

New lower bounds to the output entropy of multi-mode quantum Gaussian channels

Giacomo De Palma

Abstract—We prove that quantum thermal Gaussian input states minimize the output entropy of the multi-mode quantum Gaussian attenuators and amplifiers that are entanglement breaking and of the multi-mode quantum Gaussian phase contravariant channels among all the input states with a given entropy. This is the first time that this property is proven for a multi-mode channel without restrictions on the input states. A striking consequence of this result is a new lower bound on the output entropy of all the multi-mode quantum Gaussian attenuators and amplifiers in terms of the input entropy. We apply this bound to determine new upper bounds to the communication rates in two different scenarios. The first is classical communication to two receivers with the quantum degraded Gaussian broadcast channel. The second is the simultaneous classical communication, quantum communication and entanglement generation or the simultaneous public classical communication, private classical communication and quantum key distribution with the Gaussian quantum-limited attenuator.

Index Terms—Quantum Gaussian channels, entropic inequalities, broadcast channel, trade-off coding.

I. INTRODUCTION

ATTENUATION and noise unavoidably affect electromagnetic communications through wires, optical fibers and free space. Quantum effects become relevant for low-intensity signals as in the case of satellite communications, where the receiver can be reached by only few photons for each bit of information [1]. Quantum Gaussian channels provide the mathematical model for the attenuation and the noise affecting electromagnetic signals in the quantum regime [2]–[7].

The maximum achievable communication rate of a channel depends on the minimum noise achievable at its output, which is quantified by the output entropy [4], [8]. The determination of the maximum rates allowed by quantum mechanics for the communication to two receivers with the quantum degraded Gaussian broadcast channel [9], [10] relies on a minimum output entropy conjecture [11], [12] (Conjecture 1). This fundamental conjecture states that thermal quantum Gaussian input states minimize the output entropy of the quantum Gaussian attenuators, amplifiers and phase contravariant channels among all the input states with a given entropy. The same conjecture is necessary also to determine the triple trade-off region of the Gaussian quantum-limited attenuator [13], [14]. This region is constituted by all the achievable triples of rates for simultaneous classical communication, quantum communication and entanglement generation or for simultaneous public classical communication, private classical communication and quantum key distribution. So far, Conjecture 1 has been

proven only in the special case of one-mode channels [15]–[19]. The best current lower bound to the output entropy of multi-mode quantum Gaussian channels is provided by the quantum Entropy Power Inequality [20]–[27] (see Theorem 3). However, this lower bound is strictly lower than the output entropy generated by Gaussian input states, hence it is not sufficient to prove the conjecture (see the review [28] for a complete presentation of the state of the art).

We prove the minimum output entropy conjecture for the multi-mode quantum Gaussian attenuators and amplifiers that are entanglement breaking and for all the multi-mode phase contravariant quantum Gaussian channels (Corollary 5). This is the first time that the minimum output entropy conjecture is proven for a multi-mode channel without restrictions on the input states. Surprisingly, the implications of this result go beyond the quantum Gaussian channels that are entanglement breaking. Indeed, combining Corollary 5 with the quantum integral Stam inequality of Ref. [25], we prove a new lower bound to the output entropy of all the multi-mode quantum Gaussian attenuators and amplifiers (Theorem 6). This new lower bound is strictly better than the previous best lower bound provided by the quantum Entropy Power Inequality (see Figure 1 for a comparison).

We apply Theorem 6 to determine a new upper bound to the rates for classical communication to two receivers with the quantum degraded Gaussian broadcast channel (Corollary 10) and a new outer bound to the triple trade-off region of the Gaussian quantum-limited attenuator (Corollary 13). These bounds improve the best previous bounds based on the quantum Entropy Power Inequality (see Figure 2 and Figure 3 for a comparison).

The manuscript is structured as follows. We present quantum Gaussian channels in section II and the minimum output entropy conjecture in section III. In section IV we prove the minimum output entropy conjecture for the quantum Gaussian channels that are entanglement breaking, and in section V we prove the new lower bound to the output entropy of the quantum Gaussian attenuators and amplifiers. We apply this result to prove a new upper bound to the rates for classical communication to two receivers with the quantum degraded Gaussian broadcast channel in section VI and to prove a new outer bound to the triple trade-off region of the quantum-limited attenuator in section VII. We conclude in section VIII.

II. QUANTUM GAUSSIAN CHANNELS

A one-mode quantum Gaussian system is the mathematical model for a harmonic oscillator or a mode of the electromagnetic radiation. The Hilbert space of a one-mode quantum

Giacomo De Palma is with QMATH, Department of Mathematical Sciences, University of Copenhagen, Universitetsparken 5, 2100 Copenhagen, Denmark

Gaussian system is the irreducible representation of the canonical commutation relation [7], [4, Chapter 12]

$$[\hat{a}, \hat{a}^\dagger] = \hat{\mathbb{I}}, \quad (1)$$

where \hat{a} is the ladder operator. We define the Hamiltonian

$$\hat{H} = \hat{a}^\dagger \hat{a}, \quad (2)$$

that counts the number of excitations or photons. The vector annihilated by \hat{a} is the vacuum and is denoted by $|0\rangle$. A quantum Gaussian state is a quantum state proportional to the exponential of a quadratic polynomial in \hat{a} and \hat{a}^\dagger . The most important Gaussian states are the thermal Gaussian states, where the polynomial is proportional to the Hamiltonian (2):

$$\hat{\omega}_E = \frac{1}{(E+1)} \left(\frac{E}{E+1} \right)^{\hat{H}}, \quad (3)$$

where $E \geq 0$ is the average energy:

$$\text{Tr} \left[\hat{H} \hat{\omega}_E \right] = E. \quad (4)$$

We notice that $\hat{\omega}_0 = |0\rangle\langle 0|$ is the vacuum state of the system. The von Neumann entropy of $\hat{\omega}_E$ is

$$S(\hat{\omega}_E) = (E+1) \ln(E+1) - E \ln E =: g(E). \quad (5)$$

An n -mode Gaussian quantum system is the union of n one-mode Gaussian quantum systems, and its Hilbert space is the n -th tensor power of the Hilbert space of a one-mode Gaussian quantum system. Let $\hat{a}_1, \dots, \hat{a}_n$ be the ladder operators of the n modes. The Hamiltonian of the n -mode Gaussian quantum system is the sum of the Hamiltonians of each mode:

$$\hat{H} = \sum_{i=1}^n \hat{a}_i^\dagger \hat{a}_i. \quad (6)$$

Quantum Gaussian channels are the quantum channels that preserve the set of quantum Gaussian states. The most important families of quantum Gaussian channels are the beam-splitter, the squeezing, the quantum Gaussian attenuators, the quantum Gaussian amplifiers and the quantum heat semigroup. The beam-splitter and the squeezing are the quantum counterparts of the classical linear mixing of random variables, and are the main transformations in quantum optics. Let A and B be one-mode quantum Gaussian systems with ladder operators \hat{a} and \hat{b} , respectively. The *beam-splitter* of transmissivity $0 \leq \eta \leq 1$ is implemented by the unitary operator

$$\hat{U}_\eta = \exp \left(\left(\hat{a}^\dagger \hat{b} - \hat{b}^\dagger \hat{a} \right) \arccos \sqrt{\eta} \right), \quad (7)$$

and performs a linear rotation of the ladder operators [29, Section 1.4.2]:

$$\begin{aligned} \hat{U}_\eta^\dagger \hat{a} \hat{U}_\eta &= \sqrt{\eta} \hat{a} + \sqrt{1-\eta} \hat{b}, \\ \hat{U}_\eta^\dagger \hat{b} \hat{U}_\eta &= -\sqrt{1-\eta} \hat{a} + \sqrt{\eta} \hat{b}. \end{aligned} \quad (8)$$

The *squeezing* [30] of parameter $\kappa \geq 1$ is implemented by the unitary operator

$$\hat{U}_\kappa = \exp \left(\left(\hat{a}^\dagger \hat{b}^\dagger - \hat{a} \hat{b} \right) \text{arccosh} \sqrt{\kappa} \right), \quad (9)$$

and acts on the ladder operators as

$$\begin{aligned} \hat{U}_\kappa^\dagger \hat{a} \hat{U}_\kappa &= \sqrt{\kappa} \hat{a} + \sqrt{\kappa-1} \hat{b}^\dagger, \\ \hat{U}_\kappa^\dagger \hat{b} \hat{U}_\kappa &= \sqrt{\kappa-1} \hat{a}^\dagger + \sqrt{\kappa} \hat{b}. \end{aligned} \quad (10)$$

