Scuola Normale Superiore
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On the characterization of static spacetimes with positive cosmological constant

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La filosofia è scritta in questo grandissimo libro che continuamente ci sta aperto innanzi a gli occhi (io dico l'universo), ma non si può intendere se prima non s'impara a intender la lingua, e conoscer i caratteri, ne' quali è scritto. Egli è scritto in lingua matematica, e i caratteri son triangoli, cerchi, ed altre figure geometriche, senza i quali mezi è impossibile a intenderne umanamente parola; senza questi è un aggirarsi vanamente per un oscuro laberinto.

Galileo Galilei, *Il saggiatore*

Abstract

In this thesis we study *static vacuum spacetimes*. These are very special solutions of the Einstein Field Equations in General Relativity, where the Lorentzian structure disappears and we are left with the study of a system of PDEs on a Riemannian manifold. Although they represent the simplest examples of spacetimes, their study is by no means trivial. Our main focus will be on spacetimes with *positive cosmological constant*, even though we will provide a general overview of the other cases as well.

Our main contribution is the introduction of a new notion of mass (which will be called *virtual mass*) on vacuum static spacetimes with positive cosmological constant. We will show the plausibility of our definition, by proving that the virtual mass satisfies properties analogous to the well known Positive Mass Theorem and Riemannian Penrose Inequality for Riemannian manifolds with nonnegative scalar curvature.

As a consequence, we will prove a uniqueness theorem for the Schwarzschild–de Sitter spacetime. As we will discuss, this result shares some similarities with the well known Black Hole Uniqueness Theorem for the Schwarzschild spacetime.

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Introduction

In General Relativity, it is well known that a spacelike hypersurface (M, g) in a spacetime (X, γ) satisfies the *Einstein Constraints Equations*

$$\begin{aligned} \mathrm{R}_g - |K|_g^2 + (\mathrm{tr}_g K)^2 &= 2\Lambda + 16\pi\rho\,, \\ \mathrm{div}_g K - d(\mathrm{tr}_g K) &= 8\pi J\,, \end{aligned} \tag{A}$$

where K is the second fundamental form induced by γ on M, Λ is the *cosmological constant*, ρ is the *local energy density* and μ is the *local momentum density*. A triple (M, g, K), where (M, g) is a 3-dimensional Riemannian manifold and K is a symmetric $(2, 0)$–tensor on M satisfying (A), is usually called an *initial data set*. It is known that several matter fields have a well posed initial value formulation, meaning that, for any initial data set (M, g, K), there exists a unique (up to isometry) maximal spacetime (X, γ) such that $(M, g) \subset (X, \gamma)$ is a Cauchy hypersurface with second fundamental form K. The well posedness and stability of the initial value formulation of General Relativity is a huge topic in the literature, due to its importance in making Einstein's theory physically viable. Without any attempt of being complete, we mention some of the main references on this subject [FB52, CBG69, FM72, HE73], see also the discussion in [Wal84, Chapter 10]. This correspondence between initial data sets and spacetimes has an important practical use, as it is usually easier to study a physical problem in its initial value formulation, if it exists. Furthermore, some physical questions are more naturally formulated in terms of initial data sets. This is the case for the challenging problem of defining suitable notions of energy and mass. In fact, while in Special Relativity these definitions can be made quite explicitly (see for instance [Wal84, Chapter 4]), in General Relativity the question is much harder, as it is not clear how the gravitational field should contribute to the energy (see the discussion in [Bla11, Section 21.6]). However, some important breakthroughts have been made at least in the definition of the *total energy* of an *isolated system*.

To ease the discussion, it is convenient to introduce a common simplification. Let us require that the initial data set (M, g, K) is *time symmetric*, that is, the tensor K vanishes everywhere. Under this assumption, the second equation in (A) tells us that the local momentum density is zero, so that the local energy density can also be interpreted as local mass density, whereas the first equation in (A) simply reduces to

$$\mathrm{R}_g = 2\Lambda + 16\pi\rho\,.$$

Therefore, the physically natural assumption of nonnegative mass density ($\rho \geq 0$) becomes equivalent to the requirement that the scalar curvature of the initial data (M, g) satisfies $\mathrm{R}_g \geq 2\Lambda$. In particular, the Lorentzian structure of the spacetime disappears completely and we are left with the study of Riemannian manifolds with scalar curvature bounded from below by a constant $2\Lambda \in \mathbb{R}$. This problem heavily depends on the sign of Λ, so that the three cases $\Lambda = 0$, $\Lambda < 0$ and $\Lambda > 0$ require different analyses.

By far the most studied case is that of zero cosmological constant. If $\Lambda = 0$, the assumption that the system is isolated translates in the requirement that (M, g) is *asymptotically flat*. The precise definition will be given in Subsection 1.2.1, but roughly speaking we are asking that the metric g converges to the Euclidean metric suitably fast at infinity. In other words, there exists a chart at infinity sending M minus a compact set in $\mathbb{R}^n \setminus \mathbb{B}^n$, with respect to which the coefficients g_{ij} and their derivatives converge to the Kronecker delta δ_{ij} sufficiently fast. Under this assumption, one can define the *ADM mass* as

$$m_{ADM}(M, g) = \frac{1}{16\pi} \lim_{R \to +\infty} \int_{\{|x| = R\}} \sum_{\alpha,\beta} \left(\frac{\partial g_{\alpha\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\alpha}}{\partial x^\beta} \right) \nu^\beta \, \mathrm{d}\sigma\,, \tag{B}$$

where x^1, x^2, x^3 are the coordinates given by the chart at infinity. This mass was first introduced by Arnowitt, Deser and Misner [ADM62], and Bartnik [Bar86] proved its well posedness, in the sense that formula (B) does not depend on the choice of the chart at infinity. It follows that the ADM mass is indeed an invariant on asymptotically flat 3-manifolds. Furthermore, it satisfies two crucial properties, that we recall in the next theorem.

Theorem. *Let (M, g) be a 3-dimensional asymptotically flat Riemannian manifold with nonnegative scalar curvature and ADM mass $m_{ADM}(M, g)$ equal to $m \in \mathbb{R}$. Then, the following statements hold.*

Positive Mass Theorem (Schoen-Yau [SY79]). *If the boundary of M is empty then the number m is always nonnegative*

$$0 \leq m\,. \tag{C}$$

Moreover, the equality is fulfilled if and only if (M, g) is isometric to the flat Euclidean space.

Riemannian Penrose Inequality (Huisken-Ilmanen [HI01]). *If the boundary of M is nonempty and given by a connected, smooth and compact outermost minimal surface. Then, the following inequality holds*

$$\sqrt{\frac{|\partial M|}{16\pi}} \leq m\,. \tag{D}$$

Moreover, the equality is fulfilled if and only if (M, g) is isometric to the Schwarzschild solution with mass parameter equal to m

$$\left(\mathbb{R}^3 \setminus \{|x| < 2m\}\ ,\ \frac{d|x| \otimes d|x|}{1 - 2\,m\,|x|^{-1}} + |x|^2 g_{\mathbb{S}^2}\right) \tag{E}$$

These two results deserve some comments. Concerning the Positive Mass Theorem, inequality (C) tells us that a nonnegative local mass density ($\mathrm{R}_g \geq 0$) implies a nonnegative total mass ($m \geq 0$). The proof of this natural physical property is actually highly nontrivial. In fact, it took a lot of efforts before a first proof of this fact was given by Schoen and Yau in [SY79]. An alternative argument was provided by Witten in [Wit81], which was then converted into a rigorous mathematical proof by Parker and Taubes [PT82] (see also [AH82, Reu82]). It is worth pointing out that the ADM energy can be defined also when the initial data set is not time symmetric, and a Positive Energy Theorem has been proved also in this generalized setting [SY81, Wit81].

The Riemannian Penrose Inequality can be seen as a physically natural refinement of the Positive Mass Theorem. In fact, we can imagine the outermost minimal boundary as an event horizon hiding a black hole, which therefore should give a contribution to the total mass. We expect this contribution to depend on the area of the event horizon, as this is the case of the Schwarzschild solution (E), which represents the simplest model of a vacuum exterior region of a black hole. Inequality (D) tells us that the mass of our initial data should be at least equal to the mass of the Schwarzschild solution whose boundary has the same area of ∂M. The first proof of the Riemannian Penrose Inequality was given by Huisken and Ilmanen in [HI01] making rigorous an inverse mean curvature flow argument suggested by Geroch [Ger73] and Jang, Wald [JW77]. In particular, the argument of Huisken and Ilmanen can also be seen as an independent proof of the Positive Mass Theorem. An alternative approach was suggested by Bray [Bra01] using a conformal flow of metrics and a clever application of the Positive Mass Theorem. As a consequence, Bray was able to extend inequality (D) to the case of a disconnected boundary. However, we remark that a proof of the Riemannian Penrose Inequality in the case where the initial data set is not time symmetric presents considerable difficulties. In [Pen73], Penrose, using a beautiful heuristic argument, suggested what should be the natural form of (D) for general initial data sets. However, only very partial results are known on the validity of his statement, that is known as the *Penrose Inequality*. The relevance of Penrose's conjecture is that a proof of it would give indirect support on the validity of the *cosmic censorship conjecture*, see [Mar09] for a more thorough discussion.

We also mention that the Positive Mass Theorem and the Riemannian Penrose Inequality can be extended to dimensions greater than 3, see Subsection 1.2.1 for more details. In fact, a lot of the results that we will discuss hold in higher dimensions. Even if the three dimensional case is the most physically relevant, the higher dimensional case is still important, as it has applications both in physics (see the discussion in [ER08]) and in mathematics (for instance, in the study of the Yamabe problem, see [Sch84]). However, for the sake of simplicity, in this introduction we will just focus on the three dimensional case, postponing the analysis of the higher dimensions to the next chapters.

Finally, it should be noticed that the ADM mass is not the only possible definition of a total mass for isolated systems. Another well known notion is the *Bondi mass* [BVdBM62], which also is defined in any dimension $n \geq 3$ (even if its extension to $n \geq 4$ requires some work, see for instance [HI05]) and satisfies a Positive Mass Theorem (see [HP82, SY82, LV82, RT84]). We also mention that there have been quite a lot of attempts of defining a *quasi-local mass*, that is, a quantity measuring the mass of a region enclosed by an hypersurface. The most studied ones are the *Hawking mass* (whose property of monotonicity along inverse mean curvature flow was exploited by Huisken and Ilmanen in their proof of the Riemannian Penrose Inequality (D)) and the *Brown-York-Liu-Yau mass*, see [Wan15], but there are other possibilities, see the survey [Sza09]. However, in this work we will not be concerned with all these alternative definitions.

Naturally, the question arises whether a notion of total mass similar to (B) is available for time symmetric initial data sets of spacetimes with nonzero cosmological constant Λ. In the case $\Lambda < 0$, this led to the study of *asymptotically hyperbolic* manifolds. In analogy with the asymptotically flat case, these are Riemannian manifolds (M, g) such that there exists a chart at infinity sending M minus a compact set in the hyperbolic space minus a ball, and such that the metric g converges suitably at infinity to the hyperbolic metric, see Subsection 1.2.2 for a precise definition. It has been proven that an analogue definition of mass can be introduced, measuring the rate of convergence of g to the hyperbolic metric, and satisfying properties similar to those of the ADM mass. In fact, a Positive Mass Theorem is known to hold in this setting (see [ACG08, CH03, Wan01, Zha04]), and we also mention that the definition of mass has been extended in [CN01, Mae06] to initial data sets which are not necessarily time symmetric. However, it should be noticed that the case $\Lambda < 0$ seems to be significatively more complicated than the case $\Lambda = 0$. One of the reasons may be due to the fact that there exist much more explicit models in the case $\Lambda < 0$, which complicate the analysis (see for instance the static case discussed in Section 2.4). In particular, there are natural explicit models whose topology at infinity is not that of the hyperbolic space minus a ball (this is the case for instance of the Schwarzschild–Anti de Sitter solutions with flat and hyperbolic topology (2.4.7), (2.4.8)), and it is known that such models can have negative mass. In particular, we cannot expect to prove a Positive Mass Theorem for manifolds whose ends have different topology than the hyperbolic space minus a ball. Moreover, a proof of a Riemannian Penrose–like inequality for $\Lambda < 0$ seems still out of reach, see the discussion in [CS01]. Some first partial results are known: a proof in the graph case has been given in [dLG13] and a Penrose–like inequality for the renormalized volume has been established in [BC14]. Finally, recently Lee and Neves [LN15] were able to follow an approach similar to that of Huisken and Ilmanen in the asymptotically flat case. This led them to the proof of the Riemannian Penrose Inequality, but unfortunately this only works for manifolds with nonpositive mass.

On the other hand, the astronomical observations [A+16a, A+16b, BS11] seem to lead to the conclusion that the cosmological constant Λ of our universe should be positive, see also the discussion in [Bou07]. For this reason, the study of spacetimes with $\Lambda > 0$ acquires particular relevance. Notice that for spacetimes with positive cosmological constant it is physically reasonable to assume that our initial data set is compact, as this is the case for most explicit model solutions. Therefore, even if there is some study on initial data sets with cylindrical ends [CJK13], most of the literature is concerned with the compact case. In this setting, a first conserved quantity was shown by Abbott and Deser [AD82] using a perturbative analysis, and developed in [ABK14, BDBM02, KT02, Shi94, SIT01]. However, in order for this approach to work, one needs to require suitable asymptotics at time infinity, whereas we would like to have a definition based on the initial data set only, as for the cases $\Lambda = 0$ and $\Lambda < 0$ discussed above. A different approach is that in [LXZ10], where the initial metric is singularized via a conformal change in such a way to artificially create an asymptotic behavior that is then exploited to define a notion of mass. Ultimately however, it seems that there is no universally accepted definitions of mass in the case of a positive comological constant, see also the discussion in [Ann12, Wit01] for more insights on the problems posed by the case $\Lambda > 0$.

This led to the interest in the *Min-Oo's conjecture* [MO98], which proposed a characterization of the sphere, that might be interpreted as the natural analogue of the characterizations of the flat and hyperbolic space forms as the mass zero cases of the Positive Mass Theorems for asymptotically flat and hyperbolic manifolds. The Min-Oo's conjecture states that, if a metric g on the hemisphere $\mathbb{S}^3_+$ satisfies the following properties

(i) $\mathrm{R}_g \geq 6$ (this is the hypothesis of nonnegative energy density when the metric is rescaled so that $\Lambda = 3$),

(ii) The metric induced by g on the boundary $\partial\mathbb{S}^3_+ = \mathbb{S}^2$ is the standard unit round metric (this is the natural analogue of asymptotic flatness in the spherical setting),

(iii) the boundary $\partial\mathbb{S}^3_+ = \mathbb{S}^2$ is totally geodesic (this is the natural analogue of the condition of zero mass),

then g is the unit round metric. This natural conjecture was thought to be true for a long time, until it was finally disproven by Brendle, Marques and Neves in [BMN11] (see also [MN12] and Subsection 1.2.3). As a consequence, it is now clear that there is no straightforward ways of guessing the right analogue of the ADM mass (if it exists) in the spherical setting.

