OSCILLATORY SOLUTIONS TO TRANSPORT EQUATIONS

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ABSTRACT. Let $n \geq 3$. We show that there is no topological vector space $X \subset L^\infty \cap L^1_{\text{loc}}(\mathbb{R} \times \mathbb{R}^n)$ which embeds compactly in L^1_{loc}, contains $BV_{\text{loc}} \cap L^\infty$ and enjoys the following closure property: If $f \in X^n(\mathbb{R} \times \mathbb{R}^n)$ has bounded divergence and $u_0 \in X(\mathbb{R}^n)$, then there exists $u \in X(\mathbb{R} \times \mathbb{R}^n)$ which solves

$$\begin{cases} \partial_t u + \text{div}\,(uf) \;=\; 0 \\[2mm] u(0, \cdot) \;=\; u_0 \end{cases}$$

in the sense of distributions. $X(\mathbb{R}^n)$ is defined as the set of functions $u_0 \in L^\infty(\mathbb{R}^n)$ such that $\tilde{u}(t, x) := u_0(x)$ belongs to $X(\mathbb{R} \times \mathbb{R}^n)$. Our proof relies on an example of N. Depauw showing an ill–posed transport equation whose vector field is "almost BV".

1. INTRODUCTION

Consider the following system of hyperbolic conservation laws in n space dimensions

$$\begin{cases} \partial_t u + \text{div}_x(f(|u|)u) \;=\; 0 \\[2mm] u(0, \cdot) \;=\; \overline{u}(\cdot)\,. \end{cases} \tag{1}$$

System (1) is very special since it can be decoupled in a scalar conservation law for the modulus $\rho := |u|$

$$\partial_t \rho + \text{div}_x(f(\rho)\rho) \;=\; 0 \tag{2}$$

and a linear transport equation with variable coefficients for the angular part $\theta := u/|u|$:

$$\partial_t(\rho\theta) + \text{div}_x(f(\rho)\rho\theta) \;=\; 0\,. \tag{3}$$

Therefore it is natural to consider weak solutions u of (1) such that $\rho := |u|$ are Kruzkov solutions of (2). These solutions will be called *renormalized entropy solutions.* Even in the presence of this special "triangular" structure, in [4] the author proved that the Cauchy problem (1) can be ill posed in L^∞, showing initial data $\overline{u}$ which generate renormalized entropy solutions with wild oscillations. However in [3] the authors showed the well–posedness of renormalized entropy solutions in the class of maps $u \in L^\infty(\mathbb{R}^m, \mathbb{R}^k)$ such that $|u| \in BV_{\text{loc}}$. In this case Kruzkov solution ρ of (2) enjoys BV regularity. In [3] the authors used the recent results of [2] which extend the DiPerna–Lions theory of renormalized solutions of transport equations to BV coefficients with bounded divergence.

In order to handle different situations (for instance that of two transport equations coupled through some nonlinearity) it seems desirable to have a function space $X \subset L^\infty$

- which embeds compactly in L^1_{loc},
- which contains the functions with jump discontinuities,
- and such that solutions to transport equations which coefficients in X and initial data in X belong to X.

The question of the existence of such a space was originally raised by Alberto Bressan. In this note we show that, if we ask in addition that X contains BV functions, then such a space does not exist (compare with Question 1.1 and Theorem 1.2).

In what follows we always consider the spaces $L^1_{\mathrm{loc}} \cap L^\infty(\mathbb{R}^n)$ and $BV_{\mathrm{loc}} \cap L^\infty(\mathbb{R}^n)$ endowed with the following topologies:

- $u_k \to u$ in $L^1_{\mathrm{loc}} \cap L^\infty(\mathbb{R}^n)$ if $\|u_k\|_\infty$ is uniformly bounded and $\|u_k - u\|_{L^1(U)}$ converges to 0 for every open set $U \subset\subset \mathbb{R}^n$;
- $u_k \to u$ in $BV_{\mathrm{loc}} \cap L^\infty(\mathbb{R}^n)$ if $\|u_k\|_\infty$ is uniformly bounded and $\|u_k - u\|_{BV(U)}$ converges to 0 for every open set $U \subset\subset \mathbb{R}^n$.

Given a function space $\mathcal{S}(\mathbb{R} \times \mathbb{R}^n) \subset L^1_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^n)$ we define $\mathcal{S}(\mathbb{R}^n)$ as those functions $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that their trivial extension $\tilde{f}(t, x) := f(x)$ belongs to $\mathcal{S}(\mathbb{R} \times \mathbb{R}^n)$. Moreover we define $\mathcal{S}^k(\mathbb{R}^n)$ as the space of maps $u : \mathbb{R}^n \to \mathbb{R}^k$ such that each component belongs to $\mathcal{S}(\mathbb{R}^n)$.

