Rapidly rotating Bose-Einstein condensates in homogeneous traps

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We extend the results of a previous paper on the Gross-Pitaevskii description of rotating Bose-Einstein condensates in two-dimensional traps to confining potentials of the form $V(r)=r^s$, $2<s<\infty$. Writing the coupling constant as $1/\varepsilon^2$, we study the limit $\varepsilon \to 0$. We derive rigorously the leading asymptotics of the ground state energy and the density profile when the rotation velocity Ω tends to infinity as a power of $1/\varepsilon$. The case of asymptotically homogeneous potentials is also discussed. © *2007 American Institute of Physics.* [DOI: 10.1063/1.2789557]

I. INTRODUCTION

In a previous investigation of rapidly rotating Bose-Einstein condensates in two-dimensional anharmonic traps [Correggi *et al.* (2007)], we considered the case of a "flat" trap with a rigid boundary confining the condensate to a disk of finite radius. The present paper is a sequel to this work, extending the results to homogeneous trap potentials of the form $V(r)=r^s$ with $2<s<\infty$. The flat trap corresponds to the limiting case $s\to\infty$. For $s<\infty$, the system is no longer confined to a bounded region when the coupling constant tends to infinity and a suitable scaling of the variables is necessary to obtain a well-defined limit. This gives rise to additional features and requires some modifications that are dealt with in the present paper. As in Correggi *et al.* (2007), we identify three different parameter regimes depending on the way the rotation velocity is scaled with the interaction.

The present study is carried out strictly within the Gross-Pitaevskii (GP) framework in contrast to the recent paper Bru *et al.* (in print), where the main emphasis is on many-body aspects and the GP description is an auxiliary tool. The trap potentials considered there are three dimensional and not necessarily rotationally symmetric. A two-dimensional potential of the form r^s is an instructive special case that can be analyzed in more detail than the general case.

As in Correggi *et al.* (2007), where a discussion of the general context and an extensive list of references can be found, the starting point is the two-dimensional Gross-Pitaevskii energy functional for the wave function Ψ of the condensate which in our case can be written as

$$\hat{\mathcal{E}}^{\mathrm{GP}}[\Psi] \equiv \int_{\mathbb{R}^2} \mathrm{d}\mathbf{r} \left\{ |\nabla\Psi|^2 + r^s|\Psi|^2 - \Omega(\varepsilon)\Psi^* L\Psi + \frac{|\Psi|^4}{\varepsilon^2} \right\}. \tag{1.1}$$

Here, L is the third component of the angular momentum [i.e., $L=-i\partial/\partial\vartheta$ in polar coordinates (r,ϑ)], $\Omega(\varepsilon)$ the angular velocity, and ε is a non-negative, small parameter. The wave function is

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normalized so that $\int_{\mathbb{R}^2}|\Psi|^2=1$. Units have been chosen so that $\hbar=2m=1$, where m is the particle mass, and such that the coefficient in front of r^s is simply 1.

If $s<\infty$ the condensate spreads out indefinitely in the Thomas-Fermi (TF) limit $\varepsilon\to 0$ and the density $|\Psi|^2$ tends uniformly to zero. Nontrivial results can be obtained, however, by rescaling all lengths by an ε-dependent factor. We write $\mathbf{r}\equiv k\mathbf{r}'$, $\Psi(\mathbf{r})\equiv\Psi'(\mathbf{r}')/k$ with $k\equiv\varepsilon^{-2/(s+2)}$ and define

$$\mathcal{E}^{\mathrm{GP}}[\Psi']\equiv\varepsilon^{-4/(s+2)}\hat{\mathcal{E}}^{\mathrm{GP}}[\Psi]. \tag{1.2}$$

The functional $\mathcal{E}^{\mathrm{GP}}$ is the one we shall study. Dropping the primes of the arguments, it is explicitly given by

$$\mathcal{E}^{\mathrm{GP}}[\Psi]=\int_{\mathbb{R}^2}\mathrm{d}\mathbf{r}\left\{|\nabla\Psi|^2-\omega(\varepsilon)\Psi^*L\Psi+\frac{|\Psi|^2}{\varepsilon^2}(r^s+|\Psi|^2)\right\}, \tag{1.3}$$

where the scaled angular velocity is given by

$$\omega(\varepsilon)\equiv\varepsilon^{-4/(s+2)}\Omega(\varepsilon). \tag{1.4}$$

The functional (1.3) is defined on the domain

$$\mathcal{D}^{\mathrm{GP}}\equiv\{\Psi\in H^1(\mathbb{R}^2)|r^s|\Psi|^2\in L^1(\mathbb{R}^2)\}.$$

We set

$$E_\varepsilon^{\mathrm{GP}}\equiv\min_{\Psi\in\mathcal{D}^{\mathrm{GP}},\|\Psi\|_2=1}\mathcal{E}^{\mathrm{GP}}[\Psi], \tag{1.5}$$

and denote by $\Psi_\varepsilon^{\mathrm{GP}}$ a corresponding minimizer, which may not be unique [Seiringer (2002)].

In the following, we study the leading order asymptotics of the ground state energy and density for the functional $\mathcal{E}^{\mathrm{GP}}[\Psi]$ as $\varepsilon\to 0$. The ground state behavior for the original GP functional (1.1) follows by scaling: If we set $\hat{E}_\varepsilon^{\mathrm{GP}}\equiv\inf\hat{\mathcal{E}}^{\mathrm{GP}}[\Psi]$ and denote by $\hat{\Psi}_\varepsilon^{\mathrm{GP}}$ any ground state of (1.1), one has

$$\hat{E}_\varepsilon^{\mathrm{GP}}=\varepsilon^{4/(s+2)}E_\varepsilon^{\mathrm{GP}},\quad \hat{\Psi}_\varepsilon^{\mathrm{GP}}(\mathbf{r})=\varepsilon^{2/(s+2)}\Psi_\varepsilon^{\mathrm{GP}}(\varepsilon^{2/(s+2)}\mathbf{r}). \tag{1.6}$$

Note that the "flat" trap case studied in Fischer and Baym (2003) and Correggi *et al.* (2007) can be formally obtained by taking the limit $s\to\infty$ of (1.3): In this limit, the scaling factor $\varepsilon^{-2/(s+2)}$ converges to 1, the external potential to ∞ for $r>1$ and to 0 for $r<1$, and the rescaled angular velocity $\omega(\varepsilon)$ to $\Omega(\varepsilon)$. The formal limit $s\to\infty$ corresponds to a "flat" trap with Dirichlet conditions at the boundary rather than the Neumann conditions considered in Correggi *et al.* (2007). As noted in Correggi *et al.* (2007), both boundary conditions give the same results in the TF limit $\varepsilon\to 0$.

As in Correggi *et al.* (2007), we rewrite the GP functional in the form

$$\mathcal{E}^{\mathrm{GP}}[\Psi]=\int_{\mathbb{R}^2}\mathrm{d}\mathbf{r}\left\{|(\nabla-i\mathbf{A}_\varepsilon)\Psi|^2+\frac{|\Psi|^2}{\varepsilon^2}(r^s+|\Psi|^2)-\frac{\omega(\varepsilon)^2r^2|\Psi|^2}{4}\right\}, \tag{1.7}$$

with the vector potential

$$\mathbf{A}_\varepsilon(\mathbf{r})\equiv\frac{\omega(\varepsilon)}{2}\mathbf{e}_z\times\mathbf{r}, \tag{1.8}$$

where $\mathbf{e}_z$ is the unit vector in the z direction. The behavior of the GP functional as $\varepsilon\to 0$ depends on the way the angular velocity ω scales as a function of ε. We distinguish three cases: $\omega\ll 1/\varepsilon$, $\omega\sim 1/\varepsilon$, and $\omega\gg 1/\varepsilon$. It is convenient to write $\omega=\omega_0/\varepsilon$; the three cases then correspond to $\omega_0\ll 1$, $\omega_0\sim 1$, and $\omega_0\gg 1$. In the next section, we discuss for fixed ω_0 the TF functional that is obtained from the GP functional by dropping the first (kinetic) term in (1.7). This functional and its limits for $\omega_0\to 0$ and $\omega_0\to\infty$ describe the asymptotics of (1.7) for $\varepsilon\to 0$ as summarized in Sec.

III. In Sec. IV A, we present the proofs for the regime $\omega \sim 1/\varepsilon$, and in Sec. IV B, for $\omega \gg 1/\varepsilon$.

Since the length scale $\varepsilon^{-2/(s+2)}$ tends to infinity in the TF limit, it is clear that, for leading order calculations, only the asymptotic behavior of the confining trap potential for large arguments matters. In the last section, we indicate how our proofs can be extended to include potentials $V(r)$ that are asymptotically homogeneous in the sense of Lieb *et al.* (2001), i.e., such that for $\lambda \geq 1$,

$$|\lambda^{-s}V(\lambda r) - r^s| \leq c\lambda^{-\kappa}(1 + r^s), \tag{1.9}$$

for some constants $\kappa, c > 0$, uniformly in $r \in \mathbb{R}^+$.

II. THE TF FUNCTIONAL AND ITS PROPERTIES

The TF functional depends on the density alone and is, for fixed ω_0, defined as

$$\mathcal{E}^{\mathrm{TF}}[\rho] \equiv \int_{\mathbb{R}^2} \mathrm{d}\mathbf{r}\left\{\rho(r^s + \rho) - \frac{\omega_0^2 r^2 \rho}{4}\right\}, \tag{2.1}$$

on the domain

$$\mathcal{D}^{\mathrm{TF}} \equiv \{\rho \in L^2(\mathbb{R}^2) | \rho \geq 0, r^s\rho \in L^1(\mathbb{R}^2)\}. \tag{2.2}$$

By standard methods, there is a unique minimizer

$$\rho^{\mathrm{TF}}(r) \equiv \tfrac{1}{2}\left[\mu^{\mathrm{TF}} - r^s + \frac{\omega_0^2 r^2}{4}\right]_+, \tag{2.3}$$

where the chemical potential μ^{TF} is fixed by $\|\rho^{\mathrm{TF}}\|_1 = 1$. The ground state energy associated with (2.3) is

$$E^{\mathrm{TF}} \equiv \inf_{\rho \in D^{TF}, \|\rho\|_1 = 1} \mathcal{E}^{\mathrm{TF}}[\rho] = \mathcal{E}^{\mathrm{TF}}[\rho^{\mathrm{TF}}]. \tag{2.4}$$

Since $-r^s + \omega_0^2 r^2/4 \to -\infty$ as $r \to \infty$, due to the condition $s > 2$, the minimizer (2.3) is always compactly supported, i.e., $\mathrm{supp}(\rho^{\mathrm{TF}}) \subset \mathcal{B}_R$, for some $R < \infty$ depending on ω_0, where $\mathcal{B}_R$ denotes a two-dimensional ball centered at the origin with radius R.

