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Spin chain model for correlated quantum channels

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Abstract. We analyze the quality of the quantum information transmission along a correlated quantum channel by studying the average fidelity between input and output states and the average output purity, giving bounds for the entropy of the channel. Noise correlations in the channel are modeled by the coupling of each channel use with an element of a one-dimensional interacting quantum spin chain. Criticality of the environment chain is seen to emerge in the changes of the fidelity and of the purity.

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1. Introduction

The dynamics of any open quantum system [1] can be described in terms of linear superoperator transformations that map the initial density matrices of the system into their time evolved counterparts, modeling the action of the environment. These mappings are known as ‘quantum channels’ and in recent years have been widely studied in quantum communication [2]. In this context, quantum channels represent the input–output transformations of the signals that one party (say Alice) encodes in some quantum medium and sends to a distant party (say Bob) in order to transmit to him (quantum or classical) messages. To overcome the noise associated with the resulting signaling process, some strategies aimed to reduce communication errors have to be employed. These are based on complex encoding/decoding operations and on suitably tailoring the physical medium that acts as the information mediator. The latter can be as simple as an optical fiber or a complex many body quantum system [3], such as spin chains [4], Josephson arrays [5], coupled quantum dots [6] or cavities [7], which are likely subject to dissipation due to ‘unconventional’ or correlated environments.

In quantum information theory, the effect of noise on the quantum communication is typically quantified by the so-called *capacities* of the channel, that is the optimal rates at which (quantum or classical) information can be reliably transmitted in the limit of infinite channel uses [8]. The vast majority of the results obtained so far focused on the case of *memoryless* quantum channels, where the noise acts independently for each channel use. However, in real physical situations like those of [3], correlations in the noise acting between successive uses can be established. When this happens the communication line is said to be a *memory channel*, or more precisely, a *correlated channel*. The analysis of these setups is much more demanding than the memoryless case, and, at present, only a restricted class of them has been solved [9–16].

Recently, a physical model for representing correlated channels has been proposed in [11, 17], which, in the context of bosonic channels and qubit channels, respectively, has established a direct connection between these systems and many-body physics. The setup discussed in these proposals is depicted in figure 1. Here, Alice sends her messages to Bob by encoding them into an n -long sequence of information carriers S (the red dots of the figure), which model subsequent channel uses associated with n non-interacting bosonic modes [11] or n -independent spins [17]. The correlated noise of the channel is then described by assuming that each carrier interacts independently with a corresponding element of an n -party environment E (sketched with the connected black dots of the figure), which, in [11, 17], represents a multi-mode Gaussian state and a many-body spin state, respectively. Thus, given an input state ρ_S of the n carriers, the corresponding output state associated with the channel is

$$\mathcal{E}_n(\rho_S) = \text{Tr}_E[\mathcal{U}(\rho_S \otimes \sigma_E)\mathcal{U}^\dagger], \quad (1)$$

where σ_E is the joint input state of E and the partial trace is performed over the environment. In this equation \mathcal{U} represents the unitary coupling between S and E , which is expressed as

$$\mathcal{U} = \bigotimes_{\ell=1}^n U^{(\ell)}, \quad (2)$$

with $U^{(\ell)}$ being the interaction between the ℓ th carrier and its environmental counterpart (in [11] these were beam-splitter couplings, whereas in [17] they were phase-gate couplings). Within this framework, memoryless channels $\mathcal{E}_n = \mathcal{E}^{\otimes n}$ are obtained for factorizable environmental input states, whereas correlated noise models correspond to correlated environmental states σ_E .

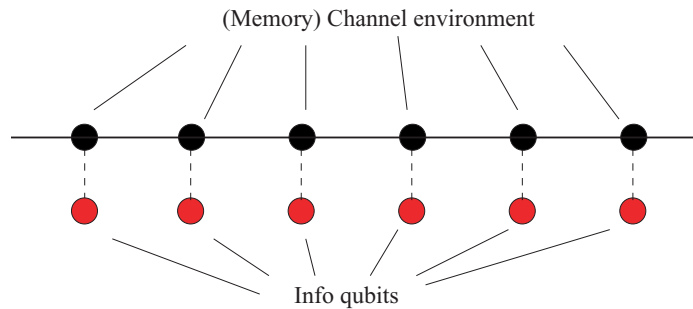


Figure 1. Model for memory channels: the red dots represent the channel uses (ordered, for instance, starting from the left to the right). The channel noise is modeled as a collection of local interactions between the channel uses and the many-body environment E (black dots).

Interestingly enough, in [17], it was shown that it is possible to relate the quantum capacity [18] of some specific channels (1) to the properties of the many-body environment E .

In this paper, we discuss a variation of the model (1), which allows us to adapt some of the techniques used in [19] for characterizing the decoherence effects induced by spin quantum baths, in order to analyze the efficiency of a class of correlated qubits channels. To do so, we consider a unitary coupling \mathcal{U} that does not factorize as in equation (2). Instead, we assume E to be a spin chain characterized by a free Hamiltonian \mathcal{H}_E , whose elements interact with the carriers S through the local Hamiltonian \mathcal{H}_{int} . With this choice, we write

$$\mathcal{U} = \exp[-i(\mathcal{H}_E + \mathcal{H}_{\text{int}})t], \quad (3)$$

with the interaction time t being a free parameter of the model. In particular, as a chain Hamiltonian, we consider a spin-1/2 XY model in a transverse field which, for suitable choices of the parameters, exhibits ground state critical properties that greatly enhanced the spin correlations [20]. In our model, the distance of the chain from criticality is expected to be nontrivially related to memory effects of the quantum communication channel. Indeed, as the latter are sensitive to the correlations present in the chain ground state, out of criticality the successive uses of the channel should exhibit a Markovian character after some typical length (related to the spatial exponential decay of the correlations). Vice versa this regime should never be reached at the critical point, as the ground state correlations exhibit no scale.

