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One-mode bosonic Gaussian channels: a full weak-degradability classification

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Abstract. A complete degradability analysis of one-mode bosonic Gaussian channels is presented. We show that apart from the class of channels which are unitarily equivalent to the channels with additive classical noise, these maps can be characterized in terms of weak- and/or anti-degradability. Furthermore a new set of channels which have null quantum capacity is identified. This is done by exploiting the composition rules of one-mode Gaussian maps and the fact that anti-degradable channels cannot be used to transfer quantum information.

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Contents

Within the context of quantum information theory [1] bosonic Gaussian channels [2]–[4] play a fundamental role. They include all the physical transformations which preserve the 'Gaussian character' of the transmitted signals and can be seen are the quantum counterpart of the Gaussian channels in the classical information theory [5]. Bosonic Gaussian channels describe most of the noise sources which are routinely encountered in optics, including those responsible for the attenuation and/or the amplification of signals along optical fibres. Moreover, due to their relatively simple structure, these channels provide an ideal theoretical playground for the study of continuous variable [6] quantum communication protocols.

Not surprisingly in recent years an impressive effort has been put forward to characterize the properties of bosonic Gaussian channels. Most of the efforts focused on the evaluation of the optimal transmission rates of these maps under the constraint on the input average energy both in the multi-mode scenario (where the channel acts on a collection of many input bosonic mode) and in the one-mode scenario (where, instead, it operates on a single input bosonic mode). In a few cases [7]–[10], the exact values of the communication capacities [11]–[13] of the channels have been computed. In the general case, however only certain bounds are available (see [3, 10], [14]–[16]). Finally various additivity issues have been analysed in [17, 18].

Recently the notions of anti-degradability and weak-degradability were proposed as a useful tool for studying the quantum capacity properties of one-mode Gaussian channels [19]. This suggested the possibility of classifying these maps in terms of a simple canonical form which was achieved in [20]. Moreover, proceeding along similar lines, the exact solution of the quantum capacity of an important subset of those channels was obtained in [21].

In this paper, we provide a complete degradability classification of one-mode Gaussian channels and exhibit a new set of channels which have null quantum capacity extending a previous result in [3].

The definition of weak- and anti-degradability of a quantum channel is similar to the definition of degradability introduced by Devetak and Shor in [22]. It is based on replacing the Stinespring dilation [23] of the channel with a representation where the ancillary system (environment) is not necessarily in a *pure* state [2, 24]. This yields a generalization of the notion of *complementary* channel from [22, 25, 26] which is named *weakly complementary* [19]. In this context weakly degradable are those channels where the modified state of the ancillary system—described by the action of the weakly complementary channel—can be recovered from the output state of the channel through the action of a third channel. Vice versa, anti-degradable channels obey the opposite rule (i.e. the output state of the channel can be obtained from the modified state of the ancilla through the action of another suitable channel). Exploiting the canonical form [20] one can show that, apart from the class B_2 consisting of the maps which are unitarily equivalent to the channels with additive classical Gaussian noise [3], all one-mode bosonic Gaussian channels are either weakly degradable or anti-degradable. As discussed in [19] the anti-degradability property allows one to simplify the analysis of the quantum capacity [13] of these channels. Indeed those maps which are anti-degradable can be shown to have null quantum capacity. On the other hand, those channels which are weakly degradable with pure ancillas (i.e. those which are degradable in the sense of [22]) have quantum capacity which can be expressed in terms of a single-letter expression. Here, we will focus mostly on the anti-degradability property and, additionally, we will show that by exploiting the composition rules of one-mode bosonic Gaussian channels, one can extend the set of maps with null quantum capacity well beyond the set of anti-degradable maps.

The paper is organized as follows. In section 1, we introduce the notion of weak-complementarity and weak-degradability in a rather general context. In section 2, we give a detailed description of the canonical decomposition of one-mode bosonic Gaussian channels. In section 3, we discuss the weak-degradability properties of one-mode channels. Finally, in section 4, we determine the new set of channels with null quantum capacity.

1. Weakly complementary and weakly degradable channels

In quantum mechanics, quantum channels describe the evolution of an open system A interacting with external degrees of freedom. In the Schrödinger picture these transformations are described by completely positive trace preserving (CPT) linear maps Φ acting on the set $\mathcal{D}(\mathcal{H}_a)$ of the density matrices ρ_a of the system. It is a well known (see e.g. [2, 24]) that Φ can be described by a unitary coupling between the system A in input state ρ_a with an external ancillary system B (describing the *environment*) prepared in some fixed *pure* state. This follows from Stinespring dilation [23] of the map which is unique up to a partial isometry. More generally, one can describe Φ as a coupling with the environment prepared in some *mixed* state ρ_b, i.e.

$$\Phi(\rho_a) = \mathrm{Tr}_b[U_{ab}(\rho_a \otimes \rho_b)U_{ab}^\dagger], \tag{1.1}$$

where $\mathrm{Tr}_b[\ldots]$ is the partial trace over the environment B, U_{ab} is a unitary operator in the composite Hilbert space $\mathcal{H}_a \otimes \mathcal{H}_b$. We call equation (1.1) a 'physical representation' of Φ

to distinguish it from the Stinespring dilation, and to stress its connection with the physical picture of the noisy evolution represented by Φ. Any Stinespring dilation gives rise to a physical representation. Moreover from any physical representation (1.1) one can construct a Stinespring dilation by purifying ρ_b with an external ancillary system C, and by replacing U_{ab} with the unitary coupling $U_{abc} = U_{ab} \otimes \mathbb{1}_c$.

Equation (1.1) motivates the following [19]

Definition 1. *For any physical representation* (1.1) *of the quantum channel* Φ *we define its* weakly complementary *as the map* $\tilde{\Phi} : \mathcal{D}(\mathcal{H}_a) \to \mathcal{D}(\mathcal{H}_b)$ *which takes the input state* ρ_a *into the state of the environment B after the interaction with A, i.e.*

$$\tilde{\Phi}(\rho_a) = \mathrm{Tr}_a[U_{ab}(\rho_a \otimes \rho_b)U_{ab}^{\dagger}]. \tag{1.2}$$

The transformation (1.2) is CPT, and it describes a quantum channel connecting systems A and B. It is a generalization of the *complementary (conjugate) channel* Φ_{com} defined in [22]–[26]. In particular, if equation (1.1) arises from a Stinespring dilation (i.e. if ρ_b of equation (1.2) is pure) the map $\tilde{\Phi}$ coincides with Φ_{com}. Hence the latter is a particular instance of a weakly complementary channel of Φ. On the other hand, by using the above purification procedure, we can always represent a weakly complementary map as a composition

$$\tilde{\Phi} = T \circ \Phi_{\mathrm{com}}, \tag{1.3}$$

where T is the partial trace over the purifying system (here '$\circ$' denotes the composition of channels). As we will see, the properties of weakly complementary and complementary maps in general differ.

