CHRYSIPPUS' LOGIC IN A NATURAL DEDUCTION SETTING

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Abstract. In this paper, we zoom in on some important features of Chrysippus' logic by framing them in a natural deduction system for relevant logic. Our view is alternative to other recent reconstructions of Chrysippus' view based on the sequent calculus. We show how, adding suitable elimination rules (akin to the five indemonstrables) that are missing in the Chrysippean system, one automatically obtains classical logic as a kind of deductive upper bound.

Keywords. Stoic logic, Natural Deduction, Relevant logic, Classical logic.

1. Introduction

It is well-known that the Stoic theory of deduction is the largest contribution to a system of propositional logic in antiquity. From the 1950s it has been at the center of a considerable body of historical and theoretical research (cf. [21, 3, 14, 24, 16, 22, 23, 4, 5, 25, 10]). Despite many general accounts of Stoic logic, however, there is little consensus among scholars about motivations and details of some of its most controversial theses. These are traditionally associated with Chrysippus of Soli (*c.* 280-206 BCE), third head of the Stoa and its leading logician. The aim of this paper is to frame in a contemporary context some salient but overlooked aspects of Chrysippus' system of deduction, which speak eloquently to our own logical concerns.

We subscribe to Susanne Bobzien's recent claim that Stoic logic "deserves more attention from contemporary logicians" [7, p. 234]. According to her, one of the reasons why Stoic logic should catch our interest is that Stoic analysis is "closest to methods of backward proof search for Gentzen-inspired substructural sequent logics" [7, p. 234]. Of course, a word of methodological caution is in order at the outset. A logical system of antiquity brings together logical commitments with distinctive metaphysical and epistemological outlooks, so that any attempt to link that system up to a contemporary proof-theoretic setting demands a considerable level of idealisation. Modulo this *caveat*, anyway, we are going to suggest that Chrysippus' system of deduction is instructive and attractive for a contemporary logician not only as anticipation of some proof-theoretic frameworks of the twentieth century, but also, more subtly and surprisingly, by virtue of the (peaceful) coexistence of what in the eyes of modern logicians are *classical* and *non-classical* patterns of inference.

In particular, we propose to implement Stoic logic within a *relevant natural deduction* setting as an alternative to other reconstructions relying on sequent calculus like the one offered by Bobzien herself ([7]; see also [8]). Moreover, we will show how adding suitable

elimination rules to Chrysippus' natural deduction system, one automatically obtains classical logic as a kind of deductive upper bound.

The plan is as follows. In section 2, we sketch Chrysippus' account of connectives. In section 3 we frame the package of Chrysippus' five indemonstrable within a relevant natural deduction system and we argue that the Stoic notion of proof also involves the commitment to the *reductio ad absurdum* rule. Section 4 traces out the distinction between "shallow" and "deeper" proofs. In section 5, we show how the underlying natural deduction system can be extended to bridge the gap with classical propositional logic. Section 6 contains some concluding remarks.

2. Chrysippus' connectives

2.1. **Conjunction.** Chrysippus' *conjunction* (συμπεπλεγμένον ἀξίωμα) is defined as one proposition "which is put together by the connective particle 'and' as for example, 'It is day and it is light"' (DL, VII, 72, [19]. Conjunction may be regarded as truth-functional as attested in this passage from Sextus Empiricus:

> [Stoics] say, just as in life we do not say that the piece of clothing that is sound in most parts, but torn in a small part, is sound (on the basis of its sound parts, which is most of them), but torn (on the basis of its small torn part), so too the conjunction, even if it has only one false component and majority of true ones, will as a whole be called false on the basis of that one [13, p. 114].

No matter how big the number of true conjuncts is, one false conjunct is enough to nullify all these truths in a conjunction.

2.2. **Negation.** Negation (ἀποφατικὸν ἀξίωμα) is truth-functional: if added to true propositions, makes them false; if added to false ones makes them true (AM, VIII 103, [13]). Negation is involutive, i.e. $\neg\neg A = A$ (DL, VII, 69, [19]) (see *infra*).