As in the case of a flat trap, the density ρ^{TF} develops a "hole" [a disk centered at the origin where the density (2.3) vanishes] when ω_0 exceeds a certain critical value $\omega_{0,c}$. Both the outer radius R_{out} of the support of ρ^{TF} and the inner radius R_{in} (if present) increase with ω_0, as discussed below. In the flat trap, only R_{in} increases with ω_0 while R_{out} is fixed from the outset.

The chemical potential μ^{TF} depends also on ω_0, and because $-r^s + \omega_0^2 r^2/4$ is monotone increasing in ω_0, it is clear that μ^{TF} is monotone decreasing in ω_0 [see also (2.12) and (2.14)]. For small ω_0, we have $\mu^{\mathrm{TF}} > 0$, whereas μ^{TF} vanishes and changes sign at $\omega_0 = \omega_{0,c}$.

A. Support of ρ^{TF} for ω_0=const

In order to study the support of ρ^{TF}, it is convenient to consider the function $f(z) \equiv \mu^{\mathrm{TF}} - z^{s/2} + (\omega_0^2/4)z$, where $z \equiv r^2 \geq 0$, so that $\rho^{\mathrm{TF}} = f(r^2)_+$. Since $\mathrm{d}^2 f/\mathrm{d}z^2 = -s(s-2)z^{(s-4)/2}/4 < 0$ for $z \in (0, \infty)$, the function f is strictly concave for $z > 0$. Moreover, $f(0) = \mu^{\mathrm{TF}}$ and $\lim_{z\to\infty} f(z) = -\infty$. Hence, if $\mu^{\mathrm{TF}} > 0$, i.e., $\omega_0 < \omega_{0,c}$, then $f(0) > 0$ and there exists a unique $z_{\mathrm{out}} > 0$ such that $f(z_{\mathrm{out}}) = 0$. The support of ρ^{TF} in the radial coordinate is the interval $[0, R_{\mathrm{out}}]$, with $R_{\mathrm{out}} \equiv \sqrt{z_{\mathrm{out}}}$. In the opposite case, $\mu^{\mathrm{TF}} < 0$, i.e., $\omega_0 > \omega_{0,c}$, we have $f(0) < 0$ but $\sup f > 0$ (since $\int \rho^{\mathrm{TF}} > 0$) and concavity implies the existence of two positive solutions of $f(z) = 0$. The support of ρ^{TF} in the radial coordinate is then an interval $[R_{\mathrm{in}}, R_{\mathrm{out}}]$ for some $0 < R_{\mathrm{in}} < R_{\mathrm{out}}$. Note also that

$$0 < f'(R_{\mathrm{in}}^2) = -\frac{s}{2}R_{\mathrm{in}}^{s-2} + \frac{\omega_0^2}{4} \quad \text{and} \quad 0 > f'(R_{\mathrm{out}}^2) = -\frac{s}{2}R_{\mathrm{out}}^{s-2} + \frac{\omega_0^2}{4}. \tag{2.5}$$

From the L^1 normalization of ρ^{TF} and

$$\mu^{\mathrm{TF}} = R_{\mathrm{out}}^s - \frac{\omega_0^2 R_{\mathrm{out}}^2}{4}, \tag{2.6}$$

we get, for $\omega_0 < \omega_{0,c}$,

$$\frac{sR_{\mathrm{out}}^{s+2}}{2(s+2)} - \frac{\omega_0^2 R_{\mathrm{out}}^4}{16} = \frac{1}{\pi}, \tag{2.7}$$

whereas, for $\omega_0 > \omega_{0,c}$, we have

$$\mu^{\mathrm{TF}} = R_{\mathrm{in}}^s - \frac{\omega_0^2 R_{\mathrm{in}}^2}{4} = R_{\mathrm{out}}^s - \frac{\omega_0^2 R_{\mathrm{out}}^2}{4}, \tag{2.8}$$

$$\frac{s(R_{\mathrm{out}}^{s+2} - R_{\mathrm{in}}^{s+2})}{2(s+2)} - \frac{\omega_0^2 (R_{\mathrm{out}}^4 - R_{\mathrm{in}}^4)}{16} = \frac{1}{\pi}. \tag{2.9}$$

The radii R_{out} and R_{in} are determined by solving (2.7), or (2.8) together with (2.9). While explicit formulas can in general not be given, the outer radius $R_{\mathrm{out},c}$ for $\mu^{\mathrm{TF}}=0$ and the critical angular velocity $\omega_{0,c}$ for the creation of a hole are easily obtained from (2.6) and (2.7),

$$R_{\mathrm{out},c} = \left(\frac{\omega_{0,c}}{2} \right)^{2/(s-2)}, \tag{2.10}$$

with

$$\omega_{0,c} = 2 \left[\frac{4(s+2)}{\pi(s-2)} \right]^{(s-2)/2(s+2)}. \tag{2.11}$$

In the "flat" trap case, i.e., for $s \to \infty$, the expression for $\omega_{0,c}$ simplifies to $4/\sqrt{\pi}$ as in Correggi *et al.* (2007).

We now show that μ^{TF} is monotonically decreasing and R_{in} and R_{out} are monotonically increasing as ω_0 increases. Consider first the case $\omega_0 < \omega_{0,c}$, i.e., $\mu^{\mathrm{TF}} > 0$. Differentiating the normalization equation $\int \rho^{\mathrm{TF}} = 1$ with respect to $t \equiv \omega_0^2/4$, using (2.6), gives

$$\partial \mu^{\mathrm{TF}} / \partial t = - R_{\mathrm{out}}^2 / 2 < 0. \tag{2.12}$$

Differentiating (2.7) gives

$$\partial R_{\mathrm{out}} / \partial t = (s R_{\mathrm{out}}^{s-2} - 2t)^{-1} R_{\mathrm{out}} / 2, \tag{2.13}$$

and, hence, $\partial R_{\mathrm{out}} / \partial t > 0$ because of (2.5).

In the case $\omega_0 > \omega_{0,c}$, we again differentiate the normalization condition for ρ^{TF}, this time using (2.8), and obtain

$$\partial \mu^{\mathrm{TF}} / \partial t = -(R_{\mathrm{out}}^2 + R_{\mathrm{in}}^2)/2 < 0. \tag{2.14}$$

Moreover, by taking the derivative of (2.8) with respect to t, we have

$$\frac{\partial \mu^{\mathrm{TF}}}{\partial t} = (s R_{\mathrm{in}}^{s-1} - 2t R_{\mathrm{in}}) \frac{\partial R_{\mathrm{in}}}{\partial t} - R_{\mathrm{in}}^2 = (s R_{\mathrm{out}}^{s-1} - 2t R_{\mathrm{out}}) \frac{\partial R_{\mathrm{out}}}{\partial t} - R_{\mathrm{out}}^2,$$

which, combined with (2.14), yields the inequalities

$$(sR_{\text{in}}^{s-1}-2tR_{\text{in}})\frac{\partial R_{\text{in}}}{\partial t}<0 \quad \text{and} \quad (sR_{\text{out}}^{s-1}-2tR_{\text{out}})\frac{\partial R_{\text{out}}}{\partial t}>0.$$

By (2.5), this implies

$$\frac{\partial R_{\text{in}}}{\partial t}>0 \quad \text{and} \quad \frac{\partial R_{\text{out}}}{\partial t}>0. \tag{2.15}$$

Altogether, we have thus seen that ρ^{TF} has compact support contained in $\mathcal{B}_{R_{\text{out}}}$. If $\omega_0<\omega_{0,c}$, the support coincides with $\mathcal{B}_{R_{\text{out}}}$, while, for $\omega_0>\omega_{0,c}$, it is the annulus $\{\mathbf{r}: R_{\text{in}}\leq r\leq R_{\text{out}}\}$. Both R_{in} and R_{out} grow as ω_0 increases.