In the second part of the paper, we generalize the previous scheme by introducing a given number of m extra spins between any two consecutive qubits, as shown in figure 2. In this case, we can use the number m to modulate the memory effects.

2. The model

As the environment E of the system in figure 1, we consider an interacting one-dimensional quantum spin-1/2 chain described by an XY exchange Hamiltonian in a transverse magnetic field:

$$\mathcal{H}_E = -\frac{J}{2} \sum_{j=1}^n [(1+\gamma)\sigma_j^x \sigma_{j+1}^x + (1-\gamma)\sigma_j^y \sigma_{j+1}^y + 2\lambda\sigma_j^z], \quad (4)$$

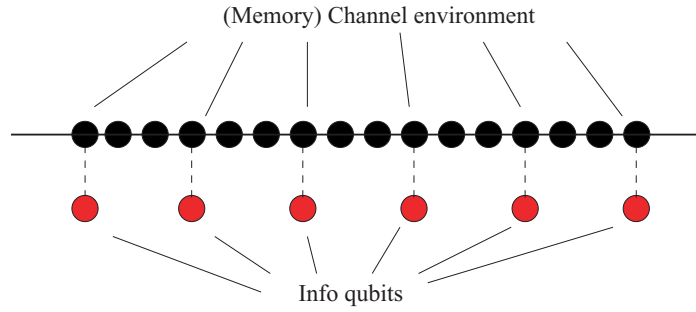


Figure 2. Generalized model of spin chain memory channels. As an example, in this figure we set $m = 2$.

where σ_j^α (with $\alpha = x, y, z$) are the Pauli matrices of the j th spin, J is the coupling strength between neighboring spins, and λ is the external field strength³. The model in equation (4) for $0 < \gamma \leq 1$ belongs to the Ising universality class, and has a critical point at $\lambda_c = 1$; for $\gamma = 0$, it reduces to the XX universality class, that is critical for $|\lambda| \leq 1$ [20].

Following [19], we then assume that each carrier qubit is coupled to one environmental spin element through the coupling Hamiltonian

$$\mathcal{H}_{\text{int}}(j) = -\varepsilon |e\rangle_{jj} \langle e| \otimes \sigma_j^z, \quad (5)$$

where $|g\rangle_j$ and $|e\rangle_j$, respectively, represent the ground and the excited state of the j th qubit. Hence the total Hamiltonian $\mathcal{H} \equiv \mathcal{H}_E + \mathcal{H}_{\text{int}}$ is given by

$$\mathcal{H} = -\frac{J}{2} \sum_{j=1}^n [(1 + \gamma) \sigma_j^x \sigma_{j+1}^x + (1 - \gamma) \sigma_j^y \sigma_{j+1}^y + 2\lambda \sigma_j^z] - \varepsilon \sum_{j=1}^n |e\rangle_{jj} \langle e| \sigma_j^z. \quad (6)$$

Finally, as in [17, 19], we suppose that at time $t = 0$ the environment chain is prepared in the ground state $|\varphi\rangle_E$ of \mathcal{H}_E . We then consider a generic input state $|\psi\rangle_S$ of the n -qubit carriers of the system (i.e. the input state of the red dots in figure 1), and write it in the computational basis:

$$|\psi\rangle_S = \sum_x \alpha_x |x\rangle_S, \quad (7)$$

where α_x are complex probability amplitudes and the sum runs over $N = 2^n$ possible choices of x , each of them being a binary string of n elements in which the j th element is represented as g or e , according to the state (ground or excited, respectively) of the corresponding j th qubit.

For each vector $|x\rangle_S$, we define \mathcal{S}_x as the set of the corresponding excited qubits (for instance, given $n = 5$ and $|x\rangle_S = |egeeg\rangle_S$, then \mathcal{S}_x contains the 1st, 3rd and 4th qubits). After a time t , the global state of the qubits and the chain will then evolve into

$$|\psi\rangle_S \otimes |\varphi\rangle_E \xrightarrow{\mathcal{U}} \sum_x \alpha_x |x\rangle_S \otimes \mathcal{U}_x |\varphi\rangle_E, \quad (8)$$

where \mathcal{U} is the global evolution operator of equation (3), whereas $\mathcal{U}_x \equiv \exp[-i \mathcal{H}_E^x t]$ is associated to the following *chain* Hamiltonian:

$$\mathcal{H}_E^x \equiv \mathcal{H}_E - \varepsilon \sum_{j \in \mathcal{S}_x} \sigma_j^z. \quad (9)$$

³ Hereafter we always use open boundary conditions, therefore we assume $\sigma_n^\alpha \sigma_1^\alpha = 0$.

According to equation (1), the channel output state is then described by the density matrix

$$\mathcal{E}_n(|\psi\rangle_{SS}\langle\psi|) = \rho'_S = \sum_{x,y} L_{xy} \alpha_x \alpha_y^* |x\rangle_{SS}\langle y|, \quad (10)$$

where

$$L_{xy} \equiv {}_E\langle\varphi| \mathcal{U}_y^\dagger \mathcal{U}_x |\varphi\rangle_E, \quad (11)$$

can be seen as a generalized Loschmidt echo, denoting the scalar product of the input environment state $|\varphi\rangle_E$ evolved with \mathcal{U}_x and \mathcal{U}_y , respectively [21]. These quantities can be evaluated by first mapping the Hamiltonian (9) into a free-fermion model via a Jordan-Wigner transformation [22]

$$c_k = \exp\left(i\pi \sum_{j=1}^{k-1} \sigma_j^+ \sigma_j^-\right) \sigma_k^-, \quad (12)$$

where $\sigma^\pm = (\sigma^x \pm i\sigma^y)/2$, and then by diagonalizing it with a Bogoliubov rotation of the Jordan-Wigner fermions $\{c_k, c_k^\dagger\}_{k=1,\dots,n}$. This allows one to find an explicit expression of the Loschmidt echo in terms of the determinant of a $2n \times 2n$ matrix (see [19] for details):

$$L_{xy} = {}_E\langle\varphi| e^{i\mathcal{H}_E^x t} e^{-i\mathcal{H}_E^y t} |\varphi\rangle_E = \det(\mathbb{I} - \rho_0 + \rho_0 e^{iH_E^x t} e^{-iH_E^y t}), \quad (13)$$

where $\mathcal{H}_E^k = \sum_{ij} [H_E^k]_{ij} \Psi_i^\dagger \Psi_j$ (the superscript k being x or y), $\Psi^\dagger = (c_1^\dagger \dots c_n^\dagger c_1 \dots c_n)$, and $[\rho_0]_{ij} = {}_E\langle\varphi| \Psi_i^\dagger \Psi_j |\varphi\rangle_E$ are the two-point correlation functions of the chain.

