a morphism $f : Z \to W$ such that $(id_{G\times_{\bar k}Z}, \varphi)\circ f = \iota$ and such that it is a section of π_Z. The morphism (ι, id_Z) is the unique morphism that satisfies these properties.

□

Lemma 3.4.22. *Let G be a simply connected linear k-group and X a k-variety endowed with a G-action. Let $B \in \operatorname{Br} X$ be an element of the Brauer group of X, and let $P \in X(k)$. Then, for every element $g \in G(k)$, we have an equality $B(P) = B(g\cdot P) \in \operatorname{Br} k$.*

Proof. We know that the Brauer group of G is constant, i.e. $\operatorname{Br} G = \operatorname{Br} k$ [Gil09]. Let $m_X : G\times_k X \to X$ be the G-action, let $m_P : G \to X$ denote the morphism defined by $g \mapsto m_X(g, P) =$ "$g\cdot P$", and let $B_P = (m_P)^*B \in \operatorname{Br} G = \operatorname{Br} k$. It is now immediate that, for every $g \in G(k)$, $B(m_X(g, P)) = B_P(g) = B_P(e) = B(P)$, as wished. □

Proof of Theorem 3.1.1. We start by showing that $X(\mathbb{A}_K)_\bullet^{\text{ét,Br}} \subset \overline{G^{scu}(K_{S_f})\cdot X(K)}$. Let $(P_v) \in X(\mathbb{A}_K)_\bullet^{\text{ét,Br}}$.

We know by Lemma 3.4.19 that there exists a right torsor $\varphi : Z \to X$, under some finite group scheme, such that Z is a left homogeneous space under G with connected geometric stabilizers, with φ being G-equivariant, and such that there exists $(Q_v) \in Z(\mathbb{A}_K)_\bullet^{\operatorname{Br} Z}$ such that $(\varphi_v(Q_v)) = (P_v)$. A theorem of Borovoi and Demarche, [BD13, Theorem 1.4], tells us that $(Q_v) \in \overline{G^{scu}(K_{S_f})\cdot Z(K)}$. Since $(\varphi_v) : Z(\mathbb{A}_K)_\bullet \to X(\mathbb{A}_K)_\bullet$ is continuous, this implies that $(P_v) = (\varphi_v(Q_v)) \in \overline{G^{scu}(K_{S_f})\cdot\varphi(Z(K))} \subset \overline{G^{scu}(K_{S_f})\cdot X(K)}$.

We now prove that $X(\mathbb{A}_K)_\bullet^{\text{ét,Br}} \supset \overline{G^{scu}(K_{S_f})\cdot X(K)}$. Since $X(\mathbb{A}_K)_\bullet^{\text{ét,Br}}$ is closed, it suffices to prove that $X(\mathbb{A}_K)_\bullet^{\text{ét,Br}} \supset G^{scu}(K_{S_f})\cdot X(K)$. Let $P \in X(K)$ and $(g_v)_{v\in S_f} \in G^{scu}(K_{S_f})$, and let $\mathbf{P_1} = (P_{1v})_{v\in M_K} \in X(\mathbb{A}_K)_\bullet$ be the adelic point defined as $P_{1v} = P_v$ if $v \notin S_f$ and $P_{1v} = g_v\cdot P_v$ if $v \in S_f$. Let $\psi : W \to X$ be a left torsor under a finite group scheme F. We know that there exists a twist $\psi^\sigma : W^\sigma \to X$, for some $\sigma \in H^1(K, F)$ such that $P = \psi^\sigma(P')$, for some $P' \in W^\sigma(K)$. Since G^{scu} is simply connected, by Lemma 3.4.21, we know that there exists a right G^{scu}-action on W^σ such that ψ^σ is G^{scu}-equivariant.

Letting $\mathbf{P'_1} = (P'_{1v})_{v\in M_K} \in W^\sigma(\mathbb{A}_K)_\bullet$ be the adelic point defined by $P'_{1v} = P'_v$ if $v \notin S_f$ and $P'_{1v} = g_v\cdot P'_v$ if $v \in S_f$, it follows from Lemma 3.4.22 that $\mathbf{P'_1} \in W^\sigma(\mathbb{A}_K)_\bullet^{\operatorname{Br} W^\sigma}$. Since $\psi(\mathbf{P'_1}) = \mathbf{P_1}$, this proves that $\mathbf{P_1} \in X(\mathbb{A}_K)_\bullet^{\psi}$. Since the argument works for any finite torsor $\psi : W \to X$, we have that $\mathbf{P_1} \in X(\mathbb{A}_K)_\bullet^{\text{ét,Br}}$, as wished.

This concludes the proof of Theorem 3.1.1. □

3.5 Compatibility of the abelianization and the Brauer–Manin pairing

In this section X denotes a quotient G/H, where G is a connected K-group, and H is a connected linear K-closed subgroup (which implies that $\bar{H}$ is connected as well).

Recall that C_X^d is defined as the dual complex of the cone of the morphism

$$[T_H^{sc} \xrightarrow{\rho_H} T_H] \xrightarrow{[\iota^{sc},\iota]} [T_G^{sc} \xrightarrow{\rho_G} SA_G],$$

where both complexes start in degree -1 (see Section 3.3.2). Note that, since $C_X = \mathrm{Cone}([T_{H^{\mathrm{sc}}} \to T_H] \to [T_{G^{\mathrm{sc}}} \to SA_G]) \xleftarrow{\sim} \mathrm{Cone}([Z_{H^{\mathrm{sc}}} \to Z_{H^{red}}] \to [T_{G^{\mathrm{sc}}} \to SA_G])$, we may also identify C_X^d with the dual of this latter complex, i.e. (as a complex of étale group sheaves over K):

$$C_X^d = \left[\widehat{T_G} \xrightarrow{(\lambda,-\widehat{\rho_G},\widehat{\iota})} \mathrm{Pic}^0\left(G^{\mathrm{ab}}\right) \oplus \widehat{T_{G^{\mathrm{sc}}}} \oplus \widehat{Z_{H^{\mathrm{red}}}} \xrightarrow{(0,\widehat{\iota^{sc}},\widehat{\rho_H})} \widehat{Z_{H^{\mathrm{sc}}}}\right],$$

where the morphism $\lambda : \widehat{T_G} \to \mathrm{Pic}^0\left(G^{\mathrm{ab}}\right)$ is the morphism arising from the construction of the dual motive of SA_G (see e.g. [HS05, Sec. 1] for details on this construction). Note that the degrees on which C_X^d is supported are $0, 1, 2$.

The goal of this section is to prove the following theorem with an explicit computation.

Theorem 3.5.1. *Let $v \in M_K$, $x \in X(K_v)$ be a local point, and $B \in \mathbb{H}^2(K_v, C_X^d)$. Then one has that:*

$$\langle x, \alpha(B)\rangle = -\langle ab^0_{K_v}(x), B\rangle, \tag{3.5.1}$$

where:

- *the first pairing is the local Brauer pairing;*
- *the second pairing is the one induced by the local pairing (3.3.6).*

3.5.1 Proof of Theorem 3.5.1

For convenience of the reader we start by recalling some standard notations that we will use in the course of the computation.

For the following discussion, and the two lemmas below, let us fix an algebraically closed field k.

For any algebraic group H/k, and any k-variety Y, endowed with a H-action $m : H \times Y \to Y$, we use the following notation (introduced by Borovoi and van Hamel [BvH06]):

$$\mathrm{UPic}_H(Y)^1 := \left\{(D, z) \in \mathrm{Div}(Y) \times k(H \times Y)^* : \left\{ \begin{array}{l} z_{h_1h_2}(y) = z_{h_1}(h_2 \cdot y) \cdot z_{h_2}(y) \\ \mathrm{div}(z) = m^*D - \mathrm{pr}_Y^* D \end{array} \right\}\right\},$$

where $z_h(y)$ stands for $z(h, y)$.

We have a natural morphism $k(Y)^*/k^* \xrightarrow{d} \mathrm{UPic}_H(Y)^1$, defined by $d(f) := \left(\mathrm{div}(f), \frac{m^*f}{\mathrm{pr}_Y^* f}\right)$. Moreover, we define:

$$\mathrm{Pic}_H(Y) := \mathrm{UPic}_H(Y)^1/d(k^*(Y)).$$

Note that there is a natural map $\mathrm{Pic}_H(Y) \to \mathrm{Pic}(Y)$.

We anticipate a lemma that we are going to need in the proof.

Lemma 3.5.2. *Let $Z := G/H$ be a homogeneous space over k, where $H \subset G^{lin}$ is a subgroup of the maximal linear subgroup G^{lin} of a connected k-group G. Let $G^{ant} \subset G$ be the maximal anti-affine subgroup of G, and let $Y := Z/G^{ant}$ (this makes sense as G^{ant} is normal in G). We denote by $\pi' : Z \to Y$, and by $\pi : Z \to G^{ab} := G/G^{lin} = Z/G^{lin}$, the two natural projections. We then have that $\mathrm{Pic}\, Z = \pi^* \mathrm{Pic}\, G^{ab} + (\pi')^* \mathrm{Pic}\, Y$.*

Proof. For an algebraic group D/k, we denote by $\chi(D)$ its group of characters $\mathrm{Hom}_{k,gr}(D, \mathbb{G}_{m,k})$. We denote by ι' the inclusion $H \hookrightarrow G^{lin}$, by η the inclusion $G^{lin}/H \hookrightarrow G/H$, by h the morphism $G^{lin} \to G^{lin}/H$.

Applying [Bri11, Prop. 3.10] to $H \subset G$ and $H \subset G^{lin}$ we obtain the two exact rows of the following commutative diagram:

$$\begin{array}{ccccccc} \chi(G^{lin}) & \xrightarrow{(c_{G^{ab}}, (\iota')^*)} & \mathrm{Pic}(G^{ab}) \times \chi(H) & \xrightarrow{(\pi^*, \mathcal{E})} & \mathrm{Pic}(G/H) & \xrightarrow{h^* \circ \eta^*} & \mathrm{Pic}(G^{lin}) \\ \downarrow = & & \downarrow pr_2 & & \downarrow \eta^* & & \downarrow = \\ \chi(G^{lin}) & \xrightarrow{(\iota')^*} & \chi(H) & \xrightarrow{\mathcal{E}} & \mathrm{Pic}(G^{lin}/H) & \xrightarrow{h^*} & \mathrm{Pic}(G^{lin}) \end{array}, \qquad (3.5.2)$$

where pr_2 denotes the projection on the second coordinate, $c_{G^{ab}}$ is the characteristic homomorphism of G (see [Bri11, p. 7]), and we refer the reader to [KKV89, Sec. 3] for the morphisms $\mathcal{E}$.

A simple diagram chase on (3.5.2) gives the following exact sequence:

$$\mathrm{Pic}(G^{ab}) \xrightarrow{\pi^*} \mathrm{Pic}(G/H) = \mathrm{Pic}(Z) \xrightarrow{\eta^*} \mathrm{Pic}(G^{lin}/H). \qquad (3.5.3)$$

Lemma 3.5.3, which follows, shows that the morphism

$$\mathrm{Pic}(Y) = \mathrm{Pic}(G/(G^{ant} \cdot H)) = \mathrm{Pic}(G^{lin}/(B \cdot H)) \to \mathrm{Pic}(G^{lin}/H),$$

where $B := \mathrm{Ker}(G^{lin} \to G/G^{ant}) = G^{lin} \cap G^{ant}$, is surjective, which, together with the exact sequence (3.5.3), is sufficient to conclude the proof of this lemma.

□

Lemma 3.5.3. *Let G be a connected linear k-group, let $B \subset G$ be a central algebraic subgroup, and let $H \subset G$ be an algebraic subgroup. We have that the following morphism is surjective:*

$$\mathrm{Pic}\left({}^{G}\!/_{B \cdot H}\right) \to \mathrm{Pic}\left({}^{G}\!/_{H}\right).$$

Proof. Let $\pi : \tilde{G} \to G$ be a central isogeny such that $\mathrm{Pic}(\tilde{G}) = 0$ (this exists by [Pop74, Thm 3]). Let $\tilde{H} := \pi^{-1}(H)$, and $\tilde{B} := \pi^{-1}(B)$. We then have the following two natural surjections (by [Pop74, Thm 4]):

$$\chi(\tilde{H}) \twoheadrightarrow \mathrm{Pic}(G/H), \quad \chi(\tilde{B} \cdot \tilde{H}) \twoheadrightarrow \mathrm{Pic}\left({}^{G}\!/_{B \cdot H}\right).$$

To conclude the proof of the lemma it is therefore enough to show that the following morphism is a surjection:

$$\chi(\tilde{B} \cdot \tilde{H}) \to \chi(\tilde{H}).$$

The above surjection follows from the three points below:

i. For all algebraic groups D/k, we have that $\chi(D) = \chi\left({}^{D}\!/_{[D, D]}\right)$.

ii. The morphism

$$ {}^{\tilde{H}}\!/_{[\tilde{H}, \tilde{H}]} \to {}^{\tilde{B} \cdot \tilde{H}}\!/_{[\tilde{B} \cdot \tilde{H}, \tilde{B} \cdot \tilde{H}]} \qquad (3.5.4)$$

is an injection: this follows from the fact that $\tilde{B}$ is central in $\tilde{G}$, as we now prove. We have that $[\tilde{B}, \tilde{G}] \subset \mathrm{Ker}(\pi)$, which is finite. By connectedness of $\tilde{G}$, this implies that $[\tilde{B}, \tilde{G}] = [\tilde{B}, \tilde{e}] = \tilde{e}$, i.e. that $\tilde{B}$ is central in $\tilde{G}$.

iii. If $G_1 \hookrightarrow G_2$ is an injection of commutative linear k-groups, then the induced morphism, between group of multiplicative type, $G_1/G_1^u \to G_2/G_2^u$ is injective (and, hence, the map $\chi(G_2) = \chi(G_2/G_2^u) \to \chi(G_1) = \chi(G_1/G_1^u)$ is surjective by [Mil17, Thm. 14.9]). To prove this, note that applying the snake lemma to the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G_1^u & \longrightarrow & G_1 & \longrightarrow & G_1/G_1^u & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & G_2^u & \longrightarrow & G_2 & \longrightarrow & G_2/G_2^u & \longrightarrow & 1 \end{array},$$

whose first two colums are exact, we obtain an injection $\mathrm{Ker}(G_1/G_1^u \to G_2/G_2^u) \to G_2^u/G_1^u$. Since the latter is unipotent, and the former is of multiplicative type, we deduce that this morphism is 0. Hence $\mathrm{Ker}(G_1/G_1^u \to G_2/G_2^u) = 0$.

□

Lemma 3.5.4. *Let $[A_1 \xrightarrow{\iota_1} \ldots \xrightarrow{\iota_{N-1}} A_N]$ be a complex, with the A_i belonging to some abelian category $\mathcal{C}$ and, for some integer $1 \le n \le N$, $A'_n \subset A_n$ be such that $A'_n \to A_n/\iota_{n-1}(A_{n-1})$ is an epimorphism. Then the following is a quasi-isomorphism (the complexes being the horizontal ones):*

$$\begin{array}{ccccccccccccc} A_1 & \to & \ldots & & A_{n-2} & \to & A'_{n-1} & \to & A'_n & \to & A_{n+1} & \to \ldots \to & A_N \\ \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A_1 & \to & \ldots & & A_{n-2} & \to & A_{n-1} & \to & A_n & \to & A_{n+1} & \to \ldots \to & A_N \end{array}, \qquad (3.5.5)$$

where $A'_{n-1} := A'_n \times_{A_n} A_{n-1} \hookrightarrow A_{n-1}$.

Proof. This follows immediately from a diagram chasing. □

Proof of Theorem 3.5.1. The proof will essentially follow the simple idea of making everything as explicit as possible in terms of Galois cocycles. The two expressions that arise from this computation are, respectively, 3.5.32 for the LHS, and 3.5.24 for the RHS of Equation (3.5.1). These two expressions are unfortunately not equal in an "obvious" manner. Hence, after these first two computations, the rest of the proof will be dedicated to show that the two obtained expressions are, in fact, equivalent in $\mathbb{H}^2(K_v, \overline{K_v}^*)$. We set $k = K_v$ (so k is no longer an algebraically closed field here), and $\Gamma = \Gamma_k$.

We recall some quasi-isomorphisms (see diagram 3.5.6 below), borrowed from [Dem11] (in the figure the complexes are the 3-term horizontal ones, and the vertical morphisms define the quasi-isomorphisms between them, and the complexes end in degree 2), which will serve to make the isomorphism α mentioned above as explicit as possible. Let $\mathrm{Div}^0(\overline{Z}) := \mathrm{Ker}(\mathrm{Div}(\overline{Z}) \to \mathrm{Pic}(\overline{SA_G}) \xleftarrow{\sim} \mathrm{Pic}(\bar{G}^{ab}) \to NS(\bar{G}^{ab}))$ (i.e. the kernel of that composition).

The vertical morphisms between the second, third and fourth row are the natural ones (see [Dem11, Sec 1.2.1] for the morphism $\mathrm{Pic}(\overline{H}') \to \mathrm{Pic}'(\overline{Z}/\overline{X})$), and the ones between the first and second row are recalled below (see the proof of Lemma 3.5.6), and they are actually isomorphisms (as proven in [Dem11]). The horizontal arrows (forming the complexes) are always the natural ones (in the first two rows these might appear with a changed sign, so we indicated the correct signs in the picture), except for the morphism $\widehat{T_G} \to \mathrm{Pic}\left(\bar{G}^{\mathrm{ab}}\right)$, which factors through $\widehat{T_G} \to \mathrm{Pic}^0\left(\bar{G}^{\mathrm{ab}}\right) \to \mathrm{Pic}\left(\bar{G}^{\mathrm{ab}}\right)$ and is the one arising from the construction of the dual motive of SA_G

(see e.g. [HS05, Sec. 1] for details on this construction).

$$\begin{array}{ccccccc}
Q'_X : & \widehat{T_G} & \xrightarrow{(-,-,+)} & \mathrm{Pic}\left(\bar G^{\mathrm{ab}}\right) \oplus \widehat{T_{G^{\mathrm{sc}}}} \oplus \widehat{Z_{\bar H^{\mathrm{red}}}} & \xrightarrow{(0,+,+)} & \widehat{Z_{H^{\mathrm{sc}}}} \\
\sim\uparrow & =\uparrow & & \sim\uparrow & & \sim\uparrow \\
Q_{mix1} : & \widehat{T_G} & \xrightarrow{(-,+,+)} & \mathrm{Pic}\left(\bar G^{\mathrm{ab}}\right) \oplus \mathrm{Pic}_{\overline{T_G}}\left(\bar G^{\mathrm{lin}}\right) \oplus \mathrm{Pic}_{\overline{T_G}}(\overline{Z'}) & \longrightarrow & \mathrm{Pic}(\overline{H'}) \\
\sim\uparrow & \uparrow & & \uparrow & & =\uparrow \\
Q_{mix} : & \widehat{T_G} \oplus \bar k(Z)^*/\bar k^* & \longrightarrow & \mathrm{UPic}_{\overline{T_G}}(\bar Z)^1 & \longrightarrow & \mathrm{Pic}(\overline{H'}) \\
\sim\downarrow & \downarrow & & \downarrow & & \sim\downarrow \\
Q'_{geom} : & \bar k(Z)^*/\bar k^* & \longrightarrow & \mathrm{Div}(\bar Z) & \longrightarrow & \mathrm{Pic}'(\bar Z/\bar X)
\end{array} \quad . \tag{3.5.6}$$

From the quasi-isomorphisms above follows that

$$\mathbb{H}^2(k, Q'_X) \cong \mathbb{H}^2(k, Q'_{geom}) \cong \mathrm{Br}_a(X, G), \tag{3.5.7}$$

where the last isomorphism is a direct consequence of (3.3.10) (note that $\bar k(Z)^* \cong \bar k(Z)^*/\bar k^* \oplus \bar k^*$, the splitting being provided by the k-rational point $e \in Z(k)$).

We also need the following quasi-isomorphisms, compatible with those appearing in the diagram above:

$$\begin{array}{ccccccc}
C^d_X : & \widehat{T_G} & \xrightarrow{(-,-,+)} & \mathrm{Pic}^0\left(\bar G^{\mathrm{ab}}\right) \oplus \widehat{T_{G^{\mathrm{sc}}}} \oplus \widehat{Z_{\bar H^{\mathrm{red}}}} & \xrightarrow{(0,+,+)} & \widehat{Z_{H^{\mathrm{sc}}}} \\
\sim\uparrow & =\uparrow & & \sim\uparrow & & \sim\uparrow \\
Q^0_{mix1} : & \widehat{T_G} & \xrightarrow{(-,+,+)} & \mathrm{Pic}^0\left(\bar G^{\mathrm{ab}}\right) \oplus \mathrm{Pic}_{\overline{T_G}}\left(\bar G^{\mathrm{lin}}\right) \oplus \mathrm{Pic}_{\overline{T_G}}(\overline{Z'}) & \longrightarrow & \mathrm{Pic}(\overline{H'}) \\
\sim\uparrow & \uparrow & & \uparrow & & =\uparrow \\
Q^0_{mix} : & \widehat{T_G} \oplus \bar k(Z)^*/\bar k^* & \longrightarrow & \mathrm{UPic}^0_{\overline{T_G}}(\bar Z)^1 & \longrightarrow & \mathrm{Pic}(\overline{H'}) \\
\sim\downarrow & \downarrow & & \downarrow & & \sim\downarrow \\
Q^{0\prime}_{geom} : & \bar k(Z)^*/\bar k^* & \longrightarrow & \mathrm{Div}^0(\bar Z) & \longrightarrow & \mathrm{Pic}'(\bar Z/\bar X)
\end{array} \quad , \tag{3.5.8}$$

where $\mathrm{Div}^0(\overline{Z}) := \mathrm{Ker}(\mathrm{Div}(\overline{Z}) \to \mathrm{Pic}(SA_G) \xleftarrow{\sim} \mathrm{Pic}(G^{ab}) \to NS(G^{ab}))$ and $\mathrm{UPic}^0_{\overline{T_G}}(\bar Z)^1 = \{(D, f) \in \mathrm{UPic}_{\overline{T_G}}(\bar Z)^1 : D \in \mathrm{Div}^0(\overline{Z})\}$.

The morphism α cited in (3.3.8) is the composition $\mathbb{H}^2(k, C^d_X) \to \mathbb{H}^2(k, Q'_X) \cong \mathbb{H}^2(k, Q'_{geom}) \cong \mathrm{Br}_a(X, G)$ (see [Dem11, Thm 2.1]).

An easy computation of non-abelian Galois cohomology, using, for instance, the explicit description of non-abelian cocycles given in Proposition 3.C.3 below (or, equivalently, using the formulas appearing in [Dem09, Sec. 4.2.1.6], which lead to the same description), gives that (using the notation of Subsection 3.3.1):

$$\mathrm{ab}^0(x) = [(\partial \bar h_\sigma; z_\sigma, t_\sigma; \xi)] \in \mathbb{H}^2(k, C_X), \tag{3.5.9}$$

where:

- for a non-abelian 1-cochain $\alpha_\sigma \in \mathrm{Fun}(\Gamma, D(\overline{k}))$ (where D denotes some k-algebraic group) we use the notation $\partial(\alpha_\sigma) := \alpha_{\sigma\eta}^{-1} \cdot \alpha_\sigma \cdot ({}^\sigma\alpha_\eta)$, and, for a non-abelian 0-cochain $\alpha \in D(\overline{k})$ we use the notation $\partial\alpha := \alpha^{-1} \cdot {}^\sigma\alpha$;

- $g \in G(\bar{k})$ is such that its projection to X is the point x;
- for all $\sigma \in \Gamma$, $h_\sigma := g^{-1} \cdot {}^\sigma g$, with $h_\sigma \in H^{red}(\bar{k})$ (we can assume wlog that $h_\sigma \in H^{red}(\bar{k})$ because H^u and all of its twists are cohomologically trivial, see also [Dem13, Lem. 2.7]);
- $g = \rho_G(\bar{g}) \cdot \xi$, with $\bar{g} \in G^{sc}(\bar{k})$ and $\xi \in SA_G(\bar{k})$;
- for all $\sigma \in \Gamma$, $h_\sigma = \rho_H(\bar{h}_\sigma) \cdot z_\sigma$, with $z_\sigma \in Z_{H_{red}}(\bar{k})$, and $\bar{h}_\sigma \in H^{sc}(\bar{k})$;
- for all $\sigma \in \Gamma$, $t_\sigma := \bar{g}^{-1} \cdot {}^g\bar{h}_\sigma^{-1} \cdot {}^\sigma\bar{g} \in T_{G^{sc}}(\bar{k})$.

Formula (3.5.9) can also be inferred from the formulas of [Dem09, Sec. 4.2.1.6], which yield $\mathrm{ab}^0(x) = [({\bar{h}_{\sigma\eta}^{-1}}{}^{h_\sigma\sigma}\bar{h}_\eta\bar{h}_\sigma; z_\sigma, \bar{g}^{-1} \cdot {}^g\bar{h}_\sigma^{-1} \cdot {}^\sigma\bar{g}; \xi)]$. In virtue of the identity ${}^z\bar{h} = \bar{h}$ for all $z \in Z(H_{red})(\bar{k}), \bar{h} \in H^{sc}(\bar{k})$, this coincides with (3.5.9).

We observe that the following identity holds:

$$\partial\xi = z_\sigma \cdot \rho_G(t_\sigma)^{-1} \in C^1(\Gamma, T_G(\bar{k})), \quad \iota^{sc}(\partial\bar{h}_\sigma)^{-1} = \partial t_\sigma \tag{3.5.10}$$

We fix the notation for the element $B \in \mathbb{H}^2(k, {C_X}^d)$ as follows. Let $\beta \in Z^2(k, Q^0_{mix})$ be such that its class in $\mathbb{H}^2(k, Q^0_{mix})$ corresponds to B via the quasi-isomorphism between first and second-last rows in (3.5.8). Using the standard notation for Galois cocycles, as defined in Subsection 3.3.1, we put:

$$\beta = ((\chi_{\sigma,\eta}, f_{\sigma,\eta}); (D_\sigma, f_\sigma); \mathcal{L}), \tag{3.5.11}$$

where $f_{\sigma,\eta}(e) = e$ (we use the notation e to denote the identity element in G, and its projection to Z), after identifying $\bar{k}(Z)^*/\bar{k}^*$ with $\{f \in \bar{k}(Z)^* : f(e) = 1\} \subset \bar{k}(Z)^*$.

Because of Lemma 3.5.2, applied to $Z = G/(H^u \cdot Z_{H^{red}})$, and Lemma 3.5.4 we may assume without loss of generality (after changing β by a coboundary), that, for every $\sigma \in \Gamma_k$, $D_\sigma \in \pi^* \operatorname{Div}^0(\overline{G}^{ab}) + (\pi')^* \operatorname{Div}(\overline{Y})$ (recall that $Y = Z/G^{ant}$, that $\pi : Z \to G^{ab}$ and that $\pi' : Z \to Y$).

By definition of $\operatorname{Div}_{\overline{T_G}}(\overline{Z})$, we have that the following identities hold:

$$\operatorname{div}((f_\sigma)_t) = t^* D_\sigma - D_\sigma \quad \forall t \in T_G(\bar{k}), \tag{3.5.12}$$

$$(f_\sigma)_{t_1 t_2} = (f_\sigma)_{t_1} \cdot t_1^*(f_\sigma)_{t_2} \quad \forall t_1, t_2 \in T_G(\bar{k}). \tag{3.5.13}$$

Moreover, by definition of cocycle, we have the following identities:

$$\operatorname{div} f_{\sigma,\eta} = -\partial D_\sigma, \tag{3.5.14}$$

$$[D_\sigma|_{\bar{H}'}] = \partial\mathcal{L}, \tag{3.5.15}$$

$$\chi_{\sigma,\eta}(t) \cdot \frac{t^* f_{\sigma,\eta}}{f_{\sigma,\eta}} = (\partial f_\sigma)_t^{-1}. \tag{3.5.16}$$

We start the computation of the RHS of (3.5.1).

Before delving in the computation, we recall (one of) the construction(s) of the duality pairing for semi-abelian varieties.

Let S be a semiabelian variety over k, e be the identity element in S, $A := S^{ab}$ and $T := \operatorname{Ker}(S \to A)$. We denote by $Z(\bar{S})$ the degree 0 zero-cycles on $\bar{S} := S \times_k \bar{k}$, and by $Z^0(\bar{S})$ the degree 0 zero-cycles $\sum_{i=0}^N n_i(P_i)$ such that $\prod P_i^{n_i} = e \in S(\bar{k})$. Moreover, let:

$$Z_{ab}(\bar{S}) := Z(\bar{S}) \Big/ \langle (t \cdot P) - (P) - (t) + (e),\ t \in T(\bar{k}),\ P \in SA_G(\bar{k}) \rangle \tag{3.5.17}$$

There is a natural morphism:

$$\begin{array}{ccc} Z(\bar{S}) & \to & S(\bar{k}), \\ \sum_{i=0}^N n_i(P_i) & \mapsto & \prod P_i^{n_i} \in S(\bar{k}), \end{array} \tag{3.5.18}$$

which factors through $Z_{ab}(\bar{S})$. We define then

$$Z^0_{ab}(\bar{S}) := \mathrm{Ker}(Z_{ab}(\bar{S}) \to S(\bar{k})) \cong Z^0(\bar{A}). \tag{3.5.19}$$

The following morphisms of complexes are quasi-isomorphisms:

$$[Z^0_{ab}(\bar{S}) \to Z_{ab}(\bar{S})] \to [Z^0(\bar{S}) \to Z(\bar{S})] \to [0 \to S(\bar{k})].$$

Let $\bar{k}(S)^*_{vert}$ be the group of rational functions on $\bar{S}$ such that their divisor is vertical with respect to the projection $\nu : \bar{S} \to \bar{A}$. The following morphism (that exists only in the derived category) is a quasi-isomorphism as well:

$$[\bar{k}(S)^*_{vert}/\bar{k}^* \to \mathrm{Div}^0(\bar{A})] \to [\widehat{T} \to A^*(\bar{k}) = \mathrm{Pic}^0(\bar{A})],$$

$$(f, D) \longmapsto (f|_T, [D]).$$

Lemma 3.5.5. *The pairing* $[\widehat{T} \to A^*(\bar{k})] \otimes^L [0 \to S(\bar{k})] \to \mathbb{G}_m[-1] := [\bar{k}^*][-1]$ *of [HS05, Sec. 1] is induced by the following pairing:*

$$[\bar{k}(S)^*_{vert}/\bar{k}^* \to \mathrm{Div}^0(\bar{A})] \otimes [Z^0_{ab}(\bar{S}) \to Z_{ab}(\bar{S})] \to \mathbb{G}_m[-1], \tag{3.5.20}$$

where

$$\bar{k}(S)^*_{vert}/\bar{k}^* \otimes Z_{ab}(\bar{S}) \to \bar{k}^* \tag{3.5.21}$$

is the pairing induced by the evaluation of a function, and the "Poincaré pairing"::

$$\mathrm{Div}^0(\bar{A}) \otimes Z^0_{ab}(\bar{S}) \to \bar{k}^* \tag{3.5.22}$$

is defined as:

$$\mathrm{Div}^0(\bar{A}) \otimes Z^0_{ab}(\bar{S}) \to \mathrm{Div}^0(\bar{A}) \otimes Z^0(\bar{A}) \to \bar{k}^*,$$

where the last arrow is defined as in [PS99, Sec. 3.2].

Proof. When S is an abelian variety, this is well-known (see [PS99, Sec. 3]). When $S = T$ is a torus, then the pairing (3.5.20) reduces to the upper row of the following commutative (in the derived category of abelian groups) diagram:

$$\begin{array}{ccccc} [\bar{k}(T)^*/\bar{k}^* \to \mathrm{Div}^0 \bar{T}] & \otimes & [Z^0_{ab}(\bar{T}) \to Z_{ab}(\bar{T})] & \longrightarrow & \mathbb{G}_m[-1] \\ \downarrow\wr & & \downarrow\wr & & \downarrow= \\ \widehat{T}[0] & \otimes & T(\bar{k})[-1] & \longrightarrow & \mathbb{G}_m[-1] \end{array}.$$

The commutativity of the above diagram shows that the pairing coincides with that of [HS05, Sec. 1] when $S = T$. (Indeed, compare with [HS05, Rmk. 1.1]: employing the notations of that remark and the previous discussion, one has that, when $M = [0 \to T] = T[0]$, then $M^* = Y^*[1] = \widehat{T}[1]$ and $\Phi_M : T[0] \otimes \widehat{T}[1] \to \mathbb{G}_m[1]$ is the pairing induced by the fact that $\widehat{T}$ represents the functor $\mathrm{Sch}_k \to \mathrm{Ab}$, $S' \mapsto \mathrm{Hom}_{S'}(T, \mathbb{G}_m)$. The induced pairing on $\bar{k}$-points coincides, up to a shift, with the second row of the above diagram.)

The proof of the general case uses a devissage argument, as follows.

Indeed, note that each of the two pairings defines a morphism, in the derived category of $Sh_{\acute{e}t,k}$, $S(\bar{k}) \to \mathrm{RHom}([\widehat{T} \to A^*(\bar{k})], \mathbb{G}_m[-1])$, and hence a morphism $S(\bar{k}) \to \tau_{\leq 0}\,\mathrm{RHom}([\widehat{T} \to A^*(\bar{k})], \mathbb{G}_m[-1])$. By [Jos09, Theorem 1.3.1(1)], this morphism is an isomorphism for the pairing of Harari and Szamuely. Hence, the composition:

$$\alpha : S(\bar{k}) \to \tau_{\leq 0}\,\mathrm{RHom}([\widehat{T} \to A^*(\bar{k})], \mathbb{G}_m[-1]) \leftarrow S(\bar{k}),$$

where the first morphism comes from pairing (3.5.20) and the second from pairing [HS05, Sec. 1], is an endomorphism of $S(\bar{k})$. Such endomorphism leaves, by functoriality, $T(\bar{k}) \subseteq S(\bar{k})$ invariant, and, since the two pairings coincide in the cases $S = T$ and $S = A$, it induces the identity on $T(\bar{k})$ and on $S(\bar{k})/T(\bar{k})$. It follows that $\alpha - id : S(\bar{k}) \to S(\bar{k})$ induces a morphism $A(\bar{k}) \to T(\bar{k})$. To conclude the proof by devissage, we would like to infer that this morphism is 0. Unfortunately, $\mathrm{Hom}(\underline{A}, \underline{T})$ is not necessarily 0 in the small étale site, where $\underline{G}$ denotes the (abelian) sheaf associated to the algebraic commutative group G (i.e., $G(\bar{k})$ in this case, but below we will use the same notation in other sites). However, since $\mathrm{Hom}(\underline{A}, \underline{T}) = 0$ in the big étale site (after applying Yoneda's lemma this holds because A is proper and T is affine), it would suffice to show that the pairing

$$[\widehat{T} \to A^*(\bar{k})] \otimes^L [0 \to S(\bar{k})] \to \mathbb{G}_m(\bar{k})[-1]$$

induced by (3.5.20) can be extended to the big site (we already know that the one defined in [HS05, Sec. 1] does, see *loc. cit.*), because then the morphism $A(\bar{k}) \to T(\bar{k})$ above would come from an algebraic morphism $A \to T$, and hence it would necessarily be 0. Since extending to the whole big site seems to be difficult, we extend instead to a slightly smaller site, which we call *big integral*, and is defined to be the étale site on the full subcategory of integral k-schemes. This makes the sought extension easier to obtain, and, in this subcategory one still has that $\mathrm{Hom}(\underline{A}, \underline{T}) = 0$ (indeed both A and T are integral, so this just follows from Yoneda's lemma and the fact that of course, even in the integral category, $\mathrm{Hom}(A, T) = 0$).

To show that this extension exists, we show that there are natural extensions of the various objects involved in formula (3.5.20). The extensions of the evaluation pairing (3.5.21) and the "Poincaré pairing" (3.5.22) are then completely natural.

We extend the sheaves as follows:

$$\bar{k}(S)^*_{vert}/\bar{k}^* \rightsquigarrow (V \mapsto Frac'_{vert}(V \times_k S)/\mathcal{O}(V)^*), \tag{3.5.23}$$

where $Frac$ denotes the function field of the variety, $vert$ indicates that the divisor of the function is vertical with respect to the projection $V \times_k S \to V \times_k A$, and the $'$ indicates that the divisor of the function is horizontal with respect to the projection $V \times_k S \to V$ (note that imposing this "horizontality" condition is what makes (3.5.23) a (pre)sheaf),

$$\mathrm{Div}^0(\bar{A}) \rightsquigarrow (V \mapsto \mathrm{Div}'^0(V \times_k A)),$$

where the prime $'$ indicates that the divisor is horizontal with respect to the projection $V \times_k A \to V$, and the 0 indicates that the divisor restricts to $Div^0(k(s') \times_k A)$ for all points $s' \in V$ (analogously as before, this horizontality is what grants what we defined is a (pre)sheaf);

$$\underline{Z^{full}_{ab}(S)} := \left((V \mapsto \mathbb{Z}^{\mathrm{Hom}(V,S)})/((t \cdot P) - (P) - (t) + (e), t \in \mathrm{Hom}(V, T), P \in \mathrm{Hom}(V, S))\right)^{sh},$$

$$Z_{ab}(\overline{S}) \rightsquigarrow \underline{Z_{ab}(S)} := \mathrm{Ker}(\underline{Z^{full}_{ab}(S)} \xrightarrow{deg} \underline{\mathbb{Z}}),$$

$$Z^0_{ab}(\overline{S}) \rightsquigarrow \underline{Z^0_{ab}(S)} := \mathrm{Ker}(\underline{Z_{ab}(S)} \to \underline{S}),$$

where the latter morphism is defined just as (3.5.18). □

We will assume throughout the rest of the proof that all the specializations of functions at the specific points appearing are $\neq 0$. This may always be done without loss of generality.