B. The quartic trap

Let us consider for illustration the quartic trap $V(r)=r^4$. The critical angular velocity and radius are given by

$$\omega_{0,c}=2\left(\frac{12}{\pi}\right)^{1/6}, \quad R_{\text{out},c}=\left(\frac{12}{\pi}\right)^{1/6}.$$

The equations for the inner and outer radii R_{in} and R_{out} reduce to equations of third order for the squares of the radii. If $\omega_0\leq\omega_{0,c}$, the chemical potential and the outer radius are

$$\mu^{\text{TF}}=\frac{1}{64}\left\{\frac{1}{2}\left(\frac{q(\omega_0)}{\pi}\right)^{1/3}+\frac{\omega_0^4}{2}\left(\frac{\pi}{q(\omega_0)}\right)^{1/3}-\frac{\omega_0^2}{2}\right\}^2-\frac{\omega_0^4}{64}$$

and

$$R_{\text{out}}=\frac{1}{4}\left\{\left(\frac{q(\omega_0)}{\pi}\right)^{1/3}+\omega_0^4\left(\frac{\pi}{q(\omega_0)}\right)^{1/3}+\omega_0^2\right\}^{1/2},$$

with $q(\omega_0)\equiv 6144+\pi\omega_0^6+64\sqrt{3}\sqrt{3072+\pi\omega_0^6}$. If $\omega_0>\omega_{0,c}$, we obtain

$$\mu^{\text{TF}}=\frac{1}{4}\left(\frac{12}{\pi}\right)^{2/3}-\frac{\omega_0^4}{64},$$

$$R_{\text{in}}=\sqrt{\frac{\omega_0^2}{8}-\frac{1}{2}\left(\frac{12}{\pi}\right)^{1/3}}, \quad R_{\text{out}}=\sqrt{\frac{\omega_0^2}{8}+\frac{1}{2}\left(\frac{12}{\pi}\right)^{1/3}}.$$

(Note that the two expressions for μ^{TF} and R_{out} are equal for $\omega_0=\omega_{0,c}$, whereas $R_{\text{in}}=0$.) For $\omega_0>\omega_{0,c}$, the TF minimizer is

$$\rho^{\text{TF}}(r)=\left[\frac{\omega_0^2}{8}\left(r^2-\frac{\omega_0^2}{16}\right)-\frac{r^4}{2}+\frac{1}{8}\left(\frac{12}{\pi}\right)^{2/3}\right]_+.$$

C. Support of ρ^{TF} for $\omega_0\to\infty$

The radius R_{m} at which the density is maximal can be explicitly calculated from (2.3),

$$R_{\text{m}}\equiv\left(\frac{\omega_0^2}{2s}\right)^{1/(s-2)}. \tag{2.16}$$

It is clear that $R_{\text{in}}<R_{\text{m}}<R_{\text{out}}$ and all radii tend to infinity if $\omega_0\to\infty$. We shall now show that $R_{\text{out}}-R_{\text{in}}$ tends to zero in this limit. For $s>4$, $R_{\text{out}}^2-R_{\text{in}}^2$ also tends to zero and the density, therefore, to infinity.

It is convenient to scale all lengths by using R_{m} as a unit, i.e, to write $r=R_{\mathrm{m}}x$. The scaled TF minimizer $\tilde{\rho}^{\mathrm{TF}}(x)\equiv R_{\mathrm{m}}^2\rho^{\mathrm{TF}}(r)$ is

$$\tilde{\rho}^{\mathrm{TF}}(x) \equiv \frac{1}{2}\left(\frac{\omega_0^2}{2s}\right)^{(s+2)/(s-2)}\left[\tilde{\mu}^{\mathrm{TF}} - x^s + \frac{sx^2}{2}\right]_+, \tag{2.17}$$

with the scaled chemical potential

$$\tilde{\mu}^{\mathrm{TF}} \equiv \left(\frac{\omega_0^2}{2s}\right)^{-s/(s-2)}\mu^{\mathrm{TF}}. \tag{2.18}$$

We also denote $x_{\mathrm{in}}\equiv R_{\mathrm{in}}/R_{\mathrm{m}}$ and $x_{\mathrm{out}}\equiv R_{\mathrm{out}}/R_{\mathrm{m}}$, so that $0\leq x_{\mathrm{in}}<1<x_{\mathrm{out}}$ and the maximum of $\tilde{\rho}^{\mathrm{TF}}$ is attained at $x=1$.

In the same way as (2.14) was derived, we have

$$\frac{\partial\tilde{\mu}^{\mathrm{TF}}}{\partial\omega_0^2} = -\left(\frac{\omega_0^2}{2s}\right)^{-2s/(s-2)}\frac{(s+2)}{\pi s(s-2)(x_{\mathrm{out}}^2 - x_{\mathrm{in}}^2)} < 0, \tag{2.19}$$

so that $\tilde{\mu}^{\mathrm{TF}}$ is a decreasing function of ω_0. Moreover, since $\tilde{\rho}^{\mathrm{TF}}(1)>0$, one has the bound

$$-\tilde{\mu}^{\mathrm{TF}} < \frac{s}{2} - 1. \tag{2.20}$$

Defining

$$h(x) \equiv x^s - (s/2)x^2 + (s/2) - 1, \tag{2.21}$$

the scaled radii x_{in} and x_{out} are determined by

$$h(x_{\mathrm{in}}) = h(x_{\mathrm{out}}), \tag{2.22}$$

together with the normalization condition for $\tilde{\rho}^{\mathrm{TF}}$ which, expressed in terms of h, is

$$\frac{h(x_{\mathrm{in}})}{2}(x_{\mathrm{out}}^2 - x_{\mathrm{in}}^2) - \int_{x_{\mathrm{in}}}^{x_{\mathrm{out}}} h(x)x\mathrm{d}x = \pi^{-1}\left(\frac{\omega_0^2}{2s}\right)^{-(s+2)/(s-2)}. \tag{2.23}$$

The right hand side of (2.23) tends to zero as $\omega_0\to\infty$, and the integral on the left hand side is always strictly less than the first term for x_{in} strictly less than x_{out}. Since $x_{\mathrm{in}}<1<x_{\mathrm{out}}$ and h is continuous with $h(x)=0$ only for $x=1$, it is clear that x_{in} and x_{out} must both tend to 1 as $\omega_0\to\infty$, and by the normalization, the density $\tilde{\rho}^{\mathrm{TF}}$ approaches a delta function concentrated on the circle with radius 1. Note also that

$$h(x_{\mathrm{in}}) = h(x_{\mathrm{out}}) = \tilde{\mu}^{\mathrm{TF}} + \frac{s}{2} - 1, \tag{2.24}$$

so that

$$\tilde{\mu}^{\mathrm{TF}} \xrightarrow[\omega_0\to\infty]{} 1 - \frac{s}{2}. \tag{2.25}$$

In order to estimate the rate of the convergence of the density to a delta function, we make a Taylor expansion of $h(x)$ around $x=1$,

$$h(x) = \tfrac{1}{2}s(s-2)(1-x)^2 + O(|1-x|^3). \tag{2.26}$$

Writing $x_{\mathrm{in}}=1-\delta+o(\delta)$, $x_{\mathrm{out}}=1+\delta+o(\delta)$, where δ is the deviation from 1 to leading order in the small parameter on the right hand side of (2.23), the normalization condition (2.23) gives

$$\delta = \left(\frac{3}{s(s-2)}\right)^{1/3}\left(\frac{\omega_0^2}{2s}\right)^{-(s+2)/3(s-2)}. \tag{2.27}$$

Multiplying δ with $R_{\mathrm{m}} \sim \omega_0^{2/(s-2)}$, we see that

$$(R_{\mathrm{out}} - R_{\mathrm{in}}) \sim \omega_0^{-2(s-1)/3(s-2)}. \tag{2.28}$$

Thus, the original density ρ^{TF} is also supported on an annulus whose thickness tend to zero. The area of the support is

$$\pi(R_{\mathrm{out}}^2 - R_{\mathrm{in}}^2) \sim \omega_0^{2(4-s)/3(s-2)}, \tag{2.29}$$

which increases for $2<s<4$ but tends to zero for $s>4$.

D. TF energy asymptotics for $\omega_0 \to \infty$

The scaled density $\tilde{\rho}^{\mathrm{TF}}$ is the minimizer of the scaled TF functional,

$$\tilde{\mathcal{E}}^{\mathrm{TF}}[\tilde{\rho}] \equiv \int_{\mathbb{R}^2} \mathrm{d}\mathbf{x}\left[\left(x^s - \frac{sx^2}{2}\right)\tilde{\rho} + \left(\frac{\omega_0^2}{2s}\right)^{-(s+2)/(s-2)}\tilde{\rho}^2\right], \tag{2.30}$$

with corresponding energy $\tilde{E}^{\mathrm{TF}} = \tilde{\mathcal{E}}^{\mathrm{TF}}[\tilde{\rho}^{\mathrm{TF}}]$. As shown in the previous subsection, $\tilde{\rho}^{\mathrm{TF}}$ converges to a delta function on the unit circle as $\omega_0 \to \infty$. The behavior of the energy in this limit is given in the following proposition.

Proposition 2.1: (TF energy for $\omega_0 \to \infty$) *For* $\omega_0 \to \infty$,

$$\tilde{E}^{\mathrm{TF}} = 1 - \frac{s}{2} + \mathcal{O}(\omega_0^{-4(s+2)/3(s-2)}). \tag{2.31}$$

Proof: The lower bound is simply obtained by neglecting the positive last term in (2.30) and using the inequality $x^s - sx^2/2 \geq 1 - s/2$. For the upper bound, we use a trial function of the form

$$\tilde{\rho}_\xi(x) \equiv \xi^{-1} j(\xi^{-1}(1-x^2)), \tag{2.32}$$

where j is a smooth non-negative function supported in $[-1/2, 1/2]$ satisfying the normalization $\pi \int \mathrm{d}r j(r) = 1$ and $0 < \xi < 1$. One can easily estimate $\|\tilde{\rho}_\xi\|_2^2 \leq C\xi^{-1}$, and exploiting the Taylor expansion of $x^s - sx^2/2$ around $x=1$, we have

$$\int_{\mathbb{R}^2} \mathrm{d}\mathbf{x}\left(x^s - \frac{sx^2}{2}\right)\tilde{\rho}_\xi \leq 1 - \frac{s}{2} + C'\xi^2,$$

so that

$$\tilde{\mathcal{E}}^{\mathrm{TF}}[\tilde{\rho}_\xi] \leq 1 - \frac{s}{2} + C'\xi^2 + C''\omega_0^{-2(s+2)/(s-2)}\xi^{-1}. \tag{2.33}$$

Optimization with respect to the parameter ξ yields the desired upper bound. □

III. MAIN RESULTS

A. The regime $\omega \ll 1/\varepsilon$

For $\omega \ll 1/\varepsilon$, the GP ground state energy and density are approximated to the leading order by the corresponding quantities in the nonrotating case, exactly as in Proposition 2.3 in Correggi *et al.* (2007). The TF functional without rotation, i.e., for $\omega_0 = 0$, is given by

$$\mathcal{E}_*^{\mathrm{TF}}[\rho] \equiv \int_{\mathbb{R}^2} \mathrm{d}\mathbf{r}\rho(r^s + \rho).$$

We denote by E_*^{TF} its ground state energy and by

$$\rho_*^{\mathrm{TF}}(r) \equiv \frac{1}{2}[\mu^{\mathrm{TF}} - r^s]_+$$

the corresponding minimizer.

