Scuola Normale Superiore di Pisa

Tesi di Perfezionamento in Matematica

On the Chow ring of the stack of rational nodal curves

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Contents

Introduction

In this thesis we start investigating the intersection theory of the Artin stack $\mathfrak{M}_0$ of nodal curves of genus 0, following a suggestion of Rahul Pandharipande.

Intersection theory on moduli spaces of stable curves has a not so long, but very intense history. It started at the beginning of the '80's with Mumford's paper [**Mum2**], where he laid the foundations and carried out the first calculations. Many people have contributed to the theory after this (such as Witten and Kontsevich), building an imposing structure.

The foundations of intersection theory on Deligne-Mumford stacks have been developed by Gillet and Vistoli. The first step towards an intersection theory on general Artin stacks (like those that arise from looking at unstable curves) was the equivariant intersection theory that Edidin and Graham developed, following an idea of Totaro. Their theory associates a commutative graded Chow ring $A^*(\mathcal{M})$ with every smooth quotient stack $\mathcal{M}$ of finite type over a field.

Unfortunately many stacks of geometric interest are not known to be quotient stacks (the general question of when a stack is a quotient stack is not well understood). Later, A. Kresch developed an intersection theory for general Artin stacks; in particular, he associates a Chow ring $A^*(\mathcal{M})$ with every smooth Artin stack $\mathcal{M}$ locally of finite type over a field, provided some technical conditions hold, which are satisfied in particular for stacks of pointed nodal curves of fixed genus.

Since there is not yet a theory of Chow rings of such stacks that extends the theory of stacks of stable curves, there does not seem to be much else to do than look at specific examples: and the first example is the stack $\mathfrak{M}_0$ of nodal connected curves of genus 0. However, even this case turns out to be extremely complicated.

In this thesis we compute the rational Chow ring of the open substack $\mathfrak{M}_0^{\leq 3}$ consisting of nodal curves of genus 0 with at most 3 nodes: it is a $\mathbb{Q}$-algebra with 10 generators and 11 relations.

The techniques that we use, and the problems that we encounter, are discussed below.

Description of contents.

Chapter 1. The material here is fairly standard. We show that $\mathfrak{M}_0$ is an Artin stack. It is not of finite type; however, for each n the open substack $\mathfrak{M}_0^{\leq n}$ consisting of families of curves whose geometric fibers have at most 2 nodes is of finite type, and its complement has codimension n. The stack $\mathfrak{M}_0^0$ of smooth rational curves is simply the classifying stack $\mathrm{B}\mathbb{P}GL_2$.

We give an example of a smooth projective surface S with a family $C \to S$ of nodal curves of genus 0 with at most 2 nodes in each fiber, in which C is an algebraic space not scheme-like. This is a simple variant on Hironaka's standard example of a smooth threefold that is not a scheme, but does not seem to be in the literature.

We introduce one of the basic tools that we use, that is the stratification by topological type. The topological type of a nodal curve of genus 0 is represented by a tree, with the vertices corresponding to components and edges corresponding to nodes. For each tree Γ, there is a natural smooth locally closed substack $\mathfrak{M}_0^\Gamma$; we show that this is a quotient stack. This allows to apply Kresch's theory and define a Chow ring $A^*(\mathfrak{M}_0)$.

Finally, we study the geometry of $\mathfrak{M}_0^\Gamma$, and in Section 1.5 we describe the normal bundle to the embedding $\mathfrak{M}_0^\Gamma \hookrightarrow \mathfrak{M}_0$.

Chapter 2. We start by recalling some of the basic facts of equivariant intersection theory, and of Kresch's intersection theory on Artin stacks. We apply the first to the calculation of all the rings $A^*(\mathfrak{M}_0^\Gamma)\otimes\mathbb{Q}$ for all trees Γ with no more than three nodes.

Denote by $\mathfrak{M}_0^n$ the smooth locally closed substack of $\mathfrak{M}_0$ consisting of curves with exactly n nodes; this is a disjoint union of the $\mathfrak{M}_0^\Gamma$ over all trees Γ with n edges. Its closure in $\mathfrak{M}_0$ is the substack $\mathfrak{M}_0^{\geq n}$ consisting of curves with at least n nodes. We describe a procedure to extend each class in $\mathfrak{M}_0^n$ to a class in $\mathfrak{M}_0^{\geq n}$.

Our technique is to compute $A^*(\mathfrak{M}_0^{\leq n})\otimes\mathbb{Q}$ for $n \leq 3$ by induction on n. For $n = 0$ we have $\mathfrak{M}_0^0 = \mathrm{B}\mathbb{P}GL_2$, and this case is well understood.

The inductive step is based on the following fact: if $n \leq 4$, then the top Chern class of the normal bundle of $\mathfrak{M}_0^n$ into $\mathfrak{M}_0^{\leq n}$ is not a 0-divisor in $A^*(\mathfrak{M}_0^n)\otimes\mathbb{Q}$. As a consequence, by an elementary algebraic Lemma we can reconstruct the ring $A^*(\mathfrak{M}_0^{\leq n})\otimes\mathbb{Q}$ from the rings $A^*(\mathfrak{M}_0^{\leq n-1})\otimes\mathbb{Q}$ and $A^*(\mathfrak{M}_0^n)\otimes\mathbb{Q}$, provided that we have an explicit way of extending each class in $A^*(\mathfrak{M}_0^{\leq n-1})\otimes\mathbb{Q}$ to a class in $A^*(\mathfrak{M}_0^{\leq n})\otimes\mathbb{Q}$, and then computing the restriction of this extension to $A^*(\mathfrak{M}_0^n)\otimes\mathbb{Q}$.

As a corollary we see that the restriction homomorphism from $A^*(\mathfrak{M}_0^{\leq 3}) \otimes \mathbb{Q}$ to the product of the $A^*(\mathfrak{M}_0^{\Gamma}) \otimes \mathbb{Q}$ over all trees Γ with at most three edges is injective.

Chapter 3. Here we define the basic classes, the classes of strata and the Mumford classes.

For each tree Γ with δ nodes we get a class γ_Γ of codimension δ in $\mathfrak{M}_0$, the class of the closure of $\mathfrak{M}_0^\Gamma$ in $\mathfrak{M}_0$. We also compute the restriction of each γ_Γ to all the rings $A^*(\mathfrak{M}_0^{\Gamma'}) \otimes \mathbb{Q}$ for any other tree Γ'.

The other kinds of classes that we need are the *Mumford classes.* These should be defined as follows: let $\mathcal{C} \xrightarrow{\Pi} \mathfrak{M}_0$ be the universal curve. Call $K \in A^1(\mathcal{C}) \otimes \mathbb{Q}$ the first Chern class of the relative dualizing sheaf $\omega_{\mathcal{C}/\mathfrak{M}_0}$, and set

$$\kappa_i = \Pi_*(K^{i+1}).$$

However, here we encounter an unfortunate technical problem: the morphism $\mathcal{C} \to \mathfrak{M}_0$ is not projective, not even represented by schemes, as Example 1.4 shows: and in Kresch's theory one does not have arbitrary proper pushforwards, only pushforwards along projective morphisms.

We are able to circumvent this problem only for curves with at most 3 nodes. This works as follows. We show that the pushforward $\Pi_*\omega^\vee_{\mathcal{C}/\mathfrak{M}_0}$ is a locally free sheaf of rank 3 on the the open substack $\mathfrak{M}_0^{\leq 3}$ (this fails for curves with 4 nodes). Hence we have Chern classes c_1, c_2 and c_3 of $\Pi_*\omega^\vee_{\mathcal{C}/\mathfrak{M}_0}$ in $A^*(\mathfrak{M}_0^{\leq 3})$.

We have a pushforward $\Pi_* A^*(\mathfrak{C}^\Gamma) \otimes \mathbb{Q} \to A^*(\mathfrak{M}_0^\Gamma) \otimes \mathbb{Q}$ along the restriction $\mathfrak{C}^\Gamma \xrightarrow{\Pi} \mathfrak{M}_0^\Gamma$ of the universal curve, because the stacks involved are quotient stacks, and arbitrary proper pushforwards exists in the theory of Edidin and Graham. This allows, with Grothendieck-Riemann-Roch, to compute the restriction of the Mumford classes to each $A^*(\mathfrak{M}_0^\Gamma) \otimes \mathbb{Q}$, even without knowing that the Mumford classes exist. It turns out that for each tree Γ with at most 3 nodes, the Mumford classes in $A^*(\mathfrak{M}_0^\Gamma) \otimes \mathbb{Q}$ are the Newton polynomials in the restrictions of c_1, c_2 and c_3, thought of as elementary symmmetric polynomials in three variables. Then we define the classes $\kappa_i \in A^*(\mathfrak{M}_0^{\leq 3}) \otimes \mathbb{Q}$ as the Newton polymomials in the c_1, c_2 and c_3.

Since $A^*(\mathfrak{M}_0^{\leq 3}) \otimes \mathbb{Q}$ injects into the product of $A^*(\mathfrak{M}_0^\Gamma) \otimes \mathbb{Q}$ over trees with at most three nodes, this gives the right definition.

Chapter 4. We put everything together, and we calculate $A^*(\mathfrak{M}_0^{\leq 3})$. We find 10 generators: the classes γ_Γ for all trees Γ with at most three nodes (they are 5), plus the Mumford class κ_2. The remaining 4 generators are somewhat unexpected.

To go beyond the case of three nodes there are two problems, the one mentioned above with the Mumford classes, and the fact that the inductive technique will break down for curves with five nodes, because the top Chern class of the normal bundle to $\mathfrak{M}_0^5$ in $\mathfrak{M}_0^{\leq 5}$ is a 0-divisor anymore. But also, it seems clear that the calculations will become quickly unwieldy, and one will need a unifying principle that is still missing.

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CHAPTER 1

Description of the stack $\mathfrak{M}_0$

1.1. Rational nodal curves

Let $\mathcal{S}ch_{\mathbb{C}}$ be the site of schemes over the complex point $\mathrm{Spec}\mathbb{C}$ equipped with the étale topology.

Definition 1.1. We define the category of rational nodal curves $\mathfrak{M}_0$ as the category over $\mathcal{S}ch_{\mathbb{C}}$ whose objects are flat and proper morphisms of finite presentation $C \xrightarrow{\pi} T$ (where C is an algebraic space over $\mathbb{C}$ and T is an object in $\mathcal{S}ch_{\mathbb{C}}$) such that for every geometric point $\mathrm{Spec}\Omega \xrightarrow{t} T$ (where Ω is an algebraically closed field) the fiber C_t

$$\begin{array}{ccc} C_t & \xrightarrow{t'} & C \\ {\scriptstyle \pi'}\downarrow & \square & \downarrow{\scriptstyle \pi} \\ \mathrm{Spec}\Omega & \xrightarrow{t} & T \end{array}$$

is a projective reduced nodal curve such that

$$\begin{aligned} h^0(C_t, \mathcal{O}_{C_t}) &= 1 \text{ (that is to say } C_t \text{ is connected;)} \\ h^1(C_t, \mathcal{O}_{C_t}) &= 0 \text{ (thus the arithmetic genus of the curve is 0).} \end{aligned}$$

Morphisms in $\mathfrak{M}_0$ are cartesian diagrams

$$\begin{array}{ccc} C' & \longrightarrow & C \\ {\scriptstyle \pi'}\downarrow & \square & \downarrow{\scriptstyle \pi} \\ T' & \longrightarrow & T \end{array}$$

The projection $pr : \mathfrak{M}_0 \to \mathcal{S}ch_{\mathbb{C}}$ is the forgetful functor

$$pr : \left\{ \begin{array}{ccc} C' & \longrightarrow & C \\ {\scriptstyle \pi'}\downarrow & \square & \downarrow{\scriptstyle \pi} \\ T' & \longrightarrow & T \end{array} \right\} \mapsto \{T' \to T\}$$

Proposition 1.2. *The category $\mathfrak{M}_0$ is fibered in groupoids over $\mathcal{S}ch_{\mathbb{C}}$ in the sense of* [**De-Mu**].

Proof. This proof is straightforward owing to the existence of fibered products for algebraic spaces (see Chapter 2 [**Knu**] Proposition 1.5) and the fact that flat and proper maps are stable for base change. $\square$

Now we study the geometric fibers of the curves in $\mathfrak{M}_0$. Given an algebraically closed field Ω, let us define a *rational tree (on Ω)* to be a connected nodal curve with finite components each of them is isomorphic to $\mathbb{P}^1_\Omega$ and which has no closed chains.

Proposition 1.3. *Let Ω be an algebraically closed field extention of $\mathbb{C}$. A curve $C \xrightarrow{\pi} Spec\Omega$ is an object of $\mathfrak{M}_0(Spec\Omega)$ if and only if it is a rational tree.*

Proof. Let $C \xrightarrow{\pi} \mathrm{Spec}\Omega$ be an object in $\mathfrak{M}_0(\mathrm{Spec}\Omega)$. Owing to the fact that π is a proper map, it has finite irreducible components. Let us consider one among these components $C_0 \subseteq C$. The curve C_0 is irreducible and by definition reduced, consequently it is normal. Let $\mathcal{I}$ to be the ideal sheaf of C_0 in C.

From the following exact sheaf sequence of sheaves

$$0 \to \mathcal{I} \to \mathcal{O}_C \to \mathcal{O}_{C_0} \to 0$$

we have the exact cohomologic sequence

$$H^1(C, \mathcal{O}_C) \to H^1(C_0, \mathcal{O}_{C_0}) \to H^2(C, \mathcal{I}).$$

Now by definition $H^1(C, \mathcal{O}_C) = 0$, furthermore $\dim_\Omega C = 1$, so we have $H^2(C, \mathcal{I}) = 0$. Consequently we have $H^1(C_0, \mathcal{O}_{C_0}) = 0$. The curve C_0 is therefore projective, normal and with genus zero, so it is isomorphic to $\mathbb{P}^1_\Omega$.

Let us suppose that the curve C has a closed chain and fix a singular point P of this chain. Consider the desingularization of C in P

$$C' \xrightarrow{\rho} C.$$

We have for all $i \geq 0$ (see Ex. 8.2 [**Hrt**])

$$H^i(C', \mathcal{O}_{C'}) \cong H^i(C, \rho_* \mathcal{O}_{C'}).$$

Then consider the following exact sheaf sequence

$$0 \to \mathcal{O}_C \to \rho_* \mathcal{O}_{C'} \to \mathcal{K}_P \to 0$$

where $\mathcal{K}_P$ is the skyscraper sheaf on P. The related cohomologic exact sequence is

$$0 \to \Omega \to H^0(C', \mathcal{O}_{C'}) \to \Omega \to H^1(C, \mathcal{O}_C) \to H^1(C', \mathcal{O}_{C'}) \to 0.$$

Owing to the fact that C' is connected, we have that the group $H^0(C', \mathcal{O}_{C'})$ maps to 0 and so we have

$$H^1(C, \mathcal{O}_C) \cong H^1(C', \mathcal{O}_{C'}) \oplus \Omega,$$

thus $H^1(C, \mathcal{O}_C)$ cannot be 0.

Consequently the curve C is a rational tree.

On the other hand let us suppose that the curve $C \xrightarrow{\pi} \mathrm{Spec}\Omega$ is a rational tree. Since the curve C is connected, we have that $h^0(C, \mathcal{O}_C) = 1$. Now we prove that

$$h^1(C, \mathcal{O}_C) = 0 \tag{1}$$

by induction on the number of components. If $C = \mathbb{P}^1_k$ we have $h^0(C, \mathcal{O}_C) = 1$ and $h^1(C, \mathcal{O}_C) = 0$.

Let us consider again the desingularization C' of C at a singular point P. We have again the exact sequence

$$0 \to \Omega \to H^0(C', \mathcal{O}_{C'}) \to \Omega \to H^1(C, \mathcal{O}_C) \to H^1(C', \mathcal{O}_{C'}) \to 0,$$

but now we have that C' has two connected components and so $h^0(C', \mathcal{O}_{C'}) = 2$, that is to say that

$$h^1(C, \mathcal{O}_C) = h^1(C', \mathcal{O}_{C'}).$$

By induction the two connected components of C' have both arithmetic genus 0, consequently also $h^1(C', \mathcal{O}_{C'})$ is equal to 0. □

It is important to notice that by definition geometric fibers of a family of curves $C \xrightarrow{\pi} T$ in $\mathfrak{M}_0$ are projective and therefore are schemes. But it is further known (see [**Knu**] V Theorem 4.9) that a curve (as an algebraic space) over an algebraically closed field Ω is a scheme.

Notwithstanding the fact that we consider families of curves whose fibers and bases are schemes we still need algebraic spaces because there exist families of rational nodal curves $C \xrightarrow{\pi} T$ where C is not a scheme.

Let us consider the following example which is based on Hironaka's example [**Hir**] of an analytic threefold which is not a scheme (this example can be found also in [**Knu**], [**Hrt**] and [**Shf**]).

Example 1.4. Let S be a projective surface over $\mathbb{C}$ and i an involution without fixed points. Suppose that there is a smooth curve C such that C meets transversally the curve $i(C)$ in two points P and Q. Let M be the product $S \times \mathbb{P}^1_{\mathbb{C}}$ and let $e : S \to M$ be the embedding $S \times 0$. On $M - e(Q)$, first blow up the curve $e(C - Q)$ and then blow up the strict transform of $e(i(C) - Q)$. On $M - e(P)$, first blow up the curve $e(i(C) - P)$ and then blow up the strict transform of $e(C - P)$. We can glue these two blown-up varieties along the inverse images of $M - e(P) - e(Q)$. The result is a nonsingular complete scheme $\widetilde{M}$.

We have an action of $\mathfrak{C}_2$ on M induced by the action of $\mathfrak{C}_2$ on S and the trivial action on $\mathbb{P}^1_{\mathbb{C}}$. Furthermore we can lift the action of $\mathfrak{C}_2$ to $\widetilde{M}$ such that we obtain an equivariant map

$$\widetilde{M} \xrightarrow{\pi} S;$$

we still call i the induced involution on $\widetilde{M}$. We have an action of $\mathfrak{C}$ on $\widetilde{M}$ which is faithful and we obtain geometric quotients f and g

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{f} & \widetilde{M}/\mathfrak{C}_2 \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\widetilde{\pi}} \\ S & \xrightarrow{g} & S/\mathfrak{C}_2 \end{array}$$

Now we notice that π is a family of rational nodal curves. In particular the generic geometric fiber is isomorphic to $\mathbb{P}^1_{\mathbb{C}}$. Geometric fibers on the two curves C and $i(C)$ have one node except those on P and Q which have two nodes. Similarly we have that also $\widetilde{\pi}$ is a family of rational nodal curves whose fibers with a node are on $g(C) = g(i(C))$ and there is only one fiber on $g(P) = g(Q)$.

Since $\mathfrak{C}_2$ acts faithfully on S and S is projective we have that $S/\mathfrak{C}_2$ is a scheme. Furthermore we mention (see [**Knu**] Chapter 4) that the category of separated algebraic spaces is stable under finite group actions. Consequently the family $\widetilde{\pi}$ belongs to $\mathfrak{M}_0(S/\mathfrak{C}_2)$. But $\widetilde{M}/\mathfrak{C}_2$ is not a scheme.

In order to prove this we follow the same argument of [**Hrt**] Appendix B Example 3.4.2. Let R be a point on the curve $C \subset S$ different from P and Q. The curve $\pi^{-1}(R) \subset \widetilde{M}$ has two rational components, we call l the one which comes from the blow up of M. In the same way we call l_0 and m_0 the two exceptional components of $\pi^{-1}(Q)$, where m_0 is the component with two nodes.

Set

$$\begin{aligned} m &:= i(l) \\ m_0' &:= i(l_0) \\ l_0' &:= i(m_0) \end{aligned}$$

and we have the following algebraic equivalences

$$\begin{aligned} l &\sim l_0 \\ m &\sim m_0' \\ l &\sim l_0' + m_0' \\ m &\sim l_0 + m_0 \end{aligned}$$

from which it follows

$$l_0 + m_0' \sim 0.$$

As $l_0 + m'_0$ is invariant for the involution we have also the following homological equivalence

$$2f(l_0) = f(l_0 + m'_0) \sim 0$$

from which $f(l_0) \sim 0$, and this show that $\widetilde{M}/\mathfrak{C}_2$ cannot be a scheme. Indeed let P be a point of $f(l_0)$. Then P has an affine neighborhood U in $\widetilde{M}/\mathfrak{C}_2$. Let H be an irreducible surface in U which passes through P but does not contain $f(l_0)$. Extend H by closure to a surface H in $\widetilde{M}/\mathfrak{C}_2$. Now H meets $f(l_0)$ in a finite nonzero number of points, so the intersection number of H with $f(l_0)$ is defined and is $\neq 0$. But the intersection number is defined on homology classes, so we cannot have $f(l_0) \sim 0$. Hence $\widetilde{M}/\mathfrak{C}_2$ is not sheme-like.

In order to complete our example, we have to exhibit a surface which satisfies the requested properties. Take the Jacobian $J_2 = \mathrm{Div}^0(C)$ of a genus 2 curve C. Let us choose on C two different points p_0 and q_0 such that $2(p_0 - q_0) \sim 0$. Let us consider the embedding

$$\begin{array}{rcl} C & \rightarrow & J_2 \\ p \in C & \mapsto & |p - q_0|. \end{array}$$

We still call C the image of the embedding. Let i be the translation on J_2 of $|p_0 - q_0|$. By definition i is an involution such that C and $i(C)$ meets transversally in two points: 0 and $|p_0 - q_0|$.

1.1.1. Stratifications. Given an object $C \xrightarrow{\pi} T$ in $\mathfrak{M}_0$, we have in C the *relative singular locus of* C, which will be denoted as C_{rs}, where geometric fibers have nodes. Further it is possible to define on T closed subschemes $\{T^{\geq k} \to T\}_{k \geq 0}$ where for each integer $k \geq 0$ the fiber product $T^{\geq k} \times_T C$ is a family of rational curves with at least k nodes. In order to give a structure of closed subspaces to C_{rs} and to the various subschemes $T^{\geq k}$, we will describe them through Fitting ideals.

We construct the closed subspace C_{rs} and the $T^{\geq k}$ locally in the Zariski topology, so we consider a family of rational nodal curve $C \xrightarrow{\pi}$ $\mathrm{Spec} A$ for some $\mathbb{C}-$algebra A.

Now we follow [**Mum1**] Lecture 8.

As the map π has relative dimension 1 the relative differential sheaf $\Omega_{C/T}$ has rank 1. Further because π is a map of finite presentation and A is a Noetherian ring, we have that the sheaf $\Omega_{C/T}$ is coherent. For every point $p \in C$ we set

$$e(p) = \dim_{k(p)}(\Omega_{C/T,p} \otimes k(p))$$

where $k(p)$ is the residual field of p.

Choose $\{a_i\}_{i=1,\dots,e(p)}$ in $\Omega_{C/T}$ whose images in $\Omega_{C/T,p} \otimes k(p)$ are a basis of this vector space, then these a_i extend to a generating system for $\Omega_{C/T,p}$. Furthermore we have an extension of this generating system to $\Omega_{C/T}$ restricted to an étale neighborhood of p. So we have a map

$$\mathcal{O}_C^{\oplus e(p)} \to \Omega_{C/T}$$

which is surjective up to a restriction to a possibly smaller neighborhood of p. At last (after a possibly further restriction) we have the following exact sequence of sheaves

$$\mathcal{O}_C^{\oplus f} \xrightarrow{A} \mathcal{O}_C^{\oplus e(p)} \to \Omega_{C/T} \to 0$$

where A is a suitable matrix $f \times e$ of local sections of $\mathcal{O}_C$. Let us indicate with $F_i(\Omega_{C/T}) \subset \mathcal{O}_C$ the ideal sheaf generated by rank $e(p) - i$ minors of A. These sheaves are known as *Fitting sheaves* and they don't depend on the choice of generators (see [**Lang**] XIX, Lemma 2.3).

Definition 1.5. We define the relative singular locus to be the closed algebraic subspace $C_{rs} \subseteq C$ associated with $F_1(\Omega_{C/T})$.

Now let us fix a point $t \in T$, we have that t belongs to $T^{\geq k}$ iff

$$F_{k-1}(\pi_*(\mathcal{O}_{C_{rs}})_t) = 0 \Longleftrightarrow \dim_{k(t)}(\pi_*(\mathcal{O}_{C_{rs}})_t \otimes k(t)) \geq k$$

so $T^{\geq k}$ is the subscheme whose geometric fibers have at least k nodes.

Set

$$T^0 := T - T^{\geq 1} \tag{2}$$

$$T^k := T^{\geq k} - T^{\geq k+1} \tag{3}$$

we have that for every $k \in \mathbb{N}$ the subscheme T^k is locally closed in T, further, given a point $t \in T$ and set

$$k := \dim_{k(t)}(\pi_*(\mathcal{O}_{C_{rs}})_t \otimes k(t)) \geq 0$$

we have that t belongs to a unique T^k.

Consequently, given a curve $C \xrightarrow{\pi} T$ where T is an affine Noetherian scheme, the family

$$\{T^k | T^k \neq \emptyset\}$$

defined above is *a stratification for* T.

Now we want to give a local description of this stratification.

First of all we consider an affine open subset Spec$A$ of T and we reduce to the restricted curve $C \xrightarrow{\pi}$ Spec$A$.

As π is of finite presentation by definition of $\mathfrak{M}_0$, from Theorem 8.9.1 [**EGA IV**] we have that there exists a cartesian diagram as follows [1]

$$\begin{array}{ccc} C & \xrightarrow{f} & C_0 \\ \pi \downarrow & \square & \downarrow \pi \\ \mathrm{Spec}A & \xrightarrow{g} & \mathrm{Spec}A_0 \end{array}$$

where A_0 is a Noetherian ring chosen from the subalgebras of A of finite type on $\mathbb{C}$ and π is a finite type morphism.

As A is the inductive limit of its sub-algebras A_λ of finite type on $\mathbb{C}$ such that

$$A_0 \subset A_\lambda,$$

for each λ we have the following

$$\begin{array}{ccccc} C & \xrightarrow{f} & C_\lambda & \xrightarrow{f} & C_0 \\ \pi \downarrow & \square & \downarrow \pi & \square & \downarrow \pi \\ \mathrm{Spec}A & \xrightarrow{g} & \mathrm{Spec}A_\lambda & \xrightarrow{g} & \mathrm{Spec}A_0 \end{array}$$

where, by abuse of notation, we call f all the arrows at the top and g all the arrows at the bottom.

For each λ we can find an open subset U_λ of $\mathrm{Spec}A_\lambda$ such that in the diagram

$$\begin{array}{ccccc} C & \xrightarrow{f} & C_\lambda & \xrightarrow{f} & C_0 \\ \pi \downarrow & \square & \downarrow \pi & \square & \downarrow \pi \\ \mathrm{Spec}A & \xrightarrow{g} & U_\lambda & \xrightarrow{g} & \mathrm{Spec}A_0 \end{array}$$

the map $C_\lambda \to U_\lambda$ is a curve in $\mathfrak{M}_0$.

If we construct for each curve $C_\lambda \xrightarrow{\pi} U_\lambda$ a closed sub space $C_{\lambda,rs} \subset C_\lambda$ such that

(*) for each $\mu \geq \lambda$ we have $C_{\mu,rs} = g^{-1}(C_{\lambda,rs})$

we obtain C_{rs} as inductive limit of the $C_{\lambda,rs}$. Similarly if we have for each $\mathrm{Spec}A_\lambda$ a closed subscheme $T_\lambda^{\geq k}$ such that we have the equivalent of property (*), we obtain canonically a subscheme $T^{\geq k}$ of T.

So we consider a family $C \xrightarrow{\pi} T$ where T is an affine noetherian scheme.

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[1] We explain here with some details what is called *reduction to the noetherian case* (see Section 8 [**EGA IV**]). In the following we just mention it when we refer to facts which in literature are proved requiring noetherianity for the base. It is anyhow fundamental that the map π be of finite presentation.

Let s be a point in C_{rs} and let us define $t := \pi(s)$. Let us write $R_t = \widehat{\mathcal{O}}_{T,t}$. As s in $C \times_T \mathrm{Spec} R_t$ is a simple node, π is flat and proper and T is noetherian, we have

$$\widehat{\mathcal{O}}_{C,s} = R_t[\![x,y]\!]/(xy - f_s)$$

where f_s belongs to R_t.

Let us consider the following diagram

$$\begin{array}{ccc} C_{rs} \times_C \mathrm{Spec}\widehat{\mathcal{O}}_{C,s} & \longrightarrow & \mathrm{Spec}\widehat{\mathcal{O}}_{C,s} \\ \downarrow & \square & \downarrow \\ C_{rs} & \longrightarrow & C \end{array}$$

$C_{rs} \times_C \mathrm{Spec}\widehat{\mathcal{O}}_{C,s}$ is defined from the ideal sheaf

$$F_1(\Omega_{C/T}) \otimes \widehat{\mathcal{O}}_{C,s} = F_1(\Omega_{C/T} \otimes \widehat{\mathcal{O}}_{C,s}).$$

Let us fix $\widehat{\Omega}_{C/T}$ as the sheaf $\Omega_{C/T} \otimes \widehat{\mathcal{O}}_{C,s}$ on $\mathrm{Spec}\widehat{\mathcal{O}}_{C,s}$. This sheaf can be written in the following way

$$\frac{\widehat{\mathcal{O}}_{C,s}\ dx \oplus \widehat{\mathcal{O}}_{C,s}\ dy}{y\ dx + x\ dy}$$

and locally admits the following exact sequence

$$\widehat{\mathcal{O}}_{C,s} \xrightarrow{A} \widehat{\mathcal{O}}^2_{C,s} \to \widehat{\Omega}_{C/T} \to 0,$$

where A is the matrix 1×2 : $(y\ \ x)$, the sheaf $F_1(\Omega_{C/T} \otimes \widehat{\mathcal{O}}_{C,P})$ is generated by minors whose rank is 1, that is to say x and y.

This means that the local neighborhood of s in C_{rs} is isomorphic as a scheme to $\mathrm{Spec}(R_t/f_s)$. This means that C_{rs} is locally isomorphic to a closed subscheme of T and the map $\pi : C_{rs} \to T$ is finite and it is not ramified.

