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Expository Article

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On Singular Liouville Equations and Systems

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Abstract: We consider some singular Liouville equations and systems motivated by uniformization problems in a non-smooth setting, as well as from models in mathematical physics. We will study the existence of solutions from a variational point of view, using suitable improvements of the Moser–Trudinger inequality. These reduce the problem to a topological one by studying the concentration property of conformal volume, which will be constrained by the functional inequalities of geometric flavour. We will mainly describe some common strategies from the papers [11, 12, 20] in simple situations to give an idea to the non-expert reader about the general methods we use.

Keywords: Geometric PDEs, Singular Liouville Equation, Variational Methods, Min-Max Schemes

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1 Introduction

One among the most classical problems in Riemannian geometry is to find *canonical metrics* on a given manifold. In two dimensions, a natural choice is to *uniformize* a surface looking for metrics of constant Gaussian curvature. One way to achieve this is to choose a *conformal representative*, namely a metric pointwise scaled by a suitable positive function. Given a compact, boundary-less surface (Σ, g) with Gaussian curvature K_g, consider the conformal change $g \mapsto \tilde{g} = e^{2w} g$, where w is a smooth function on Σ. It is known that under conformal changes the Gaussian curvature transforms according to the formula

$$-\Delta_g w + K_g = K_{\tilde{g}} e^{2w}. \tag{1.1}$$

Hence looking for constant $K_{\tilde{g}}$ amounts to solving the following PDE on Σ:

$$-\Delta_g w + K_g = \overline{K} e^{2w}, \tag{U}$$

where $\overline{K} \in \mathbb{R}$. By the Gauss–Bonnet formula, the sign of $\overline{K}$ has to be the same as that of the Euler characteristic of Σ.

We introduce next a singular version of (U). Singular objects attracted a lot of attention over the past decades since they arise in many different situations such as limits of Einstein metrics [1, 13, 65], Kähler–Einstein metrics [27] and in physical applications such as the study of interfaces or in general relativity.

One of the simplest singular objects consists of two-dimensional surfaces with finitely many conical points. The *model object* is a standard cone, that can be realized with an isometry from a planar circular sector. Isometries preserve the Gaussian curvature and hence a cone is geometrically *flat* on its side surface, but in a weak sense the curvature behaves like a measure at the conical tip. Precisely, if the opening angle θ of the cone is written as $\theta = 2\pi(1 + \alpha)$, $\alpha > -1$, the curvature at the vertex is a Dirac mass with amplitude $-2\pi\alpha$.

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With this model in mind, for points $p_1, \ldots, p_m$ in Σ and $\alpha_1, \ldots, \alpha_m > -1$, we will consider the following problem on a compact, closed surface (Σ, g) of total volume 1:

$$-\Delta_g u + K_g = \rho e^{2u} - 2\pi \sum_{j=1}^{m} \alpha_j \delta_{p_j}, \quad \rho \in \mathbb{R}. \tag{1.2}$$

Equation (1.2) is a singular version of (U), and a solution will endow Σ with a constant-curvature metric on $\Sigma \setminus \bigcup_{i=1}^{m}\{p_i\}$ and conical angles $\theta_i = 2\pi(1+\alpha_i)$ at each point p_i. All the singular structure is encoded in the *divisor*, written as a formal sum,

$$\underline{\alpha} := \sum_{i=1}^{m} \alpha_i p_i.$$

Still, by the Gauss–Bonnet formula (assuming without loss of generality that $\mathrm{Vol}_g(\Sigma) = 1$), that can be obtained by rounding-off the conical points and applying the usual Gauss–Bonnet theorem, the constant ρ should satisfy the geometric constraint

$$\rho = 2\pi\chi(\Sigma) + 2\pi \sum_{i=1}^{m} \alpha_i. \tag{1.3}$$

As we will see, equations (or systems) like (1.2) also have applications in physics and are particularly interesting for these integer values of the parameters α_i. Geometrically, these would correspond to *orbifold points* with angle greater than 2π, but for most of this paper we will limit ourselves to describe the case of negative α_i's, which is simpler to analyse. Liouville equations arise in mathematical physics as well, to describe mean field vorticity in steady flows (see [17, 23]), Chern–Simons vortices in superconductivity or electroweak theory (see [63, 70]). For these problems ρ represents a positive physical parameter, and is not assumed to satisfy (1.3). The points p_j are called *vortices*, and describe either points where vorticity is imposed by external forces [68], or *vortex points*, namely zeroes of the Higgs field with vanishing order α_i.

To study existence for (1.2), it is useful to *desingularize* the problem, as one could exploit its *variational structure*. Consider Green's function of $-\Delta_g$ on Σ with pole p, namely the solution to

$$-\Delta_g G_p(x) = \delta_p - 1 \quad \text{on } \Sigma \quad \text{with} \int_\Sigma G_p(x)\, dV_g = 0.$$

It is a standard fact that G_p has the asymptotic behavior $G_p \simeq -\frac{1}{2\pi} \log d_g(x,p)$ near the singularity, where $d_g(\cdot,\cdot)$ stands for the distance induced by the background metric g. Consider the change of variables

$$u \mapsto u + 2\pi \sum_{j=1}^{m} \alpha_j G_{p_j}(x). \tag{1.4}$$

After this, (1.2) becomes

$$-\Delta_g u = \rho\big(\tilde{h}(x)e^{2u} - \tilde{a}(x)\big) \quad \text{on } \Sigma, \quad \text{where } \tilde{h}(x) = e^{-2\pi \sum_{j=1}^{m} \alpha_j G_{p_j}(x)}. \tag{1.5}$$

Here $\tilde{a}(x)$ is a smooth function on Σ such that $\int_\Sigma \tilde{a}(x)\, dV_g = 1$, while by the asymptotics of G_{p_j} the function $\tilde{h}$ satisfies

$$\tilde{h} > 0 \text{ on } \Sigma \setminus \bigcup_j \{p_j\}, \quad \tilde{h}(x) \simeq \gamma_j d_g(x, p_j)^{2\alpha_j} \text{ near } p_j \tag{1.6}$$

for some constant $\gamma_j > 0$.

Solutions to (1.5) can be found as critical points of the Euler–Lagrange energy

$$I_{\rho,\underline{\alpha}}(u) = \int_\Sigma |\nabla_g u|^2\, dV_g + 2\rho \int_\Sigma \tilde{a}(x) u\, dV_g - \rho \log \int_\Sigma \tilde{h}(x) e^{2u}\, dV_g, \quad u \in H^1(\Sigma).$$

Let us recall that in two dimensions $H^1(\Sigma)$ embeds into $L^p(\Sigma)$ for any $p \in (1,\infty)$: the embedding can be indeed extended up to exponential class. The well-known Moser–Trudinger inequality holds and gives a quantitative estimate on exponentials of Sobolev functions

$$\log \int_\Sigma e^{2(u-\overline{u})}\, dV_g \leq \frac{1}{4\pi} \int_\Sigma |\nabla_g u|^2\, dV_g + C_{\Sigma,g}, \tag{1.7}$$

where $\overline{u} = ⨍_\Sigma u\, dV_g$ stands for the average of u on Σ.

In the singular case the Moser–Trudinger inequality on Σ has a different best constant, as was proven by Chen [26] and Troyanov [67] (see also [22]).

Proposition 1.1 ([26, 67]). *Let $\alpha_j > -1$ for all j and let $\tilde{h} : \Sigma \to \mathbb{R}$ be as in* (1.6). *Then one has the inequality*

$$\log \int_\Sigma \tilde{h}(x) e^{2(u-\overline{u})}\, dV_g \leq \frac{1}{4\pi \min\{1, 1 + \min_j \alpha_j\}} \int_\Sigma |\nabla u|^2\, dV_g + C_{\tilde{h},g} \tag{1.8}$$

for all $u \in H^1(\Sigma)$.

Notice that the multiplicative constant appearing in the last formula is determined by the most singular behavior of the function $\tilde{h}$, see (1.6), that becomes unbounded at the points p_i with negative α_i's.

Depending then on the value of ρ, we distinguish three geometric cases: the subcritical, the critical and the supercritical. In the first case, $\rho < 4\pi \min\{1, 1 + \min_j \alpha_j\}$, and the latter term in $I_{\rho,\underline{\alpha}}$ can be absorbed into the first one, giving *coercivity* of the energy. As a consequence, one always finds solutions using the direct methods of the calculus of variations, i.e. taking weak limits of minimizing sequences. See for example [50, 64, 66]. In the *regular case* (U), this situation corresponds to the negative or zero curvature case. In the second case ($\rho = 4\pi \min\{1, 1 + \min_j \alpha_j\}$), the energy $I_{\rho,\underline{\alpha}}$ is bounded below but coercivity is lost, so it is unclear whether minimizing sequences would converge. If compactness fails, a typical behavior of solutions (described in more detail later) leads to indefinite concentration of conformal volume at a finite number of points. For example, in the positive curvature case of (U) (i.e. on the sphere), the loss of compactness is caused by the action of the Möbius group, which might cause all conformal volume to concentrate to a single point but leaving the Euler–Lagrange energy for (U) invariant. A careful blow-up analysis of the minimizing sequences might still lead to existence results; we will not discuss the details here but refer the reader to [33, 58] (and to [32] specifically for the uniformization problem). The third case ($\rho > 4\pi \min\{1, 1 + \min_j \alpha_j\}$) is the most delicate one, and has no regular counterpart in (U). The fact that ρ exceeds the Chen–Troyanov constant causes unboundedness from below of the energy, so it is hopeless to try to find global minima as before. Worse than that, there are situations in which solutions do not exist: one well-known example is the *tear-drop*, namely a spherical surface with only one singularity. It is known that there is indeed no constant curvature metric on such an object (see also [19, 38, 39, 55] for more general results of this type).

The supercritical case will be the one we will mostly be interested in, and we will show that a variational approach might still give conclusions in the search of critical points for $I_{\rho,\alpha}$ of saddle type. In order to find them, as for the direct minimization methods, one fundamental condition is compactness. Concerning problem (1.2), an alternative was proved in [8] (after previous results in [15, 46, 47] for the regular case): either a sequence $(u_n)_n$ of solutions to (E_{ρ_n}) (with $\rho_n \to \overline{\rho} \in \mathbb{R}$) stays uniformly bounded, or it develops a finite number of spheres at regular points and/or *American footballs* at singular points; see Theorem 3.5 for a precise statement. An American football is obtained from a sphere (possibly covered multiple times) by cutting two meridians and by gluing the remaining edges. This results in a constant-curvature singular surface having two equal conical angles $\theta = 2\pi(1 + \alpha)$: by the modified Gauss–Bonnet formula (1.1) the total curvature of this object must be $4\pi(1 + \alpha)$.

In the blow-up alternative all the curvature is exhausted in this way and therefore ρ_n, the total curvatures of the conformal metric $\tilde{h}e^{2u_n}$, must converge to a number in the discrete set

$$\mathfrak{S} = \left\{\rho : \rho = 4\pi n + 4\pi \sum_{i \in I} (1 + \alpha_i),\ n \in \mathbb{N},\ I \subseteq \{1, \ldots, m\}\right\} \subseteq \mathbb{R}_+. \tag{1.9}$$

On the other hand, if ρ does not belong to this set, solutions have to stay compact and variational methods can be applied. Recall that in the supercritical regime the Euler–Lagrange energy is unbounded from below. However there is a way to describe how the lower bounds fail, in terms of concentration of conformal volume.

It turns out that the multiplicative constant in (1.8) improves if the conformal volume *spreads* over Σ; see Lemma 2.1. Having a better constant implies more chances to bound the energy from below, and therefore a low energy forbids too much spreading of the volume. Suppose that all the weights α_i are negative. Localizing (1.8) via cut-off functions near a regular or a singular point, we find in the denominator the value 4π or $4\pi(1+\alpha_i)$, respectively. This suggests to introduce a *weighted cardinality* χ on points of Σ as follows: Set

$$\begin{cases} \chi(q) = 4\pi & \text{if } q \in \Sigma \setminus \{p_1, \dots, p_m\}, \\ \chi(p_i) = 4\pi(1+\alpha_i) & \text{for all } i = 1, \dots, m. \end{cases}$$

Define also

$$\mathcal{P}(\Sigma) = \{\mu : \mu \text{ is a probability measure on } \Sigma\}.$$

As the total curvature we have at hand is ρ, this *counting* suggests that the limit measures for small energy should be the following:

$$\Sigma_{\rho,\underline{\alpha}} = \{\mu \in \mathcal{P}(\Sigma) : 4\pi\chi(\operatorname{supp}(\sigma)) < \rho\}.$$

Without singularities, such spaces coincide with the measures supported on a given number of points (depending on ρ), and are useful in studying problems in higher dimensions or of higher order, see, e.g., [3, 35, 36, 51].

For simplicity, we will assume here the following upper bound on ρ:

$$\rho < 4\pi \min\{1, \min_{i\neq j}(2+\alpha_i+\alpha_j)\}. \tag{1.10}$$

In this case, by setting

$$\mathcal{A} = \{p_i : 4\pi(1+\alpha_i) < \rho\}, \tag{1.11}$$

one can check that $\Sigma_{\rho,\underline{\alpha}}$ takes the simple form

$$\Sigma_{\rho,\underline{\alpha}} = \{\delta_p : p \in \mathcal{A}\}.$$

In this case, one has the following result, whose statement becomes rather simple.

Theorem 1.2 ([20], particular case). *Suppose that $\alpha_i < 0$ for all i and that ρ satisfies* (1.10). *Then* (1.5) *admits a solution provided* $\operatorname{card}(\mathcal{A}) > 1$.

While the main result in [20] is more general and deals with sets $\Sigma_{\rho,\underline{\alpha}}$ of arbitrary structure (see also [5, 53]), the proof of existence is rather simple to explain under the assumptions of Theorem 1.2, and we will treat only this case in the present paper. See also [24] for an existence result relying on degree theory.

We next discuss the *singular Toda system* arising in Chern–Simons theory, which represents a non-abelian counterpart of (1.2). Specifically, we consider the following system:

$$\begin{cases} -\Delta u_1 = 2\rho_1\Big(\dfrac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1}\,dV_g} - 1\Big) - \rho_2\Big(\dfrac{h_2 e^{u_2}}{\int_\Sigma h_2 e^{u_2}\,dV_g} - 1\Big) - 4\pi \displaystyle\sum_{j=1}^m \alpha_{1,j}(\delta_{p_j} - 1), \\ -\Delta u_2 = 2\rho_2\Big(\dfrac{h_2 e^{u_2}}{\int_\Sigma h_2 e^{u_2}\,dV_g} - 1\Big) - \rho_1\Big(\dfrac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1}\,dV_g} - 1\Big) - 4\pi \displaystyle\sum_{j=1}^m \alpha_{2,j}(\delta_{p_j} - 1), \end{cases} \tag{1.12}$$

where h_1, h_2 are smooth positive functions on Σ, and the coefficients $\alpha_{i,j}$ are again larger than -1. In geometry, (1.12) describes Frenet frames of holomorphic curves in $\mathbb{CP}^n$, see [14, 18, 28], with the p_i's standing for *ramification points* of the curves. From the physical point of view, abelian Chern–Simons vortices have been quite well studied for some time, see, e.g., [16, 21, 57, 62], while the treatment of the non-abelian case is more recent, see, e.g., [37, 44, 45, 58].