The quantum Gaussian attenuators model the attenuation and the noise affecting electromagnetic signals traveling through optical fibers or free space. The one-mode *quantum Gaussian attenuator* $\mathcal{E}_{\eta,E}$ [31, case (C) with $k = \sqrt{\lambda}$ and $N_0 = E$] can be implemented mixing the input state $\hat{\rho}$ with the one-mode thermal Gaussian state $\hat{\omega}_E$ through a beam-splitter of transmissivity $0 \leq \eta \leq 1$:

$$\mathcal{E}_{\eta,E}(\hat{\rho}) = \text{Tr}_B \left[\hat{U}_\eta (\hat{\rho} \otimes \hat{\omega}_E) \hat{U}_\eta^\dagger \right]. \quad (11)$$

If $E = 0$ the attenuator is called *quantum-limited*, and we denote

$$\mathcal{E}_{\eta,0} = \mathcal{E}_\eta. \quad (12)$$

The quantum Gaussian amplifiers model the amplification of electromagnetic signals. The one-mode *quantum Gaussian amplifier* $\mathcal{A}_{\kappa,E}$ [31, case (C) with $k = \sqrt{\kappa}$ and $N_0 = E$] can be implemented performing a squeezing of parameter $\kappa \geq 1$ on the input state $\hat{\rho}$ and the one-mode thermal Gaussian state $\hat{\omega}_E$:

$$\mathcal{A}_{\kappa,E}(\hat{\rho}) = \text{Tr}_B \left[\hat{U}_\kappa (\hat{\rho} \otimes \hat{\omega}_E) \hat{U}_\kappa^\dagger \right]. \quad (13)$$

The one-mode *Gaussian phase contravariant channel* $\tilde{\mathcal{A}}_{\kappa,E}$ [31, case (D) with $k = \sqrt{\kappa-1}$ and $N_0 = E$] is the weak complementary of $\mathcal{A}_{\kappa,E}$: for any one-mode quantum state $\hat{\rho}$,

$$\tilde{\mathcal{A}}_{\kappa,E}(\hat{\rho}) = \text{Tr}_A \left[\hat{U}_\kappa (\hat{\rho} \otimes \hat{\omega}_E) \hat{U}_\kappa^\dagger \right]. \quad (14)$$

The *displacement operator* \hat{D}_z with $z \in \mathbb{C}$ is the unitary operator that displaces the ladder operators:

$$\hat{D}_z^\dagger \hat{a} \hat{D}_z = \hat{a} + z \hat{\mathbb{I}}. \quad (15)$$

The *quantum Gaussian additive noise channel* \mathcal{N}_E [31, case (B₂) with $N_c = E$] is the quantum Gaussian channel generated by a convex combination of displacement operators with a Gaussian probability measure:

$$\mathcal{N}_E(\hat{\rho}) = \int_{\mathbb{C}} \hat{D}_{\sqrt{E}z} \hat{\rho} \hat{D}_{\sqrt{E}z}^\dagger e^{-|z|^2} \frac{dz}{\pi}, \quad E > 0. \quad (16)$$

III. THE MINIMUM OUTPUT ENTROPY CONJECTURE

Conjecture 1 (minimum output entropy conjecture). *For any $n \in \mathbb{N}$, quantum Gaussian thermal input states minimize the output entropy of the n -mode Gaussian quantum attenuators, amplifiers, phase contravariant channels and additive noise channels among all the input states with a given entropy. In other words, let $\hat{\rho}$ be a state of an n -mode Gaussian quantum system with finite entropy, and let*

$$N(\hat{\rho}) = g^{-1} \left(\frac{S(\hat{\rho})}{n} \right), \quad (17)$$

where g has been defined in (5), such that $S\left(\hat{\omega}_{N(\hat{\rho})}^{\otimes n}\right) = S(\hat{\rho})$. Then,

$$\begin{aligned} S\left(\mathcal{E}_{\eta,E}^{\otimes n}(\hat{\rho})\right) &\geq S\left(\mathcal{E}_{\eta,E}^{\otimes n}\left(\hat{\omega}_{N(\hat{\rho})}^{\otimes n}\right)\right) \\ &= ng(\eta N(\hat{\rho}) + (1-\eta)E), \\ S\left(\mathcal{A}_{\kappa,E}^{\otimes n}(\hat{\rho})\right) &\geq S\left(\mathcal{A}_{\kappa,E}^{\otimes n}\left(\hat{\omega}_{N(\hat{\rho})}^{\otimes n}\right)\right) \\ &= ng(\kappa N(\hat{\rho}) + (\kappa-1)(E+1)), \\ S\left(\tilde{\mathcal{A}}_{\kappa,E}^{\otimes n}(\hat{\rho})\right) &\geq S\left(\tilde{\mathcal{A}}_{\kappa,E}^{\otimes n}\left(\hat{\omega}_{N(\hat{\rho})}^{\otimes n}\right)\right) \\ &= ng((\kappa-1)(N(\hat{\rho})+1) + \kappa E), \\ S\left(\mathcal{N}_E^{\otimes n}(\hat{\rho})\right) &\geq S\left(\mathcal{N}_E^{\otimes n}\left(\hat{\omega}_{N(\hat{\rho})}^{\otimes n}\right)\right) \\ &= ng(N(\hat{\rho}) + E). \end{aligned} \quad (18)$$

Remark 2. Conjecture 1 has been proven only in some special cases:

- $S(\hat{\rho}) = 0$, i.e., when $\hat{\rho}$ is pure [6], [32], [33];
- $n = 1$, i.e., one-mode channels (see [16] for the quantum-limited attenuator, [18] for all the quantum attenuators, amplifiers and additive noise channels and [19] for the phase contravariant quantum Gaussian channel);
- When $\hat{\rho}$ is diagonal in some joint product basis [34].

The current best lower bound to the output entropy of multi-mode quantum Gaussian channels valid for any input state is provided by the quantum Entropy Power Inequality [20]–[24], [26], [27]:

Theorem 3. For any $n \in \mathbb{N}$ and any state $\hat{\rho}$ of an n -mode Gaussian quantum system with finite average energy,

$$S\left(\mathcal{E}_{\eta,E}^{\otimes n}(\hat{\rho})\right) \geq n \ln \left(\eta \exp \frac{S(\hat{\rho})}{n} + (1-\eta) \exp g(E) \right), \quad (19)$$

$$S\left(\mathcal{A}_{\kappa,E}^{\otimes n}(\hat{\rho})\right) \geq n \ln \left(\kappa \exp \frac{S(\hat{\rho})}{n} + (\kappa-1) \exp g(E) \right), \quad (20)$$

$$S\left(\tilde{\mathcal{A}}_{\kappa,E}^{\otimes n}(\hat{\rho})\right) \geq n \ln \left((\kappa-1) \exp \frac{S(\hat{\rho})}{n} + \kappa \exp g(E) \right) \quad (21)$$

$$S\left(\mathcal{N}_E^{\otimes n}(\hat{\rho})\right) \geq n \ln \left(\exp \frac{S(\hat{\rho})}{n} + eE \right). \quad (22)$$

Proof. The claim (19) follows from the quantum Entropy Power Inequality for the beam-splitter [22, Eq. (5)] and the representation (11) for the quantum Gaussian attenuator. The claim (20) and (21) follow from the quantum Entropy Power Inequality for the squeezing [22, Eq. (7)] and the representations (13) and (14) for the quantum phase contravariant Gaussian channel. The claim (22) follows from [27, Theorem 3]. \square

IV. GAUSSIAN STATES MINIMIZE THE OUTPUT ENTROPY OF ENTANGLEMENT BREAKING QUANTUM GAUSSIAN CHANNELS

In this Section, we prove Conjecture 1 for the phase contravariant quantum Gaussian channels and for the quantum Gaussian attenuators and amplifiers that are entanglement breaking. This result is a corollary of the following.

Theorem 4. Let A and B be quantum systems with Hilbert spaces \mathcal{H}_A and \mathcal{H}_B , and let $\Phi : A \rightarrow B$ be an entanglement breaking quantum channel such that for any quantum state $\hat{\rho}$ on \mathcal{H}_A

$$S(\Phi(\hat{\rho})) \geq f(S(\hat{\rho})), \quad (23)$$

with $f : [0, \infty) \rightarrow [0, \infty)$ increasing and convex. Then, for any $n \in \mathbb{N}$ and any quantum state $\hat{\rho}$ on $\mathcal{H}_A^{\otimes n}$,

$$S(\Phi^{\otimes n}(\hat{\rho})) \geq n f\left(\frac{S(\hat{\rho})}{n}\right). \quad (24)$$

Proof. We prove the claim by induction on n . The claim is true for $n = 1$. Let us then assume (24) for a given n . Let $\hat{\rho}_{A_1 \dots A_{n+1}}$ be a quantum state on $\mathcal{H}_A^{\otimes(n+1)}$, and let

$$\hat{\rho}_{B_1 \dots B_{n+1}} = \Phi^{\otimes(n+1)}(\hat{\rho}_{A_1 \dots A_{n+1}}). \quad (25)$$