2.3. **Conditional.** Characteristically, Chrysippus' analysis of the conditional (συνημμένον ἀξίωμα) relies on the relation of *incompatibility*. In Sextus Empiricus's presentation of four different accounts of the conditional from the Hellenistic period, the third one is commonly attributed to Chrysippus although Sextus does not mention any particular philosopher, while ascribing the "material" conditional to Philo of Megara and its omnitemporal interpretation to Diodorus Cronus. So Sextus:

> Those who introduce "connectedness" (συνάρτησις) say that a conditional is sound (ὑγιὲς) when the contradictory of its consequent conflicts (μάχηται) with its antecedent; according to them, the conditionals mentioned above [*sc. If it is day, I converse* and *If there are not indivisible elements of the things, there are indivisible elements of the things*] will be unsound, while *If it is day, it is day* will be true (PH, II, 111, [12]).

And Diogenes Laertius:

> A conditional is true in which the contradictory of the consequent conflicts with the antecedent (τὸ ἀντικείμενον τοῦ λήγοντος μάχηται τῷ ἡγουμένῳ), for example *If it is day, it is light.* This is true, for *Not it is light*, the contradictory of the consequent, conflicts with *It is day.* A conditional is false in which the contradictory of the consequent does not conflict with the antecedent, for example *If it is day, Dion is walking.* For *Not Dion is walking* does not conflict with *It is day.* (DL VII, 73, [19])

The primitive notion of incompatibility or conflict (μάχη) marks off a sort of negative connectedness between propositions. Conflict embraces not only a logical incompatibility but also an empirical one, insofar as the experience of the world provides constraints on the circumstances in which incompatibility can occur. In short, connectedness requires that the reasoner inspects and matches the specific contents of the propositions involved in the conditional statement. From the Chrysippean "connexive" conditional two consequences immediately follow:

(1) no conditional of the form $A \to \neg A$ is true, insofar as the contradictory of $\neg A$ (i.e. A) is never incompatible with A.

(2) the principle *ex falso sequitur quodlibet* —which allows one to infer anything from inconsistent premisses (i.e., $(A, \neg A) \vdash B$)— doesn't hold. For instance, from *Dion is walking; not Dion is walking* doesn't follow that *It is day* inasmuch as *Not it is day* doesn't conflict with the premisses.

2.4. **Disjunction.** The notion of conflict regulates also the behaviour of the Chrysippean disjunction (διεζευγμένον), making it essentially *exclusive* in the binary case: it is true only when exactly one disjunct is true and the disjuncts are in conflict with one another (PH, II, 191, [12]). In other words, disjunction is not truth-functional, but rather equivalent to the conjunction of the two "connexive" conditionals $A \to \neg B$ and $\neg A \to B$. This reading of disjunction is entrenched in Greek mathematical practice. Clear cases of disjunctive conflicts are given in Euclid's *Elements* by the exclusive disjunctions encapsulating conflicting properties of actual mathematical objects; for instance, "if a straight line set up on a straight line make angles, it will make <them> either two rights or equal to two rights" (*Elements* I, 13)[1].

However, according to some ancient sources (e.g. Gellius, Galen), the Stoics considered disjunctions of arity of three or greater as true when only one of the disjuncts is true. But on the exclusive reading this becomes obviously problematic since a ternary exclusive disjunction is true also when its three disjuncts are all true. We have sympathy with Jennings' hypothesis that this divergence between modern logicians and the Stoics might reside on the modern use of brackets to separate the disjuncts; for example, let $\oplus$ denote the exclusive disjunction, the ternary disjunction $(A \oplus B \oplus C)$ could also be expressed as $(A \oplus (B \oplus C))$: assume then that A is true, then $B \oplus C$ must be false and this happens

either when both B and C are false or when both B and C are *true*. Instead, according to Jennings, Stoic logicians conceived of the n-ary disjunction in terms of a prefix notation of the kind $\oplus(A, B, C)$. For instance: our disjunction $(2+2=4)\oplus((2+3=5)\oplus(2+4=6))$ is *true* because its second disjunct is false (being both of its disjunct true), whereas the Stoic disjunction $\oplus((2+2=4),(2+3=5),(2+4=6))$ is *false* because more than one of its disjuncts is true [17, pp. 256-57].