Definition 2. *Let* Φ, $\tilde{\Phi}$ *be a pair of mutually (weakly) complementary channels such that*

$$(\Psi \circ \Phi)(\rho_a) = \tilde{\Phi}(\rho_a), \tag{1.4}$$

for some channel $\Psi : \mathcal{D}(\mathcal{H}_a) \to \mathcal{D}(\mathcal{H}_b)$ *and all density matrix* $\rho_a \in \mathcal{D}(\mathcal{H}_a)$. *Then* Φ *is called* (weakly) degradable *while* $\tilde{\Phi}$ *is called* anti-degradable *(cf [19]).*

Similarly if

$$(\overline{\Psi} \circ \tilde{\Phi})(\rho_a) = \Phi(\rho_a) \ , \tag{1.5}$$

for some channel $\overline{\Psi} : \mathcal{D}(\mathcal{H}_b) \to \mathcal{D}(\mathcal{H}_a)$ and all density matrix $\rho_a \in \mathcal{D}(\mathcal{H}_a)$, then Φ is *anti-degradable* while $\tilde{\Phi}$ is *(weakly) degradable*.

In [22], the channel Φ is called *degradable* if in equation (1.4) we replace $\tilde{\Phi}$ with a complementary map Φ_{com} of Φ. Clearly any degradable channel [22] is weakly degradable but the opposite is not necessarily true. Notice, however, that due to equation (1.3), in the definition of the anti-degradable channel we can always replace weakly complementary with complementary (for this reason there is no point in introducing the notion of a weakly anti-degradable channel). This allows us to verify that if Φ is anti-degradable (1.5) then its complementary channel Φ_{com} is degradable [22] and vice versa. It is also worth pointing out that channels which are unitarily equivalent to a channel Φ which is weakly degradable (anti-degradable) are also weakly degradable (anti-degradable).

Finally an important property of anti-degradable channels is the fact that their quantum capacity [13] is null. As discussed in [19], this is a consequence of the no-cloning theorem [27] (more precisely, of the impossibility of cloning with arbitrary high fidelity [28]).

It is useful also to reformulate our definitions in the Heisenberg picture. Here, the states of the system are kept fixed and the transformation induced on the system by the channel is described by means of a linear map $\Phi_{\rm H}$ acting on the algebra $\mathcal{B}(\mathcal{H}_a)$ of all bounded operators of A so that

$$\mathrm{Tr}_a[\Phi(\rho_a)\,\Theta_a] = \mathrm{Tr}_a[\rho_a\,\Phi_{\rm H}(\Theta_a)], \tag{1.6}$$

for all $\rho_a \in \mathcal{D}(\mathcal{H}_a)$ and for all $\Theta_a \in \mathcal{B}(\mathcal{H}_a)$. From this it follows that the Heisenberg picture counterpart of the physical representation (1.1) is given by the unital channel

$$\Phi_{\rm H}(\Theta_a) = \mathrm{Tr}_b[U_{ab}^\dagger\,(\Theta_a \otimes 1\!\!1_b)\,U_{ab}\,(1\!\!1_a \otimes \rho_b)]. \tag{1.7}$$

Similarly, from (1.2) it follows that in the Heisenberg picture the weakly complementary of the channel is described by the completely positive unital map

$$\tilde{\Phi}_{\rm H}(\Theta_b) = \mathrm{Tr}_b\,[U_{ab}^\dagger(1\!\!1_a \otimes \Theta_b)\,U_{ab}\,(1\!\!1_a \otimes \rho_b)], \tag{1.8}$$

which takes bounded operators in $\mathcal{H}_b$ into bounded operators in $\mathcal{H}_a$.

Within this framework, the weak-degradability property (1.4) of the channel $\Phi_{\rm H}$ requires the existence of a channel $\Psi_{\rm H}$ taking bounded operators of $\mathcal{H}_b$ into bounded operators of $\mathcal{H}_a$, such that

$$(\Phi_{\rm H} \circ \Psi_{\rm H})(\Theta_b) = \tilde{\Phi}_{\rm H}(\Theta_b), \tag{1.9}$$

for all $\Theta_b \in \mathcal{B}(\mathcal{H}_b)$. Similarly, we say that a quantum channel $\Phi_{\rm H}$ is anti-degradable, if there exists a channel $\overline{\Psi}_{\rm H}$ from $\mathcal{B}(\mathcal{H}_a)$ to $\mathcal{B}(\mathcal{H}_b)$, such that

$$(\tilde{\Phi}_{\rm H} \circ \overline{\Psi}_{\rm H})(\Theta_a) = \Phi_{\rm H}(\Theta_a), \tag{1.10}$$

for all $\Theta_a \in \mathcal{B}(\mathcal{H}_a)$.

2. One-mode bosonic Gaussian channels

Gaussian channels arise from linear dynamics of open bosonic system interacting with a Gaussian environment via quadratic Hamiltonians. Loosely speaking, they can be characterized as CPT maps that transform Gaussian states into Gaussian states [3, 4, 29]. Here we focus on one-mode bosonic Gaussian channels which act on the density matrices of single bosonic mode A. A classification of such maps obtained recently in the paper [20] allows us to simplify the analysis of the weak-degradability property. In the following we start by reviewing the result of [20], clarifying the connection with the analysis of [19] (cf also [18]). Then we pass to the weak-degradability analysis of these channels, showing that with some important exceptions, they are either weakly degradable or anti-degradable.

2.1. General properties

Consider a single bosonic mode characterized by canonical observables Q_a, P_a obeying the canonical commutation relation $[Q_a, P_a] = i$. A consistent description of the system can be given in terms of the unitary Weyl operators $V_a(z) = \exp[i(Q_a, P_a)\cdot z]$, with $z = (x, y)^T$ being a column vector of R^2. In this framework, the canonical commutation relation is written as

$$V_a(z)\, V_a(z') = \exp\left[\frac{i}{2}\Delta(z, z')\right] V_a(z + z'),$$

where $\Delta(z, z')$ is the symplectic form

$$\Delta(z, z') = -i\, z^T \cdot \sigma_2 \cdot z' = x'y - xy', \tag{2.1}$$

with σ_2 being the second Pauli matrix. Moreover the density operators ρ_a of the system can be expressed in terms of an integral over z of the $V_a(z)$'s, i.e.

$$\rho_a = \int \frac{\mathrm{d}^2 z}{2\pi}\, \phi(\rho_a; z)\, V_a(-z), \tag{2.2}$$

with

$$\phi(\rho_a; z) = \mathrm{Tr}_a[\rho_a\, V_a(z)], \tag{2.3}$$

being the characteristic function of ρ_a.[4] Consequently a complete description of a quantum channel on A is obtained by specifying its action on the operators $V_a(z)$, or, equivalently, by specifying how to construct the characteristic function $\phi(\Phi(\rho_a); z)$ of the evolved states. In the case of Gaussian channels Φ this is done by assigning a mapping of the Weyl operators

$$\Phi_{\mathrm{H}}(V_a(z)) = V_a(K\cdot z)\, \exp[-\tfrac{1}{2}\, z^T \cdot \alpha \cdot z + \mathrm{i}\, m^T \cdot z], \tag{2.4}$$

in the Heisenberg picture, or the transformation of the characteristic functions

$$\phi(\Phi(\rho_a); z) = \phi(\rho_a; K\cdot z)\, \exp[-\tfrac{1}{2}\, z^T \cdot \alpha \cdot z + \mathrm{i}\, m^T \cdot z], \tag{2.5}$$

in the Schrödinger picture. Here m is a vector, while K and α are real matrices (the latter being symmetric and positive). Equation (2.5) guarantees that any input Gaussian characteristic function will remain Gaussian under the action of the map. A useful property of Gaussian channels is the fact that the composition of two of them (say Φ' and Φ'') is still a Gaussian channel. Indeed one can easily verify that the composite map $\Phi'' \circ \Phi'$ is of the form (2.5) with m, K and α given by

$$m = (K'')^T \cdot m' + m'', \qquad K = K'\, K'', \qquad \alpha = (K'')^T\, \alpha'\, K'' + \alpha''. \tag{2.6}$$

Here m', K' and α' belongs to Φ' while m'', K'' and α'' belongs to Φ''.