Question 1.1. *Is there a complete topological vector space $\mathcal{S}(\mathbb{R} \times \mathbb{R}^n)$ such that*

$$BV_{\mathrm{loc}} \cap L^\infty(\mathbb{R}_t \times \mathbb{R}^n_x) \quad \subset \quad \mathcal{S}(\mathbb{R}_t \times \mathbb{R}^n_x) \quad \subset \quad L^1_{\mathrm{loc}} \cap L^\infty(\mathbb{R}_t \times \mathbb{R}^n_x)$$

with the following properties?

- *The topology of $\mathcal{S}(\mathbb{R} \times \mathbb{R}^n)$ is finer than the topology of $L^1_{\mathrm{loc}} \cap L^\infty$ and coarser than the topology of $BV_{\mathrm{loc}} \cap L^\infty$;*
- *Bounded subsets of $\mathcal{S}(\mathbb{R} \times \mathbb{R}^n)$ are relatively compact in $L^1_{\mathrm{loc}} \cap L^\infty$;*
- *If $\overline{u} \in \mathcal{S}(\mathbb{R}^n)$, $g \in \mathcal{S}^n(\mathbb{R} \times \mathbb{R}^n)$, and $\mathrm{div}_x\, g \in L^\infty(\mathbb{R}_t \times \mathbb{R}^n_x)$, then there exists a (possibly non unique) $u \in L^\infty(\mathbb{R}^+, \mathcal{S}(\mathbb{R}^n))$ which solves distributionally*

$$\begin{cases} \partial_t u + \mathrm{div}_x(gu) = 0 \\ u(0, \cdot) = \overline{u}(\cdot). \end{cases} \tag{4}$$

In this paper we prove

Theorem 1.2. *The answer to Question 1.1 is negative for $n \geq 3$.*

Our proof can be described in the following way. We fix $n = 3$ and we consider a small modification of a construction of Depauw (see [6]) which yields

- an autonomous divergence free vector field $u \in L^\infty$,
- an initial data $\overline{v} \in L^\infty \cap BV$,
- and a time $T > 0$

such that in $[0, T] \times \mathbb{R}^3$ there exists a unique bounded weak solution of

$$\begin{cases} \partial_t v + \operatorname{div}_x(uv) = 0 \\ v(0, \cdot) = \overline{v}(\cdot), \end{cases} \tag{5}$$

and $u(t, \cdot)$ converges to $u(T, \cdot)$ weakly* in L^∞ but not strongly in L^1_{loc}.

Next we show the existence of coefficients $f_i \in BV \cap L^\infty$ with div $f_i \in L^\infty$ and initial data $\overline{w}_i \in BV \cap L^\infty$ such that if we consider the unique weak solutions of

$$\begin{cases} \partial_t w_i + \operatorname{div}_x(f_i w_i) = 0 \\ w_i(0, \cdot) = \overline{w}_i(\cdot), \end{cases} \tag{6}$$

then $w_i(1, \cdot)$ is the i-th component $u_i(\cdot)$ of Depauw's vector field.

It would be very interesting to understand whether one can use similar constructions to produce hyperbolic systems of conservation laws $\partial_t U + \operatorname{div}_x[F(U)] = 0$ and BV initial data with highly oscillatory admissible solutions. Slight modifications of our example produce fluxes F such that each DF_i is triangular, but the corresponding systems are not hyperbolic.

2. TRANSPORT EQUATION AND DEPAUW'S EXAMPLE OF NON–UNIQUENESS

Following some ideas of Aizenman ([1]), Depauw in [6] and Colombini, Luo and Rauch in [5] have recently given some counterexamples to the uniqueness of (4) when $g \in L^\infty$ and $\operatorname{div}_x g = 0$ (see also [4] for a related construction). The example of [6] is the starting point of the proof of Theorem 1.2. It consists of a bounded planar divergence–free vector field $a(t, x)$ with two different distributional solutions of

$$\begin{cases} \partial_t w + \operatorname{div}_x(aw) = 0 \\ w(0, \cdot) = 0. \end{cases} \tag{7}$$

First of all we define $b : [-1/2, 1/2]^2 \to \mathbb{R}^2$ as

$$b(x_1, x_2) = \begin{cases} (0, 4x_1) & \text{if } 0 < |x_2| < |x_1| < 1/4 \\ (-4x_2, 0) & \text{if } 0 < |x_1| < |x_2| < 1/4 \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

and we extend it periodically to $\mathbb{R}^2$ (see Figure 1). The field $a(t, x)$ is then given by

$$a(t, x) = \begin{cases} 0 & \text{if } t < 0 \text{ or } t > 1 \\ b(2^j x) & \text{if } t \in I_j = 2^{-j}(\frac{1}{2}, 1) \text{ for some } j \in \mathbb{N}, \end{cases}$$

and for later use we define $c(t, x) = a(1 - t, x)$. It follows immediately that a and c are bounded and divergence–free. Moreover, note that for every $t \in \mathbb{R}$ we have $a(t, \cdot) \in BV_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ but $a \notin L^1([0, t], BV_{\text{loc}})$.
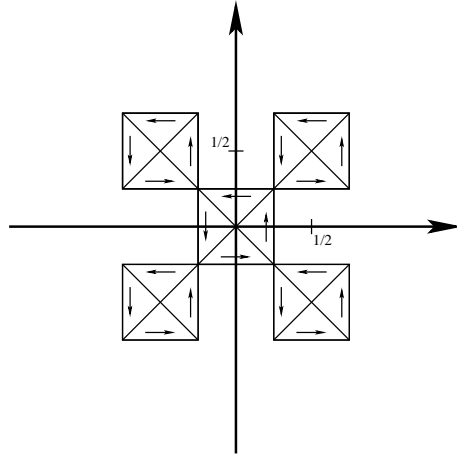
FIGURE 1. Depauw's vector field