As we have seen above, in order to have well defined C_{rs} and $T^{\geq k}$ for every scheme T we have to prove the following

Proposition 1.6. *Let $g : S \to T$ be a morphism of affine Noetherian schemes and let $C \xrightarrow{\pi} T$ be a curve on T. With reference to the following diagram*

$$\begin{array}{ccc} C_S & \xrightarrow{f} & C \\ {\scriptstyle\pi}\downarrow & \square & \downarrow{\scriptstyle\pi} \\ S & \xrightarrow{g} & T \end{array}$$

we have

$$C_{S,rs} = f^{-1}(C_{rs}) \qquad (4)$$

$$S^{\geq k} = g^{-1}(T^{\geq k}) \qquad (5)$$

Proof. The equivalence (4) follows from

$$\Omega_{C_S/S} = f^*(\Omega_{C/T}),$$

indeed

$$F_1(\Omega_{C_S/S}) = F_1(f^{-1}(\Omega_{C/T}) \otimes_{\mathcal{O}(C)} \mathcal{O}(C_S)) \cong f^{-1}(F_1(\Omega_{C/T}))\mathcal{O}_{C_S}$$

from which we have

$$\mathcal{O}_{C_S}/F_1(\Omega_{C_S/S}) \cong \big(\mathcal{O}_C/F_1(\Omega_{C/T})\big) \otimes_{\mathcal{O}_C} \mathcal{O}_{C_S}$$

In a similar way, in order to prove the equivalence (5), we have to show that for every integer $k \geq 1$ the following holds

$$\mathcal{O}_S/F_{k-1}(\pi_*(\mathcal{O}_{C_{S,rs}})) \cong (\mathcal{O}_T/F_{k-1}(\pi_*(\mathcal{O}_{C_{rs}}))) \otimes_{\mathcal{O}_T} \mathcal{O}_S.$$

We can proceed exactly as before if we have

$$\pi_*(\mathcal{O}_{C_{S,rs}}) = g^*(\pi_*(\mathcal{O}_{C_{rs}})$$

and this follows from what we have proved above

$$\mathcal{O}_{C_{S,rs}} \cong g^*(\mathcal{O}_{C_{rs}}).$$

Indeed in the following cartesian diagram

$$\begin{array}{ccc} C_{S,rs} & \longrightarrow & C_{rs} \\ \downarrow{\scriptstyle \pi} & \square & \downarrow{\scriptstyle \pi} \\ S & \xrightarrow{g} & T \end{array}$$

vertical maps are finite and consequently

$$\pi_*(g^*(\mathcal{O}_{C_{S,rs}})) \cong g^*(\pi_*(\mathcal{O}_{C_{rs}})).$$

□

1.1.2. Local projectivity. From the construction in Example 1.4 above, we have that $\widetilde{M}/\mathfrak{C}_2$ is locally in the étale topology a projective scheme. More precisely if S' is the disjoint union of $S - P$ and $S - Q$ we have

$$\begin{array}{ccc} \widetilde{M}' & \longrightarrow & \widetilde{M}/\mathfrak{C}_2 \\ \downarrow{\scriptstyle \pi'} & \square & \downarrow{\scriptstyle \widetilde{\pi}} \\ S' & \longrightarrow & S/\mathfrak{C}_2 \end{array}$$

In general we can state the following result

Proposition 1.7. *Every object in $\mathfrak{M}_0$ is locally projective in the étale topology.*

Proof. Given a curve $C \xrightarrow{\pi} T$, fix a geometric point $\mathrm{Spec}\Omega \xrightarrow{t} T$. Let k be the number of irreducible components of the fiber C_t. On each component we can choose a smooth point and we have k sections

$$\begin{array}{ccc} C_t & \longrightarrow & C \\ {\scriptstyle s_1\dots s_k}\uparrow\downarrow{\scriptstyle \pi} & \square & \downarrow{\scriptstyle \pi} \\ \mathrm{Spec}\Omega & \xrightarrow{t} & T \end{array}$$

Let U be a scheme and $U \to C$ a representable étale covering, then $U_t := \mathrm{Spec}\Omega \times_T U$ is an étale covering over the scheme C_t. Consequently we can lift each s_i to section to U_t (which we still call s_i).

We know (see [**EGA IV**] 17.16.4) that there exists an étale map $T' \to T$ with T' affine where these sections extend, more precisely, for $i = 1 \dots k$ we have the following diagram

$$\begin{array}{ccccc} U_t & \longrightarrow & U' & \longrightarrow & U \\ {\scriptstyle s_i}\uparrow\downarrow{\scriptstyle \pi} & \square & {\scriptstyle s_i}\uparrow\downarrow{\scriptstyle \pi} & \square & \downarrow{\scriptstyle \pi} \\ \mathrm{Spec}\Omega & \longrightarrow & T' & \longrightarrow & T \end{array}$$

and T' is affine.

By composition with the covering $U' \to C' := T' \times_T C$ we obtain sections

$$\begin{array}{ccc} C' & \longrightarrow & C \\ {\scriptstyle s_i}\uparrow\downarrow{\scriptstyle \pi} & \square & \downarrow{\scriptstyle \pi} \\ T' & \longrightarrow & T \end{array}$$

If for some i we have (*see* Sction 1.3 for the definition of the relative singular locus C'_{rs})

$$s_i(T') \cap C'_{rs} \neq \emptyset$$

we can restrict to a smaller neighborhood in order to have that each $s_i(T')$ lies in the relative smooth locus. Further (as we have seen in Section 1.3) we can restrict to a smaller neighborhood T' of t where all geometric fibers have at most k irreducible components (essentially because *having more than k nodes* is a closed condition).

We consider the divisor $\coprod_{i=1}^{k} s_i(T')$ which is a closed scheme-like subspace of C'. The associated ideal sheaf is an invertible sheaf; let us call $\mathfrak{L}$ its inverse. We have that $\mathfrak{L}$ is ample (with respect to π) on C'.

Indeed, since $\mathcal{L}$ is ample on each geometric fiber we have an open covering of T'

$$\{T'_\alpha \xrightarrow{f_\alpha} T'\}\alpha \in I$$

such that for each $\alpha \in I$ the sheaf $\mathcal{L}|_{\pi^{-1}(f_\alpha(U_\alpha))}$ is ample. Consequently $\mathcal{L}$ is ample on C'. □

1.2. Description of $\mathfrak{M}_0$ as an Artin stack

Let us start to prove that $\mathfrak{M}_0$ is an Artin stack. First of all we have the following

Proposition 1.8. *The category $\mathfrak{M}_0$ on $\mathcal{S}ch_{\mathbb{C}}$ (fibered in groupoids) is a stack.*

Proof. By definition (see Appendix B) $\mathfrak{M}_0$ is a stack if it satisfies the following two conditions.

A For every scheme T in $\mathcal{S}ch_{\mathbb{C}}$ and for every pair of curves C and D in $\mathfrak{M}_0(T)$ the functor $\underline{Isom}_T(C, D)$ is a sheaf for the étale topology.

In order to prove this , given an object $U \xrightarrow{f} T$ in $\mathcal{S}ch_{\mathbb{C}}/T$, let us consider an étale covering $\{\varphi_\alpha : U_\alpha \to U\}_{\alpha\in I}$. Set

$$D_U := f^*D \qquad C_U := f^*C$$
$$D_{U_\alpha} := \varphi_\alpha^* D_U \qquad C_{U_\alpha} := \varphi_\alpha^* C_U.$$

We obtain the following coverings for the curves

$$D_{U_\alpha} \xrightarrow{\psi'_\alpha} D$$
$$C_{U_\alpha} \xrightarrow{\psi_\alpha} C.$$

Given a family of isomorphisms $\{\xi_\alpha : C_{U_\alpha} \to D_{U_\alpha}\}_{\alpha\in I}$ respecting the cocycle condition, we notice that for all α the isomorphism ξ_α belongs to $\underline{Isom}_T(C, D)(U_\alpha)$.

Let us consider the family $\{\psi'_\alpha \circ \xi_\alpha\}_{\alpha\in I}$ and notice that $\psi'_\alpha \circ \xi_\alpha \in D_U(C_{U_\alpha})$. We are looking for a morphism ξ such that for every α we have that the following diagram is cartesian

$$\begin{array}{ccc} D_{U_\alpha} & \xrightarrow{\psi'_\alpha} & D_U \\ {\scriptstyle \xi_\alpha}\uparrow & \nearrow{\scriptstyle \psi'_\alpha\circ\xi_\alpha} & \uparrow{\scriptstyle \xi} \\ C_{U_\alpha} & \xrightarrow{\psi_\alpha} & C_U \end{array}$$

The family $\{\psi'_\alpha \circ \xi_\alpha : C_{U_\alpha} \to D_U\}_{\alpha\in I}$ is an étale covering.

Because D_U is an algebraic space, it is a sheaf for the étale topology, consequently we have our unique morphism $C_U \xrightarrow{\xi} D_U$.

By exchanging C and D we obtain that ξ is an isomorphism.

B For every scheme $T \in \mathcal{S}ch_{\mathbb{C}}$, given an étale covering $\{\varphi_\alpha : T_\alpha \to T\}_{\alpha \in I}$, any descent datum in $\mathfrak{M}_0$ relative to it is effective.

To see it, we recall that every descent datum is effective in the category of algebraic spaces (see [**Knu**] Chapter 2), thus for every descent datum $\{C_\alpha \xrightarrow{\pi_\alpha} T_\alpha\}_{\alpha \in I}$ there exists an algebraic space $C \xrightarrow{\pi} T$ and a family of morphisms $\{\psi_\alpha : C_\alpha \to C\}_{\alpha \in I}$ such that the following diagram is cartesian

$$\begin{array}{ccc} C_\alpha & \xrightarrow{\psi_\alpha} & C \\ {\scriptstyle \pi_\alpha}\downarrow & \square & \downarrow{\scriptstyle \pi} \\ T_\alpha & \xrightarrow{\varphi_\alpha} & T \end{array}$$

and the map π is flat and proper.

Let us consider a geometric point $\mathrm{Spec}\Omega \to T$, owing to the fact that $\{T_\alpha \to T\}_{\alpha \in I}$ is an étale covering, we have for a suitable α

$$\begin{array}{ccc} T_\alpha \times_T \mathrm{Spec}\Omega & \longrightarrow & \mathrm{Spec}\Omega \\ \downarrow & \swarrow & \downarrow \\ T_\alpha & \longrightarrow & T \end{array}$$

Indeed, when $T_\alpha \times_T \mathrm{Spec}\Omega$ is different from $\emptyset$, the map $T_\alpha \times_T \mathrm{Spec}\Omega \to \mathrm{Spec}\Omega$ is étale, so $T_\alpha \times_T \mathrm{Spec}\Omega$ is an union of finite copies of $\mathrm{Spec}\Omega$ and we obtain a map which factorises $\mathrm{Spec}\Omega \to T$. Geometric fibers of $C \xrightarrow{\pi} T$ are therefore geometric fibers of suitable T_α which are rational trees by hypothesis. $\square$

In the argument of condition **A** we have shown that the functor $\underline{Isom}_T(C, D)$ is a sheaf for the étale topology. But it has a stronger property.

Proposition 1.9. *For every scheme T in $\mathcal{S}ch_{\mathbb{C}}$ and for every pair of curves C and D in $\mathfrak{M}_0(T)$ the sheaf $\underline{Isom}_T(C, D)$ is represented by a separated algebraic space locally of finite type.*

Proof. From [**Art2**] we know that for any proper and flat morphism of algebraic spaces $Y \to X$, the functor $\underline{Hilb}_X(Y)$

$$\begin{array}{rcl} \underline{Hilb}_X(Y) : (\mathcal{S}ch_{\mathbb{C}}/X)^{\mathrm{opp}} & \to & (\mathrm{Sets}) \\ \{\widetilde{X} \to X\} & \mapsto & \left\{ \begin{array}{l} \text{closed algebraic subspaces} \\ Z \subseteq Y \times_X \widetilde{X} \text{ flat over } \widetilde{X} \end{array} \right\} \end{array}$$

is a separated algebraic space.

Therefore it is sufficient to show that $\underline{Isom}_T(C,D)$ is an open subsheaf of $\underline{Hilb}_T(C\times_T D)$ and that it is of finite type.

First of all let us notice that for every object $T'\xrightarrow{f} T$ in $\mathcal{Sch}_{\mathbb{C}}/T$ we have

$$\underline{Isom}_T(C,D)(T')=\left\{\begin{array}{ll} Z'\in \underline{Hilb}_T(C\times_T D)(T'): & \\ pr_1: Z'\to C':=C\times_T T' & \text{are both} \\ pr_2: Z'\to D':=D\times_T T' & \text{isomorphisms}\end{array}\right\}$$

where the equality is given by the map

$$\xi\in \underline{Isom}_T(C,D)(T')\mapsto \begin{array}{l}\text{graph of } \xi \\ Z'\subseteq C'\times_{T'} D'=(C\times_T D)\times_T T'\end{array}$$

In order to prove that we have an open subsheaf we must verify that for all map $T'\to \underline{Hilb}_T(C\times_T D)$, that is to say (by Yoneda) for every closed algebraic space $Z'\subseteq (C\times_T D)\times_T T'$ flat over T', the fiber product

$$\begin{array}{ccc} \widetilde{T'}:=\underline{Isom}_T(C\times_T D)\times_{\underline{Hilb}_T(C\times_T D)} T' & \longrightarrow & T' \\ \downarrow & \square & \downarrow \\ \underline{Isom}_T(C\times_T D) & \longrightarrow & \underline{Hilb}_T(C\times_T D)\end{array}$$

is an open immersion in T'. So we have to show that the sheaf $\widetilde{T'}$ is represented by an open subscheme $T_1\subseteq T'$. This is equivalent to say that there exists an open subscheme T_1 in T' which has the following property:
for all maps $T''\xrightarrow{\varphi} T'$ the natural projections

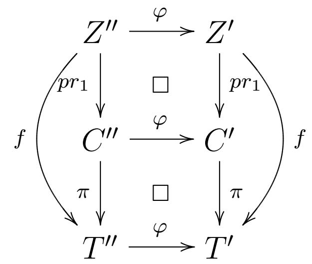
$$Z'':=Z'\times_{T'} T''\subseteq C''\times_{T'} D'' \quad \begin{array}{l} \xrightarrow{pr_1} C'' \\ \xrightarrow{pr_2} D''\end{array}$$

are isomorphisms if and only if, there exists a map $T''\to T_1$ which factorises $T''\to T'$.

The map $f: Z'\to T'$ is flat over T' for hypothesis and proper because it is the composition of two proper maps

$$Z'\xrightarrow{pr_1} C'\xrightarrow{\pi} T'.$$

Let us consider the following diagram



If $pr_1 : Z'' \to C''$ is an isomorphism we have

$$\varphi(Z'') \subseteq Z' - S$$

where

$$S := \{z \in Z' : \dim_z(pr_1^{-1} pr_1(z)) > 0\}.$$

As pr_1 is proper, for the semicontinuity property (see Theorem 7.7.5 [**EGA III**] and reduction to the noetherian case) S is closed and $f(S)$ is closed because f is proper. Now we restrict to the open subscheme (which could also be empty) of T' where pr_1 is a finite map.

Let us consider the following exact sheaf sequence

$$\mathcal{O}_{C'} \to pr_{1*}\mathcal{O}_{Z'} \to \mathcal{Q} \to 0$$

where $\mathcal{Q}$ is the quotient sheaf.

As before $\varphi(pr_1(Z''))$ is not contained in supp $\mathcal{Q}$, $\pi(\text{supp } \mathcal{Q}) \subseteq T'$ is closed, so after restricting to an open subscheme we can suppose to have the following exact sheaf sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{C'} \to pr_{1*}\mathcal{O}_{Z'} \to 0$$

where $\mathcal{I}$ is the kernel sheaf. Since f is flat we have that pr_{1*} is flat on T'. This means (see Proposition 2.1.8 [**EGA IV**]) we also have the exact sequence

$$0 \to \varphi^*(\mathcal{I}) \to \varphi^*(\mathcal{O}_{C'}) \to \varphi^*(pr_{1*}\mathcal{O}_{Z'}) \to 0$$

As pr_1 is finite we have that $\varphi * pr_{1*}(\mathcal{O}_{Z'}) = pr_{1*}\varphi^*(\mathcal{O}_{Z'})$, further pr_{1*} is an isomorphism and so $\varphi^*(\mathcal{I}) = 0$. If $\mathcal{I}$ is different from zero So we have found an open sub scheme of T', which we denote as T'_C, which factorises all morphisms φ inducing an isomorphism between Z'' and C'' and further we have by construction

$$Z' \times_{T'} T'_C \cong C' \times_{T'} T'_C.$$

In the same way we obtain T'_D and so we can define:

$$T_1 := T'_C \cap T'_D.$$

It remains to prove that the separated algebraic space $\underline{Isom}_T(C, D)$ is of locally of finite type.

As the property of being locally of finite type is local in the étale topology we can assume from Proposition (1.7) that both $C \to T$ and $D \to T$ are projective morphisms (and a fortiori C and D are schemes).

It is known that if $Y \to X$ is projective and P is a numerical polynomial, the functor

$$\begin{array}{rcl} \underline{Hilb}_{X,P}(Y) : \ (\mathcal{S}ch_{\mathbb{C}}/X)^{\text{opp}} & \to & (\text{Sets}) \\ \{\widetilde{X} \to X\} & \mapsto & \left\{ \begin{array}{l} \text{closed algebraic subspaces} \\ Z \subseteq Y \times_X \widetilde{X} \text{ flat over } \widetilde{X} \\ \text{whose Hilbert polynomial is } P \end{array} \right\} \end{array}$$

is represented by a finite type scheme.

If we prove that, given C, D, T as above, only finite Hilbert polynomial are allowed, we have done.

Let us fix a closed immersion i

$$\begin{array}{ccc} C \times_T D & \xrightarrow{i} & \mathbb{P}^N_T \\ \downarrow & \swarrow & \\ T & & \end{array}$$

For every morphism $T' \xrightarrow{\varphi} T$ we have an induced closed immersion

$$C' \times_{T'} D' \xrightarrow{i} \mathbb{P}^N_T$$

and for each closed subscheme $Z' \subset C' \times_{T'} D'$ flat on T' and equipped with isomorphisms to C' and D' we have a closed immersion in $\mathbb{P}^N_T$. We can compute its Hilbert polinomial on each complex fiber and we have a rational tree embedded in $\mathbb{P}^N_{\mathbb{C}}$ whose degree cannot be greater than N. So we have a finite number of possible Hilbert polynomial. □

Proposition 1.10. *The stack $\mathfrak{M}_0$ is an Artin stack.*

Proof. By definition $\mathfrak{M}_0$ is an Artin stack (see Appendix B) if it satisfies the following properties.

C For every pair of schemes X, Y in $\mathcal{S}ch_{\mathbb{C}}$ with maps to $\mathfrak{M}_0$ (that is to say for any pair of objects $C \to X$ and $D \to Y$ in $\mathfrak{M}_0$) the fibered product

$$\begin{array}{ccc} X \times_{\mathfrak{M}_0} Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathfrak{M}_0 \end{array}$$

is represented by a locally of finite type separated algebraic space.

We just notice that as a functor $X \times_{\mathfrak{M}_0} Y$ is the same as

$$\underline{Isom}_{X\times Y}(C \times_X (X \times Y), D \times_Y (X \times Y))$$

then we apply Proposition (1.9).

D There exists a surjective smooth map $U \to \mathfrak{M}_0$ where U is a scheme that is locally of finite type.

Let N be a positive integer, $P \in \mathbb{Z}^+[t]$ a polynomial of the form $dt+1$ and T an object in $\mathcal{Sch}_{\mathbb{C}}$. Let us consider the scheme (separated and locally of finite type)

$$\underline{Hilb}_{\mathrm{Spec}\mathbb{C},P}(\mathbb{P}^N_{\mathbb{C}}).$$

Let us define the following functor

$$\begin{array}{rcl} \mathfrak{M}_0(P,N): \ (\mathcal{Sch}_{\mathbb{C}})^{\mathrm{opp}} & \to & (\mathrm{Sets}) \\ T & \mapsto & \left\{ \begin{array}{l} \text{closed algebraic subspaces} \\ Z \subseteq \mathbb{P}^N_T \text{ flat over } T \\ \text{whose Hilbert polynomial is } P \\ \text{and such that } Z \to T \text{ is in } \mathfrak{M}_0(T) \end{array} \right\} \end{array}$$

We have an obvious map

$$\mathfrak{M}_0(P,N) \to \mathfrak{M}_0.$$

Set

$$\mathfrak{M}_0(N) := \coprod_{d\in\mathbb{Z}^+} \mathfrak{M}_0(dt+1,N).$$

In the following we prove that for each curve $C \xrightarrow{\pi} T$ in $\mathfrak{M}_0$ we have that the projection

$$\overline{T} := \mathfrak{M}_0(N) \times_{\mathfrak{M}_0} T \to T$$

is a smooth covering of T if π locally in the étale topology is projective with a closed immersion in $\mathbb{P}^N_T$. Consequently we obtain the requested smooth covering by taking the disjoint union of $\mathfrak{M}_0(N)$ over all N.

Now we invoke Grothendieck criterion of smoothness. Let A be an Artinian ring and $B \cong A/I$ where I is isomorphic to $\mathbb{K} = A/m_A$. We have to show that for each map $\mathrm{Spec}A \to T$ and for each commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}B & \longrightarrow & \overline{T} \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}A & \longrightarrow & T \end{array}$$

there exists a lifting $\mathrm{Spec}A \to \overline{T}$. This is equivalent to say that there exists a lifting of the composition

$$\mathrm{Spec}A \to \mathfrak{T} \to \mathfrak{M}_0$$

to $\mathfrak{M}_0(N)$.

The previous data give us a cartesian diagram

$$\begin{array}{ccc} C_B & \longrightarrow & C \\ \downarrow & \square & \downarrow \\ \mathrm{Spec}B & \longrightarrow & T \end{array}$$

together with a closed embedding of $C_B \xrightarrow{f_B} \mathbb{P}^N_B$. This is equivalent to have an invertible sheaf $L_B = f_B^*(\mathcal{O}(1))$ over C_B plus a basis $(s_0, \dots, s_N)$ of $H^0(C_B, L_B)$ without base points

We have to show that all these data extend to give an embedding $C \to \mathbb{P}^N_A$.

As topological spaces C_B is equal to C. Let us call C_0 the fiber of C on Spec$\mathbb{K}$. We have an isomorphism $\mathcal{O}_{C_0} \simeq 1 + I\mathcal{O}_C$ and an exact sequence

$$1 \to \mathcal{O}_{C_0} \to \mathcal{O}^*_C \to \mathcal{O}^*_{C_B} \to 1$$

which gives us

$$\mathrm{Pic}(C) \to \mathrm{Pic}(C_B) \to H^2(C_0, \mathcal{O}_{C_0}) = 0.$$

So we can extend L_B to an invertible sheaf L on C.

If we tensor the following

$$0 \to \mathbb{K} \to A \to B \to 0$$

with $\mathcal{O}_C \otimes_{\mathcal{O}_{\mathrm{Spec}\mathbb{K}}} L$ we obtain

$$0 \to L|_{C_0} \to L \to L_B \to 0$$

from which

$$H^0(L) \to H^0(L_B) \to H^1(L|_{C_0}).$$

Now we claim that $H^1(L|_{C_0}) = 0$. Indeed, as $L|_{C_0}$ is very ample, there exists a section s which does not vanishes identically on any component of C_0. So we can write

$$0 \to \mathcal{O}_{C_0} \xrightarrow{s} L|_{C_0} \to \mathcal{Q} \to 0$$

for some finite sheaf $\mathcal{Q}$. Clearly we have $H^1(\mathcal{O}_{C_0}) = H^1(\mathcal{Q}) = 0$ consequently $H^1(L|_{C_0}) = 0$.

So we can extend the sections $(s_0, \dots, s_N) \in H^0(L_B)$ to sections $(\overline{s}_1, \dots, \overline{s}_N) \in H^0(L)$ without base points. So we have a map $C \xrightarrow{f} \mathbb{P}^N_A$ whose restriction $C_B \xrightarrow{f_B} \mathbb{P}^N_B$ is an embedding. We say that f is an embedding. Indeed f is finite (that is to say proper with finite fibers). So it remains to show that the map $\mathcal{O}_{\mathbb{P}^N_A} \to f_*\mathcal{O}_C$ is surjective. In fact the quotient sheaf $\mathcal{Q}$ is such that $\mathcal{Q} \otimes \mathcal{O}_{\mathbb{P}^N_B} = 0$ and for Nakayama we have $\mathcal{Q} = 0$. □

1.3. Stratification of $\mathfrak{M}_0$ by nodes

Let $\mathcal{S}$ be an Artin stack over $\mathcal{S}ch_{\mathbb{C}}$. A family of locally closed substacks $\{\mathcal{S}^\alpha\}_{\alpha\in\mathbb{N}}$ represents a stratification for $\mathcal{S}$ if, for all morphisms

$$T \to \mathcal{S}$$

where T is a scheme, the family of locally closed subschemes

$$\{T^\alpha := \mathcal{S}^\alpha \times_{\mathcal{S}} T \neq \emptyset\}$$

is such that every point $t \in T$ is in exactly one subscheme T^α, that is to say that $\{T^\alpha\}$ is a stratification for T in the usual sense.

Definition 1.11. With reference to (2) we define $\mathfrak{M}_0^{\geq k}$ as the full subcategory of $\mathfrak{M}_0$ whose objects $C \xrightarrow{\pi} T$ are such that $T^{\geq k} = T$.

We further define $\mathfrak{M}_0^k$ to be the subcategory whose objects $C \xrightarrow{\pi} T$ are such that $T^k = T$.

Proposition 1.12. *For every $k \in \mathbb{N}$ the stack $\mathfrak{M}_0^{\geq k}$ is a closed substack of $\mathfrak{M}_0$.*

Proof. Let us consider the functorial immersion $\mathfrak{M}_0^{\geq k} \to \mathfrak{M}_0$. For every map $T \to \mathfrak{M}_0$ we have for every $T-$scheme S:

$$\begin{aligned}(\mathfrak{M}_0^{\geq k} \times_{\mathfrak{M}_0} T)(S) &= \{(C \xrightarrow{\pi} S, \varphi : S \to S) \text{ s.t.} \\ &\quad S = S^{\geq k}; \varphi \text{ is an isomorphism}\} \\ &= \{s : S \to T^{\geq k}\} = T^{\geq k}(S).\end{aligned}$$

This proves that the functorial immersion $\mathfrak{M}_0^{\geq k} \to \mathfrak{M}_0$ is representable and it is a closed immersion and so we have also that it is an Artin stack.

□

The family $\{\mathfrak{M}_0^k\}_{k\in\mathbb{N}}$ is therefore a family of locally closed substacks (a representable functor is an Artin substack). Further, given a geometric point

$$\mathrm{Spec}\Omega \xrightarrow{t} \mathfrak{M}_0$$

the associated curve $C_t \to \mathrm{Spec}\Omega$ belongs to exactly one of the $\mathfrak{M}_0^k$, so this family is a stratification for $\mathfrak{M}_0$.

Proposition 1.13. *For each $k \in \mathbb{N}$ the morphism*

$$\mathfrak{M}_0^{\geq k} \xrightarrow{k} \mathfrak{M}_0$$

is a regular embedding of codimension k.

Proof. Let us consider a smooth covering

$$U \xrightarrow{f} \mathfrak{M}_0$$

and let $C \xrightarrow{\pi} U$ be the associated curve. we have to prove that the morphism

$$U^k := \mathfrak{M}_0^k \times_{\mathfrak{M}_0} U \to U$$

is a regular embedding of codimension k.

Fix a point $p \in U^k \subset U$ and let $A := \widehat{\mathcal{O}}_{U,p}^{sh}$ be the completion of the strict henselisation of the local ring $\mathcal{O}_{U,p}$. (see [**EGA IV**] Definiton 18.8.7). Consider the following diagram

$$\begin{array}{ccc} C_A & \longrightarrow & C \\ {\scriptstyle\pi}\downarrow & \square & \downarrow{\scriptstyle\pi} \\ \mathrm{Spec} A & \longrightarrow & U \end{array}$$

As p is a point of U^k we have that $C_{A,rs}$ is the union of k nodes $q_1, \ldots, q_k$. For each $i = 1, \ldots, k$ we have

$$\widehat{\mathcal{O}}_{C_A,q_i} = A[\![x,y]\!]/(xy - f_i)$$

with $f_1, \ldots, f_k \in A$. So we can write

$$M := \mathcal{O}_{C_{A,rs}} = \prod_{i=1}^{k} (A/f_i).$$

Let us consider M as an $A-$module, we have the following exact sequence of $A-$module

$$A^k \xrightarrow{D} A^k \to M \to 0$$

where D is the diagonal matrix

$$\begin{pmatrix} f_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & f_k \end{pmatrix} \tag{6}$$

So we have $F_{k-1}(M) = (f_1, \ldots, f_k)$ and this means that

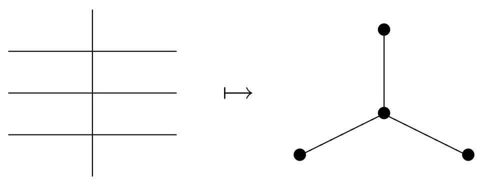
$$B := \widehat{\mathcal{O}}^{sh}_{U^{\geq k},p} = A/(f_1, \ldots, f_k).$$

From deformation theory we have that $\{f_1, \ldots, f_k\}$ is a regular sequence, consequently the map $U^{\geq k} \to U$ is regular of codimension k as claimed. $\square$

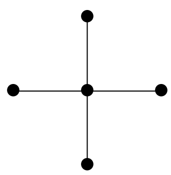
1.4. Description of strata as quotient stacks.

Now let us fix a useful notation. Given an algebraically closed field k such that $\mathrm{Spec}\Omega \in \mathcal{S}ch_{\mathbb{C}}$, let us consider an isomorphism class C in $\mathfrak{M}_0(\mathrm{Spec}\Omega)$. We define the *dual graph of* C, denoted $\Gamma(C)$ or simply Γ, to be the graph which has as many vertices as the irreducible components of C and two vertices are joined by an edge if and only if the two corresponding lines meet each other.

For example we have the following correspondence



We can associate at least one curve with every tree by this map, but such a curve is not in general unique; an example is the following tree



In the following we think of a tree Γ as a finite set of vertices with a connection law given by the set of pairs of vertices which corresponds to the edges. Given a graph Γ we call *multiplicity* of a vertex P the number $e(P)$ of edges to which it belongs and we call $E(P)$ the set of edges to which it belongs. Furthermore we call the *maximal multiplicity* of the graph Γ the maximum of multiplicities of its vertices. We also define Δ_n the set of vertex with multiplicity n.

Remark 1.14. We can associate an unique isomorphism class of curves in $\mathfrak{M}_0(\mathrm{Spec}\Omega)$ to a given tree Γ if and only if the maximal multiplicity of Γ is at most 3.

For every tree Γ, let us indicate with δ_i the number of its vertices with multiplicity i. Tree graphs classify the topological type of rational trees.

Now let us fix a stratum $\mathfrak{M}_0^k$, there are as many topological types of curves with k nodes as trees with $k+1$ vertices.

Given a curve $C \xrightarrow{\pi} T$ in $\mathfrak{M}_0^k$ where T is a connected scheme, the curves of the fibers are of the same topological type. So we can state the following

Definition 1.15. For each tree Γ with $k+1$ vertices we define $\mathfrak{M}_0^\Gamma$ as the full subcategory of $\mathfrak{M}_0^k$ whose objects are curves $C \xrightarrow{\pi} T$ such that on each connected component of T we have curves of topological type Γ.

For each Γ $\mathfrak{M}_0^\Gamma$ is an open (and closed) substack of $\mathfrak{M}_0^\Gamma$ and we can consequently write

$$\mathfrak{M}_0^k = \coprod_\Gamma \mathfrak{M}_0^\Gamma$$

where Γ varies among trees with $k+1$ vertices.