Lemma 3.5.6. *The following identity holds:*

$$\langle ab^0(x), B\rangle = \left[\widetilde{\chi}_{\sigma,\eta}(\xi) \cdot \left(({}^{\sigma}f_{\eta})_{z_\sigma}(\square)^{-1} \cdot ({}^{\sigma}f_{\eta})_{t_\sigma}(\bigstar) \cdot \frac{{}^{\sigma}\epsilon_{\eta}(\bigstar)}{{}^{\sigma}\epsilon_{\eta}(t_\sigma \cdot \bigstar)}\right) \cdot \left(\frac{{}^{\sigma\eta}\psi((\partial \bar{h}'_{\sigma})^{-1}x)}{{}^{\sigma\eta}\psi(x)}\right)\right], \tag{3.5.24}$$

for any $\square \in T_G(\bar{k})$, $\bigstar \in G^{sc}(\bar{k})$, *for any* $E_\sigma \in \mathrm{Div}^0\, \bar{G}^{ab}$, *for any* $\tilde{\chi}_{\sigma,\eta} \in \bar{k}(SA_G)^*_{vert}/\bar{k}^*$, *and for any* $\psi \in \bar{k}(\bar{H}^{sc})^*$ *such that:*

- $\widetilde{\chi}_{\sigma,\eta}|_{\overline{T_G}} = \chi_{\sigma,\eta}$,
- $[\nu^* E_\sigma] = \left[D_\sigma|_{\overline{SA_G}}\right]$ *in* $\mathrm{Pic}\,\overline{SA_G}$, *where* $\nu : \overline{SA_G} \to \bar{G}^{ab}$ *denotes the projection,*
- $\mathrm{div}\,\widetilde{\chi}_{\sigma,\eta} = \partial E_\sigma$,
- $\psi \in \bar{k}(\bar{H}^{sc})^*$ *is such that there exists a* $\overline{\mathcal{L}} \in \mathrm{Div}(\bar{H}')$ *such that* $[\overline{\mathcal{L}}] = \mathcal{L}$ *and* $\mathrm{div}\,\psi = \rho_H^* \overline{\mathcal{L}}$.

Proof. The isomorphisms appearing between the first and second lines in the diagram (3.5.6) are the following (we refer to [Dem11] for the proof that these are (iso)morphisms):

$$\begin{array}{ccc} \mathrm{Pic}_{\overline{T_G}}(\overline{Z'}) & \cong & \widehat{Z_{H_{red}}} \\ [(D,f)] & \mapsto & (z \mapsto f_z(e)) \end{array} \tag{3.5.25}$$

$$\begin{array}{ccc} \mathrm{Pic}_{\overline{T_G}}(\overline{G}^{\mathrm{lin}}) & \cong & \widehat{T_{G^{\mathrm{sc}}}} \\ [(D,f)] & \mapsto & \left(t \mapsto f_t(\star)^{-1} \cdot \frac{\epsilon(t\cdot\star)}{\epsilon(\star)}\right) \end{array} \tag{3.5.26}$$

$$\begin{array}{ccc} \mathrm{Pic}(\overline{H'}) & \cong & \widehat{Z_{\overline{H}^{\mathrm{sc}}}} \\ [D] & \mapsto & \left(z \mapsto \frac{z^*\varphi}{\varphi}\right) \end{array} \tag{3.5.27}$$

where $\epsilon \in \bar{k}(G^{sc})^*$ is such that $\mathrm{div}\,\epsilon = \rho_G^* D$, $\star$ is any element in $\bar{G}^{sc}(\bar{k})$, and $\varphi \in \bar{k}(\bar{H}^{sc})^*$ is such that $\mathrm{div}\,\varphi = \rho_H^* D$. We denote by $\hat{z}_\sigma, \hat{t}_\sigma$ and w the images of respectively, (D_σ, f_σ) under the isomorphism (3.5.25), (D_σ, f_σ) under (3.5.26), and $\mathcal{L}$ under (3.5.27).

We have the following quasi-isomorphism:

$$\begin{array}{ccccccc} C_X^d : & & \widehat{T_G} & \longrightarrow & \mathrm{Pic}^0\left(\bar{G}^{\mathrm{ab}}\right) \oplus \widehat{T_{G^{\mathrm{sc}}}} \oplus \widehat{Z_{\bar{H}^{\mathrm{red}}}} & \longrightarrow & \widehat{Z_{H^{\mathrm{sc}}}} \\ \eta\uparrow & & \uparrow & & \uparrow & & \uparrow \\ {C'_X}^d : & & \bar{k}(SA_G)^*_{vert}/\bar{k}^* & \longrightarrow & \mathrm{Div}^0(\overline{G}^{ab}) \oplus \widehat{T_{G^{\mathrm{sc}}}} \oplus \widehat{Z_{H^{red}}} & \longrightarrow & \widehat{Z_{H^{\mathrm{sc}}}} \end{array}, \tag{3.5.28}$$

where the first vertical morphism is restriction to $\overline{T_G} \subset SA_G$. We remark that technically η is only defined in the derived category.

We also have the following quasi-isomorphism:

$$\begin{array}{ccccccc} C'_X : & & Z_{H^{\mathrm{sc}}}(\bar{k}) & \longrightarrow & Z_{H^{\mathrm{red}}}(\bar{k}) \oplus T_{G^{\mathrm{sc}}}(\bar{k}) \oplus Z^0_{ab}(\overline{SA_G}) & \longrightarrow & Z_{ab}(\overline{SA_G}) \\ \downarrow v & & \downarrow & & \downarrow & & \downarrow \\ C_X : & & Z_{H^{\mathrm{sc}}}(\bar{k}) & \longrightarrow & Z_{H^{\mathrm{red}}}(\bar{k}) \oplus T_{G^{\mathrm{sc}}}(\bar{k}) & \longrightarrow & SA_G(\bar{k}) \end{array}. \tag{3.5.29}$$

We notice that $[\eta((\tilde{\chi}_{\sigma,\eta}; -E_\sigma, \hat{z}_\sigma, \hat{t}_\sigma; w))] = B \in \mathbb{H}^2(\Gamma_k, C_X^d)$, and $v((\partial\bar{h}_\sigma; z_\sigma, t_\sigma, O; (\xi))) = \mathrm{ab}^0(x)$, hence, in view of (3.5.9), (3.5.25), (3.5.26), (3.5.27) and Lemma 3.5.5, Lemma 3.3.5 (applied to $\mathcal{C} = C'_X$ and $\mathcal{C}' = {C'_X}^d$) gives:

$$\mathrm{ab}^0(x) \cup B = \left[\widetilde{\chi}_{\sigma,\eta}(\xi) \cdot \left(({}^\sigma f_\eta)_{z_\sigma}(\square)^{-1} \cdot ({}^\sigma f_\eta)_{t_\sigma}(\bigstar) \cdot \frac{{}^\sigma\epsilon_\eta(\bigstar)}{{}^\sigma\epsilon_\eta(t_\sigma \cdot \bigstar)}\right) \cdot \left(\frac{{}^{\sigma\eta}\psi((\partial\bar{h}'_\sigma)^{-1}x)}{{}^{\sigma\eta}\psi(x)}\right)\right].$$

This concludes the proof of this lemma.

□

To use (3.5.24) for our comparison purposes we make specific choices of E_σ and $\tilde{\chi}_{\sigma,\eta}$, that will lead us to identity (3.5.30) below.

We notice that, since $\overline{SA_G} \to \overline{G^{ab}}$ defines a $\mathbb{G}_m^n$-torsor (for some $n \geq 0$) on $\overline{G^{ab}}$ that is Zariski locally trivial, there exists, for each $\sigma \in \Gamma_k$, a $g_\sigma \in \bar{k}(SA_G)^*$ such that $g_\sigma(t) = (f_\sigma)_t(e)$, $\forall t \in T_G(\bar{k})$ and $\mathrm{div}(g_\sigma) = D_\sigma - \nu^* E_\sigma$, where $E_\sigma \in \mathrm{Div}(\overline{G^{ab}})$.

We now define $\tilde{\chi}_{\sigma,\eta} := f_{\sigma,\eta}^{-1} \cdot (\partial g_\sigma)^{-1}$. We notice that, by (3.5.16), $f_{\sigma,\eta} \cdot (\partial g_\sigma)$ restricts to $\chi_{\sigma,\eta}^{-1}$ on $\overline{T_G} \subset \overline{SA_G}$.

We have that, for any $t \in T(\bar{k})$, $\operatorname{div}(t^*g_\sigma/g_\sigma) = t^*D_\sigma - D_\sigma = \operatorname{div}((f_\sigma)_t)|_{SA_G}$ (since $t^*\nu^*E_\sigma = \nu^*E_\sigma$). From this we deduce that

$$t^*g_\sigma/g_\sigma = (f_\sigma)_t|_{SA_G} \cdot (\tilde{\chi}_\sigma)_t,$$

for some $(\tilde{\chi}_\sigma)_t(\star) \in \bar{k}(T_G \times SA_G)^*$ such that $(\tilde{\chi}_\sigma)_t \in \bar{k}[SA_G]^*$ for every $t \in T(\bar{k})$ where the specialization makes sense. One may easily see that $(\tilde{\chi}_\sigma)_e = 1$ and $(\tilde{\chi}_\sigma)_t(e) = 1$, from which it follows that $(\tilde{\chi}_\sigma)_t = 1 \in \bar{k}(T_G \times SA_G)^*$.

Noticing that $\partial g_\sigma(\xi) = \partial(g_\sigma(\xi)) \cdot {}^\sigma g_\eta({}^\sigma\xi)^{-1} \cdot {}^\sigma g_\eta(\xi)$, we can rewrite $\widetilde{\chi}_{\sigma,\eta}(\xi)^{-1}$, up to a 2-coboundary as follows (we remind the reader of Notation 3.3.2):

$$\widetilde{\chi}_{\sigma,\eta}(\xi)^{-1} \triangleq f_{\sigma,\eta}(\xi) \cdot {}^\sigma g_\eta({}^\sigma\xi)^{-1} \cdot {}^\sigma g_\eta(\xi) = f_{\sigma,\eta}(\xi) \cdot ({}^\sigma f_\eta)_{(\partial\xi)}(\xi)^{-1}. \tag{3.5.30}$$

To (hopefully) smoothen the computation of the LHS of (3.5.1) we introduce a second Galois pseudo-action (as defined in Section 3.2) on $\bar{k}(H^{sc})^*$ and $\operatorname{Div}(\overline{H^{sc}})$. It is defined as follows:

$$({}^{\sigma^\dagger} f)(x) := ({}^\sigma f)(\bar{h}_\sigma^{-1} \cdot x), \quad {}^{\sigma^\dagger} D := (\bar{h}_\sigma^{-1})^*({}^\sigma D), \tag{3.5.31}$$

where ${}^\sigma\star$ denotes the usual Γ_k-action on $\bar{k}(H^{sc})^*$ and $\operatorname{Div}(\overline{H^{sc}})$. We refer to this pseudo-action as *the* "twisted" Γ_k-pseudo-action (as it will be the only non-standard one appearing). To avoid confusion, we will use the notation $\partial^\dagger$ to denote a coboundary morphism taken with respect to the twisted Galois action. We notice that the restriction of the pseudo-actions (3.5.31) to $\bar{k}(H')^* \subset \bar{k}(H^{sc})^*$ and $\operatorname{Div}(\bar{H}') \subset \operatorname{Div}(\overline{H^{sc}})$ are actual actions of Γ_k. In fact, they are exactly the ones obtained by pullbacking via the isomorphism $\bar{H}' \xrightarrow{g\cdot} \overline{Z_x}$, where $Z_x := Z \times_X x \hookrightarrow Z$, the usual Γ_k-actions on $\bar{k}(Z_x)^*$ and $\operatorname{Div}(\overline{Z_x})$.

Lemma 3.5.7. *Let $\tilde{t} \in T_G(\bar{k})$ and $a \in G^{ant}(\bar{k})$ be such that $\xi = \tilde{t}a$ (note that such $\tilde{t}$ and a always exist, as one can easily prove using that $G^{ant} \subset SA_G$ and that $G^{ant} \to G^{ab}$ is surjective). Then $x^*(\alpha(B)) \in H^2(\Gamma_k, \bar{k}^*)$ is represented by the following cocycle:*

$$\left[f_{\sigma,\eta}(g \cdot \rho_H(x')) \cdot \partial^\dagger \left(\epsilon_\sigma \left({}^{\xi^{-1}}\bar{g} \cdot \iota^{sc}(x') \right) \cdot (f_\sigma)_{\tilde{t}}(\xi^{-1}\bar{g}\xi \cdot \rho_H(x')) \cdot \left(\frac{{}^{\sigma^\dagger}\psi(x')}{\psi(x')} \right)^{-1} \right) \right]_{x'=e} \in Z^2(\Gamma_k, \bar{k}^*), \tag{3.5.32}$$

where the terms appearing on the right hand side should be thought of as functions in $x' \in \overline{H^{sc}}$, the pedix $x' = e$ denotes specialization at $x' = e \in H^{sc}(\bar{k})$, and ψ is as in Lemma 3.5.6.

Proof. Let $H_x \subset G$ be the stabilizer of $x \in X(k)$. Note that x is a (trivial) left homogeneous space under H_x, and that $Z_x \to x$ is a right torsor under H', and this torsor is a form of the $\bar{H}'_x$-torsor $\overline{H_x/\tilde{H}_x} \to \bar{x}$. In particular, $Z_x \to x$ is a "Z for the trivial H_x-homogeneous space x" in the sense of Remark 3.3.9.

It then follows immediately from the fact that the morphism (3.3.9) is natural that we have the following commutative diagram:

$$\begin{array}{ccc} \tau_{\leq 2}\mathbf{R}p_{X_*}\mathbf{G}_{mX} & \longleftarrow & Q_{geom} \\ \downarrow & & \downarrow \\ \tau_{\leq 2}\mathbf{R}(p_{Z_x})_*\mathbf{G}_{mZ_x} & \longleftarrow & Q_{geom}(x, Z_x), \end{array} \tag{3.5.33}$$

where vertical morphisms are defined through pullback via the inclusion $Z_x \hookrightarrow Z$, and $Q_{geom}(x, Z_x)$ denotes the complex defined in Section 3.3.3 associated to the left H_x-homogeneous space x, and the H'-torsor $Z_x \to x$ "as Z".

Recall that we are assuming that $D_\sigma \in \pi^* \operatorname{Div}^0(\bar{G}^{ab}) + (\pi')^* \operatorname{Div} \bar{Y}$ (where $\pi : Z \to G^{ab}$ and $\pi' : Z \to Y$). Let, for each $\sigma \in \Gamma_k$, $E'_\sigma \in \operatorname{Div}^0(\bar{G}^{ab})$ and $D'_\sigma \in \operatorname{Div} \bar{Y}$ be such that $D_\sigma = \pi^* E'_\sigma + (\pi')^* D'_\sigma$.

We claim that

$$\operatorname{div}\left[\epsilon_\sigma\left({}^{\xi^{-1}}\bar{g}\cdot\iota^{sc}(x')\right)\cdot(f_\sigma)_{\tilde{t}}(\xi^{-1}\bar{g}\xi\cdot\rho_H(x'))\cdot\left(\frac{{}^{\sigma^\dagger}\psi(x')}{\psi(x')}\right)^{-1}\right]= \rho_H^*(g^*D_\sigma-\partial^\dagger\overline{\mathcal{L}})\in\rho_H^*\operatorname{Div}(\overline{H'})\subset\operatorname{Div}(\overline{H^{sc}}), \tag{3.5.34}$$

where the term in bracket square on the LHS should be thought of as a function in $x'\in\overline{H}^{sc}$, which we denote by g_σ. To obtain the above equality, note that, letting $g'=a^{-1}g$:

$$\operatorname{div}\left[\epsilon_\sigma\left({}^{\xi^{-1}}\bar{g}\cdot\iota^{sc}(x')\right)\cdot(f_\sigma)_{\tilde{t}}(\xi^{-1}\bar{g}\xi\cdot\rho_H(x'))\cdot\left(\frac{{}^{\sigma^\dagger}\psi(x')}{\psi(x')}\right)^{-1}\right]=\rho_H^*\left(((g')^*D_\sigma-\partial^\dagger\overline{\mathcal{L}})|_{\bar{H}'}\right),$$

and that

$$\begin{aligned}\pi^*(\pi(a)^*E_\sigma-E_\sigma)&=(a^*-id)\pi^*E_\sigma=(a^*-id)[\pi^*E_\sigma+(g')^*(\pi')^*D'_\sigma]\\&=(a^*-id)[(g')^*\pi^*E_\sigma+(\pi')^*D'_\sigma]=g^*D_\sigma-(g')^*D_\sigma,\end{aligned} \tag{3.5.35}$$

where the terms should be thought as divisors on $\overline{Z}$, the third equality follows from the fact that a commutes with g' and that $a^*\pi'=\pi'$, and the fourth follows from the fact that $(g')^*\pi=\pi$. As a consequence of (3.5.35), we have that $(g^*D_\sigma-(g')^*D_\sigma)\,|_{\bar{H}'}=(\pi^*(\pi(a)^*E_\sigma-E_\sigma))|_{\bar{H}'}=0$, which proves the claim.

We notice that we have the following isomorphism of complexes:

$$Q_{geom}(x,Z_x)=[\bar{k}(Z_x)^*\to\operatorname{Div}(\overline{Z_x})\to\operatorname{Pic}(\bar{H}')]\xrightarrow{g^*}[\bar{k}(H')^*\to\operatorname{Div}(\bar{H}')\to\operatorname{Pic}(\bar{H}')], \tag{3.5.36}$$

where the action on the LHS is the usual one, and the one on the RHS is the twisted one. It then follows now from Remark 3.3.4, in view of the commutativity of (3.5.33) and the isomorphism (3.5.36) that, for any choice $(g'_\sigma)\in\operatorname{Fun}(\Gamma_k,\bar{k}(H')^*)$ and $\overline{\mathcal{L}}'\in\operatorname{Div}(\bar{H}')$ such that $\operatorname{div}(g'_\sigma)=g^*D_\sigma-\partial^\dagger\overline{\mathcal{L}}'$ and $[\overline{\mathcal{L}}']=\mathcal{L}\in\operatorname{Pic}(\bar{H}')$, we have that $f'_{\sigma,\eta}=f_{\sigma,\eta}\cdot(\partial^\dagger g'_\sigma)\in Z^2(\Gamma_k,\bar{k}^*)$ is a cocycle representing $x^*\alpha$. By (3.5.34) we have that, for any such choice with $\overline{\mathcal{L}}'=\overline{\mathcal{L}}$, there exists $c_\sigma\in\operatorname{Fun}(\Gamma_k,\bar{k}^*)$ such that $\rho_H^*g'_\sigma=c_\sigma\cdot g_\sigma$. Lemma 3.5.7 follows. □

In the calculations that follow we use the following notation: any = sign with references under it stands for an equality that is justified by the operations or references under it. Most of the references will refer to Lemma 3.5.8, which appears just after the calculations, and is basically just a collection of easy-to-prove identities.

We have the following, where $M:={}^{\xi^{-1}}\bar{g}\in G^{sc}(\bar{k})$:

$$RHS=\left[f_{\sigma,\eta}(g\cdot x')\cdot\partial^\dagger\left(\epsilon_\sigma\left({}^{\xi^{-1}}\bar{g}x'\right)\cdot(f_\sigma)_{\tilde{t}}(\xi^{-1}\bar{g}\xi x')\cdot\left(\frac{{}^{\sigma^\dagger}\psi(x')}{\psi(x')}\right)^{-1}\right)\right]_{x'=e} \tag{3.5.37}$$

$$=(f_{\sigma,\eta}(\xi\cdot M)\cdot(\partial f_\sigma)_{\tilde{t}}(M)(\partial\epsilon_\sigma)(M))\cdot\left(\frac{({}^\sigma f_\eta)_{\tilde{t}}(M)}{({}^\sigma f_\eta)_{{}^\sigma\tilde{t}}\left({}^\sigma M\cdot\bar{h}_\sigma^{-1}\right)}\cdot\frac{{}^\sigma\epsilon_\eta(M)}{{}^\sigma\epsilon_\eta\left({}^\sigma M\cdot\bar{h}_\sigma^{-1}\right)}\right)^{-1}\cdot\partial^\dagger\left(\frac{{}^{\sigma^\dagger}\psi(x)}{\psi(x)}\right)^{-1} \tag{3.5.38}$$

$$\underset{3.5.8(4)}{=}\left(\frac{f_{\sigma,\eta}(\xi\cdot M)}{f_{\sigma,\eta}(M)}\cdot(\partial f_\sigma)_{\tilde{t}}(M)\right)\cdot\left(\frac{({}^\sigma f_\eta)_{\tilde{t}}(M)}{({}^\sigma f_\eta)_{{}^\sigma\tilde{t}}\left({}^\sigma M\cdot\bar{h}_\sigma^{-1}\right)}\cdot\frac{{}^\sigma\epsilon_\eta(M)}{{}^\sigma\epsilon_\eta\left({}^\sigma M\cdot\bar{h}_\sigma^{-1}\right)}\right)^{-1}\cdot\partial^\dagger\left(\frac{{}^{\sigma^\dagger}\psi(x)}{\psi(x)}\right)^{-1} \tag{3.5.39}$$

$$\underset{3.5.8(6)}{=}\left(\frac{f_{\sigma,\eta}(\xi\cdot M)}{f_{\sigma,\eta}(M)}\cdot(\partial f_\sigma)_{\tilde{t}}(M)\right)\cdot\left(({}^\sigma f_\eta)_{\partial a}(e)\cdot({}^\sigma f_\eta)_{\partial\xi^{-1}}(M)\cdot\frac{{}^\sigma\epsilon_\eta(M)}{{}^\sigma\epsilon_\eta\left({}^\sigma M\cdot\bar{h}_\sigma^{-1}\right)}\right)^{-1}\cdot\partial^\dagger\left(\frac{{}^{\sigma^\dagger}\psi(x)}{\psi(x)}\right)^{-1}. \tag{3.5.40}$$

On the other hand we have that:

$$LHS = \widetilde{\chi}_{\sigma,\eta}(\xi) \cdot \left(({}^{\sigma}f_{\eta})_{z_\sigma}(\square)^{-1} \cdot ({}^{\sigma}f_{\eta})_{t_\sigma}(\bigstar) \cdot \frac{{}^{\sigma}\epsilon_{\eta}(\bigstar)}{{}^{\sigma}\epsilon_{\eta}(t_\sigma \cdot \bigstar)} \right) \cdot \left(\frac{{}^{\sigma\eta}\psi((\partial \bar{h}'_{\sigma})^{-1}x)}{{}^{\sigma\eta}\psi(x)} \right) \tag{3.5.41}$$

$$\underset{(3.5.30)}{\triangleq} f_{\sigma,\eta}(\xi)^{-1} \cdot ({}^{\sigma}f_{\eta})_{\partial\xi}(\xi) \cdot \left(({}^{\sigma}f_{\eta})_{z_\sigma}(\square)^{-1} \cdot ({}^{\sigma}f_{\eta})_{t_\sigma}(\bigstar) \cdot \frac{{}^{\sigma}\epsilon_{\eta}(\bigstar)}{{}^{\sigma}\epsilon_{\eta}(t_\sigma \cdot \bigstar)} \right) \cdot \left(\frac{{}^{\sigma\eta}\psi((\partial \bar{h}'_{\sigma})^{-1}x)}{{}^{\sigma\eta}\psi(x)} \right) \tag{3.5.42}$$

$$\underset{\substack{\bigstar\mapsto e, \square\mapsto e \\ (3.5.10),(3.5.13)}}{=} \left(f_{\sigma,\eta}(\xi)^{-1} \cdot (\partial f_{\sigma})_{\tilde{t}}(e)^{-1} \right) \cdot (\partial f_{\sigma})_{\tilde{t}}(e) \cdot ({}^{\sigma}f_{\eta})_{\partial\xi}(\xi) \cdot \left(({}^{\sigma}f_{\eta})_{\partial\xi^{-1}}(e) \cdot \frac{{}^{\sigma}\epsilon_{\eta}(e)}{{}^{\sigma}\epsilon_{\eta}(t_\sigma)} \right) \cdot \left(\frac{{}^{\sigma\eta}\psi((\partial \bar{h}'_{\sigma})^{-1}x)}{{}^{\sigma\eta}\psi(x)} \right). \tag{3.5.43}$$

In view of points 5., 8., 1., and 7. (appearing in the order they are being used) of Lemma 3.5.8, we obtain $LHS \triangleq RHS^{-1}$, as wished. □

Lemma 3.5.8. *The following hold:*

i. *The function*

$$({}^{\sigma}f_{\eta})_{\partial\xi^{-1}}(\bigstar) \cdot \frac{{}^{\sigma}\epsilon_{\eta}(\bigstar)}{{}^{\sigma}\epsilon_{\eta}(t_\sigma \cdot ({}^{z_\sigma^{-1}}\bigstar))},$$

is constant in $\bigstar \in G^{sc}(\bar{k})$ *(we remind the reader of the convention of our notation* ${}^{g}h$*, when* $[C \to P]$ *is a crossed module, as are* $[H^{sc} \to H]$ *and* $[G^{sc} \to G]$*).*

ii. *We have that:*

$${}^{\sigma}M \cdot \bar{h}_{\sigma}^{-1} = t_\sigma \cdot {}^{z_\sigma^{-1}}M.$$

iii. *For any* $(f, D) \in \mathrm{UPic}_{\overline{T_G}}(\bar{Z})^1$*, with* $D \in \pi^* \mathrm{Div}(\bar{G}^{ab}) + (\pi')^* \mathrm{Div}(\bar{Y})$*, we have that* $a^* f_t = f_t$ *for any* $a \in G^{ant}(\bar{k})$*. Moreover, for any* $a \in (T_G \cap G^{ant})(\bar{k})$ *and* $t \in T_G(\bar{k})$*, we have that* $f_a(t) = f_a(e)$*.*

iv. $f_{\sigma,\eta}(\bigstar) = (\partial\epsilon_\sigma)^{-1}(\bigstar)$*, for any* $\bigstar \in G^{sc}(\bar{k})$*.*

v. *We have the following:*

$$\frac{f_{\sigma,\eta}(\xi \cdot \bigstar)}{f_{\sigma,\eta}(\bigstar)} \cdot (\partial f_\sigma)_{\tilde{t}}(\bigstar) = \frac{f_{\sigma,\eta}(\xi)}{f_{\sigma,\eta}(e)} \cdot (\partial f_\sigma)_{\tilde{t}}(e) = f_{\sigma,\eta}(\xi) \cdot (\partial f_\sigma)_{\tilde{t}}(e),$$

for any $\bigstar \in G^{sc}(\bar{k})$ *such that all the quantities appearing above are* $\neq 0$*.*

vi. *We have the following:*

$$\frac{({}^{\sigma}f_{\eta})_{\tilde{t}}(M)}{({}^{\sigma}f_{\eta})_{{}^{\sigma}\tilde{t}}({}^{\sigma}M \cdot \bar{h}_{\sigma}^{-1})} = ({}^{\sigma}f_{\eta})_{\partial\tilde{t}^{-1}}(M) = ({}^{\sigma}f_{\eta})_{\partial\xi^{-1}}(M) \cdot ({}^{\sigma}f_{\eta})_{\partial a}(M) = ({}^{\sigma}f_{\eta})_{\partial a}(e) \cdot ({}^{\sigma}f_{\eta})_{\partial\xi^{-1}}(M).$$

vii. *We have the following:*

$$\partial^{\dagger} \left(\frac{{}^{\sigma^{\dagger}}\psi(x)}{\psi(x)} \right) = \frac{{}^{\sigma\eta}\psi((\partial \bar{h}'_{\sigma})^{-1}x)}{{}^{\sigma\eta}\psi(x)},$$

where the pseudo-action of Γ_k *on the RHS is the one defined in (3.5.31).*

viii. *We have the following equality:*

$$(\partial f_\sigma)_{\tilde{t}}(e) \cdot ({}^{\sigma}f_{\eta})_{\partial\xi}(\xi) \triangleq ({}^{\sigma}f_{\eta})_{\partial a}(e).$$

Proof. i. The divisor of this function is null, so the first point follows from the fact that $\bar{k}[G^{sc}]^* = \bar{k}^*$.

ii. We have
$$^{\sigma}M \cdot \bar{h}_{\sigma}^{-1} = {}^{\sigma\xi^{-1}}({}^{\sigma}\bar{g}) \cdot \bar{h}_{\sigma}^{-1},$$
and
$$t_{\sigma} = \bar{g}^{-1} \cdot {}^{g}(\bar{h}_{\sigma}^{-1}) \cdot {}^{\sigma}\bar{g} = \bar{g}^{-1} \cdot {}^{^{\sigma}g}(\bar{h}_{\sigma}^{-1}) \cdot {}^{\sigma}\bar{g} = \bar{g}^{-1} \cdot {}^{\sigma}\bar{g} \cdot {}^{^{\sigma}\xi}(\bar{h}_{\sigma}^{-1}) = {}^{\sigma\xi^{-1}}\bar{g}^{-1} \cdot {}^{\sigma\xi^{-1}}({}^{\sigma}\bar{g}) \cdot \bar{h}_{\sigma}^{-1},$$
where the first equality follows from the following:
$$^{^{\sigma}g}(\bar{h}_{\sigma}^{-1}) = {}^{gh_{\sigma}}(\bar{h}_{\sigma}^{-1}) = {}^{g\cdot\rho_H(\bar{h}_{\sigma})\cdot z_{\sigma}}(\bar{h}_{\sigma}^{-1}) = {}^{g}(\bar{h}_{\sigma}^{-1}),$$
and the third equality follows from the fact that t_{σ} and ${}^{\sigma}\xi$ commute. Hence:
$$^{\sigma}M \cdot \bar{h}_{\sigma}^{-1} = {}^{\sigma\xi^{-1}}\bar{g} \cdot t_{\sigma} = {}^{(\partial\xi)^{-1}}M \cdot t_{\sigma} = t_{\sigma} \cdot {}^{z_{\sigma}^{-1}}M,$$
as wished.

iii. Let $A := G^{ant}$ and $T := T_G$, $m_A : A \times Z \to Z$ (resp. $m_T : T \times Z \to Z$) be the restriction of the G-action to A (resp. to T). Let u be the morphism $A \times T \times Z \to T \times Z$, $(a,t,z) \mapsto (t,az)$, $pr : A \times T \times Z \to T \times Z$ be the projection, and let $\tilde{f} \in \bar{k}(A \times T \times Z)^*$ be defined as u^*f_t/pr^*f_t. Note that, for every $a \in A(\bar{k}), t \in T(\bar{k})$, the restiction of $\tilde{f}$ to $\{a\} \times \{t\} \times Z$ is well-defined (i.e. $\{a\} \times \{t\} \times Z$ intersects the domain of $\tilde{f}$) and equal to a^*f_t/f_t, and $\mathrm{div}(\tilde{f}|_{\{a\}\times\{t\}\times Z}) = \mathrm{div}(a^*f_t/f_t) = (a^* - id)(t^* - id)D = 0$. It follows that the divisor of $\tilde{f}$ is trivial. Since the restriction of $\tilde{f}$ to $\{1\} \times Z$ is $f_e/f_e = 1$, it follows by Rosenlicht's lemma that $\tilde{f} \in \bar{k}[A \times T]^* \subset \bar{k}[A \times T \times Z]^*$. Since A is antiaffine, $\bar{k}[A]^* = \bar{k}^*$, and we deduce, again by Rosenlicht's lemma, that $\bar{k}[A \times T]^* = \bar{k}[T]^*$. Since $\tilde{f}(1,t,-) = 1 \in \bar{k}[Z]^*$ for every $t \in T(\bar{k})$, we deduce that $\tilde{f} \equiv 1$. Hence $a^*f_t = f_t$ for all a, t.

For the second part, we have: $f_t(e) \cdot f_a(e) = f_t(a) \cdot f_a(e) = f_{at}(e) = f_a(t) \cdot f_t(e)$.

iv. We have that $\mathrm{div}(f_{\sigma,\eta} \cdot (\partial\epsilon_{\sigma})^{-1}) = 0$. Since $\bar{k}[G^{sc}]^* = \bar{k}^*$ and $(f_{\sigma,\eta} \cdot (\partial\epsilon_{\sigma})^{-1})(e) = 1$, we conclude the sought equality.

v. It is enough to notice that the divisor of the LHS (as a function in $\bigstar \in G^{sc}$), which is $(-\xi^*(\partial D_{\sigma}) + \tilde{t}^*(\partial D_{\sigma}))|_{G^{sc}}$, is trivial since $(-\xi^*(\partial D_{\sigma}) + \tilde{t}^*(\partial D_{\sigma}))$ is the pullback of a divisor from G^{ab}.

vi. All equalities are a consequence of Point 3. of this lemma (plus Point 2. for the first equality).

vii. This follows immediately by expanding the LHS.

viii.
$$(\partial f_{\sigma})_{\tilde{t}}(e) \cdot ({}^{\sigma}f_{\eta})_{\partial\xi}(\xi) \triangleq ({}^{\sigma}f_{\eta})_{\partial a}(e).$$
We have
$$(\partial f_{\sigma})_{\tilde{t}}(e) = \partial((f_{\sigma})_{\tilde{t}})(e) \cdot ({}^{\sigma}f_{\eta})_{{}^{\sigma}t}(e)^{-1} \cdot ({}^{\sigma}f_{\eta})_{t}(e) \triangleq ({}^{\sigma}f_{\eta})_{{}^{\sigma}t}(e)^{-1} \cdot ({}^{\sigma}f_{\eta})_{t}(e) = ({}^{\sigma}f_{\eta})_{\partial t}(t)^{-1},$$
and
$$({}^{\sigma}f_{\eta})_{\partial\xi}(\xi) = ({}^{\sigma}f_{\eta})_{\partial\xi}(t) = ({}^{\sigma}f_{\eta})_{\partial t}(t) \cdot ({}^{\sigma}f_{\eta})_{\partial a}(t) = ({}^{\sigma}f_{\eta})_{\partial t}(t) \cdot ({}^{\sigma}f_{\eta})_{\partial a}(e),$$
where the first and middle equalities follow from Point 3.

□

3.6 Removing places

In what follows we are going to give a version of Theorem 3.1.1 for strong approximation outside some set of (finite) places S, see Theorem 3.6.5. As usual, we fix a number field K, and a set of finite places $S \subset M_K^{fin}$.

Let us recall, from the introduction of the thesis, that for a K-variety X we define the S-modified Brauer–Manin group of X to be the following:

$$\operatorname{Br}^S X := \operatorname{Ker}\left(\operatorname{Br} X \to \prod_{v \in S} \operatorname{Br} X_{K_v}\right).$$

We define the Brauer–Manin set *outside* S of X as follows:

$$X(\mathbb{A}_K^S)^{\operatorname{Br}^S X} := \begin{cases} \{x \in X(\mathbb{A}_K^S) : \ \langle x, B\rangle = 0 \ \text{ for all } B \in \operatorname{Br}^S X\} \text{ if } X(K_S) \neq \emptyset, \\ \emptyset \text{ otherwise.} \end{cases} \tag{3.6.1}$$

We clearly have an inclusion $\overline{X(K)}^S \subset X(\mathbb{A}_K^S)^{\operatorname{Br}^S X}$, where $\overline{\star}^S$ denotes the closure in the S-adeles.

We recall that the étale Brauer–Manin obstruction to strong approximation *outside* S on X is defined as follows:

$$X(\mathbb{A}_K^S)^{\text{ét},\operatorname{Br}^S} = \bigcap_{\substack{f: Y \xrightarrow{F} X \\ F \text{ finite} \\ \text{group scheme}}} \bigcup_{[\sigma] \in H^1(K,F)} f^\sigma(Y^\sigma(\mathbb{A}_K^S)^{\operatorname{Br}^S Y^\sigma}). \tag{3.6.2}$$

Since, by [CDX19, Proposition 6.4], $X(\mathbb{A}_K^S)^{\text{ét},\operatorname{Br}^S}$ is closed, we have that $\overline{X(K)}^S \subset X(\mathbb{A}_K^S)^{\text{ét},\operatorname{Br}^S}$.

Theorem 3.6.1. *Let G be a connected K-group, X be a G-homogeneous space with geometrically connected linear stabilizers, and S be a finite set of places of K, and $S_0 \subset S \cap M_K^{fin} := S_f$. We assume that the Tate-Shafarevich group $Ш(K, G^{ab})$ is finite and that $G^{sc}(K)$ is dense in $G^{sc}(\mathbb{A}_K^S)$. We then have that $\overline{G^{scu}(K_{S_f \setminus S_0}) \cdot X(K)}^{S_0} = X(\mathbb{A}_K^{S_0})_\bullet^{\operatorname{Br}^{S_0} X}$.*

Remark 3.6.2. In Theorem 3.6.1 above one can also substitute $\operatorname{Br}^{S_0} X \subset \operatorname{Br} X$ with its quotient by constant elements instead; i.e. the quotient $\operatorname{Br}^{S_0} X/\operatorname{Br}^{S_0} K \subset \operatorname{Br} X/\operatorname{Br} K$. Moreover, if we denote by $\operatorname{Br}_{loc}^{S_0}(X) := \operatorname{Ker}\left(\operatorname{Br} X \to \prod_{v \in S_0} \operatorname{Br} X_{K_v}/\operatorname{Br} K_v\right)$, we have that $X(\mathbb{A}_K^{S_0})^{\operatorname{Br}^{S_0} X} = X(\mathbb{A}_K^{S_0})^{\operatorname{Br}_{loc}^{S_0}(X)}$ (and the same holds for $X(\mathbb{A}_K)_\bullet$). This follows from the surjectivity $BrK \twoheadrightarrow \oplus_{v \in S_0} \operatorname{Br} K_v$, which is an immediate consequence of the Albert-Brauer-Hasse-Noether Theorem.

Remark 3.6.3. In the case that $S_0 = S \cap M_K^{fin}$, Theorem 3.6.1 yields the equality $\overline{X(K)}^{S_0} = X(\mathbb{A}_K^{S_0})_\bullet^{\operatorname{Br}^{S_0} X}$, i.e. a result of strong approximation on X, without the "G^{scu} error".

Remark 3.6.4. In the case that $S_0 = \emptyset$, Theorem 3.6.1 reproves the main theorem of [BD13]. However, it does use [Dem13, Cor. 6.3], and Theorem 3.5.1, itself somehow a complement to [Dem11]. As mentioned before, Demarche remarks in [Dem13, Rmq 6.4] that the compatibility of Theorem 3.5.1 is enough to re-prove Theorem 3.3.7 in this case.

