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STRONG UNIQUENESS FOR STOCHASTIC EVOLUTION EQUATIONS IN HILBERT SPACES PERTURBED BY A BOUNDED MEASURABLE DRIFT

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We prove pathwise (hence strong) uniqueness of solutions to stochastic evolution equations in Hilbert spaces with merely measurable bounded drift and cylindrical Wiener noise, thus generalizing Veretennikov's fundamental result on $\mathbb{R}^d$ to infinite dimensions. Because Sobolev regularity results implying continuity or smoothness of functions do not hold on infinite-dimensional spaces, we employ methods and results developed in the study of Malliavin–Sobolev spaces in infinite dimensions. The price we pay is that we can prove uniqueness for a large class, but not for every initial distribution. Such restriction, however, is common in infinite dimensions.

1. Introduction. We consider the following abstract stochastic differential equation in a separable Hilbert space H:

$$dX_t = \big(AX_t + B(X_t)\big)\,dt + dW_t, \qquad X_0 = x \in H, \tag{1}$$

where $A : D(A) \subset H \to H$ is self-adjoint, negative definite and such that $(-A)^{-1+\delta}$, for some $\delta \in (0, 1)$, is of trace class, $B : H \to H$ and $W = (W_t)$ is a cylindrical Wiener process. About B, we only assume that it is *Borel measurable and bounded*.

$$B \in B_b(H, H).$$

Our aim is to prove *pathwise uniqueness* for (1), thus gaining an infinite-dimensional generalization of the famous fundamental result of Veretennikov [33] in the case $H = \mathbb{R}^d$. We refer to [35] and [32] for the case $H = \mathbb{R}$ as well as to the generalizations of [33] to unbounded drifts in [23, 34] and also to the references therein; see [17, 18]. We note that [32] also includes the case of α-stable noise, $\alpha \geq 1$, which in turn was extended to $\mathbb{R}^d$ in [29].

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Explicit cases of parabolic stochastic partial differential equations, with space–time white noise in space-dimension one, have been solved on various levels of generality for the drift by Gyöngy and coworkers, in a series of papers; see [1, 16, 19, 20] and the references therein. The difference of the present paper with respect to these works is that we obtain a general abstract result, applicable, for instance, to systems of parabolic equations or equations with differential operators of higher order than two. As we shall see, the price to pay for this generality is a restriction on the initial conditions. Indeed, using that for $B = 0$ there exists a unique non-degenerate (Gaussian) invariant measure μ, we will prove strong uniqueness for μ-a.e. initial $x \in H$ or random H-valued x with distribution absolutely continuous with respect to μ.

At the abstract level, this work generalizes [5] devoted to the case where B is bounded and in addition Hölder continuous, but with no restriction on the initial conditions. To prove our result we use some ideas from [5, 10, 13, 14] and [23].

The extension of Veretennikov's result [33] and also of [23] to infinite dimensions has resisted various attempts of its realization for many years. The reason is that the finite-dimensional results heavily depend on advanced parabolic Sobolev regularity results for solutions to the corresponding Kolmogorov equations. Such regularity results, leading to continuity or smoothness of the solutions, however, do not hold in infinite dimensions. A technique different from [33] is used in [14]; see also [5, 10] and [29]. This technique allows us to prove uniqueness for stochastic equations with time independent coefficients by merely using elliptic (not parabolic) regularity results. In the present paper we succeed in extending this approach to infinite dimensions, exploiting advanced regularity results for elliptic equations in Malliavin–Sobolev spaces with respect to a Gaussian measure on Hilbert space. To the best of our knowledge this is the first time that an analogue of Veretennikov's result has been obtained.

Given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, a cylindrical Wiener process W and an $\mathcal{F}_0$-measurable r.v. x, we call *mild solution* to the Cauchy problem (1) a continuous $\mathcal{F}_t$-adapted H-valued process $X = (X_t)$ such that

$$X_t = e^{tA}x + \int_0^t e^{(t-s)A} B(X_s)\,ds + \int_0^t e^{(t-s)A}\,dW_s. \tag{2}$$

Existence of mild solutions on some filtered probability space is well known; see Chapter 10 in [7] and also Appendix A.1. Our main result is:

THEOREM 1. *Assume Hypothesis 1. For μ-a.e. (deterministic) $x \in H$, there is a unique (in the pathwise sense) mild solution of the Cauchy problem (1).*

Moreover, for every $\mathcal{F}_0$-measurable H-valued r.v. x with law μ_0 such that $\mu_0 \ll \mu$ and

$$\int_H \left(\frac{d\mu_0}{d\mu}\right)^{\zeta} d\mu < \infty$$

for some $\zeta > 1$, there is also a unique mild solution of the Cauchy problem.

The proof, performed in Section 3, uses regularity results for elliptic equations in Hilbert spaces, given in Section 2 where we also establish an Itô type formula involving $u(X_t)$ with u in some Sobolev space associated to μ. In comparison with the finite-dimensional case (cf. [23]), to prove such an Itô formula, we do not only need analytic regularity results, but also the fact that all transition probability functions associated with (2) are absolutely continuous with respect to μ. This result heavily depends on an infinite-dimensional version of Girsanov's theorem. Though, also under our conditions, this is a "folklore result" in the field; it seems hard to find an accessible reference in the literature. Therefore, we include a complete proof of the version we need in the Appendix for the convenience of the reader.

Concerning the proof of Theorem 1 given in Section 3, we remark that, in comparison to the finite-dimensional case (see, in particular, [9] and [10]), it is necessary to control infinite series of second derivatives of solutions to Kolmogorov equations which is much more elaborate.

Examples are given in Section 4.

1.1. *Assumptions and preliminaries*. We are given a real separable Hilbert space H and denote its norm and inner product by $|\cdot|$ and $\langle\cdot,\cdot\rangle$, respectively. We follow [4, 7, 8] and assume:

HYPOTHESIS 1. $A : D(A) \subset H \to H$ is a negative definite self-adjoint operator and $(-A)^{-1+\delta}$, for some $\delta \in (0, 1)$, is of trace class.

Since A^{-1} is compact, there exists an orthonormal basis (e_k) in H and a sequence of positive numbers (λ_k) such that

$$Ae_k = -\lambda_k e_k, \qquad k \in \mathbb{N}. \tag{3}$$

Recall that A generates an analytic semigroup e^{tA} on H such that $e^{tA}e_k = e^{-\lambda_k t}e_k$. We will consider a cylindrical Wiener process W_t with respect to the previous basis (e_k). The process W_t is formally given by "$W_t = \sum_{k\geq 1} \beta_k(t)e_k$" where $\beta_k(t)$ are independent, one-dimensional Wiener process; see [7] for more details.

By R_t we denote the Ornstein–Uhlenbeck semigroup in $B_b(H)$ (the Banach space of Borel and bounded real functions endowed with the essential supremum norm $\|\cdot\|_0$) defined as

$$R_t\varphi(x) = \int_H \varphi(y)N\big(e^{tA}x, Q_t\big)(dy), \qquad \varphi \in B_b(H), \tag{4}$$

where $N(e^{tA}x, Q_t)$ is the Gaussian measure in H of mean $e^{tA}x$ and covariance operator Q_t given by

$$Q_t = -\tfrac{1}{2}A^{-1}\big(I - e^{2tA}\big), \qquad t \geq 0. \tag{5}$$

We notice that R_t has a unique invariant measure $\mu := N(0, Q)$ where $Q = -\frac{1}{2}A^{-1}$. Moreover, since under the previous assumptions the Ornstein–Uhlenbeck semigroup is strong Feller and irreducible, we have by Doob's theorem that, for any $t > 0$, $x \in H$, the measures $N(e^{tA}x, Q_t)$ and μ are equivalent; see [8]. On the other hand, our assumption that $(-A)^{-1+\delta}$ is trace class guarantees that the OU process,

$$Z_t = Z(t, x) = e^{tA}x + \int_0^t e^{(t-s)A}\, dW_s \tag{6}$$

has a continuous H-valued version.

If H and K are separable Hilbert spaces, the Banach space $L^p(H, \mu, K)$, $p \geq 1$, is defined to consist of equivalent classes of measurable functions $f : H \to K$ such that $\int_H |f|_K^p \mu(dx) < +\infty$ [if $K = \mathbb{R}$ we set $L^p(H, \mu, \mathbb{R}) = L^p(H, \mu)$]. We also use the notation $L^p(\mu)$ instead of $L^p(H, \mu, K)$ when no confusion may arise.

The semigroup R_t can be uniquely extended to a strongly continuous semigroup of contractions on $L^p(H, \mu)$, $p \geq 1$, which we still denote by R_t, whereas we denote by L_p (or L when no confusion may arise) its infinitesimal generator, which is defined on smooth functions φ as

$$L\varphi(x) = \tfrac{1}{2}\mathrm{Tr}\big(D^2\varphi(x)\big) + \langle Ax, D\varphi(x)\rangle,$$

where $D\varphi(x)$ and $D^2\varphi(x)$ denote, respectively, the first and second Fréchet derivatives of φ at $x \in H$. For Banach spaces E and F we denote by $C_b^k(E, F)$, $k \geq 1$, the Banach space of all functions $f : E \to F$ which are bounded and Fréchet differentiable on E up to the order $k \geq 1$ with all derivatives bounded and continuous. We also set $C_b^k(E, \mathbb{R}) = C_b^k(E)$.

According to [8], for any $\varphi \in B_b(H)$ and any $t > 0$ one has $R_t\varphi \in C_b^\infty(H) = \bigcap_{k \geq 1} C_b^k(H)$. Moreover,

$$\langle DR_t\varphi(x), h\rangle = \int_H \langle \Lambda_t h, Q_t^{-1/2}y\rangle \varphi(e^{tA}x + y) N(0, Q_t)(dy), \qquad h \in H, \tag{7}$$

where Q_t is defined in (5),

$$\Lambda_t = Q_t^{-1/2}e^{tA} = \sqrt{2}(-A)^{1/2}e^{tA}\big(I - e^{2tA}\big)^{-1/2} \tag{8}$$

and $y \mapsto \langle \Lambda_t h, Q_t^{-1/2}y\rangle$ is a centered Gaussian random variable under $\mu_t = N(0, Q_t)$ with variance $|\Lambda_t h|^2$ for any $t > 0$; cf. [7], Theorem 6.2.2. Since

$$\Lambda_t e_k = \sqrt{2}(\lambda_k)^{1/2}e^{-t\lambda_k}\big(1 - e^{-2t\lambda_k}\big)^{-1/2}e_k,$$

we see that, for any $\epsilon \in [0, \infty)$, there exists $C_\epsilon > 0$ such that

$$\big\|(-A)^\epsilon \Lambda_t\big\|_{\mathcal{L}} \leq C_\epsilon t^{-1/2-\epsilon}. \tag{9}$$

In the sequel $\|\cdot\|$ always denotes the *Hilbert–Schmidt norm*; on the other hand $\|\cdot\|_{\mathcal{L}}$ indicates the *operator norm*.

By (7) we deduce

$$\sup_{x\in H}\left|(-A)^{\epsilon}DR_t\varphi(x)\right| = \left\|(-A)^{\epsilon}DR_t\varphi\right\|_0 \le C_\epsilon t^{-1/2-\epsilon}\|\varphi\|_0, \tag{10}$$

which by taking the Laplace transform yields, for $\epsilon \in [0, 1/2)$,

$$\left\|(-A)^{\epsilon}D(\lambda - L_2)^{-1}\varphi\right\|_0 \le \frac{C_{1,\epsilon}}{\lambda^{1/2-\epsilon}}\|\varphi\|_0. \tag{11}$$

Similarly, we find

$$\left\|(-A)^{\epsilon}DR_t\varphi\right\|_{L^2(\mu)} \le C_\epsilon t^{-1/2-\epsilon}\|\varphi\|_{L^2(\mu)} \tag{12}$$

and

$$\left\|(-A)^{\epsilon}D(\lambda - L_2)^{-1}\varphi\right\|_{L^2(\mu)} \le \frac{C_{1,\epsilon}}{\lambda^{1/2-\epsilon}}\|\varphi\|_{L^2(\mu)}. \tag{13}$$

Recall that the Sobolev space $W^{2,p}(H,\mu)$, $p \ge 1$, is defined in [3], Section 3, as the completion of a suitable set of smooth functions endowed with the Sobolev norm; see also [7], Section 9.2, for the case $p = 2$ and [31]. Under our initial assumptions, the following result can be found in [8], Section 10.2.1.

THEOREM 2. *Let $\lambda > 0$, $f \in L^2(H,\mu)$ and let $\varphi \in D(L_2)$ be the solution of the equation*

$$\lambda\varphi - L_2\varphi = f. \tag{14}$$

Then $\varphi \in W^{2,2}(H,\mu)$, $(-A)^{1/2}D\varphi \in L^2(H,\mu)$ and there exists a constant $C(\lambda)$ such that

$$\begin{aligned}&\|\varphi\|_{L^2(\mu)} + \left(\int_H \left\|D^2\varphi(x)\right\|^2\mu(dx)\right)^{1/2} + \left\|(-A)^{1/2}D\varphi\right\|_{L^2(\mu)}\\ &\quad\le C\|f\|_{L^2(\mu)}.\end{aligned} \tag{15}$$

The following extension to $L^p(\mu)$, $p > 1$ can be found in Section 3 of [3]; see also [2] and [26]; a finite-dimensional result analogous to this for nonsymmetric OU operators was proved in [27].

THEOREM 3. *Let $\lambda > 0$, $f \in L^p(H,\mu)$ and let $\varphi \in D(L_p)$ be the solution of the equation*

$$\lambda\varphi - L_p\varphi = f. \tag{16}$$

Then $\varphi \in W^{2,p}(H,\mu)$, $(-A)^{1/2}D\varphi \in L^p(H,\mu; H)$ and there exists a constant $C = C(\lambda, p)$ such that

$$\begin{aligned}&\|\varphi\|_{L^p(\mu)} + \left(\int_H \left\|D^2\varphi(x)\right\|^p\mu(dx)\right)^{1/p} + \left\|(-A)^{1/2}D\varphi\right\|_{L^p(\mu)}\\ &\quad\le C\|f\|_{L^p(\mu)}.\end{aligned} \tag{17}$$

2. Analytic results and an Itô-type formula.

2.1. *Existence and uniqueness for the Kolmogorov equation.* We are here concerned with the equation

$$\lambda u - L_2 u - \langle B, Du\rangle = f, \tag{18}$$

where $\lambda > 0$, $f \in B_b(H)$ and $B \in B_b(H, H)$.