With a change of variable similar to (1.4) the latter problem transforms into

$$\begin{cases} -\Delta u_1 = 2\rho_1\Big(\dfrac{\widetilde{h}_1 e^{u_1}}{\int_\Sigma \widetilde{h}_1 e^{u_1}\,dV_g} - 1\Big) - \rho_2\Big(\dfrac{\widetilde{h}_2 e^{u_2}}{\int_\Sigma \widetilde{h}_2 e^{u_2}\,dV_g} - 1\Big), \\ -\Delta u_2 = 2\rho_2\Big(\dfrac{\widetilde{h}_2 e^{u_2}}{\int_\Sigma \widetilde{h}_2 e^{u_2}\,dV_g} - 1\Big) - \rho_1\Big(\dfrac{\widetilde{h}_1 e^{u_1}}{\int_\Sigma \widetilde{h}_1 e^{u_1}\,dV_g} - 1\Big), \end{cases} \tag{1.13}$$

where the functions $\widetilde{h}_i$ satisfy

$$\widetilde{h}_i > 0 \text{ on } \Sigma \setminus \{p_1, \dots, p_m\}, \quad \widetilde{h}_i(x) \simeq d(x, p_j)^{2\alpha_{i,j}} \text{ near } p_j, \quad i = 1, 2. \tag{1.14}$$

As for the scalar case, one gains the variational structure with the Euler–Lagrange functional

$$J_{\rho,\underline{\alpha}}(u_1, u_2) = \int_\Sigma Q(u_1, u_2)\, dV_g + \sum_{i=1}^{2} \rho_i \Big(\int_\Sigma u_i \, dV_g - \log \int_\Sigma \widetilde{h}_i e^{u_i}\, dV_g \Big), \tag{1.15}$$

where $Q(u_1, u_2)$ is defined as

$$Q(u_1, u_2) = \frac{1}{3}(|\nabla u_1|^2 + |\nabla u_2|^2 + \nabla u_1 \cdot \nabla u_2).$$

For the regular Toda system a sharp Moser–Trudinger inequality was found in [43] (see also [29, 69] for other Liouville systems), where it was shown that

$$4\pi \sum_{i=1}^{2} \log \int_\Sigma e^{u_i - \overline{u}_i}\, dV_g \le \int_\Sigma Q(u_1, u_2)\, dV_g + C, \quad u \in H^1(\Sigma). \tag{1.16}$$

Notice that one always has the inequality $Q(u_1, u_2) \ge \frac{1}{4}|\nabla u_1|^2$, and hence (1.16) can be thought of as an extension of (1.7). Our next goal is to introduce the following theorem, which extends both (1.8) and (1.16).

Theorem 1.3 ([11]). *Suppose $p_1, \dots, p_m \in \Sigma$ and let $\alpha_{i,j}$, $i = 1, 2$, $j = 1, \dots, m$, satisfy $\alpha_{i,j} > -1$ for all i, j. Then, if the $\widetilde{h}_i$ satisfy* (1.14), *the following inequality holds:*

$$4\pi \sum_{i=1}^{2} \min\{1, 1 + \min_j \alpha_{i,j}\} \log \int_\Sigma \widetilde{h}_i e^{u_i - \overline{u}_i}\, dV_g \le \int_\Sigma Q(u_1, u_2)\, dV_g + C, \quad u_1, u_2 \in H^1(\Sigma). \tag{1.17}$$

The constants in the above inequality are sharp.

The above result is a first step of a variational attack for the study of (1.2). In the recent paper [10], the case of non-negative coefficients and positive genus has been treated by using simply inequality (1.16), as the corresponding functions $\widetilde{h}_i$ are uniformly bounded (see also [52, 54, 59] for the regular case and [4] for the scalar singular case). Inequality (1.17) is indeed needed in a general situation.

It is possible, by using blow-up analysis, to show that inequality (1.17) holds for any smaller couple of coefficients on the left-hand side, and moreover that there exist extremal functions for the corresponding Euler functionals (1.15). This is what we will present in this paper. One can then pass to the limit for these extremals when the parameters approach the critical ones. The presence of singularities causes a variety of blow-up behaviors (different blow-up rates for the two components, and blow-up at regular or singular points): these can be reduced to two cases only, by using a Pohozaev identity from the recent paper [48].

The above reasoning in terms of volume concentration for the scalar singular equation (see the comments before Theorem 1.2) allows to prove a related alternative for the two components of the system. As a counterpart of (1.10), we define

$$\overline{\rho}_1 := 4\pi \min\{1, \min_{m \neq m'} (2 + \alpha_{1m} + \alpha_{1m'})\}, \quad \overline{\rho}_2 := 4\pi \min\{1, \min_{m \neq m'} (2 + \alpha_{2m} + \alpha_{2m'})\}, \tag{1.18}$$

and suppose that $\rho_i < \overline{\rho}_i$. Define also

$$\mathcal{A}_i = \{p_j : 4\pi(1 + \alpha_{i,j}) < \rho_i\}, \quad i = 1, 2. \tag{1.19}$$

Under the above condition on the ρ_i's it turns out that for $J_{\rho,\underline{\alpha}}(u)$ low either $\widetilde{h}_1 e^{u_1}$ concentrates near a singular point in $\mathcal{A}_1$ or $\widetilde{h}_2 e^{u_2}$ concentrates near a singular point in $\mathcal{A}_2$. To express this (non-exclusive) alternative, it is natural to introduce the *join* of two topological spaces X and Y (see for instance [40]):

$$X \star Y := \frac{X \times Y \times [0, 1]}{\sim}, \tag{1.20}$$

where $\sim$ is the equivalence relation among triples (x, y, t) given by

$$(x, y, 0) \sim (x, y', 0) \text{ for all } x \in X,\ y, y' \in Y, \quad (x, y, 1) \sim (x', y, 1) \text{ for all } x, x' \in X,\ y \in Y.$$

The join of the sets $\mathcal{A}_1$ and $\mathcal{A}_2$ could then be used to characterize low-energy levels of $J_{\rho,\underline{\alpha}}$, with the join parameter $s \in [0, 1]$ expressing whether $\widetilde{h}_1 e^{u_1}$ is distributionally closer to a Dirac mass or whether $\widetilde{h}_2 e^{u_2}$ is closer to a Dirac mass (for example $s = \frac{1}{2}$ would describe couples with the same scale of concentration).

This description is however not optimal in general, as it does not take into account the interaction between two components u_1 and u_2 via the mixed term in the quadratic form Q, which penalizes aligned gradients. For the regular case of (1.12), in [54] it was shown that the *relative rate of concentration* of the two components plays a role in this matter.

It turns out that if u_1, u_2 concentrate near the same point and with the same scale (a more precise definition is given below), then the Moser–Trudinger constants for the system double. As a consequence of this fact, it follows that when $\rho_1, \rho_2 \in (4\pi, 8\pi)$ and no singularities occur, then join elements of the form $(x, x, \frac{1}{2}r)$, $x \in \Sigma$ have to be excluded (see [41] for higher values of ρ_1). We will present a new improved inequality from [12] for the singular system (1.12) in order to understand at the same time the effect of the interaction of the two components between themselves and with the singularities. As a consequence of this improved inequality, we deduce the following theorem, which to be stated needs the definition of the counterpart to (1.9): Let $\Gamma'_{i,\mathcal{M}} \subset \mathbb{R}_+$ be defined, for $i = 1, 2$ and $\mathcal{M} \subset \{1, \dots, m\}$, by

$$\Gamma'_{i,\mathcal{M}} := 4\pi\Big\{n + \sum_{j' \in \mathcal{M}'} (1 + \alpha_{ij'}) + \sum_{j \in \mathcal{M}} (2 + \alpha_{1j} + \alpha_{2j}) : n \in \mathbb{N},\ \mathcal{M}' \subset \{1, \dots, m\} \setminus \mathcal{M}\Big\}$$

and define also

$$\Gamma_{\underline{\alpha}_1,\underline{\alpha}_2} = \bigcup_{\mathcal{M} \subset \{1,\dots,M\}} \Big(\Gamma'_{1,\mathcal{M}} \times \Big[\sum_{j \in \mathcal{M}} 4\pi(1 + \alpha_{2j}), +\infty\Big) \cup \Big[\sum_{j \in \mathcal{M}} 4\pi(1 + \alpha_{1j}), +\infty\Big) \times \Gamma'_{2,\mathcal{M}}\Big). \tag{1.21}$$

We then set

$$\Gamma = \Gamma_{\underline{\alpha}_1,\underline{\alpha}_2}.$$

Theorem 1.4 ([12]). *Let* Γ *be as in* (1.21), *let* $(\overline{\rho}_1, \overline{\rho}_2)$ *be as in* (1.18) *and let* $\rho \in \mathbb{R}^2_+ \setminus \Gamma$ *satisfy* $\rho_i < \overline{\rho}_i$ *for both* $i = 1, 2$. *Define integer numbers* M_1, M_2, M_3 *by*

$$\begin{aligned} M_1 &:= \#\{j : 4\pi(1 + \alpha_{1j}) < \rho_1\}, \\ M_2 &:= \#\{j : 4\pi(1 + \alpha_{2j}) < \rho_2\}, \\ M_3 &:= \#\{j : 4\pi(1 + \alpha_{ij}) < \rho_i \textit{ and } \rho_i < 4\pi(2 + \alpha_{1j} + \alpha_{2j}) \textit{ for both } i = 1, 2\}. \end{aligned}$$

Then system (1.12) *admits solutions provided the following condition holds:*

$$(M_1, M_2, M_3) \notin \{(1, j, 0), (j, 1, 0), (2, 2, 1), (2, 3, 2), (3, 2, 2),\ j \in \mathbb{N}\}.$$

By the previous description, low sub-levels of $J_{\rho,\underline{\alpha}}$ can be identified with the topological join of $\mathcal{A}_1$ and of $\mathcal{A}_2$, with some points removed. Under the assumptions on the ρ_i's, this join consists of a graph $\mathcal{X}$ made of segments whose end-points belong to $\{p_1, \dots, p_m\}$. The conditions on (M_1, M_2, M_3) in the previous theorem ensure that this graph is non-contractible. It turns out that the above assumptions on the M_i's are necessary: in fact, in [12] a non-existence result for every case not covered by the theorem is proved.

2 Variational Aspects of Singular Liouville Equations: Improved Moser–Trudinger Inequalities

In this section, we treat problem (1.2) via variational methods. We first show that, in a regime where coercivity fails, low energy implies volume concentration at suitable points. We will then use this characterization to build min-max schemes leading to the existence of solutions.

We describe how improved Moser–Trudinger inequalities can be employed to deduce information on functions whose Euler–Lagrange energy is small enough. We would like to give some conditions on a function u in order to obtain lower bounds on the energy even when we are beyond the coercivity threshold. Indeed, the *spreading* of the function e^{2u} over the surface gives sufficient conditions to obtain this lower bound, deduced via some *improvement* of the Moser–Trudinger inequality. Two well-known examples were due to Moser [56] and Aubin [2]. Moser proved that one can replace $\frac{1}{4\pi}$ by $\frac{1}{8\pi}$ on the standard sphere (S^2, g_{S^2}) provided u is antipodally symmetric. Aubin showed instead that on (S^2, g_{S^2}) one can take any constant larger than $\frac{1}{8\pi}$ provided u is *balanced*, which means that

$$\int_{S^2} x_i e^{2u}\, dV_{S^2} = 0, \quad i = 1, 2, 3.$$

Here x_i stand for the Euclidean i-th coordinate function, so the balancing condition means having zero center of mass in $\mathbb{R}^3$ for the conformal volume.

Chen and Li [25] extended this argument to arbitrary surfaces by showing that if e^{2u} has integral bounded from below into two separate subsets of Σ, then the constant $\frac{1}{4\pi}$ in (1.7) can be basically divided by two. The result was then extended in [35, 36] for an arbitrary number of spreading regions.

Lemma 2.1 ([25]). *Let Ω_1, Ω_2 be subsets of Σ satisfying $d_g(\Omega_1, \Omega_2) \geq \delta_0$, where δ_0 is a positive real number, and let $\gamma_0 \in (0, \frac{1}{2})$. Then for any $\tilde{\varepsilon} > 0$ there exists a constant $C = C(\tilde{\varepsilon}, \delta_0, \gamma_0)$ such that*

$$\log \int_\Sigma e^{2(u-\overline{u})}\, dV_g \leq C + \frac{1}{8\pi - \tilde{\varepsilon}} \int_\Sigma |\nabla_g u|^2\, dV_g$$

for all the functions $u \in H^1(\Sigma)$ satisfying

$$\frac{\int_{\Omega_i} e^{2u}\, dV_g}{\int_\Sigma e^{2u}\, dV_g} \geq \gamma_0 \quad \textit{for all } i = 1, 2.$$

Proof. Assume without loss of generality that $\overline{u} = 0$: one can find two functions g_1, g_2 such that

$$\begin{cases} g_i(x) \in [0, 1] & \text{for every } x \in \Sigma, \\ g_i(x) = 1 & \text{for every } x \in \Omega_i,\ i = 1, 2, \\ g_i(x) = 0 & \text{if } d_g(x, \Omega_i) \geq \dfrac{\delta_0}{4}, \\ \|g_i\|_{C^4(\Sigma)} \leq C_{\delta_0}, \end{cases}$$

where C_{δ_0} is a positive constant depending only on δ_0. By interpolation, for any $\varepsilon > 0$ there exists C_{ε,δ_0} (depending only on ε and δ_0) such that for any $v \in H^1(\Sigma)$ and for any $i = 1, 2$ we have

$$\int_\Sigma |\nabla_g(g_i v)|^2\, dV_g \leq \int_\Sigma g_i^2 |\nabla_g v|^2\, dV_g + \varepsilon \int_\Sigma |\nabla_g v|^2\, dV_g + C_{\varepsilon,\delta_0} \int_\Sigma v^2\, dV_g. \tag{2.1}$$

We next notice that

$$\int_\Sigma e^{2u}\, dV_g \leq \frac{1}{\gamma_0} \int_{\Omega_i} e^{2u}\, dV_g \leq \int_\Sigma e^{2g_i u}\, dV.$$

Using the standard Moser–Trudinger inequality, we find

$$\log \int_\Sigma e^{2u}\, dV_g \leq \log \frac{1}{\gamma_0} + \frac{1}{4\pi} \sum_{i=1}^{2} \int_\Sigma |\nabla_g(g_i v)|^2\, dV_g + \sum_{i=1}^{2} \overline{g_i u} + C_{l,\Sigma,g}.$$

By (2.1) we then deduce

$$\log \int_\Sigma e^{2u}\, dV_g \leq \log \frac{1}{\gamma_0} + \frac{1+\varepsilon}{4\pi} \sum_{i=1}^{2} \int_\Sigma |\nabla_g u|^2\, dV_g + \sum_{i=1}^{2} \overline{g_i u} + C_{l,\Sigma,g} C_{\varepsilon,\delta_0} \int_\Sigma v^2\, dV_g.$$

Since we are assuming the average of u to be zero, the average terms in the last formula are bounded by a constant times the Dirichlet norm of u by Poincaré's inequality. Therefore, using the elementary inequality $t \le \varepsilon t^2 + \frac{1}{4\varepsilon}$, we find that

$$\log \int_\Sigma e^{2u}\, dV_g \le \log \frac{1}{\gamma_0} + \frac{1+\varepsilon}{4\pi} \sum_{i=1}^{2} \int_\Sigma |\nabla_g u|^2\, dV_g + C_{l,\Sigma,g,\varepsilon} + C_{\varepsilon,\delta_0} \int_\Sigma v^2\, dV_g.$$

It can be shown, for example by using truncations (in height or in Fourier modes), that the last term is of lower order and it can be absorbed into the Dirichlet energy multiplied by an arbitrarily small constant, concluding the proof. □

We discuss next the counterpart of the previous result in presence of singularities, for which we recall the inequality in Proposition 1.1. Again, we wish to derive some improved inequalities in terms of the spreading of the function

$$\tilde{f}_u := \frac{\tilde{h}(x)e^{2u}}{\int_\Sigma \tilde{h}(x)e^{2u}\, dV},$$

appearing in the singular Euler–Lagrange energy.