We briefly describe the flow of c. First of all, for $0 \le t_0 < t_1 < 1$ we denote by $X^{(c)}(t_0, t_1, x)$ the solution at time t_1 of the problem

$$\begin{cases} \dot{\psi}(t) = c(t, \psi(t)) \\ \psi(t_0) = x \,. \end{cases}$$

Note that $X^{(c)}$ is well defined since c is piecewise smooth on $[t_0, t_1] \times \mathbb{R}^2$. Next we let u_0 be the $\mathbb{Z}^2$-periodic function given by $u_0(x_1, x_2) = \operatorname{sgn}(x_1 x_2)$ on the square $\left(-\frac{1}{2}, \frac{1}{2}\right)^2$. If we define $u_k(x) = u_0(2^k x)$, then $u_k\left(X^{(c)}\left(1 - 2^{-k}, 1 - 2^{-k+1}, x\right)\right) = u_{k+1}(x)$ (see Figure 2).

By the semigroup property of $X^{(c)}$ we conclude that $u_0(X^{(c)}(0, 1 - 2^{-k}, x)) = u_k$. Therefore $u(t, x) := u_0\left(X^{(c)}(t, 1, x)\right)$ is a bounded distributional solution of

$$\begin{cases} \partial_t u + \operatorname{div}_x(uc) = 0 \\ u(0, \cdot) = u_0(\cdot) \,. \end{cases} \tag{9}$$

Note that $u(t, \cdot)$ converges weakly (but not strongly) to 0 as $t \uparrow 1$. Therefore w defined by $w(t, x) = u(1 - t, x)$ is a nontrivial weak solution of (7).

3. PROOF OF THEOREM 1.2

In the previous section c was defined as a vector field on $\mathbb{R}^2_x$ dependent on the time variable t. With a small abuse of notation, from now on we denote by $c : \mathbb{R}^3 \to \mathbb{R}^2$ the map defined by $c(x_1, x_2, x_3) = c(t, x_1, x_2)$ for $x_3 \in [0, 1]$ and 0 otherwise. Theorem 1.2 will be a consequence of the following two lemmas.

Lemma 3.1. *If $\mathcal{S}(\mathbb{R} \times \mathbb{R}^3)$ satisfies the first and third assumptions of Question 1.1, then $c \in \mathcal{S}^2(\mathbb{R}^3)$.*
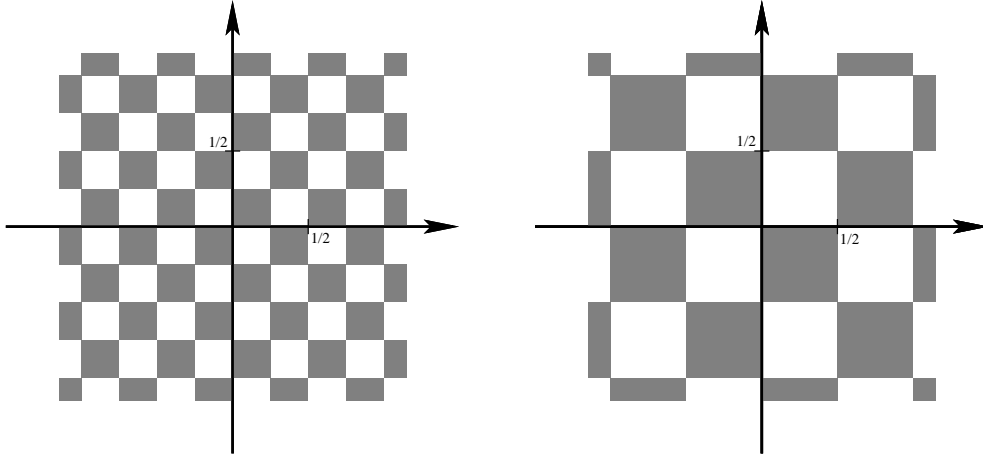
FIGURE 2. The effect of Depauw's vector field b acting for a time of $1/2$ on a chessboard of side $1/4$

Lemma 3.2. *There exists a field $\chi \in BV \cap L^\infty(\mathbb{R}_t \times \mathbb{R}^2_x, \mathbb{R}^2)$ with the following properties:*