Not all possible choices of K, α correspond to transformations Φ which are completely positive. A necessary and sufficient condition for this last property (adapted to the case of one mode) is provided by the non-negative definiteness of the following 2×2 Hermitian matrix [3, 20]

$$2\alpha - \sigma_2 + K^T\, \sigma_2\, K. \tag{2.7}$$

This matrix reduces to $2\alpha + (\mathrm{Det}[K] - 1)\,\sigma_2$ and its non-negative definiteness to the inequality

$$\mathrm{Det}[\alpha] \geqslant \left(\frac{\mathrm{Det}[K] - 1}{2}\right)^2. \tag{2.8}$$

Within the limit imposed by equation (2.8), we can use equation (2.5) to describe the whole set of the one-mode Gaussian channels.

[4] In effect an analogous decomposition (2.2) holds also for all trace class operators of A [30].

2.2. Channels with single-mode physical representation

An important subset of one-mode Gaussian channels is given by the maps Φ which possess a physical representation (1.1) with ρ_b being a Gaussian state of a *single* external Bosonic mode B and with U_{ab} being a canonical transformation of Q_a, P_a, Q_b and P_b (the latter being the canonical observables of the mode B). In particular let ρ_b be a thermal state of average photon number N, i.e.

$$\phi(\rho_b; z) = \mathrm{Tr}_b[\rho_b\, V_b(z)] = \exp\left[-(N+1/2)|z|^2/2\right], \tag{2.9}$$

and let U_{ab} be such that

$$U_{ab}^\dagger\,(Q_a, P_a, Q_b, P_b)\,U_{ab} = (Q_a, P_a, Q_b, P_b)\cdot M, \tag{2.10}$$

with M being a 4×4 symplectic matrix of block form

$$M \equiv \left(\begin{array}{c|c} m_{11} & m_{21} \\ \hline m_{12} & m_{22} \end{array}\right). \tag{2.11}$$

This yields the following evolution for the characteristic function $\phi(\rho_a; z)$,

$$\begin{aligned}\phi(\Phi(\rho_a); z) &= \mathrm{Tr}_a[\Phi(\rho_a)\, V_a(z)] = \mathrm{Tr}_a[\rho_a\, \Phi_{\mathrm{H}}(V_a(z))] \\ &= \mathrm{Tr}_{ab}[U_{ab}^\dagger\,(V_a(z)\otimes \mathbb{1})U_{ab}\,(\rho_a\otimes\rho_b)] \\ &= \mathrm{Tr}_{ab}[(V_a(m_{11}\cdot z)\otimes V_b(m_{12}\cdot z))\,(\rho_a\otimes\rho_b)] \\ &= \phi(\rho_a; m_{11}\cdot z)\,\exp\left[-(N+1/2)|m_{12}\cdot z|^2/2\right],\end{aligned} \tag{2.12}$$

which is of the form (2.5) by choosing $m = 0$, $K = m_{11}$ and $\alpha = (N+1/2)\, m_{12}^T\cdot m_{12}$. It is worth stressing that in the case of equation (2.12) the inequality (2.8) is guaranteed by the symplectic nature of the matrix M, i.e. by the fact that equation (2.10) preserves the commutation relations among the canonical operators. Indeed we have

$$\mathrm{Det}[\alpha] = (N+1/2)^2\,\mathrm{Det}[m_{12}]^2 = (N+1/2)^2\,(\mathrm{Det}[m_{11}]-1)^2 \geqslant (\mathrm{Det}[K]-1)^2/4, \tag{2.13}$$

where in the second identity the condition (2.21) was used.

As we shall see, with certain important exception one-mode Gaussian channels (2.4) are unitarily equivalent to transformations which admit physical representation with ρ_b and U_{ab} as in equations (2.9) and (2.10).

2.3. Canonical form

Following [20] any Gaussian channel (2.5) can be transformed (through unitarily equivalence) into a simple canonical form. Namely, given a channel Φ characterized by the vector m and the matrices K, α of equation (2.5), one can find unitary operators U_a and W_a such that the channel defined by the mapping

$$\rho_a \longrightarrow \Phi^{(\mathrm{can})}(\rho_a) = W_a\,\Phi(U_a\,\rho_a\,U_a^\dagger)\,W_a^\dagger \quad \text{for all } \rho_a, \tag{2.14}$$

Table 1. Canonical form for one-mode Gaussian bosonic channels. In the first columns the properties of K and α of the map Φ are reported. In last two columns instead we give the matrices K_{can} and α_{can} of the canonical form $\Phi^{(\text{can})}$ associated with Φ—see equations (2.14) and (2.15). In these expressions σ_3 is the third Pauli matrix, N_0 is a non-negative constant and κ is a positive constant. Notice that the constraint (2.8) is always satisfied. In B_1 the free parameter N_c has been set equal to 1/2—see the discussion below equation (2.17).

Channel Φ			Canonical form $\Phi^{(\text{can})}$	
Det[K]		Class	K_{can}	α_{can}
0	rank[K] = 0	A_1	0	$(N_0 + 1/2)\,\mathbb{1}$
0	rank[K] = 1	A_2	$(\mathbb{1} + \sigma_3)/2$	$(N_0 + 1/2)\,\mathbb{1}$
1	rank[α] = 1	B_1	$\mathbb{1}$	$(\mathbb{1} - \sigma_3)/4$
1	rank[α] $\neq$ 1	B_2	$\mathbb{1}$	$N_0\,\mathbb{1}$
κ^2 ($\kappa \neq 0, 1$)		C	$\kappa\,\mathbb{1}$	$\lvert\kappa^2 - 1\rvert(N_0 + 1/2)\,\mathbb{1}$
$-\kappa^2$ ($\kappa \neq 0$)		D	$\kappa\,\sigma_3$	$(\kappa^2 + 1)(N_0 + 1/2)\,\mathbb{1}$

is of the form (2.5) with $m = 0$ and with K, α replaced, respectively, by the matrices K_{can}, α_{can} of table 1, i.e.

$$\phi(\Phi^{(\text{can})}(\rho_a); z) = \phi(\rho_a; K_{\text{can}} \cdot z)\, \exp\left[-\tfrac{1}{2}\, z^T \cdot \alpha_{\text{can}} \cdot z\right]. \tag{2.15}$$

An important consequence of equation (2.15) is that to analyse the weak-degradability properties of a one-mode Gaussian channel it is sufficient to focus on the canonical map $\Phi^{(\text{can})}$ which is unitarily equivalent to it (see remark at the end of section 1). Here we will not enter into the details of the derivation of equations (2.14) and (2.15), see [20].