Similarly to Lemma 2.1, inequality (1.8) can also be localized. If some portion of $\tilde{f}_u$ is localized near a regular point, the corresponding gain in the constant will still be 4π. If instead $\tilde{f}_u$ is localized near a singular point p_i with negative weight α_i, we will gain locally a quantity of size $4\pi(1+\alpha_i)$. One therefore gets the following result.

Lemma 2.2 ([20]). *Let $n \in \mathbb{N}$, let $I \subseteq \{1,\dots,m\}$ with $n + \operatorname{card}(I) > 0$, and let $\alpha_i < 0$ for all $i \in I$. Assume there exist $r > 0$, $\delta_0 > 0$ and pairwise distinct points $\{q_1,\dots,q_n\} \subseteq \Sigma \setminus \{p_1,\dots,p_m\}$ such that the following conditions hold:*

- *For any couple $\{a,b\} \subseteq \{q_1,\dots,q_n \cup (\bigcup_{i\in I} p_i)\}$ with $a \neq b$ one has $d_g(B_r(a), B_r(b)) \ge 4\delta_0$.*
- *For any $a \in \{q_1,\dots,q_m\}$ and any $i \in \{1,\dots,m\} \setminus I$ one has $d_g(p_i, B_r(a)) \ge 4\delta_0$.*

Consider any $\gamma_0 \in (0, \frac{1}{n+\operatorname{card}(I)})$.

Then for any $\tilde{\varepsilon} > 0$ there exists a constant $C := C(\Sigma, g, n, I, r, \delta_0, \gamma_0, \tilde{\varepsilon})$ such that

$$\log \int_\Sigma \tilde{h} e^{2(u-\overline{u})}\, dV_g \le \frac{1}{4\pi(n + \sum_{i\in I}(1+\alpha_i) - \tilde{\varepsilon})} \int_\Sigma |\nabla_g u|^2\, dV_g + C$$

for all functions $u \in H^1(\Sigma)$ satisfying

$$\frac{\int_{B_r(a)} \tilde{h} e^{2u}\, dV_g}{\int_\Sigma \tilde{h} e^{2u}\, dV_g} \ge \gamma_0 \quad \text{for all } a \in \{q_1,\dots,q_n \cup (\textstyle\bigcup_{i\in I} p_i)\}.$$

The above lemma states that the more the *conformal volume* is spread over the surface, the more one gains in the Moser–Trudinger inequality, especially when some volume accumulates near regular points. In this situation, one then get lower bounds on the energy even in supercritical regimes. Therefore, if the energy gets low enough, one should expect concentration of volume. We next state two lemmas making this reasoning rigorous via a covering procedure.

Lemma 2.3. *Let $\ell \in \mathbb{N}$ and fix two positive numbers ε and r. Suppose for a non-negative function $f \in L^1(\Sigma)$ with $\|f\|_{L^1(\Sigma)} = 1$ that the following condition holds:*

$$\int_{\bigcup_{i=1}^{\ell} B_r(q_i)} f\, dV_g < 1 - \varepsilon \quad \text{for every } \ell\text{-tuple } q_1,\dots,q_\ell \in \Sigma.$$

Then there exist $\overline{\varepsilon} > 0$ and $\overline{r} > 0$, depending only on ε, r, ℓ, and Σ (but not on f), and $\ell + 1$ points $\overline{q}_1,\dots,\overline{q}_{\ell+1} \in \Sigma$ (which depend on f) satisfying

$$\int_{B_{\overline{r}}(\overline{q}_1)} f\, dV_g \ge \overline{\varepsilon}, \dots, \int_{B_{\overline{r}}(\overline{q}_{\ell+1})} f\, dV_g \ge \overline{\varepsilon}, \quad B_{2\overline{r}}(\overline{q}_i) \cap B_{2\overline{r}}(\overline{q}_j) = \emptyset \text{ for } i \neq j.$$

Proof. Arguing by contradiction, we assume that for every $\overline{\varepsilon}, \overline{r} > 0$ there exists f as in the statement and such that for every $(\ell+1)$-tuple of points $q_1, \dots, q_{\ell+1}$ in Σ we have

$$\int\limits_{B_{\overline{r}}(q_j)} f\, dV_g \geq \overline{\varepsilon} \text{ for all } j = 1, \dots, \ell+1 \quad \Longrightarrow \quad B_{2\overline{r}}(q_i) \cap B_{2\overline{r}}(q_j) \neq \emptyset \text{ for some } i \neq j. \tag{2.2}$$

Let $\overline{r} = \frac{r}{8}$, where r is given in the statement. We can find $h \in \mathbb{N}$ and h points $x_1, \dots, x_h \in \Sigma$ such that $\bigcup_{i=1}^{h} B_{\overline{r}}(x_i)$ covers Σ. For ε as in the statement of the lemma, we also define $\overline{\varepsilon} = \frac{\varepsilon}{2h}$. We remark that the choice of $\overline{r}$ and $\overline{\varepsilon}$ depends on r, ε, ℓ, and Σ only, as required.

Let $\{\tilde{x}_1, \dots, \tilde{x}_j\} \subseteq \{x_1, \dots, x_h\}$ denote the points for which $\int_{B_{\overline{r}}(\tilde{x}_i)} f\, dV_g \geq \overline{\varepsilon}$. Define $\tilde{x}_{j_1} = \tilde{x}_1$ and let A_1 denote the set

$$A_1 = \Big\{ \bigcup_i B_{\overline{r}}(\tilde{x}_i) : B_{2\overline{r}}(\tilde{x}_i) \cap B_{2\overline{r}}(\tilde{x}_{j_1}) \neq \emptyset \Big\} \subseteq B_{4\overline{r}}(\tilde{x}_{j_1}).$$

If there exists $\tilde{x}_{j_2}$ with $B_{2\overline{r}}(\tilde{x}_{j_2}) \cap B_{2\overline{r}}(\tilde{x}_{j_1}) = \emptyset$, we set

$$A_2 = \Big\{ \bigcup_i B_{\overline{r}}(\tilde{x}_i) : B_{2\overline{r}}(\tilde{x}_i) \cap B_{2\overline{r}}(\tilde{x}_{j_2}) \neq \emptyset \Big\} \subseteq B_{4\overline{r}}(\tilde{x}_{j_2}).$$

Proceeding in this way, we choose recursively points $\tilde{x}_{j_3}, \tilde{x}_{j_4}, \dots, \tilde{x}_{j_s}$ such that

$$B_{2\overline{r}}(\tilde{x}_{j_s}) \cap B_{2\overline{r}}(\tilde{x}_{j_a}) = \emptyset \quad \text{for all } 1 \leq a < s,$$

and introduce sets $A_3, \dots, A_s$ by

$$A_k = \Big\{ \bigcup_i B_{\overline{r}}(\tilde{x}_i) : B_{2\overline{r}}(\tilde{x}_i) \cap B_{2\overline{r}}(\tilde{x}_{j_k}) \neq \emptyset \Big\} \subseteq B_{4\overline{r}}(\tilde{x}_{j_k}), \quad k = 3, \dots, s.$$

Because of (2.2), the process cannot go further than $\tilde{x}_{j_\ell}$, and hence $s \leq \ell$. Using the definition of $\overline{r}$, we obtain

$$\bigcup_{i=1}^{j} B_{\overline{r}}(\tilde{x}_i) \subseteq \bigcup_{i=1}^{s} A_i \subseteq \bigcup_{i=1}^{s} B_{4\overline{r}}(\tilde{x}_{j_i}) \subseteq \bigcup_{i=1}^{s} B_r(\tilde{x}_{j_i}). \tag{2.3}$$

Then by our choice of h, $\overline{\varepsilon}$, $\{\tilde{x}_1, \dots, \tilde{x}_j\}$ and by (2.3), one has

$$\int\limits_{\Sigma \setminus \bigcup_{i=1}^{s} B_r(\tilde{x}_{j_i})} f\, dV_g \leq \int\limits_{\Sigma \setminus \bigcup_{i=1}^{j} B_{\overline{r}}(\tilde{x}_i)} f\, dV_g \leq \int\limits_{(\bigcup_{i=1}^{h} B_{\overline{r}}(x_i)) \setminus (\bigcup_{i=1}^{j} B_{\overline{r}}(\tilde{x}_i))} f\, dV_g < (h-j)\overline{\varepsilon} \leq \frac{\varepsilon}{2}.$$

Finally, if we choose $q_i = \tilde{x}_{j_i}$ for $i = 1, \dots, s$ and $q_i = \tilde{x}_{j_s}$ for $i = s+1, \dots, \ell$, we get a contradiction to the assumptions of the lemma. □

Using Lemmas 2.1 and 2.3, we can analyse the volume concentration for functions with large negative energy, showing that it has to concentrate near at most one singular point in $\mathcal{A}$, see (1.11).

Lemma 2.4. *Suppose the assumptions of Theorem 1.2 hold true. Then for any $\varepsilon > 0$ and any $r > 0$ there exists a large $L = L(\varepsilon, r)$ such that for every $u \in H^1(\Sigma)$ with $I_{\rho,\underline{\alpha}}(u) \leq -L$ there exist $p_i \in \mathcal{A}$ such that*

$$\frac{1}{\int_\Sigma \tilde{h}(x) e^{2u}\, dV_g} \int\limits_{\Sigma \setminus B_r(p_i)} \tilde{h}(x) e^{2u}\, dV_g < \varepsilon.$$

Proof. We first claim that the conformal volume of functions with low energy must concentrate near a single point. Indeed, suppose by contradiction that there exist $\varepsilon, r > 0$ and $(u_n)_n \subseteq H^1(\Sigma)$ with $I_{\rho,\underline{\alpha}}(u_n) \to -\infty$ and such that for every point $q \in \Sigma$ one has

$$\int\limits_{\bigcup_{i=1}^{k} B_r(q_i)} e^{u_n}\, dV_g < 1 - \varepsilon.$$

Noting that $I_{\rho,\underline{\alpha}}$ is invariant under adding constants, we can assume that for every n we have the normalization $\int_\Sigma \tilde{h}(x)e^{2u_n}\,dV_g = 1$. Then we can apply Lemma 2.3 with $\ell = 1$ and $f = \tilde{h}(x)e^{2u_n}$, and afterwards Lemma 2.2 with $n + \mathrm{card}(I) = 2$ and δ_0, γ_0 sufficiently small (depending on r and ε) to obtain

$$I_{\rho,\underline{\alpha}}(u_n) \geq \int_\Sigma |\nabla_g u_n|^2\,dV_g + 2\rho\int_\Sigma \tilde{a}u_n\,dV_g - C\rho - \frac{\rho}{4\pi\min_{i\neq j}(2+\alpha_i+\alpha_j)-\tilde{\varepsilon}}\int_\Sigma |\nabla_g u_n|^2\,dV_g - \rho\overline{u}_n,$$

with C and $\tilde{\varepsilon}$ independent of n ($\tilde{\varepsilon}$ arbitrarily small). Since we are assuming $\rho < 4\pi\min_{i\neq j}(2+\alpha_i+\alpha_j)$, we can choose $\tilde{\varepsilon} > 0$ so small that

$$1 - \frac{\rho}{4\pi\min_{i\neq j}(2+\alpha_i+\alpha_j)-\tilde{\varepsilon}} =: \delta > 0.$$

Hence using also the Poincaré inequality, we find

$$\begin{aligned} I_{\rho,\underline{\alpha}}(u_n) &\geq \delta\int_\Sigma |\nabla_g u_n|^2\,dV_g + 2\rho\int_\Sigma \tilde{a}(u_n - \overline{u}_n)\,dV_g - C\rho \\ &\geq \delta\int_\Sigma |\nabla_g u_n|^2\,dV_g - C\Big(\int_\Sigma |\nabla_g u_n|^2\,dV_g\Big)^{\frac{1}{2}} - C\rho \geq -C. \end{aligned}$$

This lower bound contradicts the fact that $I_{\rho,\underline{\alpha}}(u_n) \to -\infty$, and proves our claim.

To conclude the proof, we must show that the volume concentrates near a singular point in $\mathcal{A}$. In order to show this, it is sufficient to argue as before and still apply Lemma 2.2 with $I = \emptyset$ and $n = 1$, using the fact that $\rho < 4\pi$ and that the local Moser–Trudinger constant is bigger than ρ if the singular concentration point does not belong to $\mathcal{A}$. □

By Lemma 2.4, it follows that if the Euler–Lagrange energy is low enough, then the function $\tilde{h}(x)e^{2u}$, normalized in L^1, is localized near at most one singular point of Σ. Choosing ε and r sufficiently small, one can easily see that the point p_i in the statement of Lemma 2.4 must be unique, and therefore we obtain a canonical map from low-energy levels into the set of singular points.