REMARK 4. Since the corresponding Dirichlet form

$$\mathcal{E}(u, v) := \int_H \langle Du, Dv\rangle \, d\mu - \int_H \langle B, Du\rangle v \, d\mu + \lambda \int_H uv \, d\mu,$$

$u, v \in W^{1,2}(\mu)$, is weakly sectorial for λ big enough, it follows by [25], Chapter I and Subsection 3e) in Chapter II, that (18) has a unique solution in $D(L_2)$. However, we need more regularity for u.

PROPOSITION 5. *Let $\lambda \geq \lambda_0$, where*

$$\lambda_0 := 4\|B\|_0^2 C_{1,0}^2. \tag{19}$$

Then there is a unique solution $u \in D(L_2)$ of (18) given by

$$u = u_\lambda = (\lambda - L_2)^{-1}(I - T_\lambda)^{-1} f, \tag{20}$$

where

$$T_\lambda \varphi := \langle B, D(\lambda - L_2)^{-1}\varphi\rangle. \tag{21}$$

Moreover, $u \in C_b^1(H)$ with

$$\|u\|_0 \leq 2\|f\|_0, \qquad \|(-A)^\epsilon Du\|_0 \leq \frac{2C_{1,\epsilon}}{\lambda^{1/2-\epsilon}} \|f\|_0, \qquad \epsilon \in [0, 1/2), \tag{22}$$

and, for any $p \geq 2$, $u \in W^{2,p}(H, \mu)$ and, for some $C = C(\lambda, p, \|B\|_0)$,

$$\int_H \|D^2 u(x)\|^p \mu(dx) \leq C \int_H |f(x)|^p \mu(dx). \tag{23}$$

PROOF. Setting $\psi := \lambda u - L_2 u$, equation (18) reduces to

$$\psi - T_\lambda \psi = f. \tag{24}$$

If $\lambda \geq \lambda_0$ by (13), we have

$$\|T_\lambda \varphi\|_{L^2(\mu)} \leq \tfrac{1}{2}\|\varphi\|_{L^2(\mu)}, \qquad \varphi \in L^2(\mu),$$

so that (24) has a unique solution given by

$$\psi = (I - T_\lambda)^{-1} f.$$

Consequently, (18) has a unique solution $u \in L^2(H, \mu)$ given by (20). The same argument in $B_b(H)$, using (11) instead of (13) shows that

$$\|T_\lambda \varphi\|_0 \le \tfrac{1}{2}\|\varphi\|_0, \qquad \varphi \in B_b(H),$$

and that $\psi \in B_b(H)$ and hence by (20) also $u \in B_b(H)$. In particular, (22) is fulfilled by (11). To prove the last assertion we write $\lambda u - L_2 u = \langle B, Du\rangle + f$ and use estimate (22) with $\epsilon = 0$ and Theorem 3. □

2.2. *Approximations*. We are given two sequences $(f_n) \subset B_b(H)$ and $(B_n) \subset B_b(H, H)$ such that:

$$\begin{aligned} &\text{(i)} \quad f_n(x) \to f(x), \qquad B_n(x) \to B(x), \qquad \mu\text{-a.s.;} \\ &\text{(ii)} \quad \|f_n\|_0 \le M, \qquad \|B_n\|_0 \le M. \end{aligned} \tag{25}$$

PROPOSITION 6. *Let $\lambda \ge \lambda_0$, where λ_0 is defined in* (19). *Then the equation*

$$\lambda u_n - L u_n - \langle B_n, D u_n\rangle = f_n \tag{26}$$

has a unique solution $u_n \in C_b^1(H) \cap D(L_2)$ given by

$$u_n = (\lambda - L)^{-1}(I - T_{n,\lambda})^{-1} f_n, \tag{27}$$

where

$$T_{n,\lambda}\varphi := \langle B_n, D(\lambda - L_2)^{-1}\varphi\rangle. \tag{28}$$

Moreover, for any $\epsilon \in [0, 1/2)$, with constants independent of n,

$$\|u_n\|_0 \le 2M, \qquad \|(-A)^\epsilon D u_n\|_0 \le \frac{2C_{1,\epsilon}}{\lambda^{1/2-\epsilon}} M. \tag{29}$$

Finally, we have $u_n \to u$, and $Du_n \to Du$, in $L^2(\mu)$, where u is the solution to (18).

PROOF. Set

$$\psi_n := (I - T_{n,\lambda})^{-1} f_n, \qquad \psi := (I - T_\lambda)^{-1} f.$$

It is enough to show that

$$\psi_n \to \psi \qquad \text{in } L^2(H, \mu). \tag{30}$$

Let $\lambda \ge \lambda_0$, and write

$$\psi - \psi_n = T_\lambda \psi - T_{n,\lambda}\psi_n + f - f_n.$$

Then, setting $\|\cdot\|_2 = \|\cdot\|_{L^2(\mu)}$,

$$\begin{aligned} \|\psi - \psi_n\|_2 &\le \|T_{n,\lambda}\psi - T_{n,\lambda}\psi_n\|_2 + \|T_\lambda\psi - T_{n,\lambda}\psi\|_2 + \|f - f_n\|_2 \\ &\le \tfrac{1}{2}\|\psi - \psi_n\|_2 + \|T_\lambda\psi - T_{n,\lambda}\psi\|_2 + \|f - f_n\|_2. \end{aligned}$$

Consequently,

$$\|\psi - \psi_n\|_2 \leq 2\|T_\lambda \psi - T_{n,\lambda}\psi\|_2 + 2\|f - f_n\|_2.$$

We also have

$$\|T_\lambda \psi - T_{n,\lambda}\psi\|_2^2 \leq \int_H \big|B(x) - B_n(x)\big|^2 \big|D(\lambda - L_2)^{-1}\psi(x)\big|^2 \mu(dx).$$

Therefore, by the dominate convergence theorem, it follows that

$$\lim_{n\to\infty} \|T_\lambda \psi - T_{n,\lambda}\psi\|_2 = 0.$$

The conclusion follows. □

2.3. *Modified mild formulation.* For any $i \in \mathbb{N}$ we denote the ith component of B by $B^{(i)}$, that is,

$$B^{(i)}(x) := \langle B(x), e_i \rangle.$$

Then for $\lambda \geq \lambda_0$ we consider the solution $u^{(i)}$ of the equation

$$\lambda u^{(i)} - L u^{(i)} - \langle B, Du^{(i)} \rangle = B^{(i)}, \qquad \mu\text{-a.s.} \tag{31}$$

THEOREM 7. *Let X_t be a mild solution of equation* (1) *on some filtered probability space, let $u^{(i)}$ be the solution of* (31) *and set $X_t^{(i)} = \langle X_t, e_i \rangle$. Then we have*

$$\begin{aligned} X_t^{(i)} &= e^{-\lambda_i t}\big(\langle x, e_i\rangle + u^{(i)}(x)\big) - u^{(i)}(X_t) \\ &\quad + (\lambda + \lambda_i) \int_0^t e^{-\lambda_i (t-s)} u^{(i)}(X_s)\,ds \\ &\quad + \int_0^t e^{-\lambda_i (t-s)} \big(d\langle W_s, e_i\rangle + \langle Du^{(i)}(X_s), dW_s\rangle\big), \\ &\qquad\qquad t \geq 0, \qquad \mathbb{P}\text{-}a.s. \end{aligned} \tag{32}$$

PROOF. The proof uses in an essential way that, for any $t > 0$, $x \in H$, the law $\pi_t(x, \cdot)$ of $X_t = X(t, x)$ is equivalent to μ. This follows from Theorem 13 (Girsanov's theorem) in the Appendix, by which the law on $C([0, T]; H)$ of $X(\cdot, x)$ is equivalent to the law of the solution of (1) with $B = 0$, that is, it is equivalent to the law of the OU process $Z(t, x)$ given in (6). In particular, their transition probabilities are equivalent. But it is well known that the law of $Z(t, x)$ is equivalent to μ for all $t > 0$ and $x \in H$ in our case; see [7], Theorem 11.3.

Let us first describe a formal proof based on an heuristic use of Itô's formula, and then give the necessary rigorous details by approximations.

Step 1. Formal proof.

By Itô's formula we have

$$du^{(i)}(X_t) = \langle Du^{(i)}(X_t), dX_t \rangle + \tfrac{1}{2}\operatorname{Tr}\big[D^2 u^{(i)}(X_t)\big]\,dt$$

and so

$$du^{(i)}(X_t) = Lu^{(i)}(X_t)\,dt + \langle B(X_t), Du^{(i)}(X_t)\rangle dt + \langle Du^{(i)}(X_t), dW_t\rangle.$$

Now, using (31), we find that

$$du^{(i)}(X_t) = \lambda u^{(i)}(X_t)\,dt - B^{(i)}(X_t)\,dt + \langle Du^{(i)}(X_t), dW_t\rangle. \tag{33}$$

On the other hand, by (1) we deduce

$$dX_t^{(i)} = -\lambda_i X_t^{(i)}\,dt + B^{(i)}(X_t)\,dt + dW_t^{(i)}.$$

The expression for $B^{(i)}(X_t)$ that we get from this identity, we insert into (33). This yields

$$dX_t^{(i)} = -\lambda_i X_t^{(i)}\,dt + \lambda u^{(i)}(X_t)\,dt - du^{(i)}(X_t) + dW_t^{(i)} + \langle Du^{(i)}(X_t), dW_t\rangle.$$

By the variation of constants formula, this is equivalent to

$$\begin{aligned} X_t^{(i)} &= e^{-\lambda_i t}\langle x, e_i\rangle + \lambda \int_0^t e^{-\lambda_i(t-s)} u^{(i)}(X_s)\,ds \\ &\quad - \int_0^t e^{-\lambda_i(t-s)}\,du^{(i)}(X_s) + \int_0^t e^{-\lambda_i(t-s)}\big[dW_s^{(i)} + \langle Du^{(i)}(X_s), dW_s\rangle\big]. \end{aligned}$$

Finally, integrating by parts in the second integral yields (32).

Step 2. Approximation of B and u.
Set

$$B_n(x) = \int_H B\big(e^{A/n}x + y\big) N(0, Q_{1/n})(dy), \qquad x \in H. \tag{34}$$

Then B_n is of C^∞ class and all its derivatives are bounded. Moreover, $\|B_n\|_0 \le \|B\|_0$. It is easy to see that, possibly passing to a subsequence,

$$B_n \to B, \qquad \mu\text{-a.s.} \tag{35}$$

[indeed $B_n \to B$ in $L^2(H, \mu; H)$; this result can be first checked for continuous and bounded B].

Now we denote by $u_n^{(i)}$ the solution of the equation

$$\lambda u_n^{(i)} - Lu_n^{(i)} - \langle B_n, Du_n^{(i)}\rangle = B_n^{(i)}, \tag{36}$$

where $B_n^{(i)} = \langle B_n, e_i\rangle$. By Proposition 6 we have, possibly passing to a subsequence,

$$\begin{aligned} &\lim_{n\to\infty} u_n^{(i)} = u^{(i)}, \qquad \lim_{n\to\infty} Du_n^{(i)} = Du^{(i)}, \qquad \mu\text{-a.s.}, \\ &\sup_{n\ge1} \|u_n^{(i)}\|_{C_b^1(H)} = C_i < \infty, \end{aligned} \tag{37}$$

where $u^{(i)}$ is the solution of (31).

Step 3. Approximation of X_t.

For any $m \in \mathbb{N}$ we set $X_{m,t} := \pi_m X_t$, where $\pi_m = \sum_{j=1}^m e_j \otimes e_j$. Then we have

$$X_{m,t} = \pi_m x + \int_0^t A_m X_s \, ds + \int_0^t \pi_m B(X_s) \, ds + \pi_m W_t, \tag{38}$$

where $A_m = \pi_m A$.

Now we denote by $u_{n,m}^{(i)}$ the solution of the equation

$$\lambda u_{n,m}^{(i)} - L u_{n,m}^{(i)} - \langle \pi_m B_n \circ \pi_m, D u_{n,m}^{(i)} \rangle = B_n^{(i)} \circ \pi_m, \tag{39}$$

where $(B_n \circ \pi_m)(x) = B_n(\pi_m x)$, $x \in H$. Since only a finite number of variables is involved, we have, equivalently,

$$\lambda u_{n,m}^{(i)} - L_m u_{n,m}^{(i)} - \langle \pi_m B_n \circ \pi_m, D u_{n,m}^{(i)} \rangle = B_n^{(i)} \circ \pi_m$$

with

$$L_m \varphi = \tfrac{1}{2} \operatorname{Tr}[\pi_m D^2 \varphi] + \langle A_m x, D\varphi \rangle. \tag{40}$$

Moreover, since $u_{n,m}^{(i)}$ depends only on the first m variables, we have

$$u_{n,m}^{(i)}(\pi_m y) = u_{n,m}^{(i)}(y), \qquad y \in H, n, m, i \geq 1. \tag{41}$$

Applying a finite-dimensional Itô formula to $u_{n,m}^{(i)}(X_{m,t}) = u_{n,m}^{(i)}(X_t)$ yields

$$\begin{aligned} d u_{n,m}^{(i)}(X_{m,t}) = {} & \tfrac{1}{2} \operatorname{Tr}[D^2 u_{n,m}^{(i)}(X_{m,t})] \, dt \\ & + \langle D u_{n,m}^{(i)}(X_{m,t}), A_m X_t + \pi_m B(X_t) \rangle \, dt \\ & + \langle D u_{n,m}^{(i)}(X_{m,t}), \pi_m \, dW_t \rangle. \end{aligned} \tag{42}$$

On the other hand, by (39) we have

$$\begin{aligned} & \lambda u_{n,m}^{(i)}(X_{m,t}) - \tfrac{1}{2} \operatorname{Tr}[D^2 u_{n,m}^{(i)}(X_{m,t})] \\ & \quad - \langle D u_{n,m}^{(i)}(X_{m,t}), A_m X_{m,t} + \pi_m B_n(X_{m,t}) \rangle \\ & \quad = B_n^{(i)}(X_{m,t}). \end{aligned}$$

Comparing with (42) yields

$$\begin{aligned} d u_{n,m}^{(i)}(X_{m,t}) = {} & \lambda u_{n,m}^{(i)}(X_{m,t}) \, dt - B_n^{(i)}(X_{m,t}) \, dt \\ & + \langle D u_{n,m}^{(i)}(X_{m,t}), \pi_m (B(X_t) - B_n(X_{m,t})) \rangle \, dt \\ & + \langle D u_{n,m}^{(i)}(X_{m,t}), \pi_m \, dW_t \rangle. \end{aligned} \tag{43}$$

Taking into account (41), we rewrite (43) in the integral form as

$$
\begin{aligned}
&u_{n,m}^{(i)}(X_t) - u_{n,m}^{(i)}(X_r) \\
&\quad = \int_r^t \lambda u_{n,m}^{(i)}(X_s)\,ds - \int_r^t B_n^{(i)}(X_{m,s})\,ds \\
&\qquad + \int_r^t \langle Du_{n,m}^{(i)}(X_s), \big(B(X_s) - B_n(X_{m,s})\big)\rangle\,ds \\
&\qquad + \int_r^t \langle Du_{n,m}^{(i)}(X_s), dW_s\rangle,
\end{aligned} \tag{44}
$$

$t \geq r > 0$. Let us fix $n, i \geq 1$ and $x \in H$.