Since Φ is entanglement breaking, it admits a representation as a measure-prepare channel [35], i.e., there exist a complete separable metric space X , a quantum-classical channel Φ_1 that maps quantum states on A to Borel probability measures on X and a classical-quantum channel Φ_2 that maps Borel probability measures on X to quantum states on B such that

$$\Phi = \Phi_2 \circ \Phi_1. \quad (26)$$

We define the probability measure on X taking values on quantum states on $\mathcal{H}_A^{\otimes n}$

$$\hat{\rho}_{A_1 \dots A_n X} = (\mathbb{I}_{A_1 \dots A_n} \otimes \Phi_1)(\hat{\rho}_{A_1 \dots A_{n+1}}), \quad (27)$$

and the probability measure on X taking values on quantum states on $\mathcal{H}_B^{\otimes n}$

$$\hat{\rho}_{B_1 \dots B_n X} = (\Phi^{\otimes n} \otimes \mathbb{I}_X)(\hat{\rho}_{A_1 \dots A_n X}), \quad (28)$$

such that

$$\hat{\rho}_{B_1 \dots B_{n+1}} = (\mathbb{I}_{B_1 \dots B_n} \otimes \Phi_2)(\hat{\rho}_{B_1 \dots B_n X}). \quad (29)$$

We have

$$\begin{aligned} S(B_1 \dots B_n | X) &= \int_X S(B_1 \dots B_n | X = x) d\rho_X(x) \\ &\geq n \int_X f\left(\frac{S(A_1 \dots A_n | X = x)}{n}\right) d\rho_X(x) \\ &\geq n f\left(\frac{1}{n} \int_X S(A_1 \dots A_n | X = x) d\rho_X(x)\right) \\ &= n f\left(\frac{S(A_1 \dots A_n | X)}{n}\right), \end{aligned} \quad (30)$$

where we have used the inductive hypothesis (24) and Jensen's inequality applied to the convex function f . We then have

$$\begin{aligned} S(B_1 \dots B_{n+1}) &\stackrel{(a)}{=} S(B_{n+1}) + S(B_1 \dots B_n | B_{n+1}) \\ &\stackrel{(b)}{\geq} S(B_{n+1}) + S(B_1 \dots B_n | X_{n+1}) \\ &\stackrel{(c)}{\geq} f(S(A_{n+1})) + n f\left(\frac{S(A_1 \dots A_n | X_{n+1})}{n}\right) \\ &\stackrel{(d)}{\geq} f(S(A_{n+1})) + n f\left(\frac{S(A_1 \dots A_n | A_{n+1})}{n}\right) \\ &\stackrel{(e)}{\geq} (n+1) f\left(\frac{S(A_{n+1}) + S(A_1 \dots A_n | A_{n+1})}{n+1}\right) \\ &\stackrel{(f)}{=} (n+1) f\left(\frac{S(A_1 \dots A_{n+1})}{n+1}\right). \end{aligned} \quad (31)$$

(a) follows from the chain rule for the entropy; (b) follows from the data processing inequality for the channel Φ_2 ; (c) follows from the hypothesis (23) and from (30); (d) follows from the data processing inequality for the channel Φ_1 (we recall that f is increasing); (e) follows from Jensen's inequality applied to the convex function f ; (f) follows from the chain rule for the entropy. We have then proven that the claim (24) for n implies the claim (24) for $n + 1$, and by induction the claim is true for any n . \square

The following Corollary 5 proves Conjecture 1 for all the channels that are entanglement breaking. This is the first time that Conjecture 1 is proven for multi-mode channels without restrictions on the input states.

Corollary 5 (minimum output entropy conjecture for entanglement breaking channels). *Conjecture 1 holds for:*

- Any quantum Gaussian attenuator $\mathcal{E}_{\eta,E}$ with $E \geq \frac{\eta}{1-\eta}$;
- Any quantum Gaussian amplifier $\mathcal{A}_{\kappa,E}$ with $E \geq \frac{1}{\kappa-1}$;
- Any quantum Gaussian phase contravariant channel $\tilde{\mathcal{A}}_{\kappa,E}$;
- Any quantum Gaussian additive noise channel \mathcal{N}_E with $E \geq 1$.

Proof. Conjecture 1 holds for $n = 1$. From [4, Sec. 12.6.2], the conditions $E \geq \frac{\eta}{1-\eta}$, $E \geq \frac{1}{\kappa-1}$ and $E \geq 1$ imply that $\mathcal{E}_{\eta,E}$, $\mathcal{A}_{\kappa,E}$ and \mathcal{N}_E are entanglement breaking, respectively, and $\tilde{\mathcal{A}}_{\kappa,E}$ is entanglement breaking for any $E \geq 0$. From [34, Lemma 15], the functions

$$\begin{aligned} x &\mapsto g(\eta g^{-1}(x) + (1-\eta)E), \\ x &\mapsto g(\kappa g^{-1}(x) + (\kappa-1)(E+1)), \\ x &\mapsto g((\kappa-1)(g^{-1}(x)+1) + \kappa E), \\ x &\mapsto g(g^{-1}(x) + E) \end{aligned} \quad (32)$$

are increasing and convex for any $0 \leq \eta \leq 1$, $\kappa \geq 1$ and $E \geq 0$. The claim then follows from Theorem 4. \square

V. THE NEW LOWER BOUND TO THE OUTPUT ENTROPY OF QUANTUM GAUSSIAN CHANNELS

A striking consequence of Corollary 5 is the following improved lower bound for the output entropy of the multi-mode quantum Gaussian channels that are not entanglement breaking. We compare in Figure 1 this bound with the previous best bound provided by the quantum Entropy Power Inequality and with the output entropy achieved by quantum thermal Gaussian input states.

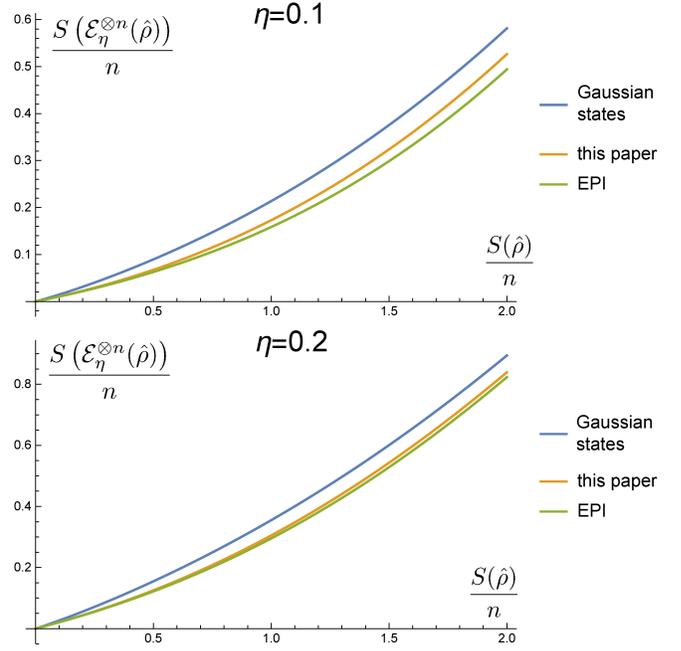


Fig. 1. Output entropy of the Gaussian quantum-limited attenuator with attenuation parameter $\eta = 0.1, 0.2$ as a function of the input entropy. The plot compares the output entropy achieved by thermal Gaussian input states with the lower bounds provided by Theorem 6 and by the quantum Entropy Power Inequality.

Theorem 6. For any $n \in \mathbb{N}$ and any state $\hat{\rho}$ of an n -mode Gaussian quantum system with finite average energy,

$$\begin{aligned} \frac{S(\mathcal{E}_{\eta,E}^{\otimes n}(\hat{\rho}))}{n} &\geq g\left(\eta g^{-1}\left(\frac{S(\hat{\rho})}{n} + g\left(\frac{\eta}{1-\eta}\right) - g(E)\right) + \eta\right) \\ &\quad + g(E) - g\left(\frac{\eta}{1-\eta}\right) \quad \forall 0 \leq E \leq \frac{\eta}{1-\eta}, \\ \frac{S(\mathcal{A}_{\kappa,E}^{\otimes n}(\hat{\rho}))}{n} &\geq g\left(\kappa g^{-1}\left(\frac{S(\hat{\rho})}{n} + g\left(\frac{1}{\kappa-1}\right) - g(E)\right) + \kappa\right) \\ &\quad + g(E) - g\left(\frac{1}{\kappa-1}\right) \quad \forall 0 \leq E \leq \frac{1}{\kappa-1}, \\ \frac{S(\mathcal{N}_E^{\otimes n}(\hat{\rho}))}{n} &\geq g\left(g^{-1}\left(\frac{S(\hat{\rho})}{n} - \ln E\right) + 1\right) + \ln E \\ &\quad \forall 0 \leq E \leq 1. \end{aligned} \quad (33)$$