We notice that Theorem 3.6.1 does not follow immediately from Theorem 3.1.1 by projecting on $X(\mathbb{A}_K^S)$, as shown in Remark 3.4.7 and Proposition 3.4.8.

The following will follow from Theorem 3.6.1 using an argument similar to that of the proof of Theorem 3.1.1. We notice that, when $S_0 = \emptyset$, Theorem 3.6.5 is a reformulation of Theorem 3.1.1.

Theorem 3.6.5. *Let G be a connected K-group, X be a G-homogeneous space with linear stabilizers, and S be a finite set of places of K, and $S_0 \subset S \cap M_K^{fin}$. We assume that the Tate-Shafarevich group $Ш(K, G^{ab})$ is finite and that $G^{sc}(K)$ is dense in $G^{sc}(\mathbb{A}_K^S)$. We then have that $\overline{G^{scu}(K_{S_f \setminus S_0}) \cdot X(K)}^{S_0} = X(\mathbb{A}_K^{S_0})_\bullet^{\text{ét},\operatorname{Br}^{S_0}}$.*

3.6.1 Lemmas on complexes

Definition 3.6.6. A *good* complex is a complex of commutative algebraic K-groups $[M_{-2} \xrightarrow{f_{-2}} M_{-1} \xrightarrow{f_{-1}} M_0]$ such that M_{-2} and M_{-1} are groups of multiplicative type, M_0 is a semi-abelian variety, $\operatorname{Ker} f_{-2}$ is a finite group, and M_i is in degree i.

For a good complex C of commutative algebraic K-groups, and a field $F \supset K$, we denote by $\mathbb{H}^i(F, C)$ the F-hypercohomology of the complex C. When F is a local field, we endow the groups $\mathbb{H}^i(F, C)$ with their natural topologies as in [Dem13, Sec 5.1].

We remind the reader that we are using the notation $\mathbb{P}^i(K, C) := \prod'_{v \in M_F} \mathbb{H}^i(F_v, C)_\bullet$, where the restricted product is taken over $\mathbb{H}^i(O_v, C) \to \mathbb{H}^i(F_v, C)$ (after an implied choice of an integral model for C has been made), and $\mathbb{H}^i(F_v, C)_\bullet$ denotes the usual hypercohomology for $v \in M_K^{fin}$ and hypercohomology modified à la Tate (as defined in [HS05, p. 103]) for $v \in M_K^\infty$.

Lemma 3.6.7. *Let C be a good complex. The topological group $\mathbb{P}^0(K, C)/\mathbb{H}^0(K, C)$ is quasi-compact. As a direct consequence, $\mathbb{P}^0(K, C)/\overline{\mathbb{H}^0(K, C)}$ is compact, where $\overline{\mathbb{H}^0(K, C)}$ denotes the closure of $\mathbb{H}^0(K, C)$ in $\mathbb{P}^0(K, C)$.*

Proof. We follow a *devissage* used by Demarche in [Dem13]. We first prove the result when the complex C is middle exact, i.e. if $C = [M_{-2} \xrightarrow{f_{-2}} M_{-1} \xrightarrow{f_{-1}} M_0]$ with $\operatorname{Ker} f_{-1} = \operatorname{Im} f_{-2}$.

We have a commutative diagram:

$$\begin{array}{ccccc} \operatorname{Ker} f_{-2} & \longrightarrow & 0 & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ M_{-2} & \longrightarrow & M_{-1} & \longrightarrow & M_0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 & \longrightarrow & \operatorname{Coker} f_{-1}. \end{array} \tag{3.6.3}$$

Denoting $\operatorname{Ker} f_{-2}$ by F and $\operatorname{Coker} f_{-1}$ by M, the commutative diagram (3.6.3) induces the following distinguished triangle:

$$F[2] \to C \to M \to F[3], \tag{3.6.4}$$

where, for an abelian group A, we also use the letter A, with a slight abuse of notation, to denote the complex $[\cdots \to 0 \to A \to 0 \to \cdots]$, where A lies in degree 0. We notice that, because of the assumption that C is good, M is a semi-abelian variety (being the quotient of a semi-abelian variey by a subgroup of multiplicative type) and both $F[2]$ and $M = M[0]$ are good.

The triangle (3.6.4) induces the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} H^2(K, F) & \longrightarrow & \mathbb{H}^0(K, C) & \longrightarrow & H^0(K, M) & \longrightarrow & H^3(K, F) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ P^2(K, F) & \longrightarrow & \mathbb{P}^0(K, C) & \xrightarrow{\alpha} & P^0(K, M) & \longrightarrow & P^3(K, F). \end{array} \tag{3.6.5}$$

All the morphisms of (3.6.5) are continuous by construction, and α is open (this easily follows from a computation of the long exact sequence associated with the distinguished triangle $\mathcal{F}[2] \to \mathcal{C} \to \mathcal{M} \to \mathcal{F}[3]$, where $\mathcal{F}, \mathcal{C}, \mathcal{M}$ are integral models of F, C and M over a ring of S_0-integers O_{K,S_0}, for S_0 sufficiently large).

The Poitou-Tate theorem [Har20, Thm. 17.13] implies that the last vertical arrow of (3.6.5) is an isomorphism of finite groups. Since $H^0(K, M) \to P^0(K, M)$ is injective, a diagram chasing of (3.6.5) yields the following exact sequence:

$$P^2(K, F)/H^2(K, F) \to \mathbb{P}^0(K, C)/\mathbb{H}^0(K, C) \xrightarrow{\bar{\alpha}} P^0(K, M)/H^0(K, M) \to F', \tag{3.6.6}$$

where F' is some finite discrete group (this follows from the finiteness of $P^3(K,F)$). Moreover, all the morphisms of (3.6.6) are continuous, and $\bar{\alpha}$ is open. As proven in [Har08, Lemme 4], the quotient $P^0(K,M)/H^0(K,M)$ is quasi-compact (actually Harari proves the compactness of $P^0(K,M)/\overline{H^0(K,M)}$, but, because of Lemma 3.A.5, this is equivalent to what we want). We also have that $P^2(K,F)/H^2(K,F) \cong H^0(K,F^d)^D$ (as topological groups) by the Poitou-Tate exact sequence. Since $H^0(K,F^d)^D$ is a finite discrete set, we deduce that $P^2(K,F)/H^2(K,F)$ is one as well. Therefore, applying Lemma 3.A.7, we deduce that $\mathbb{P}^0(K,C)/\mathbb{H}^0(K,C)$ is quasi-compact, as wished.

We now turn to the case of a general good C. Let $p : M_{-1}/\operatorname{Im}(f_{-2}) \to P$ be an embedding of the quotient $M_2/\operatorname{Im}(f_{-2})$ into a quasitrivial torus P, and let C' be the complex $[M_{-2} \to M_{-1} \to M_0 \oplus P]$, which is middle exact. We have the following distinguished triangle:

$$P \to C' \to C \to P[1], \tag{3.6.7}$$

which induces the following commutative diagram with exact rows and continuous morphisms:

$$\begin{array}{ccccccc} H^0(K,P) & \longrightarrow & \mathbb{H}^0(K,C') & \longrightarrow & \mathbb{H}^0(K,C) & \longrightarrow & H^1(K,P) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ P^0(K,P) & \longrightarrow & \mathbb{P}^0(K,C') & \longrightarrow & \mathbb{P}^0(K,C) & \longrightarrow & P^1(K,P). \end{array} \tag{3.6.8}$$

Hilbert's Theorem 90 and Shapiro's lemma imply that the last column is zero, hence we get the surjectivity of:

$$\mathbb{P}^0(K,C') \to \mathbb{P}^0(K,C),$$

which induces a (continuous) surjective morphism $\mathbb{P}^0(K,C')/\mathbb{H}^0(K,C') \to \mathbb{P}^0(K,C)/\mathbb{H}^0(K,C)$. It follows now from the quasi-compactness of $\mathbb{P}^0(K,C')/\mathbb{H}^0(K,C')$ that $\mathbb{P}^0(K,C)/\mathbb{H}^0(K,C)$ is quasi-compact as well, thus concluding the proof. □

Corollary 3.6.8. *Let C be a good complex. The groups $\mathbb{P}^0(K,C)$ and $(\mathbb{P}^0(K,C))^\wedge$ have same image in $\mathbb{H}^2(K,C^d)^D$, under the morphism $\vartheta : (\mathbb{P}^0(K,C))^\wedge \to \mathbb{H}^2(K,C^d)^D$ defined by local duality (we recall that ϑ is the map induced from local duality as in (3.3.7)).*

Proof. The compactness of $\mathbb{P}^0(K,C)/\overline{\mathbb{H}^0(K,C)}$ implies that its image in $\mathbb{P}^0(K,C)^\wedge/\mathbb{H}^0(K,C)^\wedge$ is closed. Since it is also, by the definition of profinite completion, dense, we see that its image is the whole quotient $\mathbb{P}^0(K,C)^\wedge/\mathbb{H}^0(K,C)^\wedge$.

The corollary now follows from the following commutative diagram:

$$\begin{array}{ccccc} \mathbb{H}^0(K,C) & \hookrightarrow & \mathbb{P}^0(K,C) & \xrightarrow{\vartheta} & \mathbb{H}^2(K,C^d)^D \\ \downarrow & & \downarrow & & \| \\ (\mathbb{H}^0(K,C))^\wedge & \longrightarrow & (\mathbb{P}^0(K,C))^\wedge & \xrightarrow{\vartheta} & \mathbb{H}^2(K,C^d)^D \end{array},$$

of which both rows are complexes and the second one is exact. □

Lemma 3.6.9. *Let C be a good complex, defined over a local field K_v. Then, the pairing:*

$$\mathbb{H}^0(K_v,C)^\wedge_\bullet \times \mathbb{H}^2(K_v,C^d)_\bullet \to H^2(K_v,\overline{K_v}^*) \to \mathbb{Q}/\mathbb{Z}, \tag{3.6.9}$$

is perfect.

Proof. We focus on the proof for non-archimedean v, the proof when v is archimedean follows the same pattern with regular cohomology replaced by Tate cohomology. The proof follows the same devissage as the one in Lemma 3.6.7, described in diagram (3.6.3) and (3.6.7), of which we borrow the notation. We start with the case that C is middle exact. We have the following distinguished triangles:

$$F[2] \to C \to M \to F[3],$$

and

$$F^d[-3] \to M^d \to C^d \to F^d[-2].$$

We deduce that the rows of the following commutative diagram are exact:

$$\begin{array}{ccccccccc} \mathbb{H}^{-1}(K_v, M) & \longrightarrow & H^2(K_v, F) & \longrightarrow & \mathbb{H}^0(K_v, C) & \longrightarrow & \mathbb{H}^0(K_v, M) & \longrightarrow & H^3(K_v, F) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathbb{H}^3(K_v, M^d)^D & \longrightarrow & H^0(K_v, F^d)^D & \longrightarrow & \mathbb{H}^2(K_v, C^d)^D & \longrightarrow & \mathbb{H}^2(K_v, M^d)^D & \longrightarrow & H^{-1}(K_v, F^d)^D. \end{array} \tag{3.6.10}$$

Since $H^2(K_v, F)$ and $H^3(K_v, F) (\cong 0)$ are finite groups, $\mathbb{H}^0(K_v, C)$ and $\mathbb{H}^0(K_v, M)$ are endowed with their profinite topologies, and $\mathbb{H}^{-1}(K_v, M) = \mathbb{H}^{-1}_\wedge(K_v, M) \cong 0$ (see [HS05, Sec. 2] for the definition of $\mathbb{H}^{-1}_\wedge$), we deduce that the first row of the following diagram is exact:

$$\begin{array}{ccccccccc} \mathbb{H}^{-1}_\wedge(K_v, M) & \longrightarrow & H^2(K_v, F) & \longrightarrow & \mathbb{H}^0(K_v, C)^\wedge & \longrightarrow & \mathbb{H}^0(K_v, M)^\wedge & \longrightarrow & H^3(K_v, F) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathbb{H}^3(K_v, M^d)^D & \longrightarrow & H^0(K_v, F^d)^D & \longrightarrow & \mathbb{H}^2(K_v, C^d)^D & \longrightarrow & \mathbb{H}^2(K_v, M^d)^D & \longrightarrow & H^{-1}(K_v, F^d)^D \end{array} . \tag{3.6.11}$$

Since the first and fourth columns are isomorphisms by [HS05, Thm 0.1], and the second and fifth columns are isomorphisms by local duality for finite Galois modules, we deduce that the middle column is an isomorphism as well, concluding the proof of the middle exact case. For the general case, we use the following distinguished triangle (again, we borrow the notation used in the proof of the previous lemma):

$$P \to C' \to C \to P[1], \tag{3.6.12}$$

from which we deduce the exactness of the rows of the following commutative diagram:

$$\begin{array}{ccccccc} H^0(K_v, P) & \longrightarrow & \mathbb{H}^0(K_v, C') & \longrightarrow & \mathbb{H}^0(K_v, C) & \longrightarrow & H^1(K_v, P) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^2(K_v, P^d)^D & \longrightarrow & H^2(K_v, (C')^d)^D & \longrightarrow & H^2(K_v, C^d)^D & \longrightarrow & H^1(K_v, P^d)^D. \end{array} \tag{3.6.13}$$

Since P is quasi-trivial, we deduce that the last column is 0. Moreover, since all groups appearing on the upper row are endowed with their profinite topologies, and profinite completion is a right exact functor, we deduce the exactness of the rows of the following commutative diagram:

$$\begin{array}{ccccccc} H^0(K_v, P)^\wedge & \longrightarrow & \mathbb{H}^0(K_v, C')^\wedge & \longrightarrow & \mathbb{H}^0(K_v, C)^\wedge & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^2(K_v, P^d)^D & \longrightarrow & H^2(K_v, (C')^d)^D & \longrightarrow & H^2(K_v, C^d)^D & \longrightarrow & 0. \end{array} \tag{3.6.14}$$

Since the second column is an isomorphism by the previous case, and the first is an isomorphism by local duality for tori, we deduce that the third column is an isomorphism as well. □

3.6.2 Main theorem with connected stabilizers and removed places

Let $\text{Б}_\infty(X) \subset \operatorname{Br}_{1,ur}(X)$ be defined as:

$$\text{Б}_\infty(X) := \operatorname{Ker}\left(\operatorname{Br}_1(X) \to \prod_{v \in M_K^{fin}} \operatorname{Br}(X_{K_v}) / \operatorname{Br} K_v\right). \tag{3.6.15}$$

Proof of Theorem 3.6.1. We claim that we may assume, without loss of generality that $X(K) \neq \emptyset$. In fact, we have the following inclusion:

$$X(\mathbb{A}_K)^{\mathrm{Br}^{S_0} X} \subset X(\mathbb{A}_K)^{\text{Б}_\infty(X)}.$$

In particular, if $X(\mathbb{A}_K)^{\mathrm{Br}^{S_0} X} \neq \emptyset$, then $X(\mathbb{A}_K)^{\text{Б}_\infty(X)} \neq \emptyset$, and, by Theorem 3.B.1, we deduce that $X(K) \neq \emptyset$. On the other hand, if $X(\mathbb{A}_K)^{\mathrm{Br}^{S_0} X} = \emptyset$, then, since $X(K) \subset X(\mathbb{A}_K)^{\mathrm{Br}^{S_0} X}$, there is nothing to prove. So this concludes the proof of the claim. From now on, we can and will assume that $X = G/H$, with H linear and connected. We may and will use all the abelianization paraphernalia of sections 3.3.2 and 3.3.3 (in particular Theorem 3.5.1), of which we borrow the notation as well.

We notice the following:

$$\overline{X(K)}^{S_0} = \pi^S(\overline{X(K)_{S_0}}), \tag{3.6.16}$$

where we are using the notations introduced in Section 3.2 for adelic-like objects.

We have the following commutative diagram of Hausdorff topological spaces, where the rows are exact (the first is exact in a set-wise sense, in fact the middle term is even a direct product of the other two here) and every morphism is continuous:

$$\begin{array}{ccccc}
X(K_{S_0}) & \longrightarrow & X(\mathbb{A}_K)_\bullet & \xrightarrow{\pi^{S_0}} & X(\mathbb{A}_K^{S_0})_\bullet \\
\downarrow{\scriptstyle \mathrm{ab}_{S_0}} & & \downarrow{\scriptstyle \mathrm{ab}} \quad \vartheta & & \downarrow{\scriptstyle \mathrm{ab}^{S_0}} \\
\mathbb{H}^0(K_{S_0}, C_X) & \overset{\iota}{\hookrightarrow} & \mathbb{P}^0(K, C_X) & \longrightarrow & \mathbb{P}^0_{S_0}(K, C_X) \quad \vartheta^{S_0}, \\
\downarrow{\scriptstyle \vartheta'_{S_0}} & & \downarrow{\scriptstyle \vartheta'} & & \downarrow \\
\mathbb{H}^2(K_{S_0}, C_X^d)^D & \xrightarrow{\pi^D_{S_0}} & \mathbb{H}^2(K, C_X^d)^D & \longrightarrow & \mathbb{H}^2_{S_0}(K, C_X^d)^D
\end{array} \tag{3.6.17}$$

where $\mathbb{H}^0_{S_0}(K, C_X^d) := \mathrm{Ker}(\mathbb{H}^2(K, C_X^d) \to \mathbb{H}^2(K_{S_0}, C_X^d))$. We remind the reader that there is a natural morphism $\mathbb{H}^2_{S_0}(K, C_X^d) \xrightarrow{\alpha} \mathrm{Br}^{S_0}(X)$, and that this is compatible with the Brauer–Manin obstruction in the sense of Theorem 3.5.1 . We hence have the following sequence of inclusions:

$$X(\mathbb{A}_K)_\bullet^{\alpha(\mathbb{H}^2_{S_0}(K, C_X^d))} \supset X(\mathbb{A}_K)_\bullet^{\mathrm{Br}^{S_0}(X)} \supset \overline{G^{scu}(K_{S_f \setminus S_0}) \cdot X(K)_{S_0}}.$$

Therefore, because of (3.6.16), to prove Theorem 3.6.1 it is enough to prove that $\mathrm{Ker}(\vartheta^{S_0} \circ \pi^{S_0}) = X(\mathbb{A}_K)_\bullet^{\alpha(\mathbb{H}^2_{S_0}(K, C_X^d))} \subset \overline{G^{scu}(K_{S_f \setminus S_0}) \cdot X(K)_{S_0}}$ (we use, with slight abuse of notation, the symbol Ker to denote the fiber of 0). We have that

i. $\overline{G^{scu}(K_{S_f \setminus S_0}) \cdot X(K)_{S_0}} = \overline{(\mathrm{Ker}\,\vartheta)_{S_0}}$ by Theorem 3.3.7,

ii. $(\mathrm{Ker}\,\vartheta)_{S_0} = \mathrm{ab}^{-1}((\mathrm{Ker}\,\vartheta')_{S_0})$, as it easily follows from the commutativity of (3.6.17), the fact that ab_{S_0} is surjective by [Dem13, Prop. 2.18] and Lemma 3.A.2,

iii. $\overline{(\mathrm{Ker}\,\vartheta)_{S_0}} \supset \mathrm{ab}^{-1}\left(\overline{(\mathrm{Ker}\,\vartheta')_{S_0}}\right)$ by the point above, the openness (proved in Lemma 3.3.6) of $\mathrm{ab} : X(\mathbb{A}_K)_\bullet \to \mathbb{P}^0(K, C_X)$ and Lemma 3.A.3,

iv. $\mathrm{Ker}(\vartheta^{S_0} \circ \pi^{S_0}) = \vartheta^{-1}(\mathrm{Im}(\pi^D_{S_0})) = \mathrm{ab}^{-1}(\vartheta'^{-1}(\mathrm{Im}(\pi^D_{S_0})))$ by the commutativity of (3.6.17) and the exactness of its third row.

Hence, by the points above, it is sufficient that we prove that $\vartheta'^{-1}(\mathrm{Im}(\pi^D_{S_0})) = \overline{(\mathrm{Ker}\,\vartheta')_{S_0}}$. We have the following factorization of the morphism ϑ':

$$\mathbb{P}^0(K, C_X) \to \mathbb{P}^0(K, C_X) \Big/ \overline{\mathbb{H}^0(K_{S_0}, C_X)} \to \mathbb{P}^0(K, C_X) \Big/ \mathrm{Ker}\,\vartheta' \overset{\vartheta''}{\hookrightarrow} \mathbb{H}^2(K, C_X^d)^D. \tag{3.6.18}$$

We have that:

$$\begin{aligned}\vartheta'\left(\overline{\iota(\mathbb{H}^0(K_{S_0},C_X))\cdot(\operatorname{Ker}\vartheta')}\right) &= \vartheta''\left(\overline{\iota(\mathbb{H}^0(K_{S_0},C_X))\cdot(\operatorname{Ker}\vartheta')}/(\operatorname{Ker}\vartheta')\right)\\ &= \overline{\vartheta''\left(\left(\iota(\mathbb{H}^0(K_{S_0},C_X))\cdot(\operatorname{Ker}\vartheta')\right)/(\operatorname{Ker}\vartheta')\right)}\\ &= \overline{\pi^D_{S_0}\left(\vartheta'_{S_0}(\mathbb{H}^0(K_{S_0},C_X))\right)} = \pi^D_{S_0}\left(\overline{\vartheta'_{S_0}(\mathbb{H}^0(K_{S_0},C_X))}\right)\\ &= \pi^D_{S_0}\left(\mathbb{H}^2(K_{S_0},C^d_X)^D\right) = \operatorname{Im}(\pi^D_{S_0}),\end{aligned} \tag{3.6.19}$$

where the second and fourth identity follow from Lemma 3.A.4 (whose hypothesis hold by Lemma 3.6.7 and Corollary 3.6.8 for the second identity and by the fact that the dual of a torsion group is profinite, hence compact, for the fourth identity) , the third by the commutativity of the lower-left square of (3.6.17), and the fifth one follows from the fact that ϑ'_{S_0} has dense image in $\mathbb{H}^2(K_{S_0},C^d_X)^D$ (by 3.3.6). Now, it easily follows from (3.6.19) that $\vartheta'^{-1}(\operatorname{Im}(\pi^D_{S_0})) = \overline{(\operatorname{Ker}\vartheta')_{S_0}}$. $\square$

Proof of Theorem 3.6.5. The inclusion $\overline{G^{scu}(K_{S_f\setminus S_0})\cdot X(K)}^{S_0} \subset X(\mathbb{A}_K^{S_0})_\bullet^{\acute{e}t_{S_0},\mathrm{Br}^{S_0}}$ follows from the fact that $G^{scu}(K_{S_f})\cdot X(K)\subset X(\mathbb{A}_K^{S_0})_\bullet^{\acute{e}t_{S_0},\mathrm{Br}^{S_0}}$ (which follows from Lemmas 3.4.21 and 3.4.22 as in the proof of Theorem 3.1.1) and the fact that the latter is closed.

The inclusion $\overline{G^{scu}(K_{S_f\setminus S_0})\cdot X(K)}^{S_0} \supset X(\mathbb{A}_K)_\bullet^{\acute{e}t_{S_0},\mathrm{Br}^{S_0}}$ can be proven as follows. Let $\alpha\in X(\mathbb{A}_K)_\bullet^{\acute{e}t_{S_0},\mathrm{Br}^{S_0}}$, using Lemmas 3.4.17 and 3.4.18 as in the proof of Theorem 3.1.1, we know that there is a (right) torsor $Z\xrightarrow{\varphi} X$ under a finite group scheme F, such that Z is a (left) homogeneous space under G with geometrically connected stabilizers. Since $\alpha\in X(\mathbb{A}_K)_\bullet^{\acute{e}t_{S_0},\mathrm{Br}^{S_0}}$, we may assume, up to twisting Z by some cocycle $\in H^1_{lt}(K,F)$, that there is a $\beta\in Z(\mathbb{A}_K)^{\mathrm{Br}^{S_0}Z}$ such that $\varphi(\beta)=\alpha$. Since we know, by Theorem 3.6.1, that $\beta\in\overline{G^{scu}(K_{S_f\setminus S_0})\cdot Z(K)}^{S_0}$, we deduce that $\alpha\in\varphi\left(\overline{G^{scu}(K_{S_f\setminus S_0})\cdot Z(K)}^{S_0}\right)\subset\overline{G^{scu}(K_{S_f\setminus S_0})\cdot X(K)}^{S_0}$. $\square$

With the same method of proof, one may obtain the following, which is, in some sense, a limit of Theorem 3.6.5 as S_0 grows to the whole M_K^{fin} (putting $S=S_0\cap M_K^{fin}$):

Proposition 3.6.10. *Let G be a connected K-group, X be a (left) G-homogeneous space with linear stabilizers. We assume that the Tate-Shafarevich group $Ш(K,G^{ab})$ is finite. We then have that $X(K)\neq\emptyset$ if and only if $X(\mathbb{A}_K)_\bullet^{\acute{e}t,Б(X)}\neq\emptyset$.*

Proof. The implication $X(K)\neq\emptyset\Rightarrow X(\mathbb{A}_K)_\bullet^{\acute{e}t,Б(X)}\neq\emptyset$ is clear, since $X(K)\subset X(\mathbb{A}_K)_\bullet^{\acute{e}t,Б(X)}$. On the other hand, assume that $X(\mathbb{A}_K)_\bullet^{\acute{e}t,Б(X)}\neq\emptyset$, then combining Lemma 3.4.17 and Lemma 3.4.18, we know that there exists a finite group scheme F, a right F-torsor $\varphi:Y\to X$ such that Y is (left) homogeneous space with geometrically connected stabilizers. Moreover, since there exists a $(P_v)\in X(\mathbb{A}_K)_\bullet^{\acute{e}t,Б(X)}$, we may assume (up to twisting Y) that there exists an adelic point $(Q_v)\in Y(\mathbb{A}_K)^{Б(Y)}$. Hence, by Theorem 3.B.4 (see also the discussion that follows the theorem) there exists a $Q\in Y(K)$, hence $\varphi(Q)\in X(K)\neq\emptyset$. $\square$

3.A Appendix: Topological and set-theoretic lemmas

In this appendix, we will use the following notation:

Notation 3.A.1. Let X and A be non-empty sets, if $Y\subset X\times A$, we denote by Y_A the set $\pi_A^{-1}(\pi_A(Y))$, where $\pi_A : X\times A\to A$ is the projection on the second factor.

We warn that this notation is similar to one defined in Section 3.2, which was referring to the particular case of adele-like sets. We believe that there should be no risk of confusion, since 3.A.1 is only used in this appendix.

Lemma 3.A.2. *Let X, Y, A, A' be non-empty sets and assume we have functions $f : X \to Y$, $p : A' \to A$, and a subset $Z \subset Y \times A$. If p is surjective, we have that:*

$$((f \times p)^{-1}(Z))_{A'} = (f \times p)^{-1}(Z_A),$$

where we are using the notation 3.A.1.

Proof. The proof is straightforward. □

Lemma 3.A.3. *Let $f : X \to Y$ be an open morphism of topological spaces. We have that, for any subset $Z \subset Y$, $\overline{f^{-1}(Z)} \supset f^{-1}(\overline{Z})$.*

Proof. For any $U \subset X$ disjoint from $f^{-1}(Z)$, the image $f(U) \subset Z$ is open and disjoint from Z, hence from $\overline{Z}$. Unraveling the definitions, the lemma follows. □

Lemma 3.A.4. *Let $\alpha : X \to Y$ be a continuous map of topological spaces, with X compact and Y Hausdorff. Then, for any subset $S \subset X$ we have that $\overline{\alpha(S)} = \alpha(\overline{S})$.*

Proof. This is common knowledge. □

Lemma 3.A.5. *Let B be a topological abelian group (not necessarily Hausdorff), let $0 \in B$ be the unit element, and let $D = \overline{\{0\}}$ be its closure. Then, B is quasi-compact if and only if the quotient B/D, which is Hausdorff, is compact.*

Proof. If B is quasi-compact, then B/D, being a quotient of it, is clearly quasi-compact as well.

If B/D is compact, we are going to prove the quasi-compactness of B by showing that, if $\mathcal{C} = \{C_i\}_{i \in I}$ is a collection of closed subsets of B such that $\cap_{i \in I} C_i = \emptyset$, then there exists a finite subset of indexes $I_0 \subset I$ such that $\cap_{i \in I_0} C_i = \emptyset$. In fact we notice that, whenever $P \in C_i$, then $\pi^{-1}(\pi(P)) = P + D = \overline{P} \subset C_i$, where $\pi : B \to B/D$ denotes the projection. Hence, for each $i \in I$, we have that $C_i = \pi^{-1}(\pi(C_i))$. Therefore, $\cap_{i \in I} \pi(C_i) = \emptyset$, and there exists a finite $I_0 \subset I$ such that $\cap_{i \in I_0} \pi(C_i) = \emptyset$. It follows that $\cap_{i \in I_0} C_i = \cap_{i \in I_0} \pi^{-1}(\pi(C_i)) = \pi^{-1}(\cap_{i \in I_0} \pi(C_i)) = \emptyset$, as wished. □

Lemma 3.A.6. *Let B be a topological group, and let D and K be two subsets of B, where D is closed and K is quasi-compact. Then the sum $D + K$ is closed in B.*

Proof. Since D is closed, we know that the topological quotient B/D is Hausdorff. Let $\pi : B \to B/D$ be the projection. We have that $D + K = \pi^{-1}(\pi(K))$. Since K is quasi-compact, $\pi(K) \subset B/D$ is as well. Hence, since B/D is Hausdorff, $\pi(K)$ is closed. Therefore $\pi^{-1}(\pi(K))$, i.e. $D + K$ is closed as well. □

Lemma 3.A.7. *Let $A \xrightarrow{\varphi} B \xrightarrow{\psi} C \xrightarrow{\alpha} F$ be an exact sequence of topological groups, where all the morphisms are continuous. Assume that F is finite and discrete, ψ is open, and A and C are quasi-compact. Then B is quasi-compact as well.*

Proof. First of all we may assume, up to changing C with $\operatorname{Ker} \alpha$, without loss of generality, that $F = \{1\}$.

Let us now prove that the morphism ψ is closed. Let $D \subset B$ be a closed subset, and let $D' := A' + D$ be the sum of the image A' under φ of A and D. Since A is quasi-compact, so is A'. Hence, by Lemma 3.A.6, $D' := A' + D$ is closed. Since $\psi(D) = \psi(D')$, and $\psi(B \setminus D') = C \setminus \psi(A' + D) = C \setminus \psi(D)$, and the former is open, one has that $\psi(D)$ is closed, as wished.

We now prove the compactness of B. Let I be a set of indices, and $B = \cup_{i \in I} U_i$ be a covering of B. Let, for each $c \in C$, $\{U_1^c, \ldots, U_{n(c)}^c\}$ be a finite subcovering of $\mathcal{U}$ such that $\psi^{-1}(c) \subset \cup_{i=1}^{n(c)} U_i^c$

(the subcovering may always be assumed to be finite since $\psi^{-1}(c)$, being a translate of A' is quasi-compact). Let now,

$$V_c := \left\{ c' \in C \mid \psi^{-1}(c') \subset \bigcup_{i=1}^{n(c)} U_i^c \right\} = C \setminus \psi \left(B \setminus \cup_{i=1}^{n(c)} U_i^c \right),$$

which is open (since ψ is closed).

Since C is compact and $\cup_{c \in C} V_c = C$, there exist a finite number $n \in \mathbb{N}$ and $c_1, \ldots, c_n \in C$ such that $\cup_{j=1}^{n} V_{c_j} = C$. It is straightforward to verify that then $\cup_{j=1}^{n} \cup_{i=1}^{n(c_j)} U_i^{c_j} = B$. □

3.B Appendix: Hasse principle

The following result is basically already present in [BCTS08, Appendix A]. However, since it is not explicitly stated there, we represent it here for completeness, with a proof that is just a simplified version of the proof of [BCTS08, Theorem A.1].

Theorem 3.B.1. *Let K be a number field and X be a left homogeneous space under a connected algebraic group G/K, satisfying $Ш(K, G^{ab})$ finite. We assume that the G-action on X has connected geometric stabilizers. We then have that $X(K) \neq \emptyset$ if and only if $X(\mathbb{A}_K)^{Б_\infty(X)} \neq \emptyset$ (see (3.6.15) for the definition of $Б_\infty(X)$).*

We also include a second formulation, which seems less interesting per se, but it gives a cleaner version of Proposition 3.6.10. For this second formulation, let us define by X^{ab} the maximal abelian torsor quotient of X. I.e., the (GIT) quotient X/G^{lin}. Note that there is a natural morphism $\pi_X : X \to X^{\mathrm{ab}}$.

Theorem 3.B.2. *In Theorem 3.B.1, the condition $X(K) \neq \emptyset$ is also equivalent to $X(\mathbb{A}_K)^{Б(X)} \cap \pi_X^{-1}(X^{ab}(\mathbb{A}_K)^{ét}) \neq \emptyset$.*

Note that the above condition holds, for instance, when $X(\mathbb{A}_K)^{ét, Б(X)} \neq \emptyset$ (indeed we have that $\pi_X(X(\mathbb{A}_K)^{ét}) \subseteq X^{\mathrm{ab}}(\mathbb{A}_K)^{ét}$).

Remark 3.B.3.

i. For any smooth geometrically connected variety X/K (3.6.15) describes the Brauer set that is locally constant on non-archimedean places. We notice that, if $B \in \mathrm{Br}(X)$ is locally constant for all $v \notin S$ (with S finite), then, since there exists a smooth model $\mathcal{X}$ for X over some $\operatorname{Spec} O_{K,S'}$ (with $S' \supset S$ finite), such that $B \in \mathrm{Br}(\mathcal{X})$, and, by enlarging S', we may assume, by Lang-Weil estimates, that $\mathcal{X}(O_v) \neq \emptyset$ for all $v \notin S' \cup M_K^\infty$, we have that B is necessarily 0 for all $v \notin S' \cup M_K^\infty$ (since it is constant, and its value on the integral points is automatically 0).

ii. We notice that $Б_\infty(X)$ differs from the classical $Б(X)$ just by the behaviour at M_K^∞. In particular, if K is totally imaginary, then $Б_\infty(X) = Б(X)$. So, in this case, Theorem 3.B.1 reduces to [BCTS08, Thm 3.4], so it is already stated explicitly in the paper [BCTS08].

iii. We notice that, by [Har94, Thm. 2.1.1] and the first point of this remark, for any smooth geometrically connected variety Y/K, all elements of $Б_\infty(Y)$ are unramified elements of $\mathrm{Br}(Y)$.

We will actually prove the following stronger result, which is, however, a bit more involuted in its formulation. This does not simplify the proof of the theorem, so the only reason we do so is because this stronger result is needed in Proposition 3.6.10.

We use the notation X^{ab} to denote the maximal abelian torsor quotient of X. I.e., the quotient X/G^{lin}. Note that there is a natural morphism $\pi_X : X \to X^{\mathrm{ab}}$. Moreover, we define:

$$Б_{weird} X := \{ b \in Б_\infty(X) \mid b_v \in \pi_X^* \operatorname{Br} X_{K_v}^{\mathrm{ab}} \ \forall v \in M_K^\infty \}.$$

Theorem 3.B.4. *In the setting of Theorem 3.B.1, the condition $X(K) \neq \emptyset$ is equivalent to $X(\mathbb{A}_K)^{Б_{weird}(X)} \neq \emptyset$.*

For the application to Proposition 3.6.10, we need the following lemma.

Lemma 3.B.5. $X(\mathbb{A}_K)^{ét,Б} \subseteq X(\mathbb{A}_K)^{Б_{weird}(X)}$.

Proof. To prove this, we may assume that $X(\mathbb{A}_K)^{ét,Б} \neq \emptyset$. In this case, we deduce by functoriality that $X^{\mathrm{ab}}(\mathbb{A}_K)^{ét,Б} \neq \emptyset$, hence $X^{\mathrm{ab}}(\mathbb{A}_K)^{Б(X^{\mathrm{ab}})} \neq \emptyset$. By [Har06] this implies that $X^{\mathrm{ab}}(K) \neq \emptyset$, or, in other words, $X^{\mathrm{ab}} \cong G^{\mathrm{ab}}$. In particular, there exists an isogeny $[2] : X^{\mathrm{ab}} \to X^{\mathrm{ab}}$, defined just by translating the isogeny $[2] : G^{\mathrm{ab}} \to G^{\mathrm{ab}}$.

Recall that, for a principal homogeneous space Y of an abelian variety A defined over a field F with $H^3(F, \overline{F}^*) = 0$ (such as global or local fields), one has that $\mathrm{Br}_1 Y/\operatorname{Im}\mathrm{Br}\, F \cong H^1(F, A')$, where A' is the dual abelian variety, and the isomorphism is functorial in Y. Moreover, when F is a real field, note that one has that $H^1(F, A')$ is 2-torsion (by a standard restriction-corestriction argument), and hence is annihilated by the morphism $H^1(F, A') \xrightarrow{[2]} H^1(F, A')$, where $[2] : A' \to A'$ denotes multiplication by 2.

It follows that the cover $[2] : G^{\mathrm{ab}} \to G^{\mathrm{ab}}$ satisfies the following property: for every $b \in \mathrm{Br}_1 G^{\mathrm{ab}}_{K_v}$, with v real, and every twist $[2]_\sigma : G^{\mathrm{ab}}_{K_v,\sigma} \to G^{\mathrm{ab}}_{K_v}$ by a $\sigma \in Z^1(K_v, G^{\mathrm{ab}}[2])$, we have that $[2]^*_\sigma b$ is constant.(The same holds also for v complex, but it is trivial.) It follows that, for every twist f_σ of the $G^{\mathrm{ab}}[2]$-étale cover $f : X \times_{X^{ab},[2]} X^{ab} \to X$ (by a $\sigma \in Z^1(K, G^{\mathrm{ab}}[2])$), the pullback $f^*_\sigma Б_{weird}(X)$ is contained in $Б(X)$. Hence:

$$X(\mathbb{A}_K)^{ét,Б} \subseteq \bigcup_{\sigma \in Z^1(K,G^{\mathrm{ab}}[2])} f_\sigma(X(\mathbb{A}_K)^{Б(X)}) \subseteq \bigcup_{\sigma \in Z^1(K,G^{\mathrm{ab}}[2])} f_\sigma(X(\mathbb{A}_K)^{f^*_\sigma Б_{weird}(X)}) \subseteq X(\mathbb{A}_K)^{Б_{weird}(X)},$$

as wished. □

Proof of Theorem 3.B.4. As said before, we follow step-by-step the reductions of [BCTS08, Theorem A.1].