Possibly passing to a subsequence, and taking the limit in probability (with respect to $\mathbb{P}$), from identity (44), we arrive at

$$
\begin{aligned}
du_n^{(i)}(X_t) = \lambda u_n^{(i)}(X_t)\,dt - B_n^{(i)}(X_t)\,dt& \\
+ \langle Du_n^{(i)}(X_t), \big(B(X_t) - B_n(X_t)\big)\rangle\,dt& \\
+ \langle Du_n^{(i)}(X_t), dW_t\rangle, \qquad \mathbb{P}\text{-a.s.}&
\end{aligned} \tag{45}
$$

Let us justify such assertion.

First note that in equation (39) we have the drift term $\pi_m B_n \circ \pi_m$ which converges pointwise to B_n and $B_n^{(i)} \circ \pi_m$ which converges pointwise to $B_n^{(i)}$ as $m \to \infty$. Since such functions are also uniformly bounded, we can apply Proposition 6 and obtain that, possibly passing to a subsequence (recall that n is fixed),

$$
\begin{aligned}
&\lim_{m\to\infty} u_{n,m}^{(i)} = u_n^{(i)}, \qquad \lim_{m\to\infty} Du_{n,m}^{(i)} = Du_n^{(i)}, \qquad \mu\text{-a.s.}, \\
&\sup_{m\geq 1} \|u_{n,m}^{(i)}\|_{C_b^1(H)} = C_i < \infty.
\end{aligned} \tag{46}
$$

Now we only consider the most involved terms in (44).

We have, using that the law $\pi_t(x,\cdot)$ of X_t is absolutely continuous with respect to μ,

$$
\begin{aligned}
&\mathbb{E}\int_r^t |u_{n,m}^{(i)}(X_s) - u_n^{(i)}(X_s)|\,ds \\
&\qquad = \int_r^t ds \int_H |u_{n,m}^{(i)}(y) - u_n^{(i)}(y)| \frac{d\pi_s(x,\cdot)}{d\mu}(y)\mu(dy),
\end{aligned}
$$

which tends to 0, as $m \to \infty$, by the dominated convergence theorem [using (46)].

This implies $\lim_{m\to\infty} \int_r^t \lambda u_{n,m}^{(i)}(X_s)\,ds = \int_r^t \lambda u_n^{(i)}(X_s)\,ds$ in $L^1(\Omega, \mathbb{P})$. Similarly, we prove that $u_{n,m}^{(i)}(X_t)$ and $u_{n,m}^{(i)}(X_r)$ converge, respectively, to $u_n^{(i)}(X_t)$ and $u_n^{(i)}(X_r)$ in L^1.

To show that

$$
\begin{aligned}
(47)\qquad \lim_{m\to\infty} \mathbb{E}\int_r^t \big|\langle Du_{n,m}^{(i)}(X_s), \pi_m\big(B(X_s) - B_n(X_{m,s})\big)\rangle \\
- \langle Du_n^{(i)}(X_s), \big(B(X_s) - B_n(X_s)\big)\rangle\big|\, ds = 0,
\end{aligned}
$$

it is enough to prove that $\lim_{m\to\infty} H_m + K_m = 0$, where

$$
H_m = \mathbb{E}\int_r^t \big|\langle Du_{n,m}^{(i)}(X_s) - Du_n^{(i)}(X_s), \pi_m\big(B(X_s) - B_n(X_{m,s})\big)\rangle\big|\, ds
$$

and

$$
K_m = \mathbb{E}\int_r^t \big|\langle Du_n^{(i)}(X_s), \big[\pi_m B(X_s) - B(X_s)\big] + \big[B_n(X_s) - \pi_m B_n(X_{m,s})\big]\rangle\big|\, ds.
$$

It is easy to check that $\lim_{m\to\infty} K_m = 0$. Let us deal with H_m. We have

$$
\begin{aligned}
(48)\qquad H_m &\le 2\|B\|_0 \int_r^t \mathbb{E}\big|Du_{n,m}^{(i)}(X_s) - Du_n^{(i)}(X_s)\big|\, ds \\
&\le \int_r^t ds \int_H \big|Du_{n,m}^{(i)}(y) - Du_n^{(i)}(y)\big| \frac{d\pi_s(x,\cdot)}{d\mu}(y)\mu(dy),
\end{aligned}
$$

which tends to 0 as $m \to \infty$ by the dominated convergence theorem [using (46)]. This shows (47).

It remains to prove that

$$
\lim_{m\to\infty} \int_r^t \langle Du_{n,m}^{(i)}(X_s), dW_s\rangle = \int_r^t \langle Du_n^{(i)}(X_s), dW_s\rangle \qquad \text{in } L^2(\Omega, \mathbb{P}).
$$

For this purpose we use the isometry formula together with

$$
\lim_{m\to\infty} \int_r^t \mathbb{E}\big|Du_{n,m}^{(i)}(X_s) - Du_n^{(i)}(X_s)\big|^2\, ds = 0
$$

[which can be proved arguing as in (48)]. Thus we have proved (45).

In order to pass to the limit as $n \to \infty$ in (45), we recall formula (37) and argue as before [using also that $\pi_t(x,\cdot) \ll \mu$]. We find

$$
\begin{aligned}
(49)\qquad u^{(i)}(X_t) - u^{(i)}(X_r) = \int_r^t \lambda u^{(i)}(X_s)\, ds - \int_r^t B^{(i)}(X_s)\, ds \\
+ \int_r^t \langle Du^{(i)}(X_s), dW_s\rangle,
\end{aligned}
$$

$t \ge r > 0$. Since u is continuous and trajectories of (X_t) are continuous, we can pass to the limit as $r \to 0^+$ in (49), $\mathbb{P}$-a.s., and obtain an integral identity on $[0, t]$.

But

$$
dX_t^{(i)} = -\lambda_i X_t^{(i)}\, dt + B^{(i)}(X_t)\, dt + dW_t^{(i)}, \qquad \mathbb{P}\text{-a.s.}
$$

Now we proceed as in step 1. Namely, we derive $B^{(i)}(X_t)$ from the identity above and insert in (49); this yields

$$dX_t^{(i)} = -\lambda_i X_t^{(i)}\, dt + \lambda u^{(i)}(X_t)\, dt - du^{(i)}(X_t) + dW_t^{(i)} + \langle Du^{(i)}(X_t), dW_t\rangle,$$

$\mathbb{P}$-a.s. Then we use the variation of constants formula. □

REMARK 8. Formula (49) with $r = 0$ seems to be of independent interest. As an application, one can deduce, when $x \in H$ is deterministic, the representation formula

$$\mathbb{E}[u^{(i)}(X_t)] = \int_0^\infty e^{-\lambda t}\mathbb{E}[B^{(i)}(X_t)]\, dt.$$

This follows by taking the Laplace transform in both sides of (49) (with $r = 0$) and integrating by parts with respect to t.

The next lemma shows that $u(x) = \sum_{k\geq 1} u^{(k)}(x)e_k$ [$u^{(k)}$ as in (31)] is a well-defined function which belongs to $C_b^1(H, H)$. Recall that λ_0 is defined in (19).

LEMMA 9. *For λ sufficiently large, that is, $\lambda \geq \widetilde{\lambda}$, with $\widetilde{\lambda} = \widetilde{\lambda}(A, \|B\|_0)$, there exists a unique $u = u_\lambda \in C_b^1(H, H)$ which solves*

$$u(x) = \int_0^\infty e^{-\lambda t} R_t\big(Du(\cdot)B(\cdot) + B(\cdot)\big)(x)\, dt, \qquad x \in H,$$

where R_t is the OU semigroup defined as in (4) and acting on H-valued functions. Moreover, we have the following assertions:

(i) *Let $\epsilon \in [0, 1/2[$. Then, for any $h \in H$, $(-A)^\epsilon Du(\cdot)[h] \in C_b(H, H)$ and $\|(-A)^\epsilon Du(\cdot)[h]\|_0 \leq C_{\epsilon,\lambda}|h|$;*

(ii) *for any $k \geq 1$, $\langle u(\cdot), e_k\rangle = u^{(k)}$, where $u^{(k)}$ is the solution defined in (31);*

(iii) *there exists $c_3 = c_3(A, \|B\|_0) > 0$ such that, for any $\lambda \geq \widetilde{\lambda}$, $u = u_\lambda$ satisfies*

$$\|Du\|_0 \leq \frac{c_3}{\sqrt{\lambda}}. \tag{50}$$

PROOF. Let $E = C_b^1(H, H)$, and define the operator S_λ,

$$S_\lambda v(x) = \int_0^\infty e^{-\lambda t} R_t\big(Dv(\cdot)B(\cdot) + B(\cdot)\big)(x)\, dt, \qquad v \in E, x \in H.$$

To prove that $S_\lambda : E \to E$, we take into account estimate (11) with $\epsilon = 0$. Note that to check the Fréchet differentiability of $S_\lambda v$ in each $x \in H$, we first show its Gâteaux differentiability. Then using formulas (7) and (11), we obtain the continuity of the Gâteaux derivative from H into $L(H)$. [$L(H)$ denotes the Banach space of all bounded linear operators from H into H endowed with $\|\cdot\|_{\mathcal{L}}$], and this implies, in particular, the Fréchet differentiability.

For $\lambda \geq (\lambda_0 \vee 2\|B\|_0)$, S_λ is a contraction and so there exists a unique $u \in E$ which solves $u = S_\lambda u$. Using again (11) we obtain (i). Moreover, (ii) can be deduced from the fact that, for each $k \geq 1$, $u_k = \langle u(\cdot), e_k\rangle$ is the unique solution to the equation

$$u_k(x) = \int_0^\infty e^{-\lambda t} R_t\big(\langle Du_k(\cdot), B(\cdot)\rangle + B^k(\cdot)\big)(x)\,dt, \qquad x \in H,$$

in $C_b^1(H)$ (the uniqueness follows by the contraction principle) and also the function $u^{(k)} \in C_b^1(H)$ given in (31) solves such equation. Finally (iii) follows easily from the estimate

$$\|Du\|_0 \leq \frac{C_{1,0}}{\lambda^{1/2}}(\|Du\|_0\|B\|_0 + \|B\|_0), \qquad \lambda \geq (\lambda_0 \vee \|B\|_0). \qquad \square$$

3. Proof of Theorem 1. We start now the proof of pathwise uniqueness.

Let $X = (X_t)$ and $Y = (Y_t)$ be two continuous $\mathcal{F}_t$-adapted mild solutions (defined on the same filtered probability space, solutions with respect to the same cylindrical Wiener process), starting from the same x.

For the time being, x is not specified (it may be also random, $\mathcal{F}_0$-measurable). In the last part of the proof a restriction on x will emerge.

Let us fix $T > 0$. Let $u = u_\lambda : H \to H$ be such that $u(x) = \sum_{i\geq 1} u^{(i)}(x)e_i$, $x \in H$, where $u^{(i)} = u_\lambda^{(i)}$ solve (31) for some λ large enough; see Proposition 5.

By (50) we may assume that $\|Du\|_0 \leq 1/2$. We have, for $t \in [0, T]$,

$$\begin{aligned} X_t - Y_t &= u(Y_t) - u(X_t) \\ &\quad + (\lambda - A)\int_0^t e^{(t-s)A}\big(u(X_s) - u(Y_s)\big)\,ds \\ &\quad + \int_0^t e^{(t-s)A}\big(Du(X_s) - Du(Y_s)\big)\,dW_s. \end{aligned}$$

It follows that

$$\begin{aligned} |X_t - Y_t| \leq \frac{1}{2}|X_t - Y_t| &+ \left|(\lambda - A)\int_0^t e^{(t-s)A}\big(u(X_s) - u(Y_s)\big)\,ds\right| \\ &+ \left|\int_0^t e^{(t-s)A}\big(Du(X_s) - Du(Y_s)\big)\,dW_s\right|. \end{aligned}$$

Let τ be a stopping time to be specified later. Using that $1_{[0,\tau]}(t) = 1_{[0,\tau]}(t)\cdot 1_{[0,\tau]}(s)$, $0 \leq s \leq t \leq T$, we have (cf. [7], page 187)

$$\begin{aligned} &1_{[0,\tau]}(t)|X_t - Y_t| \\ &\quad \leq C 1_{[0,\tau]}(t)\left|(\lambda - A)\int_0^t e^{(t-s)A}\big(u(X_s) - u(Y_s)\big)\,ds\right| \\ &\qquad + C\left|1_{[0,\tau]}(t)\int_0^t e^{(t-s)A}\big(Du(X_s) - Du(Y_s)\big)1_{[0,\tau]}(s)\,dW_s\right|, \end{aligned}$$

where by C we denote any constant which may depend on the assumptions on A, B and T.