Proof. Quantum Gaussian attenuators. We fix $0 \leq \lambda \leq 1$ and define for any $t \geq 0$

$$\begin{aligned} \hat{\rho}(t) &= \mathcal{N}_{\frac{\lambda t}{\eta}}^{\otimes n}(\hat{\rho}), \quad E(t) = E + \frac{1-\lambda}{1-\eta} t, \\ \phi(t) &= S\left(\mathcal{E}_{\eta,E(t)}^{\otimes n}(\hat{\rho}(t))\right) \\ &\quad - \lambda S(\hat{\rho}(t)) - (1-\lambda) S\left(\hat{\omega}(E(t))^{\otimes n}\right). \end{aligned} \quad (34)$$

From [25, Eq. (113)] we have $\phi(t) \leq \phi(0)$, hence

$$\begin{aligned} \frac{S(\mathcal{E}_{\eta,E}^{\otimes n}(\hat{\rho}))}{n} &\geq \frac{S(\mathcal{E}_{\eta,E(t)}^{\otimes n}(\hat{\rho}(t)))}{n} - \lambda \frac{S(\hat{\rho}(t)) - S(\hat{\rho})}{n} \\ &\quad - (1-\lambda)(g(E(t)) - g(E)). \end{aligned} \quad (35)$$

We set

$$t = t^* = \frac{\eta - (1 - \eta)E}{1 - \lambda}, \quad (36)$$

such that $E(t^*) = \frac{\eta}{1 - \eta}$ and the channel $\mathcal{E}_{\eta, E(t^*)}$ is entanglement breaking. Then, putting together (35) and Corollary 5 we get

$$\begin{aligned} \frac{S(\mathcal{E}_{\eta, E}^{\otimes n}(\hat{\rho}))}{n} &\geq f\left(\frac{S(\hat{\rho}(t^*))}{n}\right) - \lambda \frac{S(\hat{\rho}(t^*)) - S(\hat{\rho})}{n} \\ &\quad - (1 - \lambda) \left(g\left(\frac{\eta}{1 - \eta}\right) - g(E)\right), \end{aligned} \quad (37)$$

where for any $x \geq 0$

$$f(x) = g(\eta g^{-1}(x) + \eta). \quad (38)$$

Let

$$S_0 = \frac{S(\hat{\rho})}{n} + g\left(\frac{\eta}{1 - \eta}\right) - g(E). \quad (39)$$

From [34, Lemma 15], f is convex, hence

$$f\left(\frac{S(\hat{\rho}(t^*))}{n}\right) \geq f(S_0) + \left(\frac{S(\hat{\rho}(t^*))}{n} - S_0\right) f'(S_0). \quad (40)$$

Finally, we set $\lambda = f'(S_0)$ and get from (37) and (40)

$$\frac{S(\mathcal{E}_{\eta, E}^{\otimes n}(\hat{\rho}))}{n} \geq f(S_0) + g(E) - g\left(\frac{\eta}{1 - \eta}\right), \quad (41)$$

and the claim follows.

Quantum Gaussian amplifiers. The proof for the quantum Gaussian amplifiers is analogous to the proof for the quantum Gaussian attenuators. We fix $0 \leq \lambda \leq 1$ and define for any $t \geq 0$

$$\begin{aligned} \hat{\rho}(t) &= \mathcal{N}_{\frac{\lambda t}{\kappa}}^{\otimes n}(\hat{\rho}), \quad E(t) = E + \frac{1 - \lambda}{\kappa - 1} t, \\ \phi(t) &= S\left(\mathcal{A}_{\kappa, E(t)}^{\otimes n}(\hat{\rho}(t))\right) \\ &\quad - \lambda S(\hat{\rho}(t)) - (1 - \lambda) S(\hat{\omega}(E(t))^{\otimes n}). \end{aligned} \quad (42)$$

From [25, Eq. (113)] we have $\phi(t) \leq \phi(0)$, hence

$$\begin{aligned} \frac{S(\mathcal{A}_{\kappa, E}^{\otimes n}(\hat{\rho}))}{n} &\geq \frac{S(\mathcal{A}_{\kappa, E(t)}^{\otimes n}(\hat{\rho}(t)))}{n} - \lambda \frac{S(\hat{\rho}(t)) - S(\hat{\rho})}{n} \\ &\quad - (1 - \lambda) (g(E(t)) - g(E)). \end{aligned} \quad (43)$$

We set

$$t = t^* = \frac{1 - (\kappa - 1)E}{1 - \lambda}, \quad (44)$$

such that $E(t^*) = \frac{1}{\kappa - 1}$ and the channel $\mathcal{A}_{\kappa, E(t^*)}$ is entanglement breaking. Then, putting together (43) and Corollary 5 we get

$$\begin{aligned} \frac{S(\mathcal{A}_{\kappa, E}^{\otimes n}(\hat{\rho}))}{n} &\geq f\left(\frac{S(\hat{\rho}(t^*))}{n}\right) - \lambda \frac{S(\hat{\rho}(t^*)) - S(\hat{\rho})}{n} \\ &\quad - (1 - \lambda) \left(g\left(\frac{1}{\kappa - 1}\right) - g(E)\right), \end{aligned} \quad (45)$$

where for any $x \geq 0$

$$f(x) = g(\kappa g^{-1}(x) + \kappa). \quad (46)$$

Let

$$S_0 = \frac{S(\hat{\rho})}{n} + g\left(\frac{1}{\kappa - 1}\right) - g(E). \quad (47)$$

From [34, Lemma 15], f is convex, hence

$$f\left(\frac{S(\hat{\rho}(t^*))}{n}\right) \geq f(S_0) + \left(\frac{S(\hat{\rho}(t^*))}{n} - S_0\right) f'(S_0). \quad (48)$$

Finally, we set $\lambda = f'(S_0)$ and get from (45) and (48)

$$\frac{S(\mathcal{A}_{\kappa, E}^{\otimes n}(\hat{\rho}))}{n} \geq f(S_0) + g(E) - g\left(\frac{1}{\kappa - 1}\right), \quad (49)$$

and the claim follows.

Quantum Gaussian additive noise channels. We fix $0 \leq \lambda \leq 1$ and define for any $t \geq 0$

$$\begin{aligned} \hat{\rho}(t) &= \mathcal{N}_{\lambda t}^{\otimes n}(\hat{\rho}), \quad E(t) = E + (1 - \lambda)t, \\ \phi(t) &= S\left(\mathcal{N}_{E(t)}^{\otimes n}(\hat{\rho}(t))\right) - \lambda S(\hat{\rho}(t)) - n(1 - \lambda) \ln E(t). \end{aligned} \quad (50)$$

From the proof of Theorem 5 of Ref. [26] we have $\phi(t) \leq \phi(0)$, hence

$$\begin{aligned} \frac{S(\mathcal{N}_E^{\otimes n}(\hat{\rho}))}{n} &\geq \frac{S(\mathcal{N}_{E(t)}^{\otimes n}(\hat{\rho}(t)))}{n} \\ &\quad - \lambda \frac{S(\hat{\rho}(t)) - S(\hat{\rho})}{n} - (1 - \lambda) \ln \frac{E(t)}{E}. \end{aligned} \quad (51)$$

We set

$$t = t^* = \frac{1 - E}{1 - \lambda}, \quad (52)$$

such that $E(t^*) = 1$ and the channel $\mathcal{N}_{E(t^*)}$ is entanglement breaking. Then, putting together (51) and Corollary 5 we get

$$\begin{aligned} \frac{S(\mathcal{N}_E^{\otimes n}(\hat{\rho}))}{n} &\geq f\left(\frac{S(\hat{\rho}(t^*))}{n}\right) \\ &\quad - \lambda \frac{S(\hat{\rho}(t^*)) - S(\hat{\rho})}{n} + (1 - \lambda) \ln E, \end{aligned} \quad (53)$$

where for any $x \geq 0$

$$f(x) = g(g^{-1}(x) + 1). \quad (54)$$

Let

$$S_0 = \frac{S(\hat{\rho})}{n} - \ln E. \quad (55)$$

From [34, Lemma 15], f is convex, hence

$$f\left(\frac{S(\hat{\rho}(t^*))}{n}\right) \geq f(S_0) + \left(\frac{S(\hat{\rho}(t^*))}{n} - S_0\right) f'(S_0). \quad (56)$$

Finally, we set $\lambda = f'(S_0)$ and get from (53) and (56)

$$\frac{S(\mathcal{N}_E^{\otimes n}(\hat{\rho}))}{n} \geq f(S_0) + \ln E, \quad (57)$$

and the claim follows. \square

Remark 7. Since states with infinite average energy are unphysical, for all practical purposes the hypothesis of finite average energy in Theorem 6 is not restrictive.

