UNIQUENESS OF AXIOMATIC EXTENSIONS OF CUT-FREE CLASSICAL PROPOSITIONAL LOGIC

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Abstract. In this paper, we prove that, for any cluster of extra-logical assumptions, there exists exactly one axiomatic (i.e., minimal) extension of classical propositional logic that admits cut elimination. As a corollary, it follows that classically equivalent formulas share the same axiomatization. The moral is that cut elimination "flattens" the specific information encoded by the logical structure of proper axioms.

Keywords: proof theory, axiomatic extensions, logic of pivotal assumptions, cut elimination with proper axioms.

1. Introduction

As is well known, classical propositional logic (henceforth LK_0, in a Gentzen-style sequent formulation [9, 18]) is a Post-complete system: whenever a nontautological formula α is added to LK_0 as a new axiom schema, the resulting system LK_0^α is inconsistent. This fundamental property of LK_0 can be paraphrased by saying that the only nontrivial axiomatic extensions of LK_0 are the ones obtained by adding proper (or extra-logical) axioms, i.e. formulas that are not closed under uniform substitution (US). In short, structurality and consistency are mutually excluding properties in axiomatic extensions of LK_0.

The so-called logic of pivotal assumptions is the simplest element of the set of supraclassical logic, i.e. extensions of LK_0 via proper axioms: these logics were introduced by Makinson to 'bridge the gap' between classical and nonmonotonic reasoning through a 'logical continuous' [13, 14]. In extending classical consequence relation, supraclassical logics aim at formalizing reasoning with background assumptions, namely assumptions that are specific to the situation under consideration. In particular, the distinctive feature of the logic of pivotal assumptions is that any finite set of formulas can be taken as a set of background assumptions. Thus, α is a consequence of Γ under pivotal assumptions in Ψ if $\Gamma \cup \Psi \vdash \alpha$. Intuitively, pivotal assumptions in Ψ may be understood as encoding pieces of nontautological information which support more complex conclusions, underivable in LK_0. Makinson exclusively focuses on a semantic definition of pivotal-assumption consequence: a

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formula α is a consequence of Γ under the assumptions $\alpha_1, \ldots, \alpha_n$ if, and only if, there is no classical valuation that makes all the formulas in $\Gamma \cup \{\alpha_1\} \cup \ldots \cup \{\alpha_n\}$ true and α false (see [2] for the characterizations of pivotal consequence relations for the classical, three and four-valued logics). We instead define α to be a consequence of Γ under the assumptions $\alpha_1, \ldots, \alpha_n$ if, and only if, the sequent $\Gamma \vdash \alpha$ is derivable in the system $\mathsf{LK}_0^{\alpha_1,\ldots,\alpha_n}$, which is LK_0 extended by taking $\alpha_1, \ldots, \alpha_n$ as new proper axioms. Clearly, the two definitions are equivalent: $\Gamma, \alpha_1, \ldots, \alpha_n \vdash \alpha$ in LK_0 if, and only if, $\Gamma \vdash \alpha$ in $\mathsf{LK}_0^{\alpha_1,\ldots,\alpha_n}$. On the one hand, it is easy to verify that if $\Gamma, \alpha_1, \ldots, \alpha_n \vdash \alpha$, then $\Gamma \vdash \alpha$ by n cuts on $\vdash \alpha_1, \ldots, \vdash \alpha_n$; on the other hand, if $\Gamma \vdash \alpha$, then $\Gamma, \alpha_1, \ldots, \alpha_n \vdash \alpha$, by n weakenings.

The problem of axiomatic extensions of sequent calculus has a long tradition in proof theory, inasmuch as proper axioms may jeopardize the cut elimination theorem (see [10]). Consider, for instance, the system $\mathsf{LK}_0^{p\to q}$, which augments LK_0 with the axiom $p \to q$. Whereas the sequent $p \vdash q$ is provable in $\mathsf{LK}_0^{p\to q}$ by resorting to the cut rule

$$\dfrac{\dfrac{\dfrac{}{p \vdash p}\,ax. \quad \dfrac{}{q \vdash q}\,ax.}{p, p \to q \vdash q}\,\to\vdash \quad \dfrac{}{\vdash p \to q}\,p.ax.}{p \vdash q}\,cut,$$

this clearly does not admit a cut-free proof. A method to recover cut elimination has been proposed by Negri and von Plato in [15, 16]. It essentially consists in turning proper axioms into inference rules, which preserve cut elimination once added to a suitable Gentzen system of classical logic. The authors' focus is mainly on the proof analysis of first order axiomatic theories, so their approach is wider than the one here proposed, which is limited to the propositional fragment. However, in [15, 16], the new rules stemming from proper axioms do not necessarily preserve the subformula property and, consequently, it is not clear (or at least so it seems to us) whether their formulations 'hide' implicit applications of the cut rule.