Writing $1_{[0,\tau]}(s)X_s = \widetilde{X}_s$ and $1_{[0,\tau]}(s)Y_s = \widetilde{Y}_s$, and, using the Burkholder–Davis–Gundy inequality with a large exponent $q > 2$ which will be determined below, we obtain (recall that $\|\cdot\|$ is the Hilbert–Schmidt norm (cf. [7], Chapter 4) with $C = C_q$),

$$\begin{aligned}
&\mathbb{E}[|\widetilde{X}_t - \widetilde{Y}_t|^q] \\
&\quad \le C\mathbb{E}\biggl[e^{\lambda q t}\biggl|(\lambda - A)\int_0^t e^{(t-s)A}e^{-\lambda s}\bigl(u(X_s) - u(Y_s)\bigr)1_{[0,\tau]}(s)\,ds\biggr|^q\biggr] \\
&\qquad + C\mathbb{E}\biggl[\biggl(\int_0^t 1_{[0,\tau]}(s)\bigl\|e^{(t-s)A}\bigl(Du(X_s) - Du(Y_s)\bigr)\bigr\|^2\,ds\biggr)^{q/2}\biggr].
\end{aligned}$$

In the sequel we introduce a parameter $\theta > 0$, and C_θ will denote suitable constants such that $C_\theta \to 0$ as $\theta \to +\infty$ (the constants may change from line to line). This idea of introducing θ and C_θ is suggested by [21], page 8. Similarly, we will indicate by $C(\lambda)$ suitable constants such that $C(\lambda) \to 0$ as $\lambda \to +\infty$.

From the previous inequality we deduce, multiplying by $e^{-q\theta t}$, for any $\theta > 0$,

$$\begin{aligned}
&\mathbb{E}[e^{-q\theta t}|\widetilde{X}_t - \widetilde{Y}_t|^q] \\
&\quad \le C\mathbb{E}\biggl[\biggl|(\lambda - A)\int_0^t e^{-\theta(t-s)}e^{(t-s)A}\bigl(u(X_s) - u(Y_s)\bigr) \\
&\qquad\qquad \times e^{-\theta s}1_{[0,\tau]}(s)\,ds\biggr|^q\biggr] \\
&\qquad + C\mathbb{E}\biggl[\biggl(\int_0^t e^{-2\theta(t-s)}\bigl\|e^{(t-s)A}\bigl(Du(X_s) - Du(Y_s)\bigr)\bigr\|^2 \\
&\qquad\qquad \times e^{-2\theta s}1_{[0,\tau]}(s)\,ds\biggr)^{q/2}\biggr].
\end{aligned} \tag{51}$$

Let us deal with the first term in the right-hand side. Integrating over $[0, T]$, and assuming $\theta \ge \lambda$, we get

$$\begin{aligned}
&\int_0^T C\mathbb{E}\biggl[\biggl|(\lambda - A)\int_0^t e^{-\theta(t-s)}e^{(t-s)A}\bigl(u(X_s) - u(Y_s)\bigr)e^{-\theta s}1_{[0,\tau]}(s)\,ds\biggr|^q\biggr]dt \\
&\quad = C\mathbb{E}\biggl[\int_0^T \biggl|(\lambda - A)\int_0^t e^{-\theta(t-s)}e^{(t-s)A}\bigl(u(X_s) - u(Y_s)\bigr) \\
&\qquad\qquad \times e^{-\theta s}1_{[0,\tau]}(s)\,ds\biggr|^q\,dt\biggr] \\
&\quad \le I_1 + I_2,
\end{aligned}$$

where

$$I_1 = C2^{q-1}\mathbb{E}\biggl[\int_0^T \biggl|(\theta - A)\int_0^t e^{(t-s)(A-\theta)}\bigl(u(X_s)-u(Y_s)\bigr)e^{-\theta s}1_{[0,\tau]}(s)\,ds\biggr|^q dt\biggr],$$

$$I_2 = C\mathbb{E}\biggl[\int_0^T \biggl|\int_0^t 2\theta e^{-\theta(t-s)}e^{(t-s)A}\bigl(u(X_s)-u(Y_s)\bigr)e^{-\theta s}1_{[0,\tau]}(s)\,ds\biggr|^q dt\biggr],$$

Let us estimate I_1 and I_2 separately. To estimate I_1, we use the L^q-maximal inequality; see, for instance, [24], Section 1. This implies that, $\mathbb{P}$-a.s.,

$$\int_0^T \biggl|(\theta - A)\int_0^t e^{(t-s)(A-\theta)}\bigl(u(X_s)-u(Y_s)\bigr)e^{-\theta s}1_{[0,\tau]}(s)\,ds\biggr|^q dt$$
$$\le C_4\int_0^T e^{-\theta q s}\bigl|u(X_s)-u(Y_s)\bigr|^q 1_{[0,\tau]}(s)\,ds,$$

where it is important to remark that C_4 is independent on $\theta > 0$. To see this, look at [24], Theorem 1.6, page 74, and note that for a fixed $\alpha \in (\pi/2, \pi)$, there exists $c = c(\alpha)$ such that for any $\theta > 0$, $\mu \in \mathbb{C}$, $\mu \neq 0$, such that $|\arg(\mu)| < \alpha$, we have

$$\bigl\|\bigl(\mu - (A-\theta)\bigr)^{-1}\bigr\|_{\mathcal{L}} \le \frac{c(\alpha)}{|\mu|}. \tag{52}$$

Continuing we get

$$I_1 \le C(\lambda)\int_0^T e^{-\theta q s}|\widetilde{X}_s - \widetilde{Y}_s|^q\,ds$$

with $C(\lambda) = C_0\|Du\|_0^q \to 0$ as $\lambda \to +\infty$.

Let us deal with the term I_2. Given $t \in (0, T]$, the function $s \mapsto \theta\, e^{-\theta(t-s)}(1 - e^{-\theta t})^{-1}$ is a probability density on $[0, t]$, and thus, by Jensen's inequality,

$$\begin{aligned}
I_2 &= C2^q\mathbb{E}\biggl[\int_0^T (1-e^{-\theta t})^q \\
&\qquad \times \biggl|\int_0^t e^{(t-s)A}\bigl(u(X_s)-u(Y_s)\bigr)e^{-\theta s}1_{[0,\tau]}(s)\frac{\theta e^{-\theta(t-s)}}{1-e^{-\theta t}}\,ds\biggr|^q dt\biggr] \\
&\le \widetilde{C}\mathbb{E}\biggl[\int_0^T (1-e^{-\theta t})^q\int_0^t \bigl|u(X_s)-u(Y_s)\bigr|^q e^{-q\theta s}1_{[0,\tau]}(s)\frac{\theta e^{-\theta(t-s)}}{1-e^{-\theta t}}\,ds\,dt\biggr] \\
&\le \widetilde{C}\|Du\|_0^q\mathbb{E}\biggl[\int_0^T (1-e^{-\theta t})^{q-1}\int_0^t \theta e^{-\theta(t-s)}|\widetilde{X}_s - \widetilde{Y}_s|^q e^{-q\theta s}\,ds\,dt\biggr] \\
&= \widetilde{C}\|Du\|_0^q\mathbb{E}\biggl[\int_0^T \biggl(\int_s^T (1-e^{-\theta t})^{q-1}\theta e^{-\theta(t-s)}\,dt\biggr)|\widetilde{X}_s - \widetilde{Y}_s|^q e^{-q\theta s}\,ds\biggr] \\
&\le \widetilde{C}(\lambda)\mathbb{E}\biggl[\int_0^T |\widetilde{X}_s - \widetilde{Y}_s|^q e^{-q\theta s}\,ds\biggr],
\end{aligned}$$

because $\int_s^T (1-e^{-\theta t})^{q-1}\theta e^{-\theta(t-s)}\,dt \le 1$, for any $\theta \ge \lambda$. Thus we have found

$$\mathbb{E}\left[\left|(\lambda - A)\int_0^t e^{-\theta(t-s)}e^{(t-s)A}\big(u(X_s)-u(Y_s)\big)e^{-\theta s}1_{[0,\tau]}(s)\,ds\right|^q\right] \le C(\lambda)\mathbb{E}\left[\int_0^T |\widetilde{X}_s - \widetilde{Y}_s|^q e^{-q\theta s}\,ds\right]. \tag{53}$$

Now let us estimate the second term on the right-hand side of (51). For $t>0$ fixed, Lemma 23 from Appendix A.2 implies that $ds \otimes \mathbb{P}$-a.s. on $[0,t]\times\Omega$

$$\begin{aligned}
&\big\|e^{(t-s)A}\big(Du(X_s)-Du(Y_s)\big)\big\|^2\\
&\quad= \sum_{n\ge 1} e^{-2\lambda_n(t-s)}\big|Du^{(n)}(X_s)-Du^{(n)}(Y_s)\big|^2\\
&\quad= \sum_{k\ge 1}\sum_{n\ge 1} e^{-2\lambda_n(t-s)}\big|D_k u^{(n)}(X_s)-D_k u^{(n)}(Y_s)\big|^2\\
&\quad= \sum_{k,n\ge 1} e^{-2\lambda_n(t-s)}\left|\int_0^1 \langle DD_k u^{(n)}(Z_s^r), X_s - Y_s\rangle\,dr\right|^2\\
&\quad\le \sum_{n\ge 1} e^{-2\lambda_n(t-s)}\left(\int_0^1 \big\|D^2u^{(n)}(Z_s^r)\big\|^2\,dr\right)|X_s-Y_s|^2\\
&\quad= \int_0^1 \left(\sum_{n\ge 1} e^{-2\lambda_n(t-s)}\big\|D^2u^{(n)}(Z_s^r)\big\|^2\right)dr\,|X_s-Y_s|^2,
\end{aligned}$$

where $D_k u^{(n)} = \langle Du^{(n)}, e_k\rangle$, $D_h D_k u^{(n)} = \langle D^2u^{(n)}e_h, e_k\rangle$ and $\|D^2u^{(n)}(z)\|^2 = \sum_{h,k\ge 1}|D_h D_k u^{(n)}(z)|^2$, for μ-a.e. $z \in H$, and as before,

$$Z_t^r = Z_t^{r,x} = rX_t + (1-r)Y_t.$$

Integrating the second term in (51) in t over $[0,T]$, we thus find

$$\begin{aligned}
\Gamma_T &:= \int_0^T \mathbb{E}\left[\left(\int_0^t e^{-2\theta(t-s)}1_{[0,\tau]}(s)e^{-2\theta s}\right.\right.\\
&\qquad\qquad \left.\left.\times \big\|e^{(t-s)A}\big(Du(X_s)-Du(Y_s)\big)\big\|^2\,ds\right)^{q/2}\right]dt\\
&\le \int_0^T \mathbb{E}\left[\left(\int_0^t e^{-2\theta(t-s)}1_{[0,\tau]}(s)\right.\right.\\
&\qquad\qquad \times \int_0^1\left(\sum_{n\ge 1} e^{-2\lambda_n(t-s)}\big\|D^2u^{(n)}(Z_s^r)\big\|^2\right)dr\\
&\qquad\qquad\qquad \left.\left.\times e^{-2\theta s}|X_s-Y_s|^2\,ds\right)^{q/2}\right]dt.
\end{aligned}$$

Now we consider $\delta \in (0,1)$ such that $(-A)^{-1+\delta}$ is of finite trace. Then

$$\begin{aligned}\Gamma_T &\le \int_0^T \mathbb{E}\biggl[\biggl(\int_0^t \frac{e^{-2\theta(t-s)}}{(t-s)^{1-\delta}} 1_{[0,\tau]}(s) \\ &\qquad \times \int_0^1 \biggl(\sum_{n\ge 1} (\lambda_n(t-s))^{1-\delta} \\ &\qquad\qquad \times e^{-2\lambda_n(t-s)} \frac{\|D^2 u^{(n)}(Z_s^r)\|^2}{\lambda_n^{1-\delta}}\biggr) dr \\ &\qquad\qquad\qquad \times e^{-2\theta s}|X_s - Y_s|^2 \, ds\biggr)^{q/2}\biggr] dt \\ &\le C \int_0^T \mathbb{E}\biggl[\biggl(\int_0^t \frac{e^{-2\theta(t-s)}}{(t-s)^{1-\delta}} 1_{[0,\tau]}(s) \\ &\qquad \times \int_0^1 \biggl(\sum_{n\ge 1} \frac{1}{\lambda_n^{1-\delta}} \|D^2 u^{(n)}(Z_s^r)\|^2\biggr) dr \\ &\qquad\qquad \times e^{-2\theta s}|X_s - Y_s|^2 \, ds\biggr)^{q/2}\biggr] dt.\end{aligned}$$

Let us explain the motivation of the previous estimates: on the one side we isolate the term $\frac{e^{-2\theta(t-s)}}{(t-s)^{1-\delta}}$ which will produce a constant C_θ arbitrarily small for large θ; on the other side, we keep the term $\frac{1}{\lambda_n^{1-\delta}}$ in the series $\sum_{n\ge 1} \frac{1}{\lambda_n^{1-\delta}} \|D^2 u^{(n)}(Z_s^r)\|^2$; otherwise, later on (in the next proposition), we could not evaluate high powers of this series.