On $\mathfrak{M}_0$ we can consider the universal curve defined in the following way:

Definition 1.16. Let $\mathcal{C}$ be the fibered category on $\mathcal{S}ch_{\mathbb{C}}$ whose objects are families $C \xrightarrow{\pi} T$ of $\mathfrak{M}_0$ equipped with a section $T \xrightarrow{s} C$ and whose arrows are arrows in $\mathfrak{M}_0$ which commute with sections.

Proposition 1.17. *The morphism*

$$\mathcal{C} \xrightarrow{\Pi} \mathfrak{M}_0$$

that forgets sections is representable. Consequently $\mathcal{C}$ is an Artin stack.

Proof. Let us consider a map $T \to \mathfrak{M}_0$ (which corresponds to a curve $C \to T$), the fiber product

$$\mathcal{C} \times_{\mathfrak{M}_0} T$$

is equivalent to C. Indeed for each scheme S we have

$$\begin{aligned}
(\mathcal{C} \times_{\mathfrak{M}_0} T)(S) &= \{S \to T \text{ such that } S \times_T C \text{ admits a section} \\
&\quad + \text{a section } S \to S \times_T C\} \\
&= \{S \to T \text{ which admit a factorization through } C \\
&\quad + \text{a map } S \to C\} \\
&= C(S)
\end{aligned}$$

□

Now we restricts the universal curve to $\mathfrak{M}_0^\Gamma$

$$\mathcal{C}^\Gamma \xrightarrow{\Pi} \mathfrak{M}_0^\Gamma.$$

and consider the normalization (see [**Vis**] Definition 1.18)

$$\widehat{C}^\Gamma \xrightarrow{N} \mathfrak{M}_0^\Gamma.$$

Given a curve $C \xrightarrow{\pi} T$ in $\mathfrak{M}_0^\Gamma$ (that is to say a morphism $T \xrightarrow{f} \mathfrak{M}_0^\Gamma$) we define $\widehat{C}$ as $T \times_{\mathfrak{M}_0^\Gamma} \widehat{\mathcal{C}}_0^\Gamma$. With reference to the following cartesian diagram

$$\begin{array}{ccc}
\widehat{C} & \longrightarrow & \widehat{\mathcal{C}}^\Gamma \\
{\scriptstyle n}\downarrow & \square & \downarrow{\scriptstyle N} \\
C & \longrightarrow & \mathcal{C} \\
{\scriptstyle \pi}\downarrow & \square & \downarrow{\scriptstyle \Pi} \\
T & \xrightarrow{f} & \mathfrak{M}_0^\Gamma .
\end{array}$$

We notice that when T is a reduced and irreducible scheme $\widehat{C} \xrightarrow{n} C$ is the normalization.

The map $\pi n : \widehat{C} \to T$ is proper as $\Pi N :$ $cu^\Gamma \to \mathfrak{M}_0^\Gamma$ is. Then there exists a finite covering $\widetilde{T} \to T$ (see [**Knu**] Chapter 5, Theorem 4.1) such that we have the following commutative diagram

$$\begin{array}{ccc}
 & & \widehat{C} \\
 & \swarrow{\scriptstyle g} & \downarrow{\scriptstyle n} \\
 & & C \\
 & & \downarrow{\scriptstyle \pi} \\
\widetilde{T} & \longrightarrow & T
\end{array}$$

where the map g has connected fibers.

Definition 1.18. We define $\widetilde{\mathfrak{M}}_0^\Gamma$ as the fibered category on $\mathcal{S}ch_{\mathbb{C}}$ whose objects are rational nodal curves $C \xrightarrow{\pi} T$ in $\mathfrak{M}_0^\Gamma$ equipped with an isomorphism $\varphi : \coprod^\Gamma \widetilde{T}$ over T and maps are morphisms in $\mathfrak{M}_0^\Gamma$ that preserve isomorphisms.

Proposition 1.19. *The forgetful morphism*

$$\widetilde{\mathfrak{M}}_0^\Gamma \to \mathfrak{M}_0^\Gamma$$

is representable finite étale and surjective (consequently we have that $\widetilde{\mathfrak{M}}_0^\Gamma$ *is an Artin stack).*

Proof. We have to prove that for each morphism $T \to \mathfrak{M}_0^\Gamma$ (that is to say a curve $C \xrightarrow{\pi} T$ in $\mathfrak{M}_0^\Gamma$) the projection

$$\widetilde{\mathfrak{M}}_0^\Gamma \times_{\mathfrak{M}_0^\Gamma} T \to T$$

is a finite étale and surjective map.

First of all we notice that if $\widehat{C} \to \widetilde{T}$ is the Stein factorization described above, we have a section $\widetilde{T} \to \widehat{C}$ such that (by composition with $\widehat{C} \xrightarrow{n} C$) gives an embedding $i : \widetilde{T} \to C$ that makes commute the following diagram

$$\begin{array}{ccc} C & & \\ {\scriptstyle \pi}\downarrow & \nwarrow {\scriptstyle i} & \\ T & \longleftarrow & \widetilde{T}. \end{array}$$

For each scheme S we have that an object of $\left(\widetilde{\mathfrak{M}}_0^\Gamma \times_{\mathfrak{M}_0^\Gamma} T\right)$ is the datum of a map $S \to T$ plus, given the curve

$$C_S := C \times_T S \xrightarrow{\pi} S,$$

an isomorphism $\varphi := \coprod^\Gamma S \to \widetilde{S}$.

With reference to the following cartesian diagram

$$\begin{array}{ccc} \widetilde{S} & \longrightarrow & \widetilde{T} \\ {\scriptstyle i}\downarrow & \square & \downarrow{\scriptstyle i} \\ C_S & \longrightarrow & C \\ {\scriptstyle \pi}\downarrow & \square & \downarrow{\scriptstyle \pi} \\ S & \longrightarrow & T \end{array}$$

from the composition $\coprod^\Gamma S \xrightarrow{i \circ \varphi} C_S$ we obtain a family $s_\Gamma : S \to C_S$ of sections that separate components. On the other hand if we have Γ sections that separates components we can give an isomorphism $\varphi := \coprod^\Gamma S \to \widetilde{S}$.

Now we have that the projection

$$\widetilde{\mathfrak{M}}_0^\Gamma \times_{\mathfrak{M}_0^\Gamma} T \to T$$

is étale and surjective from the fact that the property of having Γ sections that separate components is local for the étale topology and we obtain a finite morphism with the same order of the monodromy group of the étale covering $\widetilde{T} \to T$. $\square$

Definition 1.20. We call $\mathcal{M}_{0,i}^n$ (resp. $\mathcal{M}_{0,i}^{\geq n}$) the stack of rational curves with (resp. at least) n nodes with i sections.

Proposition 1.21. *We have the following equivalence*

$$\widetilde{\mathfrak{M}}_0^\Gamma \cong \prod_{P \in \Gamma} (\mathcal{M}_{0,e(P)}^0)$$

Proof. The right term does not depend on the edges of Γ. In order to give the equivalence, we have to make a choice: for each $P \in \Gamma$ we fix an order for the sections of $\mathcal{M}_{0,e(P)}^0$ and for each edge $\lambda = (P, Q) \in \Gamma$ we choose a pair of sections s_P^λ in $\mathcal{M}_{0,e(P)}^0$ and s_Q^λ in $\mathcal{M}_{0,e(Q)}^0$.

We define a map

$$\prod_{P\in\Gamma}(\mathcal{M}^0_{0,e(P)}) \to \widetilde{\mathfrak{M}}^\Gamma_0$$

as follows. Given an object

$$\prod_{P\in\Gamma}(B_P \to T; s_1, \ldots, s_{e(P)}) \in \left(\prod_{P\in\Gamma}(\mathcal{M}^0_{0,e(P)})\right)(T)$$

we obtain an object in $\widetilde{\mathfrak{M}}^\Gamma_0(T)$ by gluing along sections which correspond by edges. On the other hand given an object in $\widetilde{\mathfrak{M}}^\Gamma_0(T)$ we obtain one in $\left(\prod_{P\in\Gamma}(\mathcal{M}^0_{0,e(P)})\right)(T)$ by normalization. □

Proposition 1.22. *For each tree Γ the stack $\mathfrak{M}^\Gamma_0$ is a quotient stack.*

Proof. We have seen above that we have a finite morphism

$$\prod_{P\in\Gamma}(\mathcal{M}^0_{0,e(P)}) \to \mathfrak{M}^\Gamma_0.$$

Each $\mathcal{M}^0_{0,E(P)}$ is a quotient stack. More precisely, if we call $\mathbb{P}^1_{\mathbb{C},E(P)}$ the curve $\mathbb{P}^1_{\mathbb{C}}$ with $e(P)$ marked points.

We conclude by noting that product of quotient stacks is still a quotient stack and that a stack which admits an étale covering (and consequently flat and projective) by quotient stack is a quotient stack itself (see Lemma 2.13 [**EHKV**]). □

Proposition 1.23. *Each $\mathfrak{M}^\Gamma_0$ is smooth.*

Proof. We have an étale covering

$$\prod_{P\in\Gamma}(\mathcal{M}^0_{0,e(P)}) \to \mathfrak{M}^\Gamma_0$$

as each $\mathcal{M}^0_{0,e(P)}$ is smooth we have that also $\mathfrak{M}^\Gamma_0$ is. □

1.4.1. Classifying spaces. In this subsection any tree Γ has maximal multiplicity ≤ 3.

Given a (linear) algebraic group G over $\mathbb{C}$, let $\mathrm{B}G$ be the classifying space $[\mathrm{Spec}\mathbb{C}/G]$.

More generally, if a group G acts on a scheme $X \in \mathcal{S}ch_{\mathbb{C}}$ let us define the quotient stack $[X/G]$ as the stack of G principal bundles equipped with an equivariant map on X; in the case X is the geometric point $\mathrm{Spec}\mathbb{C}$ we reduce to the classifying space $\mathrm{B}G$.

Now we want to prove that for every tree Γ with maximal multiplicity at most 3, the locally closed substack $\mathfrak{M}^\Gamma_0$ is equivalent to the classifying space of $\mathrm{Aut}(C_0)$ where C_0 is a curve in $\mathfrak{M}_0(\mathrm{Spec}\mathbb{C})$.

What we really need is just to verify that every object in $\mathfrak{M}_0^\Gamma$ admits a local trivialization in the étale topology (see Proposition B.3).

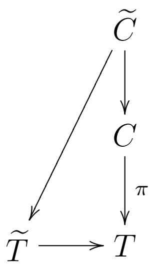
Let us start with $\mathfrak{M}_0^0$, this is the stack BS_2 of Brauer-Severi rank 2 algebraic spaces (see [**Grot1**]) and it is known its equivalence with $\mathrm{B}\mathbb{P}Gl_2$.

Theorem 1.24. [**Grot1**] *Theorem 8.2*
Given an object $C \xrightarrow{\pi} T$ of $BS_2(T)$, there exists an étale morphism $T' \to T$ such that $T' \times_T C \cong \mathbb{P}^1_{\mathbb{C}} \times_{Spec\mathbb{C}} T'$.

In order to describe the substack $\mathfrak{M}_0^\Gamma$ (where Γ has maximal multiplicity at most 3) as a classifying space we state the following

Proposition 1.25. *Every curve $C \xrightarrow{\pi} T$ in $\mathfrak{M}_0^\Gamma$ is locally trivial for the étale topology.*

Proof. Let us consider the Stein factorization



The family $\widetilde{C} \to \widetilde{T}$ is an object of $BS_2$where are fixed at most three sections on each connected component of $\widetilde{T}$, then from [**Grot1**] Theorem 8.2 it admits a local trivialization. Now we can reconstruct a trivialization for π.

□

So we have the equivalence

$$\mathfrak{M}_0^\Gamma \simeq \mathrm{BAut}(C_0).$$

In order to describe the group $\mathrm{Aut}(C_0)$ we fix coordinates $[X, Y]$ on each component of C_0 such that

- on components with one node the point $[1, 0]$ is the node,
- on components with two nodes the points $[0, 1]$ and $[1, 0]$ are the nodes,
- on components with two nodes the points $[0, 1]$, $[1, 1]$ and $[1, 0]$ are the nodes.

Let us define on each component

$$0 := [0,1]$$
$$1 := [1,1]$$
$$\infty := [1,0]$$

We have a canonical surjective morphism

$$\mathrm{Aut}(C_0) \xrightarrow{g} \mathrm{Aut}(\Gamma)$$

which sends each automorphism to the induced graph automorphism. On the other side we can fix for every element of $\mathrm{Aut}(\Gamma)$ an automorphism of C_0 such that we obtain a section s of the previous map

$$\mathrm{Aut}(C_0) \underset{g}{\overset{s}{\rightleftarrows}} \mathrm{Aut}(\Gamma)$$

Now we want to describe the section s. Given an element h of $\mathrm{Aut}(\Gamma)$, there exists a unique automorphism γ of C_0 such that

- γ permutes components of C_0 by following the permutation of vertices given by h
- on components with one node, γ makes correspond the points $0, 1, \infty$
- on components with two nodes, γ makes correspond the point 1.

Now let us consider the normal subgroup $g^{-1}(\mathrm{id})$ of automorphisms of C_0 which do not permute components. It is the direct product of groups of automorphisms of each component which fix nodes. In particular let $E \subset \mathbb{P}GL_2$ the group of automorphisms which fix ∞; if

$$\sigma := \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is an element of $\mathbb{P}GL_2$ it belongs to E if and only if $c = 0$, so we must have $a \neq 0$ and we can write

$$\sigma = \begin{bmatrix} 1 & b \\ 0 & d \end{bmatrix}.$$

The group which fixes 0 and ∞ is gives by elements of the following type

$$\begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix}.$$

and it is isomorphic to $\mathbb{G}_{\mathbf{m}}$. Through this description we have a monomorphism ψ of $\mathbb{G}_{\mathbf{m}}$ into E.

Also the additive group $\mathbb{G}_{\mathbf{a}}$ has an injection in E

$$\begin{aligned} \varphi : \mathbb{G}_{\mathbf{a}} &\to E \\ b &\mapsto \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

There is also an epimorphism

$$\begin{aligned} \rho E &\to \mathbb{G}_{\mathbf{m}} \\ \begin{bmatrix} 1 & b \\ 0 & d \end{bmatrix} &\mapsto \begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix}, \end{aligned}$$

consequently we have obtained that E is the semidirect product of $\mathbb{G}_{\mathbf{m}}$ and $\mathbb{G}_{\mathbf{a}}$

$$0 \longrightarrow \mathbb{G}_{\mathbf{a}} \xrightarrow{\varphi} E \underset{\rho}{\overset{\psi}{\rightleftarrows}} \mathbb{G}_{\mathbf{m}} \longrightarrow 1.$$

At last we have that there is only identity which fixes three points in $\mathbb{P}^1_{\mathbb{C}}$. So we can conclude that

$$g^{-1}(\mathrm{id}) = E^{\delta_1} \times \mathbb{G}_{\mathbf{m}}^{\delta_2}$$

and we have a (not canonical) injection

$$E^{\delta_1} \times \mathbb{G}_{\mathbf{m}}^{\delta_2} \to \mathrm{Aut}(C_0)$$

Then $Aut(C_0)$ is the semi-direct product given by the exact sequence

$$1 \longrightarrow \mathbb{G}_{\mathbf{m}}^{\delta_2} \times E^{\delta_1} \longrightarrow \mathrm{Aut}(C_0) \rightleftarrows Aut(\Gamma) \longrightarrow 1 \tag{7}$$

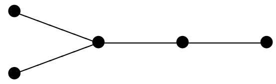
and we write it as

$$\mathrm{Aut}(\Gamma) \ltimes (\mathbb{G}_{\mathbf{m}})^{\delta_2} \times E^{\delta_1}. \tag{8}$$

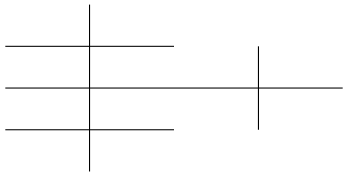
Similarly we obtain that when Γ has maximal multiplicity at most 3 the following holds

$$\widetilde{\mathfrak{M}}_0^{\Gamma} = \mathrm{B}(\mathbb{G}_{\mathbf{m}})^{\delta_2} \times E^{\delta_1}.$$

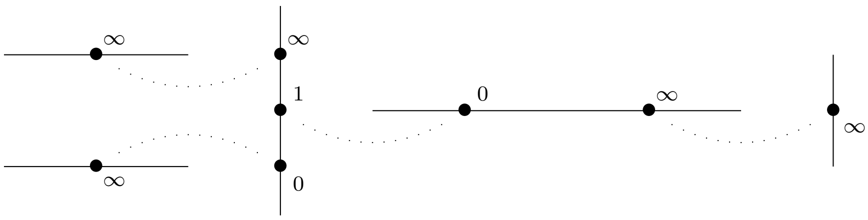
Example 1.26. Let us consider the following graph



it corresponds to curves of the following topological type



Now let us consider a coordinate system with the properties above, for example we fix on each component the following points



where we join with dotted lines points which correspond in the curve.

We notice that in our example $\mathrm{Aut}(\Gamma) = \mathfrak{C}_2$ and we have

$$\mathrm{Aut}(C_0) \cong \mathfrak{C}_2 \ltimes E^3 \times \mathbb{G}_{\mathbf{m}}$$

1.5. Dualizing and normal bundles

In this section we briefly give the descripition of basic bundles over $\mathfrak{M}_0$ and its strata whose Chern classes (in the sense of section 2.3.1) are among the natural generators of the whole Chow ring. In particular we point out their restriction to $\mathfrak{M}_0^{\leq 3}$.

On $\mathfrak{M}_0$ we consider the universal curve $\mathcal{C} \xrightarrow{\Pi} \mathfrak{M}_0^{\Gamma}$.

Let $U \to \mathfrak{M}_0$ be a smooth covering and let us consider the following cartesian diagram

$$\begin{array}{ccc} \mathcal{C}_U & \longrightarrow & \mathcal{C} \\ \downarrow \Pi & \square & \downarrow \Pi \\ U & \longrightarrow & \mathfrak{M}_0 \end{array}$$

Definition 1.27. We define the relative dualizing sheaf ω_0 of $\mathcal{C} \xrightarrow{\Pi} \mathfrak{M}_0$ as the dualizing sheaf $\omega_{\mathcal{C}_U/U}$ on $\mathcal{C}_U$.

This is a good definition because for each curve $C \xrightarrow{\pi} T$ in $\mathfrak{M}_0$ there is the dualizing sheaf $\omega_{C/T}$ and its formation commutes with the base change.

For each curve $C \to T$ in $\mathfrak{M}_0^{\leq \delta}$ we have the push forward of the sheaf $\omega_{C/T}^{\vee}$ in order to have a well defined push forward of $(\omega_0)^{\vee}$ we need that for each curve in $\mathfrak{M}_0^{\leq \delta}$ the sheaf $\pi_* \omega_{C/T}^{\vee}$ is locally free of constant rank and it should respect the base change. We can do it when δ is 3. In particular we have the following

Proposition 1.28. *Let $C \xrightarrow{\pi} T$ be a curve in $\mathfrak{M}_0^{\leq 3}$. Then $\pi_*\omega_{C/T}^\vee$ is a locally free sheaf of rank 3 and its formation commutes with base change.*

Proof. First of all we prove that it is locally free and its formation commutes with base change. It is enough to show that for every geometric point t of T

$$H^1(C_t, \omega_{C/T}^\vee) = 0.$$

From Serre duality

$$H^1(C_t, \omega_{C_t/t}^\vee) = H^0(C_t, \omega_{C/T}^{\otimes 2})^\vee.$$

When the fiber is isomorphic to $\mathbb{P}^1$ we have

$$\omega_{C_t/t}^{\otimes 2} = (\mathcal{O}(-2))^{\otimes 2} = \mathcal{O}(-4)$$

and we do not have global sections different from 0.

When C_t is singular we have that the restriction of $\omega_{C_t/t}^{\otimes 2}$ to components with a node is $(\mathcal{O}(-1))^{\otimes 2} = \mathcal{O}(-2)$ and consequently the restriction of sections to these components must be 0. The restriction to components with two nodes is $\mathcal{O}(0)$ (so restriction of sections must be constant) and restriction to components with three nodes is $\mathcal{O}(2)$ (in this case the restriction of sections is a quadratic form on $\mathbb{P}^1$).

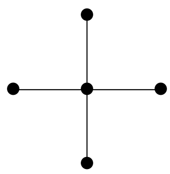
Given these conditions, global sections of $\omega_{C_t/t}^{\otimes 2}$ on curves with at most three nodes (as they have to agree on the nodes) must be zero.

Similarly we verify that $h^0(C_t, \omega_{C_t/t}^\vee) = 3$ and conclude by noting that

$$(\pi_*(\omega_{C/T}^\vee))_t = H^0(C_t, \omega_{C_t/t}^\vee).$$

□

We notice that in $\mathfrak{M}_0^{\leq 4}$ there are curves $C \xrightarrow{\pi} \mathrm{Spec}\Omega$ for which there are global sections for $\omega_{C/\mathrm{Spec}\Omega}^{\otimes 2}$. When we have a curve of topological type



the restriction of global sections on the central component are quartic forms on $\mathbb{P}_\Omega^1$ which vanishes on four points, thus we have

$$H^0(C, \omega_{C/\mathrm{Spec}\Omega}^{\otimes 2}) = \Omega.$$

1.5.1. The normal bundle. Given a tree Γ we consider the (local) regular embedding

$$\mathfrak{M}_0^\Gamma \xrightarrow{in} \mathfrak{M}_0^{\leq\delta}$$

where δ is the number of edges in Γ. From [**Kre**] Section 5 we have that there exists a relative tangent bundle $\mathcal{T}_{in}$ on $\mathfrak{M}_0^\Gamma$ which injects in $in^*(\mathcal{T}_{\mathfrak{M}_0^{\leq\delta}})$.

Definition 1.29. Let us consider the following exact sequence of sheaf

$$0 \to \mathcal{T}_{in} \to in^*(\mathcal{T}_{\mathfrak{M}_0^{\leq\delta}}) \to in^*(\mathcal{T}_{\mathfrak{M}_0^{\leq\delta}})/\mathcal{T}_{in} \to 0.$$

We define the normal bundle as the quotient sheaf on $\mathfrak{M}_0^\Gamma$

$$\mathcal{N}_{in} := in^*(\mathcal{T}_{\mathfrak{M}_0^{\leq\delta}})/\mathcal{T}_{in}$$

When Γ has maximal multiplicity at most 3, we can describe it as the quotient of first order deformations by $\mathrm{Aut}(C_0)$ in the following way. Let us consider the irreducible components C_Γ [2] of C_0. The space of first order deformations near a node P where two curves C_α and C_β meet is (see [**Ha-Mo**] p. 100)

$$T_P(C_\alpha) \otimes T_P(C_\beta).$$

Consequently the space N_Γ of first order deformations of C_0 is

$$\bigoplus^{\alpha<\beta} \#(C_\alpha \cap C_\beta) T_P(C_\alpha) \otimes T_P(C_\beta).$$

On N_Γ there is an action of $\mathrm{Aut}(C_0)$ which we will describe in Section 3.1.

---

[2]Let X and I be two sets. We define X^I to be the set

$$\prod_I X.$$

When we give an injective map $I \xrightarrow{\varphi} X$ we have a family of elements of X with indices in I

$$x_I := \{x_\alpha : x_\alpha = \varphi(\alpha), \alpha \in I\};$$

with abuse of notation we say that the set x_I has I elements. If we have two families x_I and y_I with maps $I \xrightarrow{\varphi} x_I$ and $I \xrightarrow{\varphi'} y_I$ from the same set of indices I. We denote with $x_I \xrightarrow{I} y_I$ the unique map which make commute the following diagram

$$\begin{array}{ccc} I & \xrightarrow{\varphi} & x_I \\ & \searrow^{\varphi'} & \downarrow I \\ & & y_I \end{array}$$

CHAPTER 2

Intersection theory on Artin stacks

The aim of this section is to recall some basic facts about intersection theory on Artin stacks. The main reference is [**Kre**]. This theory is useful to give a global description of the intersection ring of $\mathfrak{M}_0$ and of $\mathfrak{M}_0^{\leq n}$ for $n \geq 2$ because in this cases we don't have a description of $\mathfrak{M}_0^{\leq n}$ as a quotient stack.

Futhermore, to study the Chow ring of each $\mathfrak{M}_0^{\Gamma}$ we use the equivariant intersection theory developed by Edidin, Graham [**Ed-Gr1**] and Totaro [**Tot**].

2.1. Equivariant intersection theory

In this section we follow [**Ed-Gr1**].

Let $\mathcal{S}ch_{\mathbb{C}}^{ft}$ be the category of finite type schemes [1] over $\mathbb{C}$. For every $i \in \mathbb{N}$ we have on $\mathcal{S}ch_{\mathbb{C}}$ the functor

$$CH_i : \mathcal{S}ch_{\mathbb{C}}^{ft} \rightarrow \text{(Groups)}$$

where the evaluation on every scheme is the abelian group respectively of dimension and codimension i up to rational equivalence.

In the same way we have the Chow functor

$$CH_* : \mathcal{S}ch_{\mathbb{C}}^{ft} \rightarrow \text{(Groups)}$$

which gives the Chow group.

Let X be a scheme of dimension n in $\mathcal{S}ch_{\mathbb{C}}^{ft}$ such that X_{red} is quasi-projective and G a (linear) algebraic group G over $\mathbb{C}$ of dimension g which acts on X and whose action on X_{red} is linearized with respect to some projective embedding.

Now we define the equivariant Chow groups of X with respect to the group G.

Definition 2.1. Let V be any representation of dimension l of G over $\mathbb{C}$ such that the group G freely acts out of a closed subset $S \subset V$

[1] There is equivariant intersection theory also for algebraic spaces of finite type over $\mathbb{C}$, but in this work we apply it to schemes.

$G-$equivariant of codimension $\geq s$. For every $i < s$ let us define [2]

$$CH_{n-i}([X/G]) \ := \ CH_{n+l-g-i}((X \times U)/G)$$

where $U = V - S$.

This is a good definition in the sense that for every $i < s$ the groups $CH_{n-i}((X \times U)/G)$ is independent both of the representation V and of the closed S of codimension $\geq s$ (see 2.2 [**Ed-Gr1**]).For this reason we'll say that $(X \times U)/G$ is an *approximation of order* s *of* $[X/G]$ and we call it X_G^s (when we will write $A_{i+l-g}(X_G^s)$ we mean that $s \geq n-i$).

These equivariant Chow groups are also independent of the representation of the quotient stack $[X/G]$.

As we can choose s large enough, we obtain a well defined graded group depending on the action of G on X

$$A_*^G(X) \ := \ \bigoplus^{0 \leq i \leq \infty} CH_{n-i}([X/G]).$$

Instead of $A_*^G(\mathrm{Spec}\mathbb{C})$ we will write A_*^G.

Let $f : X \to Y$ be a $G-$equivariant map. If f is proper, then by descent, the induced map

$$f_G : X_G \to Y_G$$

is also proper. Likewise, if f is flat of relative dimension k then f_G is flat of dimension k.

Definition 2.2. Define proper pushforward

$$f_* : A_i^G(X) \to A_i^G(Y),$$

and flat pullback

$$f^* : A_i^G(Y) \to A_{i-k}^G(X)$$

as

$$\begin{aligned} f_{G*} \ &: \ A_{i+l-g}(X_G) \to A_{i+l-g}(Y_G) \\ f_G^* \ &: \ A_{i+l-g}(Y_G) \to A_{i+l-g+k}(X_G) \end{aligned}$$

respectively.

If f is smooth, then f_G is also smooth. Furthermore, if f is a regular embedding, then f_G is also a regular embedding and if f is a l.c.i morphism then f_G is also l.c.i.

[2]The quotient $(X \times U)/G$ is well defined as algebraic space.

Definition 2.3. If $f : X \to Y$ is l.c.i. of codimension d then define $f^* : A_i^G(Y) \to A_{i-d}^G(X)$ as

$$f_G^* : A_{i+l-g}(Y_G) \to A_{i+l-g-d}(X_G).$$

Equivariant Chow groups satisfy all the formal properties of ordinary Chow groups (see [**Ful**], Chapters 1-6). In particular, if X is smooth, there is an intersection product on the the equivariant Chow groups $A_*^G(X)$ which makes $\oplus A_i^G(X)$ into a graded ring.

Define equivariant operational Chow groups

$$A_G^i(Y \to X)$$

as the group of bivariant classes (see [**Ful**], Chapter 17)

$$c(Y \to X) : A_*(X_G) \to A_{*-i}(Y_G)$$

for every $G-$map $Y \to X$. As for ordinary bivariant Chow groups, these operations are compatible with the operations on equivariant Chow groups defined above (pullback for l.c.i. morphisms, proper push-forward, etc.). We define i^{th} cohomology Chow group

$$A_G^i(X) := A_G^i(X \xrightarrow{\text{id}} X).$$

Further we define

$$A_G^*(X) := \bigoplus_{i \in \mathbb{Z}} A_G^i(X).$$

For any X smooth, $A_G^*(X)$ has a ring structure. The ring $A_G^*(X)$ is graded, and $A_G^i(X)$ can be non-zero for any $i \geq 0$. [3]

We recall Proposition 4 [**Ed-Gr1**].

Proposition 2.4. *If X is smooth then for every i we have the following isomorphism between Chow groups*

$$A_G^i(X) \simeq A_{n-i}^G(X).$$

As a consequence we have a group isomorphism between $A_G^*(X)$ and $A_*^G(X)$. Furthermore the group $A_G^*(X)$ keeps a graded ring structure.

If $f : Y \to X$ is a flat map or a l.c.i. embedding between smooth schemes we have from Proposition (2.4) a group homomorphism

$$f^* : A_G^*(X) \to A_G^*(Y)$$

which is a ring homomorphism.

Now we briefly give the definition of equivariant Chern classes and the equivariant version of Grothendieck-Riemann-Roch Theorem.

---

[3] Note that by construction, the equivariant Chern classes we will define in the the following, are elements of the equivariant operational Chow ring.

Let E be a vector bundle over an algebraic space X which is equivariant for the action of a linear algebraic group G. For every approximation $(X \times U)/G$ of order s, there is a vector bundle

$$E^s_G := (E \times U)/G \to (X \times U)/G.$$

For every $i < s$ the following Chern class is well defined

$$c_i^G(E) := c_i(E^s_G).$$

If we consider an approximation of order greater than the rank r of E, we obtain the equivariant Chern roots $\alpha_1, \dots, \alpha_r$.

Definition 2.5. We have a well defined Chern character (see Example 3.2.3 [**Ful**])

$$\begin{aligned} ch^G(E) &:= \sum_{i=1}^{r} e^{\alpha_i} \\ &= r + c_1^G(E) + \frac{1}{2}(c_1^G(E)^2 - 2c_2^G(E)) + \\ &\quad + \frac{1}{6}(c_1^G(E)^3 - 3c_1^G(E)c_2^G(E) + 3c_3^G(E)) + \dots \\ &= r + \sum_{j=1}^{\infty} \frac{1}{j!} n_j^G(E). \end{aligned}$$

where the n_j are known as Newton polynomials.