- $\chi = 0$ *for* $(x_1, x_2) \notin [-1/4, 3/4]^2$;
- $\mathrm{div}_x \chi$ *is a bounded function;*
- *If b_1 is the first component of the field b defined in (8), $I = 2 + \int_{[-1/4,3/4]^2} b_1(x)\, dx$ and ω is the unique solution of*

$$\begin{cases} \partial_t \omega + \mathrm{div}_x(\omega \chi) = 0 \\ \omega(0, \cdot) = I, \end{cases} \tag{10}$$

  then

$$\omega(1, x_1, x_2) = 2 + b_1(x_1, x_2) \qquad \text{for } (x_1, x_2) \in [-1/4, 3/4]^2. \tag{11}$$

Note that Lemma 3.2 makes sense because the solution of (10) is unique and belongs to $C(\mathbb{R}^+, L^1_{\mathrm{loc}} \cap L^\infty)$, thanks to the recent results of [2]. We postpone the proof of Lemma 3.2 to the next section and we give here the proofs of Lemma 3.1 and Theorem 1.2.

Proof of Lemma 3.1. We show that there exists a vector field $\beta \in BV_{\mathrm{loc}} \cap L^\infty(\mathbb{R}_t \times \mathbb{R}^3_x, \mathbb{R}^3)$ such that the following holds. If I is the constant of Lemma 3.2 and z is the unique solution in $C(\mathbb{R}, L^1_{\mathrm{loc}} \cap L^\infty)$ of

$$\begin{cases} \partial_t z + \mathrm{div}_x(z\beta) = 0 \\ z(0, \cdot) = I, \end{cases} \tag{12}$$

then $z(1, x) = 2 + c_1(1, x)$. Since $z \in C(\mathbb{R}^+, L^1_{\mathrm{loc}} \cap L^\infty)$ and $\mathcal{S}$ is complete, we conclude $c_1 = (z(1, \cdot) - 2) \in \mathcal{S}(\mathbb{R}^3)$. Since c_2 is given by rotating c_1 of $90°$, we then get the conclusion of the lemma.

It remains to construct β. We set $\beta_3 = 0$. Next we define

$$\chi^k(t, x_1, x_2) \; := \; 2^{-k}\chi(t, 2^k x_1, 2^k x_2)\,, \tag{13}$$

where χ is as in Lemma 3.2. For $x_3 \in [1 - 2^{-k}, 1 - 2^{-(k+1)}]$ we define

$$(\beta_1, \beta_2)(t, x_1, x_2, x_3) \; := \; \chi^k(t, x_1, x_2) \qquad \text{for } x_1, x_2 \in \left[-\frac{2^{-k}}{4}, 2^{-(k+1)}\frac{3}{4}\right] \tag{14}$$

and we extend it to $(x_1, x_2) \in \mathbb{R}^2$ by $2^{-k}\mathbb{Z}^2$-periodicity. For $x_3 \notin [0, 1]$ we simply set $(\beta_1, \beta_2) = 0$.

Clearly, β is bounded. Moreover note that $\|\mathrm{div}_{x_1, x_2}\chi^k\|_\infty = \|\mathrm{div}_{x_1, x_2}\chi\|_\infty$, and therefore $\mathrm{div}_x\beta \in L^\infty$. To check that $\beta \in BV_{\mathrm{loc}}$, we partition $\mathbb{R}_t \times \mathbb{R}^2 \times [0, 1]$ in the following way: each stripe $[0, 1] \times \mathbb{R}^2 \times [1 - 2^{-k}, 1 - 2^{-k-1}]$ is subdivided in a family $\mathcal{R}_k$ of rectangles of sides 1 in the time direction, 2^{-k-1} in the x_3 direction and 2^{-k} in the x_1, x_2 directions. In the spatial directions x_1, x_2 these rectangles are centered on the points $(2^{-k}i/4, 2^{-k}j/4)$, for $(i, j) \in \mathbb{Z}^2$.

For any given $L > 0$, the BV norm of β in $\Omega = [0, 1] \times [-L, L]^2 \times [0, 1]$ can be estimated by summing the BV norm of β in the rectangles intersecting Ω and the L^1 norm of the jump of β along the surfaces of such rectangles.

Note that the number $N(k)$ of members of $\mathcal{R}_k$ intersecting Ω is of order $2^{2k}L^2$. If R is a rectangle of $\mathcal{R}_k$ then:

- $\|\beta\|_{BV(R)} \le 2^{-k+1}\|\chi^k\|_{BV} = 2^{-k+1}\left(2^{-k}\right)^2\|\chi\|_{BV}$;
- The area of the boundary of R is of order 2^{-2k}, whereas the jump of β along such surface is of order 2^{-k}; therefore the L^1 norm of the jump on the boundary of R is of order 2^{-3k}.