VI. BOUND TO THE CAPACITY REGION OF THE QUANTUM DEGRADED GAUSSIAN BROADCAST CHANNEL

Let A, B, A', B' be one-mode Gaussian quantum systems. The quantum degraded Gaussian broadcast channel [9], [10] maps a state $\hat{\rho}_A$ of A to a state $\hat{\rho}_{A'B'}$ of the joint quantum system $A'B'$ with

$$\hat{\rho}_{A'B'} = \hat{U}_\eta (\hat{\rho}_A \otimes |0\rangle_B \langle 0|) \hat{U}_\eta^\dagger, \quad (58)$$

where \hat{U}_η is the unitary operator defined in (7) and $\frac{1}{2} \leq \eta \leq 1$. The channel can be understood as follows. A encodes the information into the state of the electromagnetic radiation $\hat{\rho}_A$, and sends it through a beam-splitter of transmissivity η . A' and B' receive the transmitted and the reflected part of the signal, respectively, whose joint state is $\hat{\rho}_{A'B'}$. This channel is called degraded since the state received by B' can be obtained applying a quantum-limited attenuator to the state received by A' [10]:

$$\hat{\rho}_{B'} = \mathcal{E}_{\frac{1-\eta}{\eta}}(\hat{\rho}_{A'}). \quad (59)$$

The simplest communication strategy is time sharing, which consists in communicating only with A' for a fraction of the time and only with B' for the remaining fraction of the time. Superposition coding [10], [36], [37] is a more sophisticated strategy that achieves higher rates communicating with A' and B' simultaneously. Let $E > 0$ be the maximum average energy per mode of the input states. Superposition coding allows to achieve with the quantum degraded Gaussian broadcast channel (58) any rate pair $(R_{A'}, R_{B'})$ satisfying [10, Sec. IV]

$$R_{A'} \geq 0, \quad 0 \leq R_{B'} \leq g((1-\eta)E) - g\left(\frac{1-\eta}{\eta} g^{-1}(R_{A'})\right). \quad (60)$$

Assuming Conjecture 1 for the quantum-limited attenuator, the capacity region of the quantum degraded Gaussian broadcast channel coincides with the region identified by (60) [10], i.e., any achievable rate pair satisfies (60).

Despite Conjecture 1 still lacks a proof, the known lower bounds to the output entropy of the multi-mode quantum-limited attenuators still imply bounds to the capacity region of the quantum degraded Gaussian broadcast channel. The first of these bounds has been determined from the quantum Entropy Power Inequality [22]. The following Theorem 8 shows that any lower bound to the output entropy of the multi-mode quantum-limited attenuators in terms of the input entropy implies a bound to the capacity region of the quantum degraded Gaussian broadcast channel. We then combine Theorem 8 with Theorem 6 to obtain a new bound to this capacity region.

Theorem 8. *Let us suppose that for any $n \in \mathbb{N}$, any $0 \leq \lambda \leq 1$ and any input state $\hat{\rho}$ of an n -mode Gaussian quantum system with finite average energy*

$$S(\mathcal{E}_\lambda^{\otimes n}(\hat{\rho})) \geq n f_\lambda \left(\frac{S(\hat{\rho})}{n} \right), \quad (61)$$

where the function f_λ is increasing and convex. Then, any achievable rate pair $(R_{A'}, R_{B'})$ for the quantum degraded Gaussian broadcast channel satisfies

$$R_{A'} \geq 0, \quad 0 \leq R_{B'} \leq g((1-\eta)E) - f_{\frac{1-\eta}{\eta}}(R_{A'}), \quad (62)$$

where $E \geq 0$ is the maximum allowed average energy per mode of the input.

Proof. The capacity region of the quantum degraded Gaussian broadcast channel is the closure of the union over $n \in \mathbb{N}$ of regions of the form [10]

$$n R_{A'} \leq \sum_{i \in I} p_i^{(n)} \left(S(\hat{\rho}_i^{A'(n)}) - \sum_{j \in J} q_j^{(n)} S(\hat{\rho}_{i,j}^{A'(n)}) \right), \quad (63)$$

$$n R_{B'} \leq S(\hat{\rho}_{B'}^{(n)}) - \sum_{i \in I} p_i^{(n)} S(\hat{\rho}_i^{B'(n)}), \quad (64)$$

where $\{p_i^{(n)} q_j^{(n)}, \hat{\rho}_{i,j}^{A(n)}\}_{i \in I, j \in J}$ is an ensemble of pure encoding states on n copies of the quantum system A and

$$\hat{\rho}_A^{(n)} = \sum_{i \in I, j \in J} p_i^{(n)} q_j^{(n)} \hat{\rho}_{i,j}^{A(n)}, \quad (65)$$

$$\hat{\rho}_{i,j}^{A'B'(n)} = \hat{U}_\eta^{\otimes n} (\hat{\rho}_{i,j}^{A(n)} \otimes (|0\rangle_B \langle 0|)^{\otimes n}) \hat{U}_\eta^{\dagger \otimes n}, \quad (66)$$

$$\hat{\rho}_i^{A'B'(n)} = \sum_{j \in J} q_j^{(n)} \hat{\rho}_{i,j}^{A'B'(n)}, \quad (67)$$

$$\hat{\rho}_{B'}^{(n)} = \sum_{i \in I} p_i^{(n)} \hat{\rho}_i^{B'(n)}, \quad (68)$$

and the average state satisfies the energy constraint

$$\text{Tr} [\hat{H} \hat{\rho}_A^{(n)}] \leq n E. \quad (69)$$

Since $S(\hat{\rho}_{i,j}^{A'(n)}) \geq 0$ for any $i \in I$ and $j \in J$, we have from (63)

$$R_{A'} \leq \frac{1}{n} \sum_{i \in I} p_i^{(n)} S(\hat{\rho}_i^{A'(n)}). \quad (70)$$

The energy constraint (69) implies

$$\text{Tr} [\hat{H} \hat{\rho}_{B'}^{(n)}] \leq n(1-\eta)E, \quad (71)$$

where \hat{H} is the Hamiltonian on n copies of B' , hence

$$S(\hat{\rho}_{B'}^{(n)}) \leq n g((1-\eta)E), \quad (72)$$

where we have used that quantum thermal Gaussian states maximize the entropy among all the states with the same average energy. From (59) we have for any $i \in I$

$$\hat{\rho}_i^{B'(n)} = \mathcal{E}_{\frac{1-\eta}{\eta}}^{\otimes n} (\hat{\rho}_i^{A'(n)}). \quad (73)$$

Since the state $\hat{\rho}_A^{(n)}$ has finite average energy, $\hat{\rho}_i^{A'(n)}$ has finite average energy for any $i \in I$, and we have from the hypothesis (61)

$$S(\hat{\rho}_i^{B'(n)}) \geq n f_{\frac{1-\eta}{\eta}} \left(\frac{S(\hat{\rho}_i^{A'(n)})}{n} \right). \quad (74)$$

Since $f_{\frac{1-\eta}{\eta}}$ is convex and increasing, we have from Jensen's inequality and (70)

$$\frac{1}{n} \sum_{i \in I} p_i^{(n)} S(\hat{\rho}_i^{B'(n)}) \geq f_{\frac{1-\eta}{\eta}} \left(\frac{1}{n} \sum_{i \in I} p_i^{(n)} S(\hat{\rho}_i^{A'(n)}) \right). \quad (75)$$

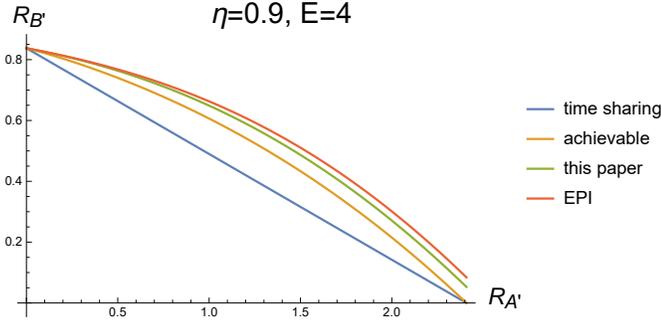


Fig. 2. Capacity region in nats for the communication to two receivers with the quantum degraded Gaussian broadcast channel with parameter $\eta = 0.9$ and input states with at most $E = 4$ average photons per mode. The plot compares the regions achievable with time sharing and with the superposition coding (60) to the outer bounds provided by the quantum Entropy Power Inequality and by Corollary 10. All the rates are expressed in nats per channel use.

Putting together (64), (72) and (75) we get

$$R_{B'} \leq g((1-\eta)E) - f_{\frac{1-\eta}{\eta}} \left(\frac{1}{n} \sum_{i \in I} p_i^{(n)} S(\hat{\rho}_i^{A'(n)}) \right), \quad (76)$$

and the claim follows from (70) and (76). \square

Corollary 9 ([10]). *Assuming Conjecture 1 for the quantum-limited attenuator, the capacity region (60) achievable with the superposition coding is optimal.*

Proof. Follows from Theorem 8. \square

The following Corollary 10 provides the new outer bound to the capacity region of the quantum degraded Gaussian broadcast channel. We compare in Figure 2 this outer bound with the previous outer bound provided by the quantum Entropy Power Inequality and with the achievable region (60).