3. The channel

The echoes (11) provide a complete characterization of the correlated channel \mathcal{E}_n . In particular, since $L_{xx} = 1$ for all x , equation (10) shows that the channel \mathcal{E}_n is *unital*, i.e. it maps the completely mixed state $\frac{1}{N} \sum_x |x\rangle_{SS}\langle x|$ into itself; moreover, it is also a *pure dephasing* channel, since every state that is diagonal in the computational basis is transmitted without errors. The $N \times N$ matrix of elements L_{xy}/N coincides with the *Choi–Jamiołkowski state* [23] of the map. The latter is defined as the output density matrix obtained when sending through the channel \mathcal{E}_n half of the canonical maximally entangled state $|+\rangle_{SA} \equiv \frac{1}{\sqrt{N}} \sum_x |x\rangle_S \otimes |x\rangle_A$ of the N -level system S , i.e.

$$J(\mathcal{E}_n) \equiv (\mathcal{E}_n \otimes \mathcal{I}_A)(|+\rangle_{SA}\langle +|) = \sum_{x,y} \frac{L_{xy}}{N} |xx\rangle_{SA}\langle yy|, \quad (14)$$

with A being a N -dimensional ancillary system and \mathcal{I}_A being the identity map. Similarly to the case analyzed in [17], this is a *maximally correlated* state [24] whose one-way distillable entanglement is known to coincide with the ‘hashing bound’ [24]–[26]:

$$D_1(J(\mathcal{E}_n)) = H(J_S(\mathcal{E}_n)) - H(J(\mathcal{E}_n)) = \log_2 N - H(J(\mathcal{E}_n)), \quad (15)$$

where $J_S(\mathcal{E}_n) \equiv \text{Tr}_A[J(\mathcal{E}_n)]$ is the reduced density matrix of $J(\mathcal{E}_n)$ associated with the system S , whereas $H(\cdot) = \text{Tr}[(\cdot) \log_2(\cdot)]$ is the von Neumann entropy. Even though we will not provide evidence that our channels (10) belong to the subclass of *forgetful* channels [14], for the sake

of completeness it is worth mentioning that at least for such class the regularized version of equation (15) can be used [17, 25] to bound the quantum capacity [8, 18] of \mathcal{E}_n . This is⁴

$$Q(\mathcal{E}_n) \geq \lim_{n \rightarrow \infty} \frac{D_1(J_S(\mathcal{E}_n))}{n} = 1 - \lim_{n \rightarrow \infty} \frac{H(J(\mathcal{E}_n))}{n}. \quad (16)$$

The quantity $H(J(\mathcal{E}_n))$ corresponds to the *entropy of the channel* \mathcal{E}_n of [28], which can be used as an estimator of the channel noise. In our case, it also has a simple interpretation in terms of the properties of the many-body system E : it measures the entropy of the ground state $|\varphi\rangle_E$ after it has evolved through a random application of the perturbed unitaries \mathcal{U}_x ⁵, i.e.

$$H(J(\mathcal{E}_n)) = H(\sigma'_E) \quad \text{with} \quad \sigma'_E \equiv \frac{1}{N} \sum_x \mathcal{U}_x |\varphi\rangle_{EE} \langle \varphi| \mathcal{U}_x^\dagger. \quad (17)$$

Unfortunately, for large n the computation of the von Neumann entropy of the state $J(\mathcal{E}_n)$ is impractical both analytically and numerically, since it requires to evaluate an exponential number of L_{xy} elements. Interestingly enough, however, we can simplify our analysis by considering the fidelity between $J(\mathcal{E}_n)$ and its input counterpart $|+\rangle_{SA}$ (see equation (14)). As discussed in the following section, this is a relevant information theoretical quantity, since it is directly related to the *average fidelity* between input and output state of the channel \mathcal{E}_n and provides us an upper bound for $H(J(\mathcal{E}_n))$. Similarly, we can compute the purity of $J(\mathcal{E}_n)$ which, on one hand, gives a lower bound for $H(J(\mathcal{E}_n))$, while, on the other hand, is directly related to the average channel output purity of the map \mathcal{E}_n .

4. Average transmission fidelity

According to equation (14), the fidelity between the Choi–Jamiołkowski state $J(\mathcal{E}_n)$ and its input counterpart coincides with the average value of the Loschmidt echoes L_{xy} , i.e.

$$\mathcal{F} \equiv {}_{SA} \langle + | J(\mathcal{E}_n) | + \rangle_{SA} = \frac{1}{N^2} \sum_{x,y} L_{xy}. \quad (18)$$

Even without computing all the L_{xy} , this quantity can be numerically evaluated by performing a sampling over N_{av} randomly chosen couples (x, y) of initial conditions, and averaging

⁴ The inequality (16) is a consequence of the fact that the quantum capacity Q of a channel does not increase if we provide the communicating parties with a one-way (from the sender to the receiver) classical side communication line [25, 27]. It is derived by constructing an explicit quantum communication protocol in which (i) Alice sends through the channel half of the maximally entangled state $|+\rangle_{SA}$ to Bob, (ii) the resulting state $J(\mathcal{E}_n)$ is then one-way distilled obtaining $D_1(J(\mathcal{E}_n))$ Bell pairs which, finally, (iii) are employed to teleport Alice's messages to Bob. It is worth noticing that for the channel analyzed in [17] the right-hand side of equation (15) was also an upper bound for Q .