Hence we get

$$\|\beta\|_{BV(\Omega)} \; \le \; \sum_k CN(k)2^{-3k} \; \le \; C\sum_k 2^{2k}2^{-3k} < \infty\,. \tag{15}$$

It remains to check that the unique solution $z \in C([0, 1], L^1_{\mathrm{loc}} \cap L^\infty)$ of (12) satisfies $z(1, x) = 2 + c_1(x)$. Let ω be the unique function in $C(\mathbb{R}, L^1_{\mathrm{loc}} \cap L^\infty)$ solving (10). We define

$$\omega^k(t, x_1, x_2) \; := \; \omega(t, 2^k x_1, 2^k x_2)\,.$$

Clearly,

$$\partial_t \omega^k + \mathrm{div}_x(\omega^k\chi^k) \; = \; 0\,.$$

We define a function $z \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^3)$ as

- Constantly equal to I outside the stripe $[0, 1] \times \mathbb{R}^2 \times [0, 1]$;
- On any rectangle of $\mathcal{R}_k$, z is the obvious translation of ω_k.

One readily checks that

- $\partial_t z + \mathrm{div}_x(\beta z) = 0$;
- $z(0, x) = I$;
- $z(1, x) = 2 + c_1(x)$;
- $z \in C([0, 1], L^1_{\mathrm{loc}} \cap L^\infty)$.

This completes the proof. $\qquad\square$

Proof of Theorem 1.2. First of all notice that it suffices to prove the theorem in the case $n = 3$, since $\mathcal{S}(\mathbb{R} \times \mathbb{R}^{n-1})$ inherits the three properties satisfied by $\mathcal{S}(\mathbb{R} \times \mathbb{R}^n)$. For $n = 3$ let $\mathcal{S}$ be any space satisfying the first and third assumptions of Question 1.1 and consider $g(x)$ given by $(g_1, g_2)(x) = c(x)$, $g_3(x) = 1$. Define

$$v_0(x_1, x_2, x_3) \;:=\; u_0(x_1, x_2)\mathbf{1}_{[-1,0]}(x_3)\,, \tag{16}$$

$$v(t, x_1, x_2, x_3) \;:=\; u(x_3, x_1, x_2)\mathbf{1}_{[-1,0]}(x_3 - t)\,, \tag{17}$$

where u_0 and u are defined as in the previous section. The following "formal" computation can be easily justified by regularizing $\mathbf{1}_{[-1,0]}$:

$$\partial_t v + \operatorname{div}_x(gv) \;=\; \mathbf{1}_{[-1,0]}(x_3 - t)\big[\partial_{x_3} u(x_3, x_1, x_2) + \operatorname{div}_{x_1,x_2}(u(x_3, x_1, x_2)c(x_3, x_1, x_2))\big] \;=\; 0\,. \tag{18}$$

We claim that the function $v(t, x)$ defined in (17) is the unique solution of (18) with initial data (16). This would conclude the proof of the theorem: since $v_0 \in BV_{\mathrm{loc}} \cap L^\infty \subset \mathcal{S}(\mathbb{R}^3)$, the claim together with Lemma 3.1 and the third property of $\mathcal{S}$ would imply $v \in L^\infty(\mathbb{R}^+, \mathcal{S}(\mathbb{R}^3))$; on the other hand $\{v(t, \cdot)\}_{t \in [0,1]}$ is not strongly precompact in $L^1_{\mathrm{loc}} \cap L^\infty$, therefore $\mathcal{S}$ would not satisfy the second assumption of Question 1.1.

Let us now prove the claim. Note that this is not in contradiction with what observed in the previous section, since the 2-dimensional transport equation of Depauw would correspond to the case $g_3 = -1$, whereas we have $g_3 = 1$.

Let us assume that $\tilde{v}$ solves $\partial_t v + \operatorname{div}_x(gv) = 0$ with initial data (16). If we set $w = \tilde{v} - v$ we find

$$\begin{cases} \partial_t w + \operatorname{div}(gw) \;=\; 0 \\[2mm] w(0, \cdot) \;=\; 0\,. \end{cases} \tag{19}$$

We define $w = 0$ on $\{t \leq 0\}$. Then w solves $\partial_t w + \operatorname{div}(gw) = 0$ in the sense of distributions on the whole $\mathbb{R}^4$.

Consider the distribution $T := \partial_t w^2 + \operatorname{div}(gw^2)$ and for every $\varepsilon > 0$ define the set

$$E_\varepsilon \;=\; \mathbb{R}_t^+ \times \mathbb{R}^3 \setminus \{|x_3 - 1| \leq \varepsilon\}\,.$$

The vector field g is divergence free and piecewise smooth on each set E_ε. Therefore the renormalization property holds and we conclude that $T = 0$ on each E_ε (for instance because of the results of [2]; however in the case at hand the renormalization property can be shown by elementary arguments). The arbitrariness of ε implies that T is a distribution concentrated on $\{x_3 = 1\}$.