Corollary 10. *Any achievable rate pair $(R_{A'}, R_{B'})$ for the quantum degraded Gaussian broadcast channel satisfies (62) with*

$$f_\lambda(x) = g \left(\lambda g^{-1} \left(x + g \left(\frac{\lambda}{1-\lambda} \right) \right) + \lambda \right) - g \left(\frac{\lambda}{1-\lambda} \right). \quad (77)$$

Proof. Follows from Theorem 8 and Theorem 6. \square

VII. BOUND TO THE TRIPLE TRADE-OFF REGION OF THE GAUSSIAN QUANTUM-LIMITED ATTENUATOR

We consider the scenario where a quantum channel is used to transmit both classical and quantum information and to generate entanglement shared between the sender and the receiver. The simplest strategy is the time sharing, which consists in sending only classical information for a fraction of the time, only quantum information for another fraction of the time, and using the channel to generate shared entanglement for the remaining fraction of the time. The trade-off coding is a more sophisticated strategy that achieves higher rates performing the three tasks simultaneously [38]–[40]. A quantum channel where the trade-off coding achieves a remarkable gain with

respect to time sharing is the quantum-limited attenuator [13]. Let $C \geq 0$, $Q \geq 0$ and $G \in \mathbb{R}$ be the rates for classical communication, quantum communication and entanglement generation, respectively, where $G < 0$ means that the shared entanglement is consumed instead of being generated. Then, the quantum-limited attenuator with attenuation parameter $\frac{1}{2} \leq \eta \leq 1$ and input states with maximum average energy per mode E can achieve all the triple of rates (C, Q, G) such that [13], [14]

$$\begin{aligned} C + 2Q &\leq g(\beta E) + g(\eta E) - g((1-\eta)\beta E), \\ Q + G &\leq g(\eta\beta E) - g((1-\eta)\beta E), \\ C + Q + G &\leq g(\eta E) - g((1-\eta)\beta E) \end{aligned} \quad (78)$$

for some $0 \leq \beta \leq 1$. Assuming Conjecture 1, the trade-off region identified by (78) is optimal [13], [14], i.e., any achievable triple of rates (C, Q, G) satisfies (78).

We consider also the scenario where a quantum channel is used to transmit both public and private classical information and to generate a secret key shared between the sender and the receiver. As before the trade-off coding achieves higher rates with respect to the time sharing. Let $C \geq 0$, $P \geq 0$ and $K \in \mathbb{R}$ be the rates for public classical communication, private classical communication and key generation, respectively, where $K < 0$ means that the shared secret key is consumed instead of being generated. Then, the quantum-limited attenuator with attenuation parameter $\frac{1}{2} \leq \eta \leq 1$ and input states with maximum average energy per mode $E > 0$ can achieve all the triple of rates (C, P, K) such that [13], [14]

$$\begin{aligned} C + P &\leq g(\eta E), \\ P + K &\leq g(\eta\beta E) - g((1-\eta)\beta E), \\ C + P + K &\leq g(\eta E) - g((1-\eta)\beta E) \end{aligned} \quad (79)$$

for some $0 \leq \beta \leq 1$. Assuming Conjecture 1, the trade-off region identified by (79) is optimal [13], [14], i.e., any achievable triple of rates (C, P, K) satisfies (79).

Similarly to the quantum degraded Gaussian broadcast channel, even if Conjecture 1 still lacks a proof we can still determine bounds to the triple trade-off regions of the quantum-limited attenuator. The first of these bounds follows from the quantum Entropy Power Inequality [41, Appendix C]. The following Theorem 11 shows that any lower bound to the output entropy of the multi-mode quantum-limited attenuators in terms of the input entropy implies a bound to their triple trade-off regions. We then combine Theorem 11 with Theorem 6 to obtain a new outer bound to the trade-off regions of the quantum-limited attenuator.

Theorem 11. *Let us suppose that for any $n \in \mathbb{N}$, any $0 \leq \lambda \leq 1$ and any quantum state $\hat{\rho}$ of an n -mode Gaussian quantum system with finite average energy,*

$$S(\mathcal{E}_\lambda^{\otimes n}(\hat{\rho})) \geq n f_\lambda \left(\frac{S(\hat{\rho})}{n} \right), \quad (80)$$

where the function f_λ is increasing and convex. Then, any achievable rate triple (C, Q, G) for the triple trade-off among

classical communication, quantum communication and entanglement generation with the Gaussian quantum-limited attenuator with attenuation parameter $\frac{1}{2} \leq \eta \leq 1$ satisfies

$$C + 2Q \leq g(\eta E) + f_{\eta}^{-1}(g(\beta \eta E)) - f_{\frac{1-\eta}{\eta}}(g(\beta \eta E)), \quad (81)$$

$$Q + G \leq g(\beta \eta E) - f_{\frac{1-\eta}{\eta}}(g(\beta \eta E)), \quad (82)$$

$$C + Q + G \leq g(\eta E) - f_{\frac{1-\eta}{\eta}}(g(\beta \eta E)) \quad (83)$$

for some $0 \leq \beta \leq 1$. Moreover, any achievable rate triple (C, P, K) for the triple trade-off among public classical communication, private classical communication and key generation satisfies

$$C + P \leq g(\eta E), \quad (84)$$

$$P + K \leq g(\beta \eta E) - f_{\frac{1-\eta}{\eta}}(g(\beta \eta E)), \quad (85)$$

$$C + P + K \leq g(\eta E) - f_{\frac{1-\eta}{\eta}}(g(\beta \eta E)) \quad (86)$$

for some $0 \leq \beta \leq 1$.

Proof. The set of the achievable triple of rates (C, Q, G) for the Gaussian quantum-limited attenuator with attenuation parameter $\frac{1}{2} \leq \eta \leq 1$ is the closure of the union over $n \in \mathbb{N}$ of regions of the form [39], [40]

$$\begin{aligned} n(C + 2Q) &\leq S(\mathcal{E}_{\eta}^{\otimes n}(\hat{\rho}^{(n)})) \\ &\quad + \sum_{i \in I} p_i^{(n)} \left(S(\hat{\rho}_i^{(n)}) - S(\tilde{\mathcal{E}}_{\eta}^{\otimes n}(\hat{\rho}_i^{(n)})) \right), \end{aligned} \quad (87)$$

$$\begin{aligned} n(Q + G) &\leq \sum_{i \in I} p_i^{(n)} \left(S(\mathcal{E}_{\eta}^{\otimes n}(\hat{\rho}_i^{(n)})) - S(\tilde{\mathcal{E}}_{\eta}^{\otimes n}(\hat{\rho}_i^{(n)})) \right), \end{aligned} \quad (88)$$

$$\begin{aligned} n(C + Q + G) &\leq S(\mathcal{E}_{\eta}^{\otimes n}(\hat{\rho}^{(n)})) \\ &\quad - \sum_{i \in I} p_i^{(n)} S(\tilde{\mathcal{E}}_{\eta}^{\otimes n}(\hat{\rho}_i^{(n)})), \end{aligned} \quad (89)$$

where for any $n \in \mathbb{N}$, $\{p_i^{(n)}, \hat{\rho}_i^{(n)}\}_{i \in I}$ is an ensemble of states of an n -mode Gaussian quantum system such that the average state

$$\hat{\rho}^{(n)} = \sum_{i \in I} p_i^{(n)} \hat{\rho}_i^{(n)} \quad (90)$$

satisfies the energy constraint

$$\text{Tr}[\hat{H} \hat{\rho}^{(n)}] \leq nE, \quad (91)$$

and $\tilde{\mathcal{E}}_{\eta}$ is the complementary channel of \mathcal{E}_{η} .