In this paper, we consider an alternative strategy, which involves the possibility to decompose any arbitrary proper axiom into a set of sequents made of atomic formulas [1]. Once such a decomposition has been closed under the cut rule, both cut elimination and subformula property can be easily rescued. More generally, this is a corollary of the so-called strong cut elimination theorem firstly remarked by Girard in [11].

As an original contribution to the proof-theoretical analysis of the logic of pivotal assumption, we prove that, for any cluster of extra-logical assumptions, there exists exactly one axiomatic (i.e., minimal) extension of LK_0 that admits cut elimination. From this, it follows as a corollary that classically equivalent formulas share the same axiomatization. This is an interesting result since it can be viewed as saying that cut elimination "flattens" the specific information encoded by the logical structure of proper axioms.

The paper is organized as follows. In §2, we show how to extend LK_0 with axioms corresponding to pivotal assumptions. Then, we describe the procedure for decomposing any pivotal assumption into a set of classically underivable sequents displaying only atomic formulas (Theorem 2.1) [1]. In passing, we provide a variant of the proof of Post-completeness for classical propositional logic based on such a decomposition (Theorem 2.3). Then, in §3, we state the cut elimination theorem for the logic of pivotal assumptions (Corollary 3.4) from which the subformula property follows (Corollary 3.6); we contextually provide a method for deciding whether a given extension of LK_0 is consistent (Theorem 3.7). In §4, we show that, for any cluster of proper axioms, there is exactly one axiomatic extension that preserves cut elimination (Therem 4.4). This fact in turn implies that classically equivalent formulas share the same axiomatization (Corollary 4.5). Finally, §5 presents our conclusions.

2. Proper axioms decomposition and Post-completeness

2.1. Sequent calculus for classical propositional logic. The language of classical propositional logic includes, as basic symbols, countably many propositional variables $p, q \ldots$ — the set $\mathcal{A}$ of atomic sentences — as well as symbols for connectives: $\neg, \wedge, \vee, \rightarrow$. The set of formulas $\mathcal{F}$ is defined standardly by the following grammar:

$$\mathcal{F} := \mathcal{A} \mid \neg\mathcal{F} \mid \mathcal{F} \wedge \mathcal{F} \mid \mathcal{F} \vee \mathcal{F} \mid \mathcal{F} \rightarrow \mathcal{F}.$$

The expression $\alpha \leftrightarrow \beta$ is to be understood as a shorthand for $(\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$. Sequents are expressions of the form $\Gamma \vdash \Delta$, where Γ and Δ are sets of formulas.[1] We write Γ, Δ and Γ, α for meaning $\Gamma \cup \Delta$ and $\Gamma \cup \{\alpha\}$, respectively. The calculus LK_0 is reported in Table 2.1.

Notation. Given a finite set of sequents $\mathscr{S} = \{\Gamma_1 \vdash \Delta_1 \ ; \ \Gamma_2 \vdash \Delta_2 \ ; \ \ldots \ ; \ \Gamma_n \vdash \Delta_n\}$ we denote with $\mathsf{LK}_0^{\mathscr{S}}$ the system obtained from LK_0 by adding all the sequents in $\mathscr{S}$ as new axioms. In the case where $\mathscr{S} = \{\vdash \alpha_1; \ldots; \vdash \alpha_n\}$, we will simply write $\mathsf{LK}_0^{\alpha_1,\ldots,\alpha_n}$. Given two formal systems S_1 and S_2 sharing a same language, we write $\mathsf{S}_1 \approx \mathsf{S}_2$ (resp. $\mathsf{S}_1 \preccurlyeq \mathsf{S}_2$) for meaning that S_1 and S_2 prove the same formulas (resp. all the formulas provable in S_1 are also theorems of S_2).

[1] We depart from the standard proof-theoretic approach in which sequents are multisets or lists. While this simplification allows us to leave out the structural rules of contraction and exchange, it does not impair the validity of our results since the proof of each one of them can be easily adapted so as to fit with Gentzen's standard formulation.

Axiom

$$\frac{}{\alpha \vdash \alpha}\ ax.$$

Cut

$$\frac{\Gamma \vdash \alpha, \Delta \qquad \Gamma', \alpha \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ cut$$

Structural Rules

$$\frac{\Gamma \vdash \Delta}{\Gamma, \alpha \vdash \Delta}\ weak. \vdash \qquad\qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \alpha, \Delta} \vdash weak.$$