Proposition 2.5. *Under the assumptions of Theorem 1.2 there exist a large constant L and a continuous map $\Psi : \{I_{\rho,\underline{\alpha}} \leq -L\} \to \mathcal{A}$ such that if $I_{\rho,\underline{\alpha}}(u_n) \to -\infty$, then*

$$\frac{1}{\int_\Sigma \tilde{h}(x)e^{2u_n}\,dV_g}\tilde{h}(x)e^{2u} \rightharpoonup \delta_{\Psi(u_n)}$$

as $n \to \infty$.

Remark 2.6. If $\mathrm{card}(\mathcal{A}) > 1$, since $\mathcal{A}$ is discrete, from the continuity of Ψ it follows that $\{I_{\rho,\underline{\alpha}} \leq -L\}$ is disconnected for L sufficiently large.

We will see in the next section how to construct a sort of *inverse map* to Ψ, which will allow to prove the existence of solutions via suitable min-max schemes.

3 Proof of Theorem 1.2

In this section, we prove our first existence result, concerning singular Liouville equations.

3.1 Test Function Estimates

For each singular point p_i and $\lambda > 0$ define the function

$$\varphi_{i,\lambda}(x) = \log\frac{\lambda}{\big(1+\lambda^2 d(x,x_i)^{2(1+\alpha_i)}\big)}.$$

This function satisfies the following properties.

Proposition 3.1. *Let p_i be a singular point in $\mathcal{A}$. Then as $\lambda \to +\infty$, the following properties hold true:*

(i) *We have*

$$\frac{1}{\int_\Sigma \tilde{h}(x)e^{2\varphi_{i,\lambda}}\,dV_g}\tilde{h}(x)e^{2\varphi_{i,\lambda}} \rightharpoonup \delta_{p_i}$$

weakly in the sense of distributions.

(ii) $I_{\rho,\underline{\alpha}}(\varphi_{i,\lambda}) \to -\infty$.

Proof. To prove (i) we first notice that since $\tilde{h}(x) \simeq d(x,p_i)^{2\alpha_i}$, one has

$$\tilde{h}(x)e^{2\varphi_{i,\lambda}(x)} \geq \frac{\lambda^{-2\alpha_i}\lambda^2}{\left(1+\lambda^2 d(x,x_i)^{2(1+\alpha_i)}\right)^2} \geq C^{-1}\frac{\lambda^{-2\alpha_i}\lambda^2}{(\lambda^2\lambda^{-2(1+\alpha_i)}r)^2} \geq C^{-1}\lambda^2 \quad \text{in } B_{2\lambda^{-1}(p_i)} \setminus B_{\lambda^{-1}(p_i)}.$$

Integrating this, we obtain

$$\int_\Sigma \tilde{h}(x)e^{2\varphi_{i,\lambda}}\,dV_g \geq \int_{B_{2\lambda^{-1}(p_i)}\setminus B_{\lambda^{-1}(p_i)}} \tilde{h}(x)e^{2\varphi_{i,\lambda}}\,dV_g \geq C^{-1}. \tag{3.1}$$

On the other hand, one has that

$$\int_{\Sigma\setminus B_{2\lambda^{-1}(p_i)}} \tilde{h}(x)e^{2\varphi_{i,\lambda}}\,dV_g \leq C\int_{2\lambda^{-1}}^{\infty} \frac{r^{2\alpha_i}\lambda^2}{(1+\lambda^2 r^{2(1+\alpha_i)})^2}\,r\,dr.$$

By the change of variables $\lambda r^{1+\alpha_i} = s^{1+\alpha_i}$, we then obtain

$$\int_{\Sigma\setminus B_{2\lambda^{-1}(p_i)}} \tilde{h}(x)e^{2\varphi_{i,\lambda}}\,dV_g \leq C\int_{2\lambda^{-\frac{\alpha_i}{1+\alpha_i}}}^{\infty} \frac{s^{2\alpha_i+1}}{(1+s^{2(1+\alpha_i)})^2}\,ds \to 0$$

as $\lambda \to +\infty$. Property (i) follows from the latter formula and (3.1).

To show (ii), we prove the following estimates:

$$\rho\int_\Sigma \varphi_{i,\lambda}\,dV_g = -(\rho + o_\lambda(1))\log\lambda \quad (o_\lambda(1) \to 0 \text{ as } \lambda \to +\infty), \tag{3.2}$$

$$\int_\Sigma |\nabla_g\varphi_{i,\lambda}|^2\,dV_g \leq 8(1+\alpha_i)\pi(1+o_\lambda(1))\log\lambda \quad \text{as } \lambda \to +\infty. \tag{3.3}$$

Once we have these, (ii) follows immediately.

Proof of (3.2). Fixing any $\delta > 0$ small, we have that

$$\log\frac{\lambda}{1+\lambda^2\operatorname{diam}(\Sigma)^{2(1+\alpha_i)}} \leq \varphi_{i,\lambda}(y) \leq \log\frac{\lambda}{1+\lambda^2\delta^{2(1+\alpha_i)}}, \quad y \in \Sigma \setminus B_\delta(p_i),$$

and

$$\log\frac{\lambda}{1+\lambda^2\delta^{2(1+\alpha_i)}} \leq \varphi_{i,\lambda}(y) \leq \log\lambda \quad \text{for } y \in B_{2\delta}(p_i).$$

From these two estimates and some elementary computations we deduce that

$$\rho\int_\Sigma \varphi_{i,\lambda}\,dV_g = \rho\big[(-\log\lambda)(1+O(\delta^2)) + O(1) + O(\delta^2)(|\log\lambda| + |\log\delta|)\big]$$

as $\lambda \to +\infty$. By the arbitrarity of δ, estimate (3.2) follows.

Proof of (3.3). We will show the following two pointwise estimates on the gradient of $\varphi_{i,\lambda}$:

$$|\nabla\varphi_{i,\lambda}(y)| \leq C\lambda \quad \text{for every } y \in \Sigma, \tag{3.4}$$

where C is a constant independent of σ and λ, and

$$|\nabla \varphi_{i,\lambda}(y)| \le \frac{2(1+\alpha_i)}{d(y,p_i)}. \tag{3.5}$$

To check (3.4) we notice that

$$\frac{\lambda^2 d(y,p_i)}{1+\lambda^2 d(y,p_i)^{2(1+\alpha_i)}} \le C\lambda^{\frac{1}{1+\alpha_i}}, \tag{3.6}$$

where C is a fixed constant (independent of λ). Moreover,

$$\nabla \varphi_{i,\lambda}(y) = -2(1+\alpha_i)\lambda^2 \frac{d(y,p_i)^{1+2\alpha_i}\nabla_y(d_i(y))}{(1+\lambda^2 d(y,p_i)^{2(1+\alpha_i)})}. \tag{3.7}$$

Using $|\nabla_y d_i(y))| \le 1$ and inserting (3.6) into (3.7), we obtain immediately (3.4). Similarly, erasing the term 1 from the denominator, we deduce (3.5).

Estimate (3.4) then implies

$$\int_{B^*(p_i)} |\nabla_g \varphi_{i,\lambda}|^2 \, dV_g \le C, \quad \text{where } B^* := B_{\frac{1}{\lambda^{1/(1+\alpha_i)}}} \tag{3.8}$$

for some fixed C depending only on Σ and α_i. On the other hand, using polar coordinates centred at p_i and using (3.5), we find that

$$\int_{\Sigma \setminus B^*(p_i)} |\nabla_g \varphi_{i,\lambda}|^2 \, dV_g \le 4 \int_{\Sigma \setminus B^*(p_i)} \frac{(1+\alpha_i)^2}{d_g(y,x_i)^2} \, dV_g \le 8\pi(1+o_\lambda(1))(1+\alpha_i)\log\lambda$$

as $\lambda \to +\infty$. From (3.8) and the last formula we finally deduce (3.3). □

3.2 Min-Max Scheme and Existence

We next introduce a variational scheme for obtaining the existence of solutions for (1.5). First, let L be so large that Proposition 2.5 applies with $\frac{L}{4}$, and then choose two distinct points p_{i_1}, p_{i_2} such that $4\pi(1+\alpha_i) < \rho$.

Next choose $\overline{\lambda}$ be so large that $I_{\rho,\underline{\alpha}}(\varphi_{i_1,\overline{\lambda}}) \le -L$ and $I_{\rho,\underline{\alpha}}(\varphi_{i_2,\overline{\lambda}}) \le -L$ (see Proposition 3.1 (ii)). Fixing this value of $\overline{\lambda}$, we define the family of maps

$$\Pi_{\overline{\lambda}} = \{\varpi : [0,1] \to H^1(\Sigma) : \varpi \text{ is continuous and } \varpi(0) = \varphi_{i_1,\overline{\lambda}}, \varpi(1) = \varphi_{i_2,\overline{\lambda}}\}. \tag{3.9}$$

Lemma 3.2. *The family $\Pi_{\overline{\lambda}}$ is non-empty. Moreover, letting*

$$\overline{\Pi}_{\overline{\lambda}} = \inf_{\varpi \in \Pi_{\overline{\lambda}}} \sup_{t \in [0,1]} I_{\rho,\underline{\alpha}}(\varpi(t)),$$

we have

$$\overline{\Pi}_{\overline{\lambda}} > -\frac{L}{2}.$$

Proof. To show that $\Pi_{\overline{\lambda}} \ne \emptyset$, it suffices to consider the map

$$\overline{\varpi}(t) = (1-t)\varphi_{i_1,\overline{\lambda}} + t\varphi_{i_2,\overline{\lambda}}. \tag{3.10}$$

Arguing by contradiction, we suppose that $\overline{\Pi}_{\overline{\lambda}} \le -\frac{L}{2}$. Then there would exist a map $\varpi \in \Pi_{\overline{\lambda}}$ with

$$\sup_{t\in[0,1]} I_{\rho,\underline{\alpha}}(\varpi(t)) \le -\frac{3}{8}L.$$

Since by our choice of L Proposition 2.5 applies with $\frac{L}{4}$, the composition

$$t \mapsto \Psi \circ \varpi(t)$$

is well defined and continuous. However, by Proposition 3.1 (i) and Proposition 2.5 one has that

$$\Psi \circ \varpi(0) = p_1, \quad \Psi \circ \varpi(1) = p_2,$$

which contradicts the continuity of this map. □

By the statement of Lemma 3.2 and standard variational arguments, one can find a *Palais–Smale sequence* $(u_n)_n$ for $I_{\rho,\underline{\alpha}}$ at level $\overline{\Pi}_{\overline{\lambda}}$, namely a sequence for which

$$I_{\rho,\underline{\alpha}}(u_n) \to \overline{\Pi}_{\overline{\lambda}}, \quad I'_\rho(u_n) \to 0.$$

Unfortunately it is not known whether Palais–Smale sequences admit converging subsequences. To show this property, recall first that $u \mapsto e^{2u}$ is compact from $H^1(\Sigma)$ to $L^p(\Sigma)$, which by Hölder's inequality implies the compactness of $u \mapsto \tilde{h}(x)e^{2u}$ from $H^1(\Sigma)$ into $L^1(\Sigma)$. Therefore, it would be sufficient to show that a Palais–Smale sequence is bounded.

This indeed can be proved indirectly, following an argument in [60], by slightly modifying the value of the parameter ρ. We choose a small $\rho_0 > 0$, and allow ρ to vary in the interval $[1-\rho_0, 1+\rho_0]$. We consider then the functional $I_{\rho,\underline{\alpha}}$ for these values of ρ. If ρ_0 is sufficiently small, the interval $[1-\rho_0, 1+\rho_0]$ will be compactly contained in the complement of the set $\mathfrak{S}$; see (1.9). Following the previous estimates with minor changes, we easily check that the min-max scheme applies uniformly for $\rho \in [1-\rho_0, 1+\rho_0]$ and for $\overline{\lambda}$ sufficiently large. Precisely, given any large $L > 0$, there exist ρ_0 sufficiently small and $\overline{\lambda}$ so large that for $\rho \in [1-\rho_0, 1+\rho_0]$ we have

$$\sup_{t\in\{0,1\}} I_{\rho,\underline{\alpha}}(\varpi(m)) < -2L, \quad \overline{\Pi}_\rho := \inf_{\varpi\in\Pi_{\overline{\lambda}}} \sup_{t\in[0,1]} I_{\rho,\underline{\alpha}}(\varpi(t)) > -\frac{L}{2}, \tag{3.11}$$

where $\Pi_{\overline{\lambda}}$ is defined in (3.9). Moreover, by using for example the test map (3.10), one shows that for ρ_0 sufficiently small there exists a large constant $\overline{L}$ such that

$$\overline{\Pi}_\rho \leq \overline{L} \quad \text{for every } \rho \in [1-\rho_0, 1+\rho_0].$$

We have the following result, regarding the dependence in ρ of the min-max value $\overline{\Pi}_\rho$; see [34].

Lemma 3.3. *Let $\overline{\lambda}$ be so large and ρ_0 be so small that* (3.11) *holds. Then the function*

$$\rho \mapsto \overline{\Pi}_\rho/\rho \quad \textit{is non-increasing in } [1-\rho_0, 1+\rho_0].$$

Proof. For $\rho' \geq \rho$ we have

$$\frac{I_{\rho,\underline{\alpha}}(u)}{\rho} - \frac{I_{\rho'}(u)}{\rho'} = \frac{1}{2}\Big(\frac{1}{\rho} - \frac{1}{\rho'}\Big) \int_\Sigma |\nabla_g u|^2 \, dV_g \geq 0,$$

which clearly implies $\overline{\Pi}_\rho/\rho \geq \overline{\Pi}_{\rho'}/\rho'$. □

From Lemma 3.3 we deduce that the function $\rho \mapsto \overline{\Pi}_\rho/\rho$ is differentiable almost everywhere, and we obtain the following corollary.

Corollary 3.4. *Let $\overline{\lambda}$ and ρ_0 be as in Lemma 3.3, and let $\Lambda \subset [1-\rho_0, 1+\rho_0]$ be the (dense) set of ρ for which the function $\overline{\Pi}_\rho/\rho$ is differentiable. Then for $\rho \in \Lambda$ the functional $I_{\rho,\underline{\alpha}}$ possesses a bounded Palais–Smale sequence $(u_l)_l$ at level $\overline{\Pi}_\rho$, weakly converging to a solution of* (1.5).

Proof. The existence of a Palais–Smale sequence $(u_l)_l$ follows from Lemma 3.2, and the boundedness is proved exactly as in [34, Lemma 3.2]. □

From the above result we obtained a sequence $\rho_k \to \rho$ such that $I_{\rho_k,\underline{\alpha}}$ has a critical point.