The dependence on the matrix K_{can} of $\Phi^{(\text{can})}$ upon the parameters of Φ can be summarized as follows,

$$K_{\text{can}} = \begin{cases} \begin{cases} \sqrt{\text{Det}[K]}\,\mathbb{1} & \text{Det}[K] \geqslant 0 \\ \sqrt{|\text{Det}[K]|}\,\sigma_3 & \text{Det}[K] < 0 \end{cases} & \text{rank}[K] \neq 1, \\ \\ \qquad (\mathbb{1} + \sigma_3)/2 & \text{rank}[K] = 1, \end{cases} \tag{2.16}$$

with σ_3 being the third Pauli matrix. Analogously for α_{can} we have

$$\alpha_{\text{can}} = \begin{cases} \sqrt{\text{Det}[\alpha]}\,\mathbb{1} & \text{rank}[\alpha] \neq 1, \\ \\ N_c\,(\mathbb{1} - \sigma_3)/2 & \text{rank}[\alpha] = 1. \end{cases} \tag{2.17}$$

The quantity N_c is a free parameter which can set to any positive value upon properly calibrating the unitaries U_a and W_a of equation (2.14). Following [20] we will assume $N_c = 1/2$. Notice also that from equation (2.8), rank[α] = 1 is only possible for Det[K] = 1.

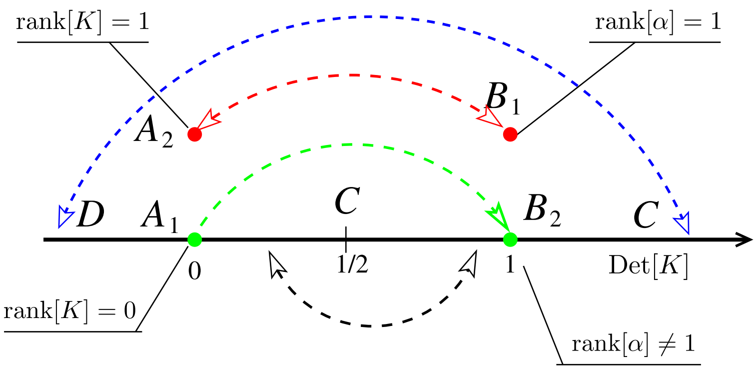

Figure 1. Pictorial representation of the classification in terms of canonical forms of table 1. Depending on the values of Det[K], rank[K] and rank[α] any one-mode Gaussian channel can be transformed to one of the channels of the scheme through unitary transformations as in equation (2.14). The points on the thick oriented line for Det[K] < 0 represent the maps of D, those with Det[K] > 0 and Det[K] $\neq 1$ represent C. The classes $A_{1,2}$ and $B_{1,2}$ are represented by the four coloured points of the graph. Notice that the channel B_2 and A_1 can be obtained as limiting cases of D and C. The dotted arrows connect channels which are weakly complementary (1.2) of each other with respect to the physical representations introduced in section 2.4. For instance the weakly complementary of B_1 is channel of the class A_2 (and vice versa)—see subsection 3.1 and table 2 for details. Notice that the weakly complementary channel of A_1 belongs to B_2. However, not all the channels of B_2 have weakly complementary channels which are in A_1—see section 2.5.

Equations (2.16) and (2.17) show that only the determinant and the rank of K and α are relevant for defining $K_{\rm can}$ and $\alpha_{\rm can}$. Indeed one can verify that $K_{\rm can}$ and $\alpha_{\rm can}$ maintain the same determinant and rank of the original matrices K and α, respectively. This is a consequence of the fact the Φ and $\Phi^{(\rm can)}$ are connected through a symplectic transformation for which Det[K], Det[α], rank[K], and rank[α] are invariant quantities. (In particular Det[K] is directly related with the invariant quantity q analysed in [19].)

The six inequivalent canonical forms of table 1 follow by parametrizing the value of $\sqrt{\text{Det}[\alpha]}$ to account for the constraints imposed by the inequality (2.8). It should be noticed that to determine which class a certain channel belongs to, it is only necessary to know if Det[K] is null, equal to 1, negative or positive ($\neq 1$). If Det[K] $= 0$ the class is determined by the rank of the matrix. If Det[K] $= 1$ the class is determined by the rank of α (see figure 1). Within the various classes, the specific expression of the canonical form depends then upon the effective values of Det[K] and Det[α]. We observe also that the class A_1 can be obtained as a limiting case (for $\kappa \to 0$) of the maps of class C or D. Analogously the class B_2 can be obtained as a limiting case of the maps of class C. Indeed consider the channel with $K_{\rm can} = \kappa \mathbb{1}$ and $\alpha_{\rm can} = |\kappa^2 - 1|(N_0' + 1/2)\mathbb{1}$ with $N_0' = N_0/(|\kappa^2 - 1|) - 1/2$, with N_0 and κ positive ($\kappa \neq 0, 1$). For κ sufficiently close to 1, N_0' is positive and the maps belongs to the class C of table 1. Moreover in the limit of $\kappa \to 1$ this channel yields the map B_2.

Finally it is interesting to study how the canonical forms of table 1 compose under the product (2.6). A simple calculation shows that the following rules apply:

$$\begin{array}{c|cccccc} \circ & A_1 & A_2 & B_1 & B_2 & C & D \\ \hline A_1 & A_1 & A_1 & A_1 & A_1 & A_1 & A_1 \\ A_2 & A_1 & A_2 & A_2 & A_2 & A_2 & A_2 \\ B_1 & A_1 & A_2 & B_1 & B_1/B_2 & C & D \\ B_2 & A_1 & A_2 & B_1/B_2 & B_2 & C & D \\ C & A_1 & A_2 & C & C & B_2/C & D \\ D & A_1 & A_2 & D & D & D & C \end{array} \tag{2.18}$$

In this table, for instance, the element on row 2 and column 3 represents the class (i.e. A_2) associated to the product $\Phi'' \circ \Phi'$ between a channel Φ' of B_1 and a channel Φ'' of A_2. Notice that the canonical form of the products $B_1 \circ B_2$, $B_2 \circ B_1$ and $C \circ C$ is not uniquely defined. In the first case, in fact, even though the determinant of the matrix K of equation (2.6) is one, the rank of the corresponding α might be one or different from one depending on the parameters of the two 'factor' channels: consequently the $B_1 \circ B_2$ and $B_2 \circ B_1$ might belong either to B_1 or to B_2. In the case of $C \circ C$ instead it is possible that the resulting channel will have $\mathrm{Det}[K] = 1$ making it a B_2 map. Typically however $C \circ C$ will be a map of C. Composition rules analogous to those reported here have been extensively analysed in [16, 17, 19].