Next we show that $w = 0$ for $x_3 \leq 1$. Fix $\tau, L > 0$, $\gamma \in]0, 1[$ and consider the trapezoid $\Omega \subset \mathbb{R} \times \mathbb{R}^3$ given by

$$\Big([-\tau, \tau] \times [-L, L]^2 \times [-L, \gamma]\Big) \;\cup\; \Big\{(x_1, x_2) \in [-L, L]^2\,, \; x_3 \leq -L\,, \; -\tau \leq t \leq (x_3 + 3L)/2\Big\} \tag{20}$$

(see Figure 3).
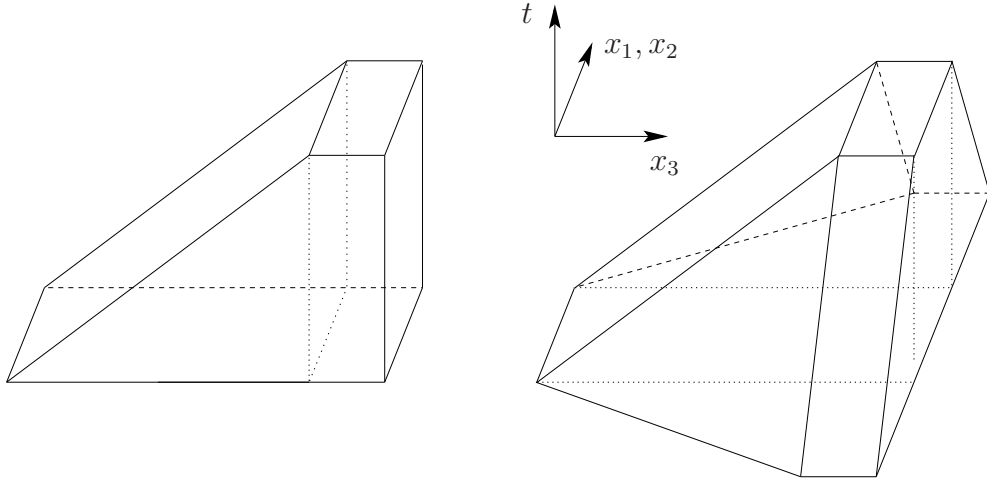
FIGURE 3. The polytopes Ω and Ω'.

Denote by ν the exterior unit normal to $\partial\Omega$. Clearly $\partial\Omega$ is the union of a finite number of prisms and one can easily check that

$$\nu \cdot (1, g) \ \geq \ 0 \qquad \text{on } \{x_3 = \gamma\} \cup \{2t = x_3 + 3L\}$$

and

$$\nu \cdot (1, g) \ = \ 1 \qquad \text{on } \{t = \tau\}.$$

Similarly it is not difficult to see that one can find a larger polytope Ω' of the form shown in Figure 3 such that

$$\nu \cdot (1, g) \ \geq \ 0 \qquad \text{on } \partial\Omega' \setminus \{t = -\tau\}.$$

If w and g were smooth, since $w^2 = 0$ on $\{t = -\tau\}$ and $\partial_t w^2 + \mathrm{div}_x(w^2 g) = 0$ on Ω', we could integrate by parts to conclude

$$\int_{x \in [-L,L] \times [-L,\gamma]} w^2(\tau, x)\, dx \ = \ -\int_{\partial\Omega' \setminus \{t=\tau,-\tau\}} w^2(1, g) \cdot \nu \ \leq \ 0. \tag{21}$$

Therefore we would conclude $w^2 = 0$ on $\{\tau\} \times [-L, L]^2 \times [\gamma, L]$. Choosing γ, L, and τ arbitrarily, we would conclude $w = 0$ on $\{x_3 \leq 1\}$. In order to make this argument rigorous we fix a standard family of mollifiers ρ_ε supported on the ball $B_\varepsilon(0) \subset \mathbb{R}^4$. Applying the same argument above to the vector field $(w^2 * \rho_\varepsilon, (w^2 g) * \rho_\varepsilon))$, we conclude that $w^2 * \rho_\varepsilon = 0$ on $\{\tau\} \times [-L, L]^2 \times [-L, \gamma]$ for every $\tau > \varepsilon$, $L > 0$ and $\gamma \in]0, 1[$. Therefore we conclude $w^2 * \rho_\varepsilon = 0$ on $\{0 \leq x_3 < 1\} \cap \{t > \varepsilon\}$. Letting $\varepsilon \downarrow 0$ we get $w^2 = 0$ on $x_3 < 1$.

Summarizing:

- w solves (19);
- $w = 0$ on $\{x_3 < 1\}$;
- $g(t, x) = (0, 0, 1)$ on $\{x_3 > 1\}$.

Therefore w solves, in the sense of distributions, the following mixed boundary value problem:

$$\begin{cases} \partial_t w + \partial_{x_3} w = 0 & \text{on } x_3 > 1, t > 0 \\ \\ w(0, x) = 0 & \text{for } x_3 > 1 \\ w(t, x_1, x_2, 1) = 0 & \text{for } t > 0 \text{ and } (x_1, x_2) \in \mathbb{R}^2. \end{cases} \tag{22}$$

Clearly, this implies $w = 0$ also on $x_3 \geq 1$ and hence completes the proof. $\qquad\square$