⁵ This is a trivial consequence of the fact that $J(\mathcal{E}_n)$ is the reduced density matrix of the pure state $\frac{1}{\sqrt{N}} \sum_x |xx\rangle_{SA} \otimes \mathcal{U}_x |\varphi\rangle_E$ tracing out the environment, and of the fact that the von Neumann entropies of the reduced density matrices of a pure bipartite system coincide.

over them⁶:

$$\mathcal{F} \approx \mathcal{F}_{\text{av}} \equiv \frac{1}{N_{\text{av}}} \sum_{(x,y)=1}^{N_{\text{av}}} \text{Re}[L_{xy}], \quad (19)$$

(where we used the fact that $L_{yx} = L_{xy}^*$).

For memoryless channels \mathcal{F} has a characteristic exponential behavior f^n (with f being the single-use Choi–Jamiołkowski input–output fidelity): thus, at least in principle, the study of \mathcal{F} as a function of n provides a simple (but not always successful) method to detect the presence of correlations among successive channel uses. The quantity \mathcal{F} yields also an upper bound for $H(J(\mathcal{E}_n))$ through the quantum Fano inequality [2], i.e.

$$H(J(\mathcal{E}_n)) \leq h_2(\mathcal{F}) + (1 - \mathcal{F}) \log_2(4^n - 1) \leq h_2(\mathcal{F}) + 2n(1 - \mathcal{F}), \quad (20)$$

where $h_2(\cdot) = -(\cdot) \log_2(\cdot) - [1 - (\cdot)] \log_2[1 - (\cdot)]$ is the binary entropy function⁷. Furthermore \mathcal{F} is directly related to the average transmission fidelity $\langle F \rangle$ of the map \mathcal{E}_n . For a given pure input state (7), the transmission fidelity is

$$F(\psi) \equiv {}_S \langle \psi | \mathcal{E}_n(|\psi\rangle_S {}_S \langle \psi|) | \psi \rangle_S = \sum_{x,y} L_{xy} |\alpha_x|^2 |\alpha_y|^2. \quad (21)$$

Taking the average with respect to all possible inputs, we get

$$\langle F \rangle = \sum_{x,y} L_{xy} p_{xy}, \quad (22)$$

where $p_{xy} = \langle |\alpha_x|^2 |\alpha_y|^2 \rangle$ with $\langle \dots \rangle$ being the average with respect to the uniform Haar measure. The probability distribution p_{xy} can be computed by using simple geometrical arguments [30]. As shown in appendix A, this yields $p_{xy} = \frac{1+2\delta_{x,y}}{N(N+2)}$ and hence:

$$\langle F \rangle = \left(\frac{2}{N(N+2)} \sum_{x>y} \text{Re}[L_{xy}] \right) + \frac{3}{N+2}, \quad (23)$$

where we used the fact that $L_{xx} = 1$ and $L_{xy} = L_{yx}^*$. Therefore, from equation (18) we get

$$\langle F \rangle = \frac{N}{N+2} \mathcal{F} + \frac{2}{N+2}. \quad (24)$$

The fidelities \mathcal{F} and $\langle F \rangle$ are not directly related to the channel quantum capacity, nonetheless, as in equation (20), they can be used to derive bounds for Q . In particular if the channel is forgetful from equations (20) and (16) one gets $Q \geq -1 + 2 \lim_{n \rightarrow \infty} \mathcal{F}$. More generally, values near to unity of the fidelity between the output channel states and their corresponding

⁶ We numerically checked the convergence of \mathcal{F} with N_{av} . We first considered a situation with a few number of qubits ($n \leq 10$), to compare sampled averages, \mathcal{F}_{av} , with exact averages over all possible events, $\mathcal{F}_{\text{ex}} = \mathcal{F}$. We found that, already at $N_{\text{av}} = 10^4$, absolute differences $|\mathcal{F}_{\text{av}} - \mathcal{F}_{\text{ex}}|$ are always less than 2×10^{-2} , while at $N_{\text{av}} = 5 \times 10^4$ the error is less than 5×10^{-3} , independently of the values of the interaction time t , the transverse field λ and the system size n . Secondly, we simulated systems with definitely larger sizes ($n \approx 50$) and simply checked the convergence of \mathcal{F}_{av} with N_{av} . Differences between fidelities with $N_{\text{av}} = 10^4$ and $N_{\text{av}} = 5 \times 10^4$ are of the same order as the deviation of the curve with $N_{\text{av}} = 10^4$ from the exact one for small sizes. Therefore, we can reliably affirm that fidelity results with $N_{\text{av}} = 5 \times 10^4$ are exact, up to an absolute error of order 5×10^{-3} .

⁷ Equation (20) can be easily derived by noticing that $H(J(\mathcal{E}_n))$ and \mathcal{F} coincide, respectively, with the *exchange entropy* and *entanglement fidelity* of the channel \mathcal{E}_n [29] associated with the maximally mixed state $\mathbb{I}_S/2^n$ of S .