Proposition 3.1: (GP energy and density asymptotics for $\omega \ll 1/\varepsilon$) *For any $\omega(\varepsilon)$ such that* $\lim_{\varepsilon\to 0}\varepsilon\omega(\varepsilon)=0$ *and for ε tending to zero,*

$$\varepsilon^2 E_\varepsilon^{\mathrm{GP}} = E_*^{\mathrm{TF}} + o(1), \tag{3.1}$$

$$\| |\Psi_\varepsilon^{\mathrm{GP}}|^2 - \rho_*^{\mathrm{TF}} \|_{L^2(\mathbb{R}^2)} = o(1). \tag{3.2}$$

Proof: The lower bound for the ground state energy is actually trivial, since it is sufficient to neglect the first positive term in (1.7) to obtain

$$\varepsilon^2 E_\varepsilon^{\mathrm{GP}} \geqslant E_*^{\mathrm{TF}} - C\varepsilon^2\omega(\varepsilon)^2.$$

In order to get an appropriate upper bound, we test the GP functional on the (real) GP minimizer for $\omega=0$ and obtain $E_\varepsilon^{\mathrm{GP}} \leqslant E_\varepsilon^{\mathrm{GP}}|_{\omega=0}$. The result is then a consequence of Lemma 2.3 in Lieb *et al.* (2001) and we get the bound [see Eq. (2.18) in Lieb *et al.* (2001)]

$$\varepsilon^2 E_\varepsilon^{\mathrm{GP}} \leqslant E_*^{\mathrm{TF}} + C\varepsilon^{2/3}. \tag{3.3}$$

The density convergence is a simple corollary [see the proof of Proposition 2.3 in Correggi *et al.* (2007) and Theorem 2.1 in Lieb *et al.* (2001)]. □

B. The regime $\omega \sim 1/\varepsilon$

We now assume that $\omega(\varepsilon)=\omega_0/\varepsilon$, with $\omega_0>0$ a finite constant. The analogs of Theorem 2.1 and Corollary 2.1 in Correggi *et al.* (2007) are the following:

Theorem 3.1: (GP energy asymptotics for $\omega \sim 1/\varepsilon$) *For any $\omega_0>0$ and for ε tending to zero,*

$$\varepsilon^2 E_\varepsilon^{\mathrm{GP}} = E^{\mathrm{TF}} + \mathcal{O}(\varepsilon|\log \varepsilon|). \tag{3.4}$$

Corollary 3.1: (GP density asymptotics for $\omega \sim 1/\varepsilon$) *For any $\omega_0>0$ and for ε tending to zero,*

$$\| |\Psi_\varepsilon^{\mathrm{GP}}|^2 - \rho^{\mathrm{TF}} \|_{L^2(\mathbb{R}^2)} = \mathcal{O}(\sqrt{\varepsilon|\log \varepsilon|}). \tag{3.5}$$

The asymptotics of the energy and density for the original functional (1.1) quantities is then given by

$$\varepsilon^{-2s/(s+2)}\hat{E}_\varepsilon^{\mathrm{GP}} = E^{\mathrm{TF}} + \mathcal{O}(\varepsilon|\log\varepsilon|) \quad \text{and} \quad \varepsilon^{-4/(s+2)}|\hat{\Psi}_\varepsilon^{\mathrm{GP}}(\varepsilon^{-2/(s+2)}\mathbf{r})|^2 \xrightarrow[\varepsilon\to 0]{} \rho^{\mathrm{TF}}(r),$$

where the convergence of the density is in the norm topology of $L^2(\mathbb{R}^2)$.

For $s<\infty$, the condensate is not confined to a bounded region and $|\Psi_\varepsilon^{\mathrm{GP}}|^2$ is a function supported on the whole of $\mathbb{R}^2$. From Corollary 3.1, it follows immediately that $|\Psi_\varepsilon^{\mathrm{GP}}|^2$ is small outside the support of ρ^{TF} in L^2 norm, i.e.,

$$\int_{\mathbb{R}^2\setminus\mathrm{supp}(\rho^{\mathrm{TF}})} d\mathbf{r}|\Psi_\varepsilon^{\mathrm{GP}}|^4 = \mathcal{O}(\varepsilon|\log \varepsilon|), \tag{3.6}$$

but much more can be shown, namely, that $|\Psi_\varepsilon^{\mathrm{GP}}|$ is pointwise exponentially small outside the support of ρ^{TF}.

Theorem 3.2: (Exponential smallness of the GP density, $\omega \sim 1/\varepsilon$) *For any* $\omega_0 > 0$, $\mathbf{r} \in \mathcal{T}_\varepsilon^{\mathrm{out}} \equiv \{\mathbf{r} \in \mathbb{R}^2 | r \geq R_{\mathrm{out}} + \varepsilon^{1/3}\}$ *and for* ε *sufficiently small,*

$$|\Psi_\varepsilon^{\mathrm{GP}}(\mathbf{r})|^2 \leq C_{\omega_0}\varepsilon^{1/6}\sqrt{|\log \varepsilon|}\exp\left[-\frac{C'_{\omega_0}\mathrm{dist}(\mathbf{r},\partial\mathcal{T}_\varepsilon^{\mathrm{out}})^2}{\varepsilon^{5/6}}\right]. \tag{3.7}$$

Furthermore, for any $\omega_0 > \omega_{0,c}$, *the same estimate holds for* $\mathbf{r} \in \mathcal{T}_\varepsilon^{\mathrm{in}} \equiv \{\mathbf{r} \in \mathbb{R}^2 | r \leq R_{\mathrm{in}} - \varepsilon^{1/3}\}$ *and* $\partial\mathcal{T}_\varepsilon^{\mathrm{out}}$ *replaced with* $\partial\mathcal{T}_\varepsilon^{\mathrm{in}}$.

C. The regime $\omega \gg 1/\varepsilon$

For convenience, in particular, for the statement of Theorem 3.4 below, and comparison with Correggi *et al.* (2007), we assume that ω increases as a power of $1/\varepsilon$, i.e., that $\omega(\varepsilon) = \omega_1/\varepsilon^{1+\alpha}$, with some constants $\omega_1, \alpha > 0$. This means that we take $\omega_0 = \varepsilon\omega(\varepsilon) = \omega_1/\varepsilon^{\alpha}$. Theorem 3.3 holds true for general $\omega_0(\varepsilon) = \varepsilon\omega(\varepsilon) \to \infty$ if $\omega_1/\varepsilon^{\alpha}$ is replaced by $\omega_0(\varepsilon)$.

In the regime $\omega \gg 1/\varepsilon$, the limiting functional is still given by (2.1), but since ω_0 now depends on ε, this is also the case for the TF ground state energy and density. We thus use the notations $E_\varepsilon^{\mathrm{TF}}$ and $\rho_\varepsilon^{\mathrm{TF}}$. Proposition 2.1 yields the ground state energy asymptotics for the functional $E_\varepsilon^{\mathrm{TF}}$, i.e.,

$$\varepsilon^{2\alpha s/(s-2)}E_\varepsilon^{\mathrm{TF}} = \left(\frac{\omega_1^2}{2s}\right)^{s/(s-2)}\left(1-\frac{s}{2}\right) + \mathcal{O}(\varepsilon^{4\alpha(s+2)/3(s-2)}). \tag{3.8}$$

The following theorem describes the GP ground state energy asymptotics.

Theorem 3.3: (GP energy asymptotics for $\omega \gg 1/\varepsilon$) *For any* $\omega_1, \alpha > 0$ *and* ε *tending to zero,*

$$\varepsilon^2\varepsilon^{2\alpha s/(s-2)}E_\varepsilon^{\mathrm{GP}} = \left(\frac{\omega_1^2}{2s}\right)^{s/(s-2)}\left(1-\frac{s}{2}\right) + \mathcal{O}(\varepsilon^{4\alpha(s+2)/3(s-2)}) + \mathcal{O}(\varepsilon\varepsilon^{\alpha(s+2)/(s-2)}). \tag{3.9}$$

Note the occurrence of two remainders in (3.9): The first one, of order $\varepsilon^{4\alpha(s+2)/3(s-2)}$, is actually the expected optimal one, since it coincides with the (optimal) error term in (3.8). Therefore, as long as $\alpha \leq 3(s-2)/(s+2)$, the second term is just a higher order correction and the result is optimal as far as the order of the error term is concerned. However, for larger α, the leading correction in (3.9) is given by the second error term and it is due to the particular form of the trial function involved in the proof (see Sec. IV B).

In order to state a pointwise estimate analogous to (3.7), it is convenient to rescale the GP minimizer in the same way as when $\tilde{\rho}^{\mathrm{TF}}$ was obtained from ρ^{TF} by scaling. Thus, we define [see also (4.19)]

$$\tilde{\Psi}_\varepsilon^{\mathrm{GP}}(\mathbf{x}) \equiv R_{\mathrm{m}}\Psi_\varepsilon^{\mathrm{GP}}(R_{\mathrm{m}}\mathbf{x}), \tag{3.10}$$

with $\mathbf{x} \equiv R_{\mathrm{m}}^{-1}\mathbf{r}$. The scaled minimizer $\tilde{\Psi}_\varepsilon^{\mathrm{GP}}$ is concentrated in a neighborhood of $x=1$ and exponentially small everywhere else.

Theorem 3.4: (Exponential smallness of the GP density, $\omega \gg 1/\varepsilon$) *Set*

$$\beta \equiv \min\left[\frac{4\alpha(s+2)}{3(s-2)}, 1 + \frac{\alpha(s+2)}{s-2}\right]. \tag{3.11}$$

For any $\alpha, \omega_1 > 0$, $\mathbf{r} \in \mathcal{T}_\varepsilon \equiv \{\mathbf{r} \in \mathbb{R}^2 || 1 - r| \geq \varepsilon^{\beta/3}\}$ *and for* ε *tending to zero,*

$$|\widetilde{\Psi}_\varepsilon^{\mathrm{GP}}(\mathbf{x})|^2 \leq C_{\omega_1}\varepsilon^{\beta/6}\varepsilon^{-\alpha(s+2)/(s-2)} \exp\left[-\frac{C'_{\omega_1}\mathrm{dist}(\mathbf{x},\partial\mathcal{T}_\varepsilon)^2}{\varepsilon^\gamma}\right], \quad (3.12)$$

where

$$\gamma \equiv 1 - \frac{\beta}{3} + \frac{\alpha(s+2)}{s-2}. \quad (3.13)$$

(Note that, for both possible values of β, the exponent γ is positive.) Furthermore, the density $|\widetilde{\Psi}_\varepsilon^{\mathrm{GP}}(\mathbf{x})|^2$ converges in the sense of distributions to a Dirac delta function concentrated at $x=1$.