Logical Rules

$$\frac{\Gamma \vdash \alpha, \Delta}{\Gamma, \neg\alpha \vdash \Delta}\ \neg \vdash \qquad\qquad \frac{\Gamma, \alpha \vdash \Delta}{\Gamma \vdash \neg\alpha, \Delta} \vdash \neg$$

$$\frac{\Gamma, \alpha, \beta \vdash \Delta}{\Gamma, \alpha \wedge \beta \vdash \Delta}\ \wedge \vdash \qquad\qquad \frac{\Gamma \vdash \alpha, \Delta \qquad \Gamma' \vdash \beta, \Delta'}{\Gamma, \Gamma' \vdash \alpha \wedge \beta, \Delta, \Delta'} \vdash \wedge$$

$$\frac{\Gamma, \alpha \vdash \Delta \qquad \Gamma', \beta \vdash \Delta'}{\Gamma, \Gamma', \alpha \vee \beta \vdash \Delta, \Delta'}\ \vee \vdash \qquad\qquad \frac{\Gamma \vdash \alpha, \beta, \Delta}{\Gamma \vdash \alpha \vee \beta, \Delta} \vdash \vee$$

$$\frac{\Gamma \vdash \alpha, \Delta \qquad \Gamma', \beta \vdash \Delta'}{\Gamma, \Gamma', \alpha \to \beta \vdash \Delta, \Delta'}\ \to\vdash \qquad\qquad \frac{\Gamma, \alpha \vdash \beta, \Delta}{\Gamma \vdash \alpha \to \beta, \Delta} \vdash\to$$

Table 1. The sequent calculus LK_0.

2.2. Complementary sequents.

Definition 1 (complementary sequent, decomposition). A sequent $\Gamma \vdash \Delta$ is said to be complementary[2] in the case where:

(1) it displays only atomic formulas,
(2) $\Gamma \cap \Delta = \varnothing$.

[2] The term ‘complementary’ is suggested by the fact that unprovable, atomic sequents are exactly the axioms of complementary classical logic, that is the system that proves all, and only, the nontheorems of LK_0 [19, 20, 6]. Complementary sequents are called normal clauses in [1] and regular in [16, p. 51].

Given a formula α, we say that a set of complementary sequents $\mathscr{S}$ is a decomposition for α if $\mathsf{LK}_0^\alpha \approx \mathsf{LK}_0^{\mathscr{S}}$.

Remark 1. Due to condition (2), complementary sequents are not provable in LK_0.

Theorem 2.1. Any formula α admits a decomposition $\mathscr{S}_\alpha$.

Proof. We need to show that, for any α, there is a $\mathscr{S}_\alpha$ such that $\mathsf{LK}_0^\alpha \approx \mathsf{LK}_0^{\mathscr{S}_\alpha}$. The first part of the proof provides an algorithm for associating, with any given formula α, a set of complementary sequents $\mathscr{S}_\alpha$. The second shows that $\mathsf{LK}_0^\alpha \approx \mathsf{LK}_0^{\mathscr{S}_\alpha}$. The algorithm hinges on the fact that any formula can be turned into an equivalent one in conjunctive normal form. In particular, the procedure consists of the following four steps.

(1) Turn α into its conjunctive normal form $\texttt{cnf}(\alpha) = \alpha_1 \wedge \ldots \wedge \alpha_n$, where each clause α_i is a disjunction of literals.
(2) For each one of the clauses of $\texttt{cnf}(\alpha)$, build a new sequent by removing any occurrence of the disjunction connective, i.e. for all $\alpha_i = l_1 \vee \ldots \vee l_k$ the corresponding sequent is $\vdash l_1, \ldots, l_k$.
(3) Shift all the negative literals on the left and remove the negation, i.e. any sequent of the form $\vdash p_1, \ldots, p_n, \neg q_1, \ldots, \neg q_m$ becomes $q_1, \ldots, q_m \vdash p_1, \ldots, p_n$.
(4) Remove identity sequents, namely sequents of the form $\Gamma, p \vdash \Delta, p$.

Now, we need to prove that $\mathsf{LK}_0^\alpha \approx \mathsf{LK}_0^{\mathscr{S}_\alpha}$. This is an easy consequence of the invertibility of left-conjunction, right-disjunction and right-negation rules in LK_0 [12].

- LK_0 proves $\vdash \alpha \wedge \beta$ iff it proves both $\vdash \alpha$ and $\vdash \beta$. The leftwards implication is trivial: simply apply the right-conjunction rule. Below, it is proved that $\vdash \alpha \wedge \beta$ implies $\vdash \alpha$. In a similar way, we can derive $\vdash \beta$ from the same assumption:

$$\dfrac{\vdash \alpha \wedge \beta \qquad \dfrac{\dfrac{\dfrac{}{\alpha \vdash \alpha}\,ax.}{\alpha, \beta \vdash \alpha}\,weak \vdash}{\alpha \wedge \beta \vdash \alpha}\,\wedge \vdash}{\vdash \alpha}\,cut.$$