Using the (triple) Hölder inequality in the integral with respect to s, with $\frac{2}{q} + \frac{1}{\beta} + \frac{1}{\gamma} = 1$, $\gamma > 1$ and $\beta > 1$ such that $(1-\delta)\beta < 1$, and Jensen's inequality in the integral with respect to r, we find

$$\Gamma_T \le \widetilde{C}_\theta \mathbb{E}\biggl[\Lambda_T \int_0^T e^{-q\theta s} |\widetilde{X}_s - \widetilde{Y}_s|^q \, ds\biggr], \tag{54}$$

where

$$\widetilde{C}_\theta = \biggl(\int_0^T \frac{e^{-2\beta\theta r}}{r^{(1-\delta)\beta}} \, dr\biggr)^{q/2\beta}$$

(which converges to zero as $\theta \to \infty$) and

$$\Lambda_T := \int_0^T \biggl(\int_0^t 1_{[0,\tau]}(s) \int_0^1 \biggl(\sum_{n\ge 1} \frac{1}{\lambda_n^{1-\delta}} \|D^2 u^{(n)}(Z_s^r)\|^2\biggr)^\gamma dr \, ds\biggr)^{q/2\gamma} dt.$$

We may choose $\gamma = \frac{q}{2}$ so that $\frac{q}{2\gamma} = 1$. This is compatible with the other constraints, namely $q > 2$, $\frac{2}{q} + \frac{1}{\beta} + \frac{1}{\gamma} = 1$, $\beta > 1$ such that $(1-\delta)\beta < 1$, because we may choose $\beta > 1$ arbitrarily close to 1 and then solve $\frac{4}{q} + \frac{1}{\beta} = 1$ for q, which would require $q > 4$. So, from now on we fix $q \in (4, \infty)$ and $\gamma = q/2$. Hence

$$\Lambda_T := \int_0^T \int_0^t 1_{[0,\tau]}(s) \int_0^1 \Big(\sum_{n \ge 1} \frac{1}{\lambda_n^{1-\delta}} \| D^2 u^{(n)}(Z_s^r) \|^2 \Big)^\gamma \, dr \, ds \, dt$$
$$\le T \cdot \int_0^{T \wedge \tau} \int_0^1 \Big(\sum_{n \ge 1} \frac{1}{\lambda_n^{1-\delta}} \| D^2 u^{(n)}(Z_s^r) \|^2 \Big)^\gamma \, dr \, ds.$$

Define now, for any $R > 0$, the stopping time

$$\tau_R^x = \inf\Big\{ t \in [0, T] : \int_0^t \int_0^1 \Big(\sum_{n \ge 1} \frac{1}{\lambda_n^{1-\delta}} \| D^2 u^{(n)}(Z_s^r) \|^2 \Big)^\gamma \, dr \, ds \ge R \Big\}$$

and $\tau_R^x = T$ if this set is empty. Take $\tau = \tau_R^x$ in the previous expressions and collect the previous estimates. Using also (53) we get from (51), for any $\theta \ge \lambda$,

$$\begin{aligned}
&\int_0^T e^{-q\theta t} \mathbb{E}|\widetilde{X}_t - \widetilde{Y}_t|^q \, dt \\
&\qquad \le C(\lambda) \int_0^T e^{-q\theta s} \mathbb{E}|\widetilde{X}_s - \widetilde{Y}_s|^q \, ds \\
&\qquad\quad + \widetilde{C}_\theta R \int_0^T e^{-q\theta s} \mathbb{E}|\widetilde{X}_s - \widetilde{Y}_s|^q \, ds.
\end{aligned}$$

Now we fix λ large enough such that $C(\lambda) < 1$ and consider θ greater of such λ.

For sufficiently large $\theta = \theta_R$, *depending on* R,

$$\mathbb{E}\Big[\int_0^T e^{-q\theta_R t} 1_{[0,\tau_R]}(t) |X_t - Y_t|^q \, dt \Big] = \mathbb{E}\Big[\int_0^{\tau_R} e^{-q\theta_R t} |X_t - Y_t|^q \, dt \Big] = 0.$$

In other words, for every $R > 0$, $\mathbb{P}$-a.s., $X = Y$ on $[0, \tau_R]$ (identically in t, since X and Y are continuous processes). We have $\lim_{R \to \infty} \tau_R = T$, $\mathbb{P}$-a.s., because of the next proposition. Hence, $\mathbb{P}$-a.s., $X = Y$ on $[0, T]$, and the proof is complete.

PROPOSITION 10. *For μ-a.e. $x \in H$, we have $\mathbb{P}(S_T^x < \infty) = 1$, where*

$$S_T^x = \int_0^T \int_0^1 \Big(\sum_{n \ge 1} \frac{1}{\lambda_n^{1-\delta}} \| D^2 u^{(n)}(Z_s^r) \|^2 \Big)^\gamma \, dr \, ds$$

with $\gamma = q/2$. The result is true also for a random $\mathcal{F}_0$-measurable, H-valued initial condition under the assumptions stated in Theorem 1.

PROOF. We will show that, for any $x \in H$, μ-a.s.,

$$\mathbb{E}[S_T^x] < +\infty.$$

We will also show this result for random initial conditions under the specified assumptions.

Step 1. In this step $x \in H$ is given, without restriction. Moreover, the result is true for a general $\mathcal{F}_0$-measurable initial condition x without restrictions on its law.

We have

$$Z_t^r = e^{tA}x + \int_0^t e^{(t-s)A}\bar{B}_s^r\,ds + \int_0^t e^{(t-s)A}\,dW_s,$$

where

$$\bar{B}_s^r = \bigl[rB(X_s) + (1-r)B(Y_s)\bigr], \qquad r \in [0,1].$$

Define

$$\rho_r = \exp\biggl(-\int_0^t \bar{B}_s^r\,dW_s - \frac{1}{2}\int_0^t |\bar{B}_s^r|^2\,ds\biggr).$$

We have, since $|\bar{B}_s^r| \le \|B\|_0$,

$$\mathbb{E}\biggl[\exp\biggl(k\int_0^T |\bar{B}_s^r|^2\,ds\biggr)\biggr] \le C_k < \infty \tag{55}$$

for all $k \in \mathbb{R}$, independently of x and r, simply because B is bounded. Hence an infinite-dimensional version of Girsanov's theorem with respect to a cylindrical Wiener process (the proof of which is included in the Appendix; see Theorem 13) applies and gives us that

$$\widetilde{W}_t := W_t + \int_0^t \bar{B}_s^r\,ds$$

is a cylindrical Wiener process on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \widetilde{\mathbb{P}}_r)$ where $\frac{d\widetilde{\mathbb{P}}_r}{d\mathbb{P}}|_{\mathcal{F}_T} = \rho_r$. Hence

$$Z_t^r = e^{tA}x + \int_0^t e^{(t-s)A}\,d\widetilde{W}_s$$

is the sum of a stochastic integral which is Gaussian with respect to $\widetilde{\mathbb{P}}_r$, plus the independent (because $\mathcal{F}_0$-measurable) random variable $e^{tA}x$. Its law is uniquely determined by A, r and the law of x.

Denote by $W_A(t)$ the process

$$W_A(t) := \int_0^t e^{(t-s)A}\,dW_s.$$

We have $e^{\cdot A}x + W_A(\cdot) = Z^r$ in law. We have

$$\begin{aligned}\mathbb{E}[S_T^x] &= \mathbb{E}\biggl[\int_0^T\int_0^1\biggl(\sum_{n\ge1}\frac{1}{\lambda_n^{1-\delta}}\|D^2u^{(n)}(Z_s^r)\|^2\biggr)^{\gamma}\,dr\,ds\biggr]\\ &= \int_0^T\int_0^1 \mathbb{E}\biggl[\biggl(\sum_{n\ge1}\frac{1}{\lambda_n^{1-\delta}}\|D^2u^{(n)}(Z_s^r)\|^2\biggr)^{\gamma}\biggr]\,dr\,ds.\end{aligned} \tag{56}$$

Applying the Girsanov theorem, we find, for $r \in [0, 1]$,

$$\begin{aligned}\mathbb{E}\Big[\rho_r^{-1/2}\rho_r^{1/2}\Big(\sum_{n\geq 1}\frac{1}{\lambda_n^{1-\delta}}\|D^2u^{(n)}(Z_s^r)\|^2\Big)^{\gamma}\Big]\\ \leq (\mathbb{E}[\rho_r^{-1}])^{1/2}\Big(\mathbb{E}\Big[\rho_r\Big(\sum_{n\geq 1}\frac{1}{\lambda_n^{1-\delta}}\|D^2u^{(n)}(Z_s^r)\|^2\Big)^{2\gamma}\Big]\Big)^{1/2}\\ \leq \mathbb{E}[\rho_r^{-1}] + \mathbb{E}\Big[\Big(\sum_{n\geq 1}\frac{1}{\lambda_n^{1-\delta}}\|D^2u^{(n)}(e^{sA}x + W_A(s))\|^2\Big)^{2\gamma}\Big].\end{aligned}$$

By (56) it follows that

$$\begin{aligned}\mathbb{E}[S_T^x] &\leq T\int_0^1 \mathbb{E}[\rho_r^{-1}]\,dr\\ &\quad + \int_0^T \mathbb{E}\Big[\Big(\sum_{n\geq 1}\frac{1}{\lambda_n^{1-\delta}}\|D^2u^{(n)}(e^{sA}x + W_A(s))\|^2\Big)^{2\gamma}\Big]\,ds.\end{aligned}\tag{57}$$

Step 2. We have $\mathbb{E}[\rho_r^{-1}] \leq C < \infty$ independently of $x \in H$ (also in the case of an $\mathcal{F}_0$-measurable x) and $r \in [0, 1]$. Indeed,

$$\begin{aligned}\mathbb{E}[\rho_r^{-1}] &= \mathbb{E}\Big[\exp\Big(\int_0^t \bar{B}_s^r\,dW_s + \frac{1}{2}\int_0^t |\bar{B}_s^r|^2\,ds\Big)\Big]\\ &= \mathbb{E}\Big[\exp\Big(\int_0^t \bar{B}_s^r\,dW_s + \Big(\frac{3}{2}-1\Big)\int_0^t |\bar{B}_s^r|^2\,ds\Big)\Big]\\ &\leq \mathbb{E}\Big[\exp\Big(\int_0^t 2\bar{B}_s^r\,dW_s - \frac{1}{2}\int_0^t |2\bar{B}_s^r|^2\,ds\Big)\Big]^{1/2} C_3^{1/2}\end{aligned}$$

by (55). But

$$\mathbb{E}\Big[\exp\Big(\int_0^t 2\bar{B}_s^r\,dW_s - \frac{1}{2}\int_0^t |2\bar{B}_s^r|^2\,ds\Big)\Big] = 1,$$

because of Girsanov's theorem. Therefore, $\mathbb{E}[\rho_r^{-1}]$ is bounded uniformly in x and r.

Step 3. Let us come back to (57). To prove that $\mathbb{E}[S_T^x] < +\infty$ and hence finish the proof, it is enough to verify that

$$\int_0^T \mathbb{E}\Big[\Big(\sum_{n\geq 1}\frac{1}{\lambda_n^{1-\delta}}\|D^2u^{(n)}(e^{sA}x + W_A(s))\|^2\Big)^{2\gamma}\Big]\,ds < \infty.\tag{58}$$

If μ_s^x denotes the law of $e^{sA}x + W_A(s)$, we have to prove that

$$\int_0^T \int_H \Big(\sum_{n\geq 1}\frac{1}{\lambda_n^{1-\delta}}\|D^2u^{(n)}(y)\|^2\Big)^{2\gamma}\mu_s^x(dy)\,ds < \infty.\tag{59}$$

Now we check (58) for deterministic $x \in H$. In step 4 below, we will consider the case where x is an $\mathcal{F}_0$-measurable r.v.

We estimate

$$\left(\sum_{n\geq 1}\frac{1}{\lambda_n^{1-\delta}}\|D^2u^{(n)}(y)\|^2\right)^{2\gamma} \leq \left(\sum_{n\geq 1}\frac{1}{\lambda_n^{(1-\delta)(2\gamma/(2\gamma-1))}}\right)^{2\gamma-1}\sum_{n\geq 1}\|D^2u^{(n)}(y)\|^{4\gamma}.$$

Since $\frac{2\gamma}{2\gamma-1} > 1$ we have $\sum_{n\geq 1}\frac{1}{\lambda_n^{(1-\delta)(2\gamma/(2\gamma-1))}} < \infty$. Hence we have to prove that

$$\int_0^T\int_H\sum_{n\geq 1}\|D^2u^{(n)}(y)\|^{4\gamma}\mu_s^x(dy)\,ds < \infty. \tag{60}$$

Unfortunately, we cannot verify (60) for an individual deterministic $x \in H$. On the other hand, by (23) we know that, for any $\eta \geq 2$,

$$\int_H\|D^2u^{(n)}(z)\|^{\eta}\mu(dz) \leq C_\eta\int_H|B^{(n)}(x)|^{\eta}\mu(dx),$$

where C_η is independent of n. Hence we obtain

$$\begin{aligned}\int_H\sum_{n\geq 1}\|D^2u^{(n)}(y)\|^{4\gamma}\mu(dy) &\leq C_{4\gamma}\int_H\sum_{n\geq 1}|B^{(n)}(y)|^{4\gamma}\mu(dy)\\ &\leq C_{4\gamma}\|B\|_0^{4\gamma-2}\int_H|B(x)|^2\mu(dx)\\ &\leq C_{4\gamma}\|B\|_0^{4\gamma}.\end{aligned}$$

This estimate is clearly related to (60) since the law μ_s^x is equivalent to μ for every $s > 0$ and x. The problem is that $\frac{d\mu_s^x}{d\mu}$ degenerates too strongly at $s = 0$. Therefore we use the fact that

$$\int_H\left(\int_H f(z)\mu_s^x(dz)\right)\mu(dx) = \int_H f(z)\mu(dz)$$

for all $s \geq 0$, for every nonnegative measurable function f. Thus, for any $s \geq 0$ with $f(y) = \sum_{n\geq 1}\|D^2u^{(n)}(y)\|^{4\gamma}$, we get

$$\begin{aligned}&\int_H\left(\int_H\sum_{n\geq 1}\|D^2u^{(n)}(y)\|^{4\gamma}\mu_s^x(dy)\right)\mu(dx)\\ &\quad= \int_H\sum_{n\geq 1}\|D^2u^{(n)}(y)\|^{4\gamma}\mu(dy)\\ &\quad\leq C_{4\gamma}\|B\|_0^{4\gamma} < \infty.\end{aligned}$$

Step 4. We prove (58) in the case of a random initial condition x $\mathcal{F}_0$-measurable with law μ_0 such that $\mu_0 \ll \mu$ and $\int_H h_0^\zeta\,d\mu < \infty$ for some $\zeta > 1$, where $h_0 := \frac{d\mu_0}{d\mu}$.