2.4. Single-mode physical representation of the canonical forms

Apart from the case B_2, which will be treated separately (see next section), all canonical transformations of table 1 can be expressed as in equation (2.12), i.e. through a physical representation (1.1) with ρ_b being a thermal state (2.9) of a single external Bosonic mode B and U_{ab} being a linear transformation (2.10).[5] To show this it is sufficient to verify that, for each of the classes of table 1 but B_2, there exists a non-negative number N and a symplectic matrix M such that equation (2.12) gives the mapping (2.15). This yields the conditions

$$m_{11} = K_{\mathrm{can}}, \tag{2.19}$$

$$m_{12} = O\sqrt{\frac{\alpha_{\mathrm{can}}}{N + 1/2}}, \tag{2.20}$$

with $O^T = O^{-1}$ being an orthogonal 2×2 matrix to be determined through the symplectic condition

$$\mathrm{Det}[m_{11}] + \mathrm{Det}[m_{12}] = 1, \tag{2.21}$$

which guarantees that $U_{ab}^{\dagger} Q_a U_{ab}$ and $U_{ab}^{\dagger} P_a U_{ab}$ satisfy canonical commutation relations. It is worth noticing that once m_{11} and m_{12} are determined within the constraint (2.21) the remaining blocks (i.e. m_{21} and m_{22}) can always be found in order to satisfy the remaining symplectic conditions of M. An explicit example will be provided in few paragraphs. For the classes A_1, A_2, B_1, D and C with $\kappa < 1$, equations (2.20) and (2.21) can be solved by choosing $O = \mathbb{1}$

[5] The exceptional role of B_2 corresponds to the fact that any one-mode bosonic Gaussian channel can be represented as a unitary coupling with a single-mode environment plus an additive classical noise (see next section and [4]).

and $N = N_0$. Indeed for B_1 the latter setting is not necessary. Any non-negative number will do the job: thus we choose $N = 0$ making the density matrix ρ_b of equation (2.9) the vacuum of the B. For C with $\kappa > 1$ instead a solution is obtained by choosing $O = \sigma_3$ and again $N = N_0$. The corresponding transformations (2.10) for Q_a and P_a (together with the choice for N) are summarized below.

Class		ρ_b	$U_{ab}^\dagger Q_a U_{ab}$	$U_{ab}^\dagger P_a U_{ab}$
A_1		Thermal$(N = N_0)$	Q_b	P_b
A_2		Thermal$(N = N_0)$	$Q_a + Q_b$	P_b
B_1		Vacuum$(N = 0)$	Q_a	$P_a + P_b$
C	$\kappa < 1$	Thermal$(N = N_0)$	$\kappa Q_a + \sqrt{1-\kappa^2}\, Q_b$	$\kappa P_a + \sqrt{1-\kappa^2}\, P_b$
C	$\kappa > 1$	Thermal$(N = N_0)$	$\kappa Q_a + \sqrt{\kappa^2-1}\, Q_b$	$\kappa P_a - \sqrt{\kappa^2-1}\, P_b$
D		Thermal$(N = N_0)$	$\kappa Q_a + \sqrt{\kappa^2+1}\, Q_b$	$-\kappa P_a + \sqrt{\kappa^2+1}\, P_b$.

To complete the definition of the unitary operators U_{ab} we need to provide also the transformations of Q_b and P_b. This corresponds to fixing the blocks m_{21} and m_{22} of M and cannot be done uniquely: one possible choice is presented in the following table:

Class		$U_{ab}^\dagger Q_b U_{ab}$	$U_{ab}^\dagger P_b U_{ab}$
A_1		Q_a	P_a
A_2		Q_a	$P_a - P_b$
B_1		$Q_a - Q_b$	$-P_b$
C	$\kappa < 1$	$\sqrt{1-\kappa^2}\, Q_a - \kappa Q_b$	$\sqrt{1-\kappa^2}\, P_a - \kappa P_b$
C	$\kappa > 1$	$\sqrt{\kappa^2-1}\, Q_a + \kappa Q_b$	$-\sqrt{\kappa^2-1}\, P_a + \kappa P_b$
D		$\sqrt{\kappa^2+1}\, Q_a + \kappa Q_b$	$\sqrt{\kappa^2+1}\, P_a - \kappa P_b$.

The above definitions make explicit the fact that the canonical form C represents attenuator ($\kappa < 1$) and amplifier ($\kappa > 1$) channel [3]. We will see in the following sections that the class D is formed by the weakly complementary of the amplifier channels of the class C. For the sake of clarity the explicit expression for the matrices M of the various classes has been reported in appendix.

Finally it is important to notice that the above physical representations are equivalent to Stinespring representations only when the average photon number N of ρ_b nullifies. In this case the environment B is represented by a pure input state (i.e. the vacuum). According to our definitions this is always the case for the canonical form B_1 while for the canonical forms A_1, A_2, C and D it happens for $N_0 = 0$.

2.5. The class B_2: additive classical noise channel

As mentioned in the previous section the class B_2 of table 1 must be treated separately. The map B_2 corresponds[6] to the additive classical noise channel [3] defined by

$$\Phi(\rho_a) = \int d^2z\, p(z)\, V_a(z)\, \rho_a\, V_a(-z), \tag{2.22}$$

[6] This can be seen for instance by evaluating the characteristic function of the state (2.22) and comparing it with equation (2.15).

with $p(z) = (2\pi N_0)^{-1} \exp[-|z|^2/(2N_0)]$ which, in the Heisenberg picture, can be seen as a random shift of the annihilation operator a.

These channels admit a natural physical representation which involve two environmental modes in a pure state (see [20] for details) but do not have a physical representations (1.1) involving a single environmental mode. This can be verified by noticing that in this case, from equations (2.19) and (2.20) we get

$$m_{11} = \mathbb{1}, \tag{2.23}$$

$$m_{12} = \sqrt{N_0/(N+1/2)}\, O, \tag{2.24}$$

which yields

$$\mathrm{Det}[m_{11}] + \mathrm{Det}[m_{12}] = 1 \pm N_0/(N+1/2)\,, \tag{2.25}$$

independently of the choice of the orthogonal matrix O.[7] Therefore, apart from the trivial case $N_0 = 0$, the only solution to the constraint (2.21) is by taking the limit $N \to \infty$. This would correspond to representing the channel B_2 in terms of a linear coupling with a single-mode thermal state ρ_b of 'infinite' temperature. Unfortunately this is not a well-defined object. However, we can use the 'asymptotic' representation described at the end of subsection 2.3 where it was shown how to obtain B_2 as limiting case of C class maps, to claim at least that there exists a one-parameter family of one-mode Gaussian channels which admits single-mode physical representation and which converges to B_2.

3. Weak-degradability of one-mode Gaussian channels

In the previous section, we have seen that all one-mode Gaussian channels are unitarily equivalent to one of the canonical forms of table 1. Moreover we verified that, with the exception of the class B_2, all the canonical forms admits a physical representation (1.1) with ρ_b being a thermal state of a single environmental mode and U_{ab} being a linear coupling. Here we will use such representations to construct the weakly complementary (1.2) of these channels and to study their weak-degradability properties.

3.1. Weakly complementary channels

In this section we construct the weakly complementary channels $\tilde{\Phi}$ of the class A_1, A_2, B_1, C and D starting from their single-mode physical representations (1.1) of subsection 2.4. Because of the linearity of U_{ab} and the fact that ρ_b is Gaussian, the channels $\tilde{\Phi}$ are Gaussian. This can be seen for instance by computing the characteristic function (2.3) of the output state $\tilde{\Phi}(\rho_a)$

$$\begin{aligned}\phi(\tilde{\Phi}(\rho_a); z) &= \mathrm{Tr}_b[\tilde{\Phi}(\rho_a)\, V_b(z)] = \mathrm{Tr}_b[\rho_a\, \tilde{\Phi}_{\mathrm{H}}(V_b(z))] \\ &= \phi(\rho_a; m_{21}\cdot z)\, \exp[-\tfrac{1}{2}(N+1/2)\,|m_{22}\cdot z|^2],\end{aligned} \tag{3.1}$$

[7] This follows from the fact that $\mathrm{Det}[O] = \pm 1$ since $O^T = O^{-1}$.