Similarly we have the Todd class

$$\begin{aligned} Td^G(E) &:= \prod_{i=1}^{r} \frac{\alpha_i}{1 - e^{-\alpha_i}} \\ &= 1 + \frac{1}{2}c_1^G(E) + \frac{1}{12}(c_1^G(E)^2 + c_2^G(E)) + \frac{1}{24}c_1^G(E)c_2^G(E) + \dots \end{aligned}$$

Now we can state the equivariant version of the **Grothendieck-Riemann-Roch** theorem ([**Ed-Gr2**]).

Theorem 2.6. *Let $f : X \to Y$ be a G-equivariant morphism of locally complete intersection schemes or a smooth and $G-$equivariantly quasi-projective morphism and let E be a $G-$equivariant vector bundle over X, then we have*

$$ch^G(f_*E) = f_*(Td^G(T_f)ch^G(E)),$$

where T_f is the relative tangent bundle.

2.1.1. The category of actions. In the following we describe a natural way to relate Chow rings of different group actions. In particular we define a pull-back map between Chow groups.

In order to do that we define the *category of actions* $\mathcal{Act}_{\mathbb{C}}$ whose objects are triples (X, G, γ) where $\gamma : G \times X \to X$ is the action of an algebraic group G over an $X \in \mathcal{Sch}_{\mathbb{C}}^{ft}$ and whose morphisms $(f, \phi) : (X', G', \gamma') \to (X, G, \gamma)$ are pairs where $\phi : G' \to G$ is a morphism of algebraic groups and $f : X' \to X$ is a $\phi-$equivariant morphism.

When X is smooth we call A^* the functor such that

$$A^*(X, G, \gamma) := A^*_G(X)$$

where in the right term we forget the action; furthermore, given a morphism (f, ϕ) where f is flat or l.c.i. we give a morphism of graded rings

$$(f, \phi)^* : A^*_G(X) \to A^*_{G'}(X').$$

Let $X^s_G := (X \times U)/G$ and $X'^s_{G'} := (X' \times U')/G'$ be two approximations of order s respectively of $[X/G]$ and $[X'/G']$. In order to have the pull-back morphism we give another approximation $\widehat{X'}^s_{G'}$ of $[X'/G']$ and a morphism $\widehat{X'}^s_{G'} \to X^s_G$ (flat or l.c.i.). First of all we define an action of G' over $U' \times U$

$$(g', u', u) \mapsto (g'u', \phi(g')u);$$

in order that the morphism

$$\begin{array}{rcl} X' \times U' \times U & \to & X \times U \\ (x', u', u) & \mapsto & (f(x'), u) \end{array}$$

is $\phi-$equivariant. Consequently it induces a map

$$\widehat{X'}^s_{G'} := (X' \times U' \times U)/G' \xrightarrow{(f,\phi)_{G',G}} X^s_G.$$

Similarly as above, if f is flat (resp. l.c.i.), the map $(f, \phi)^s_{G',G}$ is flat (resp. l.c.i.).

Consequently we have a well defined pull-back morphism

$$(f, \phi)^* : A^*_G(X) \to A^*_{G'}(X').$$

which is a ring homomorphism.

If we have $f = \mathrm{id}_X$ (resp. $\phi = \mathrm{id}_G$) we'll write ϕ^* (resp. f^*) instead of $(f, \phi)^*$.

Let $(f, \phi) : (X', G', \gamma') \to (X, G, \gamma)$ be a morphism in $\mathcal{Act}_{\mathbb{C}}$. When $(f, \phi)^s_{G',G}$ is proper we have a well defined push-forward

$$(f, \phi)_* : A^*_{G'}(X') \to A^*_G(X).$$

If we have $f = \mathrm{id}_X$ (resp. $\phi = \mathrm{id}_G$) we'll write ϕ_* (resp. f_*) instead of $(f, \phi)_*$.

Remark 2.7. If ϕ is injective (that is to say if G' is a subgroup of G) we can chose $\widehat{X'}^s_{G'} := X' \times U/G'$. Moreover if $f = \mathrm{id}_X$ and X is smooth and G' is finite in G, then the map $\widehat{X'}^s_{G'} \to X^s_G$ is proper and so ϕ_* is a well defined morphism of graded groups.

2.2. Some results on particular classifying spaces

As we have seen in Section 1.4, if Γ is a tree with $k+1$ edges and maximal multiplicity at most three, we have a natural identification of $\mathfrak{M}_0^\Gamma$ with the classifying space $\mathrm{BAut}(C_0)$ where C_0 is a fixed curve of $\mathfrak{M}_0^\Gamma(\mathrm{Spec}\mathbb{C})$. Let us call respectively $\Delta_1, \Delta_2, \Delta_3$ the sets of vertices with multiplicity $1, 2, 3$. We recall that

$$\mathrm{Aut}(C_0) = \mathrm{Aut}(\Gamma) \ltimes \left(E^{\Delta_1} \times \mathbb{G}_{\mathbf{m}}^{\Delta_2}\right).$$

We also have that the group $E^{\Delta_1} \times \mathbb{G}_{\mathbf{m}}^{\Delta_2}$ is a normal subgroup of $\mathrm{Aut}(C_0)$.

In the following we mainly refer to [**Vez**]. Due to the following Proposition, we can state that

$$A^*_{\mathrm{Aut}(C_0)} \otimes \mathbb{Q} \cong (A^*_{E^{\Delta_1} \times (\mathbb{G}_{\mathbf{m}})^{\Delta_2}} \otimes \mathbb{Q})^{\mathrm{Aut}(\Gamma)}$$

Proposition 2.8. *Given an exact sequence of algebraic groups over* $\mathbb{C}$

$$1 \longrightarrow H \xrightarrow{\phi} G \xrightarrow{\psi} F \longrightarrow 1.$$

with F finite and H normal in G, we have

$$A^*_G \otimes \mathbb{Q} \cong (A^*_H \otimes \mathbb{Q})^F$$

Proof. With $(A^*_H \otimes \mathbb{Q})^F$ we mean the subring of $A^*_H \otimes \mathbb{Q}$ whose elements are invariant for the action of F. In the proof we render explicit the action of F on A^*_H induced by the exact sequence.

Let σ be the cardinality of F and let $R^s_G := U/G$ be an approximation of $\mathrm{B}G$ with s large enough.

Now we define an action of F over $R^s_H := U/H$ (as H is a sub group of G it acts on U). Given an $f \in F$, we have that there exists a $g \in G$ such that $\psi(g) = f$. Given a point $[u]_H \in U/H$, we set

$$f([u]_H) := [g(u)]_H.$$

This is a goog definition because if g' is another element of G such that $\psi(g') = f$ there exists an $h \in H$ such that $g' = hg$. This means that

$$[g(u)]_H = [g'(u)]_H.$$

The action of F on R^s_H induces an action on $A^*(R^s_H) \otimes \mathbb{Q}$.

In order to verify the requested isomorphism, it is sufficient to prove the following

$$A^*(U/G) \otimes \mathbb{Q} \cong (A^*(U/H) \otimes \mathbb{Q})^F.$$

Furthermore the morphism (which we call ϕ^s)

$$U/H \xrightarrow{\phi^s_{H,G}} U/G$$

is an F-principal bundle for normality of H. In particular f is a finite proper map. So we have the following morphisms of groups

$$A^*(U/G) \otimes \mathbb{Q} \xrightarrow{\phi^{s*}} A^*(U/H) \otimes \mathbb{Q}$$
$$A^*(U/H) \otimes \mathbb{Q} \xrightarrow{\phi^s_*} A^*(U/G) \otimes \mathbb{Q}$$

and by the projection formula (see p.323 [**Ful**])

$$\begin{array}{rcl} \phi^s_*\phi^{s*} : A^*(U/H) \otimes \mathbb{Q} & \to & A^*(U/H) \otimes \mathbb{Q} \\ a & \mapsto & \sigma a \end{array}$$

this means that $\phi^s_*\phi^{s*}$ is a (group) isomorphism and in particular ϕ^{s*} is injective (as ring homomorphism).

The image of ϕ^{s*} is exactly $(A^*(U/H) \otimes \mathbb{Q})^F$ as we need. Indeed if α is an element of $A^*_G \otimes \mathbb{Q}$ we have that $\phi^{s*}(\alpha)$ is invariant for the action of F. On the other hand if a is an element of $(A^*(U/H) \otimes \mathbb{Q})^F$, it follows that

$$\phi^{s*}\phi^s_*(a) = \sigma a$$

□

So we have reduced to compute $A^*_{E^{\Delta_1} \times (\mathbb{G}_m)^{\Delta_2}}$ and the action of $\mathrm{Aut}(\Gamma)$ on it.

Proposition 2.9. *The group* $A^*_{E^{\Delta_1} \times (\mathbb{G}_m)^{\Delta_2}} \otimes \mathbb{Q}$ *is* $\mathbb{Q}[x_{\Delta_1}, y_{\Delta_2}]$ [4]*, that is to say it is algebraically generated by* $\Delta_1 \cup \Delta_2$ *independent generators of degree 1.*

Proof. We use the following fact (see [**Vez**] Proposition 2.8):

---

[4] Let R be a $\mathbb{Q}$-algebra and let s_I be a subset of R for some set of indices I. If f_J are J polynomials (for some set J) in $\mathbb{Q}[x_I]$, we write

$$R = \mathbb{Q}[s_I]/(f_J(s_I))$$

to indicate that the ring R is generated by s_I and the kernel of the evaluation map

$$\begin{array}{rcl} \mathbb{Q}[x_I] & \to & R \\ x_I & \xrightarrow{I} & s_I \end{array}$$

is generated by f_J. When there are no f_J this means that R is a free $\mathbb{Q}$-algebra in the f_J.

Claim 2.10. *For every linear algebraic group G, we have*

$$A^*_{G\times\mathbb{G}_m} \cong A^*_G \otimes_{\mathbb{Z}} A^*_{\mathbb{G}_m}.$$

By recalling that $A^*_{\mathbb{G}_\mathbf{m}} \otimes \mathbb{Q}$ is isomorphic to $\mathbb{Q}[r]$, where r is an order one class, we have

$$A^*_{E^{\Delta_1}\times(\mathbb{G}_\mathbf{m})^{\Delta_2}} \otimes \mathbb{Q} \cong A^*_{E^{\Delta_1}} \otimes_{\mathbb{Q}} \mathbb{Q}[y_{\Delta_2}].$$

As we have seen in Section 1.4, the group E is the semidirect product

$$0 \longrightarrow \mathbb{G}_\mathbf{a} \xrightarrow{\varphi} E \underset{\rho}{\overset{\psi}{\rightleftarrows}} \mathbb{G}_\mathbf{m} \longrightarrow 1,$$

consequently, for each $n \in \mathbb{N}^+$, the group E^n is the semidirect product

$$0 \longrightarrow (\mathbb{G}_\mathbf{a})^n \xrightarrow{\varphi^n} E^n \underset{\rho^n}{\overset{\psi^n}{\rightleftarrows}} (\mathbb{G}_\mathbf{m})^n \longrightarrow 1 \tag{9}$$

Let $R := U/E$ an approximation of $\mathrm{B}E$ of order as large as needed. In the same way we have $R^n := U^n/E^n$ as an approximation of right order of $\mathrm{B}E^n$. Now we notice that

$$\frac{U^n}{(\mathbb{G}_\mathbf{a})^n} \cong \left(\frac{U}{\mathbb{G}_\mathbf{a}}\right)^n \xrightarrow{\varphi_{\mathbb{G}^n_\mathbf{a},E^n}} \frac{U^n}{E^n} \cong \left(\frac{U}{E}\right)^n$$

is a $(\mathbb{G}_\mathbf{m})^n-$principal bundle. Let $W \to (U/E)^n$ be the vector bundle of rank n associated with the principal bundle above. We call $L \to U/E$ the line bundle associated to the $\mathbb{G}_\mathbf{m}-$principal bundle $U/\mathbb{G}_\mathbf{a} \to U/E$. We clearly have

$$W = \prod^n L.$$

If we define for every $1 \le i \le n$

$$L_i := (U/E) \times \cdots \times L \times \cdots \times (U/E)$$

where L is in the i^{th} position, we have the following isomorphism of vector bundles over $(U/E)^n$

$$W \cong \bigoplus_{i=1}^n L_i.$$

From [**Grot2**], if we have a linear algebraic group G and a G-principal bundle $P \xrightarrow{\pi} X$ over an algebraic space X and $V \to X$ is its associated line bundle, we have that the pullback map

$$A^*(X) \xrightarrow{\pi^*} A^*(P)$$

is surjective and its kernel is generated by the Chern roots of V. Consequently we obtain

$$A^*\left(\frac{U^n}{(\mathbb{G}_{\mathbf{a}})^n}\right) \cong \frac{A^*(U/E)^n}{(c_1(L_1),\ldots,c_1(L_n))}.$$

Because $(\mathbb{G}_{\mathbf{a}})^n$ is unipotent we have that $A^*_{(\mathbb{G}_{\mathbf{a}})^n} \cong \mathbb{Z}$ (see [**Vez**] Proposition 2.6), consequently $A^*_{E^n}$ is generated by $(c_1(L_1),\ldots,c_1(L_n))$. In the same way we notice that $A^*_{(\mathbb{G}_{\mathbf{m}})^n}$ is generated by n classes whose images through $(\rho^n)^*$ corresponds to each $c_1(L_i)$, so we have that $(\rho^n)^*$ is surjective. The injectivity of $(\rho^n)^*$ follows from the fact that the sequence (9) splits. So we have that the pullback morphism $(\rho^n)^*$ is an isomorphism and we can write

$$A^*_{E^{\Delta_1}} \otimes \mathbb{Q} \cong \mathbb{Q}[x_{\Delta_1}].$$

□

Now we describe the action of $\mathrm{Aut}(\Gamma)$ on

$$A^*_{E^{\Delta_1}\times\mathbb{G}_{\mathbf{m}}^{\Delta_2}} \cong \mathbb{Q}[x_{\Delta_1}, y_{\Delta_2}].$$

At this point we need a more explicit description of the classes $x_{\Delta_1}, y_{\Delta_2}$.

Let us fix a curve C_0 of topological type Γ and consider the étale covering

$$\widetilde{\mathfrak{M}}_0^\Gamma \xrightarrow{\phi} \mathfrak{M}_0^\Gamma$$

defined in Section 1.3. We have seen that this map corresponds to

$$(\mathcal{M}^0{}_{0,1})^{\Delta_1} \times (\mathcal{M}^0_{0,2})^{\Delta_2} \times (\mathcal{M}^0_{0,3})^{\Delta_3} \quad \xrightarrow{\phi} \quad \mathfrak{M}_0^\Gamma.$$

Since $\mathcal{M}^0_{0,3} \simeq \mathrm{Spec}\mathbb{C}$ we have

$$(\mathcal{M}^0{}_{0,1})^{\Delta_1} \times (\mathcal{M}^0_{0,2})^{\Delta_2} \quad \xrightarrow{\phi} \quad \mathfrak{M}_0^\Gamma.$$

For what we have seen in Section 1.4

$$\begin{aligned} \mathcal{M}^0{}_{0,1} &\simeq \mathrm{B}E \\ \mathcal{M}^0{}_{0,2} &\simeq \mathrm{B}\mathbb{G}_{\mathbf{m}} \end{aligned}$$

consequently we can write

$$\mathrm{B}(E^{\Delta_1} \times \mathbb{G}_{\mathbf{m}}^{\Delta_2}) \quad \xrightarrow{\phi} \quad \mathrm{BAut}(C_0)$$

Let P be a vertex of Γ such that $e(P) = 1$ or 2. On the component $\mathcal{M}^0_{0,e(P)}$ let us consider the universal curve

$$\mathcal{C}^P \xrightarrow{\widetilde{\Pi}} \mathcal{M}^0_{0,e(P)}.$$

On $\mathbb{P}^1_{\mathbb{C}}$ we fix coordinates and we define the points

$$\begin{aligned} z_\infty &:= [1,0] \\ z_0 &:= [0,1] \end{aligned}$$

We can write (see Section 1.4)

- $\mathcal{C}^P \simeq [\mathbb{P}^1_{\mathbb{C}}/E]$ when $e(P)=1$
- $\mathcal{C}^P \simeq [\mathbb{P}^1_{\mathbb{C}}/\mathbb{G}_{\mathbf{m}}]$ when $e(P)=2$

Let us consider on $\mathbb{P}^1_{\mathbb{C}}$ the linear bundles $\mathcal{O}(z_\infty)$ and $\mathcal{O}(z_0)$. We have a natural action of $\mathbb{G}_{\mathbf{m}}$ on global sections of both of them induced by the action of $\mathbb{G}_{\mathbf{m}}$ (that, we recall, fixes z_∞ and z_0) on $\mathbb{P}^1_{\mathbb{C}}$. Similarly we have an action of the group E on global sections of $\mathcal{O}(z_\infty)$. When $e(P)=1$ set

$$\psi^{e(P)}_{\infty,P} := c_1^E(H^0(\mathcal{O}(z_\infty),\mathbb{P}^1_{\mathbb{C}}));$$

while, if $e(P)=2$ set

$$\begin{aligned} \psi^{e(P)}_{\infty,P} &:= c_1^{\mathbb{G}_{\mathbf{m}}}(H^0(\mathcal{O}(z_\infty),\mathbb{P}^1_{\mathbb{C}})) \\ \psi^{e(P)}_{0,P} &:= c_1^{\mathbb{G}_{\mathbf{m}}}(H^0(\mathcal{O}(z_0),\mathbb{P}^1_{\mathbb{C}})). \end{aligned}$$

Clearly, when $e(P)=2$, we have

$$\psi^2_{\infty,P}+\psi^2_{0,P}=0$$

Now we define

$$\begin{aligned} t_P &:= \psi^1_{\infty,P} \text{ when } e(P)=1 \\ r_P &:= \psi^2_{\infty,P} \text{ when } e(P)=2 \end{aligned}$$

We can write for each P such that $e(P)=2$

$$r_P=\frac{\psi^2_{\infty,P}-\psi^2_{0,P}}{2}.$$

Clearly all the classes t_{Δ_1} and r_{Δ_2} are of order one and indipendent. From what we have seen above these classes generates the ring $A^*(\widetilde{\mathfrak{M}}^\Gamma_0)\otimes\mathbb{Q}$ and we have

$$A^*(\widetilde{\mathfrak{M}}^\Gamma_0)\otimes\mathbb{Q}\simeq\mathbb{Q}[t_{\Delta_1},r_{\Delta_2}].$$

Now we can describe the action of $\mathrm{Aut}\Gamma$ on $\mathbb{Q}[t_{\Delta_1},r_{\Delta_2}]$. An element $g\in\mathrm{Aut}(\Gamma)$ acts on C_0 with a permutation g_1 on the components with one node and a permutation g_2 of components with two nodes. As we have chosen coordinates on C_0 such that ∞ corresponds to the node of the terminal components, we make g_1 act directly to the set $\{t_{\Delta_1}\}$. We make g_2 act similarly on the set $\{r_{\Delta_2}\}$ but we have in addition to consider the sign, that is to say that when g_2 sends a vertex P of Γ to another vertex Q (such that $e(P)=e(Q)=2$), we have two possibilities

- the automorphism g exchange coordinates 0 and ∞ and so we have
$$g(r_P) = g\left(\frac{\psi^2_{\infty,P} - \psi^2_{0,P}}{2}\right) = \frac{\psi^2_{0,Q} - \psi^2_{\infty,Q}}{2} = -r_Q$$
- the automorphism g sends 0 in 0 and ∞ in ∞; in this case we have
$$g(r_P) = r_Q.$$

2.3. Chow groups for Artin stacks

Let X be an integral stack of finite type over $\mathcal{Sch}_{\mathbb{C}}$ [5]. We set $Z_i(X)$ the free abelian group of the set of integral closed substacks of X of dimension i.

Set
$$Z_*(X) \quad := \quad \bigoplus_{i\in\mathbb{Z}} Z_i(X),$$

A *rational function* on X is a morphism $U \to \mathbb{A}^1_{\mathbb{C}}$ defined on a nonempty open substack U of X. Rational functions form a field $\mathbb{C}(X)$ called *quotient field* of X.

Now we can define the group of *rational equivalences* on X as
$$W_i(X) := \bigoplus_{V\subseteq X} \mathbb{C}(V)^*$$
where the sum is taken over all integral substacks V of X of dimension $i+1$. Set
$$W_*(X) := \bigoplus_{i\in\mathbb{Z}} W_i(X).$$
For every i there is a map
$$\partial_i : W_i(X) \to Z_i(X)$$
which locally for the smooth topology sends a rational function to the corresponding Weyl divisor. Futhermor we have the map
$$\partial_* : W_*(X) \to Z_*(X).$$

Following the notation of [**Kre**] we define the *naive Chow groups* of X as
$$A^\circ_i(X) \quad := \quad Z_i(X)/\partial W_i(X)$$
$$A^\circ_*(X) \quad := \quad \bigoplus_{i\in\mathbb{Z}} A^\circ_i(X).$$

[5]Throughout this section all stacks are integral of finite type over $\mathbb{C}$

For naive Chow groups we have the pullback defined as follow. Let $f : Y \to X$ be a flat representable morphism of constant relative dimension and let $[V]$ an element of $A^\circ_*(X)$, we define

$$f^*[V] := [Y \times_X V].$$

If $f : Y \to X$ is a proper representable morphism we define [6]

$$f_*[V] := \deg(V/W)[W]$$

where W is the image of V in X. From Proposition 3.7 [**Vis**] these are a well defined group homomorphisms.

In order to define intersection classes for Artin stacks, we need at first a straightforward generalization of the Edidin-Graham-Totaro classes which are defined for quotient stacks.

Definition 2.11. The Edidin-Graham-Totaro Chow groups are defined, for X connected by

$$\widehat{A}_i(X) := \varinjlim_{\mathfrak{B}_X} A^\circ_{i+\mathrm{rk}}(E)$$

(where $\mathfrak{B}_X$ is the set of isomorphism classes of vector bundles over X , partially ordered by declaring $E \preceq F$ whenever there exists a surjection of vector bundles $F \to E$), and for $X = X_1 \coprod \cdots \coprod X_r$ by

$$\widehat{A}_i(X) := \bigoplus_{j=1}^{r} \widehat{A}_i(X_j).$$

Set

$$\widehat{A}_*(X) := \bigoplus_{i\in\mathbb{Z}} \widehat{A}_i(X).$$

Given a flat morphism of constant [7] relative dimension d, $f : Y \to X$ of stacks, let $E \to X$ be a vector bundle over X. We consider the following cartesian diagram

$$\begin{array}{ccc} F & \xrightarrow{\widetilde{f}} & E \\ \downarrow & \square & \downarrow \\ Y & \xrightarrow{f} & X \end{array}$$

[6]The degree of a representable proper morphism $Y \to X$ is a straightforward extension of Definition 1.15 [**Vis**]. Given a smooth covering $U \to X$ we define $\deg(Y/X)$ as $\deg(Y \times_X U/U)$.

[7]In [**Kre**] we have a good definition of flat pullback also for morphisms of *locally* relative constant dimension.

the map f is still a flat morphism of constant relative dimension d, so it remains defined a flat pullback

$$\widetilde{f}^* : A^\circ_*(E) \to A^\circ_{*+d}(F)$$

which define a pullback map

$$f^* : \widehat{A}_*(X) \to \widehat{A}_{*+d}(Y).$$

Similarly we define the pushforward for representable proper maps.

Given a morphism $f : Y \to X$ with Y connected, we define the restricted Edidin- Graham-Totaro Chow groups to be the groups

$$\widehat{A}^f_i(Y) := \varinjlim_{\mathfrak{B}_X} A^\circ_{i+\mathrm{rk}}(f^*E)$$

If $Y = Y_1 \coprod \cdots \coprod Y_r$ with each Y_j connected, we set

$$\widehat{A}^f_i(Y) := \bigoplus_{j=1}^{r} \widehat{A}^f_i(Y_j)$$

Let X be an Artin stack. We call $\mathfrak{U}_X$ the set of isomorphism classes of projective $X-$stacks with the partial ordering given by inclusion of components.

If $p_1 : T \to Y$ and $p_2 : T \to Y$ are projective morphisms, Then we set

$$\widehat{B}^{p_1,p_2}_k(Y) := \{p_{2*}\beta_2 - p_{1*}\beta_1 | \quad (\beta_1, \beta_2) \in \widehat{A}^{p_1}_k(T) \oplus \widehat{A}^{p_2}_k(T) \text{ and } \iota_{p_1}(\beta_1) = \iota_{p_2}(\beta_2)\}$$

which is a subgroup of $\widehat{A}_k(Y)$.

If we have a stack morphism as above $f : Y \to X$, we call $\widehat{B}_k(Y)$ the union of all subgroups $\widehat{B}^{p_1,p_2}_k(Y)$ such that the map $f \circ p_1$ is 2-isomorphic to $f \circ p_2$. So we can state the following:

Definition 2.12. Let X be an Artin stack. We call Chow groups of X

$$A_k(X) := \varinjlim_{\mathfrak{U}_X} \widehat{A}_k(Y)/\widehat{B}_k(Y).$$

At last we define flat pullback for the above intersection groups (see Section 2.2 [**Kre**]). Given a flat morphism $f : Y \to X$ of constant relative dimension d, let $h : X' \to X$ be a projective morphism. We consider the following

$$\begin{array}{ccc} Y' & \xrightarrow{\widetilde{f}} & X' \\ {\scriptstyle h}\downarrow & \square & \downarrow{\scriptstyle h} \\ Y & \xrightarrow{f} & X \end{array}$$

Since flat pullback on A_*° commutes with projective pushforward, the map

$$\widetilde{f}^* : \widehat{A}_*(X') \to \widehat{A}_*(Y')$$

descends to give a map

$$\frac{\widehat{A}_*(X')}{\widehat{B}_*(X')} \to \frac{\widehat{A}_*(Y')}{\widehat{B}_*(Y')}$$

which passes to the limit to give us

$$f^* : A_*(X) \to A_*(Y).$$

We redirect to **Theorem 2.1.12** [**Kre**] for the main properties of this functor group. We just point out that a ring structure is given for a stack which is smooth and can be stratified by locally closed substacks which are each isomorphic to the quotient stack of an algebraic group acting on a scheme.

We will denote a tipical element of $A_*(X)$ as a pair (f, α) where f is a projective map $Y \to X$ and α is a class in $\widehat{A}_k(Y)$. On the other hand we forget the map f when it is the identity map.

The definition of projective pushforward is tautological. Given a projective morphism $g : X \to Z$, we define

$$g_*(f, \alpha) := (g \circ f, \alpha).$$

Further we have that in a fiber diagram projective pushforward on A_* commutes with flat pullback.

With reference to Section 6 [**Kre**] we have that if X is a smooth Artin stack of finite type over $\mathbb{C}$, we have an intersection product on X. In this situation we define

$$A^i(X) := A_{n-i}(X)$$

where n is the dimension [8] of X and

$$A^*(X) := \bigoplus_{i\in\mathbb{Z}} A^i(X).$$

We give to $A^*(X)$ the graded ring structure given by the intersection product. From Proposition 3.4.2 [**Kre**] we have

$$A^i(X) = 0$$

for each $i < 0$.

Remark 2.13. Except from the finite type condition, we have seen in the first Chapter that $\mathfrak{M}_0$ has the above properties. In order to apply Kresh's theory we notice that for any $k \in \mathbb{N}$ the stack $\mathfrak{M}_0^{\leq k}$ is of finite type (as it comes out from proof of Proposition 1.10). Further, for each $r \geq k$ we have

$$A^k(\mathfrak{M}_0^{\leq k}) = A^k(\mathfrak{M}_0^{\leq r});$$

consequently we can define

$$A^k(\mathfrak{M}_0) = A^k(\mathfrak{M}_0^{\leq r})$$

for each $r \geq k$.

In this work we compute $A^*(\mathfrak{M}_0 \leq 3)$ and in particular we have

$$A^{\leq 3}(\mathfrak{M}_0) = A^{\leq 3}(\mathfrak{M}_0^{\leq 3}).$$

2.3.1. Chern classes. One of the main tools we need to do intersection theory is the top Chern class operation.

Let X be a connected Artin stack, and let $E \to X$ be a vector bundle on X of rank r.

Then we have a map

$$\widehat{A}_j(X) \to \widehat{A}_{j-r}(X)$$

which, for any vector bundle $F \to X$, with rank k, sends $\alpha \in A^\circ_{j+k}(F)$ to $s_*\alpha \in \widehat{A}_{j+k}(F \oplus E)$, where s is the zero section of $F \oplus E \to F$. We

[8]For the definition of dimension of an Artin stack X we refer to [**L-MB**]. If

$$U \xrightarrow{f} X$$

is a smooth covering we have the notion of relative dimension of f. For each morphism $T \to X$ (where T is a scheme), the projection map

$$U \times_X T \xrightarrow{f'} T$$

is smooth and we call $\dim(f)$ the relative dimension of f'. This definition is independent from $T \to X$. So we can define

$$\dim(X) = \dim(U) - \dim(f).$$

Note that stacks could have negative dimension.

denote the image class in $\widehat{A}_{j-r}(X)$ by α^E. When Y is not connected we take for α^E the sum on each connected component.

Definition 2.14. Let X be a stack, and let $E \to X$ be a vector bundle of rank r. We define the top Chern class operation

$$c_{\mathrm{top}}(E) \cap - : A_j(X) \to A_{j-r}(X)$$

by $(f, \alpha) \to (f, \alpha^{f^*E})$.

This is a good definition because the map respects equivalence.

With reference to Section 3 of [**Kre**] we have from top Chern classes the usual Segre and Chern classes, the Whitney formula for exact sequence of vector bundles and the projection formula.

2.4. Gysin maps and excess intersection formula

Now we mainly follow [**Kre**] and use definitions of cone and normal bundles for Artin stacks given in Section 1.5.

Let $i : F \to G$ be a regular local immersion of Artin stacks of codimension d, and let $g : G' \to G$ an arbitrary morphism. Let us consider the following diagram

$$\begin{array}{ccccc}
C_{F'}G' & & & & \\
{\scriptstyle p}\downarrow & \searrow & & & \\
C_F G \times_F F' & \longrightarrow & F' & \xrightarrow{j} & G' \\
\downarrow & \square & {\scriptstyle f}\downarrow & \square & \downarrow{\scriptstyle g} \\
C_F G & \longrightarrow & F & \xrightarrow{i} & G
\end{array}$$

As i is a local immersion, also the map $j : F' \to G'$ is a local immersion, and we may form the deformation space $M^{\circ}_{F'}G' \to \mathbb{P}^1$ with general fiber G' and special fiber over the infinity point $C_{F'}G'$. In particular we have an isomorphism

$$G' \times (\mathbb{P}^1\mathbb{C}/(\infty)) \xrightarrow{\sim} M^{\circ}_{F'}G'/C_{F'}G';$$

we also call s the closed immersion $C_{F'}G' \to M^{\circ}_{F'}G'$. Now we have a map

$$\varphi : A_*(G') \to A_*(C_F G \times_F F')$$

defined as the composition

$$\begin{array}{ccccccc}
A_*(G') & \longrightarrow & A_{*+1}(G' \times (\mathbb{P}^1\mathbb{C}/(\infty)) & & & & \\
 & & \sim\downarrow & & & & \\
 & & A_{*+1}(M^{\circ}_{F'}G'/C_{F'}G') & \xrightarrow{s^*} & A_*(C_{F'}G') & \xrightarrow{p_*} & A_*(C_F G \times_F F')
\end{array}$$

where the last map is the pushforward via the closed immersion p.