Note that, due to the exclusive character of disjunction, Chrysippus' rejection of *ex falso* is compatible with disjunctive syllogism (DS), alias the fifth indemonstrable argument (ἀναπόιδεκτος λόγος) ascribed to Chrysippus by Diogenes Laertius and Sextus (DL, VII, 80-81 [19]; PH, II, 157-158, [12]) (see *infra*). In the twentieth century, logicians of relevantist persuasions have instead renounced *ex falso* by claiming the invalidity of DS associated with an *inclusive* reading of disjunction. This is the moral they draw from a venerable argument *promoting ex falso*, which goes back at least to Alexander Neckam's *De naturis rerum* (c. 1200) [26] and revived in C. I. Lewis and C.H. Langford's *Symbolic Logic* (1932) [20, pp. 250-52]. Namely,

1.	A	Assumption.
2.	$\neg A$	Assumption.
3.	$A \vee B$	1, $\vee$-intro.
4.	B	2,3, DS.

Relevantists tend to block the use of DS in order to short-circuit the above argument, instead of disputing either the disjunction-introduction, or the very transitivity of implication which the argument incorporates [2].

3. Chrysippus' logic as a natural deduction system.

In this section, we provide a proof-theoretic reconstruction of Stoic logic in terms of a kind of natural deduction system and compare it with the reconstruction given by Bobzien [7] in terms of Gentzen-style sequent calculus. In most cases proofs are omitted or informal and sketchy arguments are given instead.

Here are Chrysippus's five "indemonstrables" in schematic form, as reported by Sextus [PH, II, 157f [12], AM, VIII, 227f, [13]]:

(1) *If A, then B; but A; therefore B.*

(2) *If A, then B; but not B; therefore not A.*

(3a) *Not both A and B; but A; therefore not B.*

(3b) *Not both A and B; but B; therefore not A.*

(4a) *Either A or B; but A; therefore not B.*

(4b) *Either A or B; but B; therefore not A.*

(5a) *Either A or B; but not A; therefore B.*

(5b) *Either A or B; but not B; therefore A.*

These inference schemes can be seen as *elimination rules* of a kind of natural deduction system:

$$\frac{A \to B \qquad A}{B} \quad (1) \qquad\qquad \frac{A \to B \qquad \neg B}{\neg A} \quad (2)$$

$$\frac{\neg(A \wedge B) \qquad A}{\neg B} \quad (3a) \qquad\qquad \frac{\neg(A \wedge B) \qquad B}{\neg A} \quad (3b)$$

$$\frac{A \oplus B \qquad A}{\neg B} \quad (4a) \qquad\qquad \frac{A \oplus B \qquad B}{\neg A} \quad (4b)$$

$$\frac{A \oplus B \qquad \neg A}{B} \quad (5a) \qquad\qquad \frac{A \oplus B \qquad \neg B}{A} \quad (5b)$$

Observe that: (i) all the rules have two premisses, (ii) there are no "discharge" rules, (iii) there are elimination rules only for certain types of "non-simple" propositions, that the Stoics called "mode-forming" (τροπικόν).

Moreover, the Stoics stated at least four metatheorems (θέματα), of which only the first and the third are extant.

> There is also another proof common to all syllogisms, even the indemonstrable, called "(reduction) to the impossible" and by the Stoics termed "first metatheorem" or "first exposition". It is formulated thus: "If some third is deduced from two, one of the two together with the opposite of the conclusion yields the opposite of the other" [9, 22.12].

So, the first metatheorem corresponds to a contraposition principle for proofs. In modern notation:

$$\text{(T1)} \qquad \frac{A, B \vdash C}{A, C^* \vdash B^*}$$

where A^* is the complement of A. The complement of A is B if $A = \neg B$ and $\neg A$ otherwise. The notion of complement is necessary to account for the double negation rule that does not appear explicitly among the Stoic indemonstrables, although, as we said, they certainly accepted the idea that a double negation of A is equivalent to A: "The supernegative is the negation of the negative; e.g., 'not – it is not day'. This posits 'it is day'"[DL, VII, 69f, [19]].

The other extant metatheorem corresponds to what is probably the most characteristic principle of logical deduction, namely the principle of *transitivity* (or "Cut" in the terminology of Gentzen's sequent calculus).

> The essentials of the so-called third metatheorem look like this: if some third is deduced from two and one (of the two) can be deduced syllogistically from others, the third is yielded by the rest and those others [9, 22.13].