- LK_0 proves $\vdash \Gamma, \alpha \vee \beta$ iff it proves $\vdash \Gamma, \alpha, \beta$. The leftwards implication is trivial: just apply the right-disjunction rule. Below, it is proved that $\vdash \Gamma, \alpha \vee \beta$ implies $\vdash \Gamma, \alpha, \beta$:

$$\dfrac{\vdash \Gamma, \alpha \vee \beta \qquad \dfrac{\dfrac{}{\alpha \vdash \alpha}\,ax. \qquad \dfrac{}{\beta \vdash \beta}\,ax.}{\alpha \vee \beta \vdash \alpha, \beta}\,\vee \vdash}{\vdash \Gamma, \alpha, \beta}\,cut.$$

- LK_0 proves $\Gamma \vdash \Delta, \neg p$ iff it proves $\Gamma, p \vdash \Delta$. The leftwards implication is trivial again: it suffices to apply the right-negation rule. Below, it is proved the converse implication:

$$\dfrac{\Gamma \vdash \Delta, \neg p \qquad \dfrac{\dfrac{}{p \vdash p}\ ax.}{p, \neg p \vdash}\ \neg\vdash}{\Gamma, p \vdash \Delta}\ cut.$$

□

Example 2.1. We show that, if

$$\alpha = ((p_0 \leftrightarrow p_1) \to (p_2 \to p_3)) \wedge (p_1 \to \neg(p_0 \wedge p_2)),$$

then $\mathscr{S}_\alpha = \{p_2 \vdash p_0, p_1, p_3 \ ;\ p_0, p_1, p_2 \vdash p_3 \ ;\ p_0, p_1, p_2 \vdash\}$. We follow step by step the procedure indicate above:

(1) Turn α into its conjunctive normal form:

$$\begin{aligned}&\texttt{cnf}(\alpha) = (p_0 \vee p_1 \vee \neg p_2 \vee p_3) \wedge (p_1 \vee \neg p_1 \vee \neg p_2 \vee p_3) \wedge \\ &\wedge (p_0 \vee \neg p_0 \vee \neg p_2 \vee p_3) \wedge (\neg p_0 \vee \neg p_1 \vee \neg p_2 \vee p_3) \wedge \\ &\wedge (\neg p_0 \vee \neg p_1 \vee \neg p_2).\end{aligned}$$

(2) Split $\texttt{cnf}(\alpha)$ into the following set of sequents:

$$\begin{aligned}\{ &\vdash p_0, p_1, \neg p_2, p_3 \\ &\vdash p_1, \neg p_1, \neg p_2, p_3 \\ &\vdash p_0, \neg p_0, \neg p_2, p_3 \\ &\vdash \neg p_0, \neg p_1, \neg p_2, p_3 \\ &\vdash \neg p_0, \neg p_1, \neg p_2 \}.\end{aligned}$$

(3) Shift all negative literals on the left so as to obtain the new set:

$$\begin{aligned}\{ &p_2 \vdash p_0, p_1, p_3 \\ &p_1, p_2 \vdash p_1, p_3 \\ &p_0, p_2 \vdash p_0, p_3 \\ &p_0, p_1, p_2 \vdash p_3 \\ &p_0, p_1, p_2 \vdash \}.\end{aligned}$$

(4) Shorten the set by removing identity sequents:

$$\begin{aligned}\{ &p_2 \vdash p_0, p_1, p_3 \\ &p_0, p_1, p_2 \vdash p_3 \\ &p_0, p_1, p_2 \vdash \}.\end{aligned}$$

Remark 2. It is easy to check that $\mathscr{S}_\alpha$ is not necessarily the unique possible decomposition for α since any decomposition may be arbitrarily expanded by weakening applications. If $\Gamma \vdash \Delta \in \mathscr{S}_\alpha$, then $\{\Gamma, \Gamma' \vdash \Delta, \Delta'\} \cup \mathscr{S}_\alpha$ is a decomposition for α as well, provided that $\Gamma, \Gamma' \vdash \Delta, \Delta'$ is complementary.

2.3. **Post-completeness: just another look.** Recall that a (uniform) substitution is a function $\sigma : \mathcal{F} \mapsto \mathcal{F}$, which commutes w.r.t. connectives, i.e. $\sigma(\neg\alpha) = \neg\sigma(\alpha)$ and $\sigma(\alpha \circ \beta) = \sigma(\alpha) \circ \sigma(\beta)$, for $\circ \in \{\wedge, \vee, \rightarrow\}$. The result of applying a substitution σ to a set Γ is the set $\{\sigma(\alpha) \mid \alpha \in \Gamma\}$. A logical system S is said to be closed under US when, for any substitution σ, if $\Gamma \vdash \Delta$ is provable in S, so is $\sigma(\Gamma) \vdash \sigma(\Delta)$.

Our proofs of Lemma 2.2 and Theorem 2.3 below are inspired by the proof of Proposition 1.4.4 given in [5, p. 19].