Theorem 3.5 ([6–8]). *Let Σ be a compact surface and let u_i solve* (1.5) *with $\tilde{h}$ as in* (1.6), *$\rho = \rho_i$, $\rho_i \to \overline{\rho}$. Suppose that $\int_\Sigma f_{u_i} \, dV_g \leq \overline{C}$ for some fixed $\overline{C} > 0$. Then along a subsequence u_{i_k} one of the following alternatives holds:*
(i) *u_{i_k} is uniformly bounded from above on Σ.*
(ii) *$\max_\Sigma(2u_{i_k} - \log \int_\Sigma f_{u_{i_k}} \, dV_g) \to +\infty$ and there exists a finite blow-up set $S = \{q_1, \dots, q_l\} \in \Sigma$ such that*
 (a) *for any $s \in \{1, \dots, l\}$ there exist $x_k^s \to q_s$ such that $u_{i_k}(x_k^s) \to +\infty$ and $u_{i_k} \to -\infty$ uniformly on the compact sets of $\Sigma \setminus S$,*

(b) $\rho_{i_k}\tilde{f}_{u_{i_k}} \rightharpoonup \sum_{s=1}^{l}\beta_s\delta_{q_s}$ *in the sense of measures, with* $\beta_s = 4\pi$ *for* $q_s \neq \{p_1, \ldots, p_m\}$, *or* $\beta_s = 4\pi(1+\alpha_j)$ *if* $q_s = p_j$ *for some* $j = \{1, \ldots, m\}$. *In particular, one has that*

$$\overline{\rho} = 4\pi n + 4\pi\sum_{j\in J}(1+\alpha_j),$$

for some $n \in \mathbb{N}\cup 0$ *and* $J \subseteq \{1, \ldots, m\}$ *(possibly empty) satisfying* $n + \operatorname{card}(J) > 0$.

Proof of Theorem 1.2. By Corollary 3.4 there exists a sequence $\rho_k \to \rho$ such that $I_{\rho_k,\underline{\alpha}}$ has a critical point u_k. By Theorem 3.5, since $\rho \notin \mathfrak{S}$, the sequence u_k must then converge to a solution of (1.2). □

We also refer to [34, 61] for previous results on surfaces with positive genus concerning the regular case of (1.2). The above method can actually be used to find multiplicity results as well for generic data; see [30, 31].

4 A Moser–Trudinger Inequality for Singular Systems

In this section, we are going to prove the following Moser–Trudinger-type inequality. It is a weaker version of Theorem 1.3, but anyway sufficient to prove the existence result Theorem 1.4. We denote by x^- the negative part of a real number x, that is,

$$x^- := \begin{cases} 0 & \text{if } x \geq 0, \\ -x & \text{if } x \leq 0. \end{cases}$$

For $i \in \{1, 2\}$ we set

$$\widetilde{\alpha}_i = -\max_{j\in\{1,\ldots,m\}} \alpha_{i,j}{}^-. \tag{4.1}$$

Proposition 4.1. *Let* Σ *be a closed surface with area* $|\Sigma| = 1$, *let* $\widetilde{h}_i$ *be as in* (1.14) *and let* $\widetilde{\alpha}_i$ *be as in* (4.1). *Then for any* $\rho = (\rho_1, \rho_2) \in \mathbb{R}^2_+$ *satisfying* $\rho_i < 4\pi(1+\widetilde{\alpha}_i)$ *for both* $i \in \{1, 2\}$ *there exists* $C(\rho) > 0$ *such that the Euler–Lagrange functional* (1.15) *verifies*

$$J_{\rho,\underline{\alpha}}(u) > -C(\rho) \quad \text{for all } u \in H^1(\Sigma)^2.$$

Definition 4.2. As in [43], we define the set of admissible parameters Λ as

$$\Lambda := \{\rho \in \mathbb{R}^2_+ : J_{\rho,\underline{\alpha}} \text{ is bounded from below}\}.$$

Clearly, Λ preserves the partial order of $\mathbb{R}^2_+$, that is if $\rho \in \Lambda$, then $\widetilde{\rho} \in \Lambda$ until $\widetilde{\rho}_i \leq \rho_i$ for both $i \in \{1, 2\}$; in these terms, Proposition 4.1 is equivalent to saying

$$(0, 4\pi(1+\widetilde{\alpha}_1)) \times (0, 4\pi(1+\widetilde{\alpha}_2)) \subset \Lambda.$$

Remark 4.3. One can easily see that Λ is not empty: since there holds

$$\frac{|\nabla u_1|^2 + |\nabla u_2|^2}{6} \leq Q(u_1, u_2),$$

one can apply the scalar Moser–Trudinger inequality (1.8) to both components to get

$$\left(0, \frac{8}{3}\pi(1+\widetilde{\alpha}_1)\right) \times \left(0, \frac{8}{3}\pi(1+\widetilde{\alpha}_2)\right) \subset \Lambda.$$

To prove Proposition 4.1, some lemmas will be needed. First of all, we notice that when the parameter ρ is in the interior of the set Λ, then the energy functional is not only bounded from below, but even coercive and it has a minimizer; on the other hand, if ρ is on the boundary of Λ, then $J_{\rho,\underline{\alpha}}$ cannot be coercive.

Lemma 4.4. *For any $\rho \in \mathring{\Lambda}$ there exists a constant C such that*

$$J_{\rho,\underline{\alpha}}(u) \geq \frac{\int_\Sigma (|\nabla u_1|^2 + |\nabla u_2|^2)\,dV_g}{C} - C.$$

Moreover, $J_{\rho,\underline{\alpha}}$ admits a minimizer $u = (u_1, u_2)$ that solves (1.13).

Proof. Taking $\delta \in (0, \frac{d(\rho,\partial\Lambda)}{\sqrt{2}})$, we have $(1+\delta)\rho \in \Lambda$, so $J_{(1+\delta)\rho,\underline{\alpha}}(u) \geq -C$. Therefore, we can write

$$\begin{aligned} J_{\rho,\underline{\alpha}}(u) &= \frac{\delta}{1+\delta}\int_\Sigma Q(u_1,u_2)\,dV_g + \frac{J_{(1+\delta)\rho,\underline{\alpha}}(u)}{1+\delta} \\ &\geq \frac{\delta}{6(1+\delta)}\int_\Sigma (|\nabla u_1|^2 + |\nabla u_2|^2)\,dV_g - C \end{aligned}$$

and the first claim follows.

To prove the rest, we notice that if we restrict ourselves to the subset of $H^1(\Sigma)^2$ consisting of all functions satisfying $\int_\Sigma \widetilde{h}_i e^{u_i}\,dV_g = 1$, the energy is coercive since, from Poincaré's inequality and (1.8) we have

$$\begin{aligned} \int_\Sigma u_i^2\,dV_g &= \int_\Sigma (u_i - \overline{u_i})^2\,dV_g + (\overline{u_i})^2 \\ &\leq C\int_\Sigma |\nabla u_i|^2\,dV_g + \left(C + \frac{1}{16\pi(1+\widetilde{\alpha}_i)}\int_\Sigma |\nabla u_i|^2\,dV_g\right)^2 \\ &\leq C\left(1 + \int_\Sigma |\nabla u_i|^2\,dV_g\right)^2. \end{aligned}$$

By $J_{\rho,\underline{\alpha}}$ being weakly lower semi-continuous as well, the existence of a minimizer follows from the direct methods of calculus of variations. □

Lemma 4.5. *For any $\rho \in \partial\Lambda$ there exists a sequence $\{\widetilde{u}_k\}_{k\in\mathbb{N}} \subset H^1(\Sigma)^2$ verifying*

$$\int_\Sigma (|\nabla \widetilde{u}_{1,k}|^2 + |\nabla \widetilde{u}_{2,k}|^2)\,dV_g \xrightarrow{k\to+\infty} +\infty, \qquad \lim_{k\to+\infty} \frac{J_{\rho,\underline{\alpha}}(\widetilde{u}_k)}{\int_\Sigma (|\nabla \widetilde{u}_{1,k}|^2 + |\nabla \widetilde{u}_{2,k}|^2)\,dV_g} \leq 0.$$

Proof. Suppose by contradiction that

$$\int_\Sigma (|\nabla u_{1,k}|^2 + |\nabla u_{2,k}|^2)\,dV_g \xrightarrow{k\to+\infty} +\infty \quad \Longrightarrow \quad \frac{J_{\rho,\underline{\alpha}}(u_k)}{\int_\Sigma (|\nabla u_{1,k}|^2 + |\nabla u_{2,k}|^2)\,dV_g} \geq \theta > 0$$

for any choice of $\{u_k\}$. This would mean that

$$J_{\rho,\underline{\alpha}}(u) \geq \frac{\theta}{2}\int_\Sigma (|\nabla u_1|^2 + |\nabla u_2|^2)\,dV_g - C.$$

Hence for any small δ we would get

$$\begin{aligned} J_{(1+\delta)\rho,\underline{\alpha}}(u) &= (1+\delta)J_{\rho,\underline{\alpha}}(u) - \delta\int_\Sigma Q(u_1,u_2)\,dV_g \\ &\geq \left((1+\delta)\frac{\theta}{2} - \frac{\delta}{2}\right)\int_\Sigma (|\nabla u_1|^2 + |\nabla u_2|^2)\,dV_g - C \\ &\geq -C. \end{aligned}$$

Thus $(1+\delta)\rho \in \Lambda$, whereas one clearly has $(1-\delta)\rho \in \Lambda$; this is in contradiction to $\rho \in \partial\Lambda$. □

Next, we need a basic calculus lemma. Its proof will be omitted, as it can be found in [43] (following an idea of Ding).

Lemma 4.6 ([43, Lemma 4.4]). *Let $\{a_k\}_{k\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{N}}$ be two sequences of real numbers satisfying*

$$a_k \xrightarrow{k\to+\infty} +\infty \quad \text{and} \quad \lim_{k\to+\infty} \frac{b_k}{a_k} \le 0.$$

Then there exists a smooth function $F : [0,+\infty) \to \mathbb{R}$ satisfying, up to subsequences,

$$0 < F'(t) < 1 \text{ for any } t \ge 0, \quad F'(t) \xrightarrow{t\to+\infty} 0, \quad F(a_k) - b_k \xrightarrow{k\to+\infty} +\infty.$$

Lemma 4.6 will be applied to the sequences

$$a_k = \int_\Sigma Q(\tilde{u}_{1,k}, \tilde{u}_{2,k})\, dV_g, \quad b_k = J_{\rho,\underline{\alpha}}(\tilde{u}_k),$$

where $\tilde{u}_k$ is as in Lemma 4.5, and we will consider the auxiliary functional

$$\tilde{J}_{\rho,\underline{\alpha}}(u) := J_{\rho,\underline{\alpha}}(u) - F\Big(\int_\Sigma Q(u_1,u_2)\, dV_g\Big),$$

whose behavior is described by the following lemma.

Lemma 4.7. *For any $\rho \in \mathring{\Lambda}$ the functional $\tilde{J}_{\rho,\underline{\alpha}}$ is bounded from below on $H^1(\Sigma)^2$ and its infimum is achieved by a function satisfying*

$$\begin{cases} -\Big(1 - \dfrac{2}{3}\mathfrak{g}(u)\Big)\Delta u_i + \dfrac{\mathfrak{g}(u)}{3}\Delta u_{3-i} = 2\rho_i(\tilde{h}_i e^{u_i} - 1) - \rho_{3-i}(\tilde{h}_{3-i}e^{u_{3-i}} - 1), \\ \displaystyle\int_\Sigma \tilde{h}_i e^{u_i}\, dV_g = 1, \end{cases}$$

where $\mathfrak{g}(u) = F'(\int_\Sigma Q(u_1,u_2)\, dV_g)$. On the other hand, if $\rho \in \partial\Lambda$, then $\inf_{H^1(\Sigma)^2} \tilde{J}_{\rho,\underline{\alpha}} = -\infty$.

Proof. For $\rho \in \mathring{\Lambda}$ one can argue as in Lemma 4.4, yielding lower semi-continuity from the regularity of F and coercivity from the behavior of F' at infinity.

For $\rho \in \partial\Lambda$, taking $\tilde{u}_k$ as in Lemma 4.5 and applying Lemma 4.6, we get

$$\tilde{J}_{\rho,\underline{\alpha}}(\tilde{u}_k) = b_k - F(a_k) \xrightarrow{k\to+\infty} -\infty.$$

This concludes the proof. □

Next, we need the following two results. The first one is from [11], and its proof is rather similar to [43, Theorem 3.1].

Theorem 4.8 ([11]). *Let $\tilde{h}_i$ be as in (1.14), let $u_k = (u_{1,k}, u_{2,k}) \in H^1(\Sigma)^2$ be solutions of*

$$\begin{cases} 7 - 0\Delta u_{i,k} = 2V_{i,k}\tilde{h}_i e^{u_{i,k}} - V_{3-i,k}\tilde{h}_{3-i}e^{u_{3-i,k}} + \psi_{i,k}, \\ \displaystyle\int_\Sigma \tilde{h}_i e^{u_{i,k}}\, dV_g \le C, \\ \|\psi_{i,k}\|_{L^p(\Sigma)} \le C, \\ V_{i,k} \xrightarrow{k\to+\infty} 1 \text{ in } L^\infty(\Sigma), \end{cases}$$

$i \in \{1,2\}$, for some $p > 1$, $C > 0$ and define the sets S_i as

$$S_i := \Big\{p \in \Sigma : \text{there exists } x_k \xrightarrow{k\to+\infty} p \text{ such that } u_{i,k}(x_k) \xrightarrow{k\to+\infty} +\infty\Big\}.$$

Then, after taking subsequences, one of the following alternatives occurs:

(i) *For each $i \in \{1,2\}$ either $u_{i,k}$ is bounded in $L^\infty(\Sigma)$ or it tends uniformly to $-\infty$.*

(ii) $S_i \neq \emptyset$ *for some* $i \in \{1,2\}$. *In this case,* S_i *is finite and either* $u_{j,k}$ *is bounded in* $L^\infty_{\mathrm{loc}}(\Sigma \setminus (S_1 \cup S_2))$ *or it converges to* $-\infty$ *in* $L^\infty_{\mathrm{loc}}(\Sigma \setminus (S_1 \cup S_2))$ *for each* $j \in \{1,2\}$. *Moreover, if* $S_i \setminus S_{3-i} \neq \emptyset$, *then the latter alternative occurs for* $u_{i,k}$.