If $X(K) \neq \emptyset$, then $\emptyset \neq X(K) \subset X(\mathbb{A}_K)^{Б_{weird}(X)}$. So we focus now on proving the other direction. Namely, we assume $X(\mathbb{A}_K)^{Б_{weird}(X)} \neq \emptyset$.

We do a first reduction to show that it is enough to prove the result for G such that G^{lin} is reductive.

Let $Y \coloneqq G^u\backslash X$, $G' \coloneqq G/G^u$, so that Y is a G' homogeneous space (this is because G^u is normal in G). Let $\varphi_G : G \to G'$ be the standard projection. We have a canonical morphism $\varphi : X \to Y$, that is φ_G-equivariant. We assume that we already know the result for (Y, G'). We notice that $\varphi^* Б_{weird}(Y) \subset Б_{weird}(X)$. In particular, if $(x_v) \in X(\mathbb{A}_K)^{Б_{weird}(X)}$, then $(\varphi(x_v)) \in Y(\mathbb{A}_K)^{Б_{weird}(Y)}$. So, we deduce, from the reduction assumption, that $Y(K) \neq \emptyset$. Let $y_0 \in Y(K)$. We then have that $X_{y_0} \to y_0 \cong \operatorname{Spec} K$ is a homogeneous space of G^u. In particular, by [Bor95, Lem 3.2], we deduce that $X_{y_0}(K) \neq \emptyset$, concluding the proof of this reduction step.

For the second reduction step, we know by [BCTS08, Proposition 3.1] that there exists a group $\tilde{G}$ such that X may be regarded as a homogeneous space under $\tilde{G}$, with linear connected stabilizers and such that $\tilde{G}^{ss}$ is semisimple simply connected. Moreover, by [BCTS08, Lemma A.3] we still have that $Ш(K, \tilde{G}^{ab})$ is finite. So we can and will assume from now on that G is such that $G^u = \{1\}$, $\tilde{G}^{ss}$ is semisimple simply connected, and the geometric stabilizer $\bar{H} \subset G^{lin}$ is connected.

The homogeneous space X defines a K-form M of $\bar{H}^{mult} = \bar{H}/\bar{H}^{ssu}$ (see [Bor95, Sec. 4.1]), the largest quotient of $\bar{H}$ of multiplicative type, and a natural homomorphism $\chi_X : M \to G^{sab} \coloneqq G/G^{ss}$.

We treat the case where χ_X is injective first (i.e. $\bar{H}^{ssu} = \bar{G} \cap \bar{G}^{ss}$). In this case, let $Y' \coloneqq X/G^{ss}$, and $\psi : X \to Y'$ be the standard projection. We have that Y' is a homogeneous space with linear stabilizers under $G^{sab} = G/G^{ss}$, a semi-abelian variety, and is therefore, a torsor under a semi-abelian variety G'' itself. Moreover, since Y' has linear stabilizers by the G^{sab}-action, we have that $G^{ab} \cong (G'')^{ab}$, and, consequently $Ш((G'')^{ab}) \cong Ш(G^{ab})$, so that $Ш(G'')$ is finite.

Let $(x_v) \in X(\mathbb{A}_K)^{Б_{weird}(X)}$, and let $\mathcal{U} \subset X(\mathbb{A}_K)^{Б_{weird}(X)}$ be the open subset defined as $\prod'_{v\in M_K^{fin}} X(K_v) \times \prod_{v\in M_K^\infty} C_{x_v}$, where $\prod'$ denotes the usual restricted product defining the adele sets (see Section 3.2), and $C_{x_v} \subset X(K_v)$ denotes, for an archimedean v, the connected component in which lies x_v. Since the Brauer–Manin pairing is constant on the connected components of the archimedean places, we have that $\mathcal{U} \subset X(\mathbb{A}_K)^{Б_{weird}(X)}$. We have by [BCTS08, Lem. A.2], that, for each $v \in M_K^\infty$, $\psi(C_{x_v}) \subset Y(K_v)$ is a connected component of $Y(K_v)$. So, if we define $\mathcal{V} \subset Y(\mathbb{A}_K)$ to be $\prod'_{v\in M_K^{fin}} Y(K_v) \times \prod_{v\in M_K^\infty} \psi(C_{x_v})$, we have that $\emptyset \neq \mathcal{V} \subset Y(\mathbb{A}_K)^{Б_{weird}(Y)}$ (since $\psi^* Б_{weird}(Y) \subset Б_{weird}(X)$). By the lemma below this implies that $\emptyset \neq \mathcal{V} \subset Y(\mathbb{A}_K)^{Б_\infty(Y)}$, and, by [Har06], there exists a $y_1 \in \mathcal{V} \cap Y(K)$.

Lemma 3.B.6. *We have that* $Б_{weird}(Y) = Б_\infty(Y)$.

Proof. According with our notation, we denote by $\pi_Y : Y \to Y^{ab}$ the maximal abelian torsor quotient of Y.

Taking the long exact sequence of cohomology associated to the following exact triangle:

$$[\overline{K}(Y^{ab})^*/\overline{K}^* \to \mathrm{Div}\,\overline{Y^{ab}}] \xrightarrow{\pi_Y^*} [\overline{K}(Y)^*/\overline{K}^* \to \mathrm{Div}\,\overline{Y}] \cong [\overline{K}(Y)^*_{vert}/\overline{K}^* \to \mathrm{Div}_{vert}\,\overline{Y}]$$
$$\to [\widehat{T} \to 0] \to [\overline{K}(Y^{ab})^*/\overline{K}^* \to \mathrm{Div}\,\overline{Y^{ab}}][1],$$

where *vert* stands for vertical with respect to π_Y, and the second morphism is defined by restriction on the fibers of π_Y, and using the isomorphism [HS08, Lemma 2.1] for Y and Y^{ab}, we deduce that there is an exact sequence

$$\mathrm{Br}_a(Y^{ab}) \xrightarrow{\pi_Y^*} \mathrm{Br}_a Y \xrightarrow{\alpha} H^2(K, \widehat{T}).$$

Note that $\alpha(Б_\infty(Y)) \subseteq Ш^2_\infty(K, \widehat{T}) := \mathrm{Ker}(H^2(K, \widehat{T}) \to \prod_{v\in M_K^{fin}} H^2(K_v, \widehat{T}))$.

We claim that $Ш^2_\infty(K, \widehat{T}) = Ш^2(K, \widehat{T})$. This follows from a more-or-less standard Chebotarev argument, which we present for the sake of completeness. Since $H^2(K, \widehat{T}) = \lim_L H^2(L/K, \widehat{T})$, where L varies among finite Galois extension of K that split T, we may prove that $Ш^2_\infty(L/K, \widehat{T}) = Ш^2(L/K, \widehat{T})$ (the notation here is the intuitive one). Since $H^2(F, \widehat{T}) \cong H^1(F, \widehat{T} \otimes \mathbb{Q}/\mathbb{Z})$, functorially in the field F, we have to prove that $Ш^1_\infty(L/K, \widehat{T} \otimes \mathbb{Q}/\mathbb{Z}) = Ш^1(L/K, \widehat{T} \otimes \mathbb{Q}/\mathbb{Z})$. Letting $\Gamma := \mathrm{Gal}(L/K)$, we have that $Ш^1(L/K, \widehat{T} \otimes \mathbb{Q}/\mathbb{Z}) = \mathrm{Ker}(H^1(\Gamma, \widehat{T}(L) \otimes \mathbb{Q}/\mathbb{Z}) \to \prod_{v\in M_K} H^1(\Gamma_v, \widehat{T}(L) \otimes \mathbb{Q}/\mathbb{Z}))$, where Γ_v denotes the decomposition group of L/K at v. We may express $Ш^1_\infty$ analogously by replacing $\prod_{v\in M_K}$ with $\prod_{v\in M_K^{fin}}$. For all archimedean places v, Γ_v is a cyclic subgroup of Γ. It follows by Chebotarev's density theorem that there are infinitely many unramified **finite** places w such that $\Gamma_w = \Gamma_v$. In particular, it immediately follows that $Ш^1_\infty = Ш^1$, thus concluding the proof of the claim.

It follows that for all $b \in Б_\infty(Y)$, $\alpha(b_v) = 0$ for all $v \in M_K^\infty$, and hence $b_v \in \mathrm{Im}\,\mathrm{Br}_1(Y^{ab}_{K_v})$ for these v. Hence, since certainly $b_v \in \mathrm{Im}\,\mathrm{Br}\,K_v$ for all $v \in M_K^{fin}$, $b \in Б_{weird}(Y)$. We have just proves the inclusion $Б_\infty(Y) \subseteq Б_{weird}(Y)$. Since the other inclusion is clear, this concludes the proof. □

The fiber X_{y_1} is a homogeneous space under the semisimple simply connected group G^{ss}, with geometric stabilizers isomorphic to $\bar{H}^{ssu}$. Moreover, by construction, X_{y_1} has real points in all real places. Hence, by [Bor93, Cor. 7.4], there exists a rational point $x' \in X_{y_1}(K)$, concluding this case.

We turn now to the general case.

We construct as in the proof of [BCTS08, Theorem A.5] a quasi-trivial torus P, a P-torsor $\varphi : Z \to X$, such that Z is a $G \times P$ homogeneous space, and φ is equivariant by $\pi : G \times P \to G$. Moreover, as in [BCTS08, Theorem A.5], we may and will assume that $(Z, G \times P)$ satisfies all of the reductions above, that the geometric stabilizers are still isomorphic to $\bar{H}$, and that the homomorphism $M \cong M_Z \to (G \times P)^{sab} \cong G^{sab} \times P$ is injective (here M_Z denotes the K-form of $\bar{H}^{mult}$ defined by Z, which happens to be, in this case, isomorphic to M). Since, by Hilbert Theorem 90 and Shapiro's Lemma, the P_{K_v}-torsors $Z_{x_v} \to x_v$ are trivial for each $v \in M_K$, there exists an adelic point $(z_v) \in Z(A_K)$ such that $(\varphi(z_v)) = (x_v)$. Moreover, combining Lemma 3.B.7 with the fact that $X^{ab} = Z^{ab}$, we deduce that $\varphi^* : \text{Б}_{weird}(X) \to \text{Б}_{weird}(Z)$ is an isomorphism, hence, $(\varphi(z_v)) \in Z(\mathbb{A}_K)^{\text{Б}_{weird}(Z)}$. Since Z satisfies the assumption of the previous case, we already know that there exists a point $z_0 \in Z(K)$. In particular, $\varphi(z_0) \in X(K)$, from which we conclude. □

The following lemma is a slightly modified version of [BCTS08, Lemma A.4]:

Lemma 3.B.7. *Let $\varphi : Z \to X$ be a torsor under a quasi-trivial torus P, where Z and X are smooth geometrically connected varieties over a number field K. Then there is an induced homomorphism $\varphi^* : \text{Б}_\infty(X) \to \text{Б}_\infty(Z)$ and it is an isomorphism.*

Proof. Let $\varphi^c : Z^c \to X^c$ be smooth compactifications of φ, Z and X. We have the following commutative diagram (see Remark 3.B.3(ii) for the rows), where the columns are defined by the pullback $(\varphi^c)^*$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Б}_\infty(X) & \longrightarrow & \mathrm{Br}_1(X^c) & \longrightarrow & \prod_{v\in M_K^{fin}} \mathrm{Br}_1(X^c_{K_v})/\operatorname{Br} K_v \\ \downarrow & & \downarrow & & \downarrow\wr & & \downarrow\wr \\ 0 & \longrightarrow & \text{Б}_\infty(Z) & \longrightarrow & \mathrm{Br}_1(Z^c) & \longrightarrow & \prod_{v\in M_K^{fin}} \mathrm{Br}_1(Z^c_{K_v})/\operatorname{Br} K_v, \end{array} \qquad (3.B.1)$$

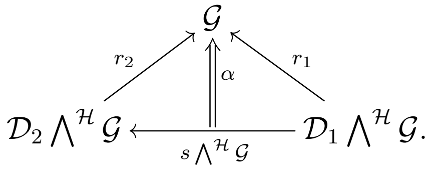
where the last two columns are isomorphisms by [BCTS08, Lemma A.4]. Hence the morphism in the first column is an isomorphism as well, concluding the proof of the lemma. □

3.C Appendix: 2-torsors

In this appendix, let $\iota : H \hookrightarrow G$ be an embedding of connected groups, defined over a field k of characteristic 0, such that H is linear and reductive and G^{lin} is reductive. We use the notation $C_H := [H^{sc} \to H]$, and $C_G := [G^{sc} \to G]$. Let $\mathcal{H}$ (resp. $\mathcal{G}$) be the k-group stack associated (in the sense of [Bre07, p. 419]) to the crossed module C_H (resp. C_G). For a k-group stack $\mathcal{C}$, we let $\mathrm{TORS}(\mathcal{C})$ denote the 2-category of left $\mathcal{C}$-torsors (in the sense of [Bre07, Sec. 6.1]).

We recall the following definition (present e.g. in [Dem13]):

Definition 3.C.1. The set $H^0(k, [C_H \to C_G])$ is the set of couples $(\mathcal{D}, r : \mathcal{D} \bigwedge^{\mathcal{H}} \mathcal{G} \xrightarrow{\sim} \mathcal{G}) \in \mathrm{TORS}(\mathcal{H}) \times \mathrm{Mor}(\mathrm{TORS}(\mathcal{G}))$ up to the following equivalence. Two elements $(\mathcal{D}_1, r_1 : \mathcal{D}_1 \bigwedge^{\mathcal{H}} \mathcal{G} \xrightarrow{\sim} \mathcal{G})$ and $(\mathcal{D}_2, r_2 : \mathcal{D}_2 \bigwedge^{\mathcal{H}} \mathcal{G} \xrightarrow{\sim} \mathcal{G})$ are equivalent if there exists a morphism $\mathrm{Mor}(\mathrm{TORS}(\mathcal{H})) \ni s : \mathcal{D}_1 \xrightarrow{\sim} \mathcal{D}_2$ and a 2-morphism $\alpha \in \mathrm{Mor}^2(\mathrm{TORS}(\mathcal{G}))$:



The formulas appearing in Proposition 3.C.3 below are just the ones coming from making Definition 3.C.1 explicit using cocycle formulas, such as the ones that one may find in [Bre07, Sec. 6] (see especially subsections 6.2 and 6.3 of *loc. cit.*).

Before coming to Proposition 3.C.3, giving an explicit description of the set $H^0(k, [C_H \to C_G])$ in terms of cocyles, we recall the following notation (which is in essence borrowed from [Bor96]).

We define $T := \operatorname{Spec} k$, $S := \operatorname{Spec} \bar{k}$ and $\Gamma := \Gamma_k$. We are going to denote by S^n the scheme $S \times_T \ldots \times_T S$, where the product is taken n times.

We recall that we have isomorphisms $\varphi_n : S^n \xrightarrow{\sim} \coprod_{(\gamma_1,\ldots,\gamma_{n-1})\in\Gamma^{n-1}} S$ defined as follows:

$$\varphi_n^{-1}|_{S_{(\gamma_1,\ldots,\gamma_{n-1})}} : S \to S^n \text{ is the map } (id, \gamma_1, \gamma_1 \cdot \gamma_2, \ldots, \gamma_1 \cdots \gamma_{n-1});$$

where $S_{(\gamma_1,\ldots,\gamma_{n-1})}$ denotes the copy of S in $\coprod_{(\gamma_1,\ldots,\gamma_{n-1})\in\Gamma^{n-1}} S$ indexed by $(\gamma_1, \ldots, \gamma_{n-1}) \in \Gamma^{n-1}$, and $\gamma_i : S \xrightarrow{\gamma_i} S$ denotes the $\operatorname{Spec} k$-morphism $\operatorname{Spec}(\operatorname{Spec} {}^{\gamma_i}\star : \operatorname{Spec} \bar{k} \to \operatorname{Spec} \bar{k})$. We notice that φ_n is a morphism of S-schemes, if we endow $\coprod_{(\gamma_1,\ldots,\gamma_{n-1})\in\Gamma^{n-1}} S$ with its natural S-scheme structure, and S^n with the S-scheme structure coming from the projection on the first coordinate.

For an a T-scheme Y, sometimes we denote, with a slight abuse of notation, the elements of $Y(S^n)$ by $y_{\gamma_1,\ldots,\gamma_{n-1}} \in \operatorname{Fun}(\Gamma^{n-1}, Y(S))$, to actually denote the composition:

$$S^n \xrightarrow{\varphi_n} \coprod_{(\gamma_1,\ldots,\gamma_{n-1})\in\Gamma^{n-1}} S \xrightarrow{y_{\gamma_1,\ldots,\gamma_{n-1}}} Y.$$

For a crossed module
$$\mathcal{C} := [\overset{-1}{C} \xrightarrow{\rho} \overset{0}{P}]$$
we define $C^1_{\mathcal{C}}(S) := C(S^3) \times P(S^2)$, $C^0_{\mathcal{C}}(S) := C(S^2) \times P(S)$, and $C^{-1}_{\mathcal{C}}(S) := C(S)$. Using the notation above, we denote elements of $C^1_{\mathcal{C}}(S)$ (resp. $C^0_{\mathcal{C}}(S), C^{-1}_{\mathcal{C}}(S)$) by $(c_{\sigma,\eta}, p_\sigma)$ (resp. $(\bar{c}_\sigma, \bar{p}), \tilde{c}$). For each of $C^i_{\mathcal{C}}(S), i = -1, 0, 1$, we denote by $e \in C^i_{\mathcal{C}}(S)$ the trivial cocycle. We recall that $C^0_{\mathcal{C}}(S)$ and $C^{-1}_{\mathcal{C}}(S)$ have group operations (denoted by $\circ_1$ and $\circ_2$) defined as follows:

$$(\bar{c}^1_\sigma, \bar{p}^1) \circ_1 (\bar{c}^2_\sigma, \bar{p}^2) := ({}^{\bar{p}^1}\bar{c}^2_\sigma \cdot \bar{c}^1_\sigma, \bar{p}^1 \cdot \bar{p}^2), \tag{3.C.1}$$

$$\tilde{c}_1 \circ_2 \tilde{c}_2 := \tilde{c}_1 \cdot \tilde{c}_2. \tag{3.C.2}$$

We define $Z^1_{\mathcal{C}}(S)$ as follows:

$$Z^1_{\mathcal{C}}(S) := \left\{ (c_{\sigma,\eta}, p_\sigma) \in C^1_{\mathcal{C}}(S), \quad \begin{cases} p_{\sigma\eta} = \rho(c_{\sigma,\eta}) p_\sigma {}^\sigma p_\eta, \\ c_{\sigma\eta,\nu} c_{\sigma,\eta} = c_{\sigma,\eta\nu} {}^{p_\sigma \sigma} c_{\eta,\nu}. \end{cases} \right\} \tag{3.C.3}$$

We have a right action of $C^0_{\mathcal{C}}(S)$ on $Z^1_{\mathcal{C}}(S)$, and one of $C^{-1}_{\mathcal{C}}(S)$ on $C^0_{\mathcal{C}}(S) \times Z^1_{\mathcal{C}}(S)$, defined as follows:

$$\star_1 : Z^1_{\mathcal{C}}(S) \times C^0_{\mathcal{C}}(S) \to Z^1_{\mathcal{C}}(S), \ (c_{\sigma,\eta}, p_\sigma) \star_1 (\bar{c}_\sigma, \bar{p}) := \left({}^{\bar{p}^{-1}} \left[\bar{c}_{\sigma\eta} \cdot c_{\sigma,\eta} \cdot {}^{p_\sigma \sigma} \bar{c}_\eta^{-1} \cdot \bar{c}_\sigma^{-1} \right], \ \bar{p}^{-1} \cdot \rho(\bar{c}_\sigma) \cdot p_\sigma \cdot {}^\sigma \bar{p} \right), \tag{3.C.4}$$

$$\begin{array}{rccc} \star_0 : & (C^0_{\mathcal{C}}(S) \times Z^1_{\mathcal{C}}(S)) \times C^{-1}_{\mathcal{C}}(S) & \to & C^0_{\mathcal{C}}(S) \times Z^1_{\mathcal{C}}(S), \\ & ((\bar{c}_\sigma, \bar{p}), (c_{\sigma,\eta}, p_\sigma)) \star_0 \tilde{c} & := & ((\tilde{c}^{-1} \cdot \bar{c}_\sigma \cdot {}^{p_\sigma \sigma} \tilde{c}, \rho(\tilde{c})^{-1} \cdot \bar{p}), (c_{\sigma,\eta}, p_\sigma)). \end{array} \tag{3.C.5}$$

We also use the notation $(a, x) \star_0 b =: (a \star_0^x b, x)$.

The actions above satisfy the following properties:

$$(a \circ_1 b) \star_0^x c = (a \star_0^x c) \circ_1 b, \quad (a \star_1 b) = a \star_1 (b \star_0^a c). \tag{3.C.6}$$

Remark 3.C.2. When C and P are commutative, we may identify $C^i_{\mathcal{C}}(S)$ with $C^i(\Gamma, \mathcal{C}(S))$ (see Subsection 3.3.1), where $\mathcal{C}(S) := [\overset{-1}{C(S)} \xrightarrow{\rho} \overset{0}{P(S)}]$. Under these identifications $Z^1_{\mathcal{C}}(S)$ corresponds to $Z^1(\Gamma, \mathcal{C}(S))$. In this case, the actions $\star_1$ and $\star_0$ correspond to the following operations:

$$a \star_1 b = a + \partial(\xi_1 b), \quad (a, c) \star_0 b = (a - \xi_1 \partial \xi_0 b, c),$$

where the ∂ on the right hand sides is the one appearing in Subsection 3.3.1, and ξ_1 (resp. ξ_0) $: C(S^2) \times P(S) \to C(S^2) \times P(S)$ (resp. $: C(S) \to C(S)$) is the map $(-id, id)$ (resp. $-id$).

Proposition 3.C.3. *Keeping the notation above, $T = \operatorname{Spec} k$, $S = \operatorname{Spec} \overline{k}$, we have a natural isomorphism between $H^0(k, [C_H \xrightarrow{\iota_*} C_G])$ and the following set:*

$$(u_{\sigma,\eta}, \psi_\sigma, a_\sigma, g) \in Z^1_{C_H}(S) \times C^0_{C_G}(S) = H^{sc}(S^3) \times H(S^2) \times G^{sc}(S^2) \times G(S),$$

$$\text{s.t.} \begin{cases} \psi_{\sigma\eta} = \rho_H(u_{\sigma,\eta}) \psi_\sigma {}^\sigma\psi_\eta, \\ u_{\sigma\eta,\nu} u_{\sigma,\eta} = u_{\sigma,\eta\nu} {}^{\psi_\sigma \sigma} u_{\eta,\nu}, \\ g \cdot \iota(\psi_\sigma) = \rho_G(a_\sigma) \cdot {}^\sigma g, \\ {}^g[\iota^{sc}(u_{\sigma,\eta})] = a_{\sigma\eta} \cdot ({}^\sigma a_\eta)^{-1} \cdot a_\sigma{}^{-1}. \end{cases}, \quad \text{i.e. with } \alpha = (u_{\sigma,\eta}, \psi_\sigma) \text{ and } \beta = (a_\sigma, g),\ \iota(\alpha) \star_1 \beta^{-1} = e;$$

quotiented by the following equivalence relation:

$$(\alpha_1, \beta_1) \sim (\alpha_2, \beta_2) \in Z^1_{C_H}(S) \times C^0_{C_G}(S), \quad \text{if there exists}$$
$$(c, d) \in C^0_{C_H}(S) \times C^{-1}_{C_G}(S), \quad \text{s.t.} \begin{cases} \alpha_1 = \alpha_2 \star_1 c, \\ \beta_1 = (\beta_2 \circ_1 \iota(c)) \star^e_0 d. \end{cases}$$

Moreover, the image under the natural morphism $H^0(k, [H \xrightarrow{\iota} G]) \to H^0(k, [C_H \xrightarrow{\iota} C_G])$ of the element $(h_\sigma, g) \in Z^1_{[1\to H]}(S) \times C^0_{[1\to G]}(S) = H(S^2) \times G(S)$ (where $g \cdot h_\sigma = {}^\sigma g$) is represented by (e, h_σ, e, g) (where the e's denote constant cocyles valued in the identity element).

Sketch of Proof. A more detailed proof may appear in a future version of this work or in other work. However, the proof is in essence an easy calculation from [Bre07, Sec. 6] (keeping in mind that, since we are over the étale site of the spectrum of a field, the hypercoverings appearing in [Bre07, Sec. 6] may always be dominated by Čech coverings by [AM69, Example 9.11]). □

Remark 3.C.4. Our definitions coincide with those of Demarche [Dem09, Sec. 4.2.1.4], in the following sense. Our definitions of Z^1, C^0 coincide with those in [Dem09, Sec. 4.2.1.4] with $[F \to G] = [C \to P]$, and our $C^{-1}_{\mathcal{C}}$ coincides with Demarche's $F(\bar{K})$. His Definition 4.2.5 coincides (under the identifications $M_1 = C_H$ and $M_2 = C_G$) with our explicit cocycle description of $Z^1_{C_H}(S) \times C^0_{C_G}(S)$ appearing in the proposition above.

Moreover, his action

$$* : Z^0(K, [M_1 \to M_2]) \times \left(C^0(K, M_1) \times F_2(\bar{K})\right) \to Z^0(K, [M_1 \to M_2])$$

in [Dem09, p. 150] coincides (under the identifications $C_H = M_1$, $C_G = M_2$ and $Z^0(K, [M_1 \to M_2]) = Z^1_{C_H}(S) \times C^0_{C_G}(S)$), with:

$$(\alpha_2, \beta_2), (c, d) \mapsto (\alpha_2 \star_1 c, (\beta_2 \circ_1 \iota(c)) \star^e_0 d),$$

which the reader may easily verify to be a group action using the properties (3.C.6).

Chapter 4

Ramified descent

4.1 Introduction

Recall that in this chapter we look for a link between the Brauer–Manin obstruction and "ramified descent". In particular, for a cover $\psi : Y \to X$ of K-varieties (K being a number field) that is generically a torsor under a finite group scheme G/K, we construct a subgroup $\operatorname{Br}_\psi X \subseteq \operatorname{Br} X$ (see Definition 4.4.5) such that the following holds.

Theorem 4.1.1. *The inclusion $X(\mathbb{A}_K)^\psi \subseteq X(\mathbb{A}_K)^{\operatorname{Br}_\psi X}$ holds. Moreover, even when G is commutative, the group $\operatorname{Br}_\psi X$ is not necessarily algebraic, and the transcendental part may provide non-trivial obstruction.*

The motivation for this theorem comes, apart from its interest in the matter of "ramified descent", also from the following question of Harari:

Question 4.1.2. Could the descent set for ramified covers be linked to a non-algebraic Brauer–Manin obstruction?

Moreover, we also prove:

Proposition 4.1.3. *Let X be a smooth, geometrically connected variety over K. Then $\overline{X(K)} \subseteq X(\mathbb{A}_K)^\psi$, the closure being in $X(\mathbb{A}_K)$.*

Note that, in particular, when X is proper, the above proposition says that the (ramified) descent set $X(\mathbb{A}_K)^\psi$ provides an obstruction to the Hasse principle and weak approximation.

Structure of the chapter In Section 4.2 we settle our notation. In Section 4.3 we formally define the "descent set" of a ramified cover, show some basic properties, and then show how this connects to the question of Harari mentioned in the introduction. In Section 4.4, we introduce the Brauer subgroup $\operatorname{Br}_\psi X$, prove that this provides an obstruction to ramified descent, and then compare it with the "classical" algebraic descent obstruction (showing, in particular, that $\operatorname{Br}_\psi X$ contains the "classical" algebraic obstruction). In Section 4.5, we prove that the descent set provides an obstruction to the Hasse Principle and weak approximation on the whole X. In Section 4.6, we provide an example where $\operatorname{Br}_\psi X$ is purely transcendental. Appendix 4.A contains some elementary lemmas that are used in Section 4.6. Appendix 4.B talks briefly about other already existing works containing the idea of "ramified descent".

4.2 Notation

Fields Unless specified otherwise, F will always denote a perfect field, k a field of characteristic 0 and K a number field.

M_K (resp. M_K^f, M_K^∞) denotes the set of (non-archimedean, archimedean) places of K.

For a place $v \in M_K$ (resp. $v \in M_K^f$), K_v (resp. O_v) denotes the v-adic completion of K (resp. the v-adic integers).

$\mathbb{A}_K$ (resp. $\mathbb{A}_K^S$, for a subset $S \subset M_K$) denotes the topological ring of adeles of K (resp. S-adeles), i.e. the topological ring $\prod'_{v\in M_K} K_v$ (resp. $\prod'_{v\in M_K\setminus S} K_v$), the restricted product being on $O_v \subseteq K_v$.

For a finite subset $S \subseteq M_K$, K_S denotes the product $\prod_{v\in S} K_v$. We let K_Ω denote the product $\prod_{v\in M_K} K_v$

For a Galois extension L/K, $\mathrm{Gal}(L/K)$ denotes the Galois group of the extension. For a field k with algebraic closure $\overline{k}$, $\Gamma_k := \mathrm{Gal}(\overline{k}/k)$.

Abelian groups For a group M of multiplicative type over a field k (i.e. a commutative group scheme, which is an extension of a finite group by a torus), $\widehat{M} := \mathrm{Hom}_{\overline{k}}(M_{\overline{k}}, \mathbb{G}_{m,\overline{k}})$ denotes the Γ_k-module of characters.

For a torsion abelian group A, A^D will denote the profinite abelian group $\mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$ endowed with the compact-open topology. If A is a profinite abelian group, A^D will denote the torsion group $\mathrm{Hom}_{cont}(A, \mathbb{Q}/\mathbb{Z})$, where $\mathbb{Q}/\mathbb{Z}$ is endowed with its discrete topology. We recall that, by Pontryagin duality, if A is torsion or profinite, there is a canonical isomorphism $A \cong (A^D)^D$.

Geometry All schemes appearing in this chapter are separated, therefore, we always tacitly assume this hypothesis throughout the chapter.

A *variety* X over a field k is a scheme of finite type over a field k (not necessarily integral).

If X is an integral scheme, we denote by η_X or $\eta(X)$ its generic point.

For a k-scheme X, we denote the residue field of a point $\xi \in X$ by $k(\xi)$.

Groups and torsors Group actions will be assumed to be right actions unless specified otherwise. Let S be a scheme, G be a group scheme over S and X be an S-scheme. A right G-*torsor* over X is an X-scheme $Y \to X$, endowed with a right G-action $m : Y \times_S G \to Y$, that is X-equivariant (i.e. such that the composition $Y \times_S G \xrightarrow{m} Y \to X$ is equal to the composition $Y \times_S G \xrightarrow{pr_1} Y \to X$) such that there exists an étale covering $X' \to X$ and an X'-isomorphism $Y \times_X X' \cong G \times_X X'$ that is G-equivariant.

For an abstract group N, and a scheme S (resp. a field F), we denote by N_S (resp. N_F) the S-scheme (resp. F-scheme) $\sqcup_{n\in N} S$, endowed with the natural S(resp. F)-group scheme structure endowed from the group structure of N. If X is an S-scheme, a torsor $Y \to X$ under an abstract group G is a torsor under the constant group G_S.

If G/k is an algebraic group, and $k \subseteq F$ is a field extension, we will use the notation $H^i(F, G)$ (with $i \in \mathbb{N}$ and $i = 0, 1$ if G is not commutative) to denote the cohomology group/set $H^i(\Gamma_F, G(\overline{F})) = Z^i(\Gamma_F, G(\overline{F}))/B^i(\Gamma_F, G(\overline{F}))$ (where $B^i(\Gamma_F, G(\overline{F}))$ is a subgroup when G is commutative and is just an equivalence relation otherwise).

If G is not commutative the set of cocycles $Z^1(\Gamma_F, G(\overline{F}))$ is the one of non-abelian (1-)cocycles, i.e., those functions $g_\sigma : \Gamma_F \to G(\overline{F})$ that satisfy $g_{\sigma\tau} = g_\sigma {}^\sigma g_\tau$. The set $H^1(\Gamma_F, G(\overline{F}))$ is the quotient of 1-cocycles by the equivalence relation $B^1(\Gamma_F, G(\overline{F})) : g_\sigma \sim g'_\sigma$ if there exists $g \in G(\overline{F})$ such that $g'_\sigma = g^{-1} g_\sigma {}^\sigma g$. Note that these cocycles correspond to (right) G-torsors through the standard correspondence [Sko01, p.18, 2.10].

If $\xi \in Z^1(K, G)$, we use the notation G^ξ to denote the inner twist of G by ξ, and G_ξ to denote the left principal homogeneous space of G obtained by twisting G by the cocycle ξ. This twist is naturally endowed with a right action of G^ξ. See [Sko01, p. 12-13] for more details on these constructions.

If X is a quasi-projective k-scheme endowed with a (left) G-action, and $\xi \in Z^1(k, G)$, we use the notation X_ξ to denote the twisted k-variety $(G_\xi \times^G X)$. [We refer the reader to [Sko01, p. 20], [Ser94, Sec. I.5.3] and [Ser94, Sec. III.1.3] for the existence of the twist and immediate properties of the twisting operation]. The variety X_ξ is naturally endowed with a (left) G^ξ-action. We recall that there exists always an isomorphism $X_\xi \times_k \overline{k} \cong X \times_k \overline{k}$.

If X is endowed with a right G-action we may still do the twisting operations, by taking the corresponding left action, using the canonical isomorphism $G \cong G^{op}, g \mapsto g^{-1}$.

Brauer group Recall that the Brauer group of a scheme X is defined to be the étale cohomology group $H^2_{ét}(X, \mathbb{G}_m)$, and that, when X is a variety defined over a number field K, this provides an obstruction, known as the *Brauer–Manin obstruction* to local-global principles, in the following sense. There is a pairing (the Brauer–Manin pairing):

$$X(\mathbb{A}_K) \times \operatorname{Br} X \to \mathbb{Q}/\mathbb{Z},$$

defined as sending $((P_v), B)$ to $((P_v), B) := \sum_v \operatorname{inv}_v B(P_v)$, where $\operatorname{inv}_v : H^2(\Gamma_{K_v}, \overline{K_v}^*) \to \mathbb{Q}/\mathbb{Z}$ is the usual invariant map (see e.g. [Har20, Thm 8.9] for a definition). Whenever $B \in \operatorname{Im} \operatorname{Br} K$ or $(P_v) \in X(K)$ (diagonally immersed in $X(\mathbb{A}_K)$), we have that the pairing $((P_v), B)$ is zero by the Albert-Brauer-Noether-Hasse theorem. In particular, it follows that $X(K)$ is a subset of

$$X(\mathbb{A}_K)^{\operatorname{Br} X} := \{(P_v) \mid ((P_v, B)) = 0 \text{ for all } B \in \operatorname{Br} X\}.$$

For a scheme X over a field F, we adopt the usual notation $\operatorname{Br}_1 X := \operatorname{Ker}(\operatorname{Br} X \to \operatorname{Br} X_{\overline{F}})$ and $\operatorname{Br}_0 X := \operatorname{Im}(p^* : \operatorname{Br} F \to \operatorname{Br} X)$, where $p : X \to \operatorname{Spec} F$ denotes the structural morphism, and $\operatorname{Br}_a X := \operatorname{Br}_1 X / \operatorname{Br}_0 X$.

To avoid too cumbersome notation, we will usually identify, when X is a smooth variety over a characteristic 0 field k, the group $\operatorname{Br} X$ with its image in $\operatorname{Br} k(X)$ under the pullback map, which is injective by [CTS21, Theorem 3.5.5]. Accordingly, for any open subscheme $U \subseteq X$, we will, with a slight abuse of notation, denote the pullback $\operatorname{Br} X \to \operatorname{Br} U$ with an inclusion $\operatorname{Br} X \subseteq \operatorname{Br} U$ ($\subseteq \operatorname{Br} k(X)$). We will say that an element $\beta \in \operatorname{Br}(k(X))$ is *unramified over* X if $\operatorname{res}_X(\beta) = 0$, where res_X denotes the residue map (defined e.g. in [CTS21, Theorem 3.7.3]):

$$\operatorname{Br} K(X) \xrightarrow{\operatorname{res}_X} \bigoplus_{\substack{D \subseteq X \\ D \text{ irreducible divisor}}} H^1(k(D), \mathbb{Q}/\mathbb{Z}),$$

or, equivalently, if it belongs to $\operatorname{Br} X \subseteq \operatorname{Br} k(X)$. We say that β is *unramified* if $\beta \in \operatorname{Br} X^c$, for one (or, equivalently, all, by [CTS21, Prop. 3.7.10]) smooth compactification(s) X^c of X. We denote the subgroup of unramified elements by $\operatorname{Br}_{nr}(k(X))$ or $\operatorname{Br}_{nr} X$.

Cohomology For a scheme X, an étale sheaf $\mathcal{F}$ on X and every $n \geq 0$, the notation $H^n(X, \mathcal{F})$ will always denote the étale cohomology $H^n_{ét}(X, \mathcal{F})$.

Equivariant commutative diagrams Let S be a scheme. For S-group schemes G_1, G_2, equipped with a morphism $G_1 \to G_2$ (usually this morphism will be implicit) and torsors $Z_1 \xrightarrow{G_1} W_1, Z_2 \xrightarrow{G_2} W_2$, we will say that a diagram

$$\begin{array}{ccc} Z_1 & \longrightarrow & Z_2 \\ \downarrow{\scriptstyle G_1} & & \downarrow{\scriptstyle G_2} \\ W_1 & \longrightarrow & W_2 \end{array} \tag{4.2.1}$$

commutes if the underlying commutative diagram is commutative and if the morphism $Z_1 \to Z_2$ is $(G_1 \to G_2)$-equivariant.