Table 2. Description of the weakly complementary (1.2) of the canonical forms A_1, A_2, B_1, C and D of table 1 constructed from the physical representations (1.1) given in subsection 2.4. The first column indicates the class of Φ. In the central columns instead is given a description of $\tilde{\Phi}$ in terms of the representation (2.5). Finally the last column reports the canonical form corresponding to the map $\tilde{\Phi}$. In all cases the identification is immediate: for instance the canonical form of the map $\tilde{\Phi}_{A_1}$ belongs to the class B_2, while the canonical form of the map $\tilde{\Phi}_D$ is the class C with $\mathrm{Det}[K_{\mathrm{can}}] > 1$. In the case of $\tilde{\Phi}_{A_2}$ the identification with the class B_1 was done by exploiting the possibility freely varying N_c of equation (2.17)—see [20]. A pictorial representation of the above weak-degradability connections is given in figure 1.

		Weak complementary channel $\tilde{\Phi}$		
Class of Φ		K	α	Class of $\tilde{\Phi}$
A_1		$\mathbb{1}$	0	B_2
A_2		$\mathbb{1}$	$(N_0+1/2)\,(\mathbb{1}-\sigma_3)/2$	B_1
B_1		$(\mathbb{1}+\sigma_3)/2$	$\mathbb{1}/2$	A_2
C	$\kappa < 1$	$\sqrt{1-\kappa^2}\,\mathbb{1}$	$k^2(N_0+1/2)\mathbb{1}$	$C\ (\kappa < 1)$
C	$\kappa > 1$	$\sqrt{\kappa^2-1}\,\sigma_3$	$\kappa^2(N_0+1/2)\,\mathbb{1}$	D
D		$\sqrt{\kappa^2+1}\,\mathbb{1}$	$\kappa^2(N_0+1/2)\,\mathbb{1}$	$C(\kappa > 1)$

where m_{21}, m_{22} are the blocks elements of the matrix M of equation (2.1) associated with the transformations U_{ab}, and with N being the average photon number of ρ_b (the values of these quantities are given in the tables of subsection 2.4—see also the appendix). By setting $m = 0$, $K = m_{21}$ and $\alpha = (N+1/2)\, m_{22}^T\, m_{22}$, equation (3.1) has the same structure (2.5) of the one-mode Gaussian channel of A. Therefore by cascading $\tilde{\Phi}$ with an isometry which exchanges A with B (see [19, 31]) we can then treat $\tilde{\Phi}$ as an one-mode Gaussian channel operating on A (this is possible because both A and B are bosonic one-mode systems). With the help of table 1 we can then determine which classes can be associated with the transformation (3.1). This is summarized in table 2.

3.2. Weak-degradability properties

Using the composition rules of equations (2.6) and (2.18) it is easy to verify that the canonical forms A_1, A_2, D and C with $\kappa \leqslant \sqrt{1/2}$ are anti-degradable (1.10). Vice versa, one can verify that the canonical forms B_1 and C with $\kappa \geqslant \sqrt{1/2}$ are weakly degradable (1.9)—for C, D and A_1 these results have been proved in [19]. Through unitary equivalence this can be summarized by saying that all one-mode Gaussian channels (2.5) having $\mathrm{Det}[K] \leqslant 1/2$ are anti-degradable, while the others (with the exception of the channels belonging to B_2) are weakly degradable (see figure 2).

In the following we verify the above relations by explicitly constructing the connecting channels Ψ and $\overline{\Psi}$ of equations (1.9) and (1.10) for each of the mentioned canonical forms.

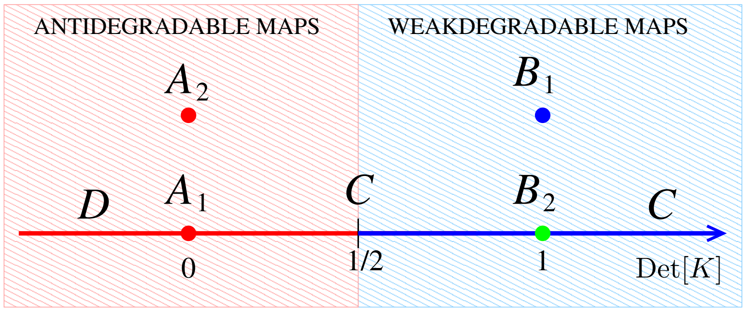

Figure 2. Pictorial representation of the weakly degradability regions for one-mode Gaussian channels. All canonical forms with $\mathrm{Det}[K] \leqslant 1/2$ are anti-degradable: this includes the classes A_1, A_2, D and part of the C. The remaining (with the exception of B_2) are instead weakly degradable. Moreover B_1 is also degradable in the sense of [22]. The same holds for channels of canonical form C with $N_0 = 0$: the exact expression for the quantum capacity of these channels has been given in [21].

Indeed one has:

- For a channel Φ of standard form A_1 or A_2, anti-degradability can be shown by simply taking $\overline{\Psi}$ of equation (1.10) coincident with the channel Φ. The result immediately follows from the composition rule (2.6).
- For a channel Φ of B_1, weak-degradability comes by assuming the map Ψ to be equal to the weakly complementary channel $\tilde{\Phi}$ of Φ (see table 2). As pointed out in [20], this also implies the degradability of Φ in the sense of [22]. Let us remind that for B_1 the physical representation given in subsection 2.4 was constructed with an environmental state ρ_b initially prepared in the vacuum state, which is pure. Therefore in this case our representation gives rise to a Stinespring dilation.
- For a channel Φ of the class C with $K_{\mathrm{can}} = \kappa\, \mathbb{1}$ and $\alpha_{\mathrm{can}} = |\kappa^2 - 1|(N_0 + 1/2)\mathbb{1}$ we have the following three possibilities:
  - If $\kappa \leqslant \sqrt{1/2}$ the channel is anti-degradable and the connecting map $\overline{\Psi}$ is a channel of C characterized by $K_{\mathrm{can}} = \kappa'\, \mathbb{1}$ and $\alpha_{\mathrm{can}} = (1 - (\kappa')^2)(N_0 + 1/2)\mathbb{1}$ with $\kappa' = \kappa/\sqrt{1-\kappa^2} < 1$.
  - If $\kappa \in [\sqrt{1/2}, 1[$ the channel is weakly degradable and the connecting map Ψ is again a channel of C defined as in the previous case but with $\kappa' = \sqrt{1-\kappa^2}/\kappa < 1$. For $N_0 = 0$ the channel is also degradable [22] since our physical representation is equivalent to a Stinespring representation.
  - If $\kappa > 1$ the channel is weakly degradable and the connecting map Ψ is a channel of D with $K_{\mathrm{can}} = \kappa'\, \mathbb{1}$ and $\alpha_{\mathrm{can}} = ((\kappa')^2 - 1)(N_0 + 1/2)\mathbb{1}$ with $\kappa' = \sqrt{k^2 - 1}/k$. As in the previous case, for $N_0 = 0$ the channel is also degradable [22].