This third metatheorem then amounts to the following proof rule:[1]

$$\text{(T3)} \qquad \frac{A, B \vdash C \quad \Gamma \vdash A(B)}{\Gamma, B(A) \vdash C}$$

the peculiar form of the first premiss sequent being probably due to the fact that all the indemonstrables have exactly two premisses. According to many commentators, the stoics also used other variants of this metatheorem to be able to express full transitivity, but this is a point on which historians can only conjecture, since there is no explicit description of the other metatheorems in the ancient sources.[2] However, (T3) in itself is sufficient to justify the following notion of *direct proof*:

Definition 1. A *direct proof* of A depending on Γ is a tree such that (i) all its nodes are labelled with formulae (ii) its root is labelled with A, (ii) its leaves are labelled with the formulae in Γ, (iii) each formula in Γ labels at least one of the leaves, (iv) each node other than the leaves is labelled with a formula that results from the two formulae labelling the two nodes above by means of an application of one of the indemonstrables (1)–(5b).

Observe that in such direct proofs all formulae occurring in the leaves are actually *used* in order to obtain the conclusion.

Example 3.1. "If the first, then if the first then the second; but the first, therefore the second" [AM, VII, 230–233, [13]].

This theorem in modern logic corresponds to what in the context of structural proof theory is known as the principle of *contraction* and it amounts to allowing multiple uses of the same proposition in the proof. A direct proof is as follows:

$$\dfrac{\dfrac{A \to (A \to B) \quad A}{A \to B}\,(1) \qquad A}{B}\,(1)$$

We mention in passing that this kind of proof was called by the Stoics "homogeneous", because it makes use of only one indemonstrable. An example of a non-homogeneous proof is the following argument:

Example 3.2. "If the first and the second, then the third; not the third; the first; therefore not the second" [AM, VIII, 234–236, [13]].

It is not homogeneous because it makes use of two distinct indemonstrables. A direct proof is as follows:

$$\dfrac{\dfrac{A \wedge B \to C \quad \neg C}{\neg(A \wedge B)}\,(2) \qquad A}{\neg B}\,(3a)$$

[1] By "proof rule" in the context of this paper we mean a rule that takes proofs as premisses and conclusion, as opposed to an "inference rule" that takes propositions as premisses and conclusion.

[2] On this point see [7].

Example 3.3. "If the first, then the first; but the first; therefore the first".

This is ascribed to Chrysippus by Alexander (see [18, p. 167]) and its direct proof is as follows:

$$\frac{A \to A \quad A}{A} \quad (1)$$

The interest of this example lies is in the fact that, taken for granted that the Stoics were interested in proofs satisfying a relevance criterion [7], they adopted a rather sophisticated one. The above proof is in fact relevant, because both its premisses are *used* in the deduction, although its assumptions are redundant, since obviously the conclusion follows immediately from the first premiss. Therefore, they seem to adopt a relevance-as-use criterion similar to that underlying modern systems of relevance logic. On the other hand, it seems that the Stoics did not admit of a proper inference with only one premiss [7], so that, from a formal viewpoint, this might have appeared to be the only way of deducing the conclusion A from the assumption A (in accordance with the formal ban on one-premiss inferences) in a relevant, albeit redundant, fashion.

It can be argued that the Stoics endorsed a notion of proof that included those by *reductio ad absurdum* as well as direct proofs. The validity of a proof is construed by them in terms of a conflict between the denied conclusion and (the conjunction of) the premises, just as the truth of a conditional is construed in terms of a similar conflict between the denied consequent and the antecedent. This is in accordance with the underlying idea that in order for an argument to be valid, the (connexive) conditional whose antecedent is the conjunction of the premises and whose consequent is the conclusion of the argument, must be true.

> Some arguments are conclusive [περαντικοί], others not conclusive [ἀπέραντοι]. They are conclusive when a connected proposition [i.e. a conditional], beginning with the conjunction of the premisses of the argument and ending with the conclusion, is true [PH, II, 137, [12]].
>
> Of arguments, some are not conclusive, others conclusive. Not conclusive are those in which the contradictory opposite of the conclusion is not incompatible with the conjunction of the premises [DL, VII, 77, [19]].