Theorem 4.9 ([48, Proposition 3.1]). *Let* $u_k = (u_{1,k}, u_{2,k}) \in H^1(\Sigma)^2$ *be solutions of* (1.13), *let* $\hat{\alpha}_i(p)$ *stand for* 0 *if* p *is regular and for* $\alpha_{i,j}$ *if* $p = p_j$. *Define also*

$$\sigma_i(p) := \lim_{r\to 0} \lim_{k\to+\infty} \int_{B_r(p)} \widetilde{h}_i e^{u_{i,k}}\, dV_g.$$

Then one has

$$\sigma_1(p)^2 - \sigma_1(p)\sigma_2(p) + \sigma_2(p)^2 = 4\pi(1 + \hat{\alpha}_1(p))\sigma_1(p) + 4\pi(1 + \hat{\alpha}_2(p))\sigma_2(p). \tag{4.2}$$

We are now in a position to prove the main result of this section.

Proof of Proposition 4.1. Suppose by contradiction that

$$(0, 4\pi(1 + \widetilde{\alpha}_1)) \times (0, 4\pi(1 + \widetilde{\alpha}_2)) \not\subset \Lambda;$$

then there is some $\overline{\rho} \in \partial\Lambda$ with $\overline{\rho}_i < 4\pi(1 + \widetilde{\alpha}_i)$ for both $i \in \{1,2\}$.

Consider a sequence $\{\rho_k\}_{k\in\mathbb{N}} \in \mathring{\Lambda}$ with $\rho_k \to \overline{\rho}$, as $k \to +\infty$, and a minimizer u_k for $J_{\rho_k,\underline{\alpha}}$, as in Lemma 4.7. Then $v_k := u_k + \log \rho_k$ solves

$$\begin{cases} -\Delta v_{i,k} = 2\dfrac{6 - 5\mathfrak{g}(v_k)}{6 - 8\mathfrak{g}(v_k) + 2\mathfrak{g}(v_k)^2}(\widetilde{h}_i e^{v_{i,k}} - \rho_{i,k}) - \dfrac{3 - 4\mathfrak{g}(v_k)}{3 - 4\mathfrak{g}(v_k) + \mathfrak{g}(v_k)^2}(\widetilde{h}_{3-i} e^{v_{3-i,k}} - \rho_{3-i,k}), \\ \displaystyle\int_\Sigma \widetilde{h}_i e^{v_{i,k}}\, dV_g = \rho_{i,k}, \end{cases}$$

with

$$\frac{6 - 5\mathfrak{g}(v_k)}{6 - 8\mathfrak{g}(v_k) + 2\mathfrak{g}(v_k)^2} \quad \text{and} \quad \frac{3 - 4\mathfrak{g}(v_k)}{3 - 4\mathfrak{g}(v_k) + \mathfrak{g}(v_k)^2}$$

both uniformly converging to 1, so Theorem 4.8 can be applied to this sequence. The normalization on the integral implies that $u_{i,k}$ cannot tend to $-\infty$ for any $i \in \{1,2\}$; moreover, we can also exclude boundedness in $L^\infty(\Sigma)$ because this would imply convergence to a minimizer $\overline{u}$ of $J_{\overline{\rho},\underline{\alpha}}$, contradicting Lemma 4.7.

The only case left is the blow-up around at least one point p: Pohozaev's identity (4.2) implies that if there is a singularity of mass $\alpha_{i,j}$ on p, then $\sigma_i \geq 4\pi(1 + \alpha_{i,j})$ for some $i \in \{1,2\}$, whereas if p is a regular point, then there is a component with a mass of at least 4π around it. In both cases, for such an i we obtain

$$4\pi(1 + \widetilde{\alpha}_i) \leq \lim_{r\to 0} \lim_{k\to+\infty} \int_{B_r(p)} \widetilde{h}_i e^{v_{i,k}}\, dV_g \leq \lim_{k\to+\infty} \int_\Sigma \widetilde{h}_i e^{v_{i,k}}\, dV_g = \overline{\rho}_i < 4\pi(1 + \widetilde{\alpha}_i),$$

which is a contradiction. □

We conclude the section by showing a partial converse of Theorem 4.1, namely that for higher values of the parameter ρ the functional $J_{\rho,\underline{\alpha}}$ is unbounded from below. Estimates of this type will be needed in Section 5 and are in the spirit of Proposition 3.1.

Proposition 4.10. *If* $\rho_i > 4\pi(1 + \widetilde{\alpha}_i)$ *for some* $i \in \{1,2\}$, *then* $\inf_{H^1(\Sigma)^2} J_{\rho,\underline{\alpha}} = -\infty$, *that is,*

$$\Lambda \subset (0, 4\pi(1 + \widetilde{\alpha}_1)] \times (0, 4\pi(1 + \widetilde{\alpha}_2)].$$

Proof. We will give the proof for $i = 1$; it is nearly identical for $i = 2$.

Choosing a point p_1 such that $\hat{\alpha}_1(p_1) = \widetilde{\alpha}_i$, we define for large λ the functions

$$\varphi_{1,\lambda}(x) = \log\Big(\frac{\lambda^{1+\widetilde{\alpha}_1}}{1 + (\lambda d(x, p_1))^{2(1+\widetilde{\alpha}_1)}}\Big)^2, \quad \varphi_{2,\lambda}(x) = -\frac{1}{2}\log\Big(\frac{\lambda^{1+\widetilde{\alpha}_1}}{1 + (\lambda d(x, p_1))^{2(1+\widetilde{\alpha}_1)}}\Big)^2.$$

Using the fact that

$$\left|\nabla(d(x,p_1)^{2(1+\tilde{\alpha}_1)})\right| \le 2(1+\tilde{\alpha}_1)d(x,p_1)^{1+2\tilde{\alpha}_1},$$

we obtain

$$|\nabla\varphi_{1,\lambda}(x)| = \left|\frac{-2\lambda^{2(1+\tilde{\alpha}_1)}|\nabla(d(x,p_1)^{2(1+\tilde{\alpha}_1)})|}{1+(\lambda d(x,p_1))^{2(1+\tilde{\alpha}_1)}}\right| \le \frac{4(1+\tilde{\alpha}_1)\lambda^{2(1+\tilde{\alpha}_1)}d(x,p_1)^{1+2\tilde{\alpha}_1}}{1+(\lambda d(x,p_1))^{2(1+\tilde{\alpha}_1)}} \le \min\left\{C\lambda, \frac{4(1+\tilde{\alpha}_1)}{d(x,p_1)}\right\},$$

and therefore

$$\begin{aligned}\int_\Sigma Q(\varphi_{1,\lambda},\varphi_{2,\lambda})\,dV_g &= \frac{1}{4}\int_\Sigma |\nabla\varphi_{1,\lambda}|^2\,dV_g \\ &\le C\lambda^2 \int_{B_{\frac{1}{\lambda}}(p_1)} dV_g + 4(1+\tilde{\alpha}_1)^2 \int_{\Sigma\setminus B_{\frac{1}{\lambda}}(p_1)} \frac{dV_g}{d(\cdot,p_1)^2} \\ &\le C + 8\pi(1+\tilde{\alpha}_1)^2\log\lambda.\end{aligned}$$

Moreover, by

$$\max\{1,(\lambda d(x,p_1))^{2(1+\tilde{\alpha}_1)}\} \le 1+(\lambda d(x,p_1))^{2(1+\tilde{\alpha}_1)} \le C\max\{1,(\lambda d(x,p_1))^{2(1+\tilde{\alpha}_1)}\} \tag{4.3}$$

one gets the following estimate on the average of $\varphi_{1,\lambda}$:

$$\overline{\varphi_{1,\lambda}} = \int_\Sigma \left(\max\{2(1+\tilde{\alpha}_1)\log\lambda, -2(1+\tilde{\alpha}_1)(\log\lambda + 2\log d(\cdot,p_1))\} + O(1)\right) dV_g.$$

Further, dividing Σ into the two regions where the above maximum is attained and using the integrability of $\log d(\cdot,p_1)$ in two dimensions, we get

$$\begin{aligned}\overline{\varphi_{1,\lambda}} &= 2(1+\tilde{\alpha}_1)\log\lambda \int_{B_{\frac{1}{\lambda}}(p_1)} dV_g - 2(1+\tilde{\alpha}_1)\log\lambda \int_{\Sigma\setminus B_{\frac{1}{\lambda}}(p_1)} dV_g - 4(1+\tilde{\alpha}_1)\int_{\Sigma\setminus B_{\frac{1}{\lambda}}(p_1)} \log d(\cdot,p_1)\,dV_g + O(1) \\ &= -2(1+\tilde{\alpha}_1)\log\lambda + O(1),\end{aligned} \tag{4.4}$$

and clearly also

$$\overline{\varphi_{2,\lambda}} = (1+\tilde{\alpha}_1)\log\lambda + O(1).$$

For a small but fixed $\delta > 0$ we have

$$\begin{aligned}\int_\Sigma \tilde{h}_1 e^{\varphi_{1,\lambda}}\,dV_g &\ge C \int_{B_\delta(p_1)\setminus B_{\frac{1}{\lambda}}(p_1)} d(\cdot,p_1)^{2\tilde{\alpha}_1} e^{\varphi_{1,\lambda}}\,dV_g \\ &\ge \frac{C}{\lambda^{2(1+\tilde{\alpha}_1)}} \int_{B_\delta(p_1)\setminus B_{\frac{1}{\lambda}}(p_1)} \frac{dV_g}{d(\cdot,p_1)^{4+2\tilde{\alpha}_1}} \ge C,\end{aligned} \tag{4.5}$$

again by (4.3). On the other hand, we can write

$$\int_\Sigma \tilde{h}_2 e^{\varphi_{2,\lambda}}\,dV_g \ge C\lambda^{1+\tilde{\alpha}_1} \int_{\Sigma\setminus B_{\frac{1}{\lambda}}(p_1)} \tilde{h}_2 d(\cdot,p_1)^{2(1+\tilde{\alpha}_1)}\,dV_g \ge C\lambda^{1+\tilde{\alpha}_1}. \tag{4.6}$$

Therefore, from (3.7) and (4.4)–(4.6) we conclude that

$$J_{\rho,\underline{\alpha}}(u) \le 2(1+\tilde{\alpha}_1)(4\pi(1+\tilde{\alpha}_1)-\rho_1)\log\lambda + O(1) \xrightarrow{\lambda\to\infty} -\infty,$$

as desired. □

5 Improved Vectorial Inequalities

First of all, we recall the following result from [9], which extends Lemma 2.2 to the vectorial case.

Lemma 5.1 ([9, Lemma 4.3]). *Let $\delta > 0, J_1, K_1, J_2, K_2 \in \mathbb{N}$ be given, let*

$$\{m_{11}, \dots, m_{1J_1}, m_{21}, \dots, m_{2J_2}\} \subset \{1, \dots, m\} \quad \text{and} \quad \{\Omega_{ij}\}_{i=1,2}^{j=1,\dots,J_i+K_i}$$

be open subsets of Σ such that

$$\begin{aligned} \alpha_{im_{ij}} &\le 0 \quad \text{for all } i = 1, 2,\ j = 1, \dots, J_i, \\ d(\Omega_{ij}, \Omega_{ij'}) &\ge \delta \quad \text{for all } i = 1, 2,\ j, j' = 1, \dots, J_i + K_i,\ j \neq j', \\ d(p_j, \Omega_{ij}) &\ge \delta \quad \text{for all } i = 1, 2,\ j = 1, \dots, K_i + M_i \text{ and all } j = 1, \dots, m,\ j \neq m_{ij}, \end{aligned}$$

and let $u \in H^1(\Sigma)^2$ satisfy

$$\int_{\Omega_{ij}} f_{i,u}\, dV_g \ge \delta \quad \text{for all } i = 1, 2,\ j = 1, \dots, J_i + K_i.$$

Then for any $\varepsilon > 0$ there exists $C = C_{\Sigma,\delta,J_1,K_1,J_2,K_2,\varepsilon} > 0$ such that

$$4\pi \sum_{i=1}^{2} \Big(K_i + \sum_{j=1}^{J_i} (1 + \alpha_{im_{ij}})\Big)\Big(\log \int_\Sigma \tilde{h}_i e^{u_i}\, dV_g - \int_\Sigma u_i\, dV_g\Big) \le (1 + \varepsilon) \int_\Sigma Q(u)\, dV_g + C.$$

Next, we will consider some improved functional inequalities that take into account the relative *speeds of concentration* of the two components of the system. Let us first set

$$\mathcal{L} := \Big\{f \in L^1(\Sigma) : f > 0 \text{ a.e. in } \Sigma,\ \int_\Sigma f\, dV_g = 1\Big\}. \tag{5.1}$$

We will define, for each $f \in \mathcal{L}$, a *center of mass* and a *scale of concentration*, inspired by [54, Proposition 3.1] but such that the center of mass belongs to a given finite set $\mathcal{F} \subset \Sigma$ (which will be, in our applications, a subset of the singular points). As in [54], we will map $\mathcal{L}$ onto the topological cone over $\mathcal{F}$ of height δ, which is defined by

$$\mathcal{C}_\delta \mathcal{F} := \frac{\mathcal{F} \times [0, \delta]}{\sim}, \tag{5.2}$$

where the equivalence relation $\sim$ is given by $(x, \delta) \sim (x', \delta)$ for any $x \in \Sigma$. The meaning of such an identification is the following: if a function $f \in \mathcal{L}$ does not concentrate around any point $x \in \mathcal{F}$, then we may not be able to define a center of mass. In this case, the equivalence relation in the definition of the cone leaves it undetermined.

Lemma 5.2. *Let $\mathcal{F} := \{x_1, \dots, x_K\} \subset \Sigma$ be a given finite set and let $\mathcal{L}$, $\mathcal{C}_\delta$ be defined by* (5.1) *and* (5.2)*. Then for $\delta > 0$ small enough there exists a map $\psi = (\beta, \varsigma) = (\beta_{\mathcal{F}}, \varsigma_{\mathcal{F}}) : \mathcal{L} \to \mathcal{C}_\delta \mathcal{F}$ such that the following hold:*

- *If $\varsigma(f) = \delta$, then either*

$$\int_{\Sigma \setminus \bigcup_{x \in \mathcal{F}} B_\delta(x)} f\, dV_g \ge \delta$$

or there exist $x', x'' \in \mathcal{F}$ with $x' \neq x''$ and

$$\int_{B_\delta(x')} f\, dV_g \ge \delta, \qquad \int_{B_\delta(x'')} f\, dV_g \ge \delta.$$

- *If $\varsigma(f) < \delta$, then*

$$\int_{B_{\varsigma(f)}(\beta(f))} f\, dV_g \ge \delta, \qquad \int_{\Sigma \setminus B_{\varsigma(f)}(\beta(f))} f\, dV_g \ge \delta.$$

Moreover, if $f^n \to \delta_x$ as $n \to +\infty$ for some $x \in \mathcal{F}$, then $(\beta(f^n), \varsigma(f^n)) \to (x, 0)$ as $n \to +\infty$.