The main goal of the present work is that of proposing a different approach to this problem. Our strategy will be to "take a step back", starting our analysis in an easier setting, and then, *a posteriori*, infer what the right notion of mass should be in the general spherical case. From this point of view, this work can be seen as the first step of our program, as it is concerned with the definition of a suitable mass in the most basic case, which is that of *vacuum static spacetimes*. These are vacuum spacetimes (X, γ) admitting a global irrotational timelike Killing vector field, see the discussion in Subsection 2.1.3. This implies that the spacetime splits as

$$X = \mathbb{R} \times M\,, \qquad \gamma = -u^2 dt \otimes dt + g_0\,,$$

where (M, g_0) is a 3-dimensional Riemannian manifold and $u : M \to \mathbb{R}$ is a smooth function called *static potential*. Under this assumption, the Einstein Field Equations force the static potential u to satisfy the following system of differential equations

$$\begin{cases} u \,\mathrm{Ric} \,=\, \mathrm{D}^2 u \,+\, \Lambda\, u\, g_0, & \text{in } M, \\ \quad \Delta u \,=\, -\Lambda\, u, & \text{in } M, \end{cases} \tag{F}$$

where $\mathrm{Ric}, \mathrm{D}^2, \Delta$ are the Ricci tensor, hessian and laplacian with respect to g_0. It is also convenient and not restrictive to assume that u is positive in the interior of M and zero at the boundary ∂M. Such triples (M, g_0, u) will be called *static triples*, and they are our main subject of study. Static triples have some important properties that will help us in the analysis. Among them, we recall that their scalar curvature R is constant and equal to 2Λ. Moreover, the quantity $|\mathrm{D}u|$ is locally constant on the boundary ∂M, a fact that leads to the definition of surface gravity, as explained below (see (G)).

As anticipated, we will focus in particular on the study of static triples with $\Lambda > 0$ and we will assume that our manifold M is compact. This means that the static potential assumes its maximum in the interior of M, on a set that we will call $\mathrm{MAX}(u)$. This set shares important analogies with the ends of asymptotically flat and asymptotically hyperbolic manifolds, as it was noticed in [BH96]. Therefore, we may expect that a suitable definition of mass should take the behavior of u on $\mathrm{MAX}(u)$ into account. To explain the process leading to our definition, let us start by considering a *region* N of M, that is, a connected component of $M \setminus \mathrm{MAX}(u)$. It is easily seen that any region N has necessarily a nonempty boundary $\partial N = \partial M \cap N$ (this is a consequence of the No Island Lemma 3.5.1). For the sake of argument, let us suppose for simplicity that ∂N is connected. It is then well known that the norm of the gradient of the static potential u is constant on ∂N. This gives rise to the notion of *surface gravity* of ∂N, that, according to [BH96, PK17], is defined as

$$\kappa(\partial N) \,=\, \frac{|\mathrm{D}u|_{|\partial N}}{\max_M(u)}\,. \tag{G}$$

The surface gravity $\kappa(\partial N)$ admits an interpretation as the force experienced by a test particle resting on ∂N (see [Wal84, Section 12.5]), so it is natural to imagine it to be related to the mass. This consideration leads us to define the mass via a comparison of the surface gravities with some reference models, which are the well known *Schwarzschild–de Sitter triples* [Kot18]

$$\begin{gathered} M \,=\, [r_-(m), r_+(m)] \times \mathbb{S}^2\,, \quad g_0 \,=\, \frac{dr \otimes dr}{1 - \frac{\Lambda}{3}\, r^2 \,-\, 2m\, r^{-1}} + r^2 g_{\mathbb{S}^2} \\ u \,=\, \sqrt{1 - \frac{\Lambda}{3}\, r^2 \,-\, 2m\, r^{-1}}\,, \end{gathered} \tag{H}$$

where m is a parameter and $[r_-(m), r_+(m)]$ is the interval in which the potential u is well defined. The Schwarzschild–de Sitter triple is one of the simplest solutions to (F) when $\Lambda > 0$, it has two boundary components ∂M_+ and ∂M_-

corresponding to the two values $r_+(m)$ and $r_-(m)$ of the radial coordinate r, and the parameter m appearing in its definition is usually interpreted in the physical literature as its mass. It is also worth noticing that, taking the limit of (H) as $m \to 0^+$, the boundary ∂M_- collapses and we obtain the *de Sitter triple* [DS17]

$$\begin{aligned} M \,=\, \left\{x \in \mathbb{R}^3 \,:\, |x|^2 \leq \frac{3}{\Lambda}\right\}\,, \quad g_0 \,=\, \frac{d|x| \otimes d|x|}{1-\frac{\Lambda}{3}\,|x|^2} + r^2 g_{\mathbb{S}^2} \\ u \,=\, \sqrt{1-\frac{\Lambda}{3}\,|x|^2}\,. \end{aligned} \tag{I}$$

Notice that the manifold (M, g_0) in the de Sitter triple is just an hemisphere with its standard spherical metric. Coming back to the solution (H), we also notice that, for u to be well defined in a nonempty interval, one needs to ask that the mass m in (H) is less than the threshold

$$m_{\max} \,=\, \frac{1}{3\sqrt{\Lambda}}\,.$$

In particular, there is another significative limit of the Schwarzschild–de Sitter triple, that corresponds to $m \to m_{\max}$. This is the *Nariai triple* [Nar51]

$$\begin{aligned} M \,=\, [0,\pi] \times \mathbb{S}^2\,, \quad g_0 \,=\, \frac{1}{\Lambda}\left[dr \otimes dr + g_{\mathbb{S}^2}\right]\,, \\ u \,=\, \sin(r)\,. \end{aligned} \tag{J}$$

For further insights on these model solutions we refer the reader to Section 2.5.

We would like our definition of mass to coincide with the parameter m in (H) when our solution is isometric to the Schwarzschild–de Sitter triple. This consideration leads us to define the *virtual mass* of a region $N \subset M \setminus \mathrm{MAX}(u)$ of a general static triple (M, g_0, u) as the value of the parameter m in (H) which would induce on one of the boundaries $\partial M_+, \partial M_-$ the surface gravity $\kappa(\partial N)$ (for a more precise explanation see Definition 3.2.4).

In the case where ∂N is not connected, the definition is similar: one first takes the maximum of the surface gravities of the components of ∂N, and then compares this value with the model solution (H). Our first main result shows that this definition is well posed and that the virtual mass satisfies a Positive Mass Statement.

Theorem 3.2.5. *Let (M, g_0, u) be a solution to problem (F) for some $\Lambda > 0$, with $u = 0$ on ∂M and $u > 0$ in the interior of M, and let N be a connected component of $M \setminus \mathrm{MAX}(u)$. Then its virtual mass $\mu(N, g_0, u)$ is well defined, it is nonnegative by definition, and it is zero if and only if the whole solution (M, g_0, u) is isometric to the de Sitter triple (I).*

In particular, since we have already observed that the underlying manifold (M, g_0) of the de Sitter triple (I) is isometric to an hemisphere, the above theorem can be interpreted as a characterization of the spherical metric within the realm of static solutions. This result concludes the first part of our program, that ultimately, we hope, will lead to the definition of a mass on general compact

Riemannian manifolds with scalar curvature bounded from below by a positive constant.

It is also interesting to point out that the strategy used to prove Theorem 3.2.5 can be adapted to prove a characterization result for the *Anti de Sitter triple*

$$M = \mathbb{R}^3 \,, \quad g_0 = \frac{d|x| \otimes d|x|}{1 + \frac{|\Lambda|}{3}\,|x|^2} + |x|^2 g_{\mathbb{S}^2} \qquad \text{(K)}$$
$$u = \sqrt{1 + \frac{|\Lambda|}{3}\,|x|^2}\,,$$

which is a solution to problem (F) when $\Lambda < 0$. Using analogue ideas and similar computations, we obtain the following result.

Theorem 3.3.2. *Let (M, g_0, u) be a solution to problem (F), such that M has empty boundary, $u > 0$ in M and $u \to +\infty$ as we approach the ends of M. Let $\mathrm{MIN}(u) \subset M$ be the set of points where u attains its minimum value $u_{\min}$ and let N be a connected component of $M \setminus \mathrm{MIN}(u)$. Then*

$$\liminf_{x \in N,\, x \to \infty} \left(u^2 - u_{\min}^2 - |\mathrm{D}u|^2\right)(x) \;\leq\; 0\,.$$

Moreover, if the equality holds then the triple (M, g_0, u) is isometric to the Anti de Sitter solution (K).

Unfortunately, in the case $\Lambda < 0$ we are not able to exploit this result in order to provide a notion of mass. As it will be explained in more details in Chapter 3, this seems to be mainly related to the fact that the above theorem only works on complete manifolds, whereas the main massive model of static triple with negative cosmological constant (the Schwarzschild–Anti de Sitter triple (2.4.5)) has nonempty boundary. However, it is worth noticing that, for static triples with $\Lambda < 0$, a definition of mass in the same spirit as ours has been given by Chrusciel and Simon [CS01], see the discussion in Subsection 2.4.2.

Theorems 3.2.5 and 3.3.2 will be the main subjects of Chapter 3, whereas in Chapter 4 we will focus on the case $\Lambda > 0$ and we will study the properties of the virtual mass, with the aim of showing more evidences in support of its relevance. In particular, we will prove that our Positive Mass Statement can be strenghtened when we have some more informations on the area of the boundaries of our spatial manifold. To introduce the following result, let us notice that inequality (D) can be rewritten as

$$|\partial M| \;\leq\; 4\pi r_0^2(m)\,,$$

where m is the ADM mass and $r_0(m) = 2m$ is the radius of the Schwarzschild black hole (E). The analogy is clear with the next theorem, which therefore should be interpreted as a Riemannian Penrose–like Inequality for the virtual mass of static vacuum spacetimes with positive cosmological constant.

Theorem 4.1.1. *Let (M, g_0, u) be a solution to problem (F) with $\Lambda > 0$, with $u = 0$ on ∂M and $u > 0$ in the interior of M. Consider a connected component*

N of $M \setminus \mathrm{MAX}(u)$ with connected smooth compact boundary ∂N and denote by $\kappa(\partial N)$ its surface gravity. Finally, let

$$m = \mu(N, g_0, u),$$

be the virtual mass of N. Then, ∂N is diffeomorphic to the sphere $\mathbb{S}^2$. Moreover, the following inequalities hold:

(i) *Cosmological Area Bound. If $\kappa(\partial N) < \sqrt{\Lambda}$, then*

$$|\partial N| \leq 4\pi r_+^2(m),$$

and the equality is fulfilled if and only if the triple (M, g_0, u) is isometric to the Schwarzschild–de Sitter triple (H) with mass m.

(ii) *Riemannian Penrose Inequality. If $\kappa(\partial N) > \sqrt{\Lambda}$, then*

$$|\partial N| \leq 4\pi r_-^2(m),$$

and the equality is fulfilled if and only if the triple (M, g_0, u) is isometric to the Schwarzschild–de Sitter triple (H) with mass m.

(iii) *Cylindrical Area Bound. If $\kappa(\partial N) = \sqrt{\Lambda}$, then*

$$|\partial N| \leq \frac{4\pi}{\Lambda},$$

and the equality is fulfilled if and only if the triple (M, g_0, u) is covered by the Nariai triple (J).

This theorem improves a well known inequality proven by Boucher, Gibbons and Horowitz [BGH84], see Theorem 2.5.1. A comparison with the more recent and stronger results provided by Ambrozio in [Amb15] is discussed in Subsection 4.1.2.

Finally, building on the above result, we will state a characterization of the Schwarzschild–de Sitter solution (H), which is analogue in some sense to the classical Black Hole Uniqueness Theorem for asymptotically flat static solutions with zero cosmological constant proved in [Isr67, zHRS73, Rob77, BMu87]. In order to state this result, we will need to introduce the definition of 2-*sided triple* (see Definition 4.1.6 and Figure 4.2), which is a solution (M, g_0, u) of (F) such that there exists a (possibly stratified) hypersurface $\Sigma \subset \mathrm{MAX}(u)$ separating M in two regions M_+ and M_-, such that

$$\max_{S \in \pi_0(\partial M_+)} \kappa(S) \leq \sqrt{\Lambda},$$
$$\max_{S \in \pi_0(\partial M_-)} \kappa(S) \geq \sqrt{\Lambda},$$

where we have denoted by $\kappa(S)$ the surface gravity of S, defined as in (G). Under this assumption, requiring also some additional hypotheses on the behavior of our solution near the separating hypersurface Σ, we obtain the following.

Theorem 4.1.8. *Let (M, g_0, u) be a 2-sided solution to problem (F) with $\Lambda > 0$, with $u = 0$ on ∂M and $u > 0$ in the interior of M. Let $\Sigma \subset \mathrm{MAX}(u)$ be the stratified hypersurface separating M_+ and M_- and denote by*

$$m_+ = \mu(M_+, g_0, u)\,, \qquad m_- = \mu(M_-, g_0, u)$$

the virtual masses of M_+ and M_-. Suppose that the following conditions hold

mass compatibility	$m_+ \leq m_-$,
regularity assumption	$x \mapsto \sqrt{\max_M(u) - u(x)}$ *is* $\mathscr{C}^2$ *along* Σ,
connected cosmological horizon	∂M_+ *is connected,*
pinching assumption	$\fint_\Sigma \lvert \mathring{\mathrm{h}} \rvert^2 \mathrm{d}\sigma < \mathrm{R}$,

where $\mathrm{R} = 2\Lambda$ is the constant scalar curvature of g_0 and $\mathring{\mathrm{h}}$ is the traceless part of the second fundamental form of Σ. Then $m_+ = m_-$. Moreover, if $m_+ = m_- < m_{\max}$, then the triple (M, g_0, u) is isometric to the Schwarzschild–de Sitter triple (H) with mass $m_+ = m_-$. If $m_+ = m_- = m_{\max}$, then the triple (M, g_0, u) is isometric to the Nariai triple (J).

We stress that most of the hypotheses in the above theorem concern the behavior of the static solution near the set $\mathrm{MAX}(u)$. This is not surprising, as we have already observed that the set $\mathrm{MAX}(u)$ can be interpreted as the analogue of the ends of static solutions with $\Lambda = 0$. Therefore, the regularity and pinching assumption can be thought as the replacement of the asymptotically flat condition. For further remarks about the meaning and plausibility of these hypotheses, we refer the reader to the discussion in Subsection 4.1.3.

Before concluding this introduction, we would like to spend some words about the possible future developments of our work. We have already discussed one of our main motivations, which is that of infer the right definition of mass for general time symmetric initial data sets with positive cosmological constant. Theorem 3.2.5 tells us that our virtual mass is a natural candidate in the static case. The next step should be to drop the hypothesis that our manifold admits a static potential, and use our results as an indication of what the correct notion of mass should be.

However, other directions of study are possible. For instance, it would be interesting to understand whether results similar to Theorems 4.1.1 and 4.1.8 are available in the case $\Lambda < 0$. In fact, as anticipated, the situation in the case of a negative cosmological constant is made wilder by the presence of a number of model solutions, such as the Schwarzschild–Anti de Sitter solutions with spherical, flat and hyperbolic geometry, or the Anti Nariai triples (for the definition of all these solutions see Section 2.4), and this complicates the analysis. In fact, to the authors knowledge, no characterization of any of these model solutions in available in the literature, with only two exceptions.

(i) From the Positive Mass Theorem for asymptotically hyperbolic manifolds follows a nice uniqueness result for the Anti de Sitter triple (K) proved in [CH03, Wan05], see Theorem 2.4.6.

(ii) The Riemannian Penrose Inequality proved by Lee and Neves [LN15] led them to the proof of a uniqueness result for the Schwarzschild–Anti de Sitter triple with hyperbolic topology and negative mass, see Theorem 2.4.9.

It would be interesting to see if some other uniqueness results can be proved, at least for what appears to be the most relevant of these model solutions, that is the Schwarzschild–Anti de Sitter triple with spherical topology.

We also mention that the technique used in the proof of Theorems 4.1.1 and 4.1.8 is inspired by the work [AM16], where the set up of a cleverly chosen *cylindrical ansatz* led to an alternative proof of the Black Hole Uniqueness Theorem in the case $\Lambda = 0$. The same idea was used in [AM17b] in the study of the *capacitary problem* in the Euclidean space. It is clear from the study in [AM16, AM17b] that the two problems, although seemingly unrelated, are actually strongly connected, as the analysis and the computations are strikingly similar, see the discussion in [AM17a]. It turns out that also in the case of static metrics with nonzero cosmological constant there is a natural Euclidean analogue, which is the well known *torsion problem*. In fact, an analysis similar to that presented in Chapter 4 can be employed in order to study this Euclidean problem, obtaining some characterizations of the rotationally symmetric solutions. This will be the topic of a forthcoming work.