Lemma 2.2. *A formula α is a tautology if, and only if, $\mathscr{S}_\alpha = \varnothing$.*

Proof. To prove the left-to-right implication, assume that α is a tautology and apply the decomposition algorithm of Theorem 2.1. At step (1), $\mathtt{cnf}(\alpha) = \alpha_1 \wedge \cdots \wedge \alpha_n$ is such that for all α_i there is an atom p_i such that $p_i \vee \neg p_i$ is a subformula α_i. At steps (2) and (3) we have n sequents of the form $\Gamma_1 \vdash \Delta_1$, ... , $\Gamma_n \vdash \Delta_n$ such that for every i, $\Gamma_i \cap \Delta_i \neq \varnothing$. At step (4), they are all removed and $\mathscr{S}_\alpha = \varnothing$.

To prove the right-to-left implication, assume by reductio that α is not a tautology. By applying step (1), $\mathtt{cnf}(\alpha)$ is such that there is some α_i such that for no atom p_i, $p_i \vee \neg p_i$ is a subformula of α_i. Then, by steps (2) and (3), we get n sequents $\Gamma_1 \vdash \Delta_1$, ... , $\Gamma_n \vdash \Delta_n$ and at least one of them is not an identity sequent. Step (4) removes at most $n-1$ sequents, thus $\mathscr{S}_\alpha$ is not empty, against our hypothesis. □

Theorem 2.3 (Post-completeness)**.** *If a nontautological formula α is added to LK_0 as a new axiom schema — i.e., it is considered as closed under US — then LK_0^α is inconsistent.*

Proof. Consider the system $\mathsf{LK}_0^{\mathscr{S}_\alpha}$. If α is nontautological, then, by Lemma 2.2, there is a sequent $\Gamma \vdash \Delta \in \mathscr{S}_\alpha$, such that $\Gamma \cap \Delta = \varnothing$. Moreover, notice that, if a formula α is taken as closed under US, then each sequent in its decomposition $\mathscr{S}_\alpha$ has to be taken, in turn, as closed under US.

Clearly $\Gamma \vdash \Delta$ is derivable in $\mathsf{LK}_0^{\mathscr{S}_\alpha}$ and, given the closure under US, so is $\sigma(\Gamma) \vdash \sigma(\Delta)$, for any substitution σ. Let σ_1 be such that $\sigma_1(p) = \neg p$, for all $p \in \Gamma$, and $\sigma_1(q) = p$, for all $q \in \Delta$. Therefore $\sigma_1(\Gamma) = \neg p$ and $\sigma_1(\Delta) = p$. Hence, $\neg p \vdash p$ which is equivalent to $\vdash p$. Then, consider another substitution σ_2 such that $\sigma_2(p) = p$, for all $p \in \Gamma$, and $\sigma_2(q) = \neg p$, for all $q \in \Delta$. Therefore, $\sigma_2(\Gamma) = p$ and $\sigma_2(\Delta) = \neg p$. Hence, $p \vdash \neg p$ that is equivalent to $p \vdash$. Now, by cut on $\vdash p$ and $p \vdash$, we obtain the empty sequent $\vdash$. We observe that $\mathsf{LK}_0^{\mathscr{S}_\alpha}$ is inconsistent and so is, by Theorem 2.1, LK_0^α. □

Remark 3. All axiomatic extensions of LK_0 preserve decidability, as can be semantically ascertained in a simple way: in order to decide whether a certain formula β is provable in LK_0^{α}, one has to compute the truth table of β and ignore all the Boolean valuations v such that $v(\alpha) = 0$. If the remaining valuations verify β, then it is a theorem of LK_0^{α}.

3. Cut elimination and consistency

In this section, we observe that any extension of LK_0 (which is not, in general, cut-free) admits an equivalent formulation that satisfies cut elimination. The new system is designed to use Theorem 2.1, which asserts that any formula can be equivalently thought of as a set of complementary sequents, and adding the following property of closure under cut.

Definition 2 (closure under cut). Given a set of complementary sequents $\mathscr{S}$ its closure under cut is the smallest supraset $\mathscr{S}^*$ of $\mathscr{S}$ — i.e., $\mathscr{S} \subseteq \mathscr{S}^*$ — such that, if $\Gamma \vdash \Delta, p$ and $\Gamma', p \vdash \Delta'$ are both in $\mathscr{S}^*$ and $(\Gamma \cup \Gamma') \cap (\Delta \cup \Delta') = \varnothing$, then $\Gamma, \Gamma' \vdash \Delta, \Delta' \in \mathscr{S}^*$.

Example 3.1. If $\mathscr{S} = \{p \vdash q \ ; \ q \vdash p \ ; \ q \vdash r\}$, then it easy to check that $\mathscr{S}^* = \{p \vdash q \ ; \ q \vdash p \ ; \ q \vdash r \ ; \ p \vdash r\}$.