IV. PROOFS

In this section, we prove the main results mentioned in Sec. III.

A. The regime $\omega(\varepsilon)\sim 1/\varepsilon$

We start by proving the ground state energy asymptotics and the other results will follow as simple corollaries. The proof is quite similar to the proof of Theorem 2.1 in Correggi *et al.* (2007): Like there, the main ingredient in the derivation of the upper bound for the energy is a trial function with a large number of vortices while the differences are essentially contained in a scaling argument.

1. Proof of Theorem 3.1

The lower bound is obtained again by simply neglecting the positive "magnetic" kinetic energy in (1.7), namely,

$$\mathcal{E}^{\mathrm{GP}}[\Psi] \geq \frac{\mathcal{E}^{\mathrm{TF}}[|\Psi|^2]}{\varepsilon^2} \geq \frac{E^{\mathrm{TF}}}{\varepsilon^2}. \quad (4.1)$$

To get an upper bound, we evaluate the GP functional on a trial function of the form

$$\Psi_\varepsilon(\mathbf{r}) = c_\varepsilon\sqrt{\rho_\varepsilon(r)}\chi_\varepsilon(\mathbf{r})g_\varepsilon(\mathbf{r}), \quad (4.2)$$

where g_ε is a phase factor, χ_ε a function that vanishes at the singularities of g_ε, and ρ_ε a suitable regularization of ρ^{TF}. More precisely, ρ_ε is defined as in Lemma 2.3 in Lieb *et al.* (2001), i.e., $\rho_\varepsilon \equiv j_\varepsilon \star \rho^{\mathrm{TF}}$, with

$$j_\varepsilon(r) \equiv \frac{1}{2\pi\varepsilon^2}\exp\left\{-\frac{r}{\varepsilon}\right\}. \quad (4.3)$$

Since $\|j_\varepsilon\|_1=1$, $\sqrt{\rho_\varepsilon}$ is L^2 normalized. It is also clear that ρ_ε converges uniformly to ρ^{TF} as $\varepsilon\to 0$ and it is uniformly bounded in ε, i.e., there exists a constant C_{ω_0} such that $\rho_\varepsilon \leq C_{\omega_0}$. Furthermore, although ρ_ε is not compactly supported, it is exponentially small in ε for $\mathbf{r}$ sufficiently far from the support of ρ^{TF}: For any $\mathbf{r}\in\mathbb{R}^2$, $r>R_{\mathrm{out}}$,

$$\rho_\varepsilon(\mathbf{r}) = \frac{1}{2\pi\varepsilon^2}\int_{r\leq R_{\mathrm{out}}} d\mathbf{r}' \exp\left\{-\frac{|\mathbf{r}-\mathbf{r}'|}{\varepsilon}\right\}\rho^{\mathrm{TF}}(r') \leq \frac{1}{2\pi\varepsilon^2}\exp\left\{-\frac{r-R_{\mathrm{out}}}{\varepsilon}\right\}. \quad (4.4)$$

Moreover, two different estimates for the gradient of ρ_ε hold true: By using the fact that $|\nabla j_\varepsilon| = \varepsilon^{-1}j_\varepsilon$, one can easily prove that $|\nabla\rho_\varepsilon| \leq \varepsilon^{-1}|\rho_\varepsilon|$, whereas by exploiting the regularity of ρ^{TF}, i.e., $\|\nabla\rho^{\mathrm{TF}}\|_1 \leq C_{\omega_0}$, one has $\|\nabla\rho_\varepsilon\|_1 \leq C_{\omega_0}$. Using both estimates, we immediately get the bound

$$\|\nabla\sqrt{\rho_\varepsilon}\|_2^2 \leq C_{\omega_0}/\varepsilon. \quad (4.5)$$

The phase factor g_ε and the cutoff function χ_ε are defined as in Correggi *et al.* (2007) by placing vortices of degree 1 at the points of the square lattice

$$\mathcal{L} = \{\mathbf{r}_j = (m\ell_\varepsilon, n\ell_\varepsilon), m,n \in \mathbb{Z} | r \leqslant 2R_{\mathrm{out}} - 2\sqrt{2}\ell_\varepsilon\}, \tag{4.6}$$

with spacing $\ell_\varepsilon = \delta\sqrt{\varepsilon}$ for some $\delta > 0$: Using complex notation $\zeta = x + iy$ for $\mathbf{r} = (x,y) \in \mathbb{R}^2$, we define

$$g_\varepsilon(\zeta) = \prod_{\zeta_j \in \mathcal{L}} \frac{\zeta - \zeta_j}{|\zeta - \zeta_j|}, \tag{4.7}$$

$$\chi_\varepsilon(\mathbf{r}) = \begin{cases} 1 & \text{if} |\mathbf{r} - \mathbf{r}_j| \geqslant \varepsilon^\eta \\ \dfrac{|\mathbf{r} - \mathbf{r}_j|}{\varepsilon^\eta} & \text{if} |\mathbf{r} - \mathbf{r}_j| \leqslant \varepsilon^\eta, \end{cases} \tag{4.8}$$

for some $\eta > 5/2$. Note that the vortex lattice $\mathcal{L}$ has the same spacing as in Correggi *et al.* (2007) but it is extended to cover the whole of the support of ρ^{TF}. The number N_ε of vortices and the normalization constant c_ε satisfy the bounds $N_\varepsilon \leqslant C_{\omega_0,\delta}/\varepsilon$, due to (4.6), and $1 \leqslant c_\varepsilon^2 \leqslant 1 + o(\varepsilon^4)$, since $\chi_\varepsilon \leqslant 1$ and $\eta > 5/2$. By setting

$$\Lambda \equiv \mathcal{B}_{2R_{\mathrm{out}}} \setminus \bigcup_{j \in \mathcal{L}} \mathcal{B}_\varepsilon^j,$$

where $\mathcal{B}_\varepsilon^j$ is a ball of radius ε^η centered at $\mathbf{r}_j$, we also have

$$\|\nabla \chi_\varepsilon\|_2^2 \leqslant \frac{1}{\varepsilon^{2\eta}} \int_{\cup_{j\in\mathcal{L}} \mathcal{B}_\varepsilon^j} d\mathbf{r} |\nabla |\mathbf{r} - \mathbf{r}_j||^2 \leqslant CN_\varepsilon \leqslant \frac{C_{\omega_0,\delta}}{\varepsilon}. \tag{4.9}$$

The evaluation of the GP functional on Ψ_ε gives

$$\begin{aligned} \mathcal{E}^{\mathrm{GP}}[\Psi_\varepsilon] &\leqslant C_1 \int_{\mathbb{R}^2} d\mathbf{r} |\nabla \sqrt{\rho_\varepsilon}|^2 + C_2 \int_{\mathbb{R}^2} d\mathbf{r} |\nabla \chi_\varepsilon|^2 + \int_{\mathbb{R}^2} d\mathbf{r} \rho_\varepsilon \chi_\varepsilon^2 |(\nabla - i\mathbf{A}_\varepsilon) g_\varepsilon|^2 + \frac{\mathcal{E}^{\mathrm{TF}}[|\Psi_\varepsilon|^2]}{\varepsilon^2} \\ &\leqslant \int_{\mathbb{R}^2} d\mathbf{r} \rho_\varepsilon \chi_\varepsilon^2 |(\nabla - i\mathbf{A}_\varepsilon) g_\varepsilon|^2 + \frac{\mathcal{E}^{\mathrm{TF}}[|\Psi_\varepsilon|^2]}{\varepsilon^2} + \frac{C_{\omega_0,\delta}}{\varepsilon}, \end{aligned} \tag{4.10}$$

where we have used the bounds (4.5) and (4.9) for the kinetic energies of $\sqrt{\rho_\varepsilon}$ and χ_ε.

We can split the first term in (4.10) into the contributions from $\mathcal{B}_{2R_{\mathrm{out}}}$ and its complement, respectively. Moreover, exploiting the pointwise estimate for $r \geqslant 2R_{\mathrm{out}}$,

$$|(\nabla - i\mathbf{A}_\varepsilon) g_\varepsilon| \leqslant |\nabla g_\varepsilon| + \frac{C_{\omega_0}}{\varepsilon} \leqslant \sum_{j \in \mathcal{L}} \frac{1}{|\mathbf{r} - \mathbf{r}_j|} + \frac{C_{\omega_0}}{\varepsilon} \leqslant \frac{C_{\omega_0,\delta}}{\varepsilon^{3/2}}, \tag{4.11}$$

and the exponential smallness (4.4), one has

$$\int_{r \geqslant 2R_{\mathrm{out}}} d\mathbf{r} \rho_\varepsilon \chi_\varepsilon^2 |(\nabla - i\mathbf{A}_\varepsilon) g_\varepsilon|^2 \leqslant \frac{C_{\omega_0,\delta}}{\varepsilon^8} \int_{2R_{\mathrm{out}}}^\infty dr\, r \exp\left\{-\frac{r - R_{\mathrm{out}}}{\varepsilon}\right\} \leqslant \frac{C_{\omega_0,\delta}}{\varepsilon^6} \exp\left\{-\frac{R_{\mathrm{out}}}{\varepsilon}\right\}.$$