Denote by μ_s the law of $e^{sA}x + W_A(s)$, $s \geq 0$. We have to prove that

$$\int_0^T \int_H \biggl(\sum_{n\geq 1} \frac{1}{\lambda_n^{1-\delta}} \|D^2 u^{(n)}(y)\|^2\biggr)^{2\gamma} \mu_s(dy)\, ds < \infty. \tag{61}$$

But, since $e^{sA}x$ and $W_A(s)$ are independent, it follows that

$$\mu_s(dz) = \int_H \mu_s^y(dz)\mu_0(dy).$$

Hence, for every Borel measurable $f : H \to \mathbb{R}$, if $\frac{1}{\zeta} + \frac{1}{\zeta'} = 1$, with $\zeta > 1$, we have

$$\int_H |f(y)|\mu_s(dy) \leq \|h_0\|_{L^\zeta(\mu)} \|f\|_{L^{\zeta'}(\mu)}. \tag{62}$$

By (62), we have (similar to step 3)

$$\begin{aligned} a_T &:= \int_0^T \int_H \biggl(\sum_{n\geq 1} \frac{1}{\lambda_n^{1-\delta}} \|D^2 u^{(n)}(y)\|^2\biggr)^{2\gamma} \mu_s(dy)\, ds \\ &\leq T \|h_0\|_{L^\zeta(\mu)} \biggl(\int_H \biggl(\sum_{n\geq 1} \frac{1}{\lambda_n^{1-\delta}} \|D^2 u^{(n)}(y)\|^2\biggr)^{2\gamma\zeta'} \mu(dy)\biggr)^{1/\zeta'}. \end{aligned}$$

By

$$\begin{aligned} &\biggl(\sum_{n\geq 1} \frac{1}{\lambda_n^{1-\delta}} \|D^2 u^{(n)}(y)\|^2\biggr)^{2\gamma\zeta'} \\ &\qquad \leq \biggl(\sum_{n\geq 1} \frac{1}{\lambda_n^{(1-\delta)(2\gamma\zeta'/(2\gamma\zeta'-1))}}\biggr)^{2\gamma\zeta'-1} \sum_{n\geq 1} \|D^2 u^{(n)}(y)\|^{4\gamma\zeta'} \\ &\qquad \leq C \sum_{n\geq 1} \|D^2 u^{(n)}(y)\|^{4\gamma\zeta'}, \end{aligned}$$

we obtain

$$a_T \leq CT \|h_0\|_{L^\zeta(\mu)} \biggl(\int_H \sum_{n\geq 1} \|D^2 u^{(n)}(y)\|^{4\gamma\zeta'} \mu(dy)\biggr)^{1/\zeta'},$$

which is finite since

$$\begin{aligned} &\int_H \sum_{n\geq 1} \|D^2 u^{(n)}(y)\|^{4\gamma\zeta'} \mu(dy) \\ &\qquad \leq C_{4\gamma\zeta'} \|B\|_0^{4\gamma\zeta'-2} \int_H |B(x)|^2 \mu(dx) \\ &\qquad \leq C_{4\gamma\zeta'} \|B\|_0^{4\gamma\zeta'}. \end{aligned}$$

The proof is complete. □

REMARK 11. Let us comment on the crucial assertion (59), that is,

$$\int_0^T \int_H \biggl(\sum_{n\geq 1} \frac{1}{\lambda_n^{1-\delta}} \|D^2 u^{(n)}(y)\|^2\biggr)^{2\gamma} \mu_s^x(dy)\,ds < \infty.$$

This holds in particular if for some $p > 1$ (large enough), we have

$$\int_0^T R_s|f|(x)\,ds \leq C_{x,T,p}\|f\|_{L^p(\mu)} \tag{63}$$

for any $f \in L^p(\mu)$ [here R_t is the OU Markov semigroup; see (4)]. *Note that if this assertion holds for any $x \in H$, then we have pathwise uniqueness for all initial conditions $x \in H$.* But so far, we could not prove or disprove (63). We expect, however, that (63) in infinite dimensions is not true for all $x \in H$.

When $H = \mathbb{R}^d$ one can show that if $p > c(d)$, then (63) holds for any $x \in H$, and so we have uniqueness for all initial conditions. Therefore, in finite dimension, our method could also provide an alternative approach to the Veretennikov result. In this respect, note that in finite dimension the SDE $dX_t = b(X_t)\,dt + dW_t$ is equivalent to $dX_t = -X_t\,dt + (b(X_t) + X_t)\,dt + dW_t$ which is in the form (1) with $A = -I$, but with linearly growing drift term $B(x) = b(x) + x$. Strictly speaking, we can only recover Veretennikov's result if we realize the generalization mentioned in Remark 12(i) below. In this alternative approach, basically the elliptic L^p-estimates with respect to the Lebesgue measure used in [33] are replaced by elliptic $L^p(\mu)$-estimates using the Girsanov theorem.

Let us check (63) when $H = \mathbb{R}^d$ and $x = 0$ for simplicity. By [8], Lemma 10.3.3, we know that

$$R_t f(x) = \int_H f(y) k_t(x, y)\mu(dy)$$

and moreover, according to [8], Lemma 10.3.8, for $p' \geq 1$,

$$\begin{aligned}&\biggl(\int_H \bigl(k_t(0, y)\bigr)^{p'} \mu(dy)\biggr)^{1/p'}\\ &\qquad = \det\bigl(I - e^{2tA}\bigr)^{-1/2+1/(2p')} \det\bigl(I + (p'-1)e^{2tA}\bigr)^{-1/(2p')}.\end{aligned}$$

By the Hölder inequality (with $1/p' + 1/p = 1$),

$$\int_0^t R_r f(0)\,dr \leq \biggl(\int_H f(y)^p \mu(dy)\biggr)^{1/p} \int_0^t \biggl(\int_H \bigl(k_r(0, y)\bigr)^{p'} \mu(dy)\biggr)^{1/p'} dr.$$

Thus (63) holds with $x = 0$ if for some $p' > 1$ near 1,

$$\begin{aligned}&\int_0^t \biggl(\int_H \bigl(k_r(0, y)\bigr)^{p'} \mu(dy)\biggr)^{1/p'} dr\\ &\quad = \int_0^t \bigl[\det\bigl(I - e^{2rA}\bigr)\bigr]^{-1/2+1/(2p')} \bigl[\det\bigl(I + (p'-1)e^{2rA}\bigr)\bigr]^{-1/(2p')} dr \qquad (64)\\ &\quad < +\infty.\end{aligned}$$

It is easy to see that in $\mathbb{R}^d$ there exists $1 < K(d) < 2$ such that for $1 < p' < K(d)$, assertion (64) holds.

REMARK 12. (i) We expect to prove more generally uniqueness for $B : H \to H$ which is at most of linear growth (in particular, bounded on each balls) by using a stopping time argument.

(ii) We also expect to implement the uniqueness result to drifts B which are also time dependent. However, to extend our method we need parabolic L^p_μ-estimate involving the Ornstein–Uhlenbeck operator which are not yet available in the literature.

4. Examples.

We discuss some examples in several steps. First we show a simple one-dimensional example of wild nonuniqueness due to noncontinuity of the drift. Then we show two infinite dimensional, very natural generalizations of this example. However, both of them do not fit perfectly with our purposes, so they are presented mainly to discuss possible phenomena. Finally, in Section 4.4, we modify the previous examples in such a way to get a very large family of *deterministic problems with nonuniqueness for all initial conditions*, which fits the assumptions of our result of uniqueness by noise.

4.1. *An example in dimension one.* In dimension 1, one of the simplest and more dramatic examples of nonuniqueness is the equation

$$\frac{d}{dt} X_t = b_{\mathrm{Dir}}(X_t), \qquad X_0 = x,$$

where

$$b_{\mathrm{Dir}}(x) = \begin{cases} 1, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \\ 0, & \text{if } x \in \mathbb{Q} \end{cases}$$

(the so-called Dirichlet function). Let us call solution any continuous function X_t such that

$$X_t = x + \int_0^t b_{\mathrm{Dir}}(X_s)\, ds$$

for all $t \geq 0$. For every x, the function

$$X_t = x + t$$

is a solution: indeed, $X_s \in \mathbb{R} \setminus \mathbb{Q}$ for a.e. s, hence $b_{\mathrm{Dir}}(X_s) = 1$ for a.e. s, hence $\int_0^t b_{\mathrm{Dir}}(X_s)\, ds = t$ for all $t \geq 0$. But from $x \in \mathbb{Q}$ we have also the solution

$$\widetilde{X}_t = x,$$

because $\widetilde{X}_s \in \mathbb{Q}$ for all $s \geq 0$ and thus $b_{\mathrm{Dir}}(X_s) = 0$ for all $s \geq 0$. Therefore, we have nonuniqueness from every initial condition $x \in \mathbb{Q}$. Not only: for every x and every $\varepsilon > 0$, there are infinitely many solutions on $[0, \varepsilon]$. Indeed, one can start with the solution $X_t = x + t$ and branch at any $t_0 \in [0, \varepsilon]$ such that $x + t_0 \in \mathbb{Q}$, continuing with the constant solution. Therefore, in a sense, there is nonuniqueness from every initial condition.

4.2. *First infinite-dimensional generalization* (*not of parabolic type*). This example can be immediately generalized to infinite dimensions by taking $H = l^2$ (the space of square summable sequences),

$$B_{\mathrm{Dir}}(x) = \sum_{n=1}^{\infty} \alpha_n b_{\mathrm{Dir}}(x_n) e_n,$$

where $x = (x_n)$, (e_n) is the canonical basis of H, and α_n are positive real numbers such that $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$. The mapping B is well defined from H to H, it is Borel measurable and bounded, but of course not continuous. Given an initial condition $x = (x_n) \in H$, if a function $X(t) = (X_n(t))$ has all components $X_n(t)$ which satisfy

$$X_n(t) = x_n + \int_0^t \alpha_n b_{\mathrm{Dir}}\big(X_n(s)\big)\, ds,$$

then $X(t) \in H$ and is continuous in H (we see this from the previous identity), and satisfies

$$X(t) = \sum_{n=1}^{\infty} X_n(t) e_n = x + \int_0^t B_{\mathrm{Dir}}\big(X(s)\big)\, ds.$$

So we see that this equation has infinitely many solutions from every initial condition.

Unfortunately our theory of regularization by noise cannot treat this simple example of nonuniqueness, because we need a regularizing operator A in the equation to compensate for the regularity troubles introduced by a cylindrical noise.

4.3. *Second infinite-dimensional generalization* (*nonuniqueness only for a few initial conditions*). Let us start in the most obvious way. Namely, consider the equation in $H = l^2$

$$X(t) = e^{tA} x + \int_0^t e^{(t-s)A} B_{\mathrm{Dir}}\big(X(s)\big)\, ds,$$

where

$$Ax = -\sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle e_n$$

with $\lambda_n > 0$, $\sum_{n=1}^{\infty} \frac{1}{\lambda_n^{1-\varepsilon_0}} < \infty$. Componentwise we have

$$X_n(t) = e^{-t\lambda_n} x_n + \int_0^t e^{-(t-s)\lambda_n} b_{\mathrm{Dir}}\big(X_n(s)\big)\, ds.$$

For $x = (x_n) \in H$ with *all* nonzero components x_n, the solution is unique, with components

$$X_n(t) = e^{-t\lambda_n} x_n + \frac{1 - e^{-t\lambda_n}}{\lambda_n}$$

[we have $X_n(s) \in \mathbb{R} \setminus \mathbb{Q}$ for a.e. s, hence $b_{\mathrm{Dir}}(X_n(s)) = 1$; and it is impossible to keep a solution constant on a rational value, due to the term $e^{-t\lambda_n} x_n$ which always appears]. This is also a solution for all x.

But from any initial condition $x = (x_n) \in H$ such that at least one component x_{n_0} is zero, we have at least two solutions: the previous one and any solution such that

$$X_{n_0}(t) = 0.$$

This example fits our theory in the sense that all assumptions are satisfied, so our main theorem of "uniqueness by noise" applies. However, our theorem states only that uniqueness is restored for μ-a.e. x, where μ is the invariant Gaussian measure of the linear stochastic problem, supported on the whole H. We already know that this deterministic problem has uniqueness for μ-a.e. x: it has a unique solution for all x with all components different from zero. Therefore our theorem is not empty but not competitive with the deterministic theory, for this example.

4.4. *Infinite-dimensional examples with wild nonuniqueness*. Let H be a separable Hilbert space with a complete orthonormal system (e_n). Let A be as in the assumptions of this paper. Assume that e_1 is eigenvector of A with eigenvalue $-\lambda_1$. Let $\widetilde{H}$ be the orthogonal to e_1 in H, the span of $e_2, e_3, \dots$, and let $\widetilde{B} : \widetilde{H} \to \widetilde{H}$ be Borel measurable and bounded. Consider $\widetilde{B}$ as an operator in H, by setting $\widetilde{B}(x) = \widetilde{B}(\sum_{n=2}^{\infty} x_n e_n)$.

Let B be defined as

$$B(x) = \big((\lambda_1 x_1) \wedge 1 + b_{\mathrm{Dir}}(x_1)\big)e_1 + \widetilde{B}(x)$$

for all $x = (x_n) \in H$. Then $B : H \to H$ is Borel measurable and bounded. The deterministic equation for the first component $X_1(t)$ is, in differential form,

$$\begin{aligned}\frac{d}{dt} X_1(t) &= -\lambda_1 X_1(t) + \lambda_1 X_1(t) + b_{\mathrm{Dir}}\big(X_1(t)\big)\\ &= b_{\mathrm{Dir}}\big(X_1(t)\big)\end{aligned}$$

as soon as

$$\big|X_1(t)\big| \le 1/\lambda_1.$$

In other words, the full drift $Ax + B(x)$ is given, on $\widetilde{H}$, by a completely general scheme coherent with our assumption (which may have deterministic uniqueness or not); and along e_1 it is the Dirichlet example of Section 4.1, at least as soon as a solution satisfies $|X_1(t)| \le 1/\lambda_1$.

Start from an initial condition x such that

$$|x_1| < 1/\lambda_1.$$

Then, by continuity of trajectories and the fact that any possible solution to the equation satisfies the inequality

$$\left|\frac{d}{dt}X_1(t)\right| \le \lambda_1 |X_1(t)| + 2,$$

there exists $\tau > 0$ such that for every possible solution, we have

$$|X_1(t)| < 1/\lambda_1 \qquad \text{for all } t \in [0, \tau].$$

So, on $[0, \tau]$, all solutions solve $\frac{d}{dt}X_1(t) = b_{\mathrm{Dir}}(X_1(t))$ which has infinitely many solutions (step 1). Therefore also the infinite-dimensional equation has infinitely many solutions.

We have proved that nonuniqueness holds for all $x \in H$ such that x_1 satisfies $|x_1| < 1/\lambda_1$. This set of initial conditions has positive μ-measure; hence we have a class of examples of deterministic equations where nonuniqueness holds for a set of initial conditions with positive μ-measure. Our theorem applies and states for μ-a.e. such initial condition we have uniqueness by noise.