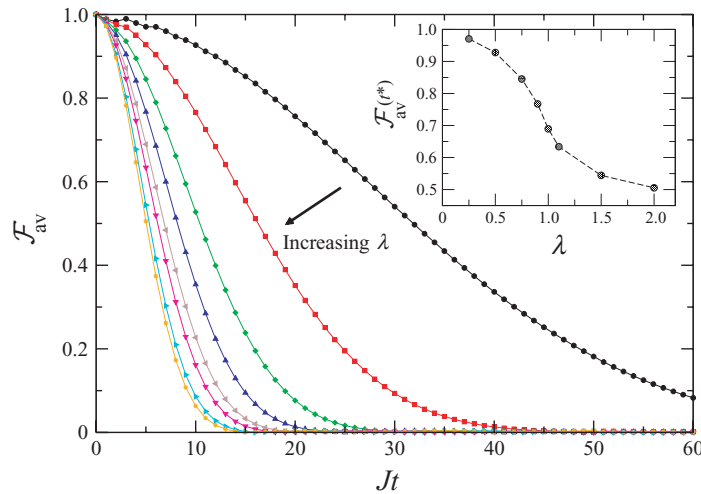


Figure 3. Averaged channel fidelity as a function of the interaction time, for different values of the transverse field λ : from right to left $\lambda = 0.25, 0.5, 0.75, 0.9, 1, 1.1, 1.5, 2$. Here, we simulated a channel of $n = 50$ qubits coupled to a $\gamma = 1$ Ising chain, and set an interaction strength $\varepsilon = 0.05$; the fidelity has been evaluated by sampling over $N_{\text{av}} = 5 \times 10^4$ randomly chosen initial conditions. In the inset, we plot the averaged fidelity at a fixed interaction time $Jt^* = 5$, as a function of λ .

input states are indicative of a fairly noiseless communication line. On the contrary, values of the transmission fidelities close to zero, while indicating output states nearly orthogonal to their input counterparts, do not necessarily imply null or low capacities, since such huge discrepancies between inputs and outputs could still be corrected by a proper encoding and decoding strategy (e.g. consider the case of a channel which simply rotates the system states).

4.1. Numerics

Equation (13) allows us to numerically compute the averaged transmission fidelity (19). An example of dependence of \mathcal{F}_{av} with respect to the free model parameter t is presented in figure 3: the plots are given for the correlated quantum channel with $n = 50$ qubits each of them coupled to one spin of a $\gamma = 1$ Ising chain with a coupling strength $\varepsilon = 0.05$. Different curves stand for different values of the transverse magnetic field λ : as it can be clearly seen, the fidelity \mathcal{F}_{av} decays as a Gaussian in time, irrespective of the field strength λ ⁸. The short-time Gaussian functional dependence of the fidelity is expected from the perturbation theory, while the fact that it lasts for longer times is probably related to the system integrability, as discussed in [31]. The signature of criticality in the environment chain can be identified by studying the function $\mathcal{F}_{\text{av}}(\lambda, t)$ for fixed interaction time t^* : the inset of figure 3 displays a nonanalytic behavior for the derivative of $\mathcal{F}_{\text{av}}(\lambda, t^*)$ with respect to λ at the critical point $\lambda_c = 1$ (such nonanalyticity will be clearer in the following where the size scaling will be considered—see figure 5). In figure 4, we

⁸ For times longer than those in the scales of figures 3 and 4, revivals of the fidelity are present. See appendix B for details.

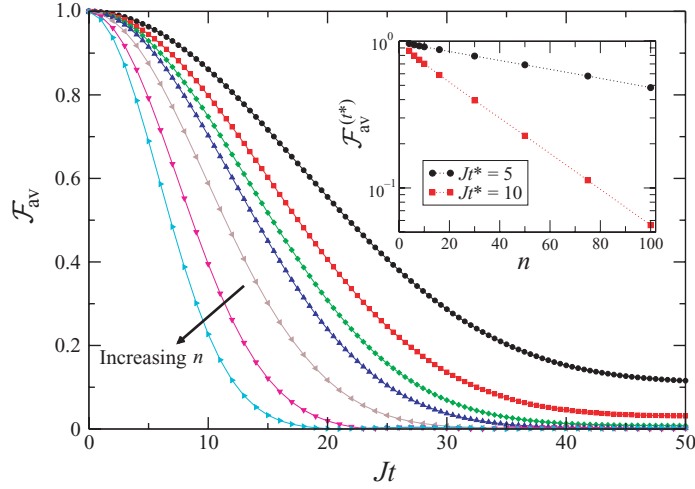


Figure 4. Fidelity for a channel coupled to an Ising chain with $\lambda = 1$, $\gamma = 1$ and different qubit numbers n : from right to left $n = 4, 6, 8, 10, 16, 30, 50$; the interaction strength is kept fixed at $\varepsilon = 0.05$; data are averaged over $N_{\text{av}} = 5 \times 10^4$ configurations. Inset: \mathcal{F}_{av} at a fixed time t^* , as a function of n .

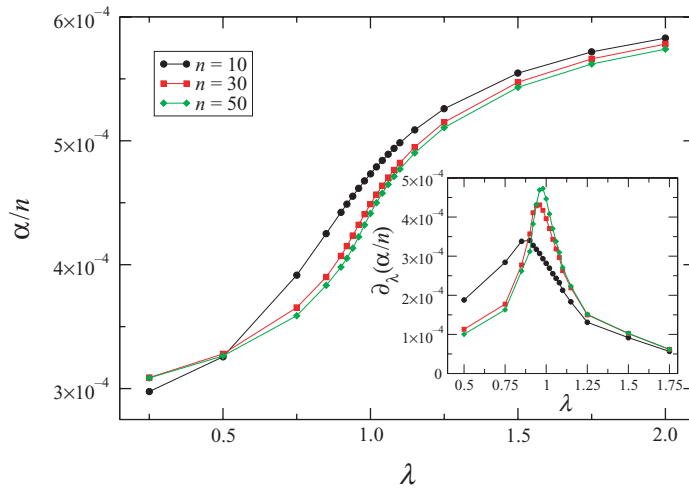


Figure 5. Short-time Gaussian decay rate α as a function of the transverse field λ (ε and γ as in figure 4). Following the fit of equation (25) in this plot we rescaled α assuming a linear dependence upon the system size n . In the inset, we plot the first derivative of the same curves in the main panel, with respect to λ .