Theorem 3.1. For any finite set of sequents $\mathscr{S}$, $\mathscr{S}^*$ is always finite.

Proof. This follows by observing that, on the one hand, the cut of two complementary sequents always produces a complementary sequent; on the other, cut applications do not introduce new atoms. □

Theorem 3.2. For any set of complementary sequents $\mathscr{S}$, $\mathsf{LK}_0^{\mathscr{S}} \approx \mathsf{LK}_0^{\mathscr{S}^*}$.

Proof. $\mathsf{LK}_0^{\mathscr{S}} \preccurlyeq \mathsf{LK}_0^{\mathscr{S}^*}$. Immediately by the fact that $\mathscr{S} \subseteq \mathscr{S}^*$.

$\mathsf{LK}_0^{\mathscr{S}^*} \preccurlyeq \mathsf{LK}_0^{\mathscr{S}}$. Given two sequents $\Gamma \vdash \Delta, p$ and $\Gamma', p \vdash \Delta'$, the sequent $\Gamma, \Gamma' \vdash \Delta, \Delta'$ is clearly derivable by a cut application. □

The possibility to restate Gentzen's Hauptsatz in a stronger way was firstly noticed by Girard in [11]. The basic idea is the following. The cut elimination algorithm works by 'pushing' cut applications upwards along proofs until both premises of each cut application are leaves of the proof tree, i.e., the cut formulas are directly introduced by logical axioms. Logical axioms are closed under cut, and hence the cut rule is dispensable. Once the calculus is enriched with a cluster of complementary proper axioms, cut applications can be pushed upwards in the same way; and, clearly, it suffices to require their closure under cut in order to eliminate cut applications. This claim is imported from [1].

Theorem 3.3 (Strong cut elimination). If a sequent $\Gamma \vdash \Delta$ is derived from a set of complementary sequents $\mathscr{S}$, then there exists a proof of the same sequent from the sequents in $\mathscr{S}$ in which every cut is made on a formula which occurs in some sequent of $\mathscr{S}$.

Now the Hauptsatz follows as a nice corollary. Let LK_{0^-} be the LK_0 sequent system without the cut rule.

Corollary 3.4 (Hauptsatz). For any set of complementary sequents $\mathscr{S}$, if $\Gamma \vdash \Delta$ is provable in $\mathsf{LK}_0^{\mathscr{S}^*}$, then it is provable in $\mathsf{LK}_{0^-}^{\mathscr{S}^*}$ as well.

Corollary 3.5. For any formula α, $\mathsf{LK}_0^{\alpha} \approx \mathsf{LK}_{0^-}^{\mathscr{S}_\alpha^*}$.

Proof. Straightforward, by combining Theorem 3.2 and Corollary 3.4. □

Corollary 3.6 (subformula property). All formulas occurring in an cut-free derivation are subformulas of the formulas from the endsequent.

Proof. As usual, by induction on the length of analytic, i.e. cut-free, proofs. □

Clearly, if you want your extension $\mathsf{LK}_0^{\alpha_1,\dots,\alpha_n}$ to be at the same time consistent and proper, each one of the new axioms has to be a contingent formula, i.e., neither a tautology nor a contradiction. This necessary condition, however, does not exclude the possibility to get a contradiction once $\alpha_1, \dots, \alpha_n$ are put together. To exclude this possibility, one has to compute the truth table of $\alpha_1 \wedge \dots \wedge \alpha_n$ and to check whether it is a contradiction. This is a semantical method for deciding the consistency of supraclassical systems. The following theorem establishes an alternative procedure, fully syntactical, for answering the same question.

Theorem 3.7. A supraclassical system $\mathsf{LK}_0^{\alpha_1,\dots,\alpha_n}$ is consistent if, and only if, the empty sequent does not belong to $\mathscr{S}^*_{\alpha_1\wedge\dots\wedge\alpha_n}$.

Proof. We appeal to the fact that $\mathsf{LK}_0^{\alpha_1,\dots,\alpha_n} \approx \mathsf{LK}_{0^-}^{\mathscr{S}^*_{\alpha_1\wedge\dots\wedge\alpha_n}}$, by Corollary 3.5. ($\Rightarrow$) Clearly, if the empty sequent belonged to $\mathscr{S}^*_{\alpha_1\wedge\dots\wedge\alpha_n}$, then it would be provable in $\mathsf{LK}_0^{\alpha_1\wedge\dots\wedge\alpha_n}$ and so this latter system would be inconsistent. ($\Leftarrow$) If $\mathsf{LK}_0^{\alpha_1,\dots,\alpha_n}$ proved the empty sequent, then it would be also provable in $\mathsf{LK}_{0^-}^{\mathscr{S}^*_{\alpha_1\wedge\dots\wedge\alpha_n}}$. Thus, by Corollary 3.6, the empty sequent would be a proper axiom from $\mathscr{S}^*_{\alpha_1\wedge\dots\wedge\alpha_n}$. □

Remark 4 (infinite bounded axiomatizations). The results achieved hitherto can be easily generalised to extensions obtained by adding an infinite number of proper axioms, provided that these involve a finite number of atoms. Since one obtains a limited number of complementary sequents by arranging a finite set of atoms, infinite axiomatizations with a finite number of atoms admit a finite treatment allowing for cut elimination.