The energy constraint (91) implies

$$\text{Tr}[\hat{H} \mathcal{E}_{\eta}^{\otimes n}(\hat{\rho}^{(n)})] \leq n\eta E, \quad (92)$$

and since thermal Gaussian states maximize the entropy among all the states with a given average energy, we have

$$S(\mathcal{E}_{\eta}^{\otimes n}(\hat{\rho}^{(n)})) \leq n g(\eta E). \quad (93)$$

The concavity of the entropy and (93) imply

$$\sum_{i \in I} p_i^{(n)} S(\mathcal{E}_{\eta}^{\otimes n}(\hat{\rho}_i^{(n)})) \leq S(\mathcal{E}_{\eta}^{\otimes n}(\hat{\rho}^{(n)})) \leq n g(\eta E), \quad (94)$$

hence there exists $0 \leq \beta_n \leq 1$ such that

$$\sum_{i \in I} p_i^{(n)} S(\mathcal{E}_{\eta}^{\otimes n}(\hat{\rho}_i^{(n)})) = n g(\beta_n \eta E). \quad (95)$$

Since the average state $\hat{\rho}^{(n)}$ has finite average energy, $\hat{\rho}_i^{(n)}$ has finite average energy for any $i \in I$. Since $\tilde{\mathcal{E}}_{\eta} = \mathcal{E}_{\frac{1-\eta}{\eta}} \circ \mathcal{E}_{\eta}$, we have from (80) for any $i \in I$

$$S(\tilde{\mathcal{E}}_{\eta}^{\otimes n}(\hat{\rho}_i^{(n)})) \geq n f_{\frac{1-\eta}{\eta}} \left(\frac{S(\mathcal{E}_{\eta}^{\otimes n}(\hat{\rho}_i^{(n)}))}{n} \right). \quad (96)$$

Since $f_{\frac{1-\eta}{\eta}}$ is convex, we have from (96) and Jensen's inequality

$$\begin{aligned} \sum_{i \in I} p_i^{(n)} S(\tilde{\mathcal{E}}_{\eta}^{\otimes n}(\hat{\rho}_i^{(n)})) &\geq n f_{\frac{1-\eta}{\eta}} \left(\frac{1}{n} \sum_{i \in I} p_i^{(n)} S(\mathcal{E}_{\eta}^{\otimes n}(\hat{\rho}_i^{(n)})) \right) \\ &= n f_{\frac{1-\eta}{\eta}}(g(\beta_n \eta E)), \end{aligned} \quad (97)$$

where in the last step we have used the definition of β_n . We have from (80) for any $i \in I$

$$S(\mathcal{E}_{\eta}^{\otimes n}(\hat{\rho}_i^{(n)})) \geq n f_{\eta} \left(\frac{S(\hat{\rho}_i^{(n)})}{n} \right), \quad (98)$$

hence

$$\begin{aligned} g(\beta_n \eta E) &= \frac{1}{n} \sum_{i \in I} p_i^{(n)} S(\mathcal{E}_{\eta}^{\otimes n}(\hat{\rho}_i^{(n)})) \\ &\geq f_{\eta} \left(\frac{1}{n} \sum_{i \in I} p_i^{(n)} S(\hat{\rho}_i^{(n)}) \right), \end{aligned} \quad (99)$$

where we have used Jensen's inequality for f_{η} . Since f_{η} is increasing, we have

$$\frac{1}{n} \sum_{i \in I} p_i^{(n)} S(\hat{\rho}_i^{(n)}) \leq f_{\eta}^{-1}(g(\beta_n \eta E)). \quad (100)$$

The claim (81) then follows from (87) together with (93), (100) and (97). The claim (82) follows from (88) together with (95) and (97). The claim (83) follows from (89) together with (93) and (97).

The set of the achievable triple of rates (C, P, K) is the closure of the union over $n \in \mathbb{N}$ of regions of the form [39], [40]

$$\begin{aligned} n(C + P) &\leq S(\mathcal{E}_{\eta}^{\otimes n}(\hat{\rho}^{(n)})) \\ &\quad - \sum_{i \in I, j \in J} p_{i,j}^{(n)} S(\mathcal{E}_{\eta}^{\otimes n}(\hat{\rho}_{i,j}^{(n)})), \end{aligned} \quad (101)$$

$$(102)$$

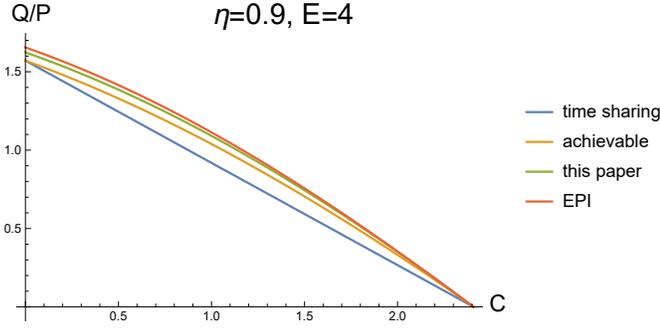


Fig. 3. Trade-off region between classical and quantum communication for the Gaussian quantum-limited attenuator with attenuation parameter $\eta = 0.9$ and input states with at most $E = 4$ average photons per mode. The plot compares the regions achievable with time sharing and with the trade-off coding (78) to the outer bounds provided by the quantum Entropy Power Inequality and by Corollary 13. These regions coincide with the corresponding regions for the trade-off between public and private classical communication, hence the plot encompasses both scenarios. All the rates are expressed in nats per channel use.

$$n(P + K) \leq \sum_{i \in I} p_i^{(n)} \left(S \left(\mathcal{E}_\eta^{\otimes n} \left(\hat{\rho}_i^{(n)} \right) \right) - S \left(\tilde{\mathcal{E}}_\eta^{\otimes n} \left(\hat{\rho}_i^{(n)} \right) \right) \right), \quad (103)$$

$$n(C + P + K) \leq S \left(\mathcal{E}_\eta^{\otimes n} \left(\hat{\rho}^{(n)} \right) \right) - \sum_{i \in I} p_i^{(n)} S \left(\tilde{\mathcal{E}}_\eta^{\otimes n} \left(\hat{\rho}_i^{(n)} \right) \right), \quad (104)$$

where for any $n \in \mathbb{N}$, $\{p_{i,j}^{(n)}, \hat{\rho}_{i,j}^{(n)}\}_{i \in I, j \in J}$ is an ensemble of pure states of an n -mode Gaussian quantum system,

$$\hat{\rho}_i^{(n)} = \sum_{j \in J} p_{j|i}^{(n)} \hat{\rho}_{i,j}^{(n)}, \quad (105)$$

and the average state

$$\hat{\rho}^{(n)} = \sum_{i \in I} p_i^{(n)} \hat{\rho}_i^{(n)} \quad (106)$$

satisfies the energy constraint (91).

Let β_n be as in (95). Then, the claim (84) follows from (101) together with (93) and the property that $S \left(\mathcal{E}_\eta^{\otimes n} \left(\hat{\rho}_{i,j}^{(n)} \right) \right) \geq 0$ for any $i \in I$ and any $j \in J$. The claim (85) follows from (103) together with (95) and (97). The claim (86) follows from (104) together with (93) and (97). \square

Corollary 12 ([13], [14]). *Assuming Conjecture 1 for the quantum-limited attenuator, the achievable trade-off regions (78) and (79) are optimal.*

Proof. Follows from Theorem 11. \square

The following Corollary 13 provides the new outer bound to the triple trade-off region of the Gaussian quantum-limited attenuator. In Figure 3, we compare this bound with the previous bound based on the quantum Entropy Power Inequality and with the achievable region (78).

Corollary 13. *Any achievable rate triple (C, Q, G) or (C, P, K) for the trade-off coding with the quantum-limited*

attenuator satisfies for some $0 \leq \beta \leq 1$ (81), (82), (83) or (84), (85), (86), respectively, with f_λ as in (77).

Proof. Follows from Theorem 11 and Theorem 6. \square

VIII. CONCLUSIONS

We have proven that quantum thermal Gaussian input states minimize the output entropy of the multi-mode quantum Gaussian attenuators and amplifiers that are entanglement breaking and of the quantum Gaussian phase contravariant channels among all the input states with a given entropy (Corollary 5). This result proves the minimum output entropy conjecture (Conjecture 1) for the above channels. This is the first time that Conjecture 1 is proven for multi-mode channels without restrictions on the input states, hence this result significantly extends the cases where the conjecture is known to hold. We have exploited Corollary 5 to prove a new lower bound to the output entropy of all the multi-mode quantum Gaussian attenuators and amplifiers (Theorem 6). This bound strongly constrains the possible violations of Conjecture 1. Then, Corollary 5 and Theorem 6 together provide extremely strong evidence for the general validity of Conjecture 1.

We have applied Theorem 6 to prove new outer bounds to the capacity region of the quantum degraded Gaussian broadcast channel (Corollary 10) and to the triple trade-off region of the Gaussian quantum-limited attenuator (Corollary 13). The conjectured optimal outer bounds would follow from Conjecture 1, whose proof will be the subject of future work.

ACKNOWLEDGEMENTS

We thank Mark Wilde for useful comments.

 This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 792557.

REFERENCES

- [1] J. Chen, J. L. Habif, Z. Dutton, R. Lazarus, and S. Guha, "Optical codeword demodulation with error rates below the standard quantum limit using a conditional nulling receiver," *Nature Photonics*, vol. 6, no. 6, pp. 374–379, 2012.
- [2] V. W. Chan, "Free-space optical communications," *Lightwave Technology, Journal of*, vol. 24, no. 12, pp. 4750–4762, 2006.
- [3] S. L. Braunstein and P. Van Loock, "Quantum information with continuous variables," *Reviews of Modern Physics*, vol. 77, no. 2, p. 513, 2005.
- [4] A. S. Holevo, *Quantum Systems, Channels, Information: A Mathematical Introduction*, ser. De Gruyter Studies in Mathematical Physics. De Gruyter, 2013.
- [5] C. Weedbrook, S. Pirandola, R. Garcia-Patron, N. J. Cerf, T. C. Ralph, J. H. Shapiro, and S. Lloyd, "Gaussian quantum information," *Reviews of Modern Physics*, vol. 84, no. 2, p. 621, 2012.
- [6] A. S. Holevo, "Gaussian optimizers and the additivity problem in quantum information theory," *Russian Mathematical Surveys*, vol. 70, no. 2, p. 331, 2015.
- [7] A. Serafini, *Quantum Continuous Variables: A Primer of Theoretical Methods*. CRC Press, 2017.
- [8] M. M. Wilde, *Quantum Information Theory*. Cambridge University Press, 2017.
- [9] S. Guha and J. H. Shapiro, "Classical information capacity of the bosonic broadcast channel," in *Information Theory, 2007. ISIT 2007. IEEE International Symposium on*. IEEE, 2007, pp. 1896–1900.
- [10] S. Guha, J. H. Shapiro, and B. I. Erkmen, "Classical capacity of bosonic broadcast communication and a minimum output entropy conjecture," *Physical Review A*, vol. 76, no. 3, p. 032303, 2007.