study the behavior of \mathcal{F}_{av} with the number n of qubits, at a fixed value of transverse magnetic field $\lambda = 1$. As it is shown in the inset, at a given interaction time t^* , the fidelity $\mathcal{F}_{\text{av}}(\lambda_c, t^*)$ decays exponentially with n (the same behavior is found when $\lambda \neq 1$). These curves admit a Gaussian fit of the form

$$\mathcal{F}_{\text{av}} \sim e^{-\alpha t^2} \quad \text{with} \quad \alpha \propto n; \quad (25)$$

indicating a decay rate with an approximative extensive character. Indeed, since the fidelity is a global quantity that describes the evolution of the whole n -body system, it should start

decaying as a Gaussian (at least at small times) [32] with a reasonably extensive decay rate. This prediction is confirmed by the results of figure 5, where we report the rescaled decay rate α/n as a function of the transverse magnetic field for different system sizes n . Interestingly enough, the linear dependence of α upon n is in part compromised in proximity of the critical point: here α/n appears to have a more pronounced dependence upon n . Even though a quantitative characterization of such a nonextensivity is difficult (as it would require a detailed study of the dependence of \mathcal{F}_{av} upon n which, at present, is impractical), one can use such a discrepancy from a purely extensive behavior as evidence of the correlations that are present in the associated channel model (in a memoryless channel, \mathcal{F} should scale exponentially with n).

It is finally worth noticing that in the proximity of the critical point, the decay rate undergoes a sudden change which becomes more evident when increasing the system size. This is indicative of a drastic enhancement of the channel response to a perturbation of the environment, yielding a much less predictable behavior than in the presence of a noncorrelated environment. Such enhancement somehow resembles the static properties of critical ground states investigated in [33]. The signature of criticality at $\lambda_c = 1$ and the finite size effects can be better analyzed by looking at the derivative of the decay rate with respect to the transverse field: the inset of figure 5 clearly shows that $\partial_\lambda \alpha$ exhibits a nonanalytic behavior at the critical point λ_c , at the thermodynamic limit. Notice also that, due to finite size effects, the maximum of $\partial_\lambda \alpha$ does not coincide exactly with the critical point that can be rigorously defined only at the thermodynamic limit, but occurs at a slightly smaller value of λ . However, we checked that a finite size scaling gives the right prediction of the critical point located at $\lambda = 1$.

5. Average output purity

Another quantity that can be evaluated with relatively little numerical effort is the purity of the Choi–Jamiołkowski state $J(\mathcal{E}_n)$, i.e.

$$\mathcal{P}_2 \equiv \text{Tr}[J(\mathcal{E}_n)^2] = \frac{1}{N^2} \sum_{xy} |L_{xy}|^2. \quad (26)$$

As in the case of \mathcal{F} , this can be computed by approximating the summation with a random sampling, i.e.

$$\mathcal{P}_2 \approx \mathcal{P}_{\text{av}} \equiv \frac{1}{N_{\text{av}}} \sum_{(x,y)=1}^{N_{\text{av}}} |L_{xy}|^2. \quad (27)$$

The quantity (26) provides us two important pieces of information. First of all, it yields a useful bound on the channel entropy $H(J(\mathcal{E}_n))$. This follows from the inequality [34]

$$H(J(\mathcal{E}_n)) \geq H_2(J(\mathcal{E}_n)) = -\log_2 \mathcal{P}_2, \quad (28)$$

where $H_2(\cdot) \equiv -\log_2 \text{Tr}[(\cdot)^2]$ is the Rényi entropy of order 2. Furthermore \mathcal{P}_2 is directly related to the average output purity $\langle P_2 \rangle$ of the channel \mathcal{E}_n . This is obtained by averaging the purity of the output state $\mathcal{E}_n(|\psi\rangle_{SS}\langle\psi|)$ over all possible inputs $|\psi\rangle_S$, i.e.

$$\langle P_2 \rangle \equiv \langle \text{Tr}[(\mathcal{E}_n(|\psi\rangle_{SS}\langle\psi|))^2] \rangle = \sum_{x,y} |L_{xy}|^2 \langle |\alpha_x|^2 |\alpha_y|^2 \rangle = \sum_{x,y} |L_{xy}|^2 p_{xy}, \quad (29)$$

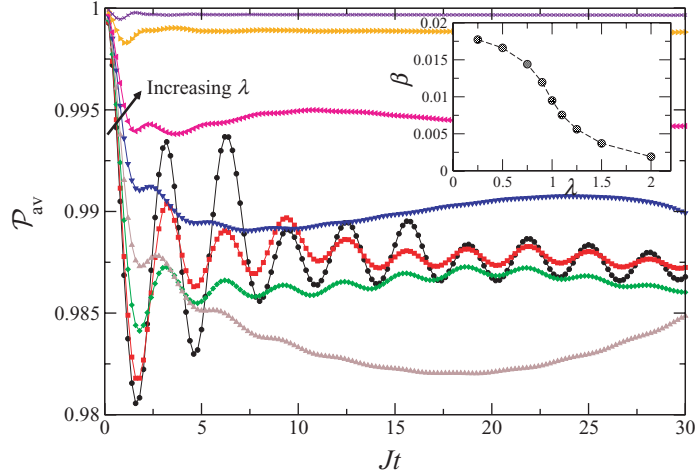


Figure 6. Averaged purity of the channel output state as a function of the interaction time, for various transverse field strengths: from left to right $\lambda = 0.25$ (black circles), 0.5 (red squares), 0.75 (green diamonds), 0.9 (brown triangles up), 1 (blue triangles down), 1.1 (magenta triangles left), 1.5 (orange triangles right) and 2 (violet crosses). We simulated a channel of $n = 30$ qubits coupled to a $\gamma = 1$ Ising chain, set an interaction strength $\varepsilon = 0.05$, and averaged over $N_{\text{av}} = 5 \times 10^4$ random initial conditions. In the inset, we plot the short-time Gaussian decay rate β as a function of λ .

where we used equation (10) and where p_{xy} are the probabilities defined in equation (A.2). According to equation (26) this yields,

$$\langle P_2 \rangle = \frac{N}{N+2} \mathcal{P}_2 + \frac{2}{N+2}. \quad (30)$$

The average purity is a rather fair indicator of the noise induced by the coupling to the environment: if the carrier qubits get strongly entangled with the environment, P_2 is greatly reduced from the unit value (for large n it will tend to zero); on the other hand, a channel which simply unitarily rotates the carrier states has a unit purity. However, we should stress that also the purity may intrinsically fail as a transmission quality quantifier: there are strongly noisy channels with very high output purity (consider, for example, the channel which maps each input state into the same pure output state).