4.3 Basic definitions and properties

4.3.1 Descent set

Descent set for torsors Let K be a number field, G be a finite group scheme over K, W be a smooth geometrically connected variety over K, and $\varphi : Z \to W$ be a G-torsor.

To recall the definition of the descent set of a torsor, let us first recall the definition (and immediate properties) of the twist of a torsor by a cocycle.

For every cohomological class $\xi \in H^1(K, G)$, there exists a twisted torsor $\varphi_\xi : Z_\xi \to W$ of the torsor φ under the twisted form G^ξ of G. The class $[\varphi_\xi] \in H^1(K, G^\xi)$ is given by the image of $[\varphi] \in H^1(K, G)$ under the well-known isomorphism $H^1(K, G) \to H^1(K, G^\xi), [Z] \mapsto [Z_\xi]$ (see e.g. [Sko01, p.20, 21]). When G is commutative, we have that $G^\xi \cong G$ (canonically), and the morphism $H^1(K, G) \to H^1(K, G^\xi), [Z] \mapsto [Z_\xi]$ becomes $[Z] \mapsto [Z] - [\xi]$.

Recall that the descent obstruction set $W(\mathbb{A}_K)^\varphi$ associated to φ is defined as follows:

$$W(\mathbb{A}_K)^\varphi := \bigcup_{\xi \in H^1(K,G)} \varphi_\xi(Z_\xi(\mathbb{A}_K)) \subseteq W(\mathbb{A}_K), \tag{4.3.1}$$

As proven in [CDX19, Prop. 6.4], $W(\mathbb{A}_K)^\varphi$ is closed in $W(\mathbb{A}_K)$ in the adelic topology. Moreover, for completeness, we remind the reader that there is an inclusion $W(K) \subseteq W(\mathbb{A}_K)^\varphi$ (see e.g. [Sko01, Section 5.3]), although this inclusion is irrelevant for the purpose of this thesis.

We are interested in defining and studying an analogue set of (4.3.1), when the morphism φ is a G-cover rather than a G-torsor (this means that we will allow ramification but demand the harmless assumption that Z is integral). We introduce the suitable setting in the next paragraph.

Covers and G-covers A morphism $\psi : Y \to X$ is a *cover* if X and Y are normal (we do not include integrality in our definition of normality), ψ is finite, X is integral and every connected component of Y surjects onto X. If G is a finite étale group scheme over a perfect field F (étaleness is automatic if char $F = 0$), a *G-cover* $\psi : Y \to X$ is a cover where both X and Y are F-varieties and such that there is a X-invariant G-action $Y \times_k G \to Y$ such that there is a non-empty open subscheme $U \subset X$ over which ψ is an (étale) G-torsor.

Remark 4.3.1. Equivalently, a cover $\psi : Y \to X$ of a normal F-variety X is a G-cover if the generic fiber is a torsor under $G_{F(X)}$. Indeed, Y is the relative normalization of X in the generic fiber $Y_{F(X)}$ [Sta20, Tag 0BAK], hence a G-torsor structure on the generic fiber extends uniquely to a G-action on the whole Y by the universal property of relative normalization [Sta20, Tag 035I]. This is clear if G is constant and follows by étale descent in general.

Descent set for G-covers Let X be a smooth geometrically connected K-variety, and let $\psi : Y \to X$ be a G-cover.

Let $U \subset X$ be an open subscheme such that $\psi^{-1}(U) \to U$ is an étale G-torsor. Let $V := \psi^{-1}(U)$ and let $\varphi = \psi|_V : V \to U$. We have a descent set $U(\mathbb{A}_K)^\varphi$ in $U(\mathbb{A}_K)$, defined as in (4.3.1).

Recall from the introduction:

Definition 4.3.2. We define the *descent set* for the G-cover $Y \to X$ as the closure $X(\mathbb{A}_K)^\psi := \overline{U(\mathbb{A}_K)^\varphi}$ in $X(\mathbb{A}_K)$.

Note that the topology here is the adelic one for $X(\mathbb{A}_K)$ (X might not be proper). However, in the end, we will use all of this mainly in the proper setting, where the adelic topology coincides with the product topology on $X(\mathbb{A}_K) = X(K_\Omega) = \prod_{v \in M_K} X(K_v)$.

We show in Lemma 4.3.4 below that Definition 4.3.2 is independent from the choice of U. Before proving that, let us remark why there is no conflict of notation with (4.3.1) (which might arise when $Y \to X$ itself is an étale G-torsor).

Remark 4.3.3. When the morphism $Y \to X$ itself is étale (and, hence, an étale G-torsor), let $X(\mathbb{A}_K)^{\psi,1}$ be the set $X(\mathbb{A}_K)^{\psi}$ defined through (4.3.1) applied to the G-torsor $Y \to X$, and $X(\mathbb{A}_K)^{\psi,2}$ be the set defined in Definition 4.3.2. Taking $U = X$ in Definition 4.3.2 (recall that we are going to show, in Lemma 4.3.4, that this definition is independent from the choice of U), we see that $X(\mathbb{A}_K)^{\psi,2} = \overline{X(\mathbb{A}_K)^{\psi,1}}$. However, by [CDX19, Prop. 6.4], $X(\mathbb{A}_K)^{\psi,1}$ is closed, hence $X(\mathbb{A}_K)^{\psi,1} = X(\mathbb{A}_K)^{\psi,2}$ and the notation $X(\mathbb{A}_K)^{\psi}$ is unambiguous.

Warning. Note that, as the continuous map $U(\mathbb{A}_K) \hookrightarrow X(\mathbb{A}_K)$ is not a topological immersion, the set $X(\mathbb{A}_K)^{\psi} \cap U(\mathbb{A}_K)$ might very well be bigger than $U(\mathbb{A}_K)^{\varphi}$. The reader may verify that, in the example given in Section 4.6, this is exactly the case.

The following lemma shows that the above definition does not depend on the choice of U:

Lemma 4.3.4. *Let $\nu : Y^{sm} \to Y$ be a G-equivariant desingularization of Y, and let r be the composition $\psi \circ \nu : Y^{sm} \to X$. Note that, for every $\xi \in H^1(K,G)$, there is a twisted form of $r : Y^{sm} \to X$ with respect to ξ, which we denote by $r_\xi : Y^{sm}_\xi \to X$. We have that:*

$$X(\mathbb{A}_K)^{\psi} = \overline{\bigcup_{\xi \in H^1(K,G)} r_\xi(Y^{sm}_\xi(\mathbb{A}_K))}.$$

Note that, when Y is smooth, the above identity holds then with Y instead of Y^{sm}.

Proof. First of all, note that a G-equivariant desingularization $\nu : Y^{\mathrm{sm}} \to Y$ always exists because of the existence of strong resolution of singularities in characteristic 0, see e.g. [EH02].

Let $V' := \nu^{-1}(V)$, note that $V' \xrightarrow{\nu} V$ is an isomorphism (since V is regular). We claim that:

$$\overline{\bigcup_{\xi \in H^1(K,G)} \varphi_\xi(V_\xi(\mathbb{A}_K))} = \overline{\bigcup r_\xi(V'_\xi(\mathbb{A}_K))} = \overline{\bigcup \overline{r_\xi(V'_\xi(\mathbb{A}_K))}} = \overline{\bigcup r_\xi(\overline{V'_\xi(\mathbb{A}_K)})} = \overline{\bigcup_{\xi \in H^1(K,G)} r_\xi(Y^{\mathrm{sm}}_\xi(\mathbb{A}_K))},$$

where the union is over $\xi \in H^1(K,G)$ everywhere, and in the third term, $\overline{V'_\xi(\mathbb{A}_K)}$ denotes the closure in $Y^{\mathrm{sm}}_\xi(\mathbb{A}_K))$. In fact, the first two identities are immediate, the third follows from the fact that r_ξ is proper, and the fourth holds because, for each $v \in M_K$, $V'_\xi(K_v)$ is dense in $Y^{\mathrm{sm}}_\xi(K_v)$ (keeping in mind that Y^{sm}_ξ is smooth, this follows from [CTS21, Theorem 10.5.1]). □

Setting From now on we fix, until Section 4.5 (included), a number field K, a finite group scheme G/K, a G-cover $\psi : Y \to X$, an open subscheme $U \subseteq X$ such that $V := \psi^{-1}(U) \to U$ is étale. We denote this last G-torsor by $\varphi : V \to U$.

We are interested in giving an explicit description (for instance, in terms of a Brauer–Manin obstruction) of the set $X(\mathbb{A}_K)^{\psi}$. As explained in the introduction, ideally, we would like a (possibly explicit) answer to the following question:

Question 4.3.5. Does there exist a $B \subseteq \mathrm{Br}(X)$ such that $X(\mathbb{A}_K)^{\psi} = X(\mathbb{A}_K)^B$?

We will mainly be interested in the question above in the case when X is proper. In this work we will not answer the question in any specific instance, but rather, in the end, provide a non-algebraic B such that $X(\mathbb{A}_K)^{\psi} \subseteq X(\mathbb{A}_K)^B$ (and also, for which it might seem conjecturally reasonable that the inclusion is actually an equality, at least when G is commutative).

Obstruction to existence and weak density of rational points Note that it easily follows from the definition that $X(\mathbb{A}_K)^{\psi}$ provides an obstruction to the existence and (weak) density of K-rational points on U. In fact, we have, by standard descent theory, that:

$$U(K) \subseteq U(\mathbb{A}_K)^{\varphi},$$

and, hence:

$$\overline{U(K)} \subseteq \overline{U(\mathbb{A}_K)^\varphi} = X(\mathbb{A}_K)^\psi,$$

where the closure is taken inside $X(\mathbb{A}_K)$ (so, when X happens to be proper, this is the closure with respect to the weak topology).

Remark 4.3.6. We will prove in Section 4.5 that actually, we even have that $\overline{X(K)} \subseteq X(\mathbb{A}_K)^\psi$, proving that the subset $X(\mathbb{A}_K)^\psi \subseteq X(\mathbb{A}_K)$ provides an obstruction to the Hasse principle and weak approximation not only on U, but on the whole X as well. Let us remark that, although the "ramified descent set" was already implicitly defined in [CS20, Section 13], it seems that the authors do not prove that this set contains the whole $X(K)$, and that thus defines an obstruction to the Hasse principle and weak approximation for K-points on X (and not just those of U).

By the Zariski purity theorem (see [Sza09, Theorem 5.2.13]), whenever a cover $Y \to X$, with X regular, is unramified, it is étale. Since, as we have seen in Remark 4.3.3, in this case the set $X(\mathbb{A}_K)^\psi$ reduces to the well-studied set defined through (4.3.1), the real interest resides in the case where $Y \to X$ is ramified. For this reason, we refer to the question above as the "ramified descent problem" for the G-cover ψ.

4.3.2 A reformulation in terms of Galois cohomology

In this subsection, we reformulate the setting as that of a question asked by David Harari at the "Rational Points 2019 workshop" in 2019.

We assume for this subsection that G is commutative and that X is proper. Let $[\varphi] \in H^1(U, G)$ be the element representing the G-torsor $V \to U$. For any $v \in M_K$, there is a map $U(K_v) \to H^1(K_v, G)$, sending a point $P : \operatorname{Spec} K_v \to U$ to the class $[\varphi|_P] \in H^1(K_v, G)$ of the restricted (pullback) torsor $\varphi|_P$. We define:

$$E_v := \operatorname{Im}([\varphi|_-] : U(K_v) \to H^1(K_v, G)).$$

We have the following, which gives a more cohomological description of the set $X(\mathbb{A}_K)^\varphi$ (and is the main proposition of this subsection), in terms of the sets E_v just defined:

Proposition 4.3.7. *Let $(P_v) \in U(K_\Omega) = \prod_{v \in M_K} U(K_v)$. For any $\beta \in H^1(K, G)$, we denote by β_v the image of β in $H^1(K_v, G)$. The following are equivalent:*

i. $(P_v) \in X(\mathbb{A}_K)^\varphi$*;*

ii. For every finite $S \subseteq M_K$ there exists $\beta \in H^1(K, G)$ such that:

$$\begin{cases} \beta_v = [\varphi|_{P_v}] & \forall\ v \in S, \\ \beta_v \in E_v & \forall\ v \notin S. \end{cases} \tag{4.3.2}$$

This connects us to the problem of Harari, which we represent here.

The question is the following:

Question 4.3.8 (Harari). Let S be a finite subset of M_K, and let, for each $v \in S$, α_v be an element of E_v. Assuming the necessary condition (NC) below, does there exist a $\beta \in H^1(K, G)$ such that

$$\begin{cases} \beta_v = \alpha_v & \forall\ v \in S, \\ \beta_v \in E_v & \forall\ v \notin S \end{cases} ?$$

Necessary condition (NC). Let G' be the Cartier dual of G. We are going to formulate the necessary condition (NC) in three equivalent ways:

$(\mathrm{NC})_1$ there exists a $\beta \in H^1(K, G)$ such that $\beta_v = \alpha_v$ for $v \in S$, and $\beta_v \in \langle E_v \rangle$ for $v \notin S$;

For the other two equivalent formulations, we recall a well-known consequence of the Poitou-Tate sequence. We define:

$$H^1(K,G)_S := \{\alpha \in H^1(K,G) \mid \alpha_v \in < E_v > \text{ for all } v \notin S\} \tag{4.3.3}$$
$$H^1(K,G')_S := \{\alpha \in H^1(K,G') \mid \alpha_v \in < E_v >^\perp \text{ for all } v \notin S\}. \tag{4.3.4}$$

Lemma 4.3.9. *There is an exact sequence:*

$$H^1(K,G)_S \to \prod_{v\in S} H^1(K_v,G) \to H^1(K,G')_S^D, \tag{4.3.5}$$

where the pairing

$$\prod_{v\in S} H^1(K_v,G) \times H^1(K,G')_S \to \mathbb{Q}/\mathbb{Z} \tag{4.3.6}$$

that defines the last map in (4.3.5) is defined by $((\alpha_v)_{v\in S},\gamma) \mapsto \sum_{v\in S} \mathrm{inv}_v(\alpha_v \cup \gamma_v)$*, where* γ_v *is the image of* γ *under* $H^1(K,G') \to H^1(K_v,G')$*, and the cup product is* $-\cup- : H^1(K_v,G) \times H^1(K_v,G') \to H^2(K_v,\overline{K_v}^*)$*.*

Proof. Note that the fact that the sequence (4.3.5) is a complex follows simply from the fact that, if there exists $\alpha \in H^1(K,G)$ with restriction α_v for all $v \in S$, then:

$$\sum_{v\in S} \mathrm{inv}_v(\alpha_v \cup \gamma_v) = \sum_{v\in M_K} \mathrm{inv}_v(\alpha_v \cup \gamma_v) = \sum_{v\in M_K} \mathrm{inv}_v(\alpha\cup\gamma)_v = 0,$$

where the first follows from the fact that, for $v \notin S$, $\alpha_v \in < E_v >$ and $\gamma_v \in < E_v >^\perp$, and hence $\mathrm{inv}_v(\alpha_v \cup \gamma_v) = 0$, while the last follows from the Albert-Brauer-Hasse-Noether theorem (as formulated e.g. in [Har20, Theorem 14.11]).

The exactness of the sequence (4.3.5) may be easily inferred from the long exact sequence of Poitou-Tate (which can be found e.g. in [NSW08, Theorem 8.6.10]). □

The other two equivalent formulations are:

$(\mathrm{NC})_2$ $E_v \neq \emptyset$ (i.e., $U(K_v) \neq \emptyset$) for all $v \notin S$ and $(\alpha_v)_{v\in S}$ is orthogonal to $H^1(K,G')_S$ (via the pairing (4.3.6));

$(\mathrm{NC})_3$ $U(K_v) \neq \emptyset$ for all $v \notin S$ and, if $(P_v)_{v\in S}$ is such that $([\varphi|_{P_v}])_{v\in S} = (\alpha_v)_{v\in S}$, then for one, or, equivalently, all adelic points $(Q_v)_{M_K}$ such that $Q_v = P_v$ for $v \in S$, the point $(Q_v) \in U(K_\Omega)$ is Brauer–Manin-orthogonal to the intersection of $\mathrm{Br}\, K + (H^1(K,G')_S \cup [V]) \subseteq \mathrm{Br}_1(U)$ with $\mathrm{Br}\, X$ (or, in other words, the "relevant" part of $\mathrm{Br}_1(U)$ for the obstruction to this problem), where the cup product refers to the map

$$H^1(K,G') \times H^1(U,G) \xrightarrow{(p^*,id)} H^1(U,G') \times H^1(U,G) \xrightarrow{-\cup-} H^2(U,\mathbb{G}_m),$$

where $p : U \to \mathrm{Spec}\, K$ denotes the projection map.

Proposition 4.3.10. *The "one, or, equivalently, all" in* $(NC)_3$ *holds, and the three formulations* $(NC)_1$*,* $(NC)_2$ *and* $(NC)_3$ *are equivalent.*

Proof. The equivalence of $(\mathrm{NC})_2$ with $(\mathrm{NC})_1$ is a direct consequence of the exactness of (4.3.5) (so, basically, of Poitou-Tate's long exact sequence).

For the equivalence of $(\mathrm{NC})_2$ with $(\mathrm{NC})_3$ and the "one, or, equivalently, all" bit in $(\mathrm{NC})_3$, we notice that the following diagram and the corresponding collection of local diagrams commute:

$$\begin{array}{ccccc} \prod'_{v\in M_K} H^1(K_v,G) & \times & H^1(K,G') & \longrightarrow & \mathbb{Q}/\mathbb{Z} \\ \uparrow{\scriptstyle [\varphi|-]} & & \downarrow{\scriptstyle -\cup[V]} & & \downarrow{\scriptstyle =} \\ U(\mathbb{A}_K) & \times & \mathrm{Br}_1(U) & \longrightarrow & \mathbb{Q}/\mathbb{Z} \end{array} . \tag{4.3.7}$$

This follows by the commutativity of the local diagrams, which itself can be deduced by the functoriality of cup product.

The "one, or, equivalently, all" in $(\mathrm{NC})_3$ now holds because, for $\gamma \in H^1(K, G')_S$, $\gamma_v \in < E_v >^{\perp}$ for all $v \notin S$, and hence, by the commutativity of (4.3.7), $\gamma \in [V] \in \mathrm{Br}_1 U$ gives local constant pairings for all $v \notin S$.

The equivalence of $(\mathrm{NC})_2$ with $(\mathrm{NC})_3$ is an immediate consequence of the commutativity of (4.3.7). □

Remark 4.3.11. If $(H^1(K, G') \cup [V]) \cap \mathrm{Br}_{nr} U$ is finite, then, for S big enough, this is equal to $H^1(K, G')_S \cup [V] \subseteq \mathrm{Br}_1(U)$. This is a consequence of Harari's formal lemma. In fact, by Harari's formal lemma [Har94, Thm. 2.1.1] (see also the remark following the theorem in *loc.cit.*), for $\gamma \in H^1(K, G')$, $B = \gamma \cup [V]$ is unramified if and only if the local pullback function $U(K_v) \xrightarrow{B|-} \mathrm{Br}\, K_v \cong \mathbb{Q}/\mathbb{Z}$ is constantly zero for $v >> 0$, which, by the compatibility (4.3.7), happens if and only if $E_v \xrightarrow{\gamma_v \cup -} \mathrm{Br}\, K_v \cong \mathbb{Q}/\mathbb{Z}$ is constantly zero for $v >> 0$ which happens if and only if $\gamma_v \in < E_v >^{\perp}$ for $v >> 0$. The just proven equivalence $\gamma \cup [V] \in \mathrm{Br}_{nr} U \Leftrightarrow \gamma_v \in < E_v >^{\perp}$ for $v >> 0$ implies that we always have an inclusion $H^1(K, G')_S \cup [V] \subseteq (H^1(K, G') \cup [V]) \cap \mathrm{Br}_{nr} U$. On the other hand, when the latter is finite, the equivalence proves that the other inclusion holds as well if S is big enough.

We denote the subgroup $(H^1(K, G') \cup [V]) \cap \mathrm{Br}_{nr} X \subseteq \mathrm{Br}_1(X)$ by $\mathrm{Br}^a_{\varphi} X$. In Section 4.6 we will prove that the answer to Question 4.3.8 is negative, giving an explicit counterexample. More precisely, we will define (see Remark 4.4.15) an adelic point $(P_v) \in U(K_\Omega)^{\mathrm{Br}^a_{\varphi} X} \setminus X(\mathbb{A}_K)^{\varphi}$. In particular, by Proposition 4.3.7 this proves that there exists a finite $S \subseteq M_K$ for which the collection of local cohomological classes $(\alpha_v)_{v \in S}, \alpha_v := [\varphi|_{P_v}]$ <u>does not</u> satisfy the condition of question 4.3.8. On the other hand, since $(P_v) \in X(\mathbb{A}_K)^{\mathrm{Br}^a_{\varphi} X}$, we have that (P_v) is orthogonal to $(H^1(K, G) \cup [V]) \cap \mathrm{Br}\, X \subseteq \mathrm{Br}_1(X)$. By Remark 4.3.11 (in the counterexample $\mathrm{Br}^a_{\varphi} X = 0$, so the required finiteness certainly holds), this means that, after possibly enlarging S again, $(\alpha_v)_{v \in S}$ satisfies the "necessary condition" (in the form $(\mathrm{NC})_3$) of Question 4.3.8.

We turn back to proving Proposition 4.3.7.

We recall a useful and easy lemma:

Lemma 4.3.12. *Let F be a perfect field and H be a smooth group scheme over F. For an H-torsor $Y \to \mathrm{Spec}\, F$, of class $[Y] \in H^1(F, H)$, and for every $\xi \in H^1(F, H)$, the class $[Y_\xi] \in H^1(K_v, H^\xi)$ (representing the H^ξ-torsor Y_ξ) is trivial if and only if $\xi = [Y]$.*

Proof. See [Ser94, Proposition 35]. □

Lemma 4.3.13. *i. For each $v \in M_K$, we have that $E_v = \{\xi \in H^1(K_v, G) \mid V_\xi(K_v) \neq \emptyset\}$*

ii. Let S be a finite subset of M_K^f, such that there exists a smooth model $\mathcal{U} \to \mathrm{Spec}\, O_{K,S}$ for $U \to \mathrm{Spec}\, K$ and an étale group-scheme model $\mathcal{G} \to \mathrm{Spec}\, O_{K,S}$ for the étale group scheme $G \to \mathrm{Spec}\, K$. For almost all $v \in M_K^f$, the image of $\mathcal{U}(O_v) \subseteq U(K_v)$ under $[\varphi|\_]$ is $H^1(O_v, \mathcal{G}) \subseteq H^1(K_v, G)$.

Proof. i. We have that P lies in $\varphi_\xi(V_\xi(F))$ if and only if the G^ξ-torsor $V_\xi|_P = (V|_P)_\xi \xrightarrow{\varphi_\xi|_P = (\varphi|_P)_\xi} P$ is trivial, which, by Lemma 4.3.12, holds if and only if $\xi = [\varphi|_P]$. Now point i. is an immediate consequence.

ii. After enlarging S we may assume that there exists a model $\psi : \mathcal{V}/\mathrm{Spec}\, O_{K,S} \to \mathcal{U}/\mathrm{Spec}\, O_{K,S}$ for φ and that the morphism $\mathcal{V} \to \mathrm{Spec}\, O_{K,S}$ is smooth with geometrically integral fibers. For $v \notin S$, and $\mathcal{X} = \mathcal{U}, \mathcal{V}$ or $\mathcal{G}$, we denote by $\mathcal{X}_v$ the O_v-scheme $\mathcal{X} \times_{\mathrm{Spec}\, O_{K,S}} \mathrm{Spec}\, O_v$. For any $v \notin S$ and $\xi \in H^1(O_v, \mathcal{G})$ we denote by $\iota(\xi)$ the image of ξ in $H^1(K_v, G)$, and by $(\mathcal{V}_v)_\xi$ the twist of the $\mathcal{G}_v$-torsor $\mathcal{V}_v \to \mathcal{U}_v$ by ξ.

We claim that, for almost all $v \in M_K^f$ and any $\xi \in H^1(O_v, \mathcal{G})$, $(\mathcal{V}_v)_\xi(O_v) \neq \emptyset$. To prove the claim note that, since, for any $v \notin S$ and $\xi \in H^1(O_v, \mathcal{G})$, $(\mathcal{V}_v)_\xi \times_{O_v} \overline{\mathbb{F}_v} \cong \mathcal{V}_v \times_{O_v} \overline{\mathbb{F}_v} = \mathcal{V} \times_{O_{K,S}} \overline{\mathbb{F}_v}$, and the latter is integral, a standard argument through Lang-Weil bounds shows that $(\mathcal{V}_v)_\xi(\mathbb{F}_v) \neq \emptyset$ when v is large enough. Since $(\mathcal{V}_v)_\xi \to \operatorname{Spec} O_v$ is a smooth morphism, the fact that $(\mathcal{V}_v)_\xi(\mathbb{F}_v) \neq \emptyset$ implies that $(\mathcal{V}_v)_\xi(O_v) \neq \emptyset$, finishing the proof of the claim.

The same argument as in point i. (substituting "F-point" with "O_v-section") shows that, if $P \in \mathcal{U}(O_v) \subset U(K_v)$ lies in the image of $(\mathcal{V}_v)_\xi(O_v) \to \mathcal{U}(O_v)$, then ξ is equal to $[\psi|_P]$. In particular $\operatorname{Im}([\psi|\_] : \mathcal{U}(O_v) \to H^1(O_v, G))$ contains $\{\xi \in H^1(O_v, G) \mid (\mathcal{V}_v)_\xi(O_v) \neq \emptyset\}$. By the claim above, this last set is equal to the whole $H^1(O_v, G)$ for almost all $v \in M_K^f$.

By the commutativity of the following diagram:

$$\begin{array}{ccc} \mathcal{U}(O_v) & \xrightarrow{[\psi|\_]} & H^1(O_v, \mathcal{G}) \\ \downarrow & & \downarrow \\ U(K_v) & \xrightarrow{[\varphi|\_]} & H^1(K_v, G), \end{array}$$

it follows that $\operatorname{Im}([\varphi|\_] : \operatorname{Im}\mathcal{U}(O_v) \to H^1(K_v, G)) = \operatorname{Im}(\operatorname{Im}([\psi|\_] : \mathcal{U}(O_v) \to H^1(O_v, G)))$. We showed in the last paragraph that $\operatorname{Im}([\psi|\_] : \mathcal{U}(O_v) \to H^1(O_v, G)) = H^1(O_v, G)$ for v large enough, so this concludes the proof of point ii.

□

Proof of Proposition 4.3.7. We divide the proof in two steps.

Step 1. Let us prove that $(P_v) \in U(\mathbb{A}_K)^\varphi \Leftrightarrow \exists \beta \in H^1(K, G) \mid (\beta_v)_{v \in M_K} = ([\varphi|_{P_v}])_{v \in M_K}$

This is standard (compare e.g. with Definition 5.3.1 in [Sko01]). However, the proof is quite short, so we include it.

We claim that for any point $(P_v) \in U(\mathbb{A}_K)$ and $\beta \in H^1(K, G)$, $(P_v) \in \varphi_\beta(V_\beta(\mathbb{A}_K))$ if and only if for all v, $[\varphi|_{P_v}] = \beta_v$. In fact, for every $v \in M_K$, Lemma 4.3.12 (recall also the proof of Lemma 4.3.13.i) implies that $P_v \in \varphi_\beta(V_\beta(K_v))$ if and only if $[\varphi|_{P_v}] = \beta_v$. Hence $(P_v) \in \varphi_\beta(V_\beta(K_\Omega))$ if and only if for all v, $[\varphi|_{P_v}] = \beta_v$. On the other hand, since $\varphi_\beta : V_\beta \to U$ is finite, hence proper, the induced morphism $V_\beta(\mathbb{A}_K) \to U(\mathbb{A}_K)$ is proper as well, hence this tells us that $(P_v) \in \varphi_\beta(V_\beta(K_\Omega))$ if and only if $(P_v) \in \varphi_\beta(V_\beta(\mathbb{A}_K))$.

Using the claim above, we deduce that

$$(P_v) \in U(\mathbb{A}_K)^\varphi \Leftrightarrow \exists \beta \in H^1(K, G) \mid (P_v) \in V_\beta(\mathbb{A}_K) \Leftrightarrow \exists \beta \in H^1(K, G) \mid (\beta_v)_{v \in M_K} = ([\varphi|_{P_v}])_{v \in M_K}.$$

This concludes the proof of Step 1.

Step 2. Proof of the proposition. We first show that i. implies ii.Let $(P_v) \in X(\mathbb{A}_K)^\varphi$, and let $(Q_{v,n})_{v \in M_K, n \in \mathbb{N}}$ be a sequence of adelic points such that for each $n \in \mathbb{N}$, $(Q_{v,n})_v \in U(\mathbb{A}_K)^\varphi$, and $(Q_{v,n})_v \to (P_v)_v$ in the adelic topology of X as $n \to \infty$. In particular, for each finite $S \subseteq M_K$, there exists an $n_0 \in \mathbb{N}$ such that Q_{v,n_0} is arbitrarily near to P_v for all $v \in S$. Since $(Q_{v,n_0}) \in U(\mathbb{A}_K)^\varphi$, we deduce by the first step that there exists $\beta \in H^1(K, G)$ such that for all v, $\beta_v = [\varphi|_{Q_{v,n_0}}]$. In particular, $\beta_v = [\varphi|_{Q_{v,n_0}}] = [\varphi|_{P_v}]$ for all $v \in S$ (as Q_{v,n_0} is arbitrarily near to P_v and the function $[\varphi|\_] : U(K_v) \to E_v \subseteq H^1(K_v, G)$ is locally constant), and $\beta_v \in E_v$ for all $v \notin S$. This finishes the proof of this implication.

Let us show now that ii. implies i.Let $(P_v)_v$, S and β be as in ii.. Let for all $v \notin S$, $Q_v \in U(K_v)$ be such that $\beta_v = [\varphi|_{Q_v}]$. By Lemma 4.3.13.ii. we may assume that for almost all $v \in M_K^f$, $Q_v \in \mathcal{U}(O_v)$ (where, as usual $\mathcal{U} \to \operatorname{Spec} O_{K,S'}$ is a model for $U \to \operatorname{Spec} K$).

Let $(P'_v(S))_v \in U(\mathbb{A}_K)$ (we want to emphasize the dependence on the set S) be the adelic point defined by

$$\begin{cases} P'_v(S) = P_v \text{ for } v \in S; \\ P'_v(S) = Q_v \text{ for } v \notin S. \end{cases}$$

Note that, by Step 1, $(P'_v(S))_v \in U(\mathbb{A}_K)^\varphi$. As S becomes bigger we have that $(P'_v(S))_v$ tends to (in the adelic topology of X) the point (P_v), thus proving that $(P_v) \in \overline{U(\mathbb{A}_K)^\varphi}$, and concluding the proof of the implication and the proposition. □

4.3.3 A case where $E_v = H^1(K_v, G)$

In this subsection we want to give an example (actually, a family of these) where the identity $E_v = H^1(K_v, G)$ holds. We will not really use this in this work, so it is included mostly for curiosity.

For all this subsection, let k be a completion of the number field K at a non-archimedean valuation v.

Remark 4.3.14. Note that, if $U' \subseteq U$ is non-empty, then:

$$\operatorname{Im}([\varphi|_{-}] : U'(k) \to H^1(k, F)) = \operatorname{Im}([\varphi|_{-}] : U(k) \to H^1(k, F)).$$

In fact, we have that $[\varphi|_{-}] : U(k) \to H^1(k, F)$, being a continuous map into a discrete set, is locally constant. Since U is k-smooth, any element of $U(k)$ can be approximated through elements of $U'(k)$. It follows that the two images above are the same. As a consequence, the definition of E_v given in (4.3.8) does not depend on the choice of U.

In this subsection, we work in the following setting. Let k be a local field (of characteristic 0), and let μ_n be the finite k-group scheme of n^{th}-roots of unity. Let X be a smooth k-variety and $\psi : Y \to X$ be a μ_n-cover. Let us assume that Y is geometrically integral (and hence so is X). Note that then $\overline{Y} \to \overline{X}$ is a $\mu_{n,\overline{k}}$-cover where $\mu_{n,\overline{k}} := \mu_n \times_k \overline{k}$.

Let $U \subseteq X$ be a non-empty open subscheme such that $\psi^{-1}(U) \to U$ is an étale μ_n-torsor. We denote this μ_n-torsor by φ.

Recall that we have a map:

$$U(k) \to H^1(k, \mu_n), \quad [\varphi|_{-}] : P \mapsto [\varphi|_P]$$

Similarly to Subsection 4.3.2, we define:

$$E := \operatorname{Im}[\varphi|_{-}]. \tag{4.3.8}$$

The following is the main result of the subsection.

Theorem 4.3.15. *Assume that there exists a geometrically integral divisor $D \subsetneq X$, with generic point η, such that, denoting with a bar the base change to the algebraic closure $\overline{k}$:*

§. *the divisor $\overline{D} \subseteq \overline{X}$ is totally ramified in the $\mu_n(\overline{k})$-cover $\overline{Y} \to \overline{X}$, i.e., the inertia group of the DVR ring $\mathcal{O}_{\overline{X},\overline{\eta}} \subseteq \overline{k}(X)$ in the $\mu_n(\overline{k})$-Galois field extension $\overline{k}(X) \subseteq \overline{k}(Y)$ is equal to $\mu_n(\overline{k})$ (see [Ser79, Sec. I.7] for the definition and basic properties of inertia groups, note that this inertia group is well-defined because $\mu_n(\overline{k})$ is commutative);*

§. *$D^{reg}(k) \neq \emptyset$, where D^{reg} denotes the (open) subscheme of regular points of D.*

Then $E = H^1(k, \mu_n)$.

We denote by a the class $[\varphi] \in H^1(U, \mu_n)$. Kummer's exact sequence gives the following exact sequence:

$$H^0(U, \mathbb{G}_m) \xrightarrow{\hat{}n} H^0(U, \mathbb{G}_m) \xrightarrow{\delta} H^1(U, \mu_n) \to H^1(U, \mathbb{G}_m)[n]. \tag{4.3.9}$$

Let a' be the image of a under $H^1(U, \mu_n) \to H^1(U, \mathbb{G}_m)[n]$. Since $H^1(U, \mathbb{G}_m) = \operatorname{Pic} U$, and every element of $\operatorname{Pic} U$ is Zariski-locally trivial, we may assume, restricting U to an open subscheme, that $a' = 0$ and we may also assume that U is affine. Let $f \in k[U]^* = H^0(U, \mathbb{G}_m)$ be an element such that $\delta(f) = a$.

The theorem will follow almost immediately from the following proposition.

Proposition 4.3.16. *The composition* $U(k) \to k^* \to k^*/(k^*)^n$, $Q \mapsto f(Q) \mapsto [f(Q)]$ *is surjective.*

Proof. First of all, note that to prove the proposition, we may always restrict X to an open subscheme as long as D still intersects it (because then the first hypothesis of our proposition will be trivially satisfied by the restriction as well, and the second will be still satisfied because, since k is a local field, the fact that $D^{reg}(k)$ is non-empty, actually implies, by the implicit function theorem, that it is Zariski-dense in D). So, after restricting X to an open subscheme, we may assume that the only irreducible component of the divisor of f is D. Moreover, we may also assume that $U = X \setminus D$.

Lemma 4.3.17. *The vanishing order of f at D (taken to be negative if f has a pole) is coprime with n.*

Proof. Note that the vanishing order of $\overline{f}$ (the pullback of f to $\overline{X}$) at $\overline{D}$ is equal to the vanishing order of f at D. So, it suffices to prove that the former is coprime with n.

Let $R \subseteq \overline{k}(X)$ be the DVR $\mathcal{O}_{\overline{X},\overline{\eta}} \subseteq \overline{k}(X)$. Pulling back the exact sequence (4.3.9) to the generic point of $\overline{X}$, we obtain an exact sequence:

$$\overline{k}(X)^* \xrightarrow{\hat{\ }n} \overline{k}(X)^* \xrightarrow{\delta} H^1(\overline{k}(X), \mu_n) \to 0$$

(the last 0 follows from Hilbert 90), and, by functoriality of the objects involved, we have that the element $\alpha \in H^1(\overline{k}(X), \mu_n)$ corresponding to the $\mu_n(\overline{k})$-Galois field extension $\overline{k}(X) \subseteq \overline{k}(Y)$ is equal to the image of $a = [\varphi]$ in $H^1(\overline{k}(X), \mu_n)$, and we also have that $\delta(\overline{f}) = \alpha$. By a standard application of Kummer theory, we deduce that $\overline{k}(Y) = \overline{k}(X)(\sqrt[n]{\overline{f}})$.

Let $v : \overline{k}(X) \to \mathbb{Z} \cup \{\infty\}$ be the valuation associated to R. The fact that $\overline{k}(X) \subseteq \overline{k}(Y)$ is totally ramified implies that there is only one extension of the valuation v to $\overline{k}(Y)$ (keeping in mind that DVRs are just local Dedekind domains, we refer the reader to [Ser79], especially Sections I.4,7 and II.2,3 for the basic theory of discrete valuations, finite extensions and ramification) and that, if we denote this extension by w, the extension of local fields $\overline{k}(X)_v \subseteq \overline{k}(Y)_w$ is a totally ramified extension of degree n. Since $\overline{k}(Y) = \overline{k}(X)(\sqrt[n]{\overline{f}})$, we have that $\overline{k}(Y)_w = \overline{k}(X)_v(\sqrt[n]{\overline{f}})$, and since the residual characteristic of v is 0, the latter is totally ramified if and only if $v(f)$ is coprime with n. ∎

Let $(f) = rD$, where $r \in \mathbb{Z}$, which we know by the lemma above to be coprime with n, and let P be a regular k-point of D. Let u_1 be a uniformizer for D at P. After possibly restricting X again to a neighbourhood of P, we may assume that $(u_1) = D$.

Since $(r, n) = 1$, by Bezout's identity there exist $s, k \in \mathbb{Z}$ such that $rs - kn = 1$. Let $u := f^s u_1^{-kn} \in \overline{k}(X)^*$. Note that $(u) = D$ on X.

To finish the proposition we use the following lemma.