Outline of the thesis.

In Chapter 1 we recall some background material that will be used throughout the work. In particular, Section 1.2 will be dedicated to the definitions and the main properties of asymptotically flat and asymptotically hyperbolic manifolds. In Subsection 1.2.3 we will also discuss in some more details the partial results on the Min-Oo's conjecture, as well as the recent paper [BMN11] which disproves it in the general case.

Chapter 2 focuses on static spacetimes. After recalling their definition and the crucial notion of surface gravity, in Sections 2.3, 2.4 and 2.5 we will analyze the three cases $\Lambda = 0$, $\Lambda < 0$ and $\Lambda > 0$ separately, discussing the main known results. One may notice that, while in the case $\Lambda = 0$ we have strong characterizations of the model solutions, much less is known when $\Lambda < 0$ and $\Lambda > 0$.

It will also appear clear from the analysis in Chapter 2 that there are three natural candidates as massless models, which are the flat space form when $\Lambda = 0$, the hyperbolic space form when $\Lambda < 0$ and the spherical space form when $\Lambda > 0$. In Chapter 3 we will prove some uniqueness results for the three cases. The characterization in the case $\Lambda = 0$ that we will present in Section 3.1 is already known, as we will just retrace the proof in [Cas10]. However, the results for $\Lambda > 0$ (Theorem 3.2.5) and $\Lambda < 0$ (Theorem 3.3.2) are new. The main focus will be the case $\Lambda > 0$, as our characterization result will allow us to define the notion of virtual mass of a region of our manifold, and to prove that it satisfies a Positive Mass Statement. Although in this introduction we have focused on the three dimensional case only, we remark that the results in Chapter 3 hold in every dimension $n \geq 3$.

Finally, in Chapter 4 we will discuss how the notion of virtual mass can be used to improve some of the known results in the literature. In particular, as anticipated, we will prove a Riemannian Penrose–like inequality (Theorem 4.1.1) and a characterization of the Schwarzschild–de Sitter solution (Theorem 4.1.8). We will also discuss some higher dimensional analogues of these results, even though we will not be able to prove such strong theorems.

1

Background material

1.1 Preliminaries and notations

In this section we set the basic notions and notations, that will be used extensively in the rest of the work. After a brief revision of the main properties of Riemannian and Lorentzian manifolds, in Subsections 1.1.4 and 1.1.5 we will discuss real analytic functions. In particular, we will recall the Structure Theorem and the Gradient Inequality proved by Łojasiewicz. We will also prove a Reverse Gradient Inequality, which seems to be new, at least to our knowledge. After that, we will recall the Maximum Principle and the Divergence Theorem, two classical and important tools in the study of Riemannian manifolds, and we will conclude by writing some well known formulæ relating the curvature of two conformal metrics.

1.1.1 Riemannian manifolds and curvature tensors.

Throughout this work, we will assume a certain familiarity of the reader with the classical Riemannian tools. Here we rapidly introduce the definitions and the notations that we will need throughout the work. For a more careful and precise explanation we refer the reader to any classical book of Riemannian geometry, such as [GHL04, Pet06].

A *Riemannian manifold* is a pair (M, g) where M is a differentiable manifold and g is a Riemannian metric on M. The manifold M and the metric g will always be assumed to be smooth, except when explicitly stated otherwise, and the dimension of M will always be denoted by n. In this work we will deal with manifolds of dimension $n \geq 3$, although in this chapter we will occasionally refer to manifolds of dimension $n = 2$. We will always assume the Einstein convention on the sum of repeated indices.

Given any two tensors T, S of the same type (p, q), the following scalar product is defined at any point of M

$$\langle T \,|\, S \rangle \,=\, \left(\prod_{i=1}^{p} g^{\alpha_i \gamma_i} \right) \left(\prod_{j=1}^{q} g_{\beta_j \eta_j} \right) T_{\alpha_1 \cdots \alpha_p}^{\beta_1 \cdots \beta_q} \, S_{\gamma_1 \cdots \gamma_p}^{\eta_1 \cdots \eta_q} \,. \tag{1.1.1}$$

Accordingly, the norm of a tensor T at any point of M is defined as $|T| = \sqrt{\langle T\,|\,T\rangle}$.

The *Levi-Civita connection* ∇ of (M,g) is the only affine connection that preserves the metric and is torsion-free. Given a chart $(x^1,\dots,x^n)$, its Christoffel symbols Γ_{ij}^k can be computed explicitly using the following formula

$$\Gamma_{\alpha\beta}^{\gamma} = \frac{g^{\gamma\eta}}{2}\left(\frac{\partial g_{\alpha\eta}}{\partial x^\beta}+\frac{\partial g_{\beta\eta}}{\partial x^\alpha}-\frac{\partial g_{\alpha\beta}}{\partial x^\eta}\right). \tag{1.1.2}$$

We will denote by $\mathscr{C}^\infty(M)$ the family of all smooth functions $M\to\mathbb{R}$. Given a function $f\in\mathscr{C}^\infty(M)$, we define its *gradient* as the vector field $\nabla f : h \mapsto \langle df\,|\,dh\rangle$, its hessian as the $(2,0)$-tensor $\nabla^2 f = \nabla(df)$, and its *laplacian* as the function $\Delta f = \mathrm{div}(\nabla f) = \mathrm{tr}\,\nabla^2 f$. With respect to coordinates $(x^1,\dots,x^n)$, we have

$$\nabla^\alpha f = g^{\alpha\beta}\partial_\alpha f\,,\quad \nabla^2_{\alpha\beta} f = \partial^2_{\alpha\beta} f - \Gamma^\gamma_{\alpha\beta}\partial_\gamma f\,,\quad \Delta f = g^{\alpha\beta}\nabla^2_{\alpha\beta} f\,.$$

The *Riemannian curvature tensor* of the manifold (M,g) will be denoted by $\mathrm{Riem} = \mathrm{R}_{\alpha\beta\gamma\eta}\, dx^\alpha\otimes dx^\beta\otimes dx^\gamma\otimes dx^\eta$. Amongst its properties, we recall the symmetries

$$\mathrm{R}_{\alpha\beta\gamma\eta} = -\mathrm{R}_{\beta\alpha\gamma\eta} = -\mathrm{R}_{\alpha\beta\eta\gamma} = \mathrm{R}_{\eta\gamma\alpha\beta}$$

and the *Bianchi identities*

$$\mathrm{R}_{\alpha\beta\gamma\eta}+\mathrm{R}_{\alpha\gamma\eta\beta}+\mathrm{R}_{\alpha\eta\beta\gamma} = 0\,, \tag{1.1.3}$$

$$\nabla_\mu \mathrm{R}_{\alpha\beta\gamma\eta}+\nabla_\gamma \mathrm{R}_{\alpha\beta\eta\mu}+\nabla_\eta \mathrm{R}_{\alpha\beta\mu\gamma} = 0\,. \tag{1.1.4}$$

One of the main features of the Riemannian tensor, is that it allows to *interchange derivatives* using the following formula

$$\nabla^2_{\beta\alpha}\omega_\gamma = \nabla^2_{\alpha\beta}\omega_\gamma - g^{\eta\mu}\mathrm{R}_{\alpha\beta\gamma\eta}\,\omega_\mu\,, \tag{1.1.5}$$

that is fulfilled by any 1-form ω.

Thanks to the symmetries of Riem, most of its contraction are identically zero on the whole manifold. Up to sign, there is a unique non-trivial contraction of the Riemannian tensor, which is called *Ricci tensor* and is denoted by $\mathrm{Ric} = \mathrm{R}_{\alpha\beta}\,dx^\alpha\otimes dx^\beta$. To avoid the sign ambiguity we ask Ric to be positive defined when g is the spherical metric. With this convention, the Ricci tensor is defined by $\mathrm{R}_{\alpha\beta} = g^{\gamma\eta}\mathrm{R}_{\alpha\gamma\beta\eta}$. We also define the *scalar curvature* R as the function $\mathrm{tr}\,\mathrm{Ric} = g^{\alpha\beta}\mathrm{R}_{\alpha\beta}$.

Given a function $f\in\mathscr{C}^\infty(M)$, its gradient, hessian and laplacian are related to the Ricci tensor via the *Bochner formula*

$$\Delta|\nabla f|^2 = 2\,|\nabla^2 f|^2 + 2\,\mathrm{Ric}(\nabla f,\nabla f) + 2\,\langle\nabla\Delta f\,|\,\nabla f\,\rangle\,. \tag{1.1.6}$$

Tracing (1.1.4) we obtain the *contracted Bianchi identities*

$$g^{\eta\mu}\nabla_\mu \mathrm{R}_{\alpha\beta\gamma\eta}+\nabla_\alpha \mathrm{R}_{\beta\gamma}-\nabla_\beta \mathrm{R}_{\alpha\gamma} = 0\,, \tag{1.1.7}$$

$$2\,\mathrm{div}\,\mathrm{Ric}-\nabla \mathrm{R} = 0\,. \tag{1.1.8}$$

Finally, we introduce the *Weyl tensor* $\mathrm{W} = \mathrm{W}_{\alpha\beta\gamma\eta}\, dx^\alpha \otimes dx^\beta \otimes dx^\gamma \otimes dx^\eta$ defined by

$$\begin{aligned}\mathrm{W}_{\alpha\beta\gamma\eta} = \mathrm{R}_{\alpha\beta\gamma\eta} + \frac{\mathrm{R}}{(n-1)(n-2)}(g_{\alpha\gamma}g_{\beta\eta} - g_{\alpha\eta}g_{\beta\gamma}) \\ - \frac{1}{n}(\mathrm{R}_{\alpha\gamma}g_{\beta\eta} + \mathrm{R}_{\beta\eta}g_{\alpha\gamma} - \mathrm{R}_{\alpha\eta}g_{\beta\gamma} - \mathrm{R}_{\beta\gamma}g_{\alpha\eta}),\end{aligned} \quad (1.1.9)$$

where n is the dimension of the manifold M. The Weyl tensor has the same symmetries of the Riemannian curvature tensor and it is traceless, that is, any contraction of W vanishes on the whole M. Another important property of the Weyl tensor is its *conformal invariance*, namely, for any function $f \in \mathscr{C}^\infty(M)$, the Weyl tensor of fg is equal to f times the Weyl tensor of g. From the definition of the Weyl tensor and the Bianchi identity, we deduce the following formula

$$g^{\eta\mu}\nabla_\mu \mathrm{W}_{\alpha\beta\gamma\eta} = -\frac{n-3}{n-2}\mathrm{C}_{\alpha\beta\gamma}\,, \quad (1.1.10)$$

where

$$\mathrm{C}_{\alpha\beta\gamma} \,=\, \nabla_\alpha \mathrm{R}_{\beta\gamma} - \nabla_\beta \mathrm{R}_{\alpha\gamma} - \frac{1}{2(n-1)}\left(\nabla_\alpha \mathrm{R} g_{\beta\gamma} - \nabla_\beta \mathrm{R} g_{\alpha\gamma}\right)$$

is the *Cotton tensor*. In particular, it follows from (1.1.10) that $\mathrm{W} \equiv 0$ if $n \leq 3$, which means that the Weyl tensor is only relevant when the dimension is greater or equal than four. The Weyl tensor and the Cotton tensor are related to the definition of local conformal flatness, that we now recall. We say that our Riemannian manifold (M, g) is *locally conformally flat* if, for any $x \in M$, there exists a local chart $U \ni x$ such that $g_{\alpha\beta}(y) = e^{2\varphi(y)}\delta_{\alpha\beta}$ for some function $\varphi \in \mathscr{C}^\infty(U)$.

Theorem 1.1.1. *Let (M, g) be a n-dimensional Riemannian manifold.*

(i) If $n = 2$, then (M, g) is always locally conformally flat.

(ii) If $n = 3$, then (M, g) is locally conformally flat if and only if $\mathrm{C} \equiv 0$.

(iii) If $n \geq 4$, then (M, g) is locally conformally flat if and only if $\mathrm{W} \equiv 0$.

As already pointed out, in dimension $n \leq 3$ the Weyl tensor is always null. In particular, from identity (1.1.9) it follows that the Riemannian tensor is determined by the Ricci tensor. The situation is even simpler in dimension $n = 2$, where the Ricci tensor is actually forced to be a multiple of the scalar curvature. In particular, the Riemannian curvature tensor is completely determined by the scalar curvature R, which in turn is equal to twice the Gaussian curvature. We also recall that any 2-dimensional compact manifold (M, g) without boundary satisfies the well known *Gauss-Bonnet Formula*

$$\int_M \mathrm{R}\, \mathrm{d}\sigma \,=\, 4\pi\,\chi(M)\,, \quad (1.1.11)$$

which tells us that the integral of the scalar curvature on the whole manifold only depends on the topology of the manifold M, more precisely on its *Euler* *characteristic* $\chi(M)$. Identity (1.1.11) can be generalized to the case where M has a nonempty boundary or the metric g has some conical singularities. Moreover, a (more complicated) version of the Gauss-Bonnet Formula exists in all dimensions $n \geq 3$. However, in this work we will only make use of the Gauss-Bonnet Formula in its most basic form (1.1.11), and these generalizations will not be needed.

1.1.2 Hypersurfaces.

Let (M, g) be a Riemannian manifold and Σ be a smooth hypersurface. We will denote by g^Σ the metric induced by g on Σ. At any point of Σ, we can associate a unit normal vector field ν. The choice of ν at each point can be made in a continuous way, at least locally. In order to have a global continuous choice of the unit normal, one of course needs Σ to be orientable. The hypersurface Σ is called *totally geodesic* if every geodesic of (Σ, g^Σ) is also a geodesic of (M, g). The "curvature" of the embedding of Σ in M is measured by the *second fundamental form* of Σ which is defined, with respect to the chosen normal vector field ν, as

$$\mathrm{h}_{ij} \;=\; \langle \nabla_i \nu \,|\, \partial_j \rangle \,, \tag{1.1.12}$$

where the indices i, j run along the tangential coordinates to Σ. It is a well known fact that the hypersurface Σ is totally geodesic if and only if $\mathrm{h} \equiv 0$ on the whole Σ.