4. Axiomatic decompositions and uniqueness

In this section we prove that, for any cluster of proper axioms $\alpha_1, \dots, \alpha_n$, there exists exactly one axiomatic (in the sense of minimal) extension

$\mathsf{LK}_0^{\mathscr{S}}$ which is, at the same time, cut-free and equivalent to LK_0^{α} (Definition 3). To prove this, the decomposition algorithm designed in the proof of Theorem 2.1 and further refined by Definition 2 is not enough (see Remark 2). If we want decompositions to be minimal, and so unique (Theorem 4.4), we need to go through Definition 4.

Definition 3 (axiomatic sets of sequents). A set of complementary sequents $\mathscr{T}$ is said to be axiomatic in the case where:

(i) $\mathsf{LK}_0^{\mathscr{T}} \approx \mathsf{LK}_{0^-}^{\mathscr{T}}$ (cut elimination),

(ii) no sequent in $\mathscr{T}$ can be derived from the others in LK_{0^-} (minimality).

Definition 4 (reduct under weakening). Given a set of complementary sequents $\mathscr{T}$, its reduct under weakening is the largest subset $\mathscr{T}^\star$ of $\mathscr{T}$ such that, if both $\Gamma \vdash \Delta$ and $\Gamma, \Gamma' \vdash \Delta, \Delta'$ are in $\mathscr{T}$, then $\Gamma, \Gamma' \vdash \Delta, \Delta' \notin \mathscr{T}^\star$.

Example 4.1. We take the set of complementary sequents $\mathscr{S}_\alpha = \{p_2 \vdash p_0, p_1, p_3 \ ; \ p_0, p_1, p_2 \vdash p_3 \ ; \ p_0, p_1, p_2 \vdash\}$ from Example 2.1. Its reduct under weakening $\mathscr{S}_\alpha^\star$ is obtained from $\mathscr{S}_\alpha$ by removing the sequents that are derivable by weakening from the others, namely $p_0, p_1, p_2 \vdash p_3$. Thus, $\mathscr{S}_\alpha^\star = \{p_2 \vdash p_0, p_1, p_3 \ ; \ p_0, p_1, p_2 \vdash\}$.

Proposition 4.1. If α is a contradiction, then $\mathscr{S}_\alpha^{**} = \{ \vdash \}$.

Proof. Straightforwardly, by Theorem 3.7 and Definition 4. □

Lemma 4.2. Let $\mathscr{S}$ be an axiomatic set of sequents. If $\Gamma \vdash \Delta$ is a complementary sequent derivable in $\mathsf{LK}_0^{\mathscr{S}}$, then there is a proof of it that is simply a single-branching tree built up from a certain complementary sequent in $\mathscr{S}$ by possible successive applications of the weakening rules.

Proof. If $\Gamma \vdash \Delta$ is provable in $\mathsf{LK}_0^{\mathscr{S}}$ and $\mathscr{S}$ is axiomatic, then it is provable by a cut-free proof π. Being $\Gamma \vdash \Delta$ complementary and π cut-free, this latter must be a chain of sequents built up from one of the proper axioms in $\mathscr{S}$ by, possible, successive applications of the weakening rules. □

Theorem 4.3. For any α, $\mathscr{S}_\alpha^{**}$ is an axiomatic decomposition for α.

Proof. We firstly prove that $\mathscr{S}_\alpha^{**}$ is a decomposition for α by proving that $\mathsf{LK}_0^{\mathscr{S}_\alpha^{**}} \approx \mathsf{LK}^{\mathscr{S}_\alpha^{*}}$. Clearly, $\mathscr{S}_\alpha^{**} \subseteq \mathscr{S}_\alpha^{*}$ and so $\mathsf{LK}_0^{\mathscr{S}_\alpha^{**}} \preccurlyeq \mathsf{LK}^{\mathscr{S}_\alpha^{*}}$. Conversely, all the sequents in $\mathscr{S}_\alpha^{*} \backslash \mathscr{S}_\alpha^{**}$ are provable from those in $\mathscr{S}_\alpha^{**}$ by weakening applications, hence $\mathsf{LK}_0^{\mathscr{S}_\alpha^{**}} \succcurlyeq \mathsf{LK}_0^{\mathscr{S}_\alpha^{*}}$.