Furthermore, as the local immersion is also regular we have more over that the normal cone $C_F G$ (and consequently its fiber product with F') is a vector bundle over F (resp. over F'), so the pullback map:

$$\psi : A_{*-d}F \to A_*(C_F G \times_F F')$$

is an isomorphism.

Now we can state the following

Definition 2.15. Given a regular local immersion of Artin stacks $i : F \to G$ of codimension d and given an arbitrary morphism $g : G' \to G$; we call the refined Gysin map

$$f^! : A_*(G') \to A_{*-d}(F')$$

the composition

$$\psi^{-1} \circ \varphi.$$

We have the equivalent of **excess intersection formula** (see Theorem 6.3 [**Ful**]) for Artin stacks.

Proposition 2.16. *Let $X \xrightarrow{i} Y$ be a regular local embedding of Artin stacks and let $Y' \xrightarrow{g} Y$ and $Y'' \xrightarrow{q} Y'$ be two arbitrary maps. Furthermore we suppose that the projection*

$$j : X' := X \times_Y Y' \to Y'$$

is a regular embedding of Artin stacks.

With reference to the normal bundles (see Section 1.5) $N := N_X Y, N' := N_{X'} Y'$ and to the cartesian diagram

$$\begin{array}{ccc} X'' & \longrightarrow & Y'' \\ {\scriptstyle p}\downarrow & \square & \downarrow{\scriptstyle q} \\ X' & \xrightarrow{j} & Y' \\ {\scriptstyle f}\downarrow & \square & \downarrow{\scriptstyle g} \\ X & \xrightarrow{i} & Y \end{array}$$

given a class $\alpha \in A_*(Y'')$ we have the following

$$i^! \alpha = j^! \alpha \cdot c_{top}(p^*(E))$$

where E is the vector bundle

$$f^*(N)/(N').$$

Proof. We simply follow the proof in [**Ful**].

Set

$$\begin{aligned} Q &:= \mathbb{P}(p^*(N) \oplus 1) \\ Q' &:= \mathbb{P}(p^*(N') \oplus 1) \end{aligned}$$

and set ξ and ξ' respectively as the universal quotient bundles on Q and Q'.

We start by proving the Proposition when α is the class $[V]$ of an integral closed substack V of Y''. Let us define:

$$W := X'' \times_{Y''} V$$

and C as the cone $C_W V$, we have: $C \subseteq N' \subseteq N$.

Set

$$P := \mathbb{P}(C \oplus 1).$$

We have an exact sequence of bundles over Q'

$$0 \to \xi' \to \xi|_{Q'} \to r^*(p^*E) \to 0. \tag{10}$$

Claim 2.17. *(For the proof we follow p.94* [**Ful**]*)*
Let $i : F \to G$ and $g : G' \to G$ be as before. Set

$$Q := \mathbb{P}(f^*(N_F G) \oplus 1)$$

and let ξ be the universal quotient bundle on Q. We call r the projection of Q on F'. Then for every integral closed substack $V \subseteq G'$ we have

$$i^!([V]) = r_*(c_{top}(\xi) \cap [\mathbb{P}(C_W V \oplus 1)]).$$

From the Claim we have

$$i^!([V]) = r_*(c_{top}(\xi) \cdot P).$$

By applying Whitney formula we have

$$i^!([V]) = r_*(c_{top}\xi' \cdot c_{top} r^*(p^*E) \cdot P;$$

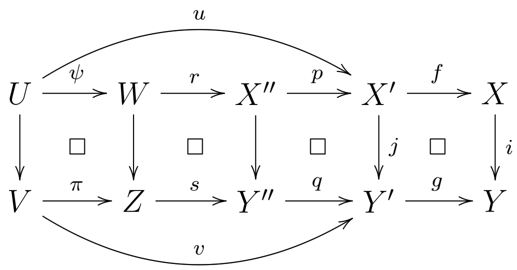
it follows from the projection formula:

$$i^!([V]) = c_{top}(p^*E) \cdot r_*(c_{top}(\xi') \cdot P)$$

and again from the Claim

$$i^!([V]) = j^!([V]) \cdot c_{top}(p^*(E)).$$

Now we consider a class $(s, \alpha) \in A_* Y''$ that is to say a projective morphism $s : Z \to Y''$ and a class $\alpha \in \widehat{A}_*(Z)/\widehat{B}_*(Z)$. Take a representative class $\beta \in \widehat{A}_*(Z)$ and a representative of it $\gamma \in A^\circ_*(V)$ for some vector bundle $\pi : V \to Z$. Now we refer to the following diagram



We can apply what we have proved above and write

$$i^! \gamma = j^! \gamma \cdot c_{top}(v^*(E))$$

this relation passes to the limit on $\mathfrak{B}_Z$ and so we have

$$i^! \beta = j^! \beta \cdot c_{top}(v^*(E)).$$

We conclude by noting that the relation above is clearly true also for α (It passes through the limit on $\mathfrak{U}_{Y''}$). ☐

2.5. Deformation of curves and extension of classes

In this section all graphs will be with maximal multiplicity at most 3.

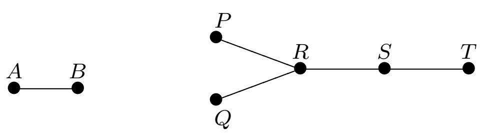
The previous formula will be applied in order to compute extension of strata classes. More precisely, given two graphs Γ and Γ' respectively with δ and δ' nodes ($\delta' > \delta$) and a polynomial a in $A^*(\mathfrak{M}_0^\Gamma) \otimes \mathbb{Q}$ we define a class in $\mathfrak{M}_0^{\leq \delta'}$ (whose restriction to $\mathfrak{M}_0^\Gamma$ is a) and compute its restriction to $\mathfrak{M}_0^{\Gamma'}$. This depends on the way of deforming Γ in Γ'.

Definition 2.18. Given two graphs as above we call an ordered deformation of Γ into Γ' any surjective map of vertices $d : \Gamma' \to \Gamma$ such that

(1) for each $P, Q \in \Gamma'$ we have $d(P) = d(Q) = A \in \Gamma'$ only if for each R in the connected path from P to Q we have $d(R) = A$;
(2) for each edge $(P, Q) \in \Gamma'$ such that $d(P) \neq d(Q)$ there must be an edge in Γ between $d(P)$ and $d(Q)$.

We denote by $\mathrm{def}_o(\Gamma, \Gamma')$ the set of deformations.

Example 2.19. Let Γ and Γ' be the following graphs



we have the following 8 ordered deformations

1)$(P,R,S,T)\mapsto A$	$Q\mapsto B$	5)$Q\mapsto A$	$(P,R,S,T)\mapsto B$
2)$(Q,R,S,T)\mapsto A$	$P\mapsto B$	6)$P\mapsto A$	$(Q,R,S,T)\mapsto B$
3)$(P,Q,R)\mapsto A$	$(S,T)\mapsto B$	7)$(S,T)\mapsto A$	$(P,Q,R)\mapsto B$
4)$(P,Q,R,S)\mapsto A$	$T\mapsto B$	8)$T\mapsto A$	$(P,Q,R,S)\mapsto B$

There exist two different equivalence relations in $\mathrm{def}_0(\Gamma,\Gamma')$. We say that two elements d_1, d_2 are in $\sim_\Gamma$ if there exists a $\gamma\in\mathrm{Aut}(\Gamma)$ such that $d_2=\gamma d_1$. Similarly we say that two elements d_1, d_2 are in $\sim_{\Gamma'}$ if there exists a $\gamma'\in\mathrm{Aut}(\Gamma')$ such that $d_2=d_1\gamma'$.

Definition 2.20. We call $\Gamma-$deformations (or simply deformations) from Γ to Γ' the set

$$\mathrm{def}_\Gamma(\Gamma,\Gamma') := \mathrm{def}_o(\Gamma,\Gamma')/\sim_\Gamma .$$

We call $\Gamma'-$deformations from Γ to Γ' the set

$$\mathrm{def}_{\Gamma'}(\Gamma,\Gamma') := \mathrm{def}_o(\Gamma,\Gamma')/\sim_{\Gamma'} .$$

In the above example we can take as representatives of $\Gamma-$deformations the first 4 ordered deformations. On the other hand we have $1\sim_{\Gamma'}2$ and $5\sim_{\Gamma'}6$.

Claim 2.21. *Let $C\xrightarrow{\pi}T$ be a family of rational nodal curves over an irreducible scheme T. Suppose further that the generic fiber has topological type Γ, then there exists a fiber of topological type $\widehat{\Gamma}$ only if there exists an ordered deformation of Γ into $\widehat{\Gamma}$.*

Proof. We may assume that T is reduced and of finite type over $\mathbb{C}$. We know (see Section 1.3) that $T-T^{>\delta}$ is open, so when $\widehat{\Gamma}$ has the same number of nodes of Γ, it must be Γ. In this case we can take as deformation any automorphism of Γ. When the geometric fiber C_t (for some geometric point $t\in T$) is of topological type $\widehat{\Gamma}$ with $\widehat{\delta}$ nodes such that $\widehat{\delta}>\delta$ we have exactly $n:=\widehat{\delta}-\delta$ nodes $P_1,\dots,P_n$ where

$$\widehat{\mathcal{O}}_{C_{rs},P_i}\neq\widehat{\mathcal{O}}_{T,t}.$$

By extending local sections we have a way (unique up to an automorphism of Γ) to send each of the two components which have P_i as a node to an irreducible component of the curve associated with Γ. Otherwise if P is a node of Γ such that

$$\widehat{\mathcal{O}}_{C_{rs},P}=\widehat{\mathcal{O}}_{T,t}$$

we send P to the corresponding edge of Γ (up to automorphism of Γ) and the components which have P as a node to the corresponding components. In this way we have obtained a deformation of Γ into $\widehat{\Gamma}$. □

Now let us consider the étale map

$$\widetilde{\mathfrak{M}}_0^\Gamma \xrightarrow{\phi} \mathfrak{M}_0^\Gamma.$$

We have given in Section 2.2 a description of $A^*\mathfrak{M}_0$ as the subring of polynomials of $A^*(\widetilde{\mathfrak{M}}_0^\Gamma)$ in the classes [9] $t_1, \ldots, t_{\delta_1}, r_1, \ldots, r_{\delta_2}$ invariant for the action of $\mathrm{Aut}(\Gamma)$.

We call $\mathcal{M}_{0,i}$ the stack of rational nodal curves with i sections. Let $\left(\widetilde{\mathfrak{M}}_0^\Gamma\right)^{\leq \delta' - \delta}$ be the substack of

$$(\mathcal{M}_{0,1})^{\Delta_1} \times (\mathcal{M}_{0,2})^{\Delta_2} \times (\mathcal{M}_{0,3})^{\Delta_3}$$

whose fibers have at most $\delta' - \delta$ nodes (the sum of nodes is taken over all the connected components). Polynomials in $\mathbb{Q}[t_1, \ldots, t_{\delta_1}, r_1, \ldots, r_{\delta_2}]$ has a natural extension to $\left(\widetilde{\mathfrak{M}}_0^\Gamma\right)^{\leq \delta' - \delta}$. Let us fix one of such polynomials a which are invariants for the action of $\mathrm{Aut}\Gamma$.

We have seen that the étale covering

$$\widetilde{\mathfrak{M}}_0^\Gamma \xrightarrow{\phi} \mathfrak{M}_0^\Gamma$$

is obtained by gluing sections in a way which depends on Γ.

By gluing sections in the same way we obtain a functor

$$\left(\widetilde{\mathfrak{M}}_0^\Gamma\right)^{\leq \delta' - \delta} \xrightarrow{\Pi} \mathfrak{M}_0^{\delta'}.$$

Proposition 2.22. *With the above notation we have that the closure* $\overline{\mathfrak{M}}_0^\Gamma$ *of* $\mathfrak{M}_0^\Gamma$ *in* $\mathfrak{M}_0^{\leq \delta'}$ *is*

$$\Pi\left(\left(\widetilde{\mathfrak{M}}_0^\Gamma\right)^{\leq \delta' - \delta}\right).$$

Proof. From claim 2.21 we have

$$\Pi\left(\left(\widetilde{\mathfrak{M}}_0^\Gamma\right)^{\leq \delta' - \delta}\right).$$

On the other hand let $C_\Omega \xrightarrow{\pi} \mathrm{Spec}\Omega$ be the image in $\mathfrak{M}_0^{\delta'}$ of a geometric point of $\left(\widetilde{\mathfrak{M}}_0^\Gamma\right)^{\leq \delta' - \delta}$. The dual graph Γ_Ω of C_Ω is a deformation of Γ. In order to show that $C_\Omega \xrightarrow{\pi} \mathrm{Spec}\Omega$ is a geometric point of $\overline{\mathfrak{M}}_0^\Gamma$, we fix a deformation $d : \Gamma_\Omega \to \Gamma$. For each vertex A of Γ, the set $d^{-1}(A)$ is a subtree of Γ_Ω. We can give a deformation $C_A \xrightarrow{\pi} T$ of C_Ω such that the generic fiber is $\mathbb{P}^1$. Furthermore we can define on $C_A \xrightarrow{\pi} T$ a family of $E(A)$ that respect d. At last we glue all C_A along sections and obtain a deformation $C \xrightarrow{\pi} T$ of $C_\Omega \xrightarrow{\pi} \mathrm{Spec}\Omega$ in $\overline{\mathfrak{M}}_0^\Gamma(\mathrm{Spec}\Omega)$. □

[9] corresponding to sections of $\widetilde{\mathfrak{M}}_0^\Gamma$

As a consequence we have that the image of Π is the closed substack

$$\mathfrak{M}_0^{\mathrm{def}(\Gamma,\delta')} \subset \mathfrak{M}_0^{\leq\delta'}.$$

Proposition 2.23. *The map*

$$\Pi : \left(\widetilde{\mathfrak{M}}_0^{\Gamma}\right)^{\delta'-\delta} \to \mathfrak{M}_0^{\leq\delta'}$$

is finite.

Proof. We have to prove that Π is representable, with finite fibers and proper. The not trivial property to verify is properness. We prove it through the valutative criterion (see [**Hrt**] p.101).

Let R be a valuation ring and $\mathbb{K}$ be its residual field. Let us consider a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}\mathbb{K} & \longrightarrow & \left(\widetilde{\mathfrak{M}}_0^{\Gamma}\right)^{\delta'-\delta} \\ \downarrow & & \downarrow \Pi \\ \mathrm{Spec}R & \longrightarrow & \mathfrak{M}_0^{\leq\delta'} \end{array}$$

we have to prove that there exist a unique morphism

$$\mathrm{Spec}R \to \left(\widetilde{\mathfrak{M}}_0^{\Gamma}\right)^{\delta'-\delta}$$

which make the diagram commute. This is equivalent to the following statement

> given a curve $C_R \to \mathrm{Spec}R$ in $\mathfrak{M}_0^{\leq\delta'}(R)$ and a lifting $\overline{C}_{\mathbb{K}}$ to $\left(\widetilde{\mathfrak{M}}_0^{\Gamma}\right)^{\delta'-\delta}(\mathbb{K})$ of its restriction $C_{\mathbb{K}} \to \mathrm{Spec}\mathbb{K}$, there exists a unique lifting $\overline{C}_R$ which induces $\widetilde{C}_{\mathbb{K}}$.

Now the normalization $\widetilde{C}_{\mathbb{K}}$ of $C_{\mathbb{K}}$ is the same of the normalization of $\overline{C}_{\mathbb{K}}$, furthermore if $\widetilde{C}_R$ is the normalization of C_R we have

$$\widetilde{C}_{\mathbb{K}} = \widetilde{C}_R \times_{\mathrm{Spec}R} \mathrm{Spec}\mathbb{K}.$$

By gluing with the same rule of $\overline{C}_{\mathbb{K}}$ we obtain a lifting $\overline{C}_R$. $\square$

Therefore the map Π is finite (of order σ) so we have the push forward

$$\Pi_* : A^*\left(\left(\widetilde{\mathfrak{M}}_0^{\Gamma}\right)^{\delta'-\delta}\right) \to A^*\left(\mathfrak{M}_0^{\mathrm{def}(\Gamma,\delta')}\right)$$

Let us consider a polynomial a in $A^*(\mathfrak{M}_0^{\Gamma}) \otimes \mathbb{Q}$. With reference to the étale covering

$$\widetilde{\mathfrak{M}}_0^{\Gamma} \xrightarrow{\phi} \mathfrak{M}_0^{\Gamma}$$

we define

$$\widetilde{a} := \frac{\phi^* a}{\sigma} \in A^*(\widetilde{\mathfrak{M}}_0^{\Gamma}) \otimes \mathbb{Q},$$

this means that $\phi_*(\widetilde{a}) = a$. Since $\widetilde{a}$ can be written as a polynomial in the classes ψ (defined in Section 2.2) that depends only from edges of Γ, it has a natural extension to $A^*\left(\left(\widetilde{\mathfrak{M}}_0^{\Gamma}\right)^{\delta'-\delta}\right) \otimes \mathbb{Q}$.

If Γ' is a deformation of Γ with δ' edges, with reference to the inclusion

$$\mathfrak{M}_0^{\Gamma'} \xrightarrow{in} \mathfrak{M}_0^{\leq \delta'},$$

we want to describe the class

$$in^*(\Pi_*(\widetilde{a})) \in A^*(\mathfrak{M}_0^{\Gamma'}) \otimes \mathbb{Q}.$$

In order to do that we consider the cartesian product

$$\Psi(\Gamma, \Gamma') := \mathfrak{M}_0^{\Gamma'} \times_{\mathfrak{M}_0^{\delta'}} \left(\widetilde{\mathfrak{M}}_0^{\Gamma}\right)^{\delta'-\delta}.$$

We refer to the corresponding diagram

$$\begin{array}{ccc} \Psi(\Gamma, \Gamma') & \xrightarrow{pr_2} & \left(\widetilde{\mathfrak{M}}_0^{\Gamma}\right)^{\leq \delta'-\delta} \\ {\scriptstyle pr_1}\downarrow & \square & \downarrow{\scriptstyle \Pi} \\ \mathfrak{M}_0^{\Gamma'} & \xrightarrow{in} & \mathfrak{M}_0^{\leq \delta'} \end{array}$$

Proposition 2.24. *We have*

$$\Psi(\Gamma, \Gamma') = \coprod_{\alpha=1}^{\mu} \Psi(\Gamma, \Gamma')_\alpha$$

where the union is taken over the set of ordered deformations up to $\sim_{\Gamma'}$.

Proof. Clearly we have that $\Psi(\Gamma, \Gamma')$ is an inclusion of components in the union of all stacks

$$\left(\widetilde{\mathfrak{M}}_0^{\Gamma}\right)^{\delta'-\delta} = \prod_{A \in \Gamma} \mathcal{M}_{0,E(A)}^{n_A}$$

where n_Γ is a set of integers such that

$$\sum_{A \in \Gamma} n_A = \delta' - \delta.$$

Let d be a deformation of Γ into Γ'. For each vertex A in Γ, the set $d^{-1}(A)$ is a subtree of Γ'. We denote with $E(d, A)$ the set of edges of $d^{-1}(A)$. We indicate with

$$\mathcal{M}(d, A)$$

the unique component of

$$\mathcal{M}_{0,E(A)}^{E(d,A)}$$

which respects the deformation d.

Therefore we can associate with each d the following connected components of $\Psi(\Gamma, \Gamma')$

$$\prod_{A\in\Gamma} \mathcal{M}(d, A).$$

On the other hand each connected components $\Psi(\Gamma, \Gamma')$ of $\Psi(\Gamma, \Gamma')$ is a connected component of

$$\left(\widetilde{\mathfrak{M}}_0^{\Gamma}\right)^{\delta'-\delta} = \prod_{A\in\Gamma} \mathcal{M}_{0,E(A)}^{n_A}$$

for some family of integer n_Γ such that

$$\sum_{A\in\Gamma} n_A = \delta' - \delta.$$

and such that gluing along edges of Γ we obtain a covering of $\mathfrak{M}_0^{\Gamma'}$ [10].

Now we want to describe all the deformations of Γ into Γ' which respects $\Psi(\Gamma, \Gamma')_\alpha$. For each $A \in \Gamma$ the connected component which we have in $\mathfrak{M}_{0,E(A)}^{n_A}$ determines the topological type of the subtree of Γ' which we can send into A. This means that the deformation is uniquely determined up to $\sim_{\Gamma'}$. □

Example 2.25.

Let Γ and Γ' be the graphs of the Example 2.19. We have $\delta' - \delta = 3$ and

$$\widetilde{\mathfrak{M}}_0^{\Gamma} = \mathcal{M}_{0,1}^{0} \times \mathcal{M}_{0,1}^{0}$$

with a double covering

$$\widetilde{\mathfrak{M}}_0^{\Gamma} \xrightarrow{\phi} \mathfrak{M}_0^{\Gamma}.$$

Clearly $\Psi(\Gamma, \Gamma')$ is an inclusion of components in

$$\coprod_{i,j:i+j=3} \left(\mathcal{M}_{0,1}^{i} \times \mathcal{M}_{0,1}^{j}\right) \tag{11}$$

We have that $\Psi(\Gamma, \Gamma')$ has 6 components which corresponds to deformations (enumerated in Example 2.19) up to $\sim_{\Gamma'}$, with the following inclusions:

- deformation 1 (which is Γ'-equivalent to 2) and 4 correspond to two connected components of $\mathcal{M}_{0,1}^{3} \times \mathcal{M}_{0,1}^{0}$
- deformation 3 corresponds to a connected component of $\mathcal{M}_{0,1}^{2} \times \mathcal{M}_{0,1}^{1}$

[10] We will determine in the following the order of this covering

- deformation 7 corresponds to a connected components of $\mathcal{M}^1_{0,1} \times \mathcal{M}^2_{0,1}$
- deformation 5 (which is Γ'-equivalent to 6) and 8 corresponds to two connected components of $\mathcal{M}^0_{0,1} \times \mathcal{M}^3_{0,1}$

For every α we have the related commutative diagram

$$\begin{array}{ccc} \Psi(\Gamma,\Gamma')_\alpha & \xrightarrow{pr_2^\alpha} & \left(\widetilde{\mathfrak{M}}^\Gamma_0\right)^{\leq \delta'-\delta} \\ \downarrow{\scriptstyle pr_1^\alpha} & & \downarrow{\scriptstyle \Pi} \\ \mathfrak{M}^{\Gamma'}_0 & \xrightarrow{in} & \mathfrak{M}^{\leq \delta'}_0 \end{array}$$

The map pr_2^α is a closed immersion of codimension $\delta' - \delta$. Let us still call $\widetilde{a}$ the pullback of the polynomial $\widetilde{a}$ through pr_2^α. By the excess intersection formula 2.16 (which we apply when p is the identity map) we have

$$in^!(\widetilde{a}) = \sum_{\alpha=1}^{\mu} (\widetilde{a} \cdot c_{top}[(pr_1^{\alpha *}\mathcal{N}_{in})/\mathcal{N}_{pr_2^\alpha}].$$

By applying the push forward along pr_1^α we write

$$\sum_{\alpha=1}^{\mu} pr^\alpha_{1*}(\widetilde{a} \cdot c_{top}[(pr_1^{\alpha *}\mathcal{N}_{in})/\mathcal{N}_{pr_2}] =: \sum_{\alpha=1}^{\mu} pr^\alpha_{1*}(\widetilde{a} \cdot c_{top}(\mathcal{N}^\alpha)).$$

For each α we have an étal covering

$$f_\alpha : \widetilde{\mathfrak{M}}^{\Gamma'} \to \Psi(\Gamma,\Gamma')_\alpha$$

that glue along sections.

For every $\alpha = 1, \ldots, \mu$ let us call σ'_α the order of pr_1^α.

We have the following commutative diagram

$$\begin{array}{ccc} \widetilde{\mathfrak{M}}^{\Gamma'}_0 & \xrightarrow{f_\alpha} & \Psi(\Gamma,\Gamma')_\alpha \\ & {\scriptstyle \phi}\searrow \quad \swarrow{\scriptstyle pr_1^\alpha} & \\ & \mathfrak{M}^{\Gamma'}_0 & \end{array}$$

from which we have

$$\mathrm{ord}\phi = \mathrm{ord} f_\alpha \cdot \mathrm{ord}\sigma'_\alpha.$$

Remark 2.26. The order of f_α is the number of $g \in \mathrm{Aut}(\Gamma')$ such that for each deformation d associated to α the deformation $d \circ g$ is Γ'-equivalent to d.

We want to write $\phi^* pr^{\alpha}_{1*}(\widetilde{a} \cdot c_{top}(\mathcal{N}^{\alpha}))$ in $A^*(\widetilde{\mathfrak{M}}^{\Gamma}_0) \otimes \mathbb{Q}$. We have

$$\phi^* pr^{\alpha}_{1*}(\widetilde{a} \cdot c_{top}(\mathcal{N}^{\alpha})) = \sum_{\alpha=1}^{\mu} \sigma'_{\alpha} f^*_{\alpha}(\widetilde{a}) \cdot c_{top}(\widetilde{\mathcal{N}}^{\alpha}) \tag{12}$$

where

$$\widetilde{\mathcal{N}}^{\alpha} := f^*_{\alpha}\mathcal{N}^{\alpha}.$$

We postpone to the following Chapter the computation of $c_{top}(\widetilde{\mathcal{N}}^{\alpha}) \in A^*(\widetilde{\mathfrak{M}}^{\Gamma}_0) \otimes \mathbb{Q}$. Anyway we still need the pullback of ψ classes along f_{α}. We clarify how to do it with the following example.

Example 2.27. Let Γ and Γ' be trees again as in Example 2.19. We have two ψ classes in $A^*(\widetilde{\mathfrak{M}}^{\Gamma}_0)$ that are t_A and t_B. Let $\Psi(\Gamma, \Gamma')_{\alpha}$ be the component associated to deformation 4, then we have

$$\begin{aligned} f^*_{\alpha}(t_A) &= t_S \\ f^*_{\alpha}(t_B) &= t_T. \end{aligned}$$

2.5.1. Algebraic lemmas. Now we recall Lemma 4.4 [**Ve-Vi**]:

Lemma 2.28. *Let A, B and C be rings, $f : B \to A$ and $g : B \to C$ ring homomorphism. Let us suppose that there exists an homomorphism of abelian groups $\phi : A \to B$ such that:*
the sequence

$$A \xrightarrow{\phi} B \xrightarrow{g} C \to 0$$

is exact; the composition $f \circ \phi : A \to A$ is multiplication by a central element $a \in A$ which is not a 0-divisor.

Then f and g induce an isomorphism of ring

$$(f, g) : B \to A \times_{A/(a)} C,$$

where the homomorphism $A \xrightarrow{p} A/(a)$ is the projection, while $C \xrightarrow{q} A/(a)$ is induced bya the isomorphism $C \simeq B/\ker g$ and the homomorphism of rings $f : B \to A$.

Proof. Owing to the fact that a is not a 0-divisor we immediately have that ρ is injective. Let us observe that the map $(f, g) : B \to A \times_{A/(a)} C$ is well defined for universal property and for commutation(?) of the diagram:

$$\begin{array}{ccc} B & \xrightarrow{g} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle q} \\ A & \xrightarrow{p} & A/(a) \end{array}$$

Now let us exhibit the inverse function $A \times_{A/(a)} C \xrightarrow{\rho} B$. Given $(\alpha, \gamma) \in A \times_{A/(a)} C$, let us chose an element $\beta \in B$ such that $g(\beta) = \gamma$. By definition of q we have that $f(\beta) - \alpha$ lives in the ideal (a) and so (it is an hypothesis on $f \circ \phi$) there exists in A an element $\tilde{\alpha}$ such that:

$$f(\beta) - \alpha = f(\phi(\tilde{\alpha}))$$

from which:

$$f(\beta - \phi(\tilde{\alpha})) = \alpha$$

we define then $\rho(\alpha, \gamma) := \beta - \phi(\tilde{\alpha})$. In order to verify that it is a good definition, let us suppose that there exist an element $\beta_0 \in B$ such that $(f, g)(\beta_0) = 0$, to be precise there exists an element $\alpha_0 \in A$ such that: $\phi(\alpha_0) = 0$ and furthermore

$$0 = f(\phi(\alpha_0)) = a\alpha_0$$

but we have that a is not a divisor by zero, so necessary we have $\alpha_0 = 0$ and $\beta_0 = 0$. we conclude by noting that from the definition of ρ we have immediately that it is an isomorphism. $\square$

The Lemma 2.28 will be used for computing the Chow ring of $\mathfrak{M}_0^{\leq n+1}$ when the rings $A * [\mathfrak{M}_0^{\leq n}] \otimes \mathbb{Q}$ and $A * [\mathfrak{M}_0^{n+1}] \otimes \mathbb{Q}$ are known. As a matter of fact, given an Artin stack $\mathcal{X}$ and a closed Artin substack $\mathcal{Y} \xrightarrow{i} \mathcal{X}$ of positive codimension, we have the exact sequence (see [**Kre**] Section 4)

$$A^*(\mathcal{Y}) \otimes \mathbb{Q} \xrightarrow{i_*} A^*(\mathcal{X}) \otimes \mathbb{Q} \xrightarrow{j^*} A^*(\mathcal{U}) \to 0$$

By using the Self-intersection Formula, it follows that:

$$i^* i_* 1 = i^*[\mathcal{Y}] = c_{top}(\mathcal{N}_{\mathcal{Y}/\mathcal{X}}),$$

when $c_{top}(\mathcal{N}_{\mathcal{Y}/\mathcal{X}})$ is not 0-divisor we can apply the Lemma.

In the following it will be used also the following algebraic Lemma:

Lemma 2.29. *Given the morphisms*

$$A_1 \xrightarrow{p_1} \overline{A}_1 \to B \leftarrow \overline{A}_2 \xleftarrow{p_2} A_2$$

in the category of rings, where the maps p_1 e p_2 are quotient respectively for ideals I_1 e I_2. Then it is defined an isomorphism

$$\overline{A}_1 \times_B \overline{A}_2 \cong \frac{A_1 \times_B A_2}{(I_1, I_2)}.$$

Proof. Let us consider the map

$$(p_1, p_2) : A_1 \times_B A_2 \to \overline{A}_1 \times_B \overline{A}_2,$$

by surjectivity of p_1 e p_2 this map is sujective, while the kernel is the ideal (I_1, I_2). $\square$

CHAPTER 3

Fundamental classes on $\mathfrak{M}_0$

In this Chapter we work on a fixed tree Γ with maximal multiplicity at most 3 and δ edges. We indicate with $\Delta_1, \Delta_2, \Delta_3$ the sets of vertices that belongs respectively to one, two and three edges (and with $\delta_1, \delta_2, \delta_3$ their cardinalities).