We also define the *mean curvature* of Σ as the trace of the second fundamental form with respect to the metric g^Σ

$$\mathrm{H} \;=\; g_\Sigma^{ij}\mathrm{h}_{ij} \;=\; \sum_{i,j=1}^{n-1} g_\Sigma^{ij}\mathrm{h}_{ij} \,. \tag{1.1.13}$$

The relation between the intrinsic and extrinsic curvature of the hypersurface Σ is governed by the *Gauss-Codazzi equation*

$$\mathrm{R} = \mathrm{R}^\Sigma + 2\,\mathrm{Ric}(\nu,\nu) + |\mathrm{h}|^2 - \mathrm{H}^2 \,, \tag{1.1.14}$$

where Ric, R are the Ricci tensor and scalar curvature of the ambient space (M, g), whereas $\mathrm{R}^\Sigma, |\mathrm{h}|$ are the scalar curvature and the second fundamental form of Σ with respect to the metric g^Σ. In particular, if Σ is totally geodesic, the above formula simplifies to

$$\mathrm{R} = \mathrm{R}^\Sigma + 2\,\mathrm{Ric}(\nu,\nu) \,. \tag{1.1.15}$$

A particular case, which will be useful in the following, is when the hypersurface Σ is a level set of a proper function $f \in \mathscr{C}^\infty(M)$. Given $s \in \mathbb{R}$, the level set $\{f = s\} \subset M$ is said to be *regular* if all its points are regular, that is, $\nabla f \neq 0$ at every point of the level set, it is said to be *critical* if it contains at least a critical point, that is a point at which $\nabla f = 0$. Suppose that, for some s and $\delta > 0$, there are no critical points of f in $(s-\delta, s+\delta)$. This means that in the tubular neighborhood $\mathcal{U}_\delta = \{s - \delta < f < s + \delta\}$ we have $|\nabla f| > 0$ so that $\mathcal{U}_\delta$ is foliated by regular level sets of f. As a consequence, $\mathcal{U}_\delta$ is diffeomorphic to $(s-\delta, s+\delta) \times \{f = s\}$ and the function f can be regarded as a coordinate in

$\mathcal{U}_\delta$. Thus, one can choose a local system of coordinates $\{f, \vartheta^1, ..., \vartheta^{n-1}\}$, where $\{\vartheta^1, ..., \vartheta^{n-1}\}$ are local coordinates on $\{f = s\}$. In such a system, the metric g can be written as

$$g = \frac{df \otimes df}{|\nabla f|^2} + g_{ij}(f, \vartheta^1, \dots, \vartheta^{n-1})\, d\vartheta^i \otimes d\vartheta^j ,$$

where the latin indices vary between 1 and $n-1$. We now fix in $\mathcal{U}_\delta$ the g-unit vector field $\nu = \nabla f / |\nabla f|$. Accordingly, the second fundamental form of the regular level sets of f with respect to the ambient metric g is given by

$$\mathrm{h}_{ij} = \frac{\nabla^2_{ij} f}{|\nabla f|} , \qquad \text{for} \quad i, j = 1, ..., n-1. \tag{1.1.16}$$

Taking the traces of the above expression we obtain the following formula for the mean curvature

$$\mathrm{H} = \frac{\Delta f}{|\nabla f|} - \frac{\nabla^2 f(\nabla f, \nabla f)}{|\nabla f|^3} . \tag{1.1.17}$$

1.1.3 Lorentzian manifolds.

In Chapter 2 we will have to work with *Lorentzian manifolds*, so here we recall their definition and their main properties. An $(n+1)$-dimensional Lorentzian manifold is a pair (X, γ) where X is a $(n+1)$-dimensional manifold and γ is a *Lorentzian metric*. This means that, at any point $p \in M$, there exists a coordinate chart $(x^0, \dots, x^n)$ centered at p such that

$$\gamma_{|_p} = - dx^0 \otimes dx^0 + \sum_{i=1}^{n} dx^i \otimes dx^i . \tag{1.1.18}$$

The scalar product induced by γ on tensors is defined as in the Riemannian case via formula (1.1.1). However, notice that, in this case, the squared norm $|T|^2_\gamma = \langle T \,|\, T \rangle_\gamma$ of a tensor T is not necessarily positive. We will say that a tensor T is

- *spacelike* if $|T|^2_\gamma > 0$,
- *timelike* if $|T|^2_\gamma < 0$,
- *null* if $|T|^2_\gamma = 0$.

For a vector field $v \in T_p M$, we will also say that v is *future-directed* or *past directed* depending on whether its x^0-component, with respect to a chart in which γ has the form (1.1.18), is positive or negative, respectively. For future convenience, we also define a *null hypersurface* as an hypersurface in (X, γ) such that its normal vector at each point is null.

The prototypical example of Lorentzian manifold is the *Minkowski space* $\mathbb{R}^{n,1}$, that is the space $\mathbb{R}^{n+1}$ equipped with the Lorentzian metric

$$\gamma_{\mathbb{R}^{n,1}} = -dx^0 \otimes dx^0 + \sum_{i=1}^{n} dx^i \otimes dx^i ,$$

where $(x^0, \ldots, x^n)$ are global coordinates on $\mathbb{R}^{n+1}$. To clarify the geometry of the Minkowski space (which is in fact the local geometry of any Lorentzian manifold), we observe that, if we shoot geodesics from the origin $0 \in \mathbb{R}^{n,1}$ in the null directions (in other words, we consider the union of all the geodesics departing from the origin and whose velocity is a null vector), we obtain the *light cone*

$$\left\{ (x^0)^2 = (x^1)^2 + \ldots + (x^n)^2 \right\} \subset \mathbb{R}^{n,1} .$$

It is clear by construction that the cone minus the origin is a smooth null hypersurface. The vectors in $T_0(\mathbb{R}^{n,1})$ that point towards the interior of the cone

$$(x^0)^2 > (x^1)^2 + \ldots + (x^n)^2 ,$$

are timelike, whereas the vectors that point towards the exterior of the cone are spacelike. The isometries of the Minkowski space are the so called Lorentzian transformation, and they preserve the attributes (timelike, spacelike, null, future-directed, past-directed) of the vectors. For future convenience, we also state the following easy property of null vectors.

Proposition 1.1.2. *Let (X, γ) be a $(n+1)$-dimensional Lorentzian manifold, let $p \in X$ be a point and let $v, w \in T_pX$ be two null vectors, $v \neq 0$, $w \neq 0$. If v and w are orthogonal, then they are necessarily proportional.*

Proof. The result that we want to prove is local, so we can choose coordinates $(x^0, x^1, \ldots, x^n)$ such that the metric γ at p looks like the Minkowski metric, that is

$$\gamma_{|_p} = -dx^0 \otimes dx^0 + dx^1 \otimes dx^1 + \cdots + dx^n \otimes dx^n$$

With respect to the same coordinates, we write

$$v = (v^0, \ldots, v^n) , \qquad w = (w^0, \ldots, w^n) ,$$

and, since v and w are null vectors, we have

$$(v^0)^2 = (v^1)^2 + \cdots + (v^n)^2 , \qquad (w^0)^2 = (w^1)^2 + \cdots + (w^n)^2 . \tag{1.1.19}$$

Moreover, v and w are orthogonal, hence

$$v^0 w^0 = \sum_{i=1}^{n} v^i w^i . \tag{1.1.20}$$

Recalling the Cauchy-Schwartz Inequality we compute

$$(v^0 w^0)^2 = \left(\sum_{i=1}^{n} v^i w^i \right)^2 \leq \left[\sum_{i=1}^{n} (v^i)^2 \right] \left[\sum_{i=1}^{n} (w^i)^2 \right] = (v^0 w^0)^2 ,$$

where in the latter identity we have used (1.1.19). Therefore, equality must hold in the Cauchy-Schwartz Inequality, which means that $(v^1, \ldots, v^n)$ and $(w^1, \ldots, w^n)$ are proportional. In other words, there exists $\lambda \in \mathbb{R}$ such that $w^i = \lambda v^i$ for all $i = 1, \ldots, n$. Substituting in (1.1.19), we find that $|w^0| = \lambda v^0$.

In particular, necessarily $\lambda \neq 0$, otherwise we would have $w = 0$, against the hypothesis. To conclude, it is enough to show that w^0 cannot be equal to $-\lambda v^0$. In fact, if this were the case, from (1.1.20) we would have

$$-(v^0)^2 \;=\; \sum_{i=1}^{n}(v^i)^2\,,$$

which would imply $v = 0$, against the hypothesis. □

1.1.4 Analytic functions.

In the following chapters, we will encounter analytic functions quite frequently. In particular, for our analysis, it will be crucial to have control over the behavior of their critical points. In this respect, we will repeatedly make use of the results discussed in this subsection.

First of all, let us recall the definition of a real analytic function on a differentiable manifold M. An *analytic covering* of M is a family of differentiable charts (U_i, ϕ_i) such that $M = \bigcup_i U_i$ and, if $U_i \cap U_j \neq \emptyset$ for some $i \neq j$, then the change of chart $\phi_i \circ \phi_j^{-1}$ is an analytic function. A metric g on M is said to be *analytic* if there exists an analytic covering (U_i, ϕ_i) of M such that the pull-back metric $(\phi_i^{-1})^* g$ is analytic on $\mathbb{R}^n$ for all i. Analogously, a function $f : M \to \mathbb{R}$ is said to be *analytic* if there exists an analytic covering (U_i, ϕ_i) such that $f \circ \phi_i^{-1} : \mathbb{R}^n \to \mathbb{R}$ is analytic for every i. Given a function $f \in \mathscr{C}^\infty(M)$, its critical set is defined as $\mathrm{Crit}(f) = \{x \in M \,:\, df = 0\}$, while its critical values are the numbers $x \in \mathbb{R}$ such that $f^{-1}(x) \cap \mathrm{Crit}(f) \neq \emptyset$. The following results allow to characterize nicely the set $\mathrm{Crit}(f)$ of an analytic function. Their proof is given in $\mathbb{R}^n$, however they generalize without modifications to differentiable manifolds.

Theorem 1.1.3 ([SS72, Theorem 1])**.** *The set of the critical values of an analytic function is discrete.*

Theorem 1.1.4 (Łojasiewicz Structure Theorem [KP02, Theorem 6.3.3], [Łoj91])**.** *Let f be an analytic function on M. Then its critical set is a stratified space, that is*

$$\mathrm{Crit}(f) \;=\; N^0 \sqcup N^1 \sqcup \cdots \sqcup N^{n-2} \sqcup N^{n-1}\,,$$

where, for all $i = 0, \ldots, n-1$, N^i is a finite union of connected analytic i-dimensional submanifolds.

As a consequence of the theorems above, we have the following structure on the level sets of an analytic function.

Proposition 1.1.5. *Let f be an analytic function on a Riemannian manifold (M, g). Then for any $s \in \mathbb{R}$ the corresponding level set decomposes as*

$$\{f = s\} \;=\; \Sigma \sqcup \Gamma\,,$$

where Σ is a $(n-1)$-dimensional smooth submanifold and Γ is a set with finite $\mathscr{H}^{n-2}$-measure. In particular, the second fundamental form and mean curvature of the level set $\{f = s\}$ are defined $\mathscr{H}^{n-1}$-almost everywhere.

Proof. If s is a regular value, then $\{f = s\}$ is a $(n-1)$-dimensional submanifold and there is nothing to prove. If s is a critical value, then from Theorem 1.1.3 we have that, for a small enough δ, it holds

$$\mathrm{Crit}(f) \cap \{s - \delta < f < s + \delta\} = \mathrm{Crit}(f) \cap \{f = s\}\,. \tag{1.1.21}$$

Thanks to Theorem 1.1.4, we have that there exists an $(n-1)$-dimensional submanifold $S \subseteq \mathrm{Crit}(f)$ such that $\mathscr{H}^{n-1}(\mathrm{Crit}(f) \setminus S) = 0$. From (1.1.21), we deduce that $S \cap \{f = s\}$ is still an $(n-1)$-dimensional submanifold. The thesis follows by setting

$$\Sigma = (S \cap \{f = s\}) \cup (\{f = s\} \setminus \mathrm{Crit}(f))\,, \qquad \Gamma = \{f = s\} \setminus \Sigma\,. \qquad \square$$

1.1.5 Łojasiewicz and reverse Łojasiewicz inequalities.

As a consequence of the Structure Theorem 1.1.4, Łojasiewicz was able to prove the following estimate on the behavior of an analytic function near a critical point.

Theorem 1.1.6 (Łojasiewicz inequality [Łoj63, Théorème 4], [KP94])**.** *Let (M, g) be a Riemannian manifold and let $f : M \to \mathbb{R}$ be an analytic function. Then for every point $p \in M$ there exists a neighborhood $U_p \ni p$ and real numbers $c_p > 0$ and $1 \leq \theta_p < 2$ such that for every $x \in U_p$ it holds*

$$|\nabla f|^2(x) \,\geq\, c_p\, |f(x) - f(p)|^{\theta_p}\,. \tag{1.1.22}$$

First of all, let us observe that the above theorem is only relevant when p is a critical point, as otherwise the proof is trivial. Another observation is that one can always set $c_p = 1$ in (1.1.22), at the cost of increasing the value of θ_p and restricting the neighborhood U_p. Nevertheless, the inequality is usually stated as in (1.1.22), because one would like to choose the optimal θ_p and not a bigger one. In general it is not easy to infer what the optimal value of θ_p is. However, for our purposes, the loose inequality

$$|\nabla f|(x) \,\geq\, |f(x) - f(p)|\,,$$

will be sufficient. Theorem 1.1.6 grants us that such an inequality holds in a neighborhood of each point p of our manifold M.

The gradient estimate (1.1.22) has found important applications in the study of gradient flows, as it allows to control the behavior of the flow near the critical points. It is also worth mentioning that an infinite-dimensional version of (1.1.22) has been proved by Simon [Sim83], who then used it to study the asymptotic behavior of parabolic equations near the critical points. For a thorough discussion of the various versions of the Łojasiewicz–Simon inequality, as well as for its applications, we refer the reader to [FM15] and the references therein.

In the future, we will also need an analogue estimate from above of the gradient near the critical points. The following result shows that such a bound can be obtained around the local maxima (or local minima, of course) under opportune assumptions.

Theorem 1.1.7 (Reverse Łojasiewicz). *Let (M, g) be a Riemannian manifold, let $f : M \to \mathbb{R}$ be a smooth function and let $p \in M$ be a local maximum point such that*

(i) $|\nabla^2 f|(p) \neq 0$,

(ii) the set $\{x \in M : f(x) = f(p)\}$ *is compact.*

Then for every $\theta < 1$, there exists a neighborhood $U_p \ni p$ and a real number $c_p > 0$ such that for every $x \in U_p$ it holds

$$|\nabla f|^2(x) \,\leq\, c_p \left[f(p) - f(x)\right]^\theta . \tag{1.1.23}$$

Proof. Let us start by defining the function

$$w \,=\, |\nabla f|^2 - c\,[f(p) - f]^\theta ,$$

where $c > 0$ is a constant that will be chosen conveniently later. We compute

$$\nabla w \,=\, \nabla|\nabla f|^2 \,+\, c\,\theta\,[f(p) - f]^{-(1-\theta)}\,\nabla f ,$$

and diverging the above formula

$$\begin{aligned}\Delta w \,&=\, \Delta|\nabla f|^2 \,+\, c\,\theta\,[f(p) - f]^{-(1-\theta)}\Delta f \,+\, c\,\theta\,(1-\theta)\,[f(p) - f]^{-(2-\theta)}\,|\nabla f|^2 \\ &=\, 2\,|\nabla^2 f|^2 \,+\, 2\,h \,+\, c\,\theta\,\frac{\Delta f}{[f(p) - f]^{1-\theta}} \,+\, c\,\theta\,(1-\theta)\,\frac{|\nabla f|^2}{[f(p) - f]^{2-\theta}} ,\end{aligned}$$

where in the second equality we have used the Bochner formula and we have denoted by h the quantity $h = \mathrm{Ric}(\nabla f, \nabla f) + \langle \nabla\Delta f | \nabla f\rangle$. Since $|\nabla f|$ goes to zero as we approach p, so does h. Moreover, from the hypothesis we have $|\nabla^2 f|^2 > 0$ in a neighborhood of p. Since $\{f = f(p)\}$ is compact by hypothesis, it follows that we can choose a neighborhood U of $\{f = f(p)\}$ such that $\overline{U}$ is compact and

$$|h| \,\leq\, |\nabla^2 f|^2 \quad \text{on } U .$$

Therefore, from the identity above we find that in U it holds

$$\begin{aligned}\Delta w \,&\geq\, c\,\theta\,\frac{\Delta f}{[f(p) - f]^{1-\theta}} \,+\, c\,\theta\,(1-\theta)\,\frac{|\nabla f|^2}{[f(p) - f]^{2-\theta}} \\ &=\, c\,\theta\,\frac{\Delta f}{[f(p) - f]^{1-\theta}} \,+\, c\,\theta\,(1-\theta)\,\frac{w}{[f(p) - f]^{2-\theta}} \,+\, c^2\theta\,(1-\theta)\,\frac{1}{[f(p) - f]^{2-2\theta}} ,\end{aligned}$$

where in the second equality we have used $|\nabla f|^2 = w + c\,[f(p) - f]^\theta$. It follows that, on U, it holds

$$\Delta w \,-\, c\,\theta\,(1-\theta)\,\frac{1}{[f(p) - f]^{2-\theta}}\,w \,\geq\, \theta\,F\,\left[\Delta f \,+\, (1-\theta)\,F\right] , \tag{1.1.24}$$

where

$$F = \frac{c}{[f(p) - f]^{1-\theta}} .$$

Now fix $0 < \eta < f(p) - \max_{\partial U} f$ and consider the set

$$V_\eta = \{f(p) - f \leq \eta\} \cap U$$

The neighborhood V_η is chosen in such a way that

$$F = \frac{c}{[f(p) - f]^{1-\theta}} \geq \frac{c}{\eta^{1-\theta}}, \quad \text{on } V_\eta .$$