Lemma 4.3.18. *Let X/k be an algebraic integral variety, and let u be a regular function on X, such that the divisor $D := (u)$ contains a regular k-point P. Then there exists an $\epsilon > 0$ such that the image of the function*

$$(X \setminus D)(k) \xrightarrow{u()} k^*$$

contains the punctured disk $\{x \in k^* \mid |x| \leq \epsilon\}$.

Proof. Let P be the regular point of $D = (u)$. The differential $\mathrm{d}\, u$ does not vanish at P. Hence, by the inverse function theorem in the k-adic setting, we deduce that there exist local analytic functions $u_2, \ldots, u_n$, a k-adic neighbourhood $A \subseteq X(k)$ of P, a $\epsilon > 0$ and an analytic diffeomorphism (with an analytic inverse):

$$\chi : A \xrightarrow{\sim} \mathbb{D}^n, \quad q \mapsto (u(q), u_2(q), \ldots, u_n(q)),$$

where $\mathbb{D} \subseteq k$ denotes the ϵ-disk $\{a \in k \mid |a| \leq \epsilon\}$. Noting that the image of $A \cap D$ under χ is $\{0\} \times \mathbb{D}^{n-1}$, and that the restriction of the function $u()$ on $A \setminus D$ is equal to the composition $pr_1 \circ \chi$ concludes the proof. ∎

Note that the above lemma implies that the composition $U(k) \to k^* \to k^*/(k^*)^n$, $Q \mapsto u(Q) \mapsto [u(Q)]$ is surjective. From this the proposition follows immediately using the fact that elevation to the r in $k^*/(k^*)^n$ is invertible (since r is invertible $\bmod n$), and the fact that $[f(Q)] = [u(Q)]^r \in k^*/(k^*)^n$ for all $Q \in U(k)$. □

Proof of Theorem 4.3.15. Finally, let us show that $E = H^1(k, \mu_n)$. Since Kummer's exact sequence is functorial, we have that the following diagram commutes:

$$\begin{array}{ccccc} U(k) & \times & H^1(U, \mu_n) & \longrightarrow & H^1(k, \mu_n) \\ \big\downarrow{=} & & \delta\big\uparrow & & \delta\big\uparrow \\ U(k) & \times & H^0(U, \mathbb{G}_m) & \longrightarrow & H^0(k, \mathbb{G}_m), \end{array}$$

where both rows are defined as $(Q, \alpha) \mapsto Q^*\alpha$, and both δ's represent the boundary of the Kummer long exact sequence.

Since $a = \delta(f)$, we deduce from the commutativity of the above diagram that the following commutes as well:

$$\begin{array}{ccccc} U(k) & \xrightarrow{Q \mapsto Q^* a} & H^1(k, \mu_n) & \xrightarrow{\quad = \quad} & H^1(k, \mu_n) \\ \big\downarrow{=} & & \delta\big\uparrow & & \delta\big\uparrow \\ U(k) & \xrightarrow{Q \mapsto f(Q)} & H^0(k, \mathbb{G}_m) & \longrightarrow & H^0(k, \mathbb{G}_m)/nH^0(k, \mathbb{G}_m) \end{array}$$

Using Proposition 4.3.16 we proved, we know that the lower composition is surjective. Since the morphism $H^0(k, \mathbb{G}_m)/nH^0(k, \mathbb{G}_m) \xrightarrow{\delta} H^1(k, \mu_n)$ is surjective (this is well-known and follows immediately from Hilbert's 90 Theorem), we deduce, with a simple diagram chase, that the morphism $U(k) \to H^1(k, \mu_n)$, $Q \mapsto Q^*a$ is surjective, finishing the proof of the lemma. □

4.4 A subgroup $B \subseteq \operatorname{Br} X$ such that $X(\mathbb{A}_K)^{\psi} \subseteq X(\mathbb{A}_K)^B$

Let us briefly recall the setting. We have a commutative diagram:

$$\begin{array}{ccc} & & Y^{\mathrm{sm}} \\ & & \big\downarrow{\nu} \\ V & \xrightarrow{\subseteq} & Y \\ G\big\downarrow{\varphi} & & G\big\downarrow{\psi} \\ U & \xrightarrow{\subseteq} & X \end{array}$$

where $\varphi : V \to U$ is an (étale) G-torsor, $Y \to X$ is a G-cover (in particular, ψ is finite, Y is normal and X is smooth), the horizontal morphisms are G-equivariant open inclusions, and $Y^{sm} \to Y$ is a G-equivariant desingularization.

We defined:

$$X(\mathbb{A}_K)^{\psi} = \overline{\bigcup_{\xi \in H^1(K,G)} \varphi_\xi(V_\xi(\mathbb{A}_K))} = \overline{\bigcup_{\xi \in H^1(K,G)} \nu_\xi \psi_\xi(Y_\xi^{\mathrm{sm}}(\mathbb{A}_K))}.$$

We describe in this subsection an explicit subgroup $\operatorname{Br}_\psi X \subseteq \operatorname{Br} X$ of the Brauer group of X such that $X(\mathbb{A}_K)^{\psi} \subseteq X(\mathbb{A}_K)^{\operatorname{Br}_\psi X}$ (see Theorem 4.4.7 below). This provides a first partial answer to Question 4.3.5.

In Section 4.6 (the one with the "counterexample") we will be mostly interested in the case where G is commutative, and show that $\operatorname{Br}_\psi X$ is not necessarily algebraic (see Section 4.6), and that the transcendental elements may indeed provide a non-trivial obstruction. This shows that the obstruction can not be reconstructed from the classical Brauer–Manin descent obstruction (see e.g. Subsection 4.4.4 for what exactly are we referring to here), which is only algebraic. I.e., to be precise, we will provide an example of a (geometrically integral) cover ψ, where $X(\mathbb{A}_K)^{\operatorname{Br}_\psi X} \subsetneq X(\mathbb{A}_K)$, and where $\operatorname{Br}_\psi X \cap \operatorname{Br}_a X$ is contained in the constant elements $\operatorname{Br} K$.

4.4.1 A map from group cohomology to étale cohomology

In all this subsection, we fix a perfect field F and a finite group scheme H over F.

Let W be an F-scheme and $Z \to W$ be an H-torsor, and let L/F be a finite Galois extension splitting H (i.e. such that $H(L) = H(\overline{F})$).

Note that there is a natural structure of $(H(L) \rtimes \operatorname{Gal}(L/F))$-torsor on $Z_L \to W$ (since $H(L) \rtimes \operatorname{Gal}(L/F)$ is an abstract group and not an F-group, recall from our convention in Section 4.2, that a $(H(L) \rtimes \operatorname{Gal}(L/F))$-torsor is just a $(H(L) \rtimes \operatorname{Gal}(L/F))_F$-torsor), defined by the action (defined in S-point notation, where S is a general object in the category of F-schemes):

$$(Z \times_F L) \times (H(L) \rtimes \operatorname{Gal}(L/F)) \to Z \times_F L,\ \ ((z,\xi),(h(-),\sigma)) \mapsto (zh(\xi),\xi\sigma), \tag{4.4.1}$$

where z (resp. ξ) is an S-point in Z (resp. $\operatorname{Spec} L$), the notation $h(-)$ indicates the natural transformation of S-points associated to the morphism $h : \operatorname{Spec} L \to H$. We leave to the reader the easy verification that the map defined above is indeed a group action.

We recall that, for an étale torsor $Y_1 \to Y_2$ under a constant group g, and an étale sheaf $\mathcal{F}$ over Y_2, the Hochschild-Serre spectral sequence writes down as follows:

$$H^i(g, H^j(Y_1,\mathcal{F})) => H^{i+j}(Y_2,\mathcal{F}).$$

In particular, this spectral sequence induces, for each $n \geq 0$, a morphism $H^n(g,\mathcal{F}(Y_1)) \to H^n(Y_2,\mathcal{F})$.

For an étale sheaf $\mathcal{F}$ over W, applying the construction above to the $(H(L) \rtimes \operatorname{Gal}(L/K))$-torsor $Z_L \to W$, we obtain a morphism:

$$\epsilon_L : H^n(H(L) \rtimes \operatorname{Gal}(L/F), \mathcal{F}(Z_L)) \to H^n(W,\mathcal{F}).$$

Since we do not want to carry this L throughout all the text, we introduce the following notations, which just serve the purpose of setting "$L = \overline{F}$".

Notation 4.4.1. We use the notation Γ_H to indicate the group $H(\overline{F}) \rtimes \Gamma_F$, where the external action is the Galois one.

Moreover, we denote the limit of the morphisms ϵ_L, where L varies along all finite subfields of $\overline{F}$ that split H, ordered by inclusion, and the transition morphisms of the groups $H^n(H(L) \rtimes \operatorname{Gal}(L/K), \mathcal{F}(Z_L))$ are the inflation maps, as:

$$\epsilon_{\overline{Z}/W} : H^n(\Gamma_H, \mathcal{F}(Z_{\overline{F}})) \to H^n(W,\mathcal{F}), \tag{4.4.2}$$

where we are using the following notation:

Notation 4.4.2. For an étale sheaf $\mathcal{F}$ on an F-variety Y, we use the notation $\mathcal{F}(Y_{\overline{F}})$ to indicate the direct limit

$$\mathcal{F}(Y_{\overline{F}}) := \varinjlim_{L} \mathcal{F}(Y_L),$$

where L varies over all finite field extensions of K contained in $\overline{K}$, ordered by inclusion.

When there is no risk of confusion, we will feel free to change the subscript "$\overline{Z}/W$" in $\epsilon_{\overline{Z}/W}$ to "W" (or to avoid using it completely).

4.4.2 Definition of $\mathrm{Br}_\psi(X)$

We put ourselves in the setting of Section 4.3. We are finally ready to define the group $\mathrm{Br}_\psi(X)$ to which we alluded at at the beginning of this section. Although we will give the definition without assuming that X is proper, the main application that the reader has to keep in mind is precisely this case. For instance, in the counterexample of Section 4.6 X is proper.

Applying the construction (4.4.2) to the G-torsor $V \to U$ with $F = K$, $H = G$ and $\mathcal{F} = \mathbb{G}_m$, we obtain a morphism:

$$H^2(\Gamma_G, \mathbb{G}_m(V_{\overline{K}})) \xrightarrow{\epsilon_{\overline{V}/U}} H^2(U, \mathbb{G}_m). \tag{4.4.3}$$

Note that $\mathbb{G}_m(V_{\overline{K}}) = \overline{K}[V]^*$, and that the implied Γ_G-action restricts to the inflation (along the projection $\Gamma_G \to \Gamma_K$) of the Γ_K-action on $\overline{K}^*$. Hence there is a natural morphism:

$$H^2(\Gamma_G, \overline{K}^*) \to H^2(\Gamma_G, \overline{K}[V]^*) = H^2(\Gamma_G, \mathbb{G}_m(V_{\overline{K}})),$$

where the implied action on the LHS is the one described above.

Definition 4.4.3. We define the subgroup $\mathrm{Br}_\varphi(U)$ of $\mathrm{Br}\, U$ as the image of the composition

$$H^2(\Gamma_G, \overline{K}^*) \to H^2(\Gamma_G, \overline{K}[V]^*) \xrightarrow{\epsilon_{\overline{V}/U}} H^2(U, \mathbb{G}_m) = \mathrm{Br}\, U.$$

It will be convenient to have a notation for the composition in Definition (4.4.3):

Notation 4.4.4. Let H be a finite étale group scheme over a perfect field F, W be an F-variety and $\varphi : Z \to W$ be an H-torsor. We denote the composition

$$H^2(\Gamma_H, \overline{F}^*) \to H^2(\Gamma_H, \overline{F}[Z]^*) \xrightarrow{\epsilon_{\overline{Z}/W}} H^2(W, \mathbb{G}_m)$$

with $u_{\varphi,F}$ or just u_φ.

Definition 4.4.5. We define $\mathrm{Br}_\psi(X) \subseteq \mathrm{Br}(X)$ as the intersection $\mathrm{Br}(X) \cap \mathrm{Br}_\varphi(U)$.

Note that the above definition is independent from U, indeed we even have that $\mathrm{Br}_\psi(X) = \mathrm{Br}(X) \cap \mathrm{Br}_\varphi(K(X))$, where $\mathrm{Br}_\varphi(K(X)) \subseteq \mathrm{Br}\, K(X)$ is defined as the image of $H^2(\Gamma_G, \overline{K}^*)$ in $H^2(K(X), \mathbb{G}_m)$ through the morphism:

$$H^2(\Gamma_G, \overline{K}^*) \to H^2(\Gamma_G, \overline{K}(Y)^*) \to H^2(K(X), \mathbb{G}_m),$$

where the second morphism is defined, after identifying $H^2(K(X), \mathbb{G}_m)$ with $H^2(\Gamma_{K(X)}, \overline{K(X)}^*)$, the Hochshild-Serre spectral sequence associated to the Γ_G-Galois field extension $K(X) \subseteq \overline{K}(Y)$.

Remark 4.4.6. By [...], the edge map $H^2(\Gamma_G, \overline{K}(Y)^*) \to H^2(\Gamma_{K(X)}, \overline{K(X)}^*)$ described above coincides with the inflation map along the natural morphism $\Gamma_{K(X)} \to \Gamma_G$. Moreover, the morphism $H^2(\Gamma_G, \mu_\infty) \to H^2(\Gamma_G, \overline{K}(Y)^*)$ is an isomorphism, as the Γ_G-module $\overline{K}(Y)^*/\mu_\infty$ is uniquely divisible. It follows that the image in $\mathrm{Br}\, K(X)$ of $\mathrm{Br}_\varphi U$ is equal to $\mathrm{Im}(\inf^{\Gamma_G}_{\Gamma_{K(X)}}(H^2(\Gamma_G, \mu_\infty)))$, and

$$\mathrm{Br}_\psi X = \mathrm{Br}\, X \cap \mathrm{Im}(\inf^{\Gamma_G}_{\Gamma_{K(X)}}(H^2(\Gamma_G, \mu_\infty))) \subseteq H^2(\Gamma_{K(X)}, \overline{K(X)}^*),$$

which was the definition that we gave in the introduction.

We will prove the following theorem in the next section.

Theorem 4.4.7. *We have an inclusion* $X(\mathbb{A}_K)^\psi \subseteq X(\mathbb{A}_K)^{\mathrm{Br}_\psi(X)}$.

4.4.3 Proof of Theorem 4.4.7

We start by fixing some notation and proving some easy statements that we will need in the proof of Theorem 4.4.7.

To introduce these notations and to prove these statements, we put ourselves again in the general setting that follows (chosen in accordance with Subsection 4.4.1). Let F be a perfect field and let H be a finite étale group scheme over F.

H^1 and sections of $\Gamma_H \to \Gamma_F$ We recall that Γ_H was defined as $H(\overline{F}) \rtimes \Gamma_F$. We have a (split) short exact sequence of groups:

$$1 \to H(\overline{F}) \to \Gamma_H \to \Gamma_F \to 1. \tag{4.4.4}$$

We denote the set of sections $\Gamma_F \to \Gamma_H$ of the projection $\Gamma_H \to \Gamma_F$ by $\operatorname{Sec}(\Gamma_H)$.

There is a well-know canonical bijection of pointed sets (see [NSW08, Section 1.2, Exercise 1]):

$$\aleph : \ \operatorname{Sec}(\Gamma_H)/\sim \ \longleftrightarrow \ H^1(F,H), \tag{4.4.5}$$

where $\sim$ denotes conjugation by an element in Γ_H, i.e. $\gamma_1, \gamma_2 : \Gamma_F \to \Gamma_H$ are equivalent if there exists a $b \in H(\overline{F})$ such that $\gamma_2(-) = b\gamma_1(-)b^{-1}$. If $s \in \operatorname{Sec}(\Gamma_H)$, and, for $\sigma \in \Gamma_F$, we write $s(\sigma) = (h_s(\sigma), \sigma) \in H(\overline{F}) \rtimes \Gamma_F$, then $\aleph(s)$ is the image in $H^1(F,H) = H^1(\Gamma_F, H(\overline{F}))$ of the cocyle $\sigma \mapsto h_s(\sigma)$.

A (non-bilinear) pairing Let M be a Γ_F-module. We endow M with the Γ_H-action induced by pulling back the Γ_F-action along the morphism $\Gamma_H \to \Gamma_F$.

Lemma 4.4.8. *For each $n \geq 0$, there exists a unique map (of sets):*

$$(.) : H^n(\Gamma_H, M) \times H^1(F,H) \to H^n(\Gamma_F, M), \tag{4.4.6}$$

that satisfies, for each $\alpha \in H^n(\Gamma_H, M)$, $\gamma \in \operatorname{Sec}(\Gamma_H)$, the equation $(\alpha, \aleph(\gamma)) = \gamma^*\alpha$.

Proof. Since $\aleph$ is surjective, uniqueness is clear. It remains to show that there exists one such map.

It suffices to show that, for every $\alpha, \gamma_1, \gamma_2$ (in $H^n(\Gamma_H, M), \operatorname{Sec}(\Gamma_H)$ and $\operatorname{Sec}(\Gamma_H)$ respectively), if $\aleph(\gamma_1) = \aleph(\gamma_2)$, then $\gamma_1^*\alpha = \gamma_2^*\alpha$.

Since (4.4.5) is a bijection, $\aleph(\gamma_1) = \aleph(\gamma_2)$ implies that there exists a $b \in H(\overline{F})$ such that, if $c(b) : \Gamma_H \to \Gamma_H$ denotes conjugation by b, then $\gamma_2 : \Gamma_F \to \Gamma_H$ is equal to the composition $\Gamma_F \xrightarrow{\gamma_1} \Gamma_H \xrightarrow{c(b)} \Gamma_H$. Hence $\gamma_2^*\alpha = \gamma_1^*(c(b)^*\alpha) \in H^n(\Gamma_F, M)$. However, by [NSW08, Prop. 1.6.3], we have that $c(b)^*\alpha = \alpha \in H^n(\Gamma_H, M)$, hence $\gamma_1^*\alpha = \gamma_2^*\alpha$, as wished. □

The following simple lemma is key in the proof of Theorem 4.4.7. As in Subsection 4.4.1, let W be an F-scheme, and $\varphi : Z \to W$ be an H-torsor.

Lemma 4.4.9. *Let $\mathcal{F}$ be a sheaf on $\operatorname{Spec} F$ (just as a reminder, recall that such a sheaf always arises from a Γ_F-module), and $p : W \to \operatorname{Spec} F$ be the structural projection. For each $n \geq 0$, the following diagram commutes:*

$$\begin{array}{ccccc} H^n(\Gamma_H, \mathcal{F}(\overline{F})) & \times & H^1(\Gamma_F, H) & \longrightarrow & H^n(\Gamma_F, \mathcal{F}(\overline{F})) \\ \downarrow{\scriptstyle \epsilon_{\overline{Z}/W}} & & {\scriptstyle [\varphi|_-]}\uparrow & & {\scriptstyle \sim}\downarrow{\scriptstyle \epsilon} \\ H^n(W, p^*\mathcal{F}) & \times & W(F) & \longrightarrow & H^n(F, \mathcal{F}) \end{array}, \tag{4.4.7}$$

where the morphism $W(F) \to H^1(\Gamma_F, H)$ is the one sending a point P to the class of the torsor $[\varphi|_P]$, the second row is $(\alpha, Q : \operatorname{Spec} F \to W) \mapsto Q^*\alpha$, and the first row is the pairing $(\alpha, \aleph(\xi)) \mapsto \xi^*\alpha$ defined in Lemma 4.4.8.

Proof. We fix a geometric point $\overline{w} \to W$. Since each point $P \in W(F)$ defines a section $P : \operatorname{Spec} F \to W$ of the structural morphism $p : W \to \operatorname{Spec} F$, to such point corresponds a map $P_* : \Gamma_F = \pi_{1,\acute{e}t}(\operatorname{Spec} F, \operatorname{Spec} \overline{F}) \to \pi_{1,\acute{e}t}(W, \overline{w})$, that is a section of $p_* : \pi_{1,\acute{e}t}(W, \overline{w}) \to \pi_{1,\acute{e}t}(\operatorname{Spec} F, \operatorname{Spec} \overline{F}) = \Gamma_F$.

We recall that the connected Γ_H-(étale) torsor $Z \times_F \overline{F} \to W$ (Γ_H is not a finite group, it is just profinite, so the word "étale-torsor" does not really make sense, what we mean is that there is a compatible system of étale-torsors under all quotients of Γ_H by an open subgroup, whose limit is $Z \times_F \overline{F}$) induces a surjective map (see e.g. [Sza09, Cor. 5.4.8]):

$$\pi_{1,\acute{e}t}(W, \overline{w}) \to \Gamma_H,$$

and the composition $\pi_{1,\acute{e}t}(W, \overline{w}) \to \Gamma_H \to \Gamma_F$ is equal to p_*.

Combining the maps from the two paragraphs above we obtain a map $\delta : W(F) \to \operatorname{Sec}(\Gamma_H)$. We leave to the reader the easy verification that $\aleph \circ \delta = [\varphi|_-]$ ($\delta(P)$ can be computed to be, up to equivalence, $\sigma \mapsto (h_\sigma, \sigma)$, where $h_\sigma \in Z^1(\Gamma_F, H(\overline{F}))$ is a cocyle representing the torsor $\varphi|_P$), i.e. the second vertical morphism appearing in Diagram (4.4.7).

We fix now a point $P \in W(F)$ and a $b \in H^n(\Gamma_H, \mathcal{F}(\overline{F}))$. We have that there exists a commutative diagram:

$$\begin{array}{ccc} \operatorname{Spec} \overline{F} & \longrightarrow & \overline{Z} \\ \downarrow{\scriptstyle \Gamma_F} & & \downarrow{\scriptstyle \Gamma_H} \\ \operatorname{Spec} F & \xrightarrow{P} & W \end{array},$$

where the implied morphism $\Gamma_F \to \Gamma_H$ is $\delta(P)$. The diagram above induces by functoriality of the Hochschild-Serre spectral sequence, the following commutative diagram:

$$\begin{array}{ccc} H^n(\Gamma_H, \mathcal{F}(\overline{F})) & \xrightarrow{\delta(P)^*} & H^n(\Gamma_F, \mathcal{F}(\overline{F})) \\ \downarrow{\scriptstyle \epsilon} & & \sim\downarrow{\scriptstyle \epsilon} \\ H^n(W, p^*\mathcal{F}) & \xrightarrow{P^*} & H^n(F, \mathcal{F}). \end{array}$$

Hence $\epsilon(\delta(P)^* b) = \epsilon(b)(P)$. Since, as noticed above, $\aleph(\delta(P)) = [\varphi|_P]$, and the calculations above hold for any $P \in W(F)$ and $b \in H^n(\Gamma_H, \mathcal{F}(\overline{F}))$, this proves the sought commutativity. □

Proof of Theorem 4.4.7. We start with applying Lemma 4.4.9 to the context in which we need it. We do so in the following lemma.

Lemma 4.4.10. *Let F be a field containing K, and let $\Gamma_{F,G} := G(\overline{F}) \rtimes \Gamma_F$. We have the following commutative diagram:*

$$\begin{array}{ccccc} H^2(\Gamma_{F,G}, \overline{F}^*) & \times & H^1(F, G) & \longrightarrow & H^2(\Gamma_F, \overline{F}^*) \\ \downarrow{\scriptstyle u_{\varphi_F}} & & {\scriptstyle [\varphi_F|_-]}\uparrow & & \downarrow{\scriptstyle \sim} \\ \operatorname{Br} U_F & \times & U(F) & \longrightarrow & \operatorname{Br} F, \end{array}$$

where the upper horizontal map comes from Lemma 4.4.8, and the lower one is just evaluation.

Proof. We denote the projection $U_F \to \operatorname{Spec} F$ by p. The lemma immediately follows from Lemma 4.4.9 with $n = 2, H = G, Z = V_F, W = U_F, \mathcal{F} = \mathbb{G}_m$, the (immediate) commutativity of the following diagram

$$\begin{array}{ccccc} H^2(U_F, p^*\mathbb{G}_m) & \times & U(F) & \longrightarrow & \operatorname{Br} F \\ \downarrow & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} \\ \operatorname{Br} U_F & \times & U(F) & \longrightarrow & \operatorname{Br} F, \end{array}$$

where $H^2(U_F, p^*\mathbb{G}_m) \to H^2(U_F, \mathbb{G}_m) = \operatorname{Br} U_F$ is the map induced by the morphism $p^*\mathbb{G}_m \to \mathbb{G}_m$ (note that we are working on the small site, so $p^*\mathbb{G}_m$ is not $\mathbb{G}_m$), and the commutativity of the following diagram (the commutativity of the square follows from the functoriality of the Hochschild-Serre spectral sequence, while the lower triangle commutes by definition):

$$\begin{array}{ccc} H^2(U_F, p^*\mathbb{G}_m) & \xleftarrow{\epsilon_{\overline{V}/U}} & H^2(\Gamma_G, \overline{F}^*) \\ \downarrow & \swarrow_{u_{\varphi,F}} & \downarrow \\ H^2(U_F, \mathbb{G}_m) & \xleftarrow{\epsilon_{\overline{V}/U}} & H^2(\Gamma_G, \overline{F}[V]^*). \end{array}$$

□

We now conclude the proof of Theorem 4.4.7.

Let us first prove the following framed claim:

$\boxed{U(\mathbb{A}_K)^\varphi \subseteq U(\mathbb{A}_K)^{\operatorname{Br}_\varphi(U)}}$. Let $(P_v)_{v\in M_K} \in U(\mathbb{A}_K)^\varphi$, and let $\xi \in H^1(K,G)$ be such that $(P_v)_{v\in M_K} \in \varphi_{\xi^{-1}}(V_{\xi^{-1}}(\mathbb{A}_K))$. By Lemma 4.3.12 this implies that, for every $v \in M_K$, $\xi_v = [\varphi|_{P_v}]$.

Let $B \in H^2(\Gamma_G, \overline{K}^*)$, and let v be a place in M_K. Let Γ_G be the semidirect product $G(\overline{K}) \rtimes \Gamma_K$, and let Γ_G^v be the fibered product $G(\overline{K_v}) \rtimes \Gamma_{K_v}$ (i.e. the Galois group of the cover $Z \times_K \overline{K_v} \to W \times_K K_v$). By Lemma 4.4.10 (with $F = K_v$), we deduce that $P_v^* u_{\varphi,K_v}(B_v) = (B_v.[\varphi|_{P_v}]) \in \operatorname{Br} K_v$.

Hence we have that $(u_{\varphi,K}(B), (P_v)_v)_{BM} = \sum_{v\in M_K} \operatorname{inv}_v P_v^* u_{\varphi,K_v}(B_v) = \sum_{v\in M_K} \operatorname{inv}_v(B_v.[\varphi|_{P_v}]) = \sum_{v\in M_K} \operatorname{inv}_v(B.\xi)_v = 0$, the latter by the Albert-Brauer-Hasse-Noether theorem. This concludes the proof of the framed claim.

We have $X(\mathbb{A}_K)^\psi = \overline{U(\mathbb{A}_K)^\varphi} \subseteq \overline{U(\mathbb{A}_K)^{\operatorname{Br}_\varphi(U)}} \subseteq \overline{X(\mathbb{A}_K)^{\operatorname{Br}_\psi(X)}} = X(\mathbb{A}_K)^{\operatorname{Br}_\psi(X)}$, where the last equality follows from the fact that, for any $B \subseteq \operatorname{Br}(X)$, $X(\mathbb{A}_K)^B$ is closed in $X(\mathbb{A}_K)$. □

Example 4.4.11. Let G be a finite group. Then, an example in which we fall in the above described setting is that of $U = SL_n/G$, $V = SL_n/G'$ (here $G' \subseteq G$ denotes a subgroup containing the derived subgroup of G), and $X = U^c$ (a smooth compactification of U). The example 4.6.9 that we will produce in Section 4.6 will fall under this setting, in the specific case of a solvable group G.

4.4.4 Comparison with classical abelian descent obstruction

The classical abelian descent theory We assume in this paragraph that G is commutative, so, for clarity, we use the letter A to denote it. I.e. $A = G$. Let $A' = \operatorname{Hom}(A, \mathbb{G}_{m,K})$ be the Cartier dual of A.

We define $\operatorname{Br}^a_\varphi(U)$ as the image of the composition:

$$p^*(-) \cup [\varphi] : H^1(K, A') \xrightarrow{p^*} H^1(U, A') \xrightarrow{-\cup[\varphi]} H^2(U, \mathbb{G}_m) = \operatorname{Br}(U),$$

where $-\cup[\varphi]$ denotes the specialization of the second variable of the cup product $H^1(U, A') \times H^1(U, A) \to H^2(U, \mathbb{G}_m)$ to $[\varphi] \in H^1(U, A)$. Note that actually $\operatorname{Br}^a_\varphi(U) \subseteq \operatorname{Br}_1(U)$.

Lemma 4.4.12. *We have:*

$$U(\mathbb{A}_K)^\varphi = U(\mathbb{A}_K)^{\operatorname{Br}^a_\varphi(U)}. \tag{4.4.8}$$

Proof. An adelic point $(P_v) \in U(\mathbb{A}_K)$ belongs to $U(\mathbb{A}_K)^\varphi$ if and only if the family $([\varphi|_{P_v}])_{v\in M_K}$ is global (i.e. comes by specializing a single $\alpha \in H^1(K, A)$), which, by the Poitou-Tate exact sequence holds if and only if $([\varphi|_{P_v}])_{v\in M_K}$ is orthogonal to $H^1(K, A')$, with respect to the pairing $P^1(K, A) \times H^1(K, A') \to \mathbb{Q}/\mathbb{Z}$ arising from the Poitou-Tate exact sequence. By the compatibility of this pairing with the Brauer–Manin pairing (in the way described in 4.3.7), we deduce that this orthogonality is equivalent to the (Brauer–Manin-)orthogonality of $(P_v)_{v\in M_K}$ with $\operatorname{Br}^a_\varphi(U) = H^1(K, A') \cup [V]$. □

Note that the lemma above immediately implies that $X(\mathbb{A}_K)^{\varphi} \subseteq X(\mathbb{A}_K)^{\mathrm{Br}^a_{\varphi}(U)\cap \mathrm{Br}\, X}$. We refer to this as the *"classical" descent obstruction* for the cover $\psi : Y \to X$.

The following proposition is the main result of this subsection:

Proposition 4.4.13. *The inclusion* $\mathrm{Br}^a_{\varphi}(U) \subseteq \mathrm{Br}_{\varphi} U$ *holds.*

Letting, $\mathrm{Br}^a_{\psi}(X) := \mathrm{Br}^a_{\varphi}(U) \cap \mathrm{Br}\, X$, we immediately get, as a corollary, that:

Corollary 4.4.14. *The inclusion* $\mathrm{Br}^a_{\psi}(X) \subseteq \mathrm{Br}_{\psi} X$ *holds. In particular, we have the following series of inclusions:*

$$X(\mathbb{A}_K)^{\psi} \subseteq X(\mathbb{A}_K)^{\mathrm{Br}_{\psi} X} \subseteq X(\mathbb{A}_K)^{\mathrm{Br}^a_{\psi}(X)}.$$

However, in contrast with what happens on U (where the inclusions $U(\mathbb{A}_K)^{\varphi} \subseteq U(\mathbb{A}_K)^{\mathrm{Br}_{\varphi} U} \subseteq U(\mathbb{A}_K)^{\mathrm{Br}^a_{\varphi}(U)}$ are actually equalities by (4.4.8)), the last inclusion in the corollary above may well be strict!

Remark 4.4.15. Example 4.6.9 will also prove the following (slightly stronger) statement: for a point $(P_v) \in \prod_v U(K_v)$ it is not enough to be orthogonal to $\mathrm{Br}^a_{\psi}(X)$ to infer that it lies in $X(\mathbb{A}_K)^{\psi}$ (or, equivalently, that it satisfies condition ii of Proposition 4.3.7). In fact, in this example, $\mathrm{Br}\, X/\,\mathrm{Br}\, K$ will be finite, hence $U(K_\Omega)^{\mathrm{Br}^a_{\psi}(X)}$ will be dense in $X(\mathbb{A}_K)^{\mathrm{Br}^a_{\psi}(X)}$, and $U(K_\Omega)^{\mathrm{Br}_{\psi}(X)}$ will be dense in $X(\mathbb{A}_K)^{\mathrm{Br}_{\psi}(X)}$. Since, moreover, we will have that $X(\mathbb{A}_K)^{\mathrm{Br}_{\psi} X} \subsetneq X(\mathbb{A}_K)^{\mathrm{Br}^a_{\psi}(X)}$, it follows that $U(K_\Omega)^{\mathrm{Br}_{\psi}(X)} \subsetneq U(K_\Omega)^{\mathrm{Br}^a_{\psi}(X)}$. Hence there exists an element $(P_v) \in U(K_\Omega)^{\mathrm{Br}^a_{\psi}(X)} \setminus U(K_\Omega)^{\mathrm{Br}_{\psi}(X)} \subseteq U(K_\Omega)^{\mathrm{Br}^a_{\psi}(X)} \setminus X(\mathbb{A}_K)^{\psi}$, as wished. It follows, in particular, keeping in mind the results of Subsection 4.3.2 and especially the discussion following Remark 4.3.11, that the answer to Harari's Question 4.3.8 is "No".

Proof of Proposition 4.4.13. To prove the proposition, we are going to prove that the morphism $H^1(K, A') \to \mathrm{Br}\, U$ (whose image is $\mathrm{Br}^a_{\varphi}(U)$) decomposes as follows:

$$H^1(K, A') \to H^2(\Gamma_A, \overline{K}^*) \xrightarrow{u_\varphi} \mathrm{Br}\, U. \tag{4.4.9}$$

Note that this immediately implies the proposition. The existence of the desired factorization is actually an immediate consequence of the following claim.

We claim that the following diagram commutes:

$$\begin{array}{ccccc} & & H^2(\Gamma_A, \overline{K}^*) & & \\ & \overset{\zeta}{\nearrow} & & \overset{u_\varphi}{\searrow} & \\ H^1(\Gamma_K, A'(\overline{K})) = H^1(K, A') & & \xrightarrow{\quad p_U^*(-)\cup[\varphi]\quad} & & \mathrm{Br}\, U, \end{array}$$

where $p_U : U \to \operatorname{Spec} K$ denotes the structural projection, and ζ is defined as the following composition:

$$\zeta : H^1(\Gamma_K, A'(\overline{K})) \xrightarrow{\inf} H^1(\Gamma_A, A'(\overline{K})) \xrightarrow{-\cup\alpha} H^2(\Gamma_A, \mu_\infty) \to H^2(\Gamma_A, \overline{K}^*),$$

where $\alpha \in H^1(\Gamma_A, A)$ is the element represented by the cocycle $A(\overline{K}) \rtimes \Gamma_K = \Gamma_A \to A(\overline{K}), (a, \sigma) \mapsto a$. To prove the claim, recall that we have morphisms:

$$\epsilon : H^1(\Gamma_A, A) \to H^1(U, A),\ \ \epsilon : H^1(\Gamma_A, A') \to H^1(U, A'), \epsilon : H^2(\Gamma_A, \mu_\infty) \to H^2(U, \mu_\infty),$$

that are compatible with cup product, i.e. $\epsilon(a \cup b) = \epsilon(a) \cup \epsilon(b)$, for $a \in H^1(\Gamma_A, A), b \in H^1(\Gamma_A, A')$, $a \cup b \in H^2(\Gamma_A, \mu_\infty)$, where $\mu_\infty \subseteq \overline{K}^*$ denotes the subgroup of roots of unity.

Note that the commutativity of the diagram above is equivalent to saying that, for every $\gamma \in H^1(\Gamma_K, A'(\overline{K}))$,

$$\iota(\epsilon(\inf(\gamma) \cup \alpha)) = u_\varphi(\zeta(\gamma)) = p_U^*(\gamma) \cup [\varphi] \in H^2(U, \mathbb{G}_m),$$

where ι denotes the morphism $H^2(U,\mu_\infty) \to H^2(U,\mathbb{G}_m)$. Noting that $p_U^*(\gamma)\cup[\varphi] = \iota(p_U^*(\gamma)\cup[\varphi])$ (in the sense that the first cup product sign refers to the map $H^1(U,A')\times H^1(U,A) \to H^2(U,\mathbb{G}_m)$ while the second refers to $H^1(U,A')\times H^1(U,A) \to H^2(U,\mu_\infty)$, and the former is the image of the latter under ι), to prove the desired equality it is sufficient that we prove that

$$\epsilon(\inf(\gamma)\cup\alpha) = p_U^*(\gamma)\cup[\varphi] \in H^2(U,\mu_\infty), \text{ for all } \gamma \in H^1(\Gamma_K, A'(\overline{K})).$$

Note that $\epsilon(\inf(\gamma)\cup\alpha) = \epsilon(\inf(\gamma))\cup\epsilon(\alpha)$ (compatibility of ϵ with cup product), and now the above equality follows from the following two facts.

- The following diagram commutes:

$$\begin{array}{ccc} H^1(\Gamma_K, A'(\overline{K})) & \xrightarrow{\epsilon=id} & H^1(K,A') \\ \big\downarrow{\scriptstyle inf} & & \big\downarrow{\scriptstyle p_U^*} \\ H^1(\Gamma_A, A'(\overline{K}) = A'(\overline{V})) & \xrightarrow{\epsilon} & H^1(U,A') \end{array},$$

  where, with a slight abuse of notation, we are using the letter A' to denote both the K-group it represents and the étale sheaf p_U^*A' on U. (The ϵ in the first line of the diagram happens to be the identity through the standard identification $H^1(\Gamma_K, A'(\overline{K})) = H^1(K,A')$). The commutativity of the diagram follows from the functoriality of the Hochschild-Serre spectral sequence, and it immediately implies that $\epsilon(\inf(\gamma)) = p_U^*(\gamma)$ for all $\gamma \in H^1(\Gamma_K, A'(\overline{K}))$.

- $\epsilon(\alpha) = [\varphi]$. This follows from an easy cocycle computation that we leave to the interested reader.

□

4.5 Obstruction to rational points on X

Let us recall the setting: X is a smooth (not necessarily proper, even though this is the main example to keep in mind) geometrically connected variety over K, G/K is a finite group scheme and $\psi : Y \to X$ is a G-cover. In particular, Y is normal.