4. Proof of the technical Lemma 3.2

Proof. We define two functions $\tilde{z}, \overline{z} \in BV(\mathbb{R}^2)$ in the following way:

$$\tilde{z}(x) := \begin{cases} 2 + b_1(x) & \text{for } x \in [-1/4, 3/4]^2 \\ I & \text{otherwise}, \end{cases} \tag{23}$$

$$\overline{z}(x) := \begin{cases} 2 + \int_{-1/4}^{3/4} b_1(x_1, \xi) \, d\xi & \text{for } x \in [-1/4, 3/4]^2 \\ I & \text{otherwise}. \end{cases} \tag{24}$$

It suffices to show the existence of two vector fields $\tilde{\chi}$ and $\overline{\chi}$ such that:

- The unique solution $\overline{\omega} \in C([0, 1], L^1_{\text{loc}} \cap L^\infty)$ of

$$\begin{cases} \partial_t \overline{\omega} + \text{div}_x(\overline{\omega}\,\overline{\chi}) = 0 \\ \\ \overline{\omega}(0, \cdot) = I \end{cases} \tag{25}$$

  satisfies $\overline{\omega}(1, \cdot) = \overline{z}$;
- The unique solution $\tilde{\omega} \in C([0, 1], L^1_{\text{loc}} \cap L^\infty)$ of

$$\begin{cases} \partial_t \tilde{\omega} + \text{div}_x(\tilde{\omega}\tilde{\chi}) = 0 \\ \\ \tilde{\omega}(0, \cdot) = \overline{z} \end{cases} \tag{26}$$

  satisfies $\tilde{\omega}(1, \cdot) = \tilde{z}$.

The ω and χ we are looking for will be finally given by

$$\omega(t, x) = \begin{cases} \overline{\omega}(2t, x) & \text{for } t \in [0, 1/2] \\ \tilde{\omega}(2t - 1, x) & \text{for } t \in [1/2, 1], \end{cases}$$

$$\chi(t, x) = \begin{cases} \overline{\chi}(2t, x) & \text{for } t \in [0, 1/2] \\ \tilde{\chi}(2t - 1, x) & \text{for } t \in [1/2, 1]. \end{cases}$$

Step 1: Construction of $\overline{\omega}$ and $\overline{\chi}$.

We set $\overline{\omega}(t, x) = (1 - t)I + t\overline{z}(x)$, $\overline{\chi}_2(x) = 0$ and

$$\overline{\chi}_1(t, x) = \frac{1}{\overline{\omega}(t, x)} \int_{-1/4}^{x_1} \big(\overline{z}(\xi, x_2) - I\big) \, d\xi.$$

Note that

- $\overline{\chi}$ is well defined because $\overline{\omega}(t, x) \geq 1$, and thus (25) holds;

- $\overline{\chi}(t, x_1, x_2) = 0$ if $x_2 \geq 3/4$ and $x_2 \leq -1/4$, because in this case $\overline{\omega}(t, \xi, x_2) = I$;
- $\overline{\chi}(t, x_1, x_2) = 0$ if $x_1 \leq -1/4$, because $\overline{\omega}(t, \xi, x_2) = I$ for $\xi \leq -1/4$;
- $\overline{\chi}(t, x_1, x_2) = 0$ for $x_1 = 3/4$, because

$$\int_{-1/4}^{3/4} \overline{z}(\xi, x_2) \, d\xi \; = \; I \, ;$$

- $\overline{\chi}(t, x_1, x_2) = 0$ for $x_1 \geq 3/4$, because in this case

$$\int_{-1/4}^{x_1} (\overline{z}(\xi, x_2) - I) \, d\xi \; = \; \int_{-1/4}^{3/4} (\overline{z}(\xi, x_2) - I) \, d\xi + \int_{-3/4}^{x_1} (\overline{z}(\xi, x_2) - I) \, d\xi \; = \; 0 \, .$$

Therefore $\overline{\chi}(t, x) = 0$ for $x \notin [-1/4, 3/4]^2$. Note that

$$\partial_{x_1} \overline{\chi}_1(t, x) \; = \; \frac{\overline{z}(x) - I}{\overline{\omega}(t, x)} - \frac{t \partial_{x_1} \overline{z}(x)}{\overline{\omega}^2(t, x)} \int_{-1/4}^{x_1} (\overline{z}(\xi, x_2) - I) \, d\xi \, .$$

Therefore $\mathrm{div}_x \overline{\chi} = \partial_{x_1} \overline{\chi}_1 \in L^\infty$.

A similar computation shows that $\partial_t \overline{\chi}_1$ is bounded. Finally, observe that $\partial_{x_2} \overline{\chi}_1$ is a bounded measure concentrated on the two segments $\{x_2 \in \{-1/4, 3/4\}, x_1 \in [-1/4, 3/4]\}$. Hence $\chi \in BV \cap L^\infty$.

Step 2: Construction of $\tilde{\omega}$ and $\tilde{\chi}$.