Moreover, Δf is continuous and thus bounded in V_η. This means that, for any c big enough, we have $(1-\theta)F + \Delta f \geq 0$ on the whole V_η. For such values of c, the right hand side of (1.1.24) is nonnegative, that is,

$$\Delta w - \theta(1-\theta)c\frac{1}{[f(p) - f]^{2-\theta}}w \geq 0, \quad \text{in } V_\eta .$$

Therefore, since $\theta < 1$, we can apply the Weak Maximum Principle [GT83, Corollary 3.2] to w in any open set where w is $\mathscr{C}^2$ – that is, on any open set of V_η that does not intersect $\{f = f(p)\}$ –. For this reason, it is convenient to consider the set $V_{\varepsilon\eta} = V_\eta \cap \{f(p) - f \geq \varepsilon\}$ for some $0 < \varepsilon < \eta$. Up to increasing the value of c, if needed, we can suppose

$$c \geq \max_{\{f = f(p) - \eta\}} \frac{|\nabla f|^2}{[f(p) - f]^\theta} = \frac{\max_{\{f = f(p)-\eta\}} |\nabla f|^2}{\eta^\theta},$$

so that $w \leq 0$ on $\{f = f(p) - \eta\}$. Now we apply the Weak Maximum Principle to the function w on the open set $V_{\varepsilon\eta}$, obtaining

$$w \leq \max_{\partial V_{\varepsilon\eta}}(w) = \max\left\{\max_{\{f=f(p)-\varepsilon\}}(w), \max_{\{f=f(p)-\eta\}}(w)\right\} \leq \max\left\{\max_{\{f=f(p)-\varepsilon\}}(w), 0\right\}.$$

Recalling the definition of w, taking the limit as $\varepsilon \to 0$, from the continuity of f and $|\nabla f|$, it follows that

$$\lim_{\varepsilon \to 0} \max_{\{f=f(p)-\varepsilon\}}(w) = 0,$$

hence we obtain $w \leq 0$ on $\{f(p) - \eta \leq f \leq f(p)\}$. Translating w in terms of f, we have obtained that the inequality

$$|\nabla f|^2 \leq c\,[f(p) - f]^\theta$$

holds in $U_p = \{f(p) - \eta \leq f < f(p)\}$, which is the neighborhood of p that we were looking for. □

At the moment, we do not know if the hypotheses of the above result can be relaxed, this will be the object of further investigations. However, Theorem 1.1.7, as stated, is enough for our purposes. In particular, what we will really need is the following simple refinement.

Corollary 1.1.8. *In the same hypotheses of Theorem 1.1.7, for any $\alpha < 1$ we have*

$$\lim_{f(x) \neq f(p),\, x \to p} \frac{|\nabla f|^2(x)}{[f(p) - f(x)]^\alpha} = 0.$$

Proof. From Theorem 1.1.7 it follows that we can choose constants $\alpha < \theta < 1$ and $c_p > 0$ such that

$$\frac{|\nabla f|^2}{[f(p) - f]^\alpha} \leq \frac{c_p\,[f(p) - f]^\theta}{[f(p) - f]^\alpha} = c_p\,[f(p) - f]^{\theta - \alpha},$$

and, since we have chosen $\theta > \alpha$, the right hand side goes to zero as we approach p. This proves the thesis. □

1.1.6 Divergence Theorem and Maximum Principle.

In this subsection we review two classical tools that will play an important role in our analysis. The first one is the well known Divergence Theorem. Several versions of this result are known in the literature. Here we report the statement, proved independently by De Giorgi and Federer, in the case of open domains whose boundary has a (not too big) nonsmooth portion.

Theorem 1.1.9 ([DG61a, DG61b, Fed45, Fed58])**.** *Let (M, g) be a n-dimensional Riemannian manifold, with $n \geq 2$, let $E \subset M$ be a bounded open subset of M with compact boundary ∂E of finite $(n-1)$-dimensional Hausdorff measure, and suppose that $\partial E = \Sigma \sqcup \Gamma$, where the subsets Σ and Γ have the following properties:*

(i) For every $x \in \Sigma$, there exists an open neighborhood U_x of x in M such that $\Sigma \cap U_x$ is a smooth regular hypersurface.

(ii) The subset Γ is compact and $\mathscr{H}^{n-1}(\Gamma) = 0$.

If X is a Lipschitz vector field defined in a neighborhood of $\bar{E}$ then the following identity holds true

$$\int_E \operatorname{div} X \,\mathrm{d}\mu = \int_\Sigma \langle X \,|\, \nu \rangle \,\mathrm{d}\sigma, \tag{1.1.25}$$

where ν denotes the exterior unit normal vector field.

This result admits a more general statement that allows for a less regular frontier using the notion of *reduced boundary*. Other possible extensions of the Divergence Theorem, for instance to less regular vector fields, are possible. The statement above is enough for our purposes, but for a more thorough discussion we refer the reader to the recent paper [CDC17], where the various versions of (1.1.25) are summarized while also providing a general unifying statement. We also remark that, as a consequence of Theorem 1.1.9, recalling Proposition 1.1.5, it follows that the Divergence Theorem is always in force when integrating the divergence of a Lipschitz vector field between any two level sets of an analytic function.

We pass now to recall the statement of the classical Maximum Principle. On a differentiable manifold M we consider operators of the form

$$Lu = a^{ij}\nabla^2_{ij}u + b^i\nabla_i u + cu \tag{1.1.26}$$

with $a^{ij}, b^i, c \in \mathscr{C}^\infty(M)$. We will assume that the operator L is *uniformly elliptic* and the coefficients a^{ij}, b_i, c are bounded, namely there exists a constant $\lambda > 0$ such that

$$\lambda^{-1}|v|^2 \le a^{ij}(x)v_i v_j \le \lambda |v|^2, \quad b^i(x) \le \lambda, \quad c(x) \le \lambda, \tag{1.1.27}$$

for all $x \in M$, $v \in T_x M$, $i, j = 1, \dots, n$. This family of operators satisfy the Weak Maximum Principle, which states that a $\mathscr{C}^2$ function u that satisfies $Lu \ge 0$ on a domain $\Omega \subset M$, necessarily achieves its maximum value on the boundary of the domain. A similar property, usually referred to as the Weak Minimum Principle, holds for functions u that satisfy $Lu \le 0$. A stronger version is the so called Strong Maximum (or Minimum) Principle, that we now state.

Theorem 1.1.10 (Strong Maximum –and Minimum– Principle, [GT83, Theorem 3.5])**.** *Let M be a differentiable manifold, let $\Omega \subset M$ be a domain and let $u \in \mathscr{C}^2(\Omega) \cap \mathscr{C}^0(\overline{\Omega})$ be a solution of $Lu \ge 0$ (respectively, $Lu \le 0$), where L is an operator of the form* (1.1.26) *satisfying conditions* (1.1.27)*.*

- *If $c = 0$, then u cannot achieve a maximum (respectively, a minimum) in the interior of Ω, unless u is constant.*

- *If $c \le 0$, then u cannot achieve a nonnegative maximum (respectively, a nonpositive minimum) in the interior of Ω, unless u is constant.*

We remark that the above version of the Maximum and Minimum Principles is not the most general (see for instance [GT83, Theorem 8.19] for an analogue statement for weakly differentiable functions), but it will be enough for our purposes.

1.1.7 Conformal metrics.

In Chapter 4 we will deal with two Riemannian metrics g_0, g on a differentiable manifold M that are *conformal*, that is, $g = e^{2f} g_0$ for some function $f \in \mathscr{C}^\infty(M)$. In that case, we will denote by $\mathrm{G}^\gamma_{\alpha\beta}$, D, D^2, Δ, $\langle \cdot \,|\, \cdot \rangle$, $| \cdot |$ and $\Gamma^\gamma_{\alpha\beta}$, ∇, ∇^2, Δ_g, $\langle \cdot \,|\, \cdot \rangle_g$, $| \cdot |_g$ the Christoffel symbols, covariant derivative, hessian, laplacian, scalar product, norm with respect to g_0 and g, respectively. The Ricci tensors and scalar curvatures of g_0, g will be denoted by $\mathrm{Ric} = \mathrm{R}_{\alpha\beta} dx^\alpha \otimes dx^\beta$, R and $\mathrm{Ric}_g = \mathrm{R}^{(g)}_{\alpha\beta} dx^\alpha \otimes dx^\beta$, R_g. Finally, given an hypersurface $\Sigma \subset M$, its second fundamental form and mean curvature with respect to g_0, g will be denoted by $\mathrm{h} = \mathrm{h}_{ij} dx^i \otimes dx^j$, H and $\mathrm{h}_g = \mathrm{h}^{(g)}_{ij} dx^i \otimes dx^j$, H_g respectively. The following result summarize the most important relationships between these objects with respects to the two different metrics.

Theorem 1.1.11 ([Bes08, Theorem 1.159])**.** *Let M be a differentiable manifold and let g_0 and g be two Riemannian metrics on M, such that $g = e^{2f} g_0$. Then,* *with the notations introduced above, the following formulæ hold*

$$\Gamma_{\alpha\beta}^{\gamma} = G_{\alpha\beta}^{\gamma} + \left[\delta_{\alpha}^{\gamma}\frac{\partial f}{\partial x^{\beta}} + \delta_{\beta}^{\gamma}\frac{\partial f}{\partial x^{\alpha}} - g_0^{\gamma\eta}(g_0)_{\alpha\beta}\frac{\partial f}{\partial x^{\eta}}\right], \tag{1.1.28}$$

$$\nabla^2_{\alpha\beta} w = D^2_{\alpha\beta} w - \left[\frac{\partial w}{\partial x^{\alpha}}\frac{\partial f}{\partial x^{\beta}} + \frac{\partial f}{\partial x^{\alpha}}\frac{\partial w}{\partial x^{\beta}} - \langle Df \,|\, Dw\rangle\, (g_0)_{\alpha\beta}\right], \tag{1.1.29}$$

$$\Delta_g w = e^{-2f}\left[\Delta w + (n-2)\langle Df \,|\, Dw\rangle\right] \,, \tag{1.1.30}$$

$$\mathrm{Ric}_g = \mathrm{Ric} - (n-2)\left(D^2 f - df \otimes df\right) - \left[\Delta f + (n-2)|Df|^2\right] g_0\,, \tag{1.1.31}$$

$$R_g = e^{-2f}\left[R - 2(n-2)\Delta f - (n-1)(n-2)|Df|^2\right]. \tag{1.1.32}$$

1.2 Mass

As discussed in the Introduction, Riemannian manifolds arise naturally in General Relativity as time symmetric initial data sets modelling isolated systems. In this context, it is then natural to ask if it is possible to introduce on them a suitable notion of mass. One of the most known is the ADM mass, introduced by Arnowitt, Deser and Misner for asymptotically flat manifolds. This definition is discussed in Subsection 1.2.1, where we also recall some of its most important properties, such as the Positive Mass Theorem and the Riemannian Penrose Inequality.

In Subsection 1.2.2 we will instead focus on asympotically hyperbolic manifolds. Despite some more complications in the analysis, a very similar procedure leads to a definition of mass which again satisfies a Positive Mass Theorem. The validity of a form of Riemannian Penrose Inequality is still an open question, even though some first results have been found.

Having in mind the asymptotically flat and asymptotically hyperbolic case, it is then natural to ask if an analogue of the Positive Mass Theorem holds in a spherical setting. This led to a famous conjecture by Min-Oo, which we recall in Subsection 1.2.3. This conjecture was thought to be true for a long time, until it was finally disproven by Brendle, Marquez and Neves.

1.2.1 Asymptotically flat manifolds.

In this subsection we recall the main properties of the well known family of *asymptotically flat manifolds*. These are noncompact Riemannian manifolds whose metric behave asymptotically as the flat metric. The precise definition follows. We precise that *from now on we will work only on manifolds of dimension $n \geq 3$*. We mention that an analogue notion (*asymptotically conical surface*) is available in dimension 2 (see for instance the discussion in [Chr10, Subsection 1.1.1] or [CDL16, Section 2.3]), but it requires a different study and we will not be interested in it.

Definition 1.2.1 (Asymptotically flat manifolds). *A n-dimensional Riemannian manifold (M, g) is said to be* asymptotically flat *if the following asymptotic behaviors are satisfied*

(i) There exists a compact set $K \subset M$ such that, for each connected component U of $M \setminus K$, there exists a diffeomorphism $x^{(U)} = (x^1, \ldots, x^n) : U \to \mathbb{R}^n \setminus \mathbb{B}^n$, where $\mathbb{B}^n$ is the unit ball. The component U is called end *of M, whereas the diffeomorphism $x^{(U)}$ is called* chart at infinity *of U.*

(ii) In each connected component U of $M \setminus K$, the metric g can be expressed with respect to the coordinates induced by the diffeomorphism $x^{(U)}$ as

$$g_{\alpha\beta} = \delta_{\alpha\beta} + \eta_{\alpha\beta} ,$$
$$\text{with } \eta_{\alpha\beta} = o(|x|^{\frac{2-n}{2}}) , \ \frac{\partial \eta_{\alpha\beta}}{\partial x^\gamma} = o(|x|^{-\frac{n}{2}}) , \ \frac{\partial^2 \eta_{\alpha\beta}}{\partial x^\gamma x^\sigma} = o(|x|^{\frac{-2-n}{2}}) ,$$

for every $\alpha, \beta, \gamma, \sigma = 1, \ldots, n$.

(iii) The scalar curvature $\mathrm{R} \in \mathscr{C}^\infty(M)$ is an integrable function.

As already remarked, asymptotically flat manifolds play an important role in the study of isolated systems in spacetimes with zero cosmological constant, see also Section 2.3 for a discussion in the special case of static spacetimes. One of their main features is that on them it is possible to introduce a meaningful notion of mass.

Definition 1.2.2 (ADM mass [ADM62]). *Let (M, g) be an asymptotically flat manifold, and let U be one of its ends. The ADM mass of U is defined as*

$$m^{(U)}_{ADM}(M, g) = \frac{1}{2(n-1)|\mathbb{S}^{n-1}|} \lim_{R \to +\infty} \int_{S_R} \sum_{\alpha,\beta} \left(\frac{\partial g_{\alpha\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\alpha}}{\partial x^\beta} \right) \nu^\beta \, \mathrm{d}\sigma , \qquad (1.2.1)$$

where $x^1, \ldots, x^n$ are the coordinates induced by the chart at infinity of the end U, and $S_R = \{|x| = R\}$ is the coordinate sphere of radius R.

First of all, it is important to remark that Definition 1.2.2 is well posed. In fact, it has been proved by Bartnik [Bar86] that the right hand side of (1.2.1) is independent of the choice of the chart at infinity. For the physical motivations behind this definition, we refer the reader to the original paper [ADM62]. In what follows we will focus on discussing its properties. First of all, the next celebrated result states that the ADM mass is always nonnegative.