We have seen in Section 1.4 that

$$\mathfrak{M}_0^\Gamma = \mathrm{B}\mathrm{Aut}(C_0)$$

where $\mathrm{Aut}(C_0)$ is the group $\mathrm{Aut}(\Gamma) \ltimes \left(E^{\Delta_1} \times \mathbb{G}_{\mathbf{m}}^{\Delta_2}\right)$. Let us call σ the order of $\mathrm{Aut}\Gamma$.

We have an étale covering of order σ

$$\begin{array}{ccc} \widetilde{\mathfrak{M}_0^\Gamma} & \xrightarrow{\phi} & \mathfrak{M}_0^\Gamma \\ \mathrm{B}H & \xrightarrow{\phi} & \mathrm{B}\mathrm{Aut}(C_0) \end{array}$$

where H is the group $E^{\Delta_1} \times \mathbb{G}_{\mathbf{m}}^{\Delta_2}$.

Let us call C_0 a rational nodal curve of topological type Γ where we have fixed coordinates as in Section 1.4. If $\mathcal{C}^\Gamma$ is the universal curve (see Section 1.5) on $\mathfrak{M}_0^\Gamma$ we have that

$$\mathcal{C}^\Gamma = [C_0/\mathrm{Aut}(C_0)]$$

Furthermore the functor

$$\Pi : \mathcal{C}^\Gamma \xrightarrow{\Pi} \mathrm{B}\mathrm{Aut}(C_0)$$

that forgets the map on C_0 is proper since it comes from the proper map $C_0 \to \mathrm{Spec}\mathbb{C}$ equivariant for $\mathrm{Aut}(C_0)$. Consequently, as seen in Section 2.1, we have the pushforward

$$\Pi_* : A^*_{\mathrm{Aut}(C_0)}(C_0) \otimes \mathbb{Q} \to A^*_{\mathrm{Aut}(C_0)}.$$

3.1. Classes of strata

We define γ_i as the class in $A^*(\mathfrak{M}_0^{\leq i})$ of $\mathfrak{M}_0^i$. We will indicate with $\gamma_i \in A^*(\mathfrak{M}_0)$ also the class of the closure of $\mathfrak{M}_0^i$ in $\mathfrak{M}_0$. Similarly we define γ_Γ as the class of the closure of $\mathfrak{M}_0^\Gamma$ in $\mathfrak{M}_0$.

For every tree Γ of maximal multiplicity at most three [1], let us consider the regular embedding (see Proposition 1.13)

$$\mathfrak{M}_0^\Gamma = \mathrm{BAut}(C_0) \xrightarrow{in} \mathfrak{M}_0^{\leq \delta}$$

where δ is the number of edges of Γ. We still call $\gamma_\Gamma \in A^*(\mathfrak{M}_0^\Gamma)$ the pullback $\phi^*(\gamma_\Gamma)$.

Let $\Delta_1, \Delta_2, \Delta_3$ be respectively the sets of points of Γ with multiplicity 1,2,3. We denote with Δ the set $\Delta_1 \cup \Delta_2$.

Let C_Γ be the components of C_0 (which we see as vertex of Γ). Let $E(\Gamma)$ be the set of edges. If $(\alpha, \beta) \in E(\Gamma)$ we call $z_{\alpha\beta}$ the common point of C_α and C_β.

Let us consider the normal bundle

$$N_\Gamma := N_{\mathfrak{M}_0^\Gamma / \mathfrak{M}_0^{\leq \delta}}.$$

From what we have seen in Section 1.5 we have that $N_\Gamma = \mathrm{def}_\Gamma$ is the space of first order deformations of $\mathfrak{M}_0^\Gamma$

$$\bigoplus^{(\alpha,\beta)\in E(\Gamma)} T_{z_{\alpha\beta}}(C_\alpha) \otimes T_{z_{\alpha\beta}}(C_\beta).$$

Now we have to describe the action of $\mathrm{Aut}(C_0)$ on N_Γ. As usual let us consider the étale covering

$$\widetilde{\mathfrak{M}}_0^\Gamma \xrightarrow{\phi} \mathfrak{M}_0^\Gamma$$

and set $\widetilde{N}_\Gamma = \phi^* N_\Gamma$. We notice that, given coordinates as in Section 1.3, the point $z_{\alpha\beta}$ on each component C_α (which we call $z_{\alpha\beta}|_\alpha$) is 0, 1 or ∞.

By using notation of Section 2.2 we have on $\widetilde{\mathfrak{M}}_0^\Gamma$

$$c_1^{\mathbb{G}_\mathbf{m}}(T_{z_{\alpha\beta}}(C_\alpha)) = \psi^{e(\alpha)}_{z_{\alpha\beta}|_\alpha, \alpha}$$

which is zero when $e(P) > 2$, consequently

$$c_{\mathrm{top}}^{(\mathbb{G}_\mathbf{m})^\Delta}(\widetilde{N}_\Gamma) = \prod^{(\alpha,\beta)\in E(\Gamma)} \left(\psi^{e(\alpha)}_{z_{\alpha\beta}|_\alpha, \alpha} + \psi^{e(\alpha)}_{z_{\alpha\beta}|_\beta, \beta} \right).$$

We racall the following relation in $A^*(\mathfrak{M}_0^\Gamma)$

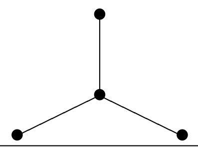
$$in^* in * [\mathfrak{M}_0^\Gamma] = c_{top}^{(\mathbb{G}_\mathbf{m})^\Delta}(N_\Gamma) \cap [\mathfrak{M}_0^\Gamma]. \tag{13}$$

and so

$$\phi^* \gamma_\Gamma = c_{\mathrm{top}}^{(\mathbb{G}_\mathbf{m})^\Delta}(\widetilde{N}_\Gamma)$$

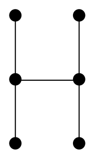
Now it is easy to compute the classes of restricted to each stratum of $\mathfrak{M}_0^{\leq 3}$ (we fix coordinates on C_0 and order components of Δ_1 and Δ_2)

[1] Except the single point

Graph (Γ)	class of stratum $\phi^*\gamma_\Gamma$
	$t_1 + t_2$
	$(t_1 - r_1)(t_2 + r_1)$
	$(t_1 - r_1)(r_1 + r_2)(t_2 - r_2)$
	$t_1 t_2 t_3$

Remark 3.1. We have also shown that these classes are not 0-divisor in $A^*(\mathfrak{M}_0^\Gamma)$ for each Γ above, so we can apply Lemma 2.28.

This fails if we consider more than four nodes. For example if we consider the following graph Γ



we have $\phi^*\gamma_\Gamma = 0$.

Proposition 3.2. *The Chow ring* $A^*(\mathfrak{M}_0^{\leq 3}) \otimes \mathbb{Q}$ *injects into the product of* $A^*(\mathfrak{M}_0^\Gamma) \otimes \mathbb{Q}$ *over trees with at most three edges.*

Proof. From the above remark, we have, for each $\delta \leq 3$, the following exact sequence of additive groups

$$0 \to A^*(\mathfrak{M}_0^\delta) \otimes \mathbb{Q} \xrightarrow{i_*^\delta} A^*(\mathfrak{M}^{\leq \delta}) \otimes \mathbb{Q} \xrightarrow{j^{*\delta}} A^*(\mathfrak{M}^{\leq (\delta-1)}).$$

Let us consider the morphism

$$A^*(\mathfrak{M}_0^{\leq 3}) \otimes \mathbb{Q} \xrightarrow{\psi} \prod_{\delta=0}^{3} \mathfrak{M}_0^\delta$$

as the product of the maps

$$i^{3*}$$
$$i^{2*}j^{3*}$$
$$i^{1*}j^{2*}j^{3*}$$
$$j^{1*}j^{2*}j^{3*}$$

Let a be an element of $A^*(\mathfrak{M}_0^{\leq 3}) \otimes \mathbb{Q}$ different from zero. If $\psi(a)$ is zero then it cannot be in the image of i_*^3 consequently $j^{3*}(a) \in A^*(\mathfrak{M}_0^{\leq 2})$ is different from zero. We can continue till we obtain that $j^{1*}j^{2*}j^{3*}$ is different from zero: absurd. $\square$

3.2. Restriction of classes of strata

Given two trees Γ and Γ' (with maximal multiplicity at most 3 and number of edges respectively equal to δ and δ') such that Γ' is a deformation of Γ, we want to describe the restriction of γ_Γ to $\mathfrak{M}_0^{\Gamma'}$. In order to do that we consider the following diagram (see Proposition 2.24).

$$\begin{array}{ccc}
\Psi(\Gamma,\Gamma') & \xrightarrow{pr_2} & \left(\widetilde{\mathfrak{M}}_0^{\Gamma}\right)^{\leq\delta'-\delta} \\
{\scriptstyle pr_1}\downarrow & \square & \downarrow{\scriptstyle \Pi} \\
\mathfrak{M}_0^{\Gamma'} & \xrightarrow{in} & \mathfrak{M}_0^{\leq\delta'}
\end{array}$$

We have

$$\Psi(\Gamma,\Gamma') = \coprod_{\alpha=1}^{\mu} \Psi(\Gamma,\Gamma')_\alpha$$

where the union is taken over the set of ordered deformations up to $\sim_{\Gamma'}$.

We refer to Section 2.26. In this case the polynomial a of equation (12) is 1, consequently $\widetilde{a}$ is $1/\sigma$. With reference to the étale covering $\widetilde{\mathfrak{M}}_0^{\Gamma'} \xrightarrow{\phi} \mathfrak{M}_0^{\Gamma'}$ and the inclusion $\mathfrak{M}_0^{\Gamma'} \xrightarrow{in} \mathfrak{M}_0^{\delta'}$, from formula 12 we obtain

$$\sum_{\alpha=1}^{\mu} \frac{\sigma'_\alpha}{\sigma} c_{top}(\widetilde{\mathcal{N}}^\alpha)$$

For each α, we have an exact sequence of sheaves

$$0 \to \widetilde{\mathcal{N}}_{pr_2^\alpha} \to \widetilde{\mathcal{N}}_{in} \to \widetilde{\mathcal{N}}^\alpha \to 0.$$

We have seen above how to compute $c_{top}(\widetilde{\mathcal{N}}_{in})$, similarly we can compute $c_{top}(\widetilde{\mathcal{N}}_{pr_2^\alpha})$ and finally we consider the following relation (that follows from the exact sequance)

$$c_{top}(\widetilde{\mathcal{N}}_{in}) = c_{top}(\widetilde{\mathcal{N}}_{pr_2^\alpha}) \cdot c_{top}(\widetilde{\mathcal{N}}^\alpha).$$

Example 3.3. Let Γ and Γ' be trees as defined in the Example 2.19. We consider the component $\Psi(\Gamma,\Gamma')_\alpha$ associated to the deformation 4. We compute

$$\begin{aligned}
c_{top}(\widetilde{\mathcal{N}}_{in}) &= -t_P t_Q r_S (t_T + r_S) \\
c_{top}(\widetilde{\mathcal{N}}_{pr_2^\alpha}) &= -t_P t_Q r_S
\end{aligned}$$

from which

$$c_{top}(\widetilde{\mathcal{N}}^\alpha) = (t_T + r_S).$$

We carry on the calculation for trees with at most three nodes in the last Chapter.

3.3. Mumford classes

On $\mathcal{C}^\Gamma$ we have the restriction of the dualizing sheaf $\omega_0 := \omega_{C/\mathfrak{M}_0}$ we have defined in Section 1.5. We call $\omega_0^\Gamma := \omega_{\mathcal{C}^\Gamma/\mathfrak{M}_0^\Gamma}$ its restriction.

We consider two kinds of classes on $A^*_{\mathrm{Aut}(C_0)}$ induced by the sheaf ω_0^Γ:

(1) the pushforward of polynomial of the Chern class [2] $K := c_1(\omega_0^\Gamma)$;
(2) the Chern classes of the pushforward of ω_0.

The first kind will give us the equivalent of Mumford classes, but on $\mathfrak{M}_0^{\leq 3}$ we can only define classes of the second type (see Section 1.5).

The aim of this section is to compute classes of the first kind for Γ with at most three nodes and to describe them as polynomials in classes of the second kind. If this description is indipentent from the graph Γ then we can define them as elements of $A^*(\mathfrak{M}_0^{\leq 3})$.

First of all we define on $\mathcal{C}^\Gamma$ and $\mathfrak{M}_0^\Gamma$ the following classes

$$\begin{aligned} K &:= c_1(\omega_0^\Gamma) \in A^1(\mathcal{C}^\Gamma), \\ \kappa_i &:= \Pi_*(K^{i+1}) \in A^i(\mathfrak{M}_0^\Gamma). \end{aligned}$$

Such classes κ_i (introduced in [**Mum2**] for the moduli spaces of stable curves) are called *Mumford classes*.

3.3.1. Mumford classes on $\mathfrak{M}_0^0$. As we have seen in Section 1.4 the stack $\mathfrak{M}_0^0$ is $\mathrm{B}\mathbb{P}Gl_2$.

Let us consider the universal curve

$$[\mathbb{P}^1_{\mathbb{C}}/\mathbb{P}Gl_2] \xrightarrow{\Pi} \mathrm{B}\mathbb{P}Gl_2.$$

We call $\overline{\Pi}$ the induced map $\mathbb{P}^1_{\mathbb{C}} \to \mathrm{Spec}\mathbb{C}$ and $\overline{\omega}_0^0$ a lifting of ω_0^0 on $\mathbb{P}^1_{\mathbb{C}}$.

We notice that $(\overline{\omega}_0^0)^\vee = T_{\overline{\Pi}}$ [3], we set

$$K := c_1^{\mathbb{P}Gl_2}(T_{\overline{\Pi}}) = -c_1^{\mathbb{P}Gl_2}(\overline{\omega}_0^0).$$

Furthermore $\overline{\Pi}_*(T_{\overline{\Pi}}) = H^0(\mathbb{P}^1_{\mathbb{C}}, T_{\overline{\Pi}}) = \mathfrak{sl}_2$ seen as adjoint representation of $\mathbb{P}Gl_2$.

---

[2] As usual, given a bundle $E \to X$ on an Artin stack we identify Chern classes $c_i(E)$ with

$$c_i(E) \cap [X] \in A^i(X)$$

[3] T_Π is the relative tangent bundle along Π

By applying the GRR Theorem we obtain

$$\begin{aligned} ch(\Pi_*(\omega_0^\vee)) &= \Pi_*(Td(T_\Pi)ch(\omega_0^\vee)) \\ ch^{\mathbb{P}Gl_2}(\mathfrak{sl}_2) &= \Pi_*(Td^{\mathbb{P}Gl_2}(T_{\overline{\Pi}})ch^{\mathbb{P}Gl_2}(T_{\overline{\Pi}})) \\ 3 - c_2^{\mathbb{P}Gl_2}(\mathfrak{sl}_2) &= \Pi_*\left[\left(e^{-(K)}\right)\left(\frac{-K}{1-e^K}\right)\right] \end{aligned}$$

By applying GRR to the trivial linear bundle we obtain:

$$1 = \Pi_*\left[\frac{-K}{1-e^K}\right]$$

If we subtract the second equation from the first, we obtain

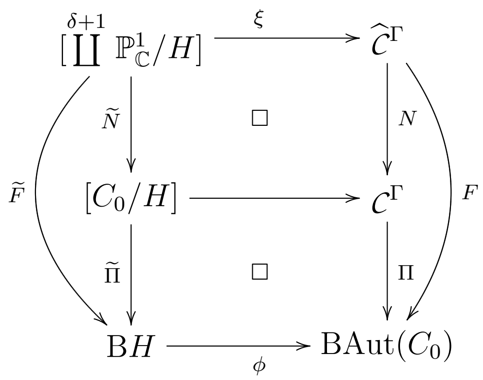
$$2 - c_2^{\mathbb{P}Gl_2}(\mathfrak{sl}_2) = \Pi_*\left[\left(\frac{1-e^K}{e^K}\right)\left(\frac{-K}{1-e^K}\right)\right] = \Pi_*\left[-Ke^{-K}\right],$$

from which we get

$$\begin{aligned} \kappa_0 &= -2 \\ \kappa_2 &= 2c_2^{\mathbb{P}Gl_2}(\mathfrak{sl}_2). \end{aligned}$$

3.3.2. Mumford classes on strata of singular curves. Now let us consider the following cartesian diagram

(14)



where $N : \widehat{\mathcal{C}}^\Gamma \to \mathcal{C}^\Gamma$ is the normalization of $\mathcal{C}^\Gamma$ described in Section 1.5.

In the following we call $\widetilde{\omega}_0^\Gamma$ the sheaf

$$\xi^* N^*(\omega_0^\Gamma).$$

and $\widetilde{K} := c_1(\omega_0)$.

Proposition 3.4. *Using the above notation, the Mumford classes $\kappa_i \in A^i(BF \ltimes H) \otimes \mathbb{Q}$ are described by the following relation*

$$\phi_* \widetilde{F}_*(\widetilde{K}^{i+1}) = \sigma \kappa_i,$$

Proof. Let us fix an index $i \in \mathbb{N}$. Since the map N is finite and generically of degree 1 we have

$$N_*N^* = \mathrm{id}$$

therefore

$$\kappa_i := \Pi_* K^{i+1} = \Pi_*(N_*N^*)K^{i+1} = F_*(N^*K^{i+1}),$$

since F is projective and ϕ is étale we can apply the projection formula and obtain

$$\phi^* F_*(N^*K^{i+1}) = \widetilde{F}_*\xi^*(N^*K^{i+1}).$$

Furthermore the map $N \circ \xi$ is finite and so

$$\xi^* N^* c_1(\omega_0^\Gamma)^{i+1} = c_1(\xi^* N^*(\omega_0^\Gamma))^{i+1} = \widetilde{K}^{i+1}.$$

We conclude by noting that $\phi_*\phi^*$ is multiplication by σ. $\square$

From the previous Proposition, we have reduced the problem to computing $\widetilde{K}$ and then writing pushforward along $\widetilde{F}$. With respect to each component of the stack

$$\widehat{\mathcal{C}}^\Gamma = \coprod^{\delta+1} [\mathbb{P}^1_{\mathbb{C}}/\mathrm{Aut}(C_0)]$$

we fix coordinates on $\mathbb{P}^1_{\mathbb{C}}$ which are compatible with coordinates chosen on C_0. By forgetting the action of $\mathrm{Aut}(\Gamma)$ we keep the same system of coordinates on $\coprod^{\delta+1} \mathbb{P}^1_{\mathbb{C}}$ when we consider $\coprod^{\delta+1} [\mathbb{P}^1_{\mathbb{C}}/H]$.

Giving a bundle on a quotient stack $[X/G]$ is equivalent to giving a bundle $U \to X$ equivariant for the action of G (see Section 2.1). By abuse of notation we still call $\widetilde{\omega}_0^\Gamma$ any lifting of $\widetilde{\omega}_0^\Gamma$ on $\coprod^{\delta+1} \mathbb{P}^1_{\mathbb{C}}$.

In order to make computations we need to render explicit the action of $H := E^{\Delta_1} \times \mathbb{G}_{\mathbf{m}}^{\Delta_2}$ on

$$\widetilde{F}_*(\widetilde{\omega}_0) = H^0\left(\coprod^{\delta+1} \mathbb{P}^1_{\mathbb{C}}, \widetilde{\omega}_0^\Gamma\right).$$

Set $\Delta := \Delta_1 \cup \Delta_2$, from the inclusion $\mathbb{G}_{\mathbf{m}} \to E$ we have a cartesian diagram of stacks

$$\begin{array}{ccc}
\mathrm{B}(\mathbb{G}_{\mathbf{m}})^\Delta & \xrightarrow{\ \phi\ } & \mathrm{B}(\mathrm{Aut}(\Gamma) \ltimes (\mathbb{G}_{\mathbf{m}})^\Delta) \\
\downarrow{\scriptstyle \Psi} & \square & \downarrow{\scriptstyle \Psi} \\
\mathrm{B}H & \xrightarrow{\ \phi\ } & \mathrm{BAut}(C_0)
\end{array}$$

Roughly speaking we can say that we obtain the stacks in the top row by fixing the point 0 on components with a node. The functor Ψ forgets these points. From what we have seen in Section 2.2 that we have the following ring isomorphisms

$$\begin{array}{ccc} A^*_H & \xrightarrow{\Psi^*} & A^*_{(\mathbb{G}_{\mathbf{m}})^\Delta} \\ A^*_{\mathrm{Aut}(C_0)} & \xrightarrow{\Psi^*} & A^*_{\mathrm{Aut}(\Gamma)\ltimes(\mathbb{G}_{\mathbf{m}})^\Delta}. \end{array}$$

We have defined the classes $t_{\Delta_1}, r_{\Delta_2}$ in A^*_H. By using the same notation of Section 2.2, the map Ψ^* is the identity on r_{Δ_2}. For each vertex P in Γ such that $e(P) = 1$ the map Ψ^* sends t_P to $c_1^{\mathbb{G}_{\mathbf{m}}}(H^0(\mathbb{P}^1_{\mathbb{C}}, \mathcal{O}(z_\infty)))$ of the same component, which we still call t_P. Consequently, with reference to the following diagram

$$\begin{array}{ccc} [\coprod^{\Delta} \mathbb{P}^1_{\mathbb{C}}/(\mathbb{G}_{\mathbf{m}})^\Delta] & & \\ \Big\downarrow{\scriptstyle \widetilde{F}} & & \\ \mathrm{B}(\mathbb{G}_{\mathbf{m}})^\Delta & \xrightarrow[\phi]{\qquad} & \mathrm{B}(\mathrm{Aut}(\Gamma) \ltimes (\mathbb{G}_{\mathbf{m}})^\Delta) \end{array}$$

we have reduced the problem to consider the action of $(\mathbb{G}_{\mathbf{m}})^\Delta$ on

$$\widetilde{F}_*(\widetilde{\omega}_0^\Gamma) = H^0\left(\coprod^{\Delta} \mathbb{P}^1_{\mathbb{C}}, \widetilde{\omega}_0^\Gamma\right)$$

where, again with abuse of notation, we call $\widetilde{\omega}_0^\Gamma$ the sheaf $\Psi^*(\widetilde{\omega}_0^\Gamma)$.

The map $\widetilde{F}$ is the union of maps

$$\widetilde{F}_P := [\mathbb{P}^1_{\mathbb{C}}/(\mathbb{G}_{\mathbf{m}})^\Delta] \to \mathrm{B}(\mathbb{G}_{\mathbf{m}})^\Delta$$

where only the component of $(\mathbb{G}_{\mathbf{m}})^\Delta$ corresponding to $P \in \Delta$ does not acts trivially on $\mathbb{P}^1_{\mathbb{C}}$ and the action is

$$\begin{array}{rcl} \gamma : \mathbb{G}_{\mathbf{m}} \times \mathbb{P}^1_{\mathbb{C}} & \to & \mathbb{P}^1_{\mathbb{C}} \\ (\lambda, [X_0, X_1]) & \mapsto & [X_0, \lambda X_1]. \end{array}$$

Set

$$P_0 := [0, 1], \quad P_1 := [1, 1], \quad P_\infty := [1, 0].$$

We have that $\mathcal{F} := (\widetilde{\omega}_0)^\vee$ is the sheaf on $\coprod^{\Delta} \mathbb{P}^1_{\mathbb{C}}$ such that restricted:

- to the components with one node it is $\mathcal{F}_P := (\omega \otimes \mathcal{O}(z_\infty))^\vee$
- to the components with two nodes it is $\mathcal{F}_P := (\omega \otimes \mathcal{O}(z_\infty + z_0))^\vee$
- to the components with three nodes it is $\mathcal{F}_P := (\omega \otimes \mathcal{O}(z_\infty + z_0 + z_1))^\vee$

where ω is the canonical bundle on $\mathbb{P}^1_{\mathbb{C}}$.

In the following, Chern classes will be equivariant for the action of $\mathbb{G}_{\mathbf{m}}$. For each $P \in \Delta$ let us indicate with K_P the class $c_1^{\mathbb{G}_{\mathbf{m}}}(\omega) \in A^1_{\mathbb{G}_{\mathbf{m}}}(\mathbb{P}^1_{\mathbb{C}})$ on each component, with R_P the class $c_1^{\mathbb{G}_{\mathbf{m}}}(\mathcal{O}(z_\infty)) \in A^1_{\mathbb{G}_{\mathbf{m}}}(\mathbb{P}^1_{\mathbb{C}})$, and with Q_P the class $c_1^{\mathbb{G}_{\mathbf{m}}}(\mathcal{O}(P_0)) \in A^1_{\mathbb{G}_{\mathbf{m}}}(\mathbb{P}^1_{\mathbb{C}})$.

We have

$$\begin{aligned} c_1^{\mathbb{G}_{\mathbf{m}}}(\mathcal{F}_{\Delta_1}) &= -K_{\Delta_1} - R_{\Delta_1} \\ c_1^{\mathbb{G}_{\mathbf{m}}}(\mathcal{F}_{\Delta_2}) &= -K_{\Delta_2} - R_{\Delta_2} - Q_{\Delta_2}. \end{aligned}$$

In order to determine Mumford classes it is then necessary to compute the pushforward classes

$$\begin{aligned} &\widetilde{F}_{\Delta_1 *}(-K_{\Delta_1} - R_{\Delta_1})^h \text{ and} \\ &\widetilde{F}_{\Delta_2 *}(-K_{\Delta_2} - R_{\Delta_2} - Q_{\Delta_2})^h \end{aligned}$$

for every natural h.

Let us start with computing the push-forward along $\widetilde{F}_\Delta$ of every power of $K_\Delta = c_1^{\mathbb{G}_{\mathbf{m}}}(\omega)$.

We have

$$\widetilde{F}_{\Delta_1 *}(\omega^\vee) = H^0(\mathbb{P}^1_{\mathbb{C}}, \omega^\vee).$$

Now it is necessary to determine the action of $(\mathbb{G}_{\mathbf{m}})$ on global sections of $\omega^\vee$. Let $z := X_1/X_0$ be the local coordinate around z_∞, the global sections of $\omega^\vee$ are generated as a vectorial space by $\frac{\partial}{\partial z}, z\frac{\partial}{\partial z}, z^2\frac{\partial}{\partial z}$. Relatively to this basis, the action of $\mathbb{G}_{\mathbf{m}}$ is given by

$$\begin{aligned} \gamma : \mathbb{G}_{\mathbf{m}} \times H^0(\mathbb{P}^1_{\mathbb{C}}, \omega^\vee) &\rightarrow H^0(\mathbb{P}^1_{\mathbb{C}}, \omega^\vee) \\ (\lambda, a\frac{\partial}{\partial z} + bz\frac{\partial}{\partial z} + cz^2\frac{\partial}{\partial z}) &\mapsto (a\lambda\frac{\partial}{\partial z} + bz\frac{\partial}{\partial z} + c\frac{1}{\lambda}z^2\frac{\partial}{\partial z}) \end{aligned}$$

the multiplicity of the action is therefore $(1, 0, -1)$, so the Chern character of $\widetilde{F}_{\Delta *}(\omega^\vee)$ is $e^{t_\Delta} + 1 + e^{-t_\Delta}$.

Let us observe that $\omega^\vee$ is the tangent bundle relative to $\widetilde{F}_\Delta$, so by applying the GRR Theorem we get

$$1 + e^{t_\Delta} + e^{-t_\Delta} = \widetilde{F}_{\Delta *}\left[\left(e^{-(K_\Delta)}\right)\left(\frac{-K_\Delta}{1 - e^{K_\Delta}}\right)\right]$$

By applying GRR to the trivial linear bundle we obtain:

$$1 = \widetilde{F}_{\Delta *}\left[\frac{-K_\Delta}{1 - e^{K_\Delta}}\right]$$

If we subtract the second equation from the first, we obtain

$$e^{t_\Delta} + e^{-t_\Delta} = \widetilde{F}_{\Delta *}\left[\left(\frac{1 - e^{K_\Delta}}{e^{K_\Delta}}\right)\left(\frac{-K_\Delta}{1 - e^{K_\Delta}}\right)\right] = \widetilde{F}_{\Delta *}\left[-K_\Delta e^{-K_\Delta}\right],$$

from this, by distinguishing between even and odd cases, it follows that

$$\begin{aligned}\widetilde{F}_{\Delta *}K_\Delta^{2h} &= 0\\ \widetilde{F}_{\Delta *}K_\Delta^{2h+1} &= -2t_\Delta^{2h}\end{aligned}$$

Let us notice that there exist two equivariant sections of $\widetilde{F}_\Delta$ given by the fixed points z_0 and z_∞, that we will call respectively s_0 and s_∞

$$\mathrm{B}(\mathbb{G}_{\mathbf{m}})^\Delta \underset{s_\infty}{\overset{s_0}{\rightrightarrows}} [\mathbb{P}^1_{\mathbb{C}}/(\mathbb{G}_{\mathbf{m}})^\Delta]$$

From the self intersection formula we have

$$\begin{aligned}&s_0^*(Q_\Delta) = -t_\Delta,\ s_0^*(R_\Delta) = 0,\ s_0^*(K_\Delta) = t_\Delta\\ &s_\infty^*(Q_\Delta) = 0,\ s_\infty^*(R_\Delta) = t_\Delta,\ s_\infty^*(K_\Delta) = -t_\Delta;\end{aligned}$$

then by applying the projection formula (see [**Ful**] p.34) for every cycle $D \in A^*_{(\mathbb{G}_{\mathbf{m}})^\Delta}(\mathbb{P}^1)$ we have

$$\begin{aligned}&Q_\Delta \cdot D = s_{0*}(1)\cdot D = s_{0*}(s_0^* D)\\ &R_\Delta \cdot D = s_{\infty *}(1)\cdot D = s_{\infty *}(s_\infty^* D).\end{aligned}$$

With reference to the previous relations, the Mumford classes are determined on every component (we separate between even and odd cases)

$$\begin{aligned}\widetilde{F}_{\Delta *}(-K_\Delta - R_\Delta)^{2h} &= \widetilde{F}_*^\Delta (K_\Delta)^{2h} + \sum_{a=1}^{2h}\binom{2h}{a}\widetilde{F}_{\Delta *}(R_\Delta^a \cdot K_\Delta^{2h-a})\\ &= \sum_{a=1}^{2h}\binom{2h}{a}(\widetilde{F}_{\Delta *}s_{\infty *})s_\infty^*(R_\Delta^{a-1}\cdot K_\Delta^{2h-a})\\ &= \sum_{a=1}^{2h}\binom{2h}{a}(-1)^{a-1}t_\Delta^{2h-1} = -t_\Delta^{2h-1}\end{aligned}$$

and in a completely analogous way, we have the following relations

$$\begin{aligned}&\widetilde{F}_{\Delta_1 *}(-K_{\Delta_1} - R_{\Delta_1})^{2h+1} = t_i^{2h}\\ &\widetilde{F}_{\Delta_2 *}(-K_{\Delta_2} - R_{\Delta_2} - Q_{\Delta_2})^h = 0.\end{aligned}$$

The last relation allows us to ignore components with two nodes. By following the notation of Proposition 3.4 we notice that

$$\widetilde{F}_*\widetilde{K}^{m+1} = \sum_{P\in\Delta_1}\widetilde{F}_{P*}(-1)^{m+1}(-K_P - R_P)^{m+1}$$

and consequently, if we order elements of Δ_1 from 1 to δ_1, we have

$$\kappa_m = -\phi_*\frac{t_1^m + \cdots + t_{\delta_1}^m}{\sigma}. \tag{15}$$

3.3.3. Definition of Mumford classes on $\mathfrak{M}_0^{\leq 3}$. Now we consider the universal curve

$$\mathcal{C}^{\leq 3} \xrightarrow{\Pi} \mathfrak{M}_0^{\leq 3}.$$

We have seen in Section 1.5 that the pushforward $\Pi_* \left(\omega_0^{\leq 3}\right)^\vee$ is a well defined rank three vector bundle. Consequently from [**Kre**] Section 3.6 we have that $c_i(\Pi_* \left(\omega_0^{\leq 3}\right)^\vee) = 0$ for $i > 3$. We fix the following notation

$$\begin{aligned}
\mathcal{c}_1 &:= c_1(\Pi_* \left(\omega_0^{\leq 3}\right)^\vee) \\
\mathcal{c}_2 &:= c_2(\Pi_* \left(\omega_0^{\leq 3}\right)^\vee) \\
\mathcal{c}_3 &:= c_3(\Pi_* \left(\omega_0^{\leq 3}\right)^\vee).
\end{aligned}$$

We still call $\mathcal{c}_1, \mathcal{c}_2, \mathcal{c}_3$ their restriction to each stratum of $\mathfrak{M}_0^{\leq 3}$.