APPENDIX

A.1. Girsanov's theorem in infinite dimensions with respect to a cylindrical Wiener process. In the main body of the paper, the Girsanov theorem for SDEs on Hilbert spaces of type (1) with cylindrical Wiener noise is absolutely crucial. Since a complete and reasonably self-contained proof is hard to find in the literature, for the convenience of the reader, we give a detailed proof of this folklore result (see, e.g., [7, 11, 15] and [12]) in our situation, but *even for at most linearly growing* B. The proof is reduced to the Girsanov theorem of general real valued continuous local martingales; see [30], (1.7) Theorem, page 329.

We consider the situation of the main body of the paper, that is, we are given a negative definite self-adjoint operator $A : D(A) \subset H \to H$ on a separable Hilbert space $(H, \langle\cdot,\cdot\rangle)$ with $(-A)^{-1+\delta}$ being of trace class, for some $\delta \in (0, 1)$, a measurable map $B : H \to H$ of at most linear growth and W a cylindrical Wiener process over H defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ represented in terms of the eigenbasis $\{e_k\}_{k\in\mathbb{N}}$ of $(A, D(A))$ through a sequence

$$W(t) = \left(\beta_k(t)e_k\right)_{k\in\mathbb{N}}, \qquad t \ge 0, \tag{65}$$

where β_k, $k \in \mathbb{N}$, are independent real valued Brownian motions starting at zero on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. Consider the stochastic equations

$$dX(t) = \left(AX(t) + B\left(X(t)\right)\right)dt + dW(t), \qquad t \in [0, T], X(0) = x, \tag{66}$$

and

$$dZ(t) = AZ(t)\,dt + dW(t), \qquad t \in [0, T], Z(0) = x, \tag{67}$$

for some $T > 0$.

THEOREM 13. *Let $x \in H$. Then* (66) *has a unique weak mild solution, and its law $\mathbb{P}_x$ on $C([0,T];H)$ is equivalent to the law $\mathbb{Q}_x$ of the solution to* (67) (*which is just the classical OU process*). *If B is bounded, x may be replaced by an $\mathcal{F}_0$-measurable H-valued random variable.*

The rest of this section is devoted to the proof of this theorem. We first need some preparation and start with recalling that because $\mathrm{Tr}[(-A)^{-1+\delta}] < \infty$, $\delta \in (0,1)$, the stochastic convolution

$$W_A(t) := \int_0^t e^{(t-s)A}\, dW(s), \qquad t \ge 0, \tag{68}$$

is a well defined $\mathcal{F}_t$-adapted stochastic process ("OU process") with continuous paths in H and

$$Z(t,x) := e^{tA}x + W_A(t), \qquad t \in [0,T], \tag{69}$$

is the unique mild solution of (66).

Let $b(t)$, $t \ge 0$, be a progressively measurable H-valued process on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ such that

$$\mathbb{E}\left[\int_0^T |b(s)|^2\, ds\right] < \infty \tag{70}$$

and

$$X(t,x) := Z(t,x) + \int_0^t e^{(t-s)A} b(s)\, ds, \qquad t \in [0,T]. \tag{71}$$

We set

$$W_k(t) := \beta_k(t) e_k, \qquad t \in [0,T], k \in \mathbb{N}, \tag{72}$$

and define

$$Y(t) := \int_0^t \langle b(s), dW(s)\rangle := \sum_{k \ge 1} \int_0^t \langle b(s), e_k\rangle\, dW_k(s), \qquad t \in [0,T]. \tag{73}$$

LEMMA 14. *The series on the right-hand side of* (73) *converges in $L^2(\Omega, \mathbb{P}; C([0,T];\mathbb{R}))$. Hence the stochastic integral $Y(t)$ is well defined and a continuous real-valued martingale, which is square integrable.*

PROOF. We have for all $n, m \in \mathbb{N}$, $n > m$, by Doob's inequality,

$$\begin{aligned} &\mathbb{E}\left[\sup_{t\in[0,T]} \left|\sum_{k=m}^n \int_0^t \langle e_k, b(s)\rangle\, dW_k(s)\right|^2\right] \\ &\quad \le 2\mathbb{E}\left[\left|\sum_{k=m}^n \int_0^T \langle e_k, b(s)\rangle\, dW_k(s)\right|^2\right] \end{aligned}$$

$$= 2 \sum_{k,l=m}^{n} \mathbb{E}\left[\int_0^T \langle e_k, b(s)\rangle \, dW_k(s) \int_0^T \langle e_l, b(s)\rangle \, dW_l(s)\right]$$

$$= 2 \sum_{k=m}^{n} \mathbb{E}\left[\int_0^T \langle e_k, b(s)\rangle^2 \, ds\right] \to 0$$

as $m, n \to \infty$ because of (70). Hence the series on the right-hand side of (73) converges in $L^2(\Omega, \mathbb{P}; C([0, T]; \mathbb{R}))$, and the assertion follows. □

REMARK 15. It can be shown that if $\int_0^t \langle b(s), dW(s)\rangle$, $t \in [0, T]$, is defined as usual, using approximations by elementary functions (see [28], Lemma 2.4.2), the resulting process is the same.

It is now easy to calculate the corresponding variation process $\langle \int_0^{\cdot} \langle b(s), dW(s)\rangle\rangle_t$, $t \in [0, T]$.

LEMMA 16. *We have*

$$\langle Y\rangle_t = \left\langle \int_0^{\cdot} \langle b(s), dW(s)\rangle \right\rangle_t = \int_0^t |b(s)|^2 \, ds, \qquad t \in [0, T].$$

PROOF. We have to show that

$$Y^2(t) - \int_0^t |b(s)|^2 \, ds, \qquad t \in [0, T],$$

is a martingale, that is, for all bounded $\mathcal{F}_t$-stopping times τ, we have

$$\mathbb{E}[Y^2(\tau)] = \mathbb{E}\left[\int_0^\tau |b(s)|^2 \, ds\right],$$

which follows immediately as in the proof of Lemma 14. □

Define the measure

$$\widetilde{\mathbb{P}} := e^{Y(T) - \langle Y\rangle_t/2} \cdot \mathbb{P} \tag{74}$$

on $(\Omega, \mathcal{F})$, which is equivalent to $\mathbb{P}$. Since $\mathcal{E}(t) := e^{Y(T) - \langle Y\rangle_t/2}$, $t \in [0, T]$, is a nonnegative local martingale, it follows by Fatou's lemma that it is a supermartingale, and since $\mathcal{E}(0) = 1$, we have

$$\mathbb{E}[\mathcal{E}(t)] \le \mathbb{E}[\mathcal{E}(0)] = 1.$$

Hence $\widetilde{\mathbb{P}}$ is a sub-probability measure.

PROPOSITION 17. *Suppose that $\widetilde{\mathbb{P}}$ is a probability measure*, *that is*,

$$\mathbb{E}[\mathcal{E}(T)] = 1. \tag{75}$$

Then

$$\widetilde{W}_k(t) := W_k(t) - \int_0^t \langle e_k, b(s)\rangle\, ds, \qquad t \in [0,T], k \in \mathbb{N},$$

are independent real-valued Brownian motions starting at 0 *on* $(\Omega, \mathcal{F}, (\mathcal{F}_t), \widetilde{\mathbb{P}})$, *that is*,

$$\widetilde{W}(t) := \big(\widetilde{W}(t)e_k\big)_{k\in\mathbb{N}}, \qquad t \in [0,T],$$

is a cylindrical Wiener process over H *on* $(\Omega, \mathcal{F}, (\mathcal{F}_t), \widetilde{\mathbb{P}})$.

PROOF. By the classical Girsanov theorem (for general real-valued martingales, see [30], (1.7) Theorem, page 329), it follows that for every $k \in \mathbb{N}$,

$$W_k(t) - \langle W_k, Y\rangle_t, \qquad t \in [0,T],$$

is a local martingale under $\widetilde{\mathbb{P}}$. Set

$$Y_n(t) := \sum_{k=1}^n \int_0^t \langle e_k, b(s)\rangle\, dW_k(s), \qquad t \in [0,T], n \in \mathbb{N}.$$

Then by Cauchy–Schwartz's inequality

$$\big|\langle W_k, Y - Y_n\rangle\big|_t = \langle W_k\rangle_t^{1/2} \langle Y - Y_n\rangle_t^{1/2}, \qquad t \in [0,T],$$

and since

$$\mathbb{E}\big[\langle Y - Y_n\rangle_t\big] = \mathbb{E}\big[(Y - Y_n)^2\big] \to 0 \qquad \text{as } n \to \infty$$

by Lemma 14, we conclude that (selecting a subsequence if necessary) $\mathbb{P}$-a.s. for all $t \in [0,T]$

$$\langle W_k, Y\rangle_t = \lim_{n\to\infty} \langle W_k, Y_n\rangle_t = \int_0^t \langle e_k, b(s)\rangle\, ds,$$

since $\langle W_k, W_l\rangle_t = 0$ if $k \neq l$, by independence. Hence each $\widetilde{W}_k$ is a local martingale under $\widetilde{\mathbb{P}}$.

It remains to show that for every $n \in \mathbb{N}$, $(\widetilde{W}_1, \dots, \widetilde{W}_n)$ is, under $\widetilde{\mathbb{P}}$, an n-dimensional Brownian motion. But $\mathbb{P}$-a.s. for $l \neq k$

$$\langle \widetilde{W}_l, \widetilde{W}_k\rangle_t = \langle W_l, W_k\rangle_t = \delta_{l,k}(t), \qquad t \in [0,T].$$

Since $\mathbb{P}$ is equivalent to $\widetilde{\mathbb{P}}$, this also holds $\widetilde{\mathbb{P}}$-a.s. Hence by Lèvy's characterization theorem ([30], (3.6) Theorem, page 150) it follows that $(\widetilde{W}_1, \dots, \widetilde{W}_n)$ is an n-dimensional Brownian motion in $\mathbb{R}^n$ for all n, under $\widetilde{\mathbb{P}}$. □

PROPOSITION 18. *Let* $W_A(t), t \in [0,T]$, *be defined as in* (68). *Then there exists* $\epsilon > 0$ *such that*

$$\mathbb{E}\Big[\exp\Big\{\epsilon \sup_{t\in[0,T]} \big|W_A(t)\big|\Big\}^2\Big] < \infty.$$

PROOF. Consider the distribution $\mathbb{Q}_0 := \mathbb{P} \circ W_A^{-1}$ of W_A on $E := C([0,T]; H)$. If $\mathbb{Q}_0$ is a Gaussian measure on E, the assertion follows by Fernique's theorem. To show that $\mathbb{Q}_0$ is a Gaussian measure on E, we have to show that for each l in the dual space E' of E, we have that $\mathbb{Q}_0 \circ l^{-1}$ is Gaussian on $\mathbb{R}$. We prove this in two steps.

Step 1. Let $t_0 \in [0,T]$, $h \in H$ and $\ell(\omega) := \langle h, \omega(t_0)\rangle$ for $\omega \in E$. To see that $\mathbb{Q}_0 \circ \ell^{-1}$ is Gaussian on $\mathbb{R}$, consider a sequence $\delta_k \in C([0,T];\mathbb{R})$, $k \in \mathbb{N}$, such that $\delta_k(t)\,dt$ converges weakly to the Dirac measure ϵ_{t_0}. Then for all $\omega \in E$,

$$\ell(\omega) = \lim_{k\to\infty} \int_0^T \langle h, \omega(s)\rangle \delta_k(s)\,ds = \lim_{k\to\infty} \int_0^T \langle h\delta_k(s), \omega(s)\rangle\,ds.$$

Since (e.g., by [4], Proposition 2.15, the law of W_A in $L^2(0,T;H)$ is Gaussian, it follows that the distribution of ℓ is Gaussian.

Step 2. The following argument is taken from [6], Proposition A.2. Let $\omega \in E$; then we can consider its Bernstein approximation

$$\beta_n(\omega)(t) := \sum_{k=1}^n \binom{n}{k} \varphi_{k,n}(t)\omega(k/n), \qquad n \in \mathbb{N},$$

where $\varphi_{k,n}(t) := t^k(1-t)^{n-k}$. But the linear map

$$H \ni x \to \ell(x\varphi_{k,n}) \in \mathbb{R}$$

is continuous in H, and hence there exists $h_{k,n} \in H$ such that

$$\ell(x\varphi_{k,n}) = \langle h_{k,n}, x\rangle, \qquad x \in H.$$

Since $\beta_n(\omega) \to \omega$ uniformly for all $\omega \in E$, it follows that for all $\omega \in E$,

$$\ell(\omega) = \lim_{n\to\infty} \ell(\beta_n(\omega)) = \lim_{n\to\infty} \sum_{k=1}^n \binom{n}{k} \langle h_{k,n}, \omega(k/n)\rangle, \qquad n \in \mathbb{N}.$$

Hence it follows by step 1 that $\mathbb{Q}_0 \circ l^{-1}$ is Gaussian. □

Now we turn to SDE (66) and define

$$\begin{aligned} M &:= e^{\int_0^T \langle B(e^{tA}x + W_A(t)), dW(t)\rangle - (1/2)\int_0^T |B(e^{tA}x+W_A(t))|^2\,dt}, \\ \widetilde{\mathbb{P}}_x &:= M\mathbb{P}. \end{aligned} \tag{76}$$

Obviously, Proposition 19 below implies (70) and that hence M is well defined.

PROPOSITION 19. *$\widetilde{\mathbb{P}}_x$ is a probability measure on $(\Omega, \mathcal{F})$, that is*, $\mathbb{E}(M) = 1$.