In this subsection, we prove that $\overline{X(K)} \subseteq X(\mathbb{A}_K)^\psi$ (the closure being in $X(\mathbb{A}_K)$). Note that, as previously remarked, one easily sees that $\overline{U(K)} \subseteq X(\mathbb{A}_K)^\psi$ (the closure again being in $X(\mathbb{A}_K)$), however, in general, one may well have that $\overline{U(K)}$ is strictly smaller than $\overline{X(K)}$. This is the case, for instance, when $U(K)$ is empty, while $(X\setminus U)(K)$ is non-empty.

Let $\nu : Y^{\mathrm{sm}} \to Y$ be a G-equivariant desingularization of Y, and let r be the composition $\psi\circ\nu : Y^{sm} \to X$. We will actually prove the stronger:

Proposition 4.5.1. *We have that*

$$X(K) \subseteq \bigcup_{\xi\in H^1(K,G)} r_\xi(Y_\xi^{sm}(K)). \tag{4.5.1}$$

Corollary 4.5.2. *The inclusion $\overline{X(K)} \subseteq X(\mathbb{A}_K)^\psi$ holds. I.e., Proposition 4.1.3 is true.*

Proof. Just use Lemma 4.3.4. □

The following proof is due to Olivier Wittenberg, which kindly suggested a proof for Proposition 4.5.1 that is much simpler than the previous one I had.

Proof of Proposition 4.5.1 (Olivier Wittenberg). Recall that $U \subseteq X$ is an open subscheme such that $V = \psi^{-1}(U) \to U$ is a(n étale) G-torsor. Let $P \in X(K)$ be a rational point, and let $C \subseteq X$ be an integral (closed) curve such that $P \in C(K)$, P is a smooth point of C, and $C \cap U \neq \emptyset$. Note that such a curve always exists. In fact, it suffices to take uniformizing parameters $u_1, \ldots, u_d$ for X at P (where $d = \dim X$), that satisfy the condition that the subvariety $\{u_1 = 0, \ldots, u_{d-1} = 0\}$ (defined in a small Zariski-open neighbourhood U_P of P in X) is not contained in $X \setminus U$ (this condition may always be attained by taking a sufficiently general K-linear invertible transformation of the parameters $u_1, \ldots, u_d$), and then define C to be the closure in X of the subvariety $\{u_1 = 0, \ldots, u_{d-1} = 0\} \subseteq U_P$ (note that we may assume that the latter is a smooth curve, after possibly restricting U_P).

Choosing a local parameter t for C at P, we get a morphism $\operatorname{Spec} K[[t]] \to C$ that specializes to P (in the sense that the morphism sends the special point of $\operatorname{Spec} K[[t]]$ to P). This morphism induces a morphism $\operatorname{Spec} K((t)) \to X$, whose set-theoretic image is the generic point of C. In particular, by construction of C, it belongs to U. Hence the G-torsor $V \to U$ gives a class in $H^1(\Gamma_{K((t))}, G)$, which we may push to $H^1(K((t^{\frac{1}{\infty}})), G)$.

The inclusion $K \subseteq K((t^{\frac{1}{\infty}}))$ induces an identification $\Gamma_{K((t^{\frac{1}{\infty}}))} = \Gamma_K$ (this follows from the algebraic-closedness of $\overline{K}((t^{\frac{1}{\infty}}))$ [Ser79, Proposition 8, Chapter IV]), and hence an identification $H^1(K((t^{\frac{1}{\infty}})), G) = H^1(K, G)$. Hence, after replacing Y with a twist, we may assume that the class in $H^1(K((t^{\frac{1}{\infty}})), G)$ is trivial. Therefore it has to be trivial already in $H^1(K((t^{\frac{1}{n}})), G)$ for some $n \geq 1$. Translated, this means that the G-torsor $\operatorname{Spec} K((t^{\frac{1}{n}})) \times_U V \to \operatorname{Spec} K((t^{\frac{1}{n}}))$ has a section. This section induces a commutative diagram as follows:

$$\begin{array}{ccc} & \nearrow & V \\ & & \downarrow \\ \operatorname{Spec} K((t^{\frac{1}{n}})) & \longrightarrow & U. \end{array}$$

By the valuative criterion of properness (applied to $Y^{\mathrm{sm}} \to X$), we may extend the diagram above to the following:

$$\begin{array}{ccc} & \nearrow & Y^{\mathrm{sm}} \\ & & \downarrow \\ \operatorname{Spec} K[[t^{\frac{1}{n}}]] & \longrightarrow & X. \end{array}$$

Since the lower morphism specializes to P, the specialization of the oblique morphism provides the sought lift of P.

□

4.6 An example where $Br_\psi X$ is purely transcendental

In this section we prove the following theorem:

Theorem 4.6.1. *There exists a smooth geometrically connected proper variety X, a finite commutative group scheme A/K, an open subscheme $U \subseteq X$, and an A-cover $\psi : Y \to X$ such that*

$$X(\mathbb{A}_K)^{\mathrm{Br}_\psi X} \neq X(\mathbb{A}_K), \tag{4.6.1}$$

and $\mathrm{Br}_1 X = \mathrm{Br}_0 X$. In particular, $\mathrm{Br}_\psi X$ is purely transcendental.

Note that an immediate consequence of the theorem is that, for the X in the theorem:

$$X(\mathbb{A}_K)^{\mathrm{Br}_\psi X} \subsetneq X(\mathbb{A}_K)^{\mathrm{Br}^a_\psi(X)}, \tag{4.6.2}$$

where we recall that, denoting by $\varphi : V \to U$ an open subcover of ψ that is an étale torsor, $\operatorname{Br}_\psi X = \operatorname{Br}_\varphi U \cap \operatorname{Br} X$, and $\operatorname{Br}^a_\psi(X) = \operatorname{Br} X \cap \operatorname{Im}(H^1(K, A') \to H^1(U, A') \xrightarrow{-\cup[\varphi]} H^2(U, \mathbb{G}_m) = \operatorname{Br} U)$. Indeed, the inclusion always holds, as proven in the last section, but, since, $\operatorname{Br}^a_\psi(X) \subseteq \operatorname{Br}_\psi(X) \cap \operatorname{Br}_1(X)$ (this was proved in the last section as well), and, in this case, $\operatorname{Br}_\psi(X) \cap \operatorname{Br}_1(X) \subseteq \operatorname{Br}_0 X$, there cannot be equality by the theorem above.

First part of the proof of Theorem 4.6.1: desired properties. In this short part of the proof, we list some desidered properties on X and ψ that would immediately imply the theorem. The rest of the section will be dedicated to proving that there actually exists an X that satisfies said properties. Properties:

i. $\operatorname{Br}_1 X = \operatorname{Br}_0 X$

ii. $X(\mathbb{A}_K) \neq \emptyset$;

iii. There exists a $v \in M_K$ and a $b \in \operatorname{Br}_\psi X$ such that the function $X(K_v) \to \mathbb{Q}/\mathbb{Z}, P \mapsto \operatorname{inv}_v(b(P))$ is non-constant.

Let us prove that any X that satisfies the above properties does indeed satisfy (4.6.2). Property i implies that $\operatorname{Br}_\psi(X)$ is purely transcendental. On the other hand, since $X(\mathbb{A}_K)$ is non-empty (property ii), property iii guarantees that $X(\mathbb{A}_K)^b \subsetneq X(\mathbb{A}_K)$, and hence that $X(\mathbb{A}_K)^{\operatorname{Br}_\psi X} \subsetneq X(\mathbb{A}_K)$, which concludes (this part of) the proof. □

Note that, as a corollary, we prove Theorem 4.1.1:

Proof of Theorem 4.1.1. The first part follows from Theorem 4.4.7. The second part follows from Theorem 4.6.1. □

A morphism $\operatorname{Hom}(\Lambda^2 A, B) \to H^2(A, B)$ Let A and B be finite commutative groups such that $\#B$ is odd.

We define a morphism:

$$\begin{array}{rcl} \iota : \operatorname{Hom}(\Lambda^2 A, B) & \to & H^2(A, B) \\ \beta & \mapsto & [(a, a') \mapsto \frac{1}{2}\beta(a' \wedge a)], \end{array}$$

where the A-action on B is trivial, and $[\star]$ denotes the element in $H^2(A, B)$ represented by the cocycle $\star \in Z^2(A, B)$ (we leave to the reader the easy verification that the one above is indeed a cocycle).

Remark 4.6.2. The definition of the morphism ι above can be easily made without appealing to cocycles when B is isomorphic to $\mathbb{F}_p$ (p is an odd prime), and A is of exponent p (note that, when $B = \mathbb{F}_p$ this is the only relevant case anyway, as $\operatorname{Hom}(\Lambda^2 A, B) = \operatorname{Hom}(\Lambda^2(A/pA), B))$. Infact, in this case, an easy cocycle computation shows that the map ι above coincides with the cup product:

$$\Lambda^2 H^1(A, \mathbb{F}_p) \to H^2(A, \mathbb{F}_p),$$

under the identification $\Lambda^2 H^1(A, \mathbb{F}_p) = \Lambda^2 \operatorname{Hom}(A, \mathbb{F}_p) = \Lambda^2 A^D = (\Lambda^2 A)^D = \operatorname{Hom}(\Lambda^2 A, \mathbb{F}_p)$. In fact, this is exactly what motivated us to choose the normalization of ι. The case of a general B is just a natural generalization.

Remark 4.6.3. Let us remark that ι defines a section of the morphism ω_C defined in [CTS07, p. 35] (with $\Gamma = A$ and $C = B$). In fact, combining the explicit description of ω_C in *loc. cit.* with [Bro94, Thm. IV.3.12] (keeping in mind equation (3.3) in [Bro94, Ch. IV]), this becomes an easy verification that we leave to the interested reader. In particular, this shows that ι is injective.

Description of our setting In this paragraph we define a number field K, a smooth geometrically connected proper variety X/K, a constant finite commutative K-group scheme A, and an A-torsor $\varphi : V \to U \subseteq X$, and we will show in the next paragraphs that the pair (X, φ) does satisfy properties i-iii.

Let K be a number field and $p \geq 5$ be a prime number such that $\mu_p \subseteq K$. We let A and B be finite (abstract) abelian groups of exponent p, and r_1 and r_2 be their rank, i.e. $A \cong (\mathbb{Z}/p\mathbb{Z})^{r_1}$ and $B \cong (\mathbb{Z}/p\mathbb{Z})^{r_2}$. We let β be an element of $\mathrm{Hom}(\Lambda^2 A, B)$, and $\overline{\beta} = \iota(\beta) \in H^2(A, B)$. We let G be a central extension of class $\overline{\beta}$, so that we have the short exact sequence:

$$1 \to B \to G \to A \to 1.$$

We assume that the injection $B \hookrightarrow G$ is an inclusion. We denote the projection $G \to A$ with π. One way to describe the extension G is the following (see [Bro94, Section IV.3]):

$$\text{as a set } G = B \times A, \text{ and the multiplication law is } (b_1, a_1) \cdot (b_2, a_2) = \left(b_1 + b_2 + \frac{1}{2}\beta(a_2 \wedge a_1), a_1 + a_2 \right).$$

We assume that β is surjective. Note that this implies that $G^{ab} = A$.

Throughout the proof we will gradually make more assumptions on A, B and β, but we prefer to postpone these to the points in the proof where they are actually used.

We identify, with a slight abuse of notation, the abstract groups A, B and G with the constant groups A_K, B_K and G_K.

We choose an embedding of G in $SL_{n,K}$ and we define U as $SL_{n,K}/G$, V as $SL_{n,K}/B$ and φ as the natural projection $SL_{n,K}/B \to SL_{n,K}/G$. Note that, as B is normal in G with quotient A, φ has a natural structure of A-torsor: the one defined by the A-action (in S-point notation) $SL_{n,K}/B \times_K A \to SL_{n,K}/B,\ (xB, a) \mapsto xaB$ for all $x \in SL_{n,K}(S)$ and $a \in A(S)$.

We let X be a smooth compactification of the K-variety U.

A morphism $\Lambda^2 A^D \to \mathrm{Br}_\varphi U$ We denote, as usual, by Γ_A the group $\Gamma_{A_K} \stackrel{\text{def}}{=} A(\overline{K}) \rtimes \Gamma_K = A \times \Gamma_K$ (recall that A is constant).

We remind the reader that $\mathrm{Br}_\varphi U$ is defined as the image of the composition

$$u_\varphi : H^2(\Gamma_A, \overline{K}^*) \to H^2(\Gamma_A, \overline{K}^*[V]) \xrightarrow{\epsilon} H^2(U, \mathbb{G}_m) = \mathrm{Br}\, U,$$

where the second morphism is defined through the Hochshild-Serre spectral sequence applied to the Γ_A-cover $\overline{V} \to U$ as in (4.4.2) and the second one is the morphism $j_U : H^2(U, p_U^* \mathbb{G}_m) \to \mathrm{Br}\, U$. Let us also recall that the restriction of the Γ_A-action on $\overline{K}^*$ to A is trivial.

Note that we have a morphism:

$$c : \Lambda^2 A^D = \Lambda^2 \mathrm{Hom}(A, \mathbb{Z}/p\mathbb{Z}) \cong \Lambda^2 H^1(A, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{-\cup-}$$

$$H^2(A, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\chi} H^2(A, \mu_p) \xrightarrow{\inf^A_{\Gamma_A}} H^2(\Gamma_A, \mu_p) \to H^2(\Gamma_A, \overline{K}^*), \quad (4.6.3)$$

where χ is a fixed isomorphism $\mathbb{Z}/p\mathbb{Z} \cong \mu_p$.

The morphism $\Lambda^2 A^D \to \mathrm{Br}_\varphi U$ that we referred to in the title of this paragraph is just the composition

$$\Lambda^2 A^D \xrightarrow{c} H^2(\Gamma_A, \overline{K}^*) \xrightarrow{u_\varphi} \mathrm{Br}\, U.$$

Description of the values of $\mathrm{Im}\, \Lambda^2 A^D \subseteq \mathrm{Br}\, U$ at local points Let $v \in M_K^{\text{fin}}$ and p be an odd prime. We have an antisymmetric bilinear pairing on $(\Gamma_{K_v}^{ab}/(\Gamma_{K_v}^{ab})^p)^D$, defined through the following composition:

$$\Lambda^2 \left(\Gamma_{K_v}^{ab}/(\Gamma_{K_v}^{ab})^p\right)^D \cong \Lambda^2 H^1(\Gamma_{K_v}, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\cup} H^2(\Gamma_{K_v}, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\chi} H^2(\Gamma_{K_v}, \mu_p) \xrightarrow{\mathrm{inv}_v} \frac{1}{p}\mathbb{Z}/\mathbb{Z},$$

which is perfect by [NSW08, Thm. 7.2.9]. We denote the above composition by B_v.

We define a pairing (linear only on the left):

$$W_\wedge : \Lambda^2 A^D \times \mathrm{Hom}(\Gamma_{K_v}^{\mathrm{ab}}, A) \to \frac{1}{p}\mathbb{Z}/\mathbb{Z},\ (\beta,\xi) \mapsto B_v(\xi^*\beta).$$

Lemma 4.6.4. *We have the following commutative diagram:*

$$\begin{array}{ccccc} \Lambda^2 A^D & \times & H^1(K_v,A) & \xrightarrow{W_\wedge} & \mathbb{Q}/\mathbb{Z} \\ \big\downarrow{\scriptstyle u_\varphi \circ c} & & {\scriptstyle [\varphi|-]}\big\uparrow & & {\scriptstyle =}\big\uparrow \\ \mathrm{Br}_\varphi(U) & \times & U(K_v) & \longrightarrow & \mathbb{Q}/\mathbb{Z} \end{array},$$

where the first row is, after the identification $H^1(K_v,A) = H^1(\Gamma_{K_v},A) = \mathrm{Hom}(\Gamma_{K_v}^{ab},A)$, the pairing $W_\wedge$, and the second one is $(B,P) \mapsto \mathrm{inv}_v(B(P))$ (i.e. the usual v-adic component of the Brauer–Manin pairing).

Proof. Let ξ be an element of $H^1(K_v,A)$, which we think also of an element of $\mathrm{Hom}(\Gamma_{K_v},A)$ and $\mathrm{Hom}(\Gamma_{K_v}^{\mathrm{ab}},A)$ through the identifications $H^1(K_v,A) = \mathrm{Hom}(\Gamma_{K_v},A) = \mathrm{Hom}(\Gamma_{K_v}^{\mathrm{ab}},A)$ (these groups are isomorphic because A is constant).

Let $g = \Gamma_{K_v}$. We set, $\Gamma_A^v := A \times g$, and we remind the reader that to the element $\xi \in H^1(K_v,A)$ we may associate a section of $\Gamma_A^v \to g$, which is unique up to conjugation by an element of A; since A is central in $A \times g$, such conjugation is always the identity. So this section is uniquely defined, and we denote it by $\aleph(\xi)$ (in accordance with the notation in (4.4.5)).

Let $\aleph_v(\xi) : g \to \Gamma_A$ be the composition of $g \xrightarrow{\aleph(\xi)} \Gamma_A^v$ with the the injection $\Gamma_A^v = A \times g \hookrightarrow A \times \Gamma_K = \Gamma_A$.

We have the following commutative diagram:

$$c: \quad \begin{array}{ccccccccc} \Lambda^2 A^D & \xrightarrow{\cong} & \Lambda^2 H^1(A,\mathbb{Z}/p\mathbb{Z}) & \xrightarrow{\cup} & H^2(A,\mathbb{Z}/p\mathbb{Z}) & \xrightarrow{\chi} & H^2(A,K^*) & \xrightarrow{\inf_{\Gamma_A}^A} & H^2(\Gamma_A,\overline{K}^*) \\ \big\downarrow{\scriptstyle \xi^*} & & \big\downarrow{\scriptstyle \xi^*} & & \big\downarrow{\scriptstyle \xi^*} & & \big\downarrow{\scriptstyle \xi^*} & & \big\downarrow{\scriptstyle \aleph_v(\xi)^*} \\ \Lambda^2((g^{\mathrm{ab}})/(g^{\mathrm{ab}})^p)^D & \xrightarrow{\cong} & \Lambda^2 H^1(g,\mathbb{Z}/p\mathbb{Z}) & \xrightarrow{\cup} & H^2(g,\mathbb{Z}/p\mathbb{Z}) & \xrightarrow{\chi} & H^2(g,K^*) & \longrightarrow & H^2(g,\overline{K_v}^*) \end{array}.$$

The commutativity is obvious except for the last square, for which it suffices to notice that, if we denote by π the projection $\Gamma_A \to A$, then the morphism $\inf_{\Gamma_A}^A$ is exactly π^*. Since $\pi \circ \chi_v(\xi) = \xi$, the last square commutes.

Now the commutativity of the external part of the above diagram for every $\xi \in H^1(K_v,A)$ implies the commutativity of the following diagram:

$$\begin{array}{ccccccccc} W_\wedge: & \Lambda^2 A^D & \times & H^1(K_v,A) & \longrightarrow & H^2(g,\overline{K_v}^*) & \xrightarrow[\sim]{\mathrm{inv}_v} & \mathbb{Q}/\mathbb{Z} \\ & \big\downarrow{\scriptstyle c} & & {\scriptstyle =}\big\uparrow & & {\scriptstyle =}\big\uparrow & & {\scriptstyle =}\big\uparrow \\ (.): & H^2(\Gamma_A,\overline{K}^*) & \times & H^1(K_v,A) & \longrightarrow & H^2(g,\overline{K_v}^*) & \xrightarrow[\sim]{\mathrm{inv}_v} & \mathbb{Q}/\mathbb{Z} \end{array},$$

where $(.)$ is the pairing $(\alpha,\xi') \mapsto (\aleph^{-1}(\xi'))^*\alpha$ defined in Lemma 4.4.8. Combining the above diagram with Lemma 4.4.10 with $F = K_v$, we obtain the commutativity of the diagram appearing in the statement of the lemma. ☐

Description of E_v in our setting We recall that $E_v \subseteq \mathrm{Hom}(\Gamma_{K_v},A)$ is defined as the image of $U(K_v) \xrightarrow{[\varphi|-]} H^1(K_v,A)$.

Lemma 4.6.5. *We have that:*

$$E_v = \{\xi \in \mathrm{Hom}(\Gamma_{K_v}, A) \mid \text{ the composition } \frac{1}{p}\mathbb{Z}/\mathbb{Z} \xrightarrow{B_v^D} \Lambda^2(\Gamma_{K_v}^{ab}/(\Gamma_{K_v}^{ab})^p) \xrightarrow{\Lambda^2\xi} \Lambda^2 A \xrightarrow{\beta} B \text{ is } 0\}.$$

Moreover, for all v, we have that $< E_v >= \mathrm{Hom}(\Gamma_{K_v}, A)$.

Proof. Remember that $U = SL_{n,K}/G$, and that $V = SL_{n,K}/B$. We denote by φ' the G-torsor $SL_{n,K} \to SL_{n,K}/G$. Note that φ is the contracted product $(SL_{n,K}) \times^G A$, so we have the commutative diagram:

$$(SL_{n,K}/G)(F) \xrightarrow{[\varphi'|-]} H^1(F, G) \longrightarrow H^1(F, A)\ , \qquad [\varphi|-] : (SL_{n,K}/G)(F) \to H^1(F,A).$$

Recall that we have, for each field F containing K, an exact sequence of pointed sets:

$$1 \to G(F) \to SL_{n,K}(F) \to (SL_{n,K}/G)(F) \xrightarrow{[\varphi'|-]} H^1(F, G) \to H^1(F, SL_{n,K}),$$

hence the map $(SL_{n,K}/G)(F) \xrightarrow{[\varphi'|-]} H^1(F, G)$ is surjective (indeed, we have that $H^1(F, SL_{n,K}) = 0$ by [PRR93, Lemma 2.3]). It follows that, for each $v \in M_K$, $E_v = E_v(\varphi) = \mathrm{Im}(H^1(K_v, G) \to H^1(K_v, A))$. Therefore, since there is an exact sequence of pointed sets:

$$\mathrm{Hom}(\Gamma_{K_v}, G) \to \mathrm{Hom}(\Gamma_{K_v}, A) \xrightarrow{\xi \mapsto (\xi^* \overline{\beta})} H^2(\Gamma_{K_v}, B),$$

(see e.g. [Bro94, Thm. 3.12], keeping in mind that the correspondence in *loc.cit.* is functorial), we infer that the following sequence of pointed sets is exact:

$$1 \to E_v \to H^1(K_v, A) = \mathrm{Hom}(\Gamma_{K_v}, A) \xrightarrow{\xi \mapsto (\xi^* \overline{\beta})} H^2(\Gamma_{K_v}, B).$$

Let us now fix a decomposition $B \cong \mathbb{F}_p^{r_2}$, and let us denote by $\pi_i : B \to \mathbb{F}_p, i = 1, \ldots, r_2$ the projections to the different factors. Let us write $\beta = \sum_i \pi_i^* \beta_i$, with $\beta_i \in \mathrm{Hom}(\Lambda^2 A, \mathbb{F}_p) = \Lambda^2 A^D$. We denote by $\overline{\beta}_i$ the image of β_i under $\iota : \mathrm{Hom}(\Lambda^2 A, \mathbb{F}_p) \to H^2(A, \mathbb{F}_p)$. Note that $\overline{\beta} = \sum_i \pi_i^* \overline{\beta_i}$. Hence we have the following commutative diagram:

$$\begin{array}{ccccc}
H^2(A, \mathbb{F}_p) & \xrightarrow{\xi^*} & H^2\left(\Gamma_{K_v}^{\mathrm{ab}}, \mathbb{F}_p\right) & \xrightarrow{\inf} & H^2\left(\Gamma_{K_v}, \mathbb{F}_p\right) \\
\cup\uparrow & & \cup\uparrow & & \sim\downarrow \mathrm{inv}_v \circ \chi \\
\Lambda^2 A^D \cong \Lambda^2 H^1(A, \mathbb{F}_p) & \xrightarrow{\Lambda^2(\xi')^D} & \Lambda^2\left(\Gamma_{K_v}^{ab}/(\Gamma_{K_v}^{ab})^p\right)^D \cong \Lambda^2 H^1(\Gamma_{K_v}, \mathbb{Z}/p\mathbb{Z}) & \xrightarrow{B_v} & \frac{1}{p}\mathbb{Z}/\mathbb{Z}
\end{array} \quad , \qquad \xi^* : H^2(A,\mathbb{F}_p) \to H^2(\Gamma_{K_v},\mathbb{F}_p) \tag{4.6.4}$$

where for the first morphism in the first row we are actually taking the pullback along the corresponding to ξ under the identification $\mathrm{Hom}(\Gamma_{K_v}, A) = \mathrm{Hom}(\Gamma_{K_v}^{ab}, A)$. For the first morphism in the second row, ξ' denotes the element corresponding to ξ under the identification $\mathrm{Hom}(\Gamma_{K_v}^{ab}, A) = \mathrm{Hom}(\Gamma_{K_v}^{ab}/(\Gamma_{K_v}^{ab})^p, A)$.

Note that the morphism $\cup : \Lambda^2 A^D \to H^2(A, \mathbb{F}_p)$ is equal to ι, as noticed in Remark 4.6.2.

From (4.6.4), keeping in mind that ι is injective (see Remark 4.6.3) we deduce that $\xi^* \overline{\beta_i} = 0 \in H^2(\Gamma_{K_v}, \mathbb{F}_p)$ if and only if the composition:

$$\frac{1}{p}\mathbb{Z}/\mathbb{Z} \xrightarrow{B_v^D} \Lambda^2(\Gamma_{K_v}^{ab}/(\Gamma_{K_v}^{ab})^p) \xrightarrow{\Lambda^2\xi} \Lambda^2 A \xrightarrow{\beta_i} \mathbb{F}_p$$

is 0. Since this equivalence holds for any $i = 1, \dots, r$, the first part of the lemma follows.

To prove that, for all v, $< E_v >= \mathrm{Hom}(\Gamma_{K_v}, A)$, note first of all that, by the first part of the lemma, for every cyclic subgroup $C \subseteq A$, we have that the subset $\mathrm{Hom}(\Gamma_{K_v}, C) \subseteq \mathrm{Hom}(\Gamma_{K_v}, A)$ is contained in E_v. Indeed, for $\xi \in \mathrm{Hom}(\Gamma_{K_v}, C)$ we have that the image of $\Lambda^2(\Gamma^{ab}_{K_v}/(\Gamma^{ab}_{K_v})^p) \xrightarrow{\Lambda^2\xi} \Lambda^2 A$ is contained in $\Lambda^2 C$, which is 0 because C is cyclic. Writing a direct sum decomposition into cyclic subgroups $A = C_1 \oplus \dots \oplus C_r$, and noticing that $\mathrm{Hom}(\Gamma_{K_v}, C_1) \oplus \dots \oplus \mathrm{Hom}(\Gamma_{K_v}, C_r) = \mathrm{Hom}(\Gamma_{K_v}, A)$, and that all of these summands are contained in E_v, we see that $< E_v >= \mathrm{Hom}(\Gamma_{K_v}, A)$, as wished. □

Description of $\mathrm{Br}_\psi X$ in our setting

Definition 4.6.6. We define $\mathfrak{Bic}(G, A) \subseteq \Lambda^2 A$ as:

$$\mathfrak{Bic}(G, A) = \{\pi(g_1) \wedge \pi(g_2), g_1 \text{ and } g_2 \text{ commute}\},$$

where we recall that G is a central extension of A by B, and π denotes the projection $G \to A$.

Warning. Note that $\mathfrak{Bic}(G, A) \subseteq \Lambda^2 A$ is only a subset (made, by definition, only of pure wedges), and not a subgroup in general.

Remark 4.6.7. Since ι is injective and functorial, we have, by [Bro94, Thm. IV.3.12] that:

$$\mathfrak{Bic}(G, A) = \{a_1 \wedge a_2, \beta(a_1 \wedge a_2) = 0\}.$$

Lemma 4.6.8. *Let $b \in \Lambda^2 A^D$. Let $c' = u_\varphi \circ c : \Lambda^2 A^D \to \mathrm{Br}\, U$. We have that $c'(b) \in \mathrm{Br}\, U$ is unramified if and only if $b(a_1 \wedge a_2) = 0$ for all $a_1 \wedge a_2 \in \mathfrak{Bic}(G, A)$.*

Proof. Using Lemma 4.6.4 and [Har94, Thm. 2.1.1], we see that $c'(b)$ is unramified if and only if, for almost all v, $W_\wedge(b, \star)$ is constant for $\star \in E_v$. Since $0 \in E_v$ for almost all v (by Lemma 4.3.13.ii) and $W_\wedge(b, 0) = 0$, we see that $W_\wedge(b, \star)$ is constant (for $\star \in E_v$) if and only if $W_\wedge(b, E_v) = 0$.

So, to prove the lemma, it suffices to prove that this last condition is equivalent to $b(a_1 \wedge a_2) = 0$ for all wedges $a_1 \wedge a_2 \in \mathfrak{Bic}(G, A)$. To do so, we are now going to study more closely the pairing $W_\wedge$ when $v \nmid p$.

We fix a non-archimedean $v \nmid p$. We have that $\Gamma^{\mathrm{ab}}_{K_v}/(\Gamma^{\mathrm{ab}}_{K_v})^p = \Gamma^{\mathrm{ab}}_{K_v,tame}/(\Gamma^{\mathrm{ab}}_{K_v,tame})^p \cong (\mathbb{Z}/p\mathbb{Z})^2$ (see [Iwa55, Thm. 2]), where $\Gamma_{K_v,tame}$ is the tame Galois group of K_v, i.e. the Galois group of the maximal tame extension of K_v, and $\Gamma^{\mathrm{ab}}_{K_v,tame}$ is its maximal abelian quotient. It follows that $\mathbb{Z}/p\mathbb{Z} \cong \Lambda^2\Gamma^{\mathrm{ab}}_{K_v}/(\Gamma^{\mathrm{ab}}_{K_v})^p$. Hence $B_v^D : (1/p)\mathbb{Z}/\mathbb{Z} \to \Lambda^2\Gamma^{\mathrm{ab}}_{K_v}/(\Gamma^{\mathrm{ab}}_{K_v})^p$, being injective (as follows from the fact that the pairing B_v is perfect), and an injective morphism between two $\mathbb{F}_p$-vector spaces of dimension 1, is an isomorphism.

Since A is abelian and of exponent p, we have an isomorphism $\mathrm{Hom}(\Gamma_{K_v}, A) \xrightarrow{\sim} \mathrm{Hom}(\Gamma^{\mathrm{ab}}_{K_v}/(\Gamma^{\mathrm{ab}}_{K_v})^p, A)$. We let γ_1, γ_2 be a basis for the two-dimensional $\mathbb{F}_p$-vector space $\Gamma^{\mathrm{ab}}_{K_v}/(\Gamma^{\mathrm{ab}}_{K_v})^p$. We define an isomorphism $\Xi_v : \mathrm{Hom}(\Gamma_{K_v}, A) \to A^2$, as the composition

$$\Xi_v : \mathrm{Hom}(\Gamma_{K_v}, A) \xrightarrow{\sim} \mathrm{Hom}(\Gamma^{\mathrm{ab}}_{K_v}/(\Gamma^{\mathrm{ab}}_{K_v})^p, A) \xrightarrow{m \mapsto (m(\gamma_1), m(\gamma_2))} A^2.$$

Since B_v^D is an isomorphism, Lemma 4.6.5 implies that:

$$E_v = \{\xi \in \mathrm{Hom}(\Gamma_{K_v}, A) \mid \text{ the composition } \Lambda^2(\Gamma^{ab}_{K_v}/(\Gamma^{ab}_{K_v})^p) \xrightarrow{\Lambda^2\xi} \Lambda^2 A \xrightarrow{\beta} B \text{ is } 0\}.$$

Hence:

$$\Xi_v(E_v) = \{(a_1, a_2) \in A^2 \mid a_1 \wedge a_2 \in \mathrm{Ker}(\beta : \Lambda^2 A \to B)\}. \tag{4.6.5}$$

We denote by ζ the isomorphism $\mathbb{Z}/p\mathbb{Z} \cong \Lambda^2\Gamma^{\text{ab}}_{K_v}/(\Gamma^{\text{ab}}_{K_v})^p$ given by $1 \mapsto a_1 \wedge a_2$. Consider now the following commutative diagram, where, to spare space, we use the notations $g := \Gamma_{K_v}$ and $g/p := \Gamma^{ab}_{K_v}/(\Gamma^{ab}_{K_v})^p$:

$$\begin{array}{lccccccc}
W_\wedge : & \Lambda^2 A^D & \times & \operatorname{Hom}(g,A) = \operatorname{Hom}(g/p,A) & \longrightarrow & \Lambda^2(g/p)^D & \xrightarrow[\sim]{B_v} & \frac{1}{p}\mathbb{Z}/\mathbb{Z} \\
& {\scriptstyle =}\big\uparrow & & \sim\big\downarrow{\scriptstyle \Xi_v} & & \sim\big\downarrow{\scriptstyle \zeta^D} & \swarrow{\scriptstyle \zeta^D\circ B_v^{-1}}\ \sim & , \\
& \Lambda^2 A^D & \times & A^2 & \longrightarrow & \mathbb{Z}/p\mathbb{Z} & &
\end{array}$$

where the last row is defined as $(f', (a_1, a_2)) \mapsto f'(a_1 \wedge a_2)$. For the commutativity, note that, if $\lambda_1 \wedge \lambda_2 \in \Lambda^2 A^D$, $\xi \in \operatorname{Hom}(g, A)$, and $(a_1, a_2) := \Xi_v(\xi) = (\xi(\gamma_1), \xi(\gamma_2)) \in A^2$, then we have that the pairing on the first row yields $\xi^*\lambda_1 \wedge \xi^*\lambda_2 \in \Lambda^2(g/p)^D$, whose image under ζ^D is $(\xi^*\lambda_1 \wedge \xi^*\lambda_2)(\gamma_1 \wedge \gamma_2)$, while the pairing on the second row yields

$$(\lambda_1 \wedge \lambda_2)(a_1 \wedge a_2) = (\lambda_1 \wedge \lambda_2)(\xi(\gamma_1) \wedge \xi(\gamma_2)) = (\xi^*\lambda_1 \wedge \xi^*\lambda_2)(\gamma_1 \wedge \gamma_2).$$

We saw at the beginning of the proof that $c'(b)$ is unramified if and only if $W_\wedge(b, E_v) = 0$ for almost all v. Since $v \mid p$ for only finitely many v, combining the above commutative diagram with (4.6.5) and Remark 4.6.7, we see that the condition $W_\wedge(b, E_v) = 0$ is equivalent to $b(a_1 \wedge a_2) = 0$ for all $a_1 \wedge a_2 \in \mathfrak{Bic}(G, A)$, as wished. □

Lemma 4.A.1 gives a family of examples of A, B and $\beta : \Lambda^2 A \to B$ such that, if $1 \to B \to G \to A \to 1$ is the extension corresponding to $\iota(\beta)$, then $\mathfrak{Bic}(G, A) = 0$. By Lemma 4.6.8 this implies that, for such a G, $\operatorname{Im} c' \subseteq \operatorname{Br}_{nr} U = \operatorname{Br} X$.

For all these examples the following assumption is satisfied.

Assumption ("Unramified" assumption)**.** $\operatorname{Im} c' \subseteq \operatorname{Br}_{nr} U$.

We say that the Brauer pairing of $\mathcal{B} \subseteq \operatorname{Br} X$ is *non-constant* on $X(K_v)$ (for some $v \in M_K$) if there exists a $b \in \mathcal{B}$ such that the local pairing $(b.-)_v : X(K_v) \to \operatorname{Br}(K_v)$ is non-constant.

Proposition 4.6.9. *Let v be a place dividing p, and let $r := \dim_{\mathbb{F}_p} \left(\Gamma^{ab}_{K_v}/(\Gamma^{ab}_{K_v})^p\right)$. We assume that $r \geq 5$ (since $\mu_p \subseteq K \subseteq K_v$, [NSW08, Thm. 7.5.11(ii)] guarantees that this holds if $p \geq 5$). For a central extension G as described in the setting paragraph, let $r_1 := \dim_{\mathbb{F}_p} A$, $r_2 := \dim_{\mathbb{F}_p} B$. There exists a function $D : \mathbb{N} \to \mathbb{N}$ such that, for G that satisfies the "unramified" assumption, the inequality $r_1 > D(r)$ and the inequality $r_2 \leq 2r_1 - 3$, there exists an element $b \in \Lambda^2 A^D$ such that the map*

$$(SL_n/G)(K_v) \to \frac{1}{p}\mathbb{Z}/\mathbb{Z}, \ \ P \mapsto \operatorname{inv}_v c'(b)(P)$$

(i.e. the v-adic component of the Brauer–Manin pairing computed on the Brauer element $c(b)$) is non-constant.

Before the proof, let us pause to say that the assumptions $p \geq 5, r_1 > D(r)$, $r_2 \leq 2r_1 - 3$ are satisfied by infinitely many examples such as in Lemma 4.A.1. Indeed, Lemma 4.A.1 gives examples, **for every** odd prime p and **every** (finite) $\mathbb{F}_p$-vector space A of dimension ≥ 4, of extensions $1 \to A \to G \to B \to 1$ that satisfy the "unramified" assumption, and all these examples satisfy $r_2 = 2r_1 - 3$.

Proof. Let $v \in M_K$ be a place dividing p. We define $C_v := \{\xi \in \operatorname{Hom}(\Gamma_{K_v}, A) \mid W_\wedge(-, \xi) : \Lambda^2 A^D \to \frac{1}{p}\mathbb{Z}/\mathbb{Z} \text{ is zero}\}$. We divide the proof in several steps (represented by the framed boxes below, each box contains the statement that we will prove in the step that follows it). The purpose of the first three steps is proving that $E_v \neq C_v$ (the first two steps are auxiliary to the third). We then show, in the fourth, how this concludes proof of the proposition.