Proof. Fix $\tau \in (\frac{1}{2}, 1)$, take

$$\delta \le \frac{\min_{x,x' \in \mathcal{F},\, x \neq x'} d(x, x')}{2}$$

and define, for $k = 1, \dots, K$,

$$I_k(f) := \int_{B_\delta(x_k)} f \, dV_g, \quad I_0(f) := \int_{\Sigma \setminus \bigcup_{x \in \mathcal{F}} B_\delta(x)} f \, dV_g = 1 - \sum_{k=1}^{k} I_k(f).$$

Choose now indices $\widetilde{k}$, $\widehat{k}$ such that

$$I_{\widetilde{k}}(f) := \max_{k \in \{0,\dots,K\}} I_k(f), \quad I_{\widehat{k}}(f) := \max_{k \neq \widetilde{k}} I_k(f).$$

We will define the map ψ depending on $\widetilde{k}$ and $I_{\widetilde{k}}(f)$:

$\widetilde{k} = 0$: Since f has little mass around each of the points x_k, we set $\varsigma(f) = \delta$ and do not define $\beta(f)$, as it would be irrelevant by the equivalence relation in (5.2). The assertion of the Lemma is verified, up to taking a smaller δ, because

$$\int_{\Sigma \setminus \bigcup_{x \in \mathcal{F}} B_\delta(x)} f \, dV_g = I_0(f) \ge \frac{1}{K+1} \ge \delta.$$

$\widetilde{k} \ge 1$, $I_{\widetilde{k}}(f) \le \frac{K\tau}{1-\tau} I_{\widehat{k}}(f)$: Here, f has still little mass around the point $x_{\widetilde{k}}$ (which could not be uniquely defined), so again we set $\varsigma(f) := \delta$. It is easy to see that $I_{\widehat{k}}(f) \ge \frac{1-\tau}{K}$, so

$$\int_{B_\delta(x_{\widetilde{k}})} f \, dV_g \ge \int_{B_\delta(x_{\widehat{k}})} f \, dV_g \ge \frac{1-\tau}{K}.$$

$\widetilde{k} \ge 1$, $I_{\widetilde{k}}(f) \ge \frac{K\tau}{1-\tau} I_{\widehat{k}}(f)$: Now, $I_{\widetilde{k}}(f) > \tau$, so one can define a scale of concentration $s(x_{\widetilde{k}}, f) \in (0, \delta)$ of f around $x_{\widetilde{k}} \in \mathcal{F}$, uniquely determined by

$$\int_{B_{s(x_{\widetilde{k}}, f)}(x_{\widetilde{k}})} f \, dV_g = \tau.$$

We can also define a center of mass $\beta(f) = x_{\widetilde{k}}$ but we have to interpolate for the scale:

Subcase $I_{\widetilde{k}}(f) \le \frac{2K\tau}{1-\tau} I_{\widehat{k}}(f)$: Setting

$$\varsigma(f) = s(x_{\widetilde{k}}, f) + \frac{I_{\widetilde{k}}(f)}{\frac{K\tau}{1-\tau} I_{\widehat{k}}(f)} (\delta - s(x_{\widetilde{k}}, f)),$$

we get $s(x_{\widetilde{k}}, f) < \varsigma(f) < \delta$. Moreover, $I_{\widehat{k}}(f) \ge \frac{1-\tau}{K(1+\tau)}$, hence

$$\int_{B_{\varsigma(f)}(\beta(f))} f \, dV_g \ge \int_{B_{s(x_{\widetilde{k}}, f)}(x_{\widetilde{k}})} f \, dV_g = \tau \ge \delta,$$

$$\int_{\Sigma \setminus B_{\varsigma(f)}(\beta(f))} f \, dV_g \ge \int_{\Sigma \setminus B_\delta(x_{\widetilde{k}})} f \, dV_g \ge \frac{1-\tau}{K(1+\tau)} \ge \delta.$$

Subcase $I_{\widetilde{k}}(f) \ge \frac{2K\tau}{1-\tau} I_{\widehat{k}}(f)$: We just set $\varsigma(f) : s(x_{\widetilde{k}}, f)$ and we get

$$\int_{B_{\varsigma(f)}(\beta(f))} f \, dV_g = \tau \ge \delta, \qquad \int_{\Sigma \setminus B_{\varsigma(f)}(\beta(f))} f \, dV_g = 1 - \tau \ge \delta.$$

To prove the final assertion, write (up to sub-sequences), $(\beta_\infty, \varsigma_\infty) = \lim_{n \to +\infty} (\beta(f^n), \varsigma(f^n))$. For large n we will have

$$\int_{\Sigma \setminus \bigcup_{x' \in \mathcal{F}} B_\delta(x')} f^n \, dV_g \le \frac{\delta}{2}, \qquad \int_{B_\delta(x'')} f^n \, dV_g \le \frac{\delta}{2} \text{ for any } x'' \in \mathcal{F} \setminus \{x\},$$

which excludes $\varsigma_\infty = \delta$. We also exclude $\varsigma_\infty \in (0, \delta)$ as it would give

$$\int_{B_{\frac{3}{2}\varsigma_\infty}(\beta_\infty)} f^n \, dV_g \geq \delta, \qquad \int_{\Sigma \setminus B_{\frac{\varsigma_\infty}{2}}(\beta_\infty)} f^n \, dV_g \geq \delta,$$

which is a contradictions since

$$\mathcal{F} \cap \left(\overline{A_{\frac{\varsigma_\infty}{2}, \frac{3}{2}\varsigma_\infty}(\beta_\infty)}\right) = \emptyset.$$

Finally, we exclude $\beta_\infty \neq x$ because we would get the following contradiction:

$$\int_{B_\delta(\beta_\infty)} f^n \, dV_g \geq \delta.$$

This concludes the proof. □

Define

$$(u_1, u_2) \mapsto \left(\frac{\tilde{h}_1 e^{u_1}}{\int_\Sigma \tilde{h}_1 e^{u_1} \, dV_g}, \frac{\tilde{h}_2 e^{u_2}}{\int_\Sigma \tilde{h}_2 e^{u_2} \, dV_g} \right) =: (f_{1,u}, f_{2,u}).$$

Combining such a map ψ with Lemma 5.1, we deduce some extra information on low sub-levels of $J_{\rho,\underline{\alpha}}$. Recall first the definition of the sets $\mathcal{A}_i$ from (1.19).

Corollary 5.3. *Let δ, ψ be as in Lemma 5.2 and define, for $u \in H^1(\Sigma)^2$,*

$$\beta_1(u) = \beta_{\mathcal{A}_1}(f_{1,u}), \quad \varsigma_1(u) = \varsigma_{\mathcal{A}_1}(f_{2,u}), \quad \beta_2(u) = \beta_{\mathcal{A}_2}(f_{2,u}), \quad \varsigma_2(u) = \varsigma_{\mathcal{A}_2}(f_{2,u}).$$

Then for any $\delta' > 0$ there exists $L_{\delta'}$ such that if $\varsigma_i(u) \geq \delta'$ for both $i = 1, 2$, then $J_{\rho,\underline{\alpha}}(u) \geq -L_{\delta'}$.

Proof. Assume first $\varsigma_1(u) = \delta$: from the statement of Lemma 5.2, we get one of the following:

$$\int_{\Sigma \setminus \bigcup_{j=1}^m B_\delta(p_j)} f_{1,u} \, dV_g \geq \frac{\delta}{2}, \tag{5.3a}$$

$$\int_{B_\delta(p_j)} f_{1,u} \, dV_g \geq \frac{\delta}{2M} \quad \text{for some } p_j \notin \mathcal{A}_1, \tag{5.3b}$$

$$\int_{B_\delta(p_j')} f_{1,u} \, dV_g \geq \delta, \qquad \int_{B_\delta(p_{j''})} f_{1,u} \, dV_g \geq \delta \quad \text{for some } j' \neq j''. \tag{5.3c}$$

Depending on which possibility occurs in (5.3), define respectively

$$\Omega_{11} := \Sigma \setminus \bigcup_{j=1}^m B_\delta(p_j), \tag{5.4a}$$

$$\Omega_{11} := B_\delta(p_j), \tag{5.4b}$$

$$\Omega_{11} := B_\delta(p_{j'}), \quad \Omega_{12} := B_\delta(p_{j''}). \tag{5.4c}$$

It is easy to verify that such sets satisfy the hypotheses of Lemma 5.1, up to eventually redefining the map ψ with a smaller

$$\delta \leq \frac{\min_{j \neq j'} d(p_j, p_{j'})}{4}.$$

In the first case, we have $J_1 = 0$ and $K_1 = 1$, in the second case either $J_1 = 0$ and $K_1 = 1$, or $J_1 = 1$ and $K_1 = 0$ but $\rho < 4\pi(1 + \alpha_{1j})$, and in the third case we have $J_1 = 2$ and $K_1 = 0$.

If $\delta' \leq \varsigma_1(u) < \delta$, then

$$\int_{\Sigma \setminus B_{\delta'}(\beta_1(u))} f_{1,u} \, dV_g \geq \delta,$$

so we have one among the following:

$$\int_{\Sigma\setminus\bigcup_{j=1}^m B_\delta(x)} f_{1,u}\,dV_g \geq \frac{\delta}{2}, \tag{5.5a}$$

$$\int_{B_\delta(\beta_1(u))} f_{1,u}\,dV_g \geq \delta, \qquad \int_{B_\delta(p_j)} f_{1,u}\,dV_g \geq \frac{\delta}{2M} \quad \text{for some } p_j \neq \beta_1(u), \tag{5.5b}$$

$$\int_{A_{\delta',\delta}(\beta_1(u))} f_{1,u}\,dV_g. \tag{5.5c}$$

Depending on which is the case in (5.5), define

$$\Omega_{11} := \Sigma\setminus\bigcup_{j=1}^m B_\delta(p_j), \tag{5.6a}$$

$$\Omega_{11} := B_\delta(u)(\beta_1(u)), \qquad \Omega_{12} := B_\delta(p_j), \tag{5.6b}$$

$$\Omega_{11} := A_{\delta',\delta}(\beta_1(u)). \tag{5.6c}$$

Repeat the same argument for u_2 to get similarly Ω_{21}, and possibly Ω_{22}. Applying Lemma 5.1, we get $J_{\rho,\underline{\alpha}}(u) \geq -L_{\delta'}$. □

With some extra work (see [12] for the details) it can be shown that the vectorial Moser–Trudinger inequality improves each time the two scales (in the sense defined by Lemma 5.2) coincide, no matter how small they are.

Proposition 5.4 ([12]). *Let $\beta_i(u)$, $\varsigma_i(u)$ be as in Corollary 5.3. There exists $L \gg 0$ such that if*

$$\begin{cases} \beta_1(u) = \beta_2(u) = p_m & \text{with } \rho_1, \rho_2 < 4\pi(2 + \alpha_{1m} + \alpha_{2m}), \\ \varsigma_1(u) = \varsigma_2(u), \end{cases}$$

then $J_{\rho,\underline{\alpha}}(u) \geq -L$.

6 Proof of Theorem 1.2

Let us introduce the space $\mathcal{X}$, which is simply a graph and will be used in our min-max scheme. It is obtained by removing some points from the join of the weighted barycenters $\mathcal{A}_1 \star \mathcal{A}_2$ defined by (1.20). The points to exclude correspond to improved inequalities for functions centered around the same point and at the same rate of concentration (see the previous section for more details). Precisely, we set

$$\mathcal{X} := \mathcal{A}_1 \star \mathcal{A}_2 \setminus \left\{\left(p_j, p_j, \frac{1}{2}\right) : \rho_1, \rho_2 < 4\pi(2 + \alpha_{1j} + \alpha_{2j})\right\}. \tag{6.1}$$

We will prove that, under the assumptions of Theorem 1.4, the space $\mathcal{X}$ is not contractible, showing that it has a non-trivial homology group. In order to do this, we will recall how to compute the homology groups of the join of two known spaces. Since the join is homotopically equivalent to a smash product of X, Y and S^1 (see [40] for details), its homology groups only depend on the homology of X and Y.

Theorem 6.1 ([40, Theorem 3.21]). *Let X and Y be two topological spaces. Then*

$$\widetilde{H}_q(X \star Y) = \sum_{q'=0}^{q} \widetilde{H}_{q'}(X) \oplus \widetilde{H}_{q-q'-1}(Y).$$

In particular, if $X = (S^{D_1})^{\vee N_1}$ and $Y = (S^{D_2})^{\vee N_2}$ are wedge sum of spheres, then $X \star Y$ has the same homology as $(S^{D_1+D_2+1})^{\vee N_1 N_2}$.

Actually, in the same book [40] it is shown that the following homotopical equivalence holds:

$$(S^{D_1})^{\vee N_1} \star (S^{D_2})^{\vee N_2} \simeq (S^{D_1+D_2+1})^{\vee N_1 N_2}.$$

We then have the following result.

Proposition 6.2. *Let $\mathcal{X}$ be as in* (6.1) *and suppose we are under the assumptions of Theorem 1.4. Then the space $\mathcal{X}$ has non-trivial homology groups and it is not contractible.*

Proof. The spaces $\mathcal{A}_i$ are discrete sets of M_i points, for $i = 1, 2$, that is a wedge sum of $M_i - 1$ copies of S^0. Therefore, by Theorem 6.1, the space $\mathcal{A}_1 \star \mathcal{A}_2$ has the same homology as $(S^1)^{\vee(M_1-1)(M_2-1)}$. The set we have to remove from the join consists of M_3 singular points $\{p_{m_1}, \dots, p_{m_{M_3}}\}$ for some $\{m_1, \dots, m_{M_3}\} \subset \{1, \dots, M\}$. Then if, for some fixed $\delta < \frac{1}{2}$, we define

$$\mathcal{Y} := \bigcup_{j=1}^{M_3} B_\delta\Big(p_{m_j}, p_{m_j}, \frac{1}{2}\Big),$$

this retracts on $\{p_{m_1}, \dots, p_{m_{M_3}}\}$. On the other hand, $\mathcal{X} \cap \mathcal{Y}$ is a disjoint union of M_3 punctured intervals, that is a discrete set of $2M_3$ points, and $\mathcal{X} \cup \mathcal{Y}$ is the whole join. Therefore, the Mayer–Vietoris sequence yields

$$\underbrace{H_1(\mathcal{X} \cap \mathcal{Y})}_{0} \to H_1(\mathcal{X}) \oplus \underbrace{H_1(\mathcal{Y})}_{0} \to \underbrace{H_1(\mathcal{X} \cup \mathcal{Y})}_{\mathbb{Z}^{(M_1-1)(M_2-1)}} \to \underbrace{\widetilde{H}_0(\mathcal{X} \cap \mathcal{Y})}_{\mathbb{Z}^{2M_3-1}} \to \widetilde{H}_0(\mathcal{X}) \oplus \underbrace{\widetilde{H}_0(\mathcal{Y})}_{\mathbb{Z}^{M_3-1}} \to \underbrace{\widetilde{H}_0(\mathcal{X} \cup \mathcal{Y})}_{0}.$$

The exactness of this sequence implies that $b_1(\mathcal{X}) - \widetilde{b}_0(\mathcal{X}) = (M_1 - 1)(M_2 - 1) - M_3$, so if the latter number is not zero, we get at least a non-trivial homology group. Algebraic computations show, under the assumption $M_1, M_2 \geq M_3$, that $(M_1 - 1)(M_2 - 1) \neq M_3$ is equivalent to the assumption of Theorem 1.4, and therefore the proof is complete. □

We will now introduce some test functions from the space $\mathcal{X}$ to arbitrarily low sub-levels of $J_{\rho,\underline{\alpha}}$. Such test functions will have a profile which resembles the entire solutions of the Liouville equation and of the Toda system. We will use suitable interpolation between each of the above three profiles depending on whether the points in $\mathcal{A}_i$ coincide or not and depending on the parameters ρ_i. The map Φ^λ will therefore be defined case by case.