Moreover, simple arguments by *reductio ad absurdum* can be justified by means of rule (1), and the metatheorems (T1) and (T3), as follows:[3]

(1) $A \to \neg B, A \vdash \neg B$ by (1)
(2) $A, B \vdash \neg(A \to \neg B)$ from 1 by(T1)
(3) $A \to B, A \vdash B$ by (1)
(4) $A \to B, A \vdash \neg(A \to \neg B)$ from 2,3 by (T3)
(5) $A \to B, A \to \neg B \vdash \neg A$, from 4 by (T1)

Hence, a notion of *indirect proof* seems to be fully justified in Stoic terms. In order to accommodate such indirect proofs in our natural deduction system, let us introduce a

[3]What is essentially the same argument in a slightly different setting presented in [15].

basic *conflict rule* that corresponds to the Aristotelian *Principle of Non-Contradiction*:

$$\text{(PNC)} \qquad \frac{A \quad \neg A}{\bot}$$

where $\bot$ is not a logical constant, as in intuitionistic logic, but simply a marker that a conflict has been revealed.

Definition 2. A *refutation* of Γ is a tree such that (i) all its nodes except the root are labelled with formulae (ii) its root is labelled with the special symbol $\bot$, (ii) its leaves are labelled with the formulae in Γ, (iii) each formula in Γ labels at least one of the leaves, (iv) each node other than the leaves is labelled with a formula that results from the two formulae labelling the two nodes above by means of an application of one of the indemonstrables (1)–(5b).

Definition 3. A *indirect proof* of A depending on Γ is a refutation of $\Gamma \cup \{A^*\}$.

In the sequel we shall call NS_0 the proof system consisting of the inference rules (1)–(5b) together with the associated notions of direct and indirect proofs. (The reason for the subscript will be apparent in Section 4 below.)

Example 3.4. "If the first, then the second; if the first, then not the second; therefore not the first".

This is clearly related to the *reductio ad absurdum* pattern that was extensively used by the Greeks. According to [18, p. 172] the Stoics probably held this theorem in high esteem, and it is mentioned by several authoritative sources, under the name of "theorem with two mode-forming premisses". Indeed, there is no indemostrable that can conclude from two mode-forming premisses, so a direct proof — as we have called it — cannot even start. Hence, its proof must be achieved by indirect means. A simple way of doing it, would consist in accepting what we have called "indirect proofs" in Definition 3 as first-class citizens in the realm of proofs.

$$\dfrac{\dfrac{A \to B \quad A(=(\neg A)^*)}{B}\,(1) \qquad \dfrac{A \to \neg B \quad A(=(\neg A)^*)}{\neg B}\,(1)}{\bot}\,(\text{PNC})$$

Another related way, illustrated in [7, p. 251], consists in making explicit use of (T1) as a rule of proof. In this case the proof takes the form of a *sequent proof*, namely a tree whose nodes are labelled with sequents, the leaves with sequents corresponding to the five indemonstrable, and all nodes except the leaves with sequents obtained from the sequents above by means of a metatheorem: T1 or T3 or a variant of T3. For a careful reconstruction of the variants of T3 the reader is referred to [7], where the author endorses the view that these variants are nothing but the missing methatheorems of the Stoics.

Here we shall loosely refer to all the variants of T3 as "Cut".

$$\dfrac{\dfrac{\dfrac{A \to B, \neg B \vdash \neg A}{A, \neg B \vdash \neg(A \to B)}\ (\text{T1}) \qquad A \to \neg B, A \vdash \neg B}{A \to \neg B, A \vdash \neg(A \to B)}\ \text{Cut}}{A \to B, A \to \neg B \vdash \neg A}\ (\text{T1})$$

A third kind of proof is illustrated in [18, pp. 171–172] via "conditionalization" (loosely speaking a Stoic version of the deduction theorem), which might be taken as an auxiliary metatheorem, using previously proven theorems as lemmas.

Indeed, there is evidence that the Stoics considered an inference valid whenever the conditional whose antecedent is the conjunction of the premises and whose consequent is the conclusion is true. This suggests that they might have had no qualms in exploiting conditionalization in their deductive practice in the following sense: if an argument was found valid, the corresponding conditional was regarded as true and could be freely assumed in another proof without counting as a genuine assumption or, if you wish, representing a sort of "unassailable" assumption. More precisely, we can distinguish two forms of conditionalization:

Weak Conditionalization. Every conditional corresponding to an indemostrable can be freely assumed in a proof.

Strong Conditionalization. Every conditional corresponding to a valid inference can be freely assumed in a proof.