5.1. Numerics

We studied the average channel purity \mathcal{P}_{av} as a function of the model free parameter, the interaction time t . The results are reported in figure 6 for a chain of $n = 30$ qubits and different values of the transverse field λ . We notice qualitatively different behaviors depending on the values of the transverse field λ : if $\lambda < \lambda_c$, the averaged purity oscillates in time and asymptotically tends to an average constant value; as far as the critical point is approached, \mathcal{P}_{av} drops to smaller values (revivals are again due to finite system size effects), reflecting the fact that at criticality correlations between the qubits and the environment are stronger. Crossing the critical point, in the z -ordered phase ($\lambda > \lambda_c$) the purity is generally higher and asymptotically

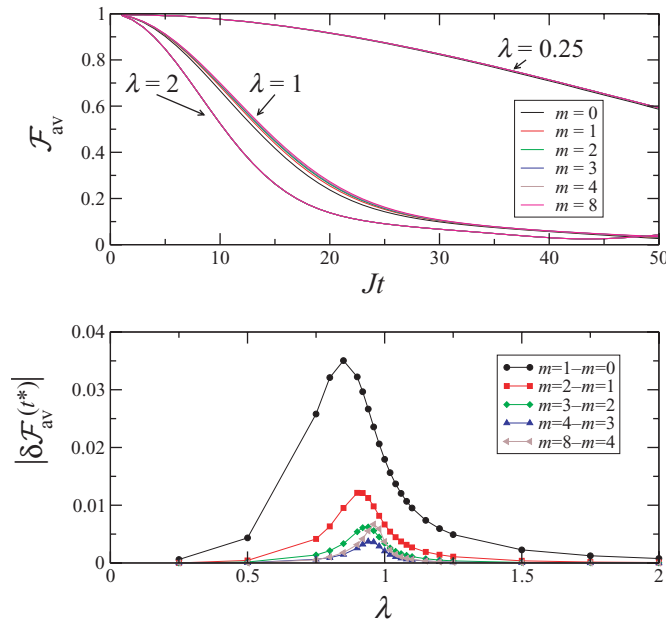


Figure 7. Upper panel: average channel fidelity for the generalized model in figure 2, with $n = 12$ qubits, $\varepsilon = 0.05$, $\gamma = 1$ (averages have been performed over $N_{\text{av}} = 10^4$ initial states). The various curves are for different numbers m of spins between two consecutive qubits, and different values of transverse magnetic fields $\lambda = 0.25, 1, 2$. Lower panel: absolute differences in the fidelities between configurations at various m , as a function of λ and at a fixed interaction time $Jt^* = 10$.

takes values very close to the unit value. This can be easily understood in the limit $\lambda \rightarrow \infty$: in this case, the spins in the chain are ‘frozen’ along the field direction and they cannot couple with anything else, resulting in a watchdog-like effect [32]. Independently of the transverse field value, the average purity \mathcal{P}_{av} decays as a Gaussian $\mathcal{P}_{\text{av}} \sim e^{-\beta t^2}$ in the short-time limit. As for the averaged fidelity \mathcal{F}_{av} , we have then analyzed the decay rate β as a function of λ : as before, β exhibits a signature of criticality via a divergence, at the thermodynamic limit, in its first derivative with respect to λ (see the inset of figure 6).

6. Generalized model

We finally concentrate on the generalized model depicted in figure 2, where a certain number m of environment spins are present between two consecutive spins coupled to the qubits. The richness of the model that is characterized by a large number of parameters, and by a global size which grows both with n and m , requires a huge numerical effort in order to simulate it, therefore we decided to analyze only the average channel fidelity \mathcal{F}_{av} . In the upper panel of figure 7, we show \mathcal{F}_{av} as a function of the interaction time t for different values of m and three values of the transverse field $\lambda = 0.25, 1, 2$; we fix a number of qubits $n = 12$ and an interaction strength $\varepsilon = 0.05$. Hereafter we will concentrate on this case as a typical result, as we performed some checks with larger numbers of channel uses ($n = 30, 50$ and $m = 0, 1, 2$), and found qualitatively analogous results. We immediately observe that differences for various

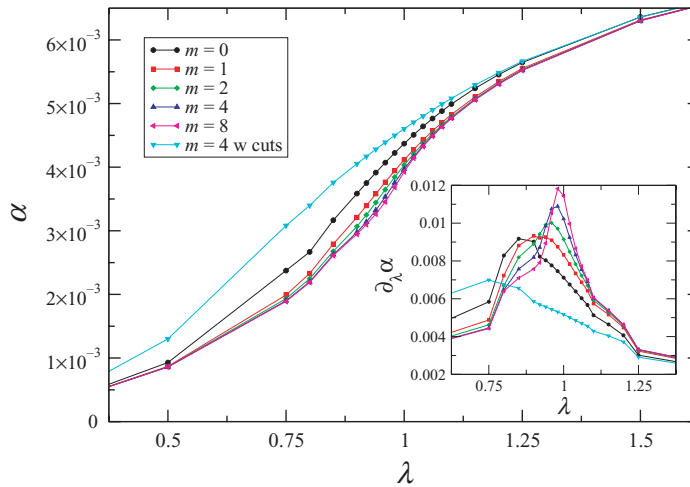


Figure 8. Gaussian decay rate of the averaged fidelity for model in figure 2, as a function of the transverse field λ . The various curves stand for different numbers of spins m between two consecutive qubits (here $n = 12$, $\varepsilon = 0.05$, $\gamma = 1$, $N_{\text{av}} = 10^4$). In the inset, we show the first derivative with respect to λ of the curves in the main panel.