- For a channel Φ of D with $K_{\mathrm{can}} = \kappa\, \sigma_3$ and $\alpha_{\mathrm{can}} = (\kappa^2 + 1)(N_0 + 1/2)\mathbb{1}$ ($\kappa > 0$ and $N_0 \geqslant 0$) we can prove anti-degradability by choosing $\overline{\Psi}$ of equation (1.10) to be yet another maps of D with $K_{\mathrm{can}} = \kappa'\, \sigma_3$ and $\alpha_{\mathrm{can}} = ((\kappa')^2 + 1)(N_0 + 1/2)\mathbb{1}$ where $\kappa' = \kappa/\sqrt{\kappa^2 + 1}$. From equation (2.6) and table 2 it then follows that $\Psi \circ \tilde{\Phi}$ is indeed equal to Φ.

Concerning the case B_2 it was shown in [20] that the channel is neither anti-degradable nor degradable in the sense of [22] (apart from the trivial case $N_0 = 0$ which corresponds to the identity map). On the other hand one can use the continuity argument given in subsection 2.5 to claim that the channel B_2 can be arbitrarily approximated with maps which are weakly degradable (those belonging to C for instance).

4. One-mode Gaussian channels with Det$[K] > 1/2$ and having null quantum capacity

In the previous section, we saw that all channels (2.5) with Det$[K] \leqslant 1/2$ are anti-degradable. Consequently these channel must have null quantum capacity [19, 31]. Here we go a little further showing that the set of the maps (2.5) which can be proved to have null quantum capacity include also some maps with Det$[K] > 1/2$. To do this we will use the following simple fact:

Let be Φ_1 a quantum channel with null quantum capacity and let be Φ_2 some quantum channel. Then the composite channels $\Phi_1 \circ \Phi_2$ and $\Phi_2 \circ \Phi_1$ have null quantum capacity.

The proof of this property follows by interpreting Φ_2 as a quantum operation performed either at the decoding or at encoding stage of the channel Φ_1. This shows that the quantum capacities of $\Phi_1 \circ \Phi_2$ and $\Phi_2 \circ \Phi_1$ cannot be greater than the capacity of Φ_1 (which is null). In the following we will present two cases where the above property turns out to provide some nontrivial results.

4.1. Composition of two class D channels

We observe that according to composition rule (2.18) the combination of any two channels Φ_1 and Φ_2 of D produces a map $\Phi_{21} \equiv \Phi_2 \circ \Phi_1$ which is in the class C. Since the class D is anti-degradable the resulting channel must have null quantum capacity. Let then $\kappa_j \sigma_3$ and $(\kappa_j^2 + 1)(N_j + 1/2)\mathbb{1}$ be the matrices K_{can} and α_{can} of the channels Φ_j, for $j = 1, 2$. From equation (2.6) one can then verify that Φ_{21} has the canonical form C with parameters

$$\kappa = \kappa_1 \kappa_2, \tag{4.1}$$

$$N_0 = \frac{(\kappa_2^2 + 1)N_2 + \kappa_2^2(\kappa_1^2 + 1)N_1}{|\kappa_1^2\kappa_2^2 - 1|} + \frac{1}{2}\left(\frac{\kappa_1^2\kappa_2^2 + 2\kappa_2^2 + 1}{|\kappa_1^2\kappa_2^2 - 1|} - 1\right). \tag{4.2}$$

Equation (4.1) shows that by varying κ_j, κ can take any positive values: in particular it can be greater than $\sqrt{1/2}$ transforming Φ_{21} into a channel which does not belong to the anti-degradable area of figure 2. On the other hand, by varying the N_j and κ_2, but keeping the product $\kappa_1\kappa_2$ fixed, the parameter N_0 can assume any value satisfying the inequality

$$N_0 \geqslant \frac{1}{2}\left(\frac{\kappa^2 + 1}{|\kappa^2 - 1|} - 1\right). \tag{4.3}$$

We can therefore conclude that all channels C with κ and N_0 as in equation (4.3) have null quantum capacity—see figure 3. A similar bound was found in a completely different way in [3].

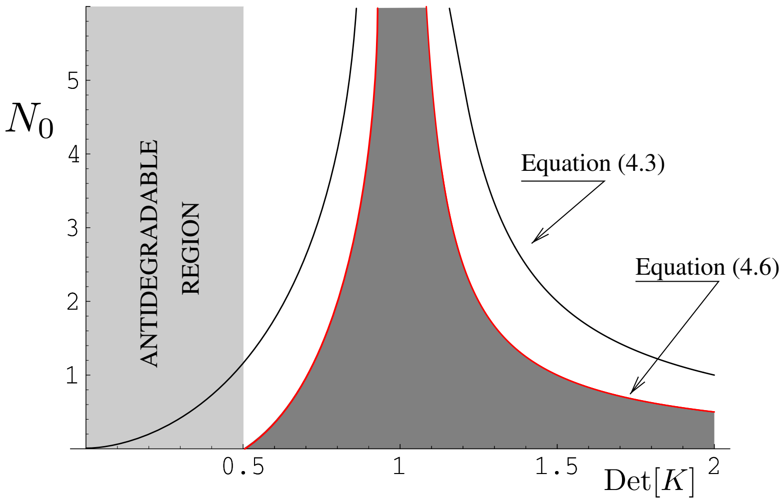

Figure 3. The dark-grey area of the plot is the region of the parameters N_0 and $\mathrm{Det}[K] = \kappa^2$ where a channel with canonical form C cannot have null quantum capacity. For $\mathrm{Det}[K] < 1/2$ the channel is anti-degradable. In the remaining white area the quantum capacity is null since these maps can be obtained by a composition of channels, one of which being anti-degradable. The curve in black refers to the bound of equation (4.3). The contour of the dark-grey area is instead given by equation (4.6).

4.2. Composition of two class C channels

Consider now the composition of two class C channels, i.e. Φ_1 and Φ_2, with one of them (say Φ_2) being anti-degradable.

Here, the canonical form of Φ_1 and Φ_2 have matrices K_{can} and α_{can} given by $K_i = \kappa_j \mathbb{1}$ and $\alpha_j = |\kappa_j^2 - 1|(N_j + 1/2)\mathbb{1}$, where for $j = 1, 2$, N_j and κ_j are positive numbers, with $\kappa_1 \neq 0, 1$ and with $\kappa_2 \in 0, \sqrt{1/2}$ (to ensure anti-degradability). From equation (2.6) follows then that the composite map $\Phi_{21} = \Phi_2 \circ \Phi_1$ has still a C canonical form with parameters

$$\kappa = \kappa_1 \kappa_2, \tag{4.4}$$

$$N_0 = \frac{|\kappa_2^2 - 1|N_2 + \kappa_2^2|\kappa_1^2 - 1|N_1}{|\kappa_1^2\kappa_2^2 - 1|} + \frac{1}{2}\left(\frac{\kappa_2^2|\kappa_1^2 - 1| + |\kappa_2^2 - 1|}{|\kappa_1^2\kappa_2^2 - 1|} - 1\right). \tag{4.5}$$

As in the previous example, κ can assume any positive value. Vice-versa keeping κ fixed, and varying $\kappa_1 > 1$ and $N_{1,2}$ it follows that N_0 can take any values which satisfy the inequality

$$N_0 \geqslant \frac{1}{2}\left(\frac{\kappa^2}{|\kappa^2 - 1|} - 1\right). \tag{4.6}$$

We can then conclude that all maps C with κ and N_0 as above must possess null quantum capacity. The result has been plotted in figure 3. Notice that the constraint (4.6) is an improvement with respect to the constraint of equation (4.3).