The remaining contribution can be estimated exactly as in Correggi *et al.* (2007) (see the proof of Theorem 2.1),

$$\int_{r\leq 2R_{\text{out}}} d\mathbf{r}\rho_\varepsilon\chi_\varepsilon^2|(\nabla - i\mathbf{A}_\varepsilon)g_\varepsilon|^2 \leq C_1\int_\Lambda d\mathbf{r}|(\nabla - i\mathbf{A}_\varepsilon)g_\varepsilon|^2 + C_2\int_{\cup_{j\in\mathcal{L}}\mathcal{B}_\varepsilon^j} d\mathbf{r}\chi_\varepsilon^2|(\nabla - i\mathbf{A}_\varepsilon)g_\varepsilon|^2,$$

and an estimate similar to (4.11) yields, for $\mathbf{r}\in\mathcal{B}_\varepsilon^j$,

$$|\nabla g_\varepsilon| \leq \sum_{k\in\mathcal{L}}\frac{1}{|\mathbf{r}-\mathbf{r_k}|} \leq \frac{1}{|\mathbf{r}-\mathbf{r_j}|} + \frac{N_\varepsilon}{\inf_{j\neq k}|\mathbf{r}_j - \mathbf{r}_k|} \leq \frac{1}{|\mathbf{r}-\mathbf{r_j}|} + \frac{N_\varepsilon}{\ell_\varepsilon},$$

so that

$$\int_{\cup_{j\in\mathcal{L}}\mathcal{B}_\varepsilon^j} d\mathbf{r}\chi_\varepsilon^2|(\nabla - i\mathbf{A}_\varepsilon)g_\varepsilon|^2 \leq 2\int_{\cup_{j\in\mathcal{L}}\mathcal{B}_\varepsilon^j} d\mathbf{r}\chi_\varepsilon^2|\nabla g_\varepsilon|^2 + 2\int_{\cup_{j\in\mathcal{L}}\mathcal{B}_\varepsilon^j} d\mathbf{r}|\mathbf{A}_\varepsilon|^2 \leq \frac{C_1|\cup_{j\in\mathcal{L}}\mathcal{B}_\varepsilon^j|}{\varepsilon^{2\eta}} + \frac{C_2N_\varepsilon^3}{\varepsilon^{1-2\eta}} + \frac{C_3N_\varepsilon}{\varepsilon^{2-2\eta}} \leq \frac{C_{\omega_0,\delta}}{\varepsilon}.$$

The bound (4.10) then becomes

$$\mathcal{E}^{\text{GP}}[\Psi_\varepsilon] \leq C\int_\Lambda d\mathbf{r}|(\nabla - i\mathbf{A}_\varepsilon)g_\varepsilon|^2 + \frac{\mathcal{E}^{\text{TF}}[|\Psi_\varepsilon|^2]}{\varepsilon^2} + \frac{C_{\omega_0,\delta}}{\varepsilon}. \tag{4.12}$$

We now observe that the upper bound estimate for the first term in the right hand side (rhs) of the above expression can be simply taken over from Theorem 3.1 in Correggi *et al.* (2007): A simple rescaling by $2R_{\text{out}}$ immediately yields

$$\int_\Lambda d\mathbf{r}|(\nabla - i\mathbf{A}_\varepsilon)g_\varepsilon|^2 \leq 4R_{\text{out}}^2\left(\frac{\pi}{2\varepsilon^2}\left(\frac{\omega_0}{2} - \frac{\pi}{\delta^2}\right)^2 + \frac{C_{\omega_0,\delta}|\log\varepsilon|}{\varepsilon}\right),$$

and, therefore, choosing $\delta = \sqrt{2\pi/\omega_0}$,

$$\int_\Lambda d\mathbf{r}|(\nabla - i\mathbf{A}_\varepsilon)g_\varepsilon|^2 \leq \frac{C_{\omega_0,\delta}|\log\varepsilon|}{\varepsilon}. \tag{4.13}$$

For the upper bound of the second term in (4.12), we proceed as in Lieb *et al.* (2001): Denoting $W(r)\equiv r^s - \omega_0^2r^2/4$, one has

$$\mathcal{E}^{\text{TF}}[|\Psi_\varepsilon|^2] - E^{\text{TF}} \leq \mathcal{E}^{\text{TF}}[\rho_\varepsilon] - \mathcal{E}^{\text{TF}}[\rho^{\text{TF}}] + o(\varepsilon) \leq \int_{\mathbb{R}^2} d\mathbf{r}\rho^{\text{TF}}(r)[j_\varepsilon \star W - W](r) + o(\varepsilon),$$

that is easily estimated using $|W(|\mathbf{r}-\varepsilon\mathbf{r}'|) - W(r)| \leq \varepsilon r' C(1+r^{s-1})$,

$$[j_\varepsilon \star W - W](r) = \frac{1}{2\pi}\int_{\mathbb{R}^2} d\mathbf{r}'[W(|\mathbf{r}-\varepsilon\mathbf{r}'|) - W(r)]e^{-r'} \leq \varepsilon C'(1+r^{s-1}). \tag{4.14}$$

We thus obtain the estimate $\mathcal{E}^{\text{TF}}[|\Psi_\varepsilon|^2] - E^{\text{TF}} \leq C_{\omega_0}\varepsilon$, and together with (4.13), this finally yields the upper bound for the GP energy, i.e., $\varepsilon^2E_\varepsilon^{\text{GP}} \leq \varepsilon^2\mathcal{E}^{\text{GP}}[\Psi_\varepsilon] \leq E^{\text{TF}} + C_{\omega_0}\varepsilon|\log\varepsilon|$. □

2. Proof of Corollary 3.1

Defining $2a(r)\equiv\mu^{\text{TF}} - r^s + (\omega_0^2/4)r^2$ for all $r\geq 0$ and using the negativity of $a(r)$ outside the support of ρ^{TF}, we have

$$\int_{\mathbb{R}^2} \mathrm{d}\mathbf{r}(|\Psi_\varepsilon^{\mathrm{GP}}|^2-\rho^{\mathrm{TF}})^2 \leqslant \int_{\mathbb{R}^2} \mathrm{d}\mathbf{r}[|\Psi_\varepsilon^{\mathrm{GP}}|^4-2a(r)|\Psi_\varepsilon^{\mathrm{GP}}|^2+(\rho^{\mathrm{TF}})^2]=\mathcal{E}^{\mathrm{TF}}[|\Psi_\varepsilon^{\mathrm{GP}}|^2]-E^{\mathrm{TF}},$$

since $\|\rho^{\mathrm{TF}}\|_2^2=\mu^{\mathrm{TF}}-E^{\mathrm{TF}}$. The inequality $\mathcal{E}^{\mathrm{TF}}[|\Psi_\varepsilon^{\mathrm{GP}}|^2]\leqslant\varepsilon^2 E_\varepsilon^{\mathrm{GP}}$ and Theorem 3.1 thus imply the result. □

Using Theorem 3.1 and Corollary 3.1 we can now show that the density of the minimizer $\Psi_\varepsilon^{\mathrm{GP}}$ is actually exponentially small outside the support of ρ^{TF}.

3. Proof of Theorem 3.2

The bound (3.7) can be derived similarly to Proposition 2.4 in Correggi *et al.* (2007) or Proposition 2 in Aftalion *et al.* (2005). We present here only the proof of the first statement, since the second one is obtained exactly in the same way.

The variational equation satisfied by $\Psi_\varepsilon^{\mathrm{GP}}$ is

$$-\Delta\Psi_\varepsilon^{\mathrm{GP}}-\frac{\omega_0}{\varepsilon}L\Psi_\varepsilon^{\mathrm{GP}}+\frac{2}{\varepsilon^2}|\Psi_\varepsilon^{\mathrm{GP}}|^2\Psi_\varepsilon^{\mathrm{GP}}+\frac{r^s}{\varepsilon^2}\Psi_\varepsilon^{\mathrm{GP}}=\mu_\varepsilon\Psi_\varepsilon^{\mathrm{GP}},$$

where the GP chemical potential μ_ε is fixed by $\|\Psi_\varepsilon^{\mathrm{GP}}\|_2=1$. Setting $U_\varepsilon\equiv|\Psi_\varepsilon^{\mathrm{GP}}|^2$ and using

$$\frac{\omega_0}{\varepsilon}|\Psi_\varepsilon^{\mathrm{GP}*}L\Psi_\varepsilon^{\mathrm{GP}}|\leqslant|\nabla\Psi_\varepsilon^{\mathrm{GP}}|^2+\frac{\omega_0^2r^2|\Psi_\varepsilon^{\mathrm{GP}}|^2}{4\varepsilon^2}, \tag{4.15}$$

we get

$$-\tfrac{1}{2}\Delta U_\varepsilon\leqslant\left[\frac{\omega_0^2r^2}{4}+\varepsilon^2\mu_\varepsilon-r^s-2U_\varepsilon\right]\frac{U_\varepsilon}{\varepsilon^2}.$$

The definition of μ_ε, Theorem 3.1, and Corollary 3.1 imply

$$\begin{aligned}\varepsilon^2\mu_\varepsilon&=\varepsilon^2\mathcal{E}^{\mathrm{GP}}[\Psi_\varepsilon^{\mathrm{GP}}]+\int_{\mathbb{R}^2}\mathrm{d}\mathbf{r}|\Psi_\varepsilon^{\mathrm{GP}}|^4\leqslant E^{\mathrm{TF}}+C_{\omega_0}\varepsilon|\log\varepsilon|+\int_{\mathbb{R}^2}\mathrm{d}\mathbf{r}|\Psi_\varepsilon^{\mathrm{GP}}|^4\\&=\mu^{\mathrm{TF}}+C_{\omega_0}\varepsilon|\log\varepsilon|+\int_{\mathbb{R}^2}\mathrm{d}\mathbf{r}[|\Psi_\varepsilon^{\mathrm{GP}}|^4-(\rho^{\mathrm{TF}})^2]\leqslant\mu^{\mathrm{TF}}+C_{\omega_0}\sqrt{\varepsilon|\log\varepsilon|}.\end{aligned} \tag{4.16}$$