Neither of them appears to be equivalent, given (T1) and Cut, to the full version of the deduction theorem, even with the restriction that the cardinality of Γ is at least 2:[4]

$$\text{(DT)} \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}$$

There seems to be no extant example of a Stoic proof of a sequent whose conclusion is a conditional, and we found no reference in the literature about how such a proof could be achieved by means of Stoic analysis.

Let us call LS the sequent calculus consisting of (i) all the sequents corresponding to the five indemonstrables as basic axioms, (ii) Cut and T1 as sequent proof rules, (iii) Weak Conditionalization as a source of auxiliary "unassailable" premisses, i.e., conditionals corresponding to indemonstrables. Let us also call LS^+ the similar system in which Weak Conditionalization is replaced by Strong Conditionalization, allowing the free introduction of conditionals corresponding to previously proven inferences.

The discussion of the last example suggests that the two approaches — the natural deduction system with direct and indirect proofs on the one hand, and the sequent calculus with T1, Cut, and Conditionalization on the other — might be equivalent. Indeed, it can be shown that LS^+ can simulate NS_0. Suppose there is a refutation of Γ, A^*; then

[4] According to most scholars, the Stoics did no allow inferences with less than two premisses; see [7].

there must be direct proofs of $\Delta_1 \vdash B$ and $\Delta_2 \vdash B$ for some B and some Δ_1, Δ_2 such that $\Delta_1 \cup \Delta_2 = \Gamma \cup \{A^*\}$. Thus, $\bigwedge \Delta_1 \to B$ and $\bigwedge \Delta_2 \to \neg B$ are true conditionals by Strong Conditionalization. Then:

$$\frac{\bigwedge \Delta_1 \to B, \neg B \vdash \neg \bigwedge \Delta_1 \quad \bigwedge \Delta_2 \to \neg B, \bigwedge \Delta_2 \vdash \neg B}{\bigwedge \Delta_1 \to B, \bigwedge \Delta_2 \to \neg B, \bigwedge \Delta_2 \vdash \neg \bigwedge \Delta_1} \text{Cut} \tag{3}$$

and exploiting the equivalence between B and its "supernegative" $\neg\neg B$,

$$\frac{\bigwedge \Delta_1 \to B, \bigwedge \Delta_1 \vdash B \quad \bigwedge \Delta_2 \to \neg B, \neg\neg B \vdash \neg \bigwedge \Delta_2}{\bigwedge \Delta_1 \to B, \bigwedge \Delta_2 \to \neg B, \bigwedge \Delta_1 \vdash \neg \bigwedge \Delta_2} \text{Cut} \tag{4}$$

Now, if $A^* \in \Delta_1$, then

$$\neg \bigwedge \Delta_1, \Delta_1 \setminus \{A^*\} \vdash A^{**} = A \tag{5}$$

is an axiomatic sequent by a plausible generalization of the second indemonstrable; and if $A^* \in \Delta_2$, then

$$\neg \bigwedge \Delta_2, A^* \vdash \neg \bigwedge \Delta_2 \setminus \{A^*\} \tag{6}$$

is also an axiomatic sequent by the same argument. So, by an application of Cut to (3) and (5) and omitting the "unassailable" conditionals corresponding to the indemonstrables

$$\Delta_1 \setminus \{A^*\}, \bigwedge \Delta_2 \vdash A. \tag{7}$$

Moreover, given that

$$\Delta_2 \vdash \bigwedge \Delta_2 \tag{8}$$

is a theorem of Stoic logic, which can be derived from the second indemonstrable by (T1), we obtain, by means of an application of cut to (7) and (8):

$$\Delta_1 \setminus \{A^*\}, \Delta_2 \vdash A. \tag{9}$$

If $A^* \notin \Delta_2$ the above sequent is the endsequent and a similar argument can be developed starting from (4) for the case in which $A^* \in \Delta_2$ but $A^* \notin \Delta_1$. Otherwise, by an application of (T1) to (7),

$$\Delta_1 \setminus \{A^*\}, A^* \vdash \neg \bigwedge \Delta_2 \tag{10}$$

and then

(11) $\Delta_1 \setminus \{A^*\}, A^* \vdash \neg \bigwedge \Delta_2 \setminus A^*$ from (6) and (10) by Cut

(12) $\Delta_1 \setminus \{A^*\}, \bigwedge \Delta_2 \setminus \{A^*\} \vdash A$ from (11) by (T1)

(13) $\Delta_1 \setminus \{A^*\}, \Delta_2 \setminus \{A^*\} \vdash A$ from (12) and (8) by Cut

and hence $\Gamma \vdash A$. The above argument establishes the following:

Proposition 3.1. *If* $\Gamma \vdash_{NS_0} A$, *then* $\Gamma \vdash_{LS^+} A$.