$\boxed{\#E_v \text{ has } p\text{-adic valuation} \geq \lceil \frac{r_1 r - 2r_2}{2} \rceil \text{ and } < \infty.}$

We choose isomorphisms $\Gamma_{K_v}^{ab}/(\Gamma_{K_v}^{ab})^p \cong \mathbb{F}_p^r$, $A \cong \mathbb{F}_p^{r_1}$ and $B \cong \mathbb{F}_p^{r_2}$ (we want to think about these p-torsion abelian groups as $\mathbb{F}_p$-vector spaces, that is why the notation $\mathbb{F}_p$ is being used). Note that, using these isomorphisms, we may identify $\mathrm{Hom}(\Gamma_{K_v}, A) = \mathrm{Hom}(\Gamma_{K_v}^{ab}/(\Gamma_{K_v}^{ab})^p, A)$ with $r_1 \times r$-matrices with coefficients in $\mathbb{F}_p$.

Moreover, using the isomorphism $\Gamma_{K_v}^{ab}/(\Gamma_{K_v}^{ab})^p \cong \mathbb{F}_p^r$, we may identify the morphism

$$\mathbb{F}_p \cong \frac{1}{p}\mathbb{Z}/\mathbb{Z} \xrightarrow{B_v^D} \Lambda^2(\Gamma_{K_v}^{ab}/(\Gamma_{K_v}^{ab})^p) \cong \Lambda^2 \mathbb{F}_p^r$$

with an antisymmetric $r \times r$ matrix, which we denote by M_P. Finally, we may identify $\beta : \Lambda^2 A \to B$ with a morphism $\Lambda^2 \mathbb{F}_p^{r_1} \to \mathbb{F}_p^{r_2}$. Identifying $\Lambda^2 \mathbb{F}_p^{r_1}$ with antisymmetric $r_1 \times r_1$-matrices (with coefficients in $\mathbb{F}_p$), this last morphism gives rise to a morphism β' from the vector space of $(r_1 \times r_1)$-$\mathbb{F}_p$ matrices to $\mathbb{F}_p^{r_2}$.

Let ξ be an element of $\mathrm{Hom}(\Gamma_{K_v}, A)$, and M_ξ the corresponding $r_1 \times r$-matrix.

The condition (equivalent, by Lemma 4.6.5, to $\xi \in E_v$)

$$\text{“ the composition } \mathbb{F}_p \cong \frac{1}{p}\mathbb{Z}/\mathbb{Z} \xrightarrow{B_v^D} \Lambda^2(\Gamma_{K_v}^{ab}/(\Gamma_{K_v}^{ab})^p) \xrightarrow{\Lambda^2\xi} \Lambda^2 A \xrightarrow{\beta} B \text{ is } 0''$$

is easily seen to be equivalent to:

$$\beta'(M_\xi M_P M_\xi^t) = 0. \tag{4.6.6}$$

In particular, if we think of $M_\xi \in M_{r_1 \times r}(\mathbb{F}_p)$ as a variable, we see that this condition is described by the zero-set of a system of r_2 quadratic (homogeneous) equations.

To conclude the proof of this step, we use the Ax-Katz theorem [Kat71]. Recall that this says that, given a system of polynomial equations:

$$\begin{cases} f_1(x_1, \dots, x_n) = 0, \\ \vdots \\ f_m(x_1, \dots, x_n) = 0, \end{cases}$$

where $x_1, \dots, x_m$ are variables in a finite field $\mathbb{F}_q$, the p-adic (where p is the radical of q) valuation of the number of solutions is:

$$\geq \left\lceil \frac{n - \sum_j d_j}{d} \right\rceil,$$

where, for $j = 1, \dots, m$, d_j is the degree of f_j and $d := \max_j d_j$.

Applying the Ax-Katz theorem to the system (4.6.6), which has $r_1 r$ variables (the entries of the matrix M_ξ) varying in $\mathbb{F}_p$ and r_2 equations, all of degree 2, we deduce that the number of solutions of this system has p-adic valuation $\geq \lceil \frac{r_1 r - 2r_2}{2} \rceil$. Moreover, note that this set of solutions is always non-empty as it contains $M_\xi = 0$, so the p-adic valuation is $< \infty$. This concludes the proof of this step.

In the next step, we prove that there is a function $C(r)$ of r such that:

$\boxed{\text{if } r_1 > v_p(C(r)), \#C_v \text{ has } p\text{-adic valuation} = v_p(C(r)).}$

We let $C(r) : \mathbb{N} \to \mathbb{N}$ be the function defined in Lemma 4.A.3 of the appendix.

Note that, for each $\xi \in \mathrm{Hom}(\Gamma_{K_v}, A)$, the morphism $W_\wedge(-, \xi) : \Lambda^2 A^D \to \frac{1}{p}\mathbb{Z}/\mathbb{Z}$ is, by definition, the composition $\Lambda^2 A^D \xrightarrow{\Lambda^2 \xi^*} \Lambda^2(\Gamma_{K_v}^{ab}/(\Gamma_{K_v}^{ab})^p) \xrightarrow{B_v} \frac{1}{p}\mathbb{Z}/\mathbb{Z}$.

Hence, for each $\xi \in \mathrm{Hom}(\Gamma_{K_v}, A)$, we have that the morphism $W_\wedge(-, \xi)$ is zero if and only if the image of 1 under $\mathbb{F}_p \to \Lambda^2(\Gamma_{K_v}^{ab}/(\Gamma_{K_v}^{ab})^p) \xrightarrow{\Lambda^2\xi} \Lambda^2 A$ is zero. Using the matrix identifications as above, this last condition is equivalent to $M_\xi M_P M_\xi^t = 0$. If $r_1 > v_p(C(r))$, Lemma 4.A.3 (next section) shows that the number of matrices M such that $M M_P M^t = 0$ has p-adic valuation equal to $v_p(C(r))$. Remembering that $\mathrm{Hom}(\Gamma_{K_v}, A) \to M_{r_1 \times r}(\mathbb{F}_p)$, $\xi \mapsto M_\xi$ is a $1:1$-correspondence, the framed claim follows.

$\boxed{C_v \subsetneq E_v}$

Recall that $C_v := \{\xi \in \mathrm{Hom}(\Gamma_{K_v}, A) \mid W_\wedge(-,\xi) : \Lambda^2 A^D \to \frac{1}{p}\mathbb{Z}/\mathbb{Z} \text{ is zero}\}$. Note that, using the matrix notation from above, $\xi \in C_v$ if and only if $MM_P M^t = 0$, while $\xi \in E_v$ if and only if $\beta'(M_\xi M_P M_\xi^t) = 0$. In particular, $C_v \subseteq E_v$.

Recall that $r \geq 5$. Hence, choosing $D(r)$ accordingly, we may assume that $\frac{r_1 r - 2r_2}{2} \geq \frac{(r-4)r_1+6}{2} > v_p(C(r))$ and $r_1 > v_p(C(r))$. In particular, by the first two framed boxes, $v_p(\#C_v) < v_p(\#E_v)$. This implies that the $C_v \neq E_v$. Since $C_v \subseteq E_v$, we deduce the framed claim.

$\boxed{\text{Conclusion}}$ Note that, by Lemma 4.6.4, the statement of (point ii of) the lemma we are trying to prove is equivalent to the fact that there exists a $b \in \Lambda^2 A^D$ such that $W_\wedge(b,-) : E_v \to \frac{1}{p}\mathbb{Z}/\mathbb{Z}$ is not constant. This is equivalent to saying that the set $W_\wedge(b, E_v)$ has at least two elements.

By the last framed box, we deduce that there exists a $\xi \in E_v$ such that $W_\wedge(-,\xi) : \Lambda^2 A^D \to \frac{1}{p}\mathbb{Z}/\mathbb{Z}$ is not zero. I.e. there exists a $b \in \Lambda^2 A^D$ such that $W_\wedge(b,\xi) \neq 0$.

Note that we have that the trivial cohomological class 0 belongs to E_v (as is clear from Lemma 4.6.5), so the set $W_\wedge(b, E_v)$ certainly contains the element $W_\wedge(b, 0) = 0$. Since $W_\wedge(b,\xi) \neq 0$, we see that $W_\wedge(b, E_v)$ contains at least the two elements 0 and ξ, thus concluding the proof. □

Proof of Theorem 4.6.1. Returning to properties i., ii. and iii. of the first part of the proof, note that property ii. is trivial, while property iii. has been proven in Proposition 4.6.9. The only one that is missing is property i. (i.e. the fact $\mathrm{Br}_1 X \cap \mathrm{Br}_\psi X = \mathrm{Br}_0 X$).

We are actually going to prove that $\mathrm{Br}_1 X = \mathrm{Br}_0 X = \mathrm{Br}\, K$ (recall that $\mathrm{Br}_0 X$ is defined to be the image of the morphism $\mathrm{Br}\, K \to \mathrm{Br}\, X$, which is injective because $X(K) \neq \emptyset$). To do this, we apply [Har07, Proposition 4] to U, thus deducing that:

$$\mathrm{Br}_1 X/\mathrm{Br}\, K = \mathrm{Br}_{1,ur} U/\mathrm{Br}\, K \cong \{\alpha \in H^1(K, M) \mid \alpha_v \perp E_v \text{ for almost all } v\},$$

where we recall that $E_v = \mathrm{Im}([\varphi|_-] : U(K_v) \to H^1(K_v, A))$, and the $\perp$ sign refers to the local pairing, and $M := \mathrm{Hom}(G, \mathbb{G}_m) = \mathrm{Hom}(G^{ab}, \mathbb{G}_m) = \mathrm{Hom}(A, \mathbb{G}_m) = A'$ (for the penultimate identity recall from the setting paragraph that the assumption that β was surjective implied that $G^{ab} = A$). Note that the condition $\alpha_v \perp E_v$ is equivalent to $\alpha_v \perp < E_v >$. However, as proved in Lemma 4.6.5, we have that $< E_v >= \mathrm{Hom}(\Gamma_{K_v}, A)$ for all v. We deduce by local duality that $\mathrm{Br}_{1,ur} U/\mathrm{Br}\, K \cong Ш^1_\omega(K, A')$. Since $\mu_p \subseteq K^*$, and A is constant, A' is constant, therefore, by Chebotarev's theorem, $Ш^1_\omega(\Gamma_K, A') = 0$. This proves the desired identity $\mathrm{Br}_1 X = \mathrm{Br}_0 X = \mathrm{Br}\, K$. □

4.A Elementary counting facts

The following lemma is an example of calculations presented in [CTS07, p. 37].

Lemma 4.A.1. *Let $p \neq 2$ be a prime. For every $\mathbb{F}_p$-vector space A of dimension $4 \leq a < \infty$, there exists an $\mathbb{F}_p$-vector space B, of dimension $b = 2a - 3$, and a (surjective) morphism*

$$\beta : \Lambda^2 A \to B, \tag{4.A.1}$$

such that, if $1 \to B \to G \xrightarrow{\pi} A \to 1$ is the extension corresponding to $\iota(\beta)$, we have that $\mathfrak{Bic}(G, A) = 0$.

We recall that $\mathfrak{Bic}(G, A)$ is defined to be the set:

$$\{\pi(g_1) \wedge \pi(g_2) \mid g_1, g_2 \in G \text{ commute}\}.$$

Proof. Let $X \subseteq \mathbb{P}_{\mathbb{F}_p}(\Lambda^2 A)$ be the scheme-theoretic image of the morphism $- \wedge - : \mathbb{P}(A) \times \mathbb{P}(A) \setminus \Delta \to \mathbb{P}(\Lambda^2 A)$. This image is isomorphic to the Grassmanian variety $\mathrm{Gr}(2, A)$. Hence, since $X(\mathbb{F}_p)$ parametrizes two-dimensional $\mathbb{F}_p$-subspaces of A:

$$\#X(\mathbb{F}_p) = \frac{(p^a - 1)(p^{a-1} - 1)}{(p^2 - 1)(p - 1)}. \tag{4.A.2}$$

By Remark 4.6.7, it suffices to show that there exists a $(2r_1 - 3)$-codimensional subspace L in $\mathbb{P}(\Lambda^2 A)$ such that $L \cap X(\mathbb{F}_p) = \emptyset$, and choose β such that $\Lambda^2 A \supseteq \mathbb{F}_p \cdot L = \operatorname{Ker} \beta$. Noting that:

$$\frac{(p^{r_1} - 1)(p^{r_1 - 1} - 1)}{(p^2 - 1)(p - 1)} < \frac{(p^{r_1} - 1)(p^{r_1 - 1} - 1)}{(p + 1)(p - 1)} \le \frac{(p^{2r_1 - 2} - 1)(p + 1)}{(p + 1)(p - 1)} = \#\mathbb{P}^{2r_1 - 3}(\mathbb{F}_p),$$

such a subspace always exists by the following lemma. □

Lemma 4.A.2. *Let $X \subseteq \mathbb{P}^N(\mathbb{F}_q)$ be a set of points, and let $0 \le n \le N$ be such that $\#X < \mathbb{P}^n(\mathbb{F}_p)$. There exists then an n-codimensional subspace $L \subseteq \mathbb{P}^N$ such that $X \cap L = \emptyset$.*

Proof. Let $k \ge 0$ be the smallest integer such that X intersects every k-dimensional subspace in $\mathbb{P}^N_{\mathbb{F}_p}$. If $k = 0$, then there is nothing to prove. Otherwise, let $L \subseteq \mathbb{P}^N_{\mathbb{F}_p}$ be a $(k-1)$-dimensional subspace such that $L \cap X = \emptyset$. Let $\pi_L : \mathbb{P}^N \setminus L \to \mathbb{P}^{N-k}$ be a projection outside of L. We know by the "smallest" assumption that $\pi_L(X(\mathbb{F}_p)) = \mathbb{P}^{N-k}(\mathbb{F}_p)$, hence $\#X(\mathbb{F}_p) \ge \#\mathbb{P}^{N-k}(\mathbb{F}_p) \Rightarrow N - k < n$, i.e. $k \ge N - n + 1$. Hence the dimension of L is $\ge N - n$ and it can be taken to be the sought subspace. □

Lemma 4.A.3. *Let $V/\mathbb{F}_p$ be a r-dimensional vector space, endowed with an alternating non-degenerate bilinear form $b : V \times V \to \mathbb{F}_p$. We assume that $p \neq 2$. Let $A/\mathbb{F}_p$ be an a-dimensional vector space ($1 \le a < \infty$). We then have that:*

$$\Xi(A, V) := \#\{\xi \in \operatorname{Hom}(A, V) \mid \xi^* b = 0\} \equiv C(r) \mod p^a,$$

where $C(r)$ is a non-zero integer depending only on r.

Proof. Let

$$M_d := \#\{\text{isotropic } d\text{-dimensional subspaces in } V\},$$
$$I_d := \#\{\text{surjective homomorphisms from } A \text{ to a } d\text{-dimensional } \mathbb{F}_p\text{-vector space}\}.$$

We then have that:

$$\Xi(A, V) = \sum_{d=0}^{\min(a, \frac{r}{2})} I_d M_d.$$

(note that the fact that V is endowed with a non-degenerate alternating linear form and $p \neq 2$ implies that r is even). One can easily see that:

$$I_d = (p^a - 1) \cdot (p^a - p) \cdots (p^a - p^{d-1}), \text{ for every } d \le a,$$
$$M_d = \frac{(p^r - 1) \cdot (p^{r-1} - p) \cdots (p^{r-d+1} - p^{d-1})}{(p^d - 1) \cdot (p^d - p) \cdots (p^d - p^{d-1})}, \text{ for every } d \le r/2.$$

In particular, $\Xi(A, V) = \Xi'(a, r)$ depends only on a and r. Note that, for a fixed r, $\Xi'(a, r)$ converges, as $a \to \infty$, p-adically to the following sum:

$$\Xi'(\infty, r) := \sum_{d=0}^{r/2} (-1)^d \frac{(p^r - 1) \cdot (p^{r-1} - p) \cdots (p^{r-d+1} - p^{d-1})}{(p - 1) \cdot (p^2 - 1) \cdots (p^d - 1)}.$$

Denoting by $a(d)$ the term multiplying the $(-1)^d$ appearing above, we notice that the sequence $a(0), \ldots, a(r/2)$ is strictly increasing, as follows by induction from the fact that $\frac{p^{r-d+1} - p^{d-1}}{p^d - 1} > 1$ for all $d \in \{0, \ldots, r/2\}$. In particular, a standard elementary calculus argument (à la Leibniz' rule) shows that $\Xi'(\infty, r) \neq 0$. □

4.B Other works where ramified descent appears

Let us mention two works where the idea of "ramified descent" has appeared. One is that of [HS16] (successor to [SSD05]), where the authors use the cyclic covers of some specific Kummer surfaces (those defined by a quotient of the product X of two genus 1 curves C^1, C^2) to prove that, under certain technical assumptions, these satisfy the Hasse principle. What they prove is that, if $\psi : X \to S$ is the μ_2-cover defining the Kummer surface, there exists an $a \in k^*/(k^*)^2 = H^1(k, \mu_2)$ such that the twist X_a has a K-rational point. They do this by first finding an a such that X_a has an adelic point, and this is equivalent to proving, with our terminology, that $S(\mathbb{A}_k)^\psi \neq \emptyset$ "in an explicit way", and then they modify the a in such a way that some Ш-groups associated to X_a are 0 (namely, the Tate-Shafarevich groups of the Jacobians of the two curves C^1_a and C^2_a), which then grants that X_a has a K-point.

The second work that we wanted to mention is [CS20], where the authors build upon Poonen's example [Poo10] to show (employing one specific S_4-cover) that the following obstruction is *stronger* than the étale-Brauer–Manin obstruction:

$$X(\mathbb{A}_K)^{\mathrm{Br},ram,sol} = \bigcap_{\substack{\psi: Y \to X \\ G-\text{cover} \\ G \text{ solvable}}} \overline{\bigcup_{\xi \in H^1(K,G)} \psi'_\xi(Y^{sm}_\xi(\mathbb{A}_K))^{\mathrm{Br}\, Y^{sm}_\xi}},$$

where the ψ'_ξ is the composition $Y^{sm}_\xi \to Y_\xi \xrightarrow{\psi_\xi} X$, on the variety X of Poonen's example.

Ringraziamenti

Questa tesi è conclusione di un lungo percorso alla Scuola Normale Superiore, prima come Studente Ordinario e poi come Perfezionando, e di un percorso di dottorato all'Université Paris-Saclay. Sarò sempre riconoscente ad entrambe le istituzioni. Sono inoltre grato anche all'Università Italo-Francese e al Max Planck Institute di Bonn per aver contribuito finanziariamente allo svolgimento del dottorato.

Ringrazio calorosamente i miei due relatori, David Harari e Umberto Zannier, che stanno al cuore di tutti i progetti di questa tesi.

Umberto Zannier è stato mio relatore non solo al dottorato, ma anche alla laurea triennale e magistrale. Di fatto è stato colui che mi ha introdotto al mondo "adulto" della matematica, e a lui devo tantissimo. Il suo modo di vedere intuitivo, sciolto ed ampio della materia è sempre stato di grande ispirazione. Lo ringrazio anche per avermi sempre spronato ad una certa competitività.

David Harari è stato il mio primo contatto con la matematica fuori da Pisa. É stato per me un'importantissima guida nel mio percorso di dottorato. La sua calma ed il suo rigore sono stati davvero un faro e mi hanno portato a poter vedere con tranquillità anche i problemi più complessi.

Ringrazio Pietro Corvaja e Cyril Demarche per avermi aiutato con interessanti discussioni collegate anche a questa tesi.

Ringrazio poi Daniel Loughran e Olivier Wittenberg, controrelatori di questa tesi, per la loro lettura e i loro preziosi commenti. Ringrazio Olivier Wittenberg anche per aver suggerito una dimostrazione della Proposizione 4.5.1 molto più semplice della mia dimostrazione originaria.

Ringrazio i miei colleghi ed amici Gioacchino Antonelli, Francesco Ballini, Matteo Barucco, Giulio Belletti, Lorenzo Benedini, Guido Bosco, Giulio Bresciani, Elia Brué, Simone Coccia, Federico Conti, Nirvana Coppola, Edoardo Fossati, Giada Franz, Federico Glaudo, Francesco Grotto, Davide Lombardo, Myrto Mavraki, Guglielmo Nocera, Giovanni Paolini, Michele Pernice, Federico Scavia, Daniele Semola, Anastasia Shepevtseva, Giorgio Stefani, Sam Streeter, Leonardo Tolomeo, Marco Trevisiol e Matteo Verzobio. Condividere con loro la mia passione è sempre stato per me il più forte stimolo in questo percorso.

Ringrazio Fabrizio Barroero, Laura Capuano, Ariyan Javanpeykar, Davide Lombardo, Amos Turchet, Francesco Veneziano e Angelo Vistoli per aver contribuito a guidare le mie scelte matematiche.

Ringrazio poi in maniera speciale la mia famiglia, ed in particolar modo mia madre, per avermi sempre supportato. Ringrazio infine gli amici a casa, per il loro costante incoroggiamento e per ricordarmi che esiste un mondo all'infuori della matematica.

Bibliography

[AM69] Michael Artin and Barry Mazur. *Etale Homotopy*. Lecture Notes in Mathematics 100. Springer-Verlag Berlin Heidelberg, 1 edition, 1969.

[BalMS] Francesca Balestrieri. Some remarks on strong approximation and applications to homogeneous spaces of linear algebraic groups. *Preprint*, https://fbalestrieri.github.io/files/SROSAAATHSOLAG.pdf, To appear in Proc. A.M.S.

[BCTS08] Mikhail V. Borovoi, Jean-Louis Colliot-Thélène, and Alexei N. Skorobogatov. The elementary obstruction and homogeneous spaces. *Duke Math. J.*, 141(2):321–364, 02 2008.

[BD13] Mikhail V. Borovoi and Cyril Demarche. Manin obstruction to strong approximation for homogeneous spaces. *Comment. Math. Helv.*, 88(1):1–54, 2013.

[Bor93] Mikhail V. Borovoi. Abelianization of the second nonabelian Galois cohomology. *Duke Math. J.*, 72(1):217–239, 10 1993.

[Bor95] Mikhail V. Borovoi. The Brauer-Manin obstructions for homogeneous spaces with connected or abelian stabilizer. *Journal für die reine und angewandte Mathematik (Crelle's Journal)*, 1996:181 – 194, 1995.

[Bor96] Mikhail V. Borovoi. Abelianization of the first Galois cohomology of reductive groups. *Internat. Math. Res. Notices*, 1996(8):401–407, 1996.

[Bre07] Lawrence Breen. *Bitorseurs et Cohomologie Non Abélienne*, pages 401–476. Birkhäuser Boston, Boston, MA, 2007.

[Bri11] Michel Brion. On the geometry of algebraic groups and homogeneous spaces. *Journal of Algebra*, 329(1):52 – 71, 2011. Special Issue Celebrating the 60th Birthday of Corrado De Concini.

[Bro94] Kenneth S. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. Corrected reprint of the 1982 original.

[BSFP14] Lior Bary-Soroker, Arno Fehm, and Sebastian Petersen. On varieties of Hilbert type. *Ann. Inst. Fourier (Grenoble)*, 64(5):1893–1901, 2014.

[BvH06] Mikhail V. Borovoi and Joost van Hamel. Extended Picard complexes for algebraic groups and homogeneous spaces. *C. R. Math. Acad. Sci. Paris*, 342(9):671–674, 2006.

[CDX19] Yang Cao, Cyril Demarche, and Fei Xu. Comparing descent obstruction and Brauer-Manin obstruction for open varieties. *Trans. Amer. Math. Soc.*, 371(12):8625–8650, 2019.

[Che89] Vladimir Chernousov. The Hasse principle for groups of type E_8. *Dokl. Akad. Nauk SSSR*, 306(5):1059–1063, 1989.

[CLX19] Yang Cao, Yongqi Liang, and Fei Xu. Arithmetic purity of strong approximation for homogeneous spaces. *J. Math. Pures Appl. (9)*, 132:334–368, 2019.

[Con02] Brian Conrad. A modern proof of Chevalley's theorem on algebraic groups. *J. Ramanujan Math. Soc.*, 17(1):1–18, 2002.

[CS20] David Corwin and Tomer Schlank. Brauer and Étale Homotopy Obstructions to Rational Points on Open Covers. *arXiv e-prints*, page arXiv:2006.11699, June 2020.

[CT90] Jean-Louis Colliot-Thélène. Surfaces rationnelles fibrées en coniques de degré 4. In *Séminaire de Théorie des Nombres, Paris 1988–1989*, volume 91 of *Progr. Math.*, pages 43–55. Birkhäuser Boston, Boston, MA, 1990.

[CTS80] Jean-Louis Colliot-Thélène and Jean-Jacques Sansuc. La descente sur les variétés rationnelles. In *Journées de Géometrie Algébrique d'Angers, Juillet 1979/Algebraic Geometry, Angers, 1979*, pages 223–237. Sijthoff & Noordhoff, Alphen aan den Rijn—Germantown, Md., 1980.

[CTS87a] Jean-Louis Colliot-Thélène and Jean-Jacques Sansuc. La descente sur les variétés rationnelles, II. *Duke Mathematical Journal*, 54(2), jan 1987.

[CTS87b] Jean-Louis Colliot-Thélène and Jean-Jacques Sansuc. Principal homogeneous spaces under flasque tori: applications. *J. Algebra*, 106(1):148–205, 1987.

[CTS07] Jean-Louis Colliot-Thélène and Jean-Jacques Sansuc. The rationality problem for fields of invariants under linear algebraic groups (with special regards to the Brauer group). In *Algebraic groups and homogeneous spaces*, volume 19 of *Tata Inst. Fund. Res. Stud. Math.*, pages 113–186. Tata Inst. Fund. Res., Mumbai, 2007.

[CTS21] Jean-Louis Colliot-Thélène and Alexei N. Skorobogatov. *The Brauer-Grothendieck group*, volume 71 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*. Springer, Cham, [2021] ©2021.

[CTX09] Jean-Louis Colliot-Thélène and Fei Xu. Brauer-Manin obstruction for integral points of homogeneous spaces and representation by integral quadratic forms. *Compos. Math.*, 145(2):309–363, 2009. With an appendix by Dasheng Wei and Xu.

[CZ17] Pietro Corvaja and Umberto Zannier. On the Hilbert property and the fundamental group of algebraic varieties. *Math. Z.*, 286(1-2):579–602, 2017.

[DAN17] Cyril Demarche, Giancarlo Lucchini Arteche, and Danny Neftin. The grunwald problem and approximation properties for homogeneous spaces. *Annales de l'institut Fourier*, 67(3):1009–1033, 2017.

[Dem09] Cyril Demarche. *Méthodes cohomologiques pour l'étude des points rationnels sur les espaces homogènes*. PhD thesis, Université de Paris-Sud. Faculté des sciences d'Orsay (Essonne), 2009.

[Dem11] Cyril Demarche. Une formule pour le groupe de Brauer algébrique d'un torseur. *Journal of Algebra*, 347(1):96 – 132, 2011.

[Dem13] Cyril Demarche. *Abélianisation des espaces homogènes et applications arithmétiques*, page 138–209. London Mathematical Society Lecture Note Series. Cambridge University Press, 2013.

[Dem17] Cyril Demarche. Obstructions de Brauer-Manin entières sur les espaces homogènes à stabilisateurs finis nilpotents. *Bull. Soc. Math. France*, 145(2):225–236, 2017.

[Dem20] Julian L. Demeio. Non-rational varieties with the Hilbert property. *Int. J. Number Theory*, 16(4):803–822, 2020.

[EH02] Santiago Encinas and Herwig Hauser. Strong resolution of singularities in characteristic zero. *Commentarii Mathematici Helvetici*, 77(4):821–845, 2002.

[Eke90] Torsten Ekedahl. An effective version of Hilbert's irreducibility theorem. In *Séminaire de Théorie des Nombres, Paris 1988–1989*, volume 91 of *Progr. Math.*, pages 241–249. Birkhäuser Boston, Boston, MA, 1990.

[Gil09] Stefan Gille. On the Brauer group of a semisimple algebraic group. *Advances in Mathematics*, 220(3):913–925, 2009.

[Gro71] Alexander Grothendieck. *Revêtements étales et groupe fondamental (SGA 1)*, volume 224 of *Lecture notes in mathematics*. Springer-Verlag, 1971.

[GS19] Alice Garbagnati and Cecília Salgado. Linear systems on rational elliptic surfaces and elliptic fibrations on K3 surfaces. *J. Pure Appl. Algebra*, 223(1):277–300, 2019.

[Har65] Günter Harder. Über die Galoiskohomologie halbeinfacher Matrizengruppen. i. *Mathematische Zeitschrift*, 90(5):404–428, 1965.

[Har66] Günter Harder. Über die Galoiskohomologie halbeinfacher Matrizengruppen. II. *Mathematische Zeitschrift*, 92(5):396–415, 1966.

[Har77] Robin Hartshorne. *Algebraic Geometry*. Springer New York, 1977.

[Har94] David Harari. Méthode des fibrations et obstruction de Manin. *Duke Mathematical Journal*, 75(1), jul 1994.

[Har06] David Harari. The Manin obstruction for torsors under connected algebraic groups. *International Mathematics Research Notices*, jan 2006.

[Har07] David Harari. Quelques propriétés d'approximation reliées à la cohomologie galoisienne d'un groupe algébrique fini. *Bulletin de la Société Mathématique de France*, 135(4):549–564, 2007.

[Har08] David Harari. Le défaut d'approximation forte pour les groupes algébriques commutatifs. *Algebra Number Theory*, 2(5):595–611, 2008.

[Har20] David Harari. *Galois Cohomology and Class Field Theory*. Springer International Publishing, 2020.

[Hil92] David Hilbert. über die Irreducibilität ganzer rationaler Functionen mit ganzzahligen Coefficienten. *Journal für die reine und angewandte Mathematik*, 110:104–129, 1892.

[HS05] David Harari and Tamás Szamuely. Arithmetic duality theorems for 1-motives. *J. Reine Angew. Math.*, 578:93–128, 2005.

[HS08] David Harari and Tamás Szamuely. Local-global principles for 1-motives. *Duke Math. J.*, 143(3):531–557, 2008.

[HS16] Yonatan Harpaz and Alexei N. Skorobogatov. Hasse principle for Kummer varieties. *Algebra Number Theory*, 10(4):813–841, 2016.

[HW20] Yonatan Harpaz and Olivier Wittenberg. Zéro-cycles sur les espaces homogènes et problème de Galois inverse. *J. Amer. Math. Soc.*, 33(3):775–805, 2020.

[Iwa55] Kenkichi Iwasawa. On Galois groups of local fields. *Transactions of the American Mathematical Society*, 80(2):448–469, feb 1955.

[Jos09] Peter Jossen. *On the arithmetic of 1-motives.* PhD thesis, 2009.

[Kat71] Nicholas M. Katz. On a theorem of Ax. *American Journal of Mathematics*, 93(2):485, apr 1971.

[KKV89] Friedrich Knop, Hanspeter Kraft, and Thierry Vust. The Picard group of a G-variety. In *Algebraische Transformationsgruppen und Invariantentheorie Algebraic Transformation Groups and Invariant Theory*, pages 77–87. Birkhäuser Basel, 1989.

[Kle05] Steven L. Kleiman. The Picard scheme. In *Fundamental algebraic geometry*, volume 123 of *Math. Surveys Monogr.*, pages 235–321. Amer. Math. Soc., Providence, RI, 2005.

[Kne65] Martin Kneser. Galois-Kohomologie halbeinfacher algebraischer Gruppen über p-adischen Körpern. i. *Mathematische Zeitschrift*, 88(1):40–47, 1965.

[Kne66a] Martin Kneser. Hasse principle for H^1 of simply connected groups. In *Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965)*, pages 159–163. Amer. Math. Soc., Providence, R.I., 1966.

[Kne66b] Martin Kneser. Strong approximation. In *Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965)*, pages 187–196. Amer. Math. Soc., Providence, R.I., 1966.

[Kne69] Martin Kneser. *Lectures on Galois Cohomology of Classical Groups.* Lectures on Mathematics 47. Tata Institute of Fundamental Research, 1969.

[Kol88] Victor A. Kolyvagin. Finiteness of $E(\mathbf{Q})$ and $Ш(E, \mathbf{Q})$ for a subclass of Weil curves. *Izv. Akad. Nauk SSSR Ser. Mat.*, 52(3):522–540, 670–671, 1988.

[Liu02] Qing Liu. *Algebraic Geometry and Arithmetic Curves.* Oxford graduate texts in mathematics. Oxford University Press, 2002.

[Lui12] Ronald van Luijk. Density of rational points on elliptic surfaces. *Acta Arith.*, 156(2):189–199, 2012.

[Mer96] Loïc Merel. Bornes pour la torsion des courbes elliptiques sur les corps de nombres. *Invent. Math.*, 124(1-3):437–449, 1996.

[Mil63] John Milnor. *Morse theory.* Based on lecture notes by M. Spivak and R. Wells. Ann. of Math. Stud., No. 51. Princeton University Press, Princeton, N.J., 1963.

[Mil17] James S. Milne. *Algebraic Groups: The Theory of Group Schemes of Finite Type over a Field.* Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2017.

[Noe17] Emmy Noether. Gleichungen mit vorgeschriebener Gruppe. *Math. Ann.*, 78(1):221–229, 1917.

[NSW08] Jürgen Neukirch, Alexander Schmidt, and Kay Wingberg. *Cohomology of number fields*, volume 323 of *Grundlehren der Mathematischen Wissenschaften.* Springer-Verlag, Berlin, second edition, 2008.

[Par99] Pierre Parent. Bornes effectives pour la torsion des courbes elliptiques sur les corps de nombres. *J. Reine Angew. Math.*, (506):85–116, 1999.

[Pla69] Vladimir P. Platonov. The problem of strong approximation and the Kneser-Tits conjecture for algebraic groups. *Mathematics of the USSR-Izvestiya*, 3(6):1139–1147, dec 1969.

[Poo10] Bjorn Poonen. Insufficiency of the Brauer-Manin obstruction applied to étale covers. *Ann. of Math. (2)*, 171(3):2157–2169, 2010.

[Pop74] Vladimir L. Popov. Picard groups of homogeneous spaces of linear algebraic groups and one-dimensional homogeneous vector bundles. *Mathematics of the USSR-Izvestiya*, 8(2):301–327, apr 1974.

[PRR93] Vladimir P. Platonov, Andrei Rapinchuk, and Rachel Rowen. *Algebraic Groups and Number Theory*. Pure and Applied Mathematics. Elsevier Science, 1993.

[PS99] Bjorn Poonen and Michael Stoll. The Cassels-Tate pairing on polarized abelian varieties. *Ann. of Math. (2)*, 150(3):1109–1149, 1999.

[Sal84] David J. Saltman. Noether's problem over an algebraically closed field. *Invent. Math.*, 77(1):71–84, 1984.

[Sch94] Claus Scheiderer. *Real and Étale cohomology*. Lecture notes in mathematics. Springer-Verlag, 1994.

[Sch99] Victor Scharaschkin. *Local-global problems and the Brauer-Manin obstruction*. ProQuest LLC, Ann Arbor, MI, 1999. Thesis (Ph.D.)–University of Michigan.

[SD01] Peter Swinnerton-Dyer. Weak approximation and R-equivalence on cubic surfaces. In Emmanuel Peyre and Yuri Tschinkel, editors, *Rational Points on Algebraic Varieties*, pages 357–404. Birkhäuser Basel, 2001.

[Ser73] Jean-Pierre Serre. *A course in arithmetic*. Graduate Texts in Mathematics, No. 7. Springer-Verlag, New York-Heidelberg, 1973. Translated from the French.

[Ser79] Jean-Pierre Serre. *Local Fields*. Springer New York, 1979.

[Ser94] Jean-Pierre Serre. Cohomologie galoisienne. Springer Berlin Heidelberg, 1994.

[Ser08] Jean-Pierre Serre. *Topics in Galois theory*, volume 1 of *Research Notes in Mathematics*. A K Peters, Ltd., MA, second edition, 2008.

[SI13] Igor' R. Shafarevich and Vasilii A. Iskovskikh. *Algebraic Geometry II: Cohomology of Algebraic Varieties. Algebraic Surfaces*. Encyclopaedia Math. Sci. Springer Berlin Heidelberg, 2013.

[Sko99] Alexei N. Skorobogatov. Beyond the Manin obstruction. *Inventiones Mathematicae*, 135(2):399–424, jan 1999.

[Sko01] Alexei N. Skorobogatov. *Torsors and Rational Points*. Cambridge Tracts in Mathematics. Cambridge University Press, 2001.

[SS91] Per Salberger and Alexei N. Skorobogatov. Weak approximation for surfaces defined by two quadratic forms. *Duke Math. J.*, 63(2):517–536, 1991.

[SSD05] Alexei N. Skorobogatov and Peter Swinnerton-Dyer. 2-descent on elliptic curves and rational points on certain Kummer surfaces. *Adv. Math.*, 198(2):448–483, 2005.

[Sta20] The Stacks Project Authors. *Stacks Project*. https://stacks.math.columbia.edu, 2020.

[Sto07a] Michael Stoll. Finite descent obstructions and rational points on curves. *Algebra Number Theory*, 1(4):349–391, 2007.

[Sto07b] Michael Stoll. Finite descent obstructions and rational points on curves. *Algebra Number Theory*, 1(4):349–391, 2007.

[Swa69] Richard G. Swan. Invariant rational functions and a problem of Steenrod. *Invent. Math.*, 7:148–158, 1969.

[Sza09] Tamás Szamuely. *Galois Groups and Fundamental Groups*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2009.

[Tei77] Bernard Teissier. *The Hunting of Invariants in the Geometry of Discriminants: Notes of a Course of Five Lectures at the Nordic Summer School, Oslo, August 1976*. Centre de mathématiques de l'Ecole Polytechnique, 1977.

[Vak] Ravi Vakil. *The rising sea*. https://math.stanford.edu/~vakil/216blog/, November 18, 2017 edition.

[Wei94] Charles A. Weibel. *An Introduction to Homological Algebra*. Cambridge University Press, apr 1994.

[WK93] Lan Wang and Masato Kuwata. Topology of rational points on isotrivial elliptic surfaces. *Int. Math. Res. Not. IMRN*, 1993(4):113–123, 02 1993.

[Zan10] Umberto Zannier. Hilbert irreducibility above algebraic groups. *Duke Math. J.*, 153(2):397–425, 2010.

[Zar58] Oscar Zariski. On the purity of the branch locus of algebraic functions. *Proc. Nat. Acad. Sci. U.S.A.*, 44:791–796, 1958.