On the other hand, since $a'(R_{\mathrm{out}})<-C_{\omega_0}<0$ [see Eq. (2.5)], a simple Taylor expansion of $a(r)$ in a neighborhood of R_{out} yields

$$a(R_{\mathrm{out}}+\varepsilon^{1/3}/2)\leqslant-C_{\omega_0}\varepsilon^{1/3}+\mathcal{O}(\varepsilon^{2/3})\leqslant-C'_{\omega_0}\varepsilon^{1/3},$$

for a possibly different constant $C'_{\omega_0}>0$. Hence,

$$-\varepsilon^2\Delta U_\varepsilon\leqslant2[2a(r)+C_{\omega_0}\sqrt{\varepsilon|\log\varepsilon|}]U_\varepsilon\leqslant Ca(r)U_\varepsilon<-C_{\omega_0}\varepsilon^{1/3}U_\varepsilon<0, \tag{4.17}$$

for any $\mathbf{r}\in\Theta_\varepsilon\equiv\{\mathbf{r}\in\mathbb{R}^2|r>R_{\mathrm{out}}+\varepsilon^{1/3}/2\}$ and ε sufficiently small. Thus, U_ε is subharmonic in Θ_ε, so that, for any $\mathbf{r}$ and ϱ with $\mathcal{B}_\varrho(\mathbf{r})\subset\Theta_\varepsilon$,

$$U_\varepsilon(\mathbf{r})\leqslant\frac{1}{\pi\varrho^2}\int_{\mathcal{B}_\varrho(r)}\mathrm{d}\mathbf{r}U_\varepsilon\leqslant\frac{1}{\sqrt{\pi}\varrho}\left[\int_{\mathcal{B}_\varrho(r)}\mathrm{d}\mathbf{r}U_\varepsilon^2\right]^{1/2}\leqslant\frac{1}{\sqrt{\pi}\varrho}\left[\int_{r\geqslant R_{\mathrm{out}}}\mathrm{d}\mathbf{r}U_\varepsilon^2\right]^{1/2}\leqslant\frac{C_{\omega_0}\sqrt{\varepsilon|\log\varepsilon|}}{\varrho},$$

where we have used (3.6). If now we take $\mathbf{r}\in\mathcal{T}_\varepsilon^{\mathrm{out}}$ and choose $\varrho=\varepsilon^{1/3}/2$, we have

$$U_\varepsilon(\mathbf{r}) \leqslant C_{\omega_0}\varepsilon^{1/6}\sqrt{|\log \varepsilon|},$$

so that $U_\varepsilon(\mathbf{r})$ converges pointwise to 0 in $\mathcal{T}_\varepsilon^{\mathrm{out}}$. Moreover, from (4.17), it follows that U_ε is a subsolution in $\mathcal{T}_\varepsilon^{\mathrm{out}}$ of

$$\begin{cases} -\Delta w + C_{\omega_0}\varepsilon^{-5/3}w = 0, \\ w(\partial\mathcal{T}_\varepsilon^{\mathrm{out}}) = C_{\omega_0}\varepsilon^{1/6}\sqrt{|\log \varepsilon|}, \end{cases} \tag{4.18}$$

whereas the rhs of (3.7) is a supersolution of the same problem for ε sufficiently small. The result is then a consequence of the comparison principle. □

B. The regime $\omega(\varepsilon) \gg 1/\varepsilon$

1. Proof of Theorem 3.3

In order to capture the leading order term in the GP energy asymptotics, it is convenient to rescale the GP functional in the following way: Setting $\tilde{\mathcal{E}}^{\mathrm{GP}}[\tilde{\Psi}] \equiv \varepsilon^2 R_{\mathrm{m}}^{-s}\mathcal{E}^{\mathrm{GP}}[\Psi]$, with $\tilde{\Psi}(\mathbf{x}) \equiv R_{\mathrm{m}}\Psi(\mathbf{r})$ and $\mathbf{x} \equiv R_{\mathrm{m}}^{-1}\mathbf{r}$, we have (remember that R_{m} depends on ε through $\omega_0(\varepsilon) = \omega_1/\varepsilon^\alpha$)

$$\tilde{\mathcal{E}}^{\mathrm{GP}}[\tilde{\Psi}] = \varepsilon^2 R_{\mathrm{m}}^{-(s+2)}\int_{\mathbb{R}^2} \mathrm{d}\mathbf{x} |(\nabla - i\mathcal{A}_\varepsilon)\tilde{\Psi}|^2 + \tilde{\mathcal{E}}_\varepsilon^{\mathrm{TF}}[|\tilde{\Psi}|^2], \tag{4.19}$$

where $\tilde{\mathcal{E}}_\varepsilon^{\mathrm{TF}}$ is the TF functional (2.30) with ω_0 replaced with $\omega_1/\varepsilon^\alpha$ and

$$\mathcal{A}_\varepsilon \equiv \frac{\omega(\varepsilon)R_{\mathrm{m}}^2}{2}\mathbf{e}_z \times \mathbf{x}.$$

The proof is thus similar to that of Proposition 2.1: By neglecting the (positive) first term in (4.19), we get the lower bound

$$\tilde{\mathcal{E}}^{\mathrm{GP}}[\tilde{\Psi}] \geqslant 1 - \frac{s}{2}. \tag{4.20}$$

In order to obtain a corresponding upper bound, we test the functional on a trial function $\tilde{\Psi}_{\xi,\varepsilon}$ similar to the one used in the proof of Proposition 2.1, i.e.,

$$\tilde{\Psi}_{\xi,\varepsilon}(\mathbf{x}) \equiv \sqrt{\tilde{\rho}_\xi(x)}\exp\left\{ i\left[\frac{\omega(\varepsilon)R_{\mathrm{m}}^2}{2}\right]\vartheta\right\}, \tag{4.21}$$

where $[\cdots]$ stands for the integer part and the density $\tilde{\rho}_\xi(x)$ is defined in (2.32) (we additionally require that $\|\nabla\sqrt{j}\|_2 < \infty$).

The estimate of the second term in (4.19) is thus already done in (2.33). It remains to bound the kinetic energy of $\tilde{\Psi}_{\xi,\varepsilon}$, i.e.,

$$\int_{\mathbb{R}^2} \mathrm{d}\mathbf{x} |(\nabla - i\mathcal{A}_\varepsilon)\tilde{\Psi}_{\xi,\varepsilon}|^2 = \int_{\mathbb{R}^2} \mathrm{d}\mathbf{x} |\nabla\sqrt{\tilde{\rho}_\xi}|^2 + \int_{\mathbb{R}^2} \mathrm{d}\mathbf{x}\left\{\frac{1}{x}\left[\frac{\omega(\varepsilon)R_{\mathrm{m}}^2}{2}\right] - \frac{\omega(\varepsilon)R_{\mathrm{m}}^2 x}{2}\right\}^2 \tilde{\rho}_\xi(x).$$

Smoothness of $\tilde{\rho}_\xi$ (and the assumption $\|\nabla\sqrt{j}\|_2 < \infty$) yields the estimate

$$\int_{\mathbb{R}^2} \mathrm{d}\mathbf{x} |\nabla\sqrt{\tilde{\rho}_\xi}|^2 \leqslant C\xi^{-2},$$

while, for ξ sufficiently small,

$$\int_{\mathbb{R}^2} \mathrm{d}\mathbf{x} \left\{ \frac{1}{x}\left[\frac{\omega(\varepsilon)R_{\mathrm{m}}^2}{2}\right] - \frac{\omega(\varepsilon)R_{\mathrm{m}}^2 x}{2} \right\}^2 \tilde{\rho}_\xi(x) \leq \omega^2(\varepsilon)R_{\mathrm{m}}^4 \int_{\mathbb{R}^2} \mathrm{d}\mathbf{x} \frac{(1-x^2)^2}{x^2}\tilde{\rho}_\xi(x) + \int_{\mathbb{R}^2} \mathrm{d}\mathbf{x} \frac{\tilde{\rho}_\xi(x)}{x^2}$$

$$\leq C_1 \omega^2(\varepsilon) R_{\mathrm{m}}^4 \xi^{-1} \int_{1-\xi/2}^{1+\xi/2} \mathrm{d}z \frac{(1-z)^2}{z} j(\xi^{-1}(1-z)) + C_2$$

$$\leq C\omega^2(\varepsilon)R_{\mathrm{m}}^4 \xi^2.$$

Altogether, we get the bound

$$\tilde{\mathcal{E}}^{\mathrm{GP}}[\tilde{\Psi}_{\xi,\varepsilon}] \leq 1 - \frac{s}{2} + \mathcal{O}(\xi^2) + \mathcal{O}(\varepsilon^{2\alpha(s+2)/(s-2)}\xi^{-1}) + \mathcal{O}(\varepsilon^2 \varepsilon^{2\alpha(s+2)/(s-2)}\xi^{-2}). \tag{4.22}$$

Optimizing with respect to the first two error terms, we obtain the same error term as in (2.31) and the last term gives only a higher order correction, as long as $\alpha \leq 3(s-2)/(s+2)$. On the other hand, for larger α, we consider the first and last terms in the above estimate and choose (in this case the second term can be neglected)

$$\xi = \sqrt{\varepsilon}\varepsilon^{\alpha(s+2)/2(s-2)},$$

which yields an overall remainder of order $\varepsilon\varepsilon^{\alpha(s+2)/s-2}$. □

2. Proof of Theorem 3.4

The first part of Theorem 3.4 can be proved exactly as in the proof of Theorem 3.2 and we omit the details. The weak convergence to a Dirac delta function supported at $x=1$ is a simple consequence of the pointwise estimate (3.12) together with the L^1 normalization of the density $|\tilde{\Psi}_\varepsilon^{\mathrm{GP}}(\mathbf{x})|^2$ (see, e.g., the discussion in Sec. II C).

V. ASYMPTOTICALLY HOMOGENEOUS POTENTIALS

We recall from Sec. I that a potential $V(r)$ is called asymptotically homogeneous if there are constants $\kappa, c > 0$ such that the estimate

$$|\lambda^{-s}V(\lambda r) - r^s| \leq c\lambda^{-\kappa}(1+r^s) \tag{5.1}$$

holds for all $\lambda \geq 1$ and all $r \in \mathbb{R}^+$. We discuss here briefly how the results for the trapping potential r^s can be extended to such potentials $V(r)$ with suitable modifications of the error terms.