We start be defining a biLipschitz map $\Psi = (\Psi_1, \Psi_2) : [-1/4, 1/4] \times \mathbb{R} \to [-1/4, 1/4] \times \mathbb{R}$ in the following way:

$$\Psi_1(x) = x_1 \text{ for every } x \text{ and } \Psi_2(x) = x_2 \text{ for } x_2 \notin [-1/4, 3/4] \, , \tag{27}$$

$$\partial_{x_2} \Psi_2(x_1, x_2) = \tilde{z}(x_1, x_2) / \overline{z}(x_1, x_2) \text{ for } (x_1, x_2) \in [-1/4, 1/4] \times [-1/4, 3/4] \, . \tag{28}$$

Note that on the set $[-1/4, 1/4] \times [-1/4, 3/4]$ the function $\overline{z}(x_1, x_2)$ depends only on x_1 and is bigger than 1. For simplicity we call this value $\rho_0(x_1)$. By definition we have

$$\int_{-1/4}^{3/4} \tilde{z}(x_1, \xi) \, d\xi \; = \; \rho_0(x_1) \, .$$

Therefore (27) and (28) are compatible. We conclude that

$$\overline{z}(\Psi) \mathrm{det} \nabla(\Psi) \; = \; \tilde{z} \, . \tag{29}$$

Next we define $\Phi : [0, 1] \times [-1/4, 1/4] \times \mathbb{R} \to [-1/4, 1/4] \times \mathbb{R}$ as

$$\Phi(t, x_1, x_2) \; = \; (1 - t)x + t\Psi^{-1}(x) \, .$$

It is easy to check that, for every $t \in [0, 1]$, $\Phi(t, \cdot)$ is biLipschitz and coincides with the identity outside $[-1/4, 1/4] \times [-1/4, 3/4]$. We denote by Φ^{-1} the map such that $\Phi^{-1}(t, \Phi(t, x)) = x$. Finally we define

$$\tilde{\chi}(t, x) \; := \; \partial_t \Phi(t, \Phi^{-1}(t, x)) \, .$$

One readily checks that $\tilde{\chi}_2 = 0$ everywhere, that $\tilde{\chi}_1 = 0$ outside $[-1/4, 1/4] \times [-1/4, 3/4]$ and that $\tilde{\chi}$ is Lipschitz. Moreover

$$\begin{cases} \dfrac{d}{dt}\Phi(t,x) = \tilde{\chi}(t, \Phi(t,x)) \\[2mm] \Phi(0,x) = x \\ \Phi(1,x) = \Psi^{-1}(x). \end{cases} \tag{30}$$

Therefore (30), (29) and Liouville's Theorem imply that the unique solution of

$$\begin{cases} \partial_t \tilde{\omega} + \mathrm{div}_x(\tilde{\omega}\tilde{\chi}) \qquad \text{for } x \in [-1/4, 1/4] \times \mathbb{R} \\[2mm] \tilde{\omega}(0, \cdot) = \overline{z} \end{cases} \tag{31}$$

satisfies $\tilde{\omega}(1, \cdot) = \tilde{z}$.

In a similar fashion we can define $\tilde{\chi}$ on $[0,1] \times [1/4, 3/4] \times \mathbb{R}$ and then extend it to $[0,1] \times \mathbb{R}^2$ by setting $\tilde{\chi} = 0$ for $x_1 \notin [-1/4, 3/4]$. Note that $\tilde{\chi}_1 = 0$ and $\partial_{x_2}\tilde{\chi}_2 \in L^\infty$, and therefore $\mathrm{div}_x \tilde{\chi} \in L^\infty$. Moreover, $\tilde{\chi}$ is compactly supported and bounded and it is Lipschitz on the open set $[0,1] \times \mathbb{R}^2 \setminus \{x_1 = -1/4, x_1 = 3/4\}$. Therefore $\tilde{\chi} \in BV \cap L^\infty$. $\qquad\square$

References

[1] Aizenman, M. *On vector fields as generators of flows: a counterexample to Nelson's conjecture.* Ann. Math., **107** (1978), 287–296.

[2] Ambrosio, L. *Transport equation and Cauchy problem for BV vector fields.* Invent. Math., **158** (2004), 227–260.

[3] Ambrosio, L.; Bouchut, F.; De Lellis, C. *Well–posedness for a class of hyperbolic systems of conservation laws in several space dimensions.* Comm. Partial Differential Equations, **29** (2004), 1635–1651.

[4] Bressan, A. *An ill posed Cauchy problem for a hyperbolic system in two space dimensions.* Rend. Sem. Mat. Univ. Padova **110** (2003), 103–117.

[5] Colombini F.; Luo T.; Rauch J. *Uniqueness and nonuniqueness for nonsmooth divergence free transport.* Séminaire Équations aux Dérivées Partielles, Exp. No. XXII, École Polytech., Palaiseau, 2003.

[6] Depauw, N. *Non–unicité du transport par un champ de vecteurs presque BV.* Sémin. Équ. Dériv. Partielles, Exp. No. XIX, École Polytech., Palaiseau, 2003.

[7] DiPerna, R. J.; Lions, P. L. *Ordinary differential equations, transport theory and Sobolev spaces.* Invent. Math., **98** (1989), 511–547.

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