In the next section we shall show that the converse holds only for a special case of LS-proofs and shall define a stronger notion of NS-proof that is indeed equivalent to that of LS^+-proof.

4. Shallow and deep arguments

The proof rule expressed by (T1), can be easily simulated by the notion of indirect NS_0-proof. Suppose there is an indirect NS_0-proof of C depending on $\{A, B\}$, namely a refutation of $\{A, B, C^*\}$. Then such a refutation can immediately be read as a proof of B^* depending on $\{A, C^*\}$. Is this the whole story? Not quite.

First, observe that in NS_0 Cut is satisfied only by direct proofs and not by indirect proofs.

Example 4.1. As shown in Example 3.4,

$$A \to B, A \to \neg B \vdash_{NS_0} \neg A$$

can be established only by means of an indirect proof. Similarly,

$$C \to D, C \to \neg D \vdash_{NS_0} \neg C.$$

Moreover, it is easy to verify that

$$\neg A, \neg C \vdash_{NS_0} \neg(A \oplus C)$$

can be established by means of an indirect proof. However,

$$A \to B, A \to \neg B, C \to D, C \to \neg D \nvdash_{NS_0} \neg(A \oplus C),$$

because if the premisses are all "mode-forming", the proof (whether direct or indirect) cannot even start.

As a consequence, indirect NS_0-proofs can simulate (T1) under the restriction that the latter is used *only as the last step in the proof*. Let LS^- be the restriction of LS allowing only proofs such that (T1) is applied only as the last step in a sequent proof. It can be shown that Weak Conditionalization does not increase the expressive power of NS_0 and so:

Proposition 4.1. *If* $\Gamma \vdash_{LS^-} A$*, then* $\Gamma \vdash_{NS_0} A$*.*

In order to achieve full equivalence between NS-proofs and LS-proofs we need to introduce a discharge rule, which might well have been acceptable by Stoic standards, although it appeals to what they would have called "hypothetical reasoning". The rule takes the form of a non-constructive dilemma rule:

$$\dfrac{\begin{array}{c}\Gamma, [A] \\ \vdots \\ \bot/B\end{array} \quad \begin{array}{c}\Gamma, [\neg A] \\ \vdots \\ \bot/B\end{array}}{\bot/B} \tag{14}$$

where the slash notation is to be understood as follows: if at least one of the conclusions of the subordinate arguments is B, then the conclusion is B, otherwise it is $\bot$. In either case, the application of the rule discharges the assumptions A and $\neg A$, so that the conclusion of the main proof no longer depends on them.

With this addition, Cut is satisfied for indirect as well as direct proofs and the proof of Example 4.1 is accomplished as follows. Let $\mathcal{T}_1$ be the following proof tree:

$$\dfrac{\dfrac{\dfrac{A \oplus C \quad [\neg A]}{C}\,(5b) \qquad \neg C}{\bot}\,\text{PNC} \qquad \dfrac{\dfrac{A \to B \quad [A]}{B}\,(1) \qquad \neg B}{\bot}\,\text{PNC}}{\bot}\,\text{PNC}$$

and $\mathcal{T}_2$ be the following proof tree:

$$\dfrac{\dfrac{C \to D \quad C}{D}\,(1) \qquad \dfrac{C \to \neg D \quad C}{\neg D}\,(1)}{\bot}\,\text{PNC}$$

Then,

$$\dfrac{\begin{array}{c}[C]\\ \mathcal{T}_2\\ \bot\end{array} \qquad \begin{array}{c}[\neg C]\\ \mathcal{T}_1\\ \bot\end{array}}{\bot}$$

is a refutation of

$$\{A \oplus C, A \to B, A \to \neg B, C \to D, C \to \neg D\}$$

and therefore an indirect proof of $\neg(A \oplus C)$ depending on

$$\{A \to B, A \to \neg B, C \to D, C \to \neg D\}.$$

Since the effect of (T1) can be simulated by the notion of indirect proof, it follows that any LS-proof can be simulated by the system NS obtained by adding to NS_0 the single discharge rule (14).