Theorem 1.2.3 (Positive Mass Theorem for Asymptotically Flat Manifolds). *Let (M, g) be an asymptotically flat manifold with empty boundary and nonnegative scalar curvature. Then for every end U of M it holds*

$$m^{(U)}_{ADM} \geq 0 .$$

Moreover, if the equality holds at one end, then M has a unique end and it is isometric to the Euclidean space.

The theorem above was first proven by Schoen and Yau in dimensions $3 \leq n \leq 7$ in [SY79] using minimal surfaces as a fundamental tool. The problem in dealing with dimensions greater than 7 was that, in principle, the minimal surfaces could become singular. A second different proof for $n = 3$ was provided by Witten in [Wit81], and this approach was shown to be easily generalizable to prove Theorem 1.2.3 for spin manifolds in any dimension $n \geq 3$. Only recently, Schoen and Yau [SY17] were able to extend their proof to every dimension $n \geq 3$, exploiting some tricks in order to deal with the formation of singularities on the minimal hypersurfaces that they were considering. Therefore, the proof of the validity of the Positive Mass Theorem for any $n \geq 3$ can now be considered complete.

The natural follow up is then to ask if Theorem 1.2.3 can be refined in the case where ∂M has a nonempty boundary. This is the content of the well known Riemannian Penrose Inequality, which we now state.

Theorem 1.2.4 (Riemannian Penrose Inequality)**.** *Let (M, g) be a Riemannian manifold of dimension $3 \leq n \leq 7$ with nonnegative scalar curvature. Suppose that (M, g) is asymptotically flat with one end with ADM mass m_{ADM}. If the boundary ∂M is a smooth compact (possibly disconnected) outermost minimal hypersurface, then*

$$m_{ADM} \geq \frac{1}{2}\left(\frac{|\partial M|}{|\mathbb{S}^{n-1}|}\right)^{\frac{n-2}{n-1}} . \tag{1.2.2}$$

Moreover, if the equality holds, then (M, g) is isometric to the Schwarzschild solution

$$\left(\mathbb{R}^n \setminus \left\{|x| < (2\, m_{ADM})^{\frac{1}{n-2}}\right\},\ \frac{d|x| \otimes d|x|}{1 - 2\, m_{ADM}\, |x|^{2-n}} + |x|^2 g_{\mathbb{S}^{n-1}}\right) . \tag{1.2.3}$$

With the requirement that ∂M is *outermost* we mean that there are no other compact minimal surfaces inside M. This hypothesis cannot be removed, as otherwise it would be easy to find counterexamples to (1.2.2) by constructing arbitrarily large minimal surfaces “hidden” behind the outermost one. The Schwarzschild solution (1.2.3) appearing in the above result as the rigidity case will be discussed in more details in Chapter 2, in particular in Section 2.3, where we will talk about static spacetimes with zero cosmological constant. Theorem 1.2.4 was first proven in dimension 3 by Huisken and Ilmanen [HI01] in the case of a connected boundary (their proof works also for disconnected boundaries, but gives a weaker inequality). The proof is based on an idea of Geroch [Ger73] and Jang, Wald [JW77], which consists in exploiting the monotonicity of the *Hawking mass*

$$m_H(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}}\left[1 - \frac{1}{16\pi}\int_\Sigma \mathrm{H}^2\, d\sigma\right]$$

of a surface $\Sigma \subset M$, under inverse mean curvature flow of Σ. The case of a disconnected boundary was then proved by Bray [Bra01] using a different approach, based on a flow of conformal metrics and a clever application of the Positive Mass Theorem. While the proof of Huisken-Ilmanen is essentially 3-dimensional, the approach of Bray has the advantage to be generalizable to higher dimensions. In fact, using the same tecnique, Bray and Lee [BL09] extended Theorem 1.2.4 to dimensions $3 \leq n \leq 7$. We are limited to dimensions less than 7, because the proof exploits the regularity of the minimal surfaces, which was also the reason why the original proof of Schoen and Yau of the Positive Mass Theorem only worked up to dimension 7. Since the issue of the regularity in the proof of Schoen and Yau has been resolved, one may wonder if it is possible to overcome the problem also in the proof of Theorem 1.2.4. For a more thorough discussion on the different proofs of the Riemannian Penrose Inequality, as well as on the partial results on the more general Penrose Inequality, we refer the reader to the survey [Mar09].

Finally, we mention that a generalization of the Positive Mass Theorem to nonsmooth metrics has been proven by Miao [Mia02] for any dimension in which the original Positive Mass Theorem works (hence now for any dimension $n \geq 3$).

Theorem 1.2.5 (Positive Mass Theorem with corners)**.** *Let (M, g) be a n-dimensional Riemannian manifold, $n \geq 3$, with one end. Let $\Sigma = \partial\Omega \subset M$ be a closed hypersurface enclosing a domain Ω. Suppose that the following conditions hold.*

(i) The restrictions g_-, g_+ of the metric g to Ω and to $M \setminus \overline{\Omega}$ are smooth, they are $\mathscr{C}^2$ up to the boundary Σ, and they induce the same metric on Σ.

(ii) Both g_+ and g_- have nonnegative scalar curvature and $(M \setminus \Omega, g_+)$ is asymptotically flat with one end with ADM mass m_{ADM}.

(iii) It holds
$$\mathrm{H}_-(p) \geq \mathrm{H}_+(p)\,, \qquad \text{for every } p \in \Sigma\,, \tag{1.2.4}$$
where $\mathrm{H}_-(p), \mathrm{H}_+(p)$ are the mean curvatures at $p \in \Sigma$ of g_-, g_+ with respect to the unit normal vector pointing towards $M \setminus \Omega$.

Then $m_{ADM} \geq 0$. Moreover, if $m_{ADM} = 0$, then (M, g) is isometric to the Euclidean space.

It should be noticed that the rigidity statement when $m_{ADM} = 0$ in Theorem 1.2.5 was proved by Miao only in dimension $n = 3$ exploiting a result of Bray and Finster [BF02], whereas in dimensions $n > 3$ his proof allowed only to deduce that the equality held in (1.2.4). The proof of the rigidity statement in arbitrary dimension was given by Shi and Tam [ST02] for spin manifolds and the spin assumption was dropped in [EMW12] and in [MS12], where an alternative proof of Theorem 1.2.5 using Ricci flow was given. As an immediate consequence of Theorem 1.2.5, Miao showed the following characterization of the Euclidean ball.

Theorem 1.2.6. *Let g be a smooth metric on the unit ball $\mathbb{B}^n \subset \mathbb{R}^n$. If the following properties are satisfied*

(i) the scalar curvature of g is nonnegative on the whole $\mathbb{B}^n$,

(ii) the metric induced by g on $\partial\mathbb{B}^n = \mathbb{S}^{n-1}$ is the standard round metric,

(iii) the mean curvature of $\partial\mathbb{B}^n = \mathbb{S}^{n-1}$ with respect to g is greater than or equal to $n - 1$,

then g is the Euclidean metric on $\mathbb{B}^n$.

This theorem is simply proven by gluing the manifold $(\mathbb{R}^n \setminus \mathbb{B}^n, g_{\mathbb{R}^n})$ to the boundary of $\mathbb{B}^n$, and then applying Theorem 1.2.5.

We also point out that other more general versions of the Positive Mass Theorem for singular metrics have been proved. For instance, Lee [Lee13] has shown the nonnegativity of the ADM mass of nonsmooth metrics under the hypothesis that the singular set has codimension at least $n/2$. Other recent versions of the Positive Mass Theorem for singular metrics are discussed in [LL15, ST16], see also [Bra11].

1.2.2 Asymptotically hyperbolic manifolds.

In this subsection we are going to discuss *asymptotically hyperbolic manifolds*. As anticipated in the Introduction, these are manifolds which modelize isolated systems in spacetimes with a negative cosmological constant. In order to introduce the definition of this family, we start by recalling the following classical definition, originally introduced by Penrose in [Pen63] (see also [HM14] and the references therein). Again, we recall that, even if not explicitly stated, we will only work with manifolds of dimension greater or equal than 3.

Definition 1.2.7. *A Riemannian manifold (M, g) is said to be* conformally compact *if the following conditions are satisfied*

(i) *The manifold $M \setminus \partial M$ is diffeomorphic to the interior of a compact manifold $\overline{M}$ with boundary $\partial\overline{M} = \partial M \cup \partial_\infty M$, with $\partial_\infty M \cap M = \emptyset$,*

(ii) *There exists a compact $\partial M \subset K \subset M$ and a function $r \in \mathscr{C}^\infty(\overline{M} \setminus K)$ such that $r \neq 0$ on M, $r = 0$ on $\partial_\infty M$, $dr \neq 0$ on $\partial_\infty M$ and the metric $\bar{g} = r^2 g$ extends smoothly to a metric on $\overline{M} \setminus K$.*

We will refer to a function with the same properties of r in (ii) as a *defining function* for the *conformal boundary* $\partial_\infty M$. It should be noticed that the defining function for $\partial_\infty M$ is not unique, but it is defined up to the multiplication by a positive function. Consequently, the metric $\bar{g}_{|_{\partial_\infty M}}$ on the conformal boundary $\partial_\infty M$ is not well defined, but its conformal class $[\bar{g}_{|_{\partial_\infty M}}]$ is. Therefore, the pair $(\partial_\infty M, [\bar{g}_{|_{\partial_\infty M}}])$ is a well defined conformal manifold, that we will call *conformal infinity* of (M, g). Notice that the conformal infinity has no reasons to be connected. However, it is interesting to remark that connectedness is granted under suitable hypotheses on the Ricci tensor and the scalar curvature.

Theorem 1.2.8 ([WY99, CG99])**.** *Let (M, g) be a conformally compact manifold such that*

$$\mathrm{Ric} + n\, g \geq 0\,, \qquad \mathrm{R} = -\, n\,(n-1) + o(r^2)$$

where r is a defining function for $\partial_\infty M$. If there exists a connected component Σ of the conformal infinity $\partial_\infty M$ such that $\mathrm{R}^\Sigma_{\bar{g}} \geq 0$ on the whole Σ, where $\mathrm{R}^\Sigma_{\bar{g}}$ is the scalar curvature of the metric induced by $\bar{g} = r^2 g$ on Σ, then $\partial_\infty M = \Sigma$, that is, $\partial_\infty M$ is connected.

In general we cannot expect the hypotheses of Theorem 1.2.8 to be satisfied, hence the manifolds that we consider are allowed to have more that one end.

A standard computation (see for instance [Gra00, Maz88]) shows that, if (M, g) is conformally compact then, in the same notations as Definition 1.2.7, the Riemannian tensor of g satisfies

$$\mathrm{R}_{\alpha\beta\gamma\eta} \,=\, -\,|dr|^2_{\bar{g}}\,(g_{\alpha\gamma}\,g_{\beta\eta}\,-\,g_{\alpha\eta}\,g_{\beta\gamma})\,+\,\mathcal{O}(r^{-3})\,. \tag{1.2.5}$$

In particular, the sectional curvatures of g go to $-|dr|^2_{\bar{g}}$ as we approach the infinity. Moreover, the quantity $|dr|^2_{\bar{g}}$ is easily seen to be independent of the choice of the defining function, so that the following definition is well posed.

Definition 1.2.9. *A Riemannian manifold (M, g) is said to be* asymptotically locally hyperbolic *if it is conformally compact and $|dr|_{\bar{g}} = 1$ on the whole $\partial_\infty M$, where r is a defining function and $\bar{g} = r^2 g$ is the corresponding conformal metric.*

The discussion above tells us that the sectional curvatures of an asymptotically locally hyperbolic manifold (M, g) converge to -1 as we approach the conformal boundary. A second important property of asymptotically locally hyperbolic manifolds is that it is possible to choose on them a canonical defining function, as the following lemma explains.

Lemma 1.2.10. *Let (M, g) be an asymptotically locally hyperbolic manifold. Then, for each choice of a metric γ on its conformal infinity, there exists a unique defining function r such that $|dr|_{\bar{g}} = 1$ on a collar of $\partial_\infty M$ and $(r^2 g)_{|_{\partial_\infty M}} = \gamma$.*

The proof is a rather easy computation, see for instance [Gra00, Lemma 2.1] for the details. The defining function in Lemma 1.2.10 will be referred to as the *special defining function* for $(\partial_\infty M, \gamma)$. The special defining function r determines a diffeomorphism between a collar of $\partial_\infty M$ and the product $\partial_\infty M \times [0, \varepsilon)$. In this collar, the metric g rewrites as

$$g \,=\, r^{-2}\big(dr \otimes dr + h_r\big)\,, \tag{1.2.6}$$

where, for all values $r \in [0, \varepsilon)$, h_r is a metric on $\partial_\infty M$ such that h_0 coincides with $\bar{g}_{|_{\partial_\infty M}}$, where $\bar{g} = r^2 g$ as usual. We are finally able to introduce a first definition of asymptotic hyperbolicity.

Definition 1.2.11. *A Riemannian manifold (M, g) is said to be* asymptotically hyperbolic *if it is asymptotically locally hyperbolic and it holds*

(i) the conformal infinity is the standard sphere, that is, $\partial_\infty M$ is diffeomorphic to the sphere $\mathbb{S}^{n-1}$, and the metric $\bar{g}_{|_{\partial_\infty M}}$, where $\bar{g} = r^2 g$ and r is a defining function, is conformal to the standard spherical metric $g_{\mathbb{S}^{n-1}}$,

(ii) the term h_r in formula (1.2.6) expands as

$$h_r = \left(1 - \frac{r^2}{4}\right)^2 \bar{g}_{|_{\partial_\infty M}} + \frac{r^n}{n}\tau + \mathcal{O}(r^{n+1})\,, \tag{1.2.7}$$

where τ is a symmetric 2-tensor on $\partial_\infty M$.

For such manifolds, Wang introduced in [Wan01] the following notion of mass.

Definition 1.2.12. *The* mass vector *of an asymptotically hyperbolic manifold* (M,g) *is defined as the vector*

$$\left(\int_{\partial_\infty M} \mathrm{tr}(\tau)\,\mathrm{d}\sigma\,,\ \int_{\partial_\infty M} \mathrm{tr}(\tau)\,x^1\,\mathrm{d}\sigma\,,\ \cdots,\ \int_{\partial_\infty M} \mathrm{tr}(\tau)\,x^n\,\mathrm{d}\sigma\right)$$

where the trace tr *and the volume form* $\mathrm{d}\sigma$ *are taken with respect to the metric* $\bar{g}_{|\partial_\infty M}$, τ *is the* 2*-tensor appearing in* (1.2.7)*,* $\partial_\infty M$ *is identified with the unit round sphere* $\mathbb{S}^{n-1}$ *with its canonical embedding in the Euclidean space* $\mathbb{R}^n$*, and* $(x^1,\dots,x^n)$ *are the standard coordinates on* $\mathbb{R}^n$*.*

The mass of (M,g) can be seen as a vector in the Minkowski space $\mathbb{R}^{1,n}$ and it is the analogue of the energy-momentum tensor for asymptotically flat manifold. It is shown in [Wan01] that, if we replace the metric $\bar{g}_{|\partial_\infty M}$ with $f\,\bar{g}_{|\partial_\infty M}$, for some positive function $f\in\mathscr{C}^\infty(\partial_\infty M)$, then the mass transforms by a Lorentz transformation, which we recall is an isometry of the Minkowski space. In particular, the norm of the mass vector

$$\begin{aligned}\left(\int_{\partial_\infty M} \mathrm{tr}(\tau)\,\mathrm{d}\sigma\right)^2 - \left|\int_{\partial_\infty M} \mathrm{tr}(\tau)\,x\,\mathrm{d}\sigma\right|^2 &= \\ = \left(\int_{\partial_\infty M} \mathrm{tr}(\tau)\,\mathrm{d}\sigma\right)^2 - \sum_{i=1}^n\left(\int_{\partial_\infty M} \mathrm{tr}(\tau)\,x^i\,\mathrm{d}\sigma\right)^2 \quad &(1.2.8)\end{aligned}$$

is independent of the choice of the conformal metric on the conformal infinity of (M,g). In order for (1.2.8) to really be an invariant of the asymptotically hyperbolic manifold (M,g), it only remains to show that the conformal compactification of an asymptotically hyperbolic manifold is unique. This is proven in [CH03]. More precisely, it is shown that, if an asymptotically hyperbolic manifold (M,g) is isometric to the interior of two compact manifolds $\overline{M}_1,\overline{M}_2$ via the inclusions $i_1: M\hookrightarrow\overline{M}_1$, $i_2: M\hookrightarrow\overline{M}_2$, and if M is conformally compact with respect to both $\overline{M}_1$ and $\overline{M}_2$, then the function $i_1\circ i_2^{-1}$ extends to a conformal diffeomorphism between $\overline{M}_1,\overline{M}_2$.