- [11] S. Guha, B. Erkmen, and J. H. Shapiro, “The entropy photon-number inequality and its consequences,” in *Information Theory and Applications Workshop, 2008*. IEEE, 2008, pp. 128–130.
- [12] S. Guha, J. H. Shapiro, and B. Erkmen, “Capacity of the bosonic wiretap channel and the entropy photon-number inequality,” in *Information Theory, 2008. ISIT 2008. IEEE International Symposium on*. IEEE, 2008, pp. 91–95.
- [13] M. M. Wilde, P. Hayden, and S. Guha, “Information trade-offs for optical quantum communication,” *Physical Review Letters*, vol. 108, no. 14, p. 140501, 2012.
- [14] —, “Quantum trade-off coding for bosonic communication,” *Physical Review A*, vol. 86, no. 6, p. 062306, 2012.
- [15] G. De Palma, D. Trevisan, and V. Giovannetti, “Passive states optimize the output of bosonic gaussian quantum channels,” *IEEE Transactions on Information Theory*, vol. 62, no. 5, pp. 2895–2906, May 2016. [Online]. Available: <http://ieeexplore.ieee.org/document/7442587/>
- [16] —, “Gaussian states minimize the output entropy of the one-mode quantum attenuator,” *IEEE Transactions on Information Theory*, vol. 63, no. 1, pp. 728–737, 2017. [Online]. Available: <http://ieeexplore.ieee.org/document/7707386/>
- [17] —, “The one-mode quantum-limited gaussian attenuator and amplifier have gaussian maximizers,” *Annales Henri Poincaré*, vol. 19, no. 10, pp. 2919–2953, Oct 2018. [Online]. Available: <https://link.springer.com/article/10.1007/s00023-018-0703-5>
- [18] —, “Gaussian states minimize the output entropy of one-mode quantum gaussian channels,” *Physical Review Letters*, vol. 118, p. 160503, Apr 2017. [Online]. Available: <https://journals.aps.org/prl/abstract/10.1103/PhysRevLett.118.160503>
- [19] H. Qi, M. M. Wilde, and S. Guha, “On the minimum output entropy of single-mode phase-insensitive gaussian channels,” *arXiv preprint arXiv:1607.05262*, 2017.
- [20] R. König and G. Smith, “The entropy power inequality for quantum systems,” *IEEE Transactions on Information Theory*, vol. 60, no. 3, pp. 1536–1548, 2014.
- [21] —, “Corrections to “the entropy power inequality for quantum systems”,” *IEEE Transactions on Information Theory*, vol. 62, no. 7, pp. 4358–4359, 2016.
- [22] G. De Palma, A. Mari, and V. Giovannetti, “A generalization of the entropy power inequality to bosonic quantum systems,” *Nature Photonics*, vol. 8, no. 12, pp. 958–964, 2014. [Online]. Available: <http://www.nature.com/nphoton/journal/v8/n12/full/nphoton.2014.252.html>
- [23] G. De Palma, A. Mari, S. Lloyd, and V. Giovannetti, “Multimode quantum entropy power inequality,” *Physical Review A*, vol. 91, no. 3, p. 032320, 2015. [Online]. Available: <http://journals.aps.org/prl/abstract/10.1103/PhysRevA.91.032320>
- [24] G. De Palma, “Gaussian optimizers and other topics in quantum information,” Ph.D. dissertation, Scuola Normale Superiore, Pisa (Italy), Sep. 2016, supervisor: Prof. Vittorio Giovannetti; arXiv:1710.09395. [Online]. Available: <https://arxiv.org/abs/1710.09395>
- [25] G. De Palma and D. Trevisan, “The conditional entropy power inequality for bosonic quantum systems,” *Communications in Mathematical Physics*, vol. 360, no. 2, pp. 639–662, Jun 2018. [Online]. Available: <https://link.springer.com/article/10.1007%2Fs00220-017-3082-8>
- [26] G. De Palma and S. Huber, “The conditional entropy power inequality for quantum additive noise channels,” *Journal of Mathematical Physics*, vol. 59, no. 12, p. 122201, 2018. [Online]. Available: <https://aip.scitation.org/doi/10.1063/1.5027495>
- [27] S. Huber, R. König, and A. Vershynina, “Geometric inequalities from phase space translations,” *Journal of Mathematical Physics*, vol. 58, no. 1, p. 012206, 2017.
- [28] G. De Palma, D. Trevisan, V. Giovannetti, and L. Ambrosio, “Gaussian optimizers for entropic inequalities in quantum information,” *Journal of Mathematical Physics*, vol. 59, no. 8, p. 081101, 2018. [Online]. Available: <https://aip.scitation.org/doi/10.1063/1.5038665>
- [29] A. Ferraro, S. Olivares, and M. Paris, *Gaussian States in Quantum Information*, ser. Napoli series on physics and astrophysics. Bibliopolis, 2005.
- [30] S. Barnett and P. Radmore, *Methods in Theoretical Quantum Optics*, ser. Oxford Series in Optical and Imaging Sciences. Clarendon Press, 2002.
- [31] A. S. Holevo, “One-mode quantum gaussian channels: Structure and quantum capacity,” *Problems of Information Transmission*, vol. 43, no. 1, pp. 1–11, 2007.
- [32] V. Giovannetti, A. Holevo, and R. García-Patrón, “A solution of gaussian optimizer conjecture for quantum channels,” *Communications in Mathematical Physics*, vol. 334, no. 3, pp. 1553–1571, 2015.
- [33] A. Mari, V. Giovannetti, and A. S. Holevo, “Quantum state majorization at the output of bosonic gaussian channels,” *Nature communications*, vol. 5, 2014.
- [34] G. De Palma, D. Trevisan, and V. Giovannetti, “Multimode gaussian optimizers for the wehrl entropy and quantum gaussian channels,” *arXiv preprint arXiv:1705.00499*, 2017. [Online]. Available: <https://arxiv.org/abs/1705.00499>
- [35] A. Kholevo, M. E. Shirokov, and R. Werner, “On the notion of entanglement in hilbert spaces,” *Russian Mathematical Surveys*, vol. 60, no. 2, pp. 359–360, 2005.
- [36] J. Yard, P. Hayden, and I. Devetak, “Quantum broadcast channels,” *Information Theory, IEEE Transactions on*, vol. 57, no. 10, pp. 7147–7162, 2011.
- [37] I. Savov and M. M. Wilde, “Classical codes for quantum broadcast channels,” *Information Theory, IEEE Transactions on*, vol. 61, no. 12, pp. 7017–7028, 2015.
- [38] M.-H. Hsieh and M. M. Wilde, “Entanglement-assisted communication of classical and quantum information,” *IEEE Transactions on Information Theory*, vol. 56, no. 9, pp. 4682–4704, 2010.
- [39] M. M. Wilde and M.-H. Hsieh, “The quantum dynamic capacity formula of a quantum channel,” *Quantum Information Processing*, vol. 11, no. 6, pp. 1431–1463, 2012.
- [40] M.-H. Hsieh and M. M. Wilde, “Trading classical communication, quantum communication, and entanglement in quantum shannon theory,” *IEEE Transactions on Information Theory*, vol. 56, no. 9, pp. 4705–4730, 2010.
- [41] H. Qi and M. M. Wilde, “Capacities of quantum amplifier channels,” *Physical Review A*, vol. 95, p. 012339, Jan 2017.

Giacomo De Palma was born in Lanciano (CH), Italy, on March 15, 1990. He received the B.S. degree in Physics and the M.S. degree in Physics from the University of Pisa (Pisa, Italy), in 2011 and 2013, respectively. He also received the “Diploma di Licenza” in Physics and the Ph.D. degree in Physics from Scuola Normale Superiore (Pisa, Italy), in 2014 and 2016, respectively. He is currently a Marie Skłodowska-Curie Individual Fellow at the University of Copenhagen (Copenhagen, Denmark).

His research areas are quantum information and mathematical physics. He is author of 22 scientific papers published in peer-reviewed journals.