5. Conclusion

In this paper, we provide a full weak-degradability classification of one-mode Gaussian channels by exploiting the canonical form decomposition of [20]. Within this context we identify those channels which are anti-degradable. By exploiting composition rules of Gaussian maps, this allows us to strengthen the bound for one-mode Gaussian channels which do not have null quantum capacity.

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Appendix A. The matrix M

Here, we give the explicit expressions of the matrix M of equation (2.11) associated with the physical representations of the classes A_1, A_2, B_1, C and D, discussed in subsection 2.4. They are,

$$M_{A_1} \equiv \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right), \quad M_{A_2} \equiv \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{array}\right), \quad M_{B_1} \equiv \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{array}\right),$$

$$M_C \equiv \left(\begin{array}{cc|cc} k & 0 & \sqrt{1-k^2} & 0 \\ 0 & k & 0 & \sqrt{1-k^2} \\ \hline \sqrt{1-k^2} & 0 & -k & 0 \\ 0 & \sqrt{1-k^2} & 0 & -k \end{array}\right) \qquad (\text{for } \kappa < 1),$$

$$M_C \equiv \left(\begin{array}{cc|cc} k & 0 & \sqrt{k^2-1} & 0 \\ 0 & k & 0 & -\sqrt{k^2-1} \\ \hline \sqrt{k^2-1} & 0 & k & 0 \\ 0 & -\sqrt{k^2-1} & 0 & k \end{array}\right) \qquad (\text{for } \kappa > 1),$$

$$M_D \equiv \left(\begin{array}{cc|cc} k & 0 & \sqrt{k^2+1} & 0 \\ 0 & -k & 0 & \sqrt{k^2+1} \\ \hline \sqrt{k^2+1} & 0 & k & 0 \\ 0 & \sqrt{k^2+1} & 0 & -k \end{array}\right).$$

References

[1] Bennett C H and Shor P W 1998 *IEEE Trans. Inform. Theory* **44** 2724
[2] Holevo A S 1972 *Probl. Inform. Transm.* **8** 63
[3] Holevo A S and Werner R F 2001 *Phys. Rev.* A **63** 032312
[4] Eisert J and Wolf M M 2005 *Preprint*quant-ph/0505151
[5] Gallager R G 1968 *Information Theory and Reliable Communication* (New York: Wiley)
[6] Braunstein S and van Look P 2005 *Rev. Mod. Phys.* **77** 513
[7] Caves C M and Drummond P D 1994 *Rev. Mod. Phys.* **66** 481
Yuen H P and Ozawa M 1993 *Phys. Rev. Lett.* **70** 363
[8] Holevo A S, Sohma M and Hirota O 1999 *Phys. Rev.* A **59** 1820
[9] Giovannetti V, Guha S, Lloyd S, Maccone L, Shapiro J H and Yuen H P 2004 *Phys. Rev. Lett.* **92** 027902
[10] Giovannetti V, Lloyd S, Maccone L and Shor P W 2003 *Phys. Rev. Lett.* **91** 047901
Giovannetti V, Lloyd S, Maccone L and Shor P W 2003 *Phys. Rev.* A **68** 062323
[11] Holevo A S 1998 *IEEE Trans. Inform. Theory* **44** 269
Hausladen P, Jozsa R, Schumacher B, Westmoreland M and Wootters W K 1996 *Phys. Rev.* A **54** 1869
Schumacher B and Westmoreland M D 1997 *Phys. Rev.* A **56** 131
[12] Holevo A S 2003 *Probab. Theory Appl.* **48** 359
Holevo A S and Shirokov M E 2004 *Commun. Math. Phys.* **249** 417
Shirovov M E 2006 *Commun. Math. Phys.* **262** 137
Holevo A S and Shirokov M E 2004 *Preprint* quant-ph/0408176
[13] Lloyd S 1997 *Phys. Rev.* A **55** 1613
Barnum H, Nielsen M A and Schumacher B 1998 *Phys. Rev.* A **57** 4153
Devetak I 2005 *IEEE Trans. Inform. Theory* **51** 44
[14] Bennett C H, Shor P W, Smolin J A and Thapliyal A V 2002 *IEEE Trans. Inform. Theory* **48** 2637
[15] Hall M J W and O'Rourke M J 1993 *Quantum Opt.* **5** 161
Hall M J W 1994 *Phys. Rev.* A **50** 3295
Shapiro J H, Giovannetti V, Guha S, Lloyd S, Maccone L and Yen B J 2004 *AIP Conf. Proc.* **734** 15
Sohma M and Hirota O 2001 *Preprint* quant-ph/0105042
[16] Giovannetti V, Guha S, Lloyd S, Maccone L and Shapiro J H 2004 *Phys. Rev.* A **70** 032315
Giovannetti V, Lloyd S, Maccone L, Shapiro J H and Yen B J 2004 *Phys. Rev.* A **70** 022328
[17] Giovannetti V and Lloyd S 2004 *Phys. Rev.* A **69** 062307
[18] Serafini A, Eisert J and Wolf M M 2005 *Phys. Rev.* A **71** 012320
[19] Caruso F and Giovannetti V 2006 *Phys. Rev.* A in press (*Preprint* quant-ph/0603257)
[20] Holevo A S 2006 *Preprint* quant-ph/0607051
[21] Wolf M M and Peréz-García D 2006 *Preprint* quant-ph/0607070
[22] Devetak I and Shor P W 2005 *Commun. Math. Phys.* **256** 287
[23] Stinespring W F 1955 *Proc. Am. Math. Soc.* **6** 211
[24] Lindblad G 1976 *Commun. Math. Phys.* **48** 116
[25] Holevo A S 2005 *Preprint* quant-ph/0509101
[26] King C, Matsumoto K, Nathanson M and Ruskai M B 2005 *Preprint* quant-ph/0509126
[27] Wootters W K and Zurek W H 1982 *Nature* **299** 802
[28] Gisin N and Massar S 1997 *Phys. Rev. Lett.* **79** 2153
Bruß D, DiVincenzo D P, Ekert A, Fuchs C A, Macchiavello C and Smolin J A 1998 *Phys. Rev.* A **57** 2368
Werner R F 1998 *Phys. Rev.* A **58** 1827
Büzek V and Hillery M 1998 *Phys. Rev. Lett.* **81** 5003
[29] Ferraro A, Olivares S and Paris M G A 2005 *Gaussian States in Quantum Information* (Napoli: Bibliopolis)
[30] Holevo A S 1982 *Probabilistic Aspects of Quantum Theory* (Amsterdam: North-Holland)
[31] Giovannetti V and Fazio R 2005 *Phys. Rev.* A **71** 032314