Therefore, Definition 1.2.12 is well posed, and the mass of an asymptotically hyperbolic manifold is well defined up to a Lorentz transformation. Moreover, the following result, which is the analogue of the Positive Mass Theorem in this context, shows that, under suitable hypotheses, the mass vector is always timelike future directed or it is zero.

Theorem 1.2.13 (Positive Mass Theorem for Asymptotically Hyperbolic Manifolds, [Wan01, Theorem 1.1])**.** *Let* (M,g) *be an asymptotically hyperbolic manifold with empty boundary. If* M *is spin and has scalar curvature* $\mathrm{R}\geq -n(n-1)$*, then*

$$\int_{\partial_\infty M} \mathrm{tr}(\tau)\,\mathrm{d}\sigma \geq \left|\int_{\partial_\infty M} \mathrm{tr}(\tau)\,x\,\mathrm{d}\sigma\right|.$$

Moreover, if the equality holds, then (M,g) *is isometric to the hyperbolic space form.*

This theorem improves a previous characterization of the hyperbolic space given by [MO89] (and then later generalized by Anderson and Dahl [AD98]). As a consequence of Theorem 1.2.13, we have that the quantity

$$\sqrt{\int_{\partial_\infty M} \mathrm{tr}(\tau)\, \mathrm{d}\sigma - \left| \int_{\partial_\infty M} \mathrm{tr}(\tau)\, x\, \mathrm{d}\sigma \right|}\,,$$

which Wang referred to as the *total mass*, is an invariant of the manifold that is always nonnegative. Another positive quantity associated to an asymptotically hyperbolic manifold (M, g) is

$$\mu = \frac{1}{16\pi} \int_{\partial_\infty M} \mathrm{tr}(\tau)\, \mathrm{d}\sigma\,, \tag{1.2.9}$$

and is usually simply called *mass*. We remark that the mass (1.2.9) is not invariant, but depends on the conformal metric chosen on $\partial_\infty M$. However, In Section 2.4 we will introduce conformally compact static triples, and we will see that for them there is a standard choice of the defining function. Therefore, for conformally compact static triples the mass μ can be canonically defined. The interest in this mass (1.2.9) comes from the fact that, in the case of the Schwarzschild–Anti de Sitter solution, it coincides with the parameter m appearing in its definition, see (2.4.5) in Subsection 2.4.1. We also point out that, in dimension three, another definition of mass that generalizes Definition 1.2.12, together with a positive mass statement, has been proposed by Zhang [Zha04].

However, both Wang's and Zhang's definitions are not completely satisfactory, as they have two limitations. The first one, is that they require the conformal infinity to be isometric to a standard sphere, whereas we would like to have more freedom on the choice of the topology. In fact, as opposed to the asymptotically flat case, in the asymptotically hyperbolic setting there are several relevant examples whose topology at infinity is not spherical, see for instance (2.4.7) and (2.4.8) in Subsection 2.4.1. This problem is not hard to overcome, and a similar definition of mass can be given more generally in the case where the conformal boundary is diffeomorphic to any space form (one needs to ask for the manifold to be *asymptotically Poincaré-Einstein*, see [GW15, BMW15] for the definitions and the details). However, we point out that we cannot expect a Positive Mass Theorem to hold for manifolds whose conformal infinity is not diffeomorphic to a sphere. In fact, in Subsection 2.4.1 we will see that the Schwarzschild–Anti de Sitter solutions with hyperbolic geometry (2.4.8) can have negative mass. Another family of asymptotically hyperbolic manifolds with flat conformal infinity and with negative mass is shown in [HM99, Section 3].

A second problem, is that it is not clear how Definitions 1.2.11, 1.2.12 represent an analogue of the notions of asymptotic flatness and ADM mass. For this reason, Chruściel and Herzlich [CH03], inspired by a previous work of Chruściel and Nagi [CN01] in the Lorentzian setting, proposed an alternative definition of asymptotic hyperbolicity, which encorporates a wider family of manifolds and which is more in the spirit of the definition of asymptotic flatness. In order to retrace the approach in [CH03], we first need to define some model metrics on a manifold of the form $N \times [R, +\infty)$, where $R > 0$ and N is an $(n-1)$-dimensional manifold admitting a metric g_N with constant scalar curvature

$R_{g_N} = (n-1)(n-2)\kappa$, with $\kappa \in \{-1, 0, 1\}$. On $N \times [R, +\infty)$ we consider the background metric

$$g_b = \frac{dr \otimes dr}{\kappa + r^2} + r^2 g_N,$$

which is easily seen to have constant scalar curvature equal to $-n(n-1)$ (moreover, it is Einstein if g_N is Einstein). Fixed a frame $\epsilon_1, \ldots, \epsilon_{n-1}$ on the tangent space of N, orthonormal with respect to the metric g_N, then

$$\partial_1 = r\,\epsilon_1, \ \ldots, \ \partial_{n-1} = r\,\epsilon_{n-1}, \ \partial_n = \sqrt{\kappa + r^2}\,\partial_r \tag{1.2.10}$$

is an orthonormal frame for the background metric.

Definition 1.2.14. *A manifold (M, g) is said to be* chart-dependent asymptotically hyperbolic *if there exists a diffeomorphism (chart at infinity)*

$$\Phi : M \setminus K \to N \times [R, +\infty),$$

where $K \subset M$ is compact, such that, with respect to the g_b-orthonormal base $\partial_1, \ldots, \partial_n$ introduced in (1.2.10), it holds

$$|g_{\alpha\beta} - \delta_{\alpha\beta}| = o(r^{-\frac{n}{2}}), \quad |\partial_\gamma g_{\alpha\beta}| = o(r^{-\frac{n}{2}}).$$

where $r \in [R, +\infty)$ is the radial coordinate introduced above on $N \times [R, +\infty)$. Moreover, we ask the function $R + n(n-1)$, where R is the scalar curvature of g, to be integrable on M.

We remark that Definition 1.2.14 is more general than Definition 1.2.11. First of all, it allows for any topology at infinity (on any N we can find a metric g_N with constant scalar curvature). Secondly, also in the case where the topology at infinity is $(N, g_N) = (\mathbb{S}^{n-1}, g_{\mathbb{S}^{n-1}})$, it is easily seen that any asymptotically hyperbolic manifold is chart-dependent asymptotically hyperbolic, while the converse is false (a chart-dependent asymptotically hyperbolic manifold has no reasons to be even conformally compact).

Now we discuss how a notion of mass can be introduced on any chart-dependent asymptotically hyperbolic manifolds. Let $\mathcal{N}_{g_b}$ be the set of functions u satisfying the following system

$$\begin{cases} u\,\mathrm{Ric}_{g_b} = \mathrm{D}^2_{g_b} u - n\,u\,g_b, & \text{in } N \times [R, +\infty), \\ \Delta_{g_b} u = n\,u, & \text{in } N \times [R, +\infty), \end{cases}$$

where $\mathrm{Ric}_{g_b}, \mathrm{D}^2_{g_b}, \Delta_{g_b}$ are the Ricci tensor, hessian and laplacian with respect to the metric g_b. The triples $(N \times [R, +\infty), g_b, u)$, with $u \in \mathcal{N}_{g_b}$ are called *static solutions* and will be discussed in more details in Section 2.4, see (2.4.1). For the moment, we only need them in order to introduce the following definition of mass.

Definition 1.2.15. *Let (M, g) be a chart-dependent asymptotically hyperbolic Riemannian manifold, so that we have a chart at infinity $\Phi : M \setminus K \to N \times$*

$[R, +\infty)$. *The* mass functional *is the function* $H : \mathcal{N}_{g_b} \to \mathbb{R}$ *defined as*

$$H(u) = \lim_{R \to +\infty} \int_{S_R} \Big[u \left(\mathrm{div}_{g_b}(g) - d\mathrm{tr}_{g_b}(g) \right) + \\ + \mathrm{tr}_{g_b}(g_b - g)\, \mathrm{D}u - \langle \mathrm{D}^{g_b} u \,|\, \cdot \rangle_{g_b - g} \Big](\nu)\, d\sigma_{g_b} \quad (1.2.11)$$

where $S_R = N \times \{R\}$*,* ν *is the outer unit normal to* S_R*.*

The following result ensures us that the functional H is well defined.

Proposition 1.2.16 ([CH03, Proposition 2.2])**.** *The limit in formula* (1.2.11) *exists and is finite. Moreover, the mass functional* H *is independent of the choice of the chart at infinity in Definition 1.2.14.*

Let us now discuss briefly the structure of $\mathcal{N}_{g_b}$ depending of N. For general pairs (N, g_N), it is only known that the space $\mathcal{N}_{g_b}$ has dimension less than or equal to $n-1$. When (N, g_N) is a quotient of the Euclidean or the hyperbolic space, then it is known that $\mathcal{N}_{g_b}$ is 1-dimensional, so in this case, the mass functional is just a complicate way to obtain a number. In the case where N is the sphere $\mathbb{S}^{n-1}$ endowed with the standard metric $g_{\mathbb{S}^{n-1}}$, the space $\mathcal{N}_{g_b}$ has dimension $n+1$, and it is generated by the functions $u^{(0)} = \cosh(r)$, $u^{(\alpha)} = x^\alpha \sinh(r)$ for $\alpha = 1, \dots, n$, where $(x^{(1)}, \dots, x^{(n)})$ are the Euclidean coordinates on $\mathbb{S}^{n-1}$ induced by the standard inclusion $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. In this case, the mass functional gives rise to the vector

$$\left(H(u^{(0)}), \dots, H(u^{(n)}) \right) . \qquad (1.2.12)$$

For asymptotically hyperbolic manifolds (in the sense of Definition 1.2.11), this vector is exactly (up to a multiplying factor) the mass vector introduced in Definition 1.2.12, hence the mass functional extends Definition 1.2.12 to a wider family of manifolds (although Wang's definition has still the practical use of being more easily computed).

Finally, also this second notion of mass satisfies a Positive Mass Theorem.

Theorem 1.2.17 (Positive Mass Theorem for Chart-Dependent Asymptotically Hyperbolic Manifolds, [CH03, Theorem 4.1])**.** *Let* (M, g) *be chart-dependent asymptotically hyperbolic in the sense of Definition 1.2.14 with respect to* $(N, g_N) = (\mathbb{S}^{n-1}, g_{\mathbb{S}^{n-1}})$*. If* M *has empty boundary, is spin, and the scalar curvature of* g *satisfies* $\mathrm{R} \geq -n(n-1)$*, then the mass vector, defined in* (1.2.12)*, is timelike future directed or it is zero. If it is zero, then* (M, g) *is isometric to the hyperbolic space.*

Theorem 1.2.17 is an interesting extension of Wang's Theorem 1.2.13 and seems to be the right analogue of the Positive Mass Theorem for asymptotically flat manifold. For further discussions on the comparison between the different Positive Mass Theorems, we address the reader to [Her05, Her16].

If one looks for extensions of Theorem 1.2.17, a natural and important direction is the study of the case where (M, g) has a nonempty boundary. A first result in this direction is proved in [CH03].

Theorem 1.2.18 ([CH03, Theorem 4.7]). *Let (M, g) be chart-dependent asymptotically hyperbolic in the sense of Definition 1.2.14 with respect to $(N, g_N) = (\mathbb{S}^{n-1}, g_{\mathbb{S}^{n-1}})$. If M is spin, has compact nonempty boundary ∂M whose mean curvature satisfies $\mathrm{H} \leq n-1$, and the scalar curvature of g verifies $\mathrm{R} \geq -n(n-1)$, then the mass is timelike future directed.*

Moreover, in [CH03], it is observed that, if the boundary is not empty, then the value of the mass is far away from zero. This is exactly what happens in the asymptotically flat case, where we know that, if the boundary of our manifold is compact minimal outer minimizing, then the Riemannian Penrose Inequality holds, see Theorem 1.2.4. We recall that the original proof of this inequality in the case $n = 3$ was given by Huisken and Ilmanen in [HI01], and was based on a weak version of the inverse mean curvature flow and on the monotonicity of the *Hawking mass* along that flow. This approach seems to be the most promising in order to try to find an analogue of Theorem 1.2.4 for asymptotically hyperbolic manifolds, at least in dimension $n = 3$. For a discussion on the problems of this approach and a conjecture about what the right Riemannian Penrose Inequality should be in the case of asymptotically hyperbolic manifolds, we refer the reader to [CS01, Section VI].

A general Riemannian Penrose Inequality for asymptotically hyperbolic manifolds seems still out of reach, but a first result in this direction has been proved by Lee and Neves in [LN15, Theorem 1.1] in the case $n = 3$ and for negative masses. As a consequence, combining their result with [CS01, Theorem I.5], Lee and Neves were able to prove a uniqueness result for 3-dimensional static spacetimes with negative mass and conformal infinity with genus grater than or equal to 2, see Theorem 2.4.9 in Subsection 2.4.2.

1.2.3 Min-Oo's conjecture.

The rigidity cases of the Positive Mass Theorems in the asymptotically flat and asymptotically hyperbolic setting, discussed in the previous subsections, can be seen as characterizations of the Euclidean and hyperbolic space forms. We have also discussed in Theorem 1.2.6 Miao's characterization of the Euclidean ball. In analogy with the these results, Min-Oo proposed the following natural conjecture.

Conjecture (Min-Oo's conjecture, cf. [MO98, Theorem 4]). *Let g be a smooth metric on the unit hemisphere $\mathbb{S}^n_+$. If the following properties are satisfied*

(i) the scalar curvature of g is greater than or equal to $n(n-1)$ on the whole $\mathbb{S}^n_+$,

(ii) the metric induced by g on $\partial\mathbb{S}^n_+ = \mathbb{S}^{n-1}$ is the standard round metric,

(iii) the boundary $\partial\mathbb{S}^n_+ = \mathbb{S}^{n-1}$ is totally geodesic with respect to g,

then g is the standard round metric on $\mathbb{S}^n_+$.

Min-Oo's conjecture can be seen as the natural analogue of the Positive Mass Theorem in the spherical setting. For several years, this conjecture was thought to be true, and several partial results have been proven. For instance, Min-Oo's conjecture holds in dimension $n = 2$ and if the scalar curvature condition is replaced by a bound on the Ricci tensor, see [HW09]. The conjecture has also been proved in dimension $n = 3$ under a suitable isoperimetric condition [Eic09], and in every dimensions for metrics conformal to the standard spherical metric [HW06]. Other partial results include the works [BM11, HW10, Lis10, Lla98], for an exhaustive discussion see the introduction of [BMN11]. However, the conjecture has finally been disproven by Brendle, Marques and Neves [BMN11], who showed the existence of counterexamples.

Theorem 1.2.19 ([BMN11, Theorem 7]). *For any $n \geq 3$, there exists a smooth metric g on the unit hemisphere $\mathbb{S}^n_+$ that satisfies the following properties*

(i) the scalar curvature of g is greater than or equal to $n(n-1)$ on the whole $\mathbb{S}^n_+$,

(ii) the scalar curvature of g is not constantly equal to $n(n-1)$,

(iii) the metric g agrees with the standard spherical metric of $\mathbb{S}^n_+$ on a collar neighborhood of the boundary $\partial\mathbb{S}^n_+ = \mathbb{S}^{n-1}$.