m are tiny, even if the fidelity generally tends to increase when increasing m ; the sensibility with m suddenly enhances at criticality ($\lambda_c = 1$), where correlations in the environment decay much slower than in the other cases. On the other hand, when λ is far from λ_c , differences between fidelities upon a variation of m are greatly suppressed, and the generalized model mostly behaves as the model in figure 1. Again, this reflects the fact that, out of criticality, each qubit is mostly influenced only by the spin that is coupled to, as the spin does not exchange correlations with the other environmental spins. The resulting channel properties are then defined only by the local properties of the chain. On the contrary, at criticality, the spins are correlated and then the resulting channel properties are influenced by the distance of the spins coupled with the qubits. In the lower panel of figure 7, we explicitly plot the differences in the fidelities for various m as a function of λ , and for a fixed interaction time; a peak in proximity of λ_c is clearly visible. We point out that, as already noted at the end of section 4 concerning the size scaling of the fidelity, the maximum in the differences does not occur exactly at the critical point.

The sensitivity to criticality is again demonstrated by the averaged fidelity Gaussian decay rate α as a function of λ , as shown in figure 8 for different values of m : the first derivative in the inset has a maximum in correspondence of a value that approaches the critical point λ_c at the thermodynamic limit. Indeed, increasing m is equivalent to approaching the thermodynamic limit of the chain, thus resulting in an increase of the quantum phase transition effects. A double check of this comes from the cyan triangles-down curve of figure 8: in this case we take $m = 4$, but we break one of the links between two intermediate spins. The environment is then formed by disconnected chains, each of them made up by 5 spins, therefore the system cannot undergo a phase transition in the limit $n \rightarrow \infty$: the signature of criticality has completely disappeared.

7. Conclusions

In conclusion, we have introduced and characterized a class of correlated quantum channels, and we have given bounds for its entropy by means of the averaged channel fidelity \mathcal{F}_{av} and purity \mathcal{P}_{av} . Even though in general these bounds might not be strict, we give a characterization of the channel in terms of quantities that have a clear meaning from the point of view of the many-body model we have introduced.

In the case of an environment defined by a quantum Ising chain, we have shown that the averaged channel purity and the fidelity depend on the environment parameters and are strongly influenced by spin correlations inside it, in particular by whether the environment is critical or not. Even though a critical environment does not particularly influence the channel properties directly, it drastically enhances the response of the channel to a perturbation of the environment. In conclusion, in the presence of a critical environment the channel behavior is much less predictable than in the presence of a noncritical one. We expect that some different environment models, such as, for example, the XY -spin chain, will behave qualitatively similarly to what was found in this work, as it belongs to the same universality class. This might not be the case for other models, like the Heisenberg chain, which will be the object of further study in the near future.

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Appendix A

The probability p_{xy} of equation (22) can be computed as follows: we first define $r_x \equiv |\alpha_x|$ and convert the string x into an integer number from 1 to $N = 2^n$ by trivially identifying $g \equiv 0$ and $e \equiv 1$. The average over a uniform distribution of all pure input states on the Bloch hypersphere for n qubits is

$$p_{xy} = C_N \int_0^1 dr_1 \cdots \int_0^1 dr_N r_x^2 r_y^2 \delta(1 - \mathbf{r}^2), \quad (\text{A.1})$$

where $r^2 = r_1^2 + \cdots + r_N^2$, and $C_N^{-1} \equiv \int_0^1 dr_1 \cdots \int_0^1 dr_N \delta(1 - \mathbf{r}^2)$ is a normalization constant. Changing the limits of integration due to the delta function and using the property of the Gamma function $\Gamma(z) = \int_0^{+\infty} y^{z-1} e^{-y} dy$, it is easy to show that $C_N = 2^N \pi^{-N/2} \Gamma(N/2)$ and

$$p_{xy} = \frac{1 + 2 \delta_{x,y}}{N(N+2)}. \quad (\text{A.2})$$

Appendix B

For times longer than those in the scales of figures 3 and 4, time revivals of the fidelity are present, i.e. the fidelity increases back towards the unit value periodically, due to the finite system size. For finite values of the transverse field revivals are not perfect, that is $\mathcal{F}_{\text{av}}(t) \neq 1$ for

$t > 0$. Anyway, as far as λ increases, the revivals are stronger and happen with period t^R which does not depend on the system size n . This can be understood in the limit $\lambda \rightarrow +\infty$, where the ground state of the environment $|\varphi\rangle_E$ is a fully z -polarized state, thus being an eigenstate of the chain Hamiltonian \mathcal{H}_E^x in equation (9): the generalized Loschmidt echo of equation (11) is then given by $L_{xy} = e^{-i\varepsilon\mathcal{N}_{x-y}t}$, where \mathcal{N}_{x-y} is the number of excited qubits in sequence x minus the one in sequence y . It is easy to see that there are $2^n \times \binom{n}{k}$ different possibilities to choose two sequences x, y such that the corresponding states differ in the state of k qubits, then $\mathcal{N}_{x-y} = \pm j$ with $j = 0, 2, \dots, k$ (if k is even) or $j = 1, 3, \dots, k$ (if k is odd). Therefore, when averaging over input states, each term contributes with $p_{xy} e^{\pm i\varepsilon j t}$. Noting that $(\mathcal{U}_y^\dagger \mathcal{U}_x)^\dagger = \mathcal{U}_x^\dagger \mathcal{U}_y$, we have

$$\langle F \rangle = 2^n p_{xx} + p_{xy} (c_0 + c_1 \cos(\varepsilon t) + c_2 \cos(2\varepsilon t) + \dots + c_n \cos(n\varepsilon t)). \quad (\text{B.1})$$

It follows a perfect revival for the fidelity at times t^R such that $\varepsilon t^R = 2\pi [\text{mod } 2\pi]$.

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