The rescaling that produced (1.2) leads in the general case to

$$\mathcal{E}_V^{\mathrm{GP}}[\Psi] = \int_{\mathbb{R}^2} \mathrm{d}\mathbf{r} \left\{ |\nabla\Psi|^2 - \omega(\varepsilon)\Psi^* L\Psi + \frac{|\Psi|^2}{\varepsilon^2}[\varepsilon^{2s/(s-2)}V(\varepsilon^{-2/(s-2)}r) + |\Psi|^2] \right\}, \tag{5.2}$$

i.e., the functional contains the rescaled external potential $\lambda^{-s}V(\lambda r)$ with $\lambda = \varepsilon^{-2/(s-2)}$.

The estimates mentioned in Sec. III A for $\omega \ll 1/\varepsilon$ generalize to asymptotically homogeneous potentials in exactly the same way as in Lieb *et al.* (2001). For the case $\omega \sim 1/\varepsilon$, we have the following

Proposition 5.1: (GP energy and density asymptotics for $\omega \sim 1/\varepsilon$) *Let the external potential $V(r) \geq 0$ be asymptotically homogeneous of degree $s > 2$ in the sense of (1.9) and let $E_{\varepsilon,V}^{\mathrm{GP}}$ and $\Psi_{\varepsilon,V}^{\mathrm{GP}}$ denote the ground state energy and wave function of the functional (5.2). Then for any $\omega_0, \kappa > 0$ fixed and ε tending to 0,*

$$\varepsilon^2 E_{\varepsilon,V}^{\mathrm{GP}} = E^{\mathrm{TF}} + \mathcal{O}(\varepsilon|\log\varepsilon|) + \mathcal{O}(\varepsilon^{2\kappa/(s-2)}). \tag{5.3}$$

Furthermore, the density $|\Psi_{\varepsilon,V}^{\mathrm{GP}}|^2$ converges to ρ^{TF} strongly in $L^1(\mathbb{R}^2)$.

Proof: The proof requires only a minor modification of the proof of Theorem 3.1: Using (1.9), we can estimate

$$\varepsilon^2 |\mathcal{E}_V^{\mathrm{GP}}[\Psi] - \mathcal{E}^{\mathrm{GP}}[\Psi]| \leq c\varepsilon^{2\kappa/(s-2)} \int_{\mathbb{R}^2} d\mathbf{r}(1+r^s)|\Psi|^2, \tag{5.4}$$

so that the appropriate upper and lower bounds to $E_{\varepsilon,V}^{\mathrm{GP}}$ can be easily obtained: By testing the functional on $\Psi_\varepsilon^{\mathrm{GP}}$, we immediately get the upper bound $E_{\varepsilon,V}^{\mathrm{GP}} \leq E_\varepsilon^{\mathrm{GP}} + o(1)$, whereas taking $\Psi = \Psi_{\varepsilon,V}^{\mathrm{GP}}$ in the above inequality, one has the lower bound

$$\varepsilon^2 E_{\varepsilon,V}^{\mathrm{GP}} \geq \varepsilon^2 E_\varepsilon^{\mathrm{GP}} - c\varepsilon^{2\kappa/(s-2)} \int_{\mathbb{R}^2} d\mathbf{r} r^s |\Psi_{\varepsilon,V}^{\mathrm{GP}}|^2, \tag{5.5}$$

which yields the expected result, provided one can show that there exists a finite constant C_{ω_0}, such that

$$\int_{\mathbb{R}^2} d\mathbf{r} r^s |\Psi_{\varepsilon,V}^{\mathrm{GP}}|^2 \leq C_{\omega_0}. \tag{5.6}$$

On the other hand, evaluating $\mathcal{E}_V^{\mathrm{GP}}$ on a smooth radial function, we see that $\varepsilon^2 E_{\varepsilon,V}^{\mathrm{GP}} \leq C'_{\omega_0}$, for some finite constant C'_{ω_0}, so that

$$\int_{\mathbb{R}^2} d\mathbf{r} \left[\varepsilon^{2s/(s-2)} V(\varepsilon^{-2/(s-2)}\mathbf{r}) - \frac{\omega_0^2 r^2}{4} \right] |\Psi_{\varepsilon,V}^{\mathrm{GP}}|^2 \leq C'_{\omega_0},$$

but, using the trivial bound $(r_0 \equiv (\omega_0^2/2)^{1/(s-2)})$,

$$\int_{\mathbb{R}^2} d\mathbf{r} r^2 |\Psi_{\varepsilon,V}^{\mathrm{GP}}|^2 \leq \left(\frac{\omega_0^2}{2}\right)^{2/(s-2)} \int_{r\leq r_0} d\mathbf{r} |\Psi_{\varepsilon,V}^{\mathrm{GP}}|^2 + \tfrac{1}{2} \int_{r\geq r_0} d\mathbf{r} r^s |\Psi_{\varepsilon,V}^{\mathrm{GP}}|^2 \leq \left(\frac{\omega_0^2}{2}\right)^{2/(s-2)} + \tfrac{1}{2} \int_{\mathbb{R}^2} d\mathbf{r} r^s |\Psi_{\varepsilon,V}^{\mathrm{GP}}|^2,$$

together with (1.9), we get (5.6), i.e.,

$$\left(\tfrac{1}{2} - o(1)\right) \int_{\mathbb{R}^2} d\mathbf{r} r^s |\Psi_{\varepsilon,V}^{\mathrm{GP}}|^2 \leq C'_{\omega_0} + \frac{\omega_0^2}{4}\left(\frac{\omega_0^2}{2}\right)^{2/(s-2)} + o(1).$$

The energy asymptotics follows then from Theorem 3.1.

In order to prove the ground state density convergence, it is sufficient to note that (5.3) implies that $|\Psi_{\varepsilon,V}^{\mathrm{GP}}|^2$ is a minimizing sequence for the TF functional $\mathcal{E}^{\mathrm{TF}}$. The statement can be thus obtained by a simple compactness argument together with identity of norms, $\|\Psi_{\varepsilon,V}^{\mathrm{GP}}\|_2^2 = \|\rho^{\mathrm{TF}}\|_1 = 1$ [see, e.g., Theorem II.2 in Lieb *et al.* (2000)]. □

For $\omega(\varepsilon) = \omega_1/\varepsilon^{1+\alpha}$ we have the following generalization of Theorems 3.3 and 3.4.

Proposition 5.2: (GP energy and density asymptotics for $\omega \gg 1/\varepsilon$) *Let the external potential V satisfy the same conditions as in Proposition 5.1.*

Then for any fixed $\omega_1, \alpha, \kappa > 0$ and ε tending to zero,

$$\varepsilon^2 \varepsilon^{2\alpha s/(s-2)} E_{\varepsilon,V}^{\mathrm{GP}} = \left(\frac{\omega_1^2}{2s}\right)^{s/(s-2)} \left(1 - \frac{s}{2}\right) + \mathcal{O}(\varepsilon^{4\alpha(s+2)/3(s-2)}) + \mathcal{O}(\varepsilon\varepsilon^{\alpha(s+2)/(s-2)}) + \mathcal{O}(\varepsilon^{2\kappa/(s-2)}\varepsilon^{2\alpha\kappa}). \tag{5.7}$$

Furthermore, the rescaled density $|\tilde{\Psi}_{\varepsilon,V}^{\mathrm{GP}}(\mathbf{x})|^2 \equiv R_{\mathrm{m}}^2 |\Psi_{\varepsilon,V}^{\mathrm{GP}}(R_{\mathrm{m}}\mathbf{x})|^2$ converges in the sense of distributions to a Dirac delta function concentrated at $x=1$.

Proof: It is sufficient to rescale the functional (5.2) as in (4.19), i.e., setting $\tilde{\mathcal{E}}_V^{\mathrm{GP}}[\tilde{\Psi}] \equiv \varepsilon^2 R_{\mathrm{m}}^{-s} \mathcal{E}_V^{\mathrm{GP}}[\Psi]$,

$$\widetilde{\mathcal{E}}_V^{\rm GP}[\widetilde{\Psi}] = \varepsilon^2 R_{\rm m}^{-(s+2)} \int_{\mathbb{R}^2} {\rm d}\mathbf{x} |(\nabla - i\mathcal{A}_\varepsilon)\widetilde{\Psi}|^2 + \int_{\mathbb{R}^2} {\rm d}\mathbf{x} \left\{ \left[R_{\rm m}^{-s} \varepsilon^{2s/(s-2)} V(\varepsilon^{-2/(s-2)} R_{\rm m} x) - \frac{s x^2}{2} \right] |\widetilde{\Psi}|^2 + R_{\rm m}^{-s-2} |\widetilde{\Psi}|^4 \right\}$$

and proceed as in the proof of Proposition 5.1 to get the estimate

$$\varepsilon^2 R_{\rm m}^{-s} |\widetilde{E}_{\varepsilon,V}^{\rm GP} - \widetilde{E}_\varepsilon^{\rm GP}| \leq C_{\omega_1} \varepsilon^{2\kappa/(s-2)} R_{\rm m}^{-\kappa}.$$

Theorem 3.3 now yields the result. □

The rescaled density $|\widetilde{\Psi}_{\varepsilon,V}^{\rm GP}(\mathbf{x})|^2 \equiv R_{\rm m}^2 |\Psi_{\varepsilon,V}^{\rm GP}(R_{\rm m}\mathbf{x})|^2$ converges in the sense of distributions to a Dirac delta function concentrated at $x=1$ by the same arguments as before (cf. the proof of Theorem 3.4 in Sec. IV B).

VI. CONCLUSION

We have analyzed in some detail the TF limit of the GP energy and density of a rapidly rotating Bose-Einstein condensate in a two-dimensional trapping potential of the form r^s, $s>2$. After discussing the scaling of the variables (that is necessary because of spreading due to the interaction and centrifugal forces), we have estimated the energy with error terms whose order in the small parameter can be expected to be optimal and proved the concentration of the GP density on the support of the TF density apart from exponentially small terms. The extension to asymptotically homogeneous potentials and the corresponding change of the error terms has also been discussed.

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