PROOF. As before we set $Z(t,x) := e^{tA}x + W_A(t)$, $t \in [0,T]$. By Proposition 18 the arguments below are standard (see, e.g., [22], Corollaries 5.14 and 5.16, pages 199 and 200). Since B is of at most linear growth, by Proposition 18 we can find $N \in \mathbb{N}$ large enough such that for all $0 \le i \le N$ and $t_i := \frac{iT}{N}$,

$$\mathbb{E}\big[e^{(1/2)\int_{t_{i-1}}^{t_i} |B(e^{tA}x+W_A(t))|^2\,dt}\big] < \infty.$$

Defining $B_i(e^{tA}x + W_A(t)) := \mathbb{1}_{(t_{i-1},t_i]}(t)B(e^{tA}x + W_A(t))$, it follows from Novikov's criterion ([30], (1.16) Corollary, page 333) that for all $1 \le i \le N$,

$$\mathcal{E}_i(t) := e^{(1/2)\int_0^t \langle B_i(e^{sA}x+W_A(s)),dW(s)\rangle - (1/2)\int_0^t |B_i(e^{sA}x+W_A(s)|^2\,ds}, \qquad t \in [0,T],$$

is an $\mathcal{F}_t$-martingale under $\mathbb{P}$. But then since $\mathcal{E}_i(t_{i-1}) = 1$, by the martingale property of each $\mathcal{E}_i$, we can conclude that

$$\begin{aligned}
&\mathbb{E}\big[e^{\int_0^t \langle B(e^{sA}x+W_A(s)),dW(s)\rangle - (1/2)\int_0^t |B(e^{sA}x+W_A(s)|^2\,ds}\big] \\
&\quad = \mathbb{E}\big[\mathcal{E}_N(t_N)\mathcal{E}_{N-1}(t_{N-1})\cdots\mathcal{E}_1(t_1)\big] \\
&\quad = \mathbb{E}\big[\mathcal{E}_N(t_{N-1})\mathcal{E}_{N-1}(t_{N-1})\cdots\mathcal{E}_1(t_1)\big] \\
&\quad = \mathbb{E}\big[\mathcal{E}_{N-1}(t_{N-1})\cdots\mathcal{E}_1(t_1)\big] \\
&\quad \vdots \\
&\quad = \mathbb{E}\big[\mathcal{E}_1(t_1)\big] = \mathbb{E}\big[\mathcal{E}_1(t_0)\big] = 1.
\end{aligned}$$

□

REMARK 20. It is obvious from the previous proof that x may always be replaced by an $\mathcal{F}_0$-measurable H-valued r.v. which is exponentially integrable, and by any $\mathcal{F}_0$-measurable H-valued r.v. if B is bounded. The same holds for the rest of the proof of Theorem 13, that is, the following two propositions.

PROPOSITION 21. *We have $\widetilde{\mathbb{P}}_x$-a.s.*

$$\begin{aligned}
Z(t,x) = e^{tA}x &+ \int_0^t e^{(t-s)A}B\big(Z(s,x)\big)\,ds \\
&+ \int_0^t e^{(t-s)A}\,d\widetilde{W}(s), \qquad t \in [0,T],
\end{aligned} \tag{77}$$

where $\widetilde{W}$ is the cylindrical Wiener process under $\widetilde{\mathbb{P}}_x$ introduced in Proposition 17 with $b(s) := B(Z(s,x))$, which applies because of Proposition 19, that is, under $\widetilde{\mathbb{P}}_x$, $Z(\cdot,x)$ is a mild solution of

$$dZ(t) = \big(AZ(t) + B\big(Z(t)\big)\big)\,dt + d\widetilde{W}(t), \qquad t \in [0,T], Z(0) = x.$$

PROOF. Since B is of at most linear growth and because of Proposition 18, to prove (77), it is enough to show that for all $k \in \mathbb{N}$ and $x_k := \langle e_k, x\rangle$ for $x \in H$ we have, since $Ae_k = -\lambda_k e_k$, that

$$dZ_k(t,x) = \big(-\lambda_k Z_k(t,x) + B_k\big(Z(t,x)\big)\big)\,dt + d\widetilde{W}_k(t), \qquad t \in [0,T], Z(0) = x.$$

But this is obvious by the definition of $\widetilde{W}_k$. $\square$

Proposition 21 settles the existence part of Theorem 13. Now let us turn to the uniqueness part and complete the proof of Theorem 13.

PROPOSITION 22. *The weak solution to* (66) *constructed above is unique and its law is equivalent to* $\mathbb{Q}_x$ *with density in* $L^p(\Omega, \mathbb{P})$ *for all* $p \geq 1$.

PROOF. Let $X(t,x)$, $t \in [0,T]$, be a weak solution to (66) on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ for a cylindrical process of type (65). Hence

$$X(t,x) = e^{tA}x + W_A(t) + \int_0^t e^{(t-s)A} B\big(X(s,x)\big)\,ds.$$

Since B is at most of linear growth, it follows from Gronwall's inequality that for some constant $C \geq 0$,

$$\sup_{t\in[0,T]} \big|X(t,x)\big| \leq C_1\Big(1 + \sup_{t\in[0,T]} \big|e^{tA}x + W_A(t)\big|\Big).$$

Hence by Proposition 18,

$$\mathbb{E}\Big[\exp\Big\{\epsilon \sup_{t\in[0,T]} \big|X(t,x)\big|\Big\}^2\Big] < \infty. \tag{78}$$

Define

$$M := e^{-\int_0^T \langle B(X(s,x)), dW(s)\rangle - (1/2)\int_0^T |B(X(s,x))|^2\,ds}$$

and $\widetilde{\mathbb{P}} := M \cdot \mathbb{P}$. Then by exactly the same arguments as above,

$$\mathbb{E}[M] = 1.$$

Hence by Proposition 17, defining

$$\widetilde{W}_k(t) := W_k(t) + \int_0^t \big\langle e_k, B\big(X(s,x)\big)\big\rangle\,ds, \qquad t \in [0,T], k \in \mathbb{N},$$

we obtain that $\widetilde{W}(t) := (\widetilde{W}_k(t)e_k)_{k\in\mathbb{N}}$ is a cylindrical Wiener process under $\widetilde{\mathbb{P}}$ and thus

$$\widetilde{W}_A(t) := \int_0^t e^{(t-s)A}\,d\widetilde{W}(s) = W_A(t) + \int_0^t e^{(t-s)A} B\big(X(s,x)\big)\,ds, \qquad t \in [0,T],$$

and therefore,

$$X(t,x) = e^{tA}x + \widetilde{W}_A(t), \qquad t \in [0,T],$$

is an Ornstein–Uhlenbeck process under $\widetilde{\mathbb{P}}$ starting at x. But since it is easy to see that

$$\int_0^T \big\langle B\big(X(s,x)\big), dW(s)\big\rangle = \int_0^T \big\langle B\big(X(s,x)\big), d\widetilde{W}(s)\big\rangle - \int_0^T \big|B\big(X(s,x)\big)\big|^2\,ds,$$

it follows that

$$\mathbb{P} = e^{\int_0^T \langle B(X(s,x)), \widetilde{W}(s)\rangle - (1/2)\int_0^T |B(X(s,x))|^2\, ds} \cdot \widetilde{\mathbb{P}}.$$

Since

$$\widetilde{W}_k(t) = \langle e_k, \widetilde{W}_A(t)\rangle + \lambda_k \int_0^t \langle e_k, \widetilde{W}_A(s)\rangle\, ds,$$

and since $X(s,x) = e^{sA}x + \widetilde{W}_A(s)$, it follows that $\int_0^T \langle B(X(s,x)), d\widetilde{W}(s)\rangle$ is measurable with respect to the σ-algebra generated by $\widetilde{W}_A$. Hence $\frac{d\mathbb{P}}{d\widetilde{\mathbb{P}}} = \rho_x(X(\cdot,x))$ for some $\rho_x \in \mathcal{B}(C([0,T];H))$, and thus setting $\mathbb{Q}_x := \widetilde{\mathbb{P}}_x \circ X(\cdot,x)^{-1}$, we get

$$\mathbb{P}_x := \mathbb{P} \circ X(\cdot,x)^{-1} = \rho_x \mathbb{Q}_x.$$

But since it is well known that the mild solution of (67) is unique in distribution, the assertion follows, because clearly $\rho_x > 0$, $\mathbb{Q}_x$-a.s. □

A.2. A useful lemma.

LEMMA 23. *Let $f \in W^{1,2}(H,\mu) \cap C_b(H)$. Let $X = (X_t)$ and $Y = (Y_t)$ be two solutions to (1) starting from a deterministic $x \in H$ or from a r.v. x as in Theorem 1. Let $t \geq 0$. Then for $dt \otimes \mathbb{P}$-a.e. (t,ω), we have*

$$\int_0^1 \big|Df\big(rX_t(\omega) + (1-r)Y_t(\omega)\big)\big|\, dr < \infty \tag{79}$$

and

$$\begin{aligned} & f\big(X_t(\omega)\big) - f\big(Y_t(\omega)\big) \\ & \quad = \int_0^1 \big\langle Df\big(rX_t(\omega) + (1-r)Y_t(\omega)\big), X_t(\omega) - Y_t(\omega)\big\rangle\, dr. \end{aligned} \tag{80}$$

PROOF. Formula (80) is meaningful if we consider a Borel representative of $Df \in L^2(\mu)$; that is, we consider a Borel function $g : H \to H$ such that $g = Df$, μ-a.e.

Clearly the right-hand side of (80) is independent of this chosen version because (setting again $Z_t^r = rX_t + (1-r)Y_t$) it is equal to

$$\left\langle \int_0^1 Df\big(Z_t^r(\omega)\big)\, dr, X_t(\omega) - Y_t(\omega)\right\rangle,$$

and for a Borel function $g : H \to H$ with $g = 0$ μ-a.e., we have, for any $T > 0$, $\epsilon \in (0,T]$,

$$\mathbb{E}\left[\int_\epsilon^T \int_0^1 \big|g\big(Z_t^r\big)\big|\, dr\, dt\right] = \int_\epsilon^T \int_0^1 \mathbb{E}\big|g\big(Z_t^r\big)\big|\, dr\, dt = 0,$$

since by the Girsanov theorem (see Theorem 13) the law of the r.v. Z_t^r, $r \in [0, 1]$, is absolutely continuous with respect to μ.

Let us prove (80). By [8], Section 9.2, there exists a sequence of functions $(f_n) \subset C_b^\infty(H)$ (each f_n is also of exponential type) such that

$$f_n \to f, \qquad Df_n \to Df \qquad \text{in } L^2(\mu) \tag{81}$$

as $n \to \infty$. We fix $t > 0$ and write, for any $n \geq 1$,

$$f_n(X_t) - f_n(Y_t) = \int_0^1 \langle Df_n\big(rX_t + (1-r)Y_t\big), X_t - Y_t\rangle\, dr. \tag{82}$$

For a fixed $T > 0$ we will show that, as $n \to \infty$, the left-hand side and the right-hand side of (82) converge in $L^1([\epsilon, T] \times \Omega, dt \otimes \mathbb{P})$, respectively, to the left-hand side and the right-hand side of (80) for all $\epsilon \in (0, T]$.

We only prove convergence of the right-hand side of (82) (the convergence of the left-hand side is similar and simpler).

Fix $\epsilon \in (0, T]$. We first consider the case in which x is deterministic. We get, using the Girsanov theorem (see Theorem 13), as in the proof of Proposition 10,

$$\begin{aligned}
a_n :=& \mathbb{E}\bigg[\int_\epsilon^T \int_0^1 \big|Df_n\big(rX_t + (1-r)Y_t\big) - Df\big(rX_t + (1-r)Y_t\big)\big| \\
&\qquad\qquad \times \big|(X_t - Y_t)\big|\, dr\, dt\bigg] \\
\leq& M \int_\epsilon^T \int_0^1 \mathbb{E}\big|Df_n\big(rX_t + (1-r)Y_t\big) - Df\big(rX_t + (1-r)Y_t\big)\big|\, dr\, dt \\
\leq& M' \int_0^1 \mathbb{E}\bigg[\rho_r^{-1/2}\rho_r^{1/2} \int_\epsilon^T \big|Df_n\big(rX_t + (1-r)Y_t\big) \\
&\qquad\qquad - Df\big(rX_t + (1-r)Y_t\big)\big|\, dt\bigg]\, dr \\
\leq& M\bigg(\int_0^1 \mathbb{E}[\rho_r^{-1}]\, dr\bigg)^{1/2}\bigg(\int_0^1 \int_\epsilon^T \mathbb{E}\big[\big|Df_n(U_t) - Df(U_t)\big|^2\big]\, dt\, dr\bigg)^{1/2} \\
\leq& C\bigg(\int_\epsilon^T \mathbb{E}\big[\big|Df_n(U_t) - Df(U_t)\big|^2\big]\, dt\bigg)^{1/2},
\end{aligned}$$

where U_t is an OU process starting at x. By [8], Lemma 10.3.3, we know that, for $t > 0$, the law of U_t has a positive density $\pi(t, x, \cdot)$ with respect to μ, bounded on $[\epsilon, T] \times H$.

It easily follows [using (81)] that $\int_\epsilon^T \mathbb{E}[|Df_n(U_t) - Df(U_t)|^2]\, dt \to 0$, as $n \to \infty$, and so $a_n \to 0$.

Similarly, one proves that

$$\int_\epsilon^T \mathbb{E}\bigg[\int_0^1 \big|Df\big(rX_t + (1-r)Y_t\big)\big|\, dr\bigg]\, dt < \infty.$$

Now since $\epsilon \in (0, T]$ was arbitrary, the assertion follows for every (nonrandom) initial condition $x \in H$.

Now let us consider the case in which x is an $\mathcal{F}_0$-measurable r.v. Using Remark 20, analogously, we find, with $1/p + 1/p' = 1$ and $1 < p < 2$,

$$\begin{aligned} a_n &\le M \int_0^1 \int_\epsilon^T \mathbb{E}[\rho_r^{-1/p} \rho_r^{1/p} |Df_n(rX_t + (1-r)Y_t) \\ &\qquad\qquad - Df(rX_t + (1-r)Y_t)|] \, dt \, dr \\ &\le M' \biggl(\int_0^1 \mathbb{E}[\rho_r^{-p'/p}] \, dr \biggr)^{1/p'} \biggl(\int_0^1 \int_\epsilon^T \mathbb{E}[|Df_n(U_t) - Df(U_t)|^p] \, dt \, dr \biggr)^{1/p} \\ &\le C \biggl(\int_\epsilon^T \mathbb{E}[|Df_n(U_t) - Df(U_t)|^p] \, dt \biggr)^{1/p}, \end{aligned}$$

where U_t is an OU process such that $U_0 = x$, $\mathbb{P}$-a.s. Using (62) with $|Df_n - Df|^p$ instead of f and $\zeta' = 2/p$, as above, we arrive at

$$a_n \le C_\epsilon \|h_0\|_{L^{2/(2-p)}(\mu)}^{1/p} \biggl(\int_H |Df_n(x) - Df(x)|^2 \mu(dx) \biggr)^{1/2},$$

where h_0 denotes the density of the law of x with respect to μ. Passing to the limit, by (82) we get $a_n \to 0$. Then analogously to the case where x is deterministic, we complete the proof. □

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