Definition 3.5. We define in $A^*(\mathfrak{M}_0^{\leq})$ Mumford classes $\mathcal{k}_1, \mathcal{k}_2, \mathcal{k}_3$ as follows

$$\begin{aligned}
\mathcal{k}_1 &:= -\mathcal{c}_1 \\
\mathcal{k}_2 &:= 2\mathcal{c}_2 - \mathcal{c}_1^2 \\
\mathcal{k}_3 &:= -\mathcal{c}_1^3 + 3\mathcal{c}_1\mathcal{c}_2 - 3\mathcal{c}_3
\end{aligned}$$

that is to say we define Mumford classes as the opposites of Newton polynomials (see Definition 2.5) in $\mathcal{c}_1, \mathcal{c}_2, \mathcal{c}_3$.

From Proposition 3.2, the Chow ring $A^*(\mathfrak{M}_0^{\leq 3}) \otimes \mathbb{Q}$ injects into the product of $A^*(\mathfrak{M}_0^{\Gamma}) \otimes \mathbb{Q}$ over trees with at most three edges. Consequently in order to verify that the above is a good definition we only need to prove that the restrictions of Mumford classes to each stratum are the opposites of Newton polynomials.

Proposition 3.6. *Let Γ be a tree with at most three edges. Then we have*

$$\begin{aligned}
\mathcal{k}_1 &= -\mathcal{n}_1(\Pi_* \left(\omega_0^{\Gamma}\right)^\vee) \\
\mathcal{k}_2 &= -\mathcal{n}_2(\Pi_* \left(\omega_0^{\Gamma}\right)^\vee) \\
\mathcal{k}_3 &= -\mathcal{n}_2(\Pi_* \left(\omega_0^{\Gamma}\right)^\vee)
\end{aligned}$$

Proof. We consider first of all the case $\mathfrak{M}_0^{\Gamma} = \mathfrak{M}_0^{0}$. Here we have

$$\begin{aligned}
\mathcal{n}_1\left(\Pi_* \left(\omega_0^{\Gamma}\right)^\vee\right) &= \mathcal{n}_1^{\mathbb{P}Gl_2}(\mathfrak{sl}_2) = 0 \\
\mathcal{n}_2\left(\Pi_* \left(\omega_0^{\Gamma}\right)^\vee\right) &= \mathcal{n}_2^{\mathbb{P}Gl_2}(\mathfrak{sl}_2) = -\mathcal{c}_2^{\mathbb{P}Gl_2}(\mathfrak{sl}_2) \\
\mathcal{n}_3\left(\Pi_* \left(\omega_0^{\Gamma}\right)^\vee\right) &= \mathcal{n}_3^{\mathbb{P}Gl_2}(\mathfrak{sl}_2) = 0
\end{aligned}$$

and we can conclude because on $\mathfrak{M}_0^0$ we have seen that $\mathscr{k}_1 = \mathscr{k}_3 = 0$ and $\mathscr{k}_2 = c_2^{\mathbb{P}Gl_2}(\mathfrak{sl}_2)$.

Now let us consider any other tree Γ with at most three edges. With reference to diagram 14, as Chern classes commutes with base changing, we have

$$\phi^* c_i(\Pi_* \left(\omega_0^\Gamma\right)^\vee) = c_i(\widetilde{\Pi}_* \left(\widetilde{\omega}_0^\Gamma\right)^\vee).$$

Given Γ we order elements of Δ_1 from 1 to δ_1. By putting together the above relation and the equation (15), we reduce to show

$$ch(\widetilde{\Pi}_* \left(\widetilde{\omega}_0^\Gamma\right)^\vee) = 3 + \sum_{m=1}^{\infty} \frac{t_1^m + \cdots + t_{\delta_1}^m}{m!}.$$

We recall that the universal curve $\widetilde{\mathcal{C}}^\Gamma$ on $\widetilde{\mathfrak{M}}_0^\Gamma = \mathrm{B}(E^{\Delta_1} \times \mathbb{G}_{\mathbf{m}}^{\Delta_2})$ is the quotient stack

$$[C_0 / E^{\Delta_1} \times \mathbb{G}_{\mathbf{m}}^{\Delta_2}].$$

Since we have a morphism

$$\Psi : \mathrm{B}(\mathbb{G}_{\mathbf{m}})^\Delta \to \widetilde{\mathfrak{M}}_0^\Gamma$$

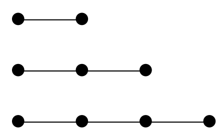
such that $\Psi^* : A^*(\widetilde{\mathfrak{M}}_0^\Gamma) \to A^*_{(\mathbb{G}_{\mathbf{m}})^\Delta}$ is an isomorphism, with reference to the cartesian diagram

$$\begin{array}{ccc} [C_0/\mathbb{G}_{\mathbf{m}}^\Delta] & \xrightarrow{\Psi} & [C_0/E^{\Delta_1} \times \mathbb{G}_{\mathbf{m}}^{\Delta_2}] \\ \widetilde{\Pi} \downarrow & \square & \downarrow \widehat{\Pi} \\ \mathrm{B}\mathbb{G}_{\mathbf{m}}^\Delta & \xrightarrow{\Psi} & \mathrm{B}E^{\Delta_1} \times \mathbb{G}_{\mathbf{m}}^{\Delta_2} \end{array}$$

we reduce to consider the pullback sheaf $\Psi^* \left(\widetilde{\omega}_0^\Gamma\right)^\vee$ which we still call $\left(\widetilde{\omega}_0^\Gamma\right)^\vee$ as its lifting to C_0. As $H^1(C_0, \left(\widetilde{\omega}_0^\Gamma\right)^\vee) = 0$ (see proof of Preposition 1.28), we have

$$ch(\widetilde{\Pi}_* \left(\widetilde{\omega}_0^\Gamma\right)^\vee) = ch^{\mathbb{G}_{\mathbf{m}}^\Delta}(H^0(C_0, \left(\widetilde{\omega}_0^\Gamma\right)^\vee))$$

On curves of topological type



we have that global sections of $\left(\widetilde{\omega}_0^\Gamma\right)^\vee$ are sections of $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(1)$ on extremal components and $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}$ on the other components, which agree on nodes. Since $H^0(\mathbb{P}^1_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(0)) = \mathbb{C}$, global sections of $\left(\widetilde{\omega}_0^\Gamma\right)^\vee$ are sections of $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(1)$ on the two extremal components which are equal on nodes.

On each extremal component we fix coordinates $[X_0', X_1']$ and $[X_0'', X_1'']$. Sections on $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(1)$ are linear forms

$$\begin{aligned} a_1 X_0' + b_1 X_1' \\ a_2 X_0'' + b_2 X_1'' \end{aligned}$$

which agree at $z_\infty = [1, 0]$. This happens if and only if $a_1 = a_2$. We have only $(\mathbb{G}_{\mathbf{m}})^{\Delta_1}$ which does not acts trivially on $H^0(C_0, \left(\widetilde{\omega}_0^\Gamma\right)^\vee)$ and the action is

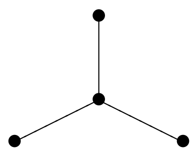
$$\begin{aligned} \gamma : (\mathbb{G}_{\mathbf{m}} \times \mathbb{G}_{\mathbf{m}}) \times H^0(C_0, \left(\widetilde{\omega}_0^\Gamma\right)^\vee) &\rightarrow H^0(C_0, \left(\widetilde{\omega}_0^\Gamma\right)^\vee) \\ (\lambda_1, \lambda_2), (a, b_1, b_2) &\mapsto (a, \lambda_1 b_1, \lambda_2 b_2) \end{aligned}$$

so the Chern character of $\widetilde{\Pi}_* \omega_0^\vee$ is

$$1 + e^{t_1} + e^{t_2} = 3 + \sum_{m=1}^{\infty} \frac{t_1^m + t_2^m}{m!}$$

as we claimed.

The last case to consider is Γ equals to



we have that global sections of $\left(\widetilde{\omega}_0^\Gamma\right)^\vee$ are sections of $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(1)$ on extremal components and $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-1)$ on the central component which agree on nodes. Since $H^0(\mathbb{P}^1_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-1)) = 0$, global sections of $\left(\widetilde{\omega}_0^\Gamma\right)^\vee$ are sections of $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(1)$ on the three extremal components which are zero on nodes. On each extremal component we fix coordinates $[X_0', X_1']$, $[X_0'', X_1'']$ and $[X_0''', X_1''']$. Sections on $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(1)$ are linear forms

$$\begin{aligned} a_1 X_0' + b_1 X_1' \\ a_2 X_0'' + b_2 X_1'' \\ a_3 X_0''' + b_3 X_1''' \end{aligned}$$

which are zero on nodes if and only if $a_1 = a_2 = a_3 = 0$. We have that only $(\mathbb{G}_{\mathbf{m}})^{\Delta_1}$ does not acts trivially on $H^0(C_0, \left(\widetilde{\omega}_0^\Gamma\right)^\vee)$ and the action is

$$\begin{aligned} \gamma : (\mathbb{G}_{\mathbf{m}} \times \mathbb{G}_{\mathbf{m}} \times \mathbb{G}_{\mathbf{m}}) \times H^0(C_0, \left(\widetilde{\omega}_0^\Gamma\right)^\vee) &\rightarrow H^0(C_0, \left(\widetilde{\omega}_0^\Gamma\right)^\vee) \\ (\lambda_1, \lambda_2, \lambda_3), (b_1, b_2, b_3) &\mapsto (\lambda_1 b_1, \lambda_2 b_2, \lambda_3 b_3) \end{aligned}$$

so the Chern character of $\widetilde{\Pi}_* \omega_0^\vee$ is

$$e^{t_1} + e^{t_2} + e^{t_3} = 3 + \sum_{m=1}^{\infty} \frac{t_1^m + t_2^m + t_3^m}{m!}$$

as we claimed. □

CHAPTER 4

The Chow ring of $\mathfrak{M}_0^{\leq 3}$

In this Chapter we calculate $A^*(\mathfrak{M}_0^{\leq 3}) \otimes \mathbb{Q}$.

4.1. The open substack $\mathfrak{M}_0^0$

$\boxed{\Gamma := \bullet}$ From Theorems (1.24) and (B.3) we have that the stack $\mathfrak{M}_0^0$ is the classifying space of $\mathbb{P}Gl_2$, owing to the fact that $\mathbb{P}Gl_2 \cong SO_3$, following [**Pan**] we have

$$A^*(\mathfrak{M}_0^0) \otimes \mathbb{Q} \cong \mathbb{Q}[c_2(\mathfrak{sl}_2)].$$

Since (see Section 3.3) $c_2(\mathfrak{sl}_2) = -(1/2)\kappa_2$ we can write

$$A^*(\mathfrak{M}_0^0) \otimes \mathbb{Q} = \mathbb{Q}[\kappa_2].$$

4.2. The first stratum

$\boxed{\Gamma := \bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet}$ We order the two components.

The automorphism group is $\mathfrak{C}_2 \ltimes (E \times E)$, (where $\mathfrak{C}_2$ is the order two multiplicative group) and the action of its generator τ over $E \times E$ exchanges the components.

Then on the ring $A^*_{E\times E} \otimes \mathbb{Q} \cong \mathbb{Q}[t_1, t_2]$(see Proposition 2.9), it exchanges the first Chern classes t_1 and t_2. The invariant polynomials are the symmetric ones which are geometrically generated by: $\{(t_1 + t_2)/2, (t_1^2+t_2^2)/2\}$. By recalling the description (15) of Mumford classes, we have

$$A^*(\mathfrak{M}_0^1) \otimes \mathbb{Q} = \mathbb{Q}[\kappa_1, \kappa_2].$$

Let us consider the two inclusions i e j (respectively closed and open immersions) and the étale covering ϕ

(16)
$$\begin{array}{ccc} \widetilde{\mathfrak{M}}_0^1 \xrightarrow{\ \phi\ } \mathfrak{M}_0^1 & & \\ & \searrow^{i} & \\ & \mathfrak{M}_0^0 \xrightarrow[j]{} & \mathfrak{M}_0^{\leq 1} \end{array}$$

we obtain the following exact sequence

$$A^*(\mathfrak{M}_0^1) \otimes \mathbb{Q} \xrightarrow{i_*} A^*(\mathfrak{M}_0^{\leq 1}) \otimes \mathbb{Q} \xrightarrow{j^*} A^*(\mathfrak{M}_0^0) \otimes \mathbb{Q} \to 0$$

for what we have seen we have:

$$\mathbb{Q}[\kappa_1,\kappa_2] \xrightarrow{i_*} A^*(\mathfrak{M}_0^{\leq 1})\otimes\mathbb{Q} \xrightarrow{j^*} \mathbb{Q}[\kappa_2] \to 0.$$

Now with reference to the paragraph (1.5) we have that the first Chern class of the normal bundle $N_{\mathfrak{M}_0^1}\mathfrak{M}_0^{\leq 1}$ is

$$i^*i_*[\mathfrak{M}_0^1] = \phi_*\frac{1}{2}(t_1+t_2) = -\kappa_1$$

Since $A^*\mathfrak{M}_0^1$ is an integral domain we can apply Lemma (2.28) and obtain the ring isomomorphism

$$A^*(\mathfrak{M}_0^{\leq 1})\otimes\mathbb{Q} \cong \mathbb{Q}[\kappa_1,\kappa_2]\times_{\mathbb{Q}[\kappa_2]}\mathbb{Q}[\kappa_2] \cong \mathbb{Q}[\kappa_1,\kappa_2].$$

where the map $q:\mathbb{Q}[\kappa_2]\to\mathbb{Q}[\kappa_1,\kappa_2]/(\kappa_1) = \mathbb{Q}[\kappa_2]$ tautologically sends κ_2 into κ_2.

4.3. The second stratum

$\boxed{\Gamma := \bullet\!\!-\!\!-\!\!\bullet\!\!-\!\!-\!\!\bullet}$ We order the two components with one node.

In this case the group of automorphism of the fiber is

$$\mathrm{Aut}(C^{\Gamma_2}) \cong \mathfrak{C}_2 \ltimes (\mathbb{G}_{\mathbf{m}}\times E\times E) =: \mathfrak{C}_2\ltimes H,$$

where the action of τ sends an element $g\in\mathbb{G}_{\mathbf{m}}$ into g^{-1} and exchange the components isomorphic to E. Following the notation of Section 2.2 we identify $A^*(\mathrm{B}(\mathbb{G}_{\mathbf{m}})^3)$ with $A^*(\widetilde{\mathfrak{M}_0^{\Gamma_2}})$ and $A^*(\mathrm{BAut}(\Gamma)\ltimes(\mathbb{G}_{\mathbf{m}})^3)$ with $A^*(\mathfrak{M}_0^{\Gamma})$. Set

$$t_1 = \psi^1(\infty,1)$$
$$t_2 = \psi^1(\infty,2)$$
$$r = \psi^2(\infty)$$

the action induced by τ on these classes is

$$\tau(r,t_1,t_2) = (-r,t_2,t_1).$$

With reference to the map

$$\mathrm{B}(\mathbb{G}_{\mathbf{m}})^3 \xrightarrow{\phi} \mathrm{BAut}(\Gamma)\ltimes(\mathbb{G}_{\mathbf{m}})^3$$

we recall that ϕ^* is an isomorphism between $A^*(\mathfrak{M}_0^1)\otimes\mathbb{Q}$ and $A^*(\mathrm{B}(\mathbb{G}_{\mathbf{m}})^3)^{\mathfrak{C}_2}$ (see Proposition 2.8).

We can describe $A^*(\mathrm{B}H)\otimes\mathbb{Q} = \mathbb{Q}[r,t_1,t_2]$ as the polynomial ring in r with coefficients in $\mathbb{Q}[t_1,t_2]$, so we write a polynomial $P(r,t_1,t_2)$ as

$$\sum_{i=0}^{k} r^i P_i(t_1,t_2).$$

The polynomial P is invariant for the action of τ iff the coefficients of the powers of r in $P(r, t_1, t_2)$ are equal to those of the polynomial $P(-r, t_2, t_1)$.

That is to say that P_i with even index are invariant for the exchange of t_1 and t_2, while those with odd index are anti-invariant. An anti-invariant polynomial Q is such that

$$Q(t_1, t_2) + Q(t_2, t_1) = 0$$

and consequently it is the product of $(t_1 - t_2)$ by an invariant polynomial. It is furthermore straightforward verifying that any such polynomial is invariant for the action of τ.

So an algebraic system of generators for $(A^*_{(\mathbb{G}_\mathbf{m})^3} \otimes \mathbb{Q})^{\mathfrak{C}_2}$ is given by

$$\begin{aligned} u_1 &:= t_1 + t_2 \\ u_2 &:= t_1^2 + t_2^2, \\ u_3 &:= r(t_1 - t_2), \\ u_4 &:= r^2. \end{aligned}$$

We know that (see Sections 3.3 and 3.1)

$$\begin{aligned} \phi^* k_1 &= -u_1 \\ \phi^* k_2 &= -u_2 \\ \phi^*(\gamma_2) &= (t_1 - r)(t_2 + r) = \frac{1}{2}(u_1^2 - u_2) + u_3 - u_4 \end{aligned}$$

where $\gamma_2 = c_2(\mathcal{N}_{\mathfrak{M}_0^2/\mathfrak{M}_0^{\leq 2}})$, and there exists a class $x \in A^2\mathfrak{M}_0^2 \otimes \mathbb{Q}$ such that

$$u_3 = \pi^* x$$

Claim 4.1. *The ideal of relations is generated on* $\mathbb{Q}[k_1, k_2, \gamma_2, x]$ *by the polynomial*

$$(2x + (2k_2 + k_1^2))^2 - (2k_2 + k_1^2)(4\gamma_2 - k_1^2) = 0 \tag{17}$$

Proof. From direct computation we have that relation (17) holds and the polynomial is irreducible. On the other hand let us consider the map

$$\begin{aligned} f : \mathbb{A}^3_\mathbb{Q} &\rightarrow \mathbb{A}^4_\mathbb{Q} \\ (t_1, t_2, r) &\mapsto (u_1, u_2, u_3, u_4) \end{aligned}$$

if the generic fiber of f is finite then $f(\mathbb{A}^3_\mathbb{Q})$ is an hypersurface in $\mathbb{A}^4_\mathbb{Q}$ and we have done. Now for semicontinuity it is sufficient to show that a fiber is finite. Let us consider the fiber on 0. We have that u_1, u_2, u_3, u_4 are simultaneously zero iff $t_1 = t_2 = r = 0$. □

Set

$$\eta := 2x + (2\kappa_2 + \kappa_1^2),$$

we have that $A^*(\mathfrak{M}_0^2) \otimes \mathbb{Q}$ is isomorphic to the graded ring

$$\mathbb{Q}[\kappa_1, \kappa_2, \gamma_2, \eta]/I.$$

where the ideal I is generated by the polynomial

$$\eta^2 - (2\kappa_2 + \kappa_1^2)(4\gamma_2 - \kappa_1^2).$$

Sinse $\phi^*\phi_*$ is multiplication by two, we also have the following relation

$$\eta = \phi_* \left(\frac{1}{2}(t_1 - t_2)(2r - t_1 + t_2) \right).$$

Let us consider the fiber diagram

$$\begin{array}{ccc} A^*(\mathfrak{M}_0^{\leq 2}) \otimes \mathbb{Q} & \xrightarrow{j^*} & \mathbb{Q}[\kappa_1, \kappa_2] \\ \downarrow i^* & \square & \downarrow q \\ \mathbb{Q}[\kappa_1, \kappa_2, \gamma_2, \eta]/I & \xrightarrow{p} & \mathbb{Q}[\kappa_1, \kappa_2, \eta]/\overline{I} \end{array}$$

where $\overline{I}$ is the ideal generated by $\eta^2 + (2\kappa_2 + \kappa_1^2)\kappa_1^2$.

The map q is injective so i^* is injective too.

We set in $A^*(\mathfrak{M}_0^{\leq 2}) \otimes \mathbb{Q}$

$$\gamma_2 := i_* 1$$
$$q := i_* \eta,$$

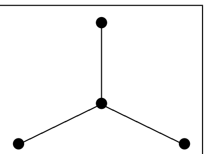
the ring we want (from injectivity of i^*) is isomorphic to the subring of $A^*(\mathfrak{M}_0^2) \otimes \mathbb{Q}$ generated by $\kappa_1, \kappa_2, \gamma_2, \gamma_2\eta$ so we have

$$A^*(\mathfrak{M}_0^{\leq 2}) \otimes \mathbb{Q} = \frac{\mathbb{Q}[\kappa_1, \kappa_2, \gamma_2, q]}{(q^2 + \gamma_2^2(2\kappa_2 + \kappa_1^2)(\kappa_1^2 - 4\gamma_2))}$$

4.4. The third stratum

The third stratum has two components.

4.4.1. The first component.

$\Gamma_3' :=$ 

We order components in Δ_1. Let us note that the component corresponding to the central vertex has three points fixed by the other three components, consequently, given a permutation of the external vertices, there is an unique automorphism related to the central vertex.

The group $\mathrm{Aut}(C^{\Gamma_3'})$ is therefore isomorphic to $S_3 \ltimes (E^3)$.

As usual from proposition (2.9) it follows that

$$A^*_{E^3} \otimes \mathbb{Q} \cong \mathbb{Q}[w_1, w_2, w_3],$$

on which S_3 acts by permuting the three classes

$$w_1 := \psi^1_{\infty,1}$$
$$w_2 := \psi^1_{\infty,2}$$
$$w_3 := \psi^1_{\infty,3}$$

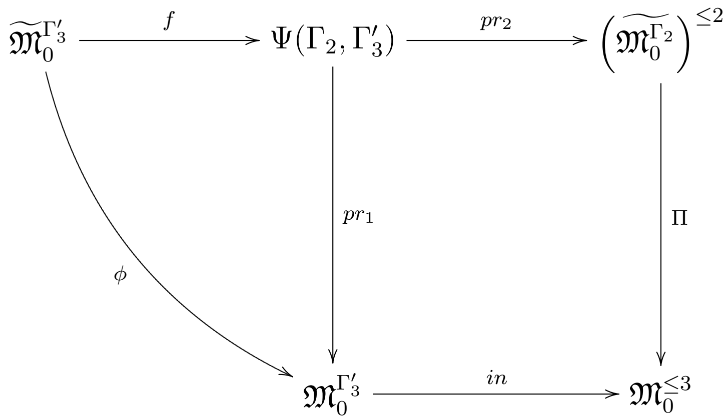
So we have

$$\begin{aligned} \phi^*\kappa_1 &= -(w_1 + w_2 + w_3), \\ \phi^*\kappa_2 &:= -(w_1^2 + w_2^2 + w_3^2), \\ \phi^*\kappa_3 &:= -(w_1^3 + w_2^3 + w_3^3); \end{aligned}$$

conesequently

$$A^*(\mathfrak{M}_0^{\Gamma'_3}) \otimes \mathbb{Q} \cong \mathbb{Q}[\kappa_1, \kappa_2, \kappa_3].$$

As before we fix the following notation



where f is the union of all f_α.

First of all let us notice that the class

$$\phi^*\gamma'_3 := \pi^* c_3(\mathcal{N}_i) = w_1 w_2 w_3$$

depends on Mumford classes in the following way

$$6\gamma'_3 = -(\kappa_1^3 - 3\kappa_1\kappa_2 + 2\kappa_3)$$

so we can write

$$A^*(\mathfrak{M}_0^{\Gamma'_3}) \otimes \mathbb{Q} = \mathbb{Q}[\kappa_1, \kappa_2, \gamma'_3].$$

The restriction of γ_2 to $A^*(\mathfrak{M}_0^{\Gamma'_3})$is

$$\begin{aligned} \gamma_2 := pr_{1*}\frac{1}{2}c_2(pr_1^*(\mathcal{N}_{in})/\mathcal{N}_{pr_2}) &= \phi_* f^* \frac{1}{2} c_2(pr_1^*(\mathcal{N}_{in})/\mathcal{N}_{pr_2}) \\ &= \frac{1}{2}\pi_*(w_1 w_3) \end{aligned}$$

from which

$$\phi^*\gamma_2 = w_1w_3 + w_1w_2 + w_2w_3$$

consequently, by writing

$$2\gamma_2 = k_1^2 - k_2$$

we have

$$A^*(\mathfrak{M}_0^{\Gamma_3'}) \otimes \mathbb{Q} = \mathbb{Q}[k_1, \gamma_2, \gamma_3'].$$

In order to restrict the class q let us notice thet we can write

$$\begin{aligned} f^*r &= 0 \\ f^*t_1 &= w_1 \\ f^*t_2 &= w_3 \end{aligned}$$

from which we have

$$f^*\left(\frac{1}{2}(t_1 - t_2)(2t - t_1 + t_2)\right) = -\frac{1}{2}(w_1 - w_3)^2$$

and so

$$\begin{aligned} \phi^*q &= \phi^*\phi_*\left(-\frac{1}{2}(w_1 - w_3)^2 w_1w_3\right) \\ &= -((w_1 - w_3)^2w_1w_3 + (w_1 - w_2)^2w_1w_2 + (w_2 - w_3)^2w_2w_3) \end{aligned}$$

we can therefore write $q = -\gamma_2(k_1^2 - 4\gamma_2) + 3\gamma_3' k_1$. With reference to the inclusions

$$\mathfrak{M}_0^{\Gamma_3'} \xrightarrow{i} \mathfrak{M}_0^{\leq 2} \cup \mathfrak{M}_0^{\Gamma_3'} \xleftarrow{j} \mathfrak{M}_0^{\leq 2}$$

we have the fiber square:

$$\begin{array}{ccc} A^*(\mathfrak{M}_0^{\leq 2} \cup \mathfrak{M}_0^{\Gamma_3'}) \otimes \mathbb{Q} & \xrightarrow{j^*} & \mathbb{Q}[k_1, k_2, \gamma_2, q]/I \\ {\scriptstyle i^*}\downarrow & \square & \downarrow{\scriptstyle \varphi} \\ \mathbb{Q}[k_1, \gamma_2, \gamma_3'] & \xrightarrow{p} & \mathbb{Q}[k_1, \gamma_2] \end{array}$$

where I is the ideal generated by the polynomial $q^2 - \gamma_2^2(2k_2 - k_1^2)(k_1^2 - 4\gamma_2)$ and the map φ is surjective and such that

$$\ker\varphi = (q + \gamma_2(k_1^2 - 4\gamma_2), \quad k_2 + 2\gamma_2 - k_1^2).$$

Now let us observe that from Lemma (2.29) the ring in question is isomorphic to

$$A/(0, I) := \frac{\mathbb{Q}[k_1, \gamma_2, \gamma_3'] \times_{\mathbb{Q}[k_1,\gamma_2]} \mathbb{Q}[k_1, k_2, \gamma_2, q]}{(0, I)}.$$

Set, with abuse of notation

$$\begin{array}{lll} k_1 := (k_1, k_1) & \gamma_2 := (\gamma_2, \gamma_2) & \gamma_3' := (\gamma_3', 0) \\ k_2 := (k_1^2 - 2\gamma_2, k_2) & q := (-\gamma_2(k_1^2 - 4\gamma_2), q) & \end{array}$$

Now we prove the following

Proposition 4.2. *The classes* $\kappa_1, \gamma_2, \gamma_3', \kappa_2, q$ *generate the ring* A.

Proof. A pair of polynomials

$$(P_1(\kappa_1, \gamma_2, \gamma_3'), P_2(\kappa_1, \gamma_2, \kappa_2, q))$$

is in A if and only if

$$P_1(\kappa_1, \gamma_2, 0) = P_2(\kappa_1, \gamma_2, \kappa_1^2 - 2\gamma_2, -\gamma_2(\kappa_1^2 - 4\gamma_2)).$$

From this it follows that the pair of polynomials can be written in A as

$$P_2(\kappa_1, \gamma_2, \kappa_2, q) + (P_1(\kappa_1, \gamma_2, \gamma_3') - P_1(\kappa_1, \gamma_2, 0), 0);$$

we end by noting that the second term can be written as

$$\gamma_3' Q(\kappa_1, \gamma_2, \gamma_3'),$$

with Q suitable polynomial. □

Let us compute the ideal of relations. Let $T(\kappa_1, \gamma_2, \gamma_3', \kappa_2, q)$ be a polynomial in $\mathbb{Q}[\kappa_1, \gamma_2, \gamma_3', \kappa_2, q]$, it is zero in A iff $T(\kappa_1, \gamma_2, \gamma_3', \kappa_1^2 - 2\gamma_2, -\gamma_2(\kappa_1^2 - 4\gamma_2))$ and $T(\kappa_1, \gamma_2, 0, \kappa_2, q)$ are respectively zero in $\mathbb{Q}[\kappa_1, \gamma_2, \gamma_3']$ and $\mathbb{Q}[\kappa_1, \gamma_2, \kappa_2, q]$: in particular this imply that T is in the ideal of γ_3'. Consequently the polynomial $T =: \gamma_3' \widehat{T}$ is zero in A iff $\widehat{T}(\kappa_1, \gamma_2, \gamma_3', \kappa_1^2 - 2\gamma_2, -\gamma_2(\kappa_1^2 - 4\gamma_2))$ is zero in $\mathbb{Q}[\kappa_1, \gamma_2, \gamma_3']$. The ideal of relations in A is so generated by

$$\gamma_3'(-\kappa_2 + 2\gamma_2 - \kappa_1^2)$$
$$\gamma_3'(q + \gamma_2(\kappa_1^2 - 4\gamma_2)).$$

Finally let us notice that the ideal $(0, I)$ is generated in A by the polynomial $q^2 + \gamma_2^2(2\kappa_2 + \kappa_1^2)(\kappa_1^2 - 4\gamma_2)$. So we can conclude that

$$A^*(\mathfrak{M}_0^{\leq 2} \cup \mathfrak{M}_0^{\Gamma_3'}) \otimes \mathbb{Q} = \mathbb{Q}[\kappa_1, \kappa_2, \gamma_2, \gamma_3', q]/J,$$

where J is the ideal generated by the polynomials

$$q^2 + \gamma_2^2(2\kappa_2 + \kappa_1^2)(\kappa_1^2 - 4\gamma_2)$$
$$\gamma_3'(-\kappa_2 + 2\gamma_2 - \kappa_1^2)$$
$$\gamma_3'(q + \gamma_2(\kappa_1^2 - 4\gamma_2))$$

4.4.2. The second component. $\Gamma''_3 :=$

The group of $\mathrm{Aut}(C^{\Gamma'_3})$ is $\mathfrak{C}_2 \ltimes (E \times E \times \mathbb{G}_{\mathbf{m}} \times \mathbb{G}_{\mathbf{m}})$. The action of τ on this group exchange simultaneously the components related to $\mathbb{G}_{\mathbf{m}}$ and those related to E.