On the other hand, (14) is a derived rule in LS^+. We have shown in the previous section that if $\Gamma, A^* \vdash_{NS_0} \bot$ then $\Gamma \vdash_{LS^+} A$. So, if $\Delta, A \vdash_{NS_0} B$, by Proposition 3.1, $\Delta, A \vdash_{LS^+} B$ and, by Cut, $\Gamma, \Delta \vdash_{LS^+} B$. The general case for NS can be shown by induction on the number of nested applications of the the rule itself in its subordinate proofs. For the case in which both subordinate proofs conclude to B, the simulation is obtained in a similar way.

It can be shown that Strong Conditionalization does not increase the deductive power of NS. In general, it holds that:

Proposition 4.2. *If* $\Gamma \vdash_{LS^+} A$*, then* $\Gamma \vdash_{NS} A$*.*

Moreover, (14) is a derived rule in LS^+, therefore:

Proposition 4.3. *If* $\Gamma \vdash_{NS} A$*, then* $\Gamma \vdash_{LS^+} A$*.*

5. Bridging the gap with classical logic

The natural deduction system NS, as well as its equivalent sequent calculus LS^+, is a relevance logic and therefore falls short of classical propositional logic. What is missing? From this point of view, it is interesting to compare NS to the classical refutation system KE [11]. The KE elimination rules for the mode-forming premisses are the same as the NS-rules, except that the primitive disjunction in KE is inclusive and not exclusive. However, the NS-rules for $\oplus$ can be easily derived in KE using the translation of exclusive disjunction in terms of inclusive disjunction, negation and conjunction. On the other hand, KE includes more elimination rules than NS. These are elimination rules for propositions of the form $\neg(A \to B)$, $A \wedge B$, $\neg(A \oplus B)$ plus an explicit double negation rule.

$$\frac{\neg(A \to B)}{A} \qquad \frac{\neg(A \to B)}{\neg B}$$

$$\frac{A \wedge B}{A} \qquad \frac{A \wedge B}{B}$$

$$\frac{\neg(A \oplus B) \qquad A}{B} \qquad \frac{\neg(A \oplus B) \qquad B}{A}$$

$$\frac{\neg(A \oplus B) \qquad \neg A}{\neg B} \qquad \frac{\neg(A \oplus B) \qquad \neg B}{\neg A}$$

$$\frac{\neg\neg A}{A}$$

The addition of the above rules to NS yields a refutation system that is sound and complete for classical propositional logic.

Interestingly enough, the missing rules are all objectionable from a Chrysippean viewpoint. The additional rules for the denied conditional, via (14) or, equivalently, via (T1), justify the following introduction rules for the conditional:

$$\frac{\neg A}{A \to B} \qquad \frac{B}{A \to B}$$

and so yield Philo's conditional with all the associated paradoxes. From the elimination rules for $A \wedge B$, again via (14) or(T1), one can easily derive the following introduction rules:

$$\frac{\neg A}{\neg(A \wedge B)} \qquad \frac{\neg B}{\neg(A \wedge B)}$$

Either of the them yields a proof of *ex falso quodlibet*. For example:

$$\dfrac{\dfrac{A \quad \dfrac{\neg A}{\neg(A \wedge \neg B)}}{\neg\neg B}}{B}$$

Finally, the rules for $\neg(A \oplus B)$ turn $\oplus$ into the Boolean exclusive disjunction and justify the following introduction rules:

$$\dfrac{A \quad \neg B}{A \oplus B} \qquad \dfrac{\neg A \quad B}{A \oplus B}$$

which are at the odds with the intensional load of Chrysippean disjunction.

6. Conclusion

We have argued that Stoic modes of inference can be sensibly reconstructed in modern terms as a natural deduction system with a *single discharge rule* expressing a form of non-constructive dilemma that governs hypothetical reasoning and allows for a natural distinction between "shallow" and "deeper" proofs. Finally, we have shown how this system can be extended to bridge the gap with classical propositional logic. This is a rational reconstruction that does not claim to be historically accurate. However, it can be seen as further and independent support to Bobzien's view that stoic logic can be seen as a forerunner of modern propositional logic and its proof-theoretic settings [6, 7]. At the same time, our reconstruction highlights exactly what is missing in Stoic logic in order to provide a classically complete system.

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