Let us start by setting $\Phi^\lambda(\zeta) = (\phi_1 - \frac{\phi_2}{2}, \phi_2 - \frac{\phi_1}{2})$ when $\zeta = (p_j, p_j m, t)$ for some j. The functions ϕ_1, ϕ_2 will be defined in different ways, depending on the relative positions of $\rho_1, \rho_2, \alpha_{1j}, \alpha_{2j}$ in $\mathbb{R}$. When dealing with the same singular point, we define ϕ_1 and ϕ_2 as follows:

(<<) $\rho_1, \rho_2 < 4\pi(2 + \alpha_{1j} + \alpha_{2j})$:

$$\phi_1 := \begin{cases} -2\log\max\{1, (\lambda d(\cdot, p_j))^{2(1+\alpha_{1j})}\} & \text{if } t < \frac{1}{2}, \\ 0 & \text{if } t > \frac{1}{2}, \end{cases}$$

$$\phi_2 := \begin{cases} 0 & \text{if } t < \frac{1}{2}, \\ -2\log\max\{1, (\lambda d(\cdot, p_j))^{2(1+\alpha_{2j})}\} & \text{if } t > \frac{1}{2}. \end{cases}$$

(<>) $\rho_1 < 4\pi(2 + \alpha_{1j} + \alpha_{2j}) < \rho_2$:

$$\phi_1 := -2\log\max\{1, \max\{1, (\lambda t)^{2(1+\alpha_{2j})}\}(\lambda d(\cdot, p_j))^{2(1+\alpha_{1j})}\},$$
$$\phi_2 := -2\log\max\{1, (\lambda t d(\cdot, p_j))^{2(2+\alpha_{1j}+\alpha_{2j})}\}.$$

(><) $\rho_2 < 4\pi(2 + \alpha_{1j} + \alpha_{2j}) < \rho_1$:

$$\phi_1 := -2\log\max\{1, (\lambda(1-t)d(\cdot, p_j))^{2(2+\alpha_{1j}+\alpha_{2j})}\},$$
$$\phi_2 := -2\log\max\{1, \max\{1, (\lambda(1-t))^{2(1+\alpha_{1j})}\}(\lambda d(\cdot, p_j))^{2(1+\alpha_{2j})}\}.$$

(>>) $\rho_1, \rho_2 > 4\pi(2 + \alpha_{1j} + \alpha_{2j})$:

$$\phi_1 := -2\log\max\Big\{1, \Big(\lambda\frac{\max\{1,\lambda t\}}{\max\{1,\lambda(1-t)\}}\Big)^{2+\alpha_{1j}+\alpha_{2j}} d(\cdot\,,p_m)^{2(1+\alpha_{1j})}, (\lambda d(\cdot\,,p_j))^{2(2+\alpha_{1j}+\alpha_{2j})}\Big\},$$

$$\phi_2 := -2\log\max\Big\{1, \Big(\lambda\frac{\max\{1,\lambda(1-t)\}}{\max\{1,\lambda t\}}\Big)^{2+\alpha_{1j}+\alpha_{2j}} d(\cdot\,,p_j)^{2(1+\alpha_{2j})}, (\lambda d(\cdot\,,p_j))^{2(2+\alpha_{1j}+\alpha_{2j})}\Big\}.$$

Let us now consider the case $x_1 \neq x_2$, $x_i \in \mathcal{A}_i$. Here, we define Φ^λ just by linearly interpolating between the test functions defined before:

$$\Phi^\lambda(x_1, x_2, t) = \Phi^{\lambda(1-t)}(x_1, x_1, 0) + \Phi^{\lambda t}(x_2, x_2, 1).$$

We then have the following result.

Proposition 6.3. *The above test functions $\{\Phi^\lambda\}_\lambda : \mathcal{X} \to H^1(\Sigma)^2$ satisfy*

$$J_{\rho,\underline{\alpha}}(\Phi^\lambda(\zeta)) \xrightarrow{\lambda\to+\infty} -\infty \quad \textit{uniformly for } \zeta \in \mathcal{X}.$$

We are finally in a position to prove Theorem 1.4. The proof will follow by showing that low sub-levels are *dominated* by the space $\mathcal{X}$ (see [40, p. 528]), which is not contractible by Proposition 6.2.

Lemma 6.4. *For $L \gg 0$ large enough there exist maps $\Phi : \mathcal{X} \to J^{-L}_{\rho,\underline{\alpha}}$ and $\Psi : J^{-L}_{\rho,\underline{\alpha}} \to \mathcal{X}$ such that $\Psi \circ \Phi$ is homotopically equivalent to $\mathrm{Id}_{\mathcal{X}}$.*

To prove Lemma 6.4 we need the following estimate. Notice that the choice of τ (see the proof of Lemma 5.2), which was not relevant in the rest of this paper, will be made in the proof of this lemma to let the following result hold true; for the proof we refer to [11].

Lemma 6.5. *Let δ be as in Lemma 5.2, let $\beta_i(u)$, $\sigma_i(u)$ be as in Corollary 5.3 and let Φ^λ be as in Theorem 6.3. Then, for a suitable choice of τ, there exist $C_0 > 0$, $\delta' \in (0, \delta)$ such that we have the following results:*

- *If either $t \geq 1 - \frac{C_0}{\lambda}$ or*

$$\begin{cases} t > \dfrac{1}{2}, \\ x_1 = x_2 =: p_j, \\ \rho_1, \rho_2 < 4\pi(2 + \alpha_{1j} + \alpha_{2j}), \end{cases}$$

  then $\sigma_1(\Phi^\lambda(\zeta)) \geq \delta'$; otherwise, $\sigma_1(\Phi^\lambda(\zeta)) < \delta$ and $\beta_1(\Phi^\lambda(\zeta)) = x_1$.
- *If either $t \leq \frac{C_0}{\lambda}$ or*

$$\begin{cases} t < \dfrac{1}{2}, \\ x_1 = x_2 =: p_j, \\ \rho_1, \rho_2 < 4\pi(2 + \alpha_{1j} + \alpha_{2j}), \end{cases}$$

  then $\sigma_2(\Phi^\lambda(\zeta)) \geq \delta'$; otherwise, $\sigma_2(\Phi^\lambda(\zeta)) < \delta$ and $\beta_2(\Phi^\lambda(\zeta)) = x_2$.

Proof of Lemma 6.4. Let δ be as in Lemma 5.2, let $\beta_i(u)$, $\varsigma_i(u)$ be as in Corollary 5.3 and let δ' be as in Lemma 6.5. Take now L so large that Corollary 5.3 and Theorem 5.4 apply.

We define $\Phi = \Phi^{\lambda_0}$ as in Theorem 6.3, with λ_0 such that $\Phi^\lambda(\mathcal{X}) \subset J^{-2L}_{\rho,\underline{\alpha}}$ for any $\lambda \geq \lambda_0$. As for $\Psi : J^{-2L}_{\rho,\underline{\alpha}} \to \mathcal{X}$, we write

$$\Psi(u) = (\beta_1(u), \beta_2(u), t'(\varsigma_1(u), \varsigma_2(u)))$$

with

$$t'(\varsigma_1(u), \varsigma_2(u)) = \begin{cases} 0 & \text{if } \varsigma_2(u) \geq \delta', \\ \dfrac{\delta' - \varsigma_2(u)}{2\delta' - \varsigma_1(u) - \varsigma_2(u)} & \text{if } \varsigma_1(u),\ \varsigma_2(u) \leq \delta', \\ 1 & \text{if } \varsigma_1(u) \geq \delta'. \end{cases}$$

Let us now verify the well-posedness of Ψ. The definition of t' makes sense because $J_{\rho,\underline{\alpha}}(u) < -L$ implies $\min\{\varsigma_1(u), \varsigma_2(u)\} \leq \delta'$ by Corollary 5.3. Moreover, if $t' > 0$ (resp. $t' < 1$), then $\varsigma_1 < \delta$ is well-defined (resp. $\varsigma_2 < \delta$ is well-defined), hence β_1 (resp. β_2) is also defined. Finally, Ψ is mapped on $\mathcal{X}$ because, from

Theorem 5.4, when $J_{\rho,\underline{\alpha}}(u) < -L$, we cannot have

$$(\beta_1(u), \beta_2(u), t'(\varsigma_1(u), \varsigma_2(u))) = \left(p_j, p_j, \frac{1}{2}\right) \quad \text{with } \rho_1, \rho_2 < 4\pi(2 + \alpha_{1j} + \alpha_{2j}).$$

To get a homotopy between the two maps, we first let λ tend to $+\infty$, in order to get x_1 and x_2, then we apply a linear interpolation for the parameter t. Writing $\Psi(\Phi^\lambda(\zeta)) = (\beta_1^\lambda(\zeta), \beta_2^\lambda(\zeta), t'^\lambda(\zeta))$, we have $F = F_2 * F_1$, with

$$\begin{aligned} F_1 &: (\zeta, s) = ((x_1, x_2, t), s) \mapsto (\beta_1^{\frac{\lambda_0}{1-s}}(\zeta), \beta_2^{\frac{\lambda_0}{1-s}}(\zeta), t'^{\lambda_0}(\zeta)) \\ F_2 &: (x_1, x_2, t'^{\lambda_0}(\zeta)) \mapsto (x_1, x_2, (1-s)t'^{\lambda_0}(\zeta) + st). \end{aligned}$$

We have to verify that all is well-defined.

If we cannot define $\beta_1^{\lambda_0/(1-s)}(\zeta)$, then by Lemma 6.5 we either have

$$t \geq 1 - \frac{C_0(1-s)}{\lambda_0} \geq 1 - \frac{C_0}{\lambda_0}$$

or we are on the first half of the punctured segment. By the same lemma, we get $\varsigma_1(\Phi^{\lambda_0}(\zeta)) \geq \delta'$, that is $t'^{\lambda_0}(\zeta) = 1$. For the same reason, if $\beta_2^{\lambda_0/(1-s)}(\zeta)$ is not defined, then $t'^{\lambda_0}(\zeta) = 0$, so $F_1 : \mathcal{X} \times [0,1] \to \mathcal{A}_1 \star \mathcal{A}_2$ makes sense. Its image is actually contained in $\mathcal{X}$ because if $x_1 = x_2$ and $\rho < 4\pi(\chi_1(x) + \chi_2(x))$, where we have set

$$\chi_i(\{x\}) := \begin{cases} 1 + \alpha_{i,j} & \text{if } x = p_j, \\ 1 & \text{if } x \notin \{p_1, \dots, p_m\}, \end{cases}$$

then from Lemma 6.5 we have $t'^{\lambda_0}(\zeta) \in \{0, 1\}$, hence in particular it does not equal $\frac{1}{2}$.

Concerning F_2, the previous lemma implies $\beta_1^{\lambda_0/(1-s)}(\zeta) = x_1$ if $t \leq 1 - \frac{C_0}{\lambda}(1-s)$ and in particular, if $t < 1$, taking the limit $s \to 1$. A similar condition holds for β_2, which gives $F_2(\cdot, 0) = F_1(\cdot, 1)$. If x_1 is not defined, then $t'^{\lambda_0}(\zeta) = 1$, hence $(1-s)t'^{\lambda_0}(\zeta) + st = 1$; similarly there are no issues when x_2 cannot be defined. Finally, by the argument used before, if $x_1 = x_2 = p_j$ and $\rho_1, \rho_2 < 4\pi(2 + \alpha_{1j} + \alpha_{2j})$, then $(1-s)t'^{\lambda_0}(\zeta) + st \neq \frac{1}{2}$. □

Concerning compactness, we have a useful result which can be deduced from minor modifications of the argument in [49]. It basically states the existence of bounded Palais–Smale sequences for ρ belonging to a dense set of $\mathbb{R}^2_+ \setminus \Gamma$, relying on the results in [42, 48]. Putting this together with the compactness result stated before, we get the following lemma.

Lemma 6.6. *Let $\rho \notin \Gamma$ be given and let $a < b$ be such that* (1.12) *has no solutions in $\{J_{\rho,\underline{\alpha}} \in [a,b]\}$. Then $\{J_{\rho,\underline{\alpha}} \leq a\}$ is a deformation retract of $\{J_{\rho,\underline{\alpha}} \leq b\}$.*

We also deduce that $J_{\rho,\underline{\alpha}}$ is uniformly bounded from above on solutions, hence we have the following corollary.

Corollary 6.7. *Let $\rho \notin \Gamma$ be given. Then there exists $L > 0$ such that $\{J_{\rho,\underline{\alpha}} \leq L\}$ is a deformation retract of $H^1(\Sigma)^2$; in particular, it is contractible.*

Proof of Theorem 1.4. Suppose by contradiction that system (1.12) has no solutions. By Lemma 6.6, we know that $\{J_{\rho,\underline{\alpha}} \leq -L\}$ is a deformation retract of $\{J_{\rho,\underline{\alpha}} \leq L\}$, hence by Corollary 6.7 it is contractible. Let $H(\zeta, s) : \mathcal{X} \times [0,1] \to \mathcal{X}$ be the homotopy equivalence defined in Lemma 6.4 and let H' be a homotopy equivalence between a constant map and $\mathrm{Id}_{\{J_{\rho,\underline{\alpha}} \leq -L\}}$. Then

$$H''(\zeta, s) = \Psi(H'(\Phi(\zeta), s)) : \mathcal{X} \times [0,1] \to \mathcal{X}$$

is an equivalence between the maps $\Psi \circ \Phi$ and a constant, and $H'' * H$ is an equivalence between $\mathrm{Id}_{\mathcal{X}}$ and a constant map. This means that $\mathcal{X}$ is contractible, contradicting Theorem 6.2. □

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