Let us now prove that $\mathscr{S}_\alpha^{**}$ is axiomatic. Clearly, the very fact of switching from $\mathscr{S}_\alpha^{*}$ to $\mathscr{S}_\alpha^{**}$ does not hamper the cut elimination algorithm, thus $\mathscr{S}_\alpha^{**}$ meets condition (i) in Definition 3. Concerning minimality, suppose there were a sequent $\Gamma \vdash \Delta$ in $\mathscr{S}_\alpha^{**}$ derivable from the others by a cut-free proof π. By Lemma 4.2, $\Gamma \vdash \Delta$ would have been obtained by weakening applications from a certain sequent in $\mathscr{S}_\alpha^{**}$ and, thus, $\mathscr{S}_\alpha^{**}$ would not be a reduct under weakening applications. Then, we conclude that $\mathscr{S}_\alpha^{**}$ meets also condition (*ii*) in Definition 3. □

Theorem 4.4. If both $\mathscr{S}$ and $\mathscr{T}$ are axiomatic and $\mathsf{LK}_0^{\mathscr{S}} \approx \mathsf{LK}_0^{\mathscr{T}}$, then $\mathscr{S} = \mathscr{T}$.

Proof. Suppose, by reductio, that $\Gamma \vdash \Delta \in \mathscr{S} \backslash \mathscr{T}$. Since $\mathsf{LK}_0^{\mathscr{S}} \approx \mathsf{LK}_0^{\mathscr{T}}$, $\Gamma \vdash \Delta$ would be provable in $\mathsf{LK}_0^{\mathscr{T}}$ as well. By Lemma 4.2, there is an $\mathsf{LK}_0^{\mathscr{T}}$ proof π of $\Gamma \vdash \Delta$ shaped as follows:

$$\cfrac{\cfrac{\cfrac{}{\Gamma' \vdash \Delta'}\text{seq. from } \mathscr{T}}{\Gamma' \vdash \Delta}\vdash \textit{weak}}{\Gamma \vdash \Delta}\textit{weak} \vdash$$

with $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ (but $\Gamma' \neq \Gamma$ or $\Delta' \neq \Delta$). Now, $\Gamma' \vdash \Delta' \notin \mathscr{S}$ and, thus, for similar reasons, $\Gamma' \vdash \Delta'$ would be provable in $\mathsf{LK}_0^{\mathscr{S}}$ by a proof π' shaped in the same way, i.e.:

$$\cfrac{\cfrac{\cfrac{}{\Gamma'' \vdash \Delta''}\text{seq. from } \mathscr{S}}{\Gamma'' \vdash \Delta'}\vdash \textit{weak}}{\Gamma' \vdash \Delta'}\textit{weak} \vdash$$

with, this time, $\Gamma'' \subseteq \Gamma'$ and $\Delta'' \subseteq \Delta'$ (but $\Gamma'' \neq \Gamma'$ or $\Delta'' \neq \Delta'$). Finally, being $\Gamma'' \subseteq \Gamma$ and $\Delta'' \subseteq \Delta$ (but, again, $\Gamma'' \neq \Gamma$ or $\Delta'' \neq \Delta$), $\Gamma \vdash \Delta$ would be derivable from $\Gamma'' \vdash \Delta''$ just by a series of weakening applications. Hence, $\Gamma \vdash \Delta$ cannot be in $\mathscr{S}$ against our assumption. □

Corollary 4.5. If α and β are equivalent in LK_0, then $\mathscr{S}_\alpha^{**} = \mathscr{S}_\beta^{**}$.

Proof. By Theorem 4.3, $\mathscr{S}_\alpha^{**}$ and $\mathscr{S}_\beta^{**}$ are both axiomatic and, clearly, $\mathsf{LK}_0^{\alpha} \approx \mathsf{LK}_0^{\beta}$. Then apply the previous theorem. □

5. Concluding remarks and future work

We are confident that the methodology expounded and developed in this paper is general enough to provide a proof-theoretic foundation of other supraclassical logics, including the nonmonotonic ones. More specifically, we plan to extend our approach to the logic of default assumptions, which is a kind of nonmonotonic logic obtained from pivotal assumption logic by shrinking the set of assumptions to those that are maximally consistent with the premises ([14, p. 30]).

From the nonmonotonic vantage point, it would be also interesting to combine the positive role of the information in the supraclassical setting with the negative information encoded by control sets, which specify an array of prohibitions preventing a given conclusion. Control sets are well-behaved with respect to proof-theoretical properties, in particular cut elimination [3, 8, 17]. More precisely, we want to investigate "controlled" calculi where control sets are injected into logical proofs by proper axioms and analyse what happens in the decomposition process.

Finally, a semantical issue to consider is the possibility of extending Carnielli's Polynomial Ring Calculus to supraclassical logics [4].

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