We have the isomorphism

$$A^*_{E\times E\times\mathbb{G}_{\mathbf{m}}\times\mathbb{G}_{\mathbf{m}}} \otimes \mathbb{Q} \cong \mathbb{Q}[v_1, \ldots, v_4]$$

where

$$\begin{aligned} v_1 &= \psi^1_{\infty,1} \\ v_2 &= \psi^1_{\infty,2} \\ v_3 &= \psi^2_{\infty,1} \\ v_4 &= \psi^2_{\infty,2} \end{aligned}$$

by gluing curves such that the two central components corresponds in the point at infinity.

It follows that the action induced by τ is

$$\tau(v_1, v_2, v_3, v_4) = (v_2, v_1, v_4, v_3).$$

Since $\mathfrak{C}_2$ has order 2, the invariant polynomials are algebraically generated by the invariant polynomials of degree at most two (see. Theorem 7.5 [**CLO**]). It is easy to see that a basis for the linear ones is given by $u_1 := v_1 + v_2$, $u_2 := v_3 + v_4$. For the vector subspace of invariant polynomials of degree two, we can compute a linear basis by using Reynolds' operator

$$\begin{array}{ll} u_3 := v_1^2 + v_2^2, & u_6 := v_1 v_3 + v_2 v_4, \\ u_4 := v_3^2 + v_4^2, & u_7 := v_1 v_2, \\ u_5 := v_1 v_4 + v_2 v_3, & u_8 := v_3 v_4 \end{array}$$

now we note that

$$\begin{aligned} u_6 &= u_1 u_2 - u_5, \\ u_7 &= (u_1^2 - u_3)/2, \\ u_8 &= (u_2^2 - u_4)/2 \end{aligned}$$

Consequently we can write

$$A^*\mathfrak{M}_0^{\Gamma'_3} \otimes \mathbb{Q} \cong \mathbb{Q}[u_1, \ldots, u_5]/I,$$

where I is the ideal generated by the polynomial

$$2u_3u_4 + 2u_1u_2u_5 - u_2^2u_3 - u_1^2u_4 - 2u_5^2 \tag{18}$$

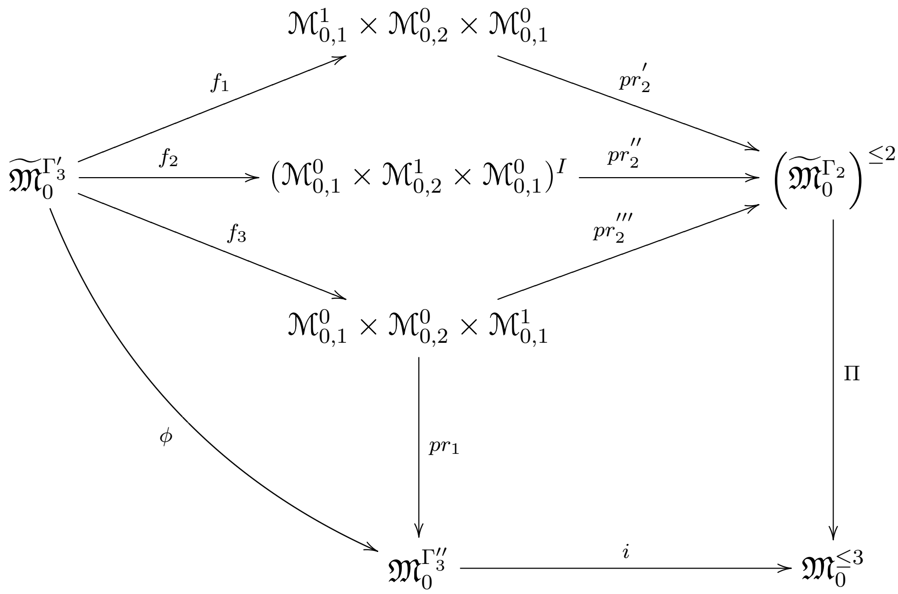
With reference to the degree two covering

$$\widetilde{\mathfrak{M}}_0^{\Gamma'_3} \xrightarrow{\phi} \mathfrak{M}_0^{\Gamma'_3}$$

we have:

$$-u_1 = \phi^*\kappa_1 \quad - u_3 = \phi^*\kappa_2.$$

In order to compute the restriction of the closure of the classes γ_2 and q of $\mathfrak{M}_0^{\leq 2}$ let us fix the notation of the following diagram



where $(\mathcal{M}_{0,1}^0 \times \mathcal{M}_{0,2}^1 \times \mathcal{M}_{0,1}^0)^I$ is the component where the marked points of the central curve (which is singular) are on different components.

As ϕ^* is an isomorphism to the algebra of polynomials which are invariants for the action of $\mathfrak{C}_2$, let us choose in $A^*(\mathfrak{M}_0^{\Gamma_3''}) \otimes \mathbb{Q}$ classes ρ, λ, μ such that

$$u_2 = \phi^*\rho \ \ , u_4 = \phi^*\lambda \ \ , u_5 = \phi^*\mu$$

First of all let us compute the restriction of the closure of $\gamma_2 \in A^*(\mathfrak{M}_0^{\leq 2}) \otimes \mathbb{Q}$, the polynomial in the classes ψ is $\frac{1}{2}$, we have

$$\begin{aligned}
c_3(\phi^*(\mathcal{N}_i)) &= (v_1 - v_3)(v_3 + v_4)(v_2 - v_4) \\
c_1(f_1^*(\mathcal{N}_{pr_2'})) &= (v_1 - v_3) \\
c_1(f_2^*(\mathcal{N}_{pr_2''})) &= (v_3 + v_4) \\
c_1(f_3^*(\mathcal{N}_{pr_2'''})) &= (v_2 - v_4)
\end{aligned}$$

from which we obtain the following relations

$$\begin{aligned}
\phi^*\gamma_3' &= (v_1 - v_3)(v_3 + v_4)(v_2 - v_4) \\
&= \phi^*\left(\rho\left(\frac{1}{2}\left(\kappa_1^2 + \rho^2 - \kappa_2 - \lambda\right) - \mu\right)\right) \\
\phi^*\gamma_2 &= ((v_3 + v_4)(v_2 - v_4) + (v_1 - v_3)(v_2 - v_4) + (v_1 - v_3)(v_3 + v_4)) \\
&= -\rho\kappa_1 - \mu + \frac{1}{2}(\kappa_1^2 - \rho^2 + \kappa_2 - \lambda)
\end{aligned}$$

In order to have γ_2 among the generators, set

$$\lambda = -2\rho\kappa_1 - 2\mu + \kappa_1^2 - \rho^2 + \kappa_2 - 2\gamma_2;$$

the equation (18) becomes

$$(19) \qquad K: \ \sigma^2 - (2\kappa_2 + \kappa_1^2)((-\kappa_1 + 3\rho)(\kappa_1 + \rho) - 4\gamma_2)$$

where we have set

$$\sigma = 2\mu - 2\kappa_2 - \kappa_1^2 + \kappa_1\rho.$$

Then we can write the ring $A^*(\mathfrak{M}_0^{\Gamma_3''}) \otimes \mathbb{Q}$ as

$$\mathbb{Q}[\kappa_1, \rho, \kappa_2, \gamma_2, \sigma]/K.$$

In the new basis we have

$$\gamma_3' = \rho(\rho^2 - \rho\kappa_1 + \gamma_2).$$

Let us restricts the closure of $q \in A^*(\widetilde{\mathfrak{M}_0^{\leq 2}}) \otimes \mathbb{Q}$ to $\mathfrak{M}_0^{\Gamma_1}$. We recall that the related polynomial in classes ψ of $\widetilde{\mathfrak{M}_0^{\Gamma_2}}$ is:

$$\frac{1}{2}(t_1 - t_2)(2r - t_1 + t_2)$$

On $\mathcal{M}_{0,1}^1 \times \mathcal{M}_{0,2}^0 \times \mathcal{M}_{0,1}^0$ we have

$$\begin{aligned} f_1^* t_1 &= v_3 \\ f_1^* t_2 &= v_2 \\ f_1^* r &= -v_4, \end{aligned}$$

on $(\mathcal{M}_{0,1}^0 \times \mathcal{M}_{0,2}^1 \times \mathcal{M}_{0,1}^0)^I$ we have

$$\begin{aligned} f_2^* t_1 &= v_1 \\ f_2^* t_2 &= v_2 \\ f_2^* r &= \frac{\psi_\infty^2 - \psi_0^2}{2} = \frac{v_3 - v_4}{2}, \end{aligned}$$

and in the end on $\mathcal{M}_{0,1}^0 \times \mathcal{M}_{0,2}^0 \times \mathcal{M}_{0,1}^1$ we have

$$\begin{aligned} f_3^* t_1 &= v_1 \\ f_3^* t_2 &= v_4 \\ f_3^* r &= v_3, \end{aligned}$$

consequently the polynomials related to the three components of $\Psi(\Gamma_2, \Gamma_3'')$ are

$$\begin{aligned} P_1 &= \frac{1}{2}(v_3 - v_2)(v_2 - v_3 - 2v_4) \\ P_2 &= \frac{1}{2}(v_1 - v_2)(v_4 - v_3 - v_1 + v_2) \\ P_3 &= \frac{1}{2}(v_4 - v_1)(v_1 - 2v_3 - v_4) \end{aligned}$$

from which

$$\begin{aligned}\phi^* q &= 2\sum_{\alpha=1}^{3}(P_\alpha N_\alpha) \\ &= \phi^*((3\rho + \kappa_1)\gamma_3'' + (\rho^2 - \gamma_2)\sigma)\end{aligned}$$

this means that the image of q in $B := A^*(\mathfrak{M}_0^{\Gamma_3''}) \otimes \mathbb{Q}/\gamma_3''$ is

$$(\rho^2 - \gamma_2)\sigma.$$

Further let us notice that the image of γ_3' in B is 0.

In order to compute $A^*(\mathfrak{M}_0^{\leq 3}) \otimes \mathbb{Q}$, let us consider its isomorphism with the fibered product

$$\begin{array}{ccc} A^*(\mathfrak{M}_0^{\leq 3}) \otimes \mathbb{Q} & \xrightarrow{j^*} & \mathbb{Q}[\kappa_1, \kappa_2, \gamma_2, \gamma_3', q]/J \\ {\scriptstyle i^*}\downarrow & \square & \downarrow{\scriptstyle \varphi} \\ \mathbb{Q}[\kappa_1, \rho, \kappa_2, \gamma_2, \sigma]/K & \xrightarrow{p} & B \end{array}$$

With reference to the fiber square

$$\begin{array}{ccc} A & \xrightarrow{j^*} & \mathbb{Q}[\kappa_1, \kappa_2, \gamma_2, \gamma_3', q] \\ {\scriptstyle i^*}\downarrow & \square & \downarrow{\scriptstyle \varphi} \\ \mathbb{Q}[\kappa_1, \rho, \kappa_2, \gamma_2, \sigma]/K & \xrightarrow{p} & B \end{array}$$

we have that the ring in question is isomorphic to the quotient $A/(0, J)$.

Now we look for generators of the ring A.

Proposition 4.3. *The following elements of A*

$$\begin{array}{lll} \kappa_1 := (\kappa_1, \kappa_1); & \kappa_2 := (\kappa_2, \kappa_2); & \gamma_2 := (\gamma_2); \\ \gamma_3' := (0, \gamma_3'); & q := ((\rho^2 - \gamma_2)\sigma, q); & \gamma_3'' := (\gamma_3'', 0); \\ r := (\gamma_3''\rho, 0); & s := (\gamma_3''\sigma, 0) & t := (\gamma_3''\rho^2, 0); \\ & u := (\gamma_3''\rho\sigma, 0) & \end{array}$$

are generators of the ring.

Proof. The image through j^* of these elements generates the whole $\mathbb{Q}[\kappa_1, \kappa_2, \gamma_2, \gamma_3', q]$, consequently we just need to show that this system, through i^*, generates the kernel of p, that is to say the ideal generated by γ_3''.

Let P be a polynomial in the ideal (γ_3''), it can be written uniquely as

$$P := \gamma_3'' \left(\sum_{i=0}^{N_1} Q_i \rho^i + \sum_{j=0}^{N_2} R_j \rho^j \sigma \right)$$

where Q_i and R_j are polynomials in $\kappa_1, \kappa_2, \gamma_2$. We can then write P in the following way:

$$P := \sum_{i=0}^{N_1} i^*(Q_i)\gamma_3''\rho^i + \sum_{j=0}^{N_2} i^*(R_j)\gamma_3''\rho^j\sigma.$$

Let us recall the following equation

$$\gamma_3'' = \rho^3 + \rho^2\kappa_1 + \rho\gamma_2,$$

so, for every $i > 2$, we have:

$$(\gamma_3'')^2\rho^{i-3} = \gamma_3''\rho^i + \gamma_3''\rho^{i-1}\kappa_1 + \gamma_3''\rho^{i-2}\gamma_2$$

from which it follows by induction that for every $i > 2$ the element $\gamma_3''\rho^i$ can be written as pull-back of a polynomial in $\kappa_1, \gamma_2, \gamma_3'', r, t$.

Now let us notice that $i^*q = \rho^2\sigma - \gamma_2\sigma$, from which for every $i > 1$ we have:

$$\gamma_3''\rho^{i-2}i^*q = \gamma_3''\rho^i\sigma - \gamma_3''\rho^{i-2}\gamma_2\sigma,$$

consequently (by considering the previous equation) we can again verify by induction that for every $i > 1$ the element $\gamma_3''\rho^i\sigma$ can be written as pull-back of a polynomial in $\kappa_1, \gamma_2, \gamma_3'', r, s, t, u$, as requested. $\square$

Now let us call $\widetilde{A}$ the ring $\mathbb{Q}[\kappa_1, \kappa_2, \gamma_2, \gamma_3', \gamma_3'', q, r, s, t, u]$. We have by fact defined a surjective homomorphism $a : \widetilde{A} \to A$; we call again j^* and i^* their composition with a, we have:

$$\ker a = \ker i^* \cap \ker j^*.$$

Now, for what we've seen, we have:

$$\begin{aligned} \ker i^* &= \left(\begin{array}{l} \gamma_3'; \\ r^2 - \gamma_3''t; \\ rs - \gamma_3''u; \\ s^2 - (\gamma_3'')^2(2\kappa_2 + \kappa_1^2)((-\kappa_1 + 2\rho)(\kappa_1 + \rho) + 4\gamma_2) \end{array}\right) \\ \ker j^* &= (\gamma_3'', r, s, t, u) \end{aligned}$$

and so:

$$\ker a = \left(\begin{array}{l} \gamma_3'\gamma_3''; \quad \gamma_3'r; \quad \gamma_3's; \quad \gamma_3't; \quad \gamma_3'u; \\ r^2 - \gamma_3''t; \quad rs - \gamma_3''u; \\ s^2 - (\gamma_3'')^2(2\kappa_2 + \kappa_1^2)((-\kappa_1 + 2\rho)(\kappa_1 + \rho) + 4\gamma_2) \end{array}\right)$$

We make the quotient of $A = A'/\ker a$ by the ideal $(0, J)$ and we obtain the following:

Theorem 4.4. *The ring $A^*(\mathfrak{M}_0^{\leq 3}) \otimes \mathbb{Q}$ is:*

$$\mathbb{Q}[\kappa_1, \kappa_2, \gamma_2, \gamma_3', \gamma_3'', q, r, s, t, u]/L,$$

where L is the ideal generated by the polynomials

$$\begin{aligned}
&\gamma_3'(-\kappa_2 + 2\gamma_2 - \kappa_1^2); \quad \gamma_3'(q + \gamma_2(\kappa_1^2 - 4\gamma_2));\\
&\gamma_3'\gamma_3''; \quad \gamma_3' r; \quad \gamma_3' s; \quad \gamma_3' t; \quad \gamma_3' u;\\
&r^2 - \gamma_3'' t; \quad rs - \gamma_3'' u;\\
&q^2 + (\gamma_2)^2(2\kappa_2 + \kappa_1^2)(\kappa_1^2 - 4\gamma_2);\\
&s^2 - (2\kappa_2 + \kappa_1^2)((-\kappa_1\gamma_3'' + 2r)(\kappa_1\gamma_3'' + r) + 4\gamma_2(\gamma_3'')^2)
\end{aligned}$$

We notice that the degree of the generators are

$$(1, 2, 2, 3, 3, 4, 4, 5, 5, 6)$$

and that the 11 relations have degree

$$(5, 7, 6, 7, 8, 8, 9, 8, 9, 8, 10).$$

APPENDIX A

Algebraic spaces

In this Appendix we give some definitions about algebraic spaces which we used throughout this work. The main reference is [**Knu**].

Definition A.1. An algebraic space X is a sheaf

$$X : (\mathcal{Sch}_{\mathbb{C}})^{opp} \to (\text{Sets})$$

in the étale topology such that there exists a scheme U and a map of sheaves $U \to X$ such that for every scheme V and every map $V \to X$ the fiber product $U \times_X V$ is representable and the projection to V is an étale surjective map. Further we request a quasiseparatedness condition: the map of schemes

$$U \times_A U \to U \times_{\mathbb{C}} U$$

is quasicompact.

A map $U \to X$ as in the definition is called a representable étale covering of X.

The structure sheaf of rings $\mathcal{O}$ on $\mathcal{Sch}_{\mathbb{C}}$ can be extended to the category of algebraic spaces over $\mathbb{C}$. Given an algebraic space X, we call $\mathcal{O}_X$ the restriction of $\mathcal{O}$ to the local étale topology of X.

An algebraic space X is *q*uasicompact if X has a covering $U \to X$ which is quasicompact.

Now we define extention to algebraic spaces of properties of the category $\mathcal{Sch}_{\mathbb{C}}$.

An algebraic space X with a representable étale covering $U \to X$ is *r*educed, nonsingular, normal, $n-$dimensional if U is.

A map $f : X \to Y$ of algebraic schemes is *s*urjective, flat, faithfully flat, universally open, étale, locally of finite presentation, locally of finite type, locally quasifinite if for some representable étale coverin $U \to Y$ and for some étale covering $V \to U \times_Y X$ the induced map $V \to U$ has the corresponding property. A map $f : X \to Y$ is *o*f finite presentation, *o*f finite type or *q*uasifinite if f is quasicompact and has the locally property. All these maps are stable for base extension.

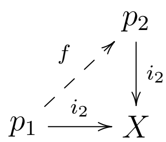
A map $f : X \to Y$ of algebraic spaces is *a*n open immersion, a closed immersion, an immersion, an affine morphism a quasiaffine morphism if for any scheme T and map $T \to Y$ the fiber product $X \times_Y T$ is a scheme (that is to say that f is representable) and the projection map $X \times_Y T \to T$ has the corresponding property. All these maps satisfy effective descent.

A map of algebraic spaces $f : X \to Y$ is *s*epareted if the induced map $X \to X \times_Y X$ is a closed immersion.

Definition A.2. Let X be an algebraic space. A *p*oint of X is a morphism

$$\mathrm{Spec} k \to X$$

where k is a field. Two points p_1, p_2



are called equivalent if there is an isomorphism f such that $i_2 f = i_1$.

The associated underlying topological space of X, $|X|$, is defined as the set of points modulo equivalence whose topological structure is given by taking a subset $V \subset |X|$ to be closed if V is of the form $|Y|$ for some closed subspace Y of X.

Let $f : X \to Y$ be a map of algebraic spaces. We say that f is *c*losed or *u*niversally closed if the induced map between the underlying topological spaces is.

A map $f : X \to Y$ is *p*roper if it is separated, of finite type and universally closed.

We can define sheaves of $\mathcal{O}_X-$modules and we have as in the case of schemes push forwards of sheaves and pull back maps of $\mathcal{O}_X-$modules. We say that a sheaf $\mathcal{F}$ of $\mathcal{O}_X-$modules is *q*uasi coherent, coherent, locally free of rank n if for some covering $\varphi : U \to X$ the sheaf $\varphi^*(\mathcal{F})$ is. A locally free $\mathcal{O}_X-$module of rank 1 is called *i*nvertible sheaf.

Let X be an algebraic space over $\mathbb{C}$. We define $\mathbb{P}^n_X$ as $\mathbb{P}^n_{\mathbb{C}} \times_{\mathbb{C}} X$, *p*rojective $n-$space over X. There is a canonical invertible sheaf $\mathcal{O}(1)$ on $\mathbb{P}^n_{\mathbb{C}}$ and we use the map $\mathbb{P}^n_X \to \mathbb{P}^n_{\mathbb{C}}$ to induce a canonical invertible sheaf $\mathcal{O}(1)$ on $\mathbb{P}^n_X$.

A map $f : X \to Y$ of algebraic spaces is *q*uasiprojective if there is an integer n and an immersion $i : X \to \mathbb{P}^n_Y$ such that f is the composition $X \to \mathbb{P}^n_Y \to Y$. The map f is *p*rojective if for some such i, i is a closed immersion.

APPENDIX B

Algebraic stacks

Now we give some basic definitions about algebraic stacks. We mainly use the following references: [**De-Mu**], [**L-MB**], Appendix [**Vis**]. A particular kind of algebaic stack are the *q*uotient stacks. We will prove Theorem B.3, which gives a condition to the points of an algebraic stack in order to be a quotiont stack.

Definition B.1. Let $\mathcal{C}$ be a category and $pr : \mathcal{S} \to \mathcal{C}$ a category over $\mathcal{C}$. For each $X \in \mathrm{Obj}(\mathcal{C})$, we denote by $\mathcal{S}(X)$ the fibre $pr^{-1}(X, \mathrm{id})$. The category $\mathcal{S}$ is fibered in groupoids over $\mathcal{C}$ if the following two conditions are verified

a For all $\varphi : X \to Y$ in $\mathcal{C}$ and $y \in \mathrm{Obj}(\mathcal{S}(Y))$

$$\begin{array}{ccc} x & \overset{f}{\dashrightarrow} & y \\ \downarrow & & \downarrow \\ X & \xrightarrow{\varphi} & Y \end{array}$$

there is a map f in $\mathcal{S}$ with $pr(f) = \varphi$.

b Given a diagram

$$\begin{array}{ccc} & & x \\ & & \downarrow{\scriptstyle f} \\ y & \xrightarrow{g} & z \end{array}$$

in $\mathcal{S}$, let

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle \varphi} \\ Y & \xrightarrow{\psi} & Z \end{array}$$

be its image in $\mathcal{C}$. Then for all $\theta : X \to Y$ such that $\varphi = \psi\theta$, there is a unique $h : x \to y$ such that $f = gh$ and $pr(h) = \theta$.

Condition **b** implies that the $f : x \to y$ whose existence is asserted in **a** is unique up to canonical isomorphism.

A *s*tack is a category $\mathcal{S}$ fibered in groupoids over a site *mathcalC* (equipped with a Grothendieck topology) such that

A for every object T in $\mathcal{C}$ and for every pair of objects x and y in $\mathcal{S}(T)$ the functor $\underline{Isom}_T(x, y)$ is a sheaf for the topology on $\mathcal{C}$;

B for every object $T \in \mathcal{C}$, given a covering $\{\varphi_\alpha : T_\alpha \to T\}_{\alpha \in I}$, any descent datum in $\mathcal{S}$ relative to it, is effective.

An *A*rtin stack is a stack $\mathcal{S}$ over $\mathcal{Sch}_{\mathbb{C}}$ equipped with the étale topology, which satisfies the following properties:

C for every pair of schemes X, Y in $\mathcal{Sch}_{\mathbb{C}}$ with maps to $\mathcal{S}$ (that is to say for any pair of objects x and y in $\mathcal{S}$) the fibered product

$$\begin{array}{ccc} X \times_{\mathcal{S}} Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathcal{S} \end{array}$$

is represented by a finite type separated algebraic space;

D there exists a surjective smooth map $U \to \mathcal{S}$ where U is a scheme.

Definition B.2. Let X be a scheme in $\mathcal{Sch}_{\mathbb{C}}$ and G a linear algebraic group which operates on X. We denote by $[X/G]$ the stack whose sections over a scheme T in $\mathcal{Sch}_{\mathbb{C}}$ is the category of principal homogeneous spaces E over T, with structural group G, provided with a $G-$morphism $\varphi E \to X$.

Theorem B.3. *Let G be a (linear) algebraic group on Spec$\mathbb{C}$ and let X be an algebraic stack such that for every algebraically closed field k which extends $\mathbb{C}$, every object in $X(Speck)$ is isomorphic to a fixed object C_0 and*

$$G = Isom_{Spec\mathbb{C}}(C_0, C_0).$$

If for every object $C \xrightarrow{\pi} T \in X$ there exists an étale covering in $\mathcal{Sch}_{\mathbb{C}}$

$$\mathcal{U} := \{T_\alpha \xrightarrow{\varphi_\alpha} T\}_{\alpha \in I}$$

such that for every α we have an isomorphism

$$\varphi_\alpha^*(C) \cong C_0 \times_{Spec\mathbb{C}} T_\alpha.$$

then X is equivalent to the classifying space BG.

Proof. We can associate to the object π the sheaf of isomorphisms

$$\mathcal{F}_C := \underline{Isom}_T(C_0 \times T, C) : \mathcal{Sch}_{\mathbb{C}}/T^{\mathrm{opp}} \to (\mathrm{Sets}).$$

By hypothesis this sheaf coincides locally in the étale topology to the sheaf $\mathcal{F}_0 := \underline{Isom}_T(C_0 \times T, C_0 \times T) : \mathcal{Sch}_{\mathbb{C}}/T^{\mathrm{opp}} \to (\mathrm{Sets})$ which is represented by the scheme $\mathrm{Aut}_{C_0} \times_{\mathrm{Spec}\mathbb{C}} T = G \times_{\mathrm{Spec}\mathbb{C}} T$, furthermore we know that an étale sheaf locally representabl is globally representable i the category of algebraic spaces. Let us call such scheme associated to the object $C \xrightarrow{\pi} T$ as $P(C) \xrightarrow{\Pi} T$. As the associated sheaf $\underline{Aut}_{C_0}$ acts to $\mathcal{F}_C$ through composition with the action of G on C_0, it follows that $P(C)$ is a G-principal bundle.

For every $T \in \mathcal{S}ch_{\mathbb{C}}$ we have then defined a map:

$$f : X(T) \to \mathrm{B}G(T)$$

which extends to a functor:

$$f : X \to \mathrm{B}G.$$

Now we show that **the functor f is an equivalence of fibered categories.**

Let us fix $T \in \mathcal{S}ch_{\mathbb{C}}$ and let $\mathcal{U} := \{T_\alpha \xrightarrow{\varphi_\alpha} T\}$ be an f.p.p.f covering of T.

Let us call the full subcategory of $\mathrm{B}G(T)$ (resp. of $X^{\mathcal{U}}(T)$) made of objects that admit a trivialization on $\mathcal{U}$ (resp. $X^{\mathcal{U}}(T)$) as $\mathrm{B}G^{\mathcal{U}}(T)$. We observe that f restricts to a functor:

$$f : \mathcal{S}^{\mathcal{U}}(T) \to \mathrm{B}G^{\mathcal{U}}(T),$$

because $T_\alpha \times_T P(C) \cong P(T_\alpha \times C)$. We have just to prove that this restriction is an equivalece of fibered categories as we can always find a simoultanous(?) trivialization.

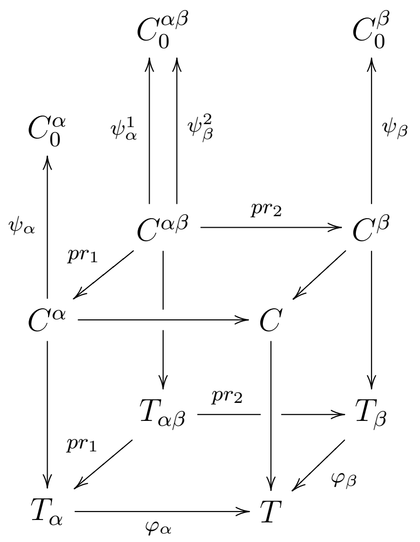
Let $C \xrightarrow{\pi} T$ be an object of X that admits a trivialization on $\mathcal{U}$ and let us choose a family of isomorphisms:

$$\{\psi_\alpha : \varphi_\alpha^*(C) =: C^\alpha \xrightarrow{\cong} C_0^\alpha := C_0 \times_{\mathrm{Spec}\,\mathbb{C}} T_\alpha\}_{\alpha \in I}.$$

Set:

$$\begin{array}{rcl} C^{\alpha\beta} & := & C \times_T T_\alpha \times_T T_\beta \\ C_0^{\alpha\beta} & := & C_0 \times_{\mathrm{Spec}\,\mathbb{C}} T_\alpha \times_T T_\beta \end{array}$$

and let us consider the following commutative diagram:



where $\psi_\alpha^1 := pr_1^*(\psi_\alpha)$ and $\psi_\beta^2 := pr_2^*(\psi_\beta)$. Let us define the isomorphism:

$$g_{\alpha\beta} := \psi_\alpha^1 \circ (\psi_\beta^2)^{-1} : C_0^{\alpha\beta} \xrightarrow{\cong} C_0^{\alpha\beta},$$

The isomorphism $g_{\alpha\beta}$ is an element of$G(T_{\alpha\beta})$ and furthermore defines a cocycle for G, that is to say an object:

$$\{g_{\alpha\beta}\}_{\alpha,\beta\in I} \in \mathrm{Coc}^{u}(G).$$

We can extends to a functor $f' : \mathcal{S}^{u}(T) \to \mathrm{Coc}^{u}(G)$.

Let us consider the following diagram:

$$\begin{array}{ccc} & & BG^{u}(T) \\ & \overset{f}{\nearrow} & \\ X^{u}(T) & & \cong \Big\downarrow g \\ & \underset{f'}{\searrow} & \\ & & \mathrm{Coc}^{u}(G) \end{array}$$

and we end by noting that there is a canonical isomorphism of f' with $g \circ f$. $\square$

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