It follows from Theorem 1.2.19 that no straightforward analogue of the Positive Mass Theorem can be possible in the spherical setting.

As anticipated in the Introduction, Theorem 1.2.19 can be seen as one of the main motivations of our work. Since the approach of replicating the ADM mass in the spherical setting seems to have failed, we suggest a different approach. We will start by analyzing a basic (even if not so simple) case, that is the case of static spacetimes, which are defined in Chapter 2. On this family of spacetimes, we will prove in Chapters 3 and 4 that a notion of mass can be defined, and that it satisfies a Positive Mass Statement and a Riemannian Penrose–like inequality. These results can be seen as evidences of the validity of our definition. The next step would be, starting from the definition in the static case, to infer the right notion of mass on a general compact initial data set. This will be the subject of further studies.

2

Static spacetimes

2.1 Definitions

In this section we are going to introduce the main topic of this work, that is, vacuum static spacetimes. In a loose sense, these are models, obeying the laws of General Relativity and in particular the Einstein Field Equations (2.1.1), which describe regions of the universe where there is no matter (or where the mass is "hidden" behind some event horizon), and that are not affected by the flow of time. The precise definitions will be given later. Here we only want to comment that vacuum static spacetimes represent indeed the most basic examples of spacetimes in General Relativity, and a clear understanding of their behavior will help in the study of more complicated models. Even in this simplified setting, the analysis is far from trivial and a classification of static vacuum spacetimes seems still out of reach. However, there are many partial interesting results, which we will discuss in this chapter. We remind the reader that in what follows it is always tacitly assumed that the dimension of our manifold is $n \geq 3$.

2.1.1 Vacuum spacetimes.

We begin by giving the definition of a general vacuum spacetime. First of all, let us recall from Subsection 1.1.3 that a Lorentzian manifold is a pair (X, γ), where X is a $(n+1)$-dimensional manifold and γ is a metric such that, at any point $p \in M$, there exists a coordinate chart $(x^0, \ldots, x^n)$ centered at p with respect to which it holds

$$\gamma_{|_p} = -\, dx^0 \otimes dx^0 + \sum_{i=1}^{n} dx^i \otimes dx^i \,.$$

A *vacuum spacetime* is a Lorentzian manifold (X, γ) that satisfies the well known *Einstein Field Equations*

$$\mathrm{Ric}_\gamma - \frac{\mathrm{R}_\gamma}{2}\,\gamma + \Lambda\,\gamma = 0\,, \tag{2.1.1}$$

where $\Lambda \in \mathbb{R}$ is the *cosmological constant*. Some comments are in order.

- The general Einstein Field Equations allow the presence of a tensor T, called *stress-energy tensor*, on the right hand side of formula (2.1.1). Since we will be interested only in the vacuum case, we will always assume $T \equiv 0$.

- By tracing the equation (2.1.1) and then substituting the result back in it, we find that formula (2.1.1) can be rewritten as

$$\mathrm{Ric}_\gamma = \frac{2\Lambda}{n-1}\gamma\,.$$

  In particular, the Einstein Field Equations are equivalent to requiring the metric γ to be Einstein. Notice that there are just three relevant distinct cases, depending on whether the cosmological constant Λ is positive, negative or null. In fact, if Λ is not zero, we can rescale the metric γ in such a way that $|\Lambda| = n(n-1)/2$. We will make use of such a rescaling later.

- At the moment, we are not making any assumptions on the topology of the spacetime X. We will see that the natural hypotheses to set vary depending on whether $\Lambda = 0$, $\Lambda > 0$ or $\Lambda < 0$, hence we will specify them later, when we will focus on the three cases separately.

2.1.2 Killing vector fields and surface gravity.

In Newtonian physics, the surface gravity of a rotationally symmetric massive body (e.g. a planet of the solar system) can be physically interpreted as the intensity of the gravitational field due to the body, as it is measured by a massless observer sitting on the surface of the body. For example the Newtonian surface gravity of the Earth is given by the well known value $\mathrm{g} = 9.8\,\mathrm{m/s}^2$, the one of Jupiter is given by $2.53\,\mathrm{g}$ and the one of the Sun is given by $28.02\,\mathrm{g}$. Of course, in the case of black holes, the Newtonian surface gravity is no longer a meaningful concept, since it becomes infinite when computed at the horizon, regardless of the mass of the black hole. To overcome this issue one is led to introduce the appropriate relativistic concept of *surface gravity*. It turns out that the surface gravity is most naturally defined on *Killing horizons*, whose definition we now recall. Although we only work in the vacuum case, we mention that most of the results that we discuss here hold more generally on any spacetime satisfying the Dominant Energy Condition.

Let (X,γ) be a $(n+1)$-dimensional vacuum spacetime, and suppose that there exists a Killing vector field K on X, that is, a vector field such that $\mathcal{L}_K\gamma = 0$ on the whole X. A *Killing horizon* $S \subset X$ is a connected n-dimensional null hypersurface, invariant under the flow of K, such that

$$|K|^2_\gamma = 0 \quad \text{and} \quad K \neq 0\,, \quad \text{on } S\,.$$

Notice that K is a null vector on S by definition, and it is tangent to S because of the invariance of S under the flow of K. Moreover, since $|K|^2_\gamma$ is constant on a Killing horizon S, the vector field $\nabla|K|^2_\gamma$, where ∇ is the covariant derivative with respect to γ, is orthogonal to S (in particular it is orthogonal to K). Since S

is a null hypersurface, it follows that $\nabla|K|^2_\gamma$ is a null vector. Therefore, K and $\nabla|K|^2_\gamma$ are orthogonal null vectors, and Proposition 1.1.2 tells us that they are necessarily proportional. Therefore we can define the *surface gravity* of a Killing horizon S as the quantity κ that satisfies

$$2\,\kappa\, K_{|_S} \;=\; -(\nabla|K|^2_\gamma)_{|_S}\,. \tag{2.1.2}$$

A priori, κ is a function on S, however, one can show (see [BCH73] or [Heu96, Theorem 7.1]) that κ is actually constant on any Killing horizon S. The proof of this fact is based on a nice computation exploiting some interesting general identities for Killing vector fields. We will avoid to show the general proof, but in Subsection 2.1.4 we will see an easy argument showing the constancy of the surface gravity in the case which is most interesting for us, that is, in the case of static spacetimes.

Concerning the physical interpretation, we point out that the surface gravity measures the acceleration a experienced along the integral curves of K on our Killing horizon S (see [Chr15, Remark 1.3.6]), in the sense that it holds $a = \kappa K$. In fact, consider a curve $\sigma : \tau \mapsto \sigma(\tau)$ such that

$$\dot\sigma(\tau) \;=\; K(\sigma(\tau))\,, \quad \sigma(\tau)\in S\,, \quad \text{for all values of } \tau\,.$$

The acceleration a of the curve σ is defined as the covariant derivative of $\dot\sigma$ along σ, so in coordinates we compute

$$a^\alpha \;=\; \frac{\nabla\dot\sigma^\alpha}{d\tau} \;=\; \dot\sigma^\beta\nabla_\beta K^\alpha \;=\; K^\beta\nabla_\beta K^\alpha \;=\; -K^\beta\nabla^\alpha K_\beta \;=\; -\frac{1}{2}\nabla^\alpha|K|^2_\gamma \;=\; \kappa K^\alpha\,,$$

as wished. An alternative way of interpreting the above computation (see for instance [FNOT96]) is observing that the value of κ measures the failure of K on being an affine null geodesic. Since the wave vector of a light signal is an affine null geodesic, the quantity κ measures the frequency decrease (*redshift*) of a light signal, or equivalently the energy loss of a photon moving along the horizon.

We now introduce the following alternative formula to compute the surface gravity, which will be useful in Subsection 2.1.4.

Proposition 2.1.1. *Let S be a Killing horizon with respect to the Killing vector field K. The surface gravity of S can be explicitly computed as*

$$\kappa^2 \;=\; -\frac{1}{2}(|\nabla K|^2_\gamma)_{|_S}\,. \tag{2.1.3}$$

Proof. We follow [Chr15] (see also [Wal84, Section 12.5]). The proof is local, and the computations are made with respect to normal coordinates $(x^0,\dots,x^n)$. First of all, since K is a Killing field, from $\mathcal{L}_K\gamma = 0$ it follows

$$\nabla_\alpha K_\beta \;=\; -\nabla_\beta K_\alpha\,. \tag{2.1.4}$$

Moreover, from the definition (2.1.2) of surface gravity, we compute

$$\kappa K^\alpha \;=\; -\frac{1}{2}\nabla^\alpha(K^\beta K_\beta) \;=\; -K^\beta\,\nabla^\alpha K_\beta\,. \tag{2.1.5}$$

To proceed in the computations, we need one last identity, which is the *Frobenius formula*

$$K_\alpha \nabla_\beta K_\eta + K_\beta \nabla_\eta K_\alpha + K_\eta \nabla_\alpha K_\beta - K_\alpha \nabla_\eta K_\beta - K_\beta \nabla_\alpha K_\eta - K_\eta \nabla_\beta K_\alpha = 0 . \quad (2.1.6)$$

To prove this identity, let us choose coordinates so that the hypersurface S coincides with the zero level set of a function f. Since we have shown that K is orthogonal to S, we have

$$K = h \nabla f , \quad \text{on } S ,$$

for a certain function h. In particular, for any $\alpha = 1, \ldots, n$ we have that the function $K_\alpha - h\nabla_\alpha f$ is zero on S, hence its gradient $\nabla(K_\alpha - h\nabla_\alpha f)$ is perpendicular to S, and in particular it is a multiple of K. It follows that, on S, it holds

$$\nabla_\alpha K_\beta = \nabla_\alpha h \nabla_\beta f + h\nabla^2_{\alpha\beta} f + K_\alpha Z_\beta = \frac{1}{h} K_\beta \nabla_\alpha h + h\nabla^2_{\alpha\beta} f + K_\alpha Z_\beta$$

for some vector field Z. Notice that the first and third term on the right hand side are multiple of K whereas the second term is symmetric. It follows that, substituting this expression in the left hand side of (2.1.6), everything vanishes. This establishes formula (2.1.6).

Now we have all the tools to prove our proposition. We start by noticing that (2.1.6) can be rewritten using (2.1.4) as

$$K_\eta \nabla_\alpha K_\beta = -K_\alpha \nabla_\beta K_\eta - K_\beta \nabla_\alpha K_\eta$$

Contracting the equation above with $\nabla^\alpha K^\beta$, and using (2.1.4) and (2.1.5) we obtain

$$\begin{aligned} K_\eta (\nabla^\alpha K^\beta)(\nabla_\alpha K_\beta) &= -K_\alpha (\nabla^\alpha K^\beta)(\nabla_\beta K_\eta) - K_\beta (\nabla^\alpha K^\beta)(\nabla_\alpha K_\eta) \\ &= -K_\alpha (\nabla^\alpha K^\beta)(\nabla_\beta K_\eta) - K_\beta (\nabla^\beta K^\alpha)(\nabla_\alpha K_\eta) \\ &= -2K_\alpha (\nabla^\alpha K^\beta)(\nabla_\beta K_\eta) \\ &= 2\kappa\, K^\beta\, \nabla_\beta K_\eta \\ &= -2\kappa^2\, K_\eta . \end{aligned}$$

This implies that $(\nabla^\alpha K^\beta)(\nabla_\alpha K_\beta) = -2\kappa^2$, as wished. □

Another notion of great importance in the study of spacetimes is that of *bifurcate Killing horizon*. This is defined as the union of a codimension-2 smooth submanifold S and four Killing horizons generated by the null geodesics departing from S in the four distinct null directions orthogonal to S. The surface gravity of each of the four Killing horizons is defined and can be computed using formula (2.1.2) or (2.1.3). Moreover, looking at the limit when one approaches the codimension-2 submanifold, one proves that the four Killing horizons have the same surface gravity. One can then define this common value as the surface gravity of the bifurcate Killing horizon. Furthermore, one can actually prove that the surface gravity of a bifurcate Killing horizon is always different from zero. The proof is again based on nice formulæ exploiting the properties of Killing vector fields and it remains valid in very general settings, see for instance [KW91], [RW96] or the survey [Chr15].

It should be noticed that the value of κ in (2.1.2) and (2.1.3) changes when we rescale the Killing vector K. Therefore, for the notion of surface gravity to be of some interest, one has to fix the normalization of the Killing vector in some canonical way. In the next subsection we will specify the normalizations for the cases of interest for this work.

2.1.3 Static Einstein System.

A spacetime (X, γ) is said to be *static* if it admits a global irrotational timelike Killing vector field. This implies that the spacetime splits as

$$X = \mathbb{R} \times M, \qquad \gamma = -u^2 dt \otimes dt + g_0,$$

where (M, g_0) is a n-dimensional Riemannian manifold and $u : M \to \mathbb{R}$ is a smooth function. It follows from [Bes08, Corollary 9.107] that identity (2.1.1) holds if and only if u satisfies the *Static Einstein System*

$$\begin{cases} u\,\mathrm{Ric} = \mathrm{D}^2 u + \dfrac{2\Lambda}{n-1}\, u\, g_0, & \text{in } M, \\ \Delta u = -\dfrac{2\Lambda}{n-1}\, u, & \text{in } M, \end{cases} \tag{2.1.7}$$

where Ric, D, and Δ represent the Ricci tensor, the Levi-Civita connection, and the Laplace-Beltrami operator of the metric g_0, respectively. If the boundary ∂M is non-empty, we will always assume that it coincides with the zero level set of u, so that, in particular, u is strictly positive in the interior of M. In the rest of the work the metric g_0 and the function u will be referred to as *static metric* and *static potential*, respectively, whereas the triple (M, g_0, u) will be called a *static solution*.

In what follows, we will be interested in the classification of static solutions up to isometry. Although this is a natural notion, to avoid ambiguities we prefer to specify exactly its meaning in the present setting.

Definition 2.1.2. *Let (M, g_0, u) and (M', g_0', u') be two static triples.*

- *We say that (M, g_0, u) and (M', g_0', u') are* isometric *if there exists a (Riemannian) isometry $F : (M, g_0) \to (M', g_0')$ such that, up to a normalization of u, it holds $u = u' \circ F$.*

- *We say that (M, g_0, u) is a* covering *of (M', g_0', u') if there exists a Riemannian covering $F : (M, g_0) \to (M', g_0')$ such that, up to a normalization of u, it holds $u = u' \circ F$.*

We now list some of the basic properties of static solutions.

- Concerning the regularity of the function u, we know from [Chr05, zH70] that u is analytic. This means that the results shown in Subsection 1.1.4 apply. In particular, the critical values of the static potential u are discrete and the critical set is a stratified $(n-1)$-dimensional submanifold.

- Taking the trace of the first equation and substituting the second one, it is immediate to deduce that the scalar curvature of the metric g_0 is constant, and more precisely it holds
$$\mathrm{R} = 2\Lambda\,. \tag{2.1.8}$$
In particular, we observe that choosing a normalization for the cosmological constant corresponds to fixing a scale for the metric g_0.

- The boundary $\partial M = \{u = 0\}$, if nonempty, satisfies some relevant properties. First of all, from (2.1.7) we have that $\mathrm{D}^2 u = 0$ on ∂M, and from this follows that the quantity $|\mathrm{D}u|$ is locally constant on ∂M. Furthermore, the constant value of $|\mathrm{D}u|$ on every connected component of ∂M cannot be zero. In fact, if $|\mathrm{D}u|$ were zero on one connected component of ∂M, then, shooting geodesics from a point p of that component and then solving the Cauchy problem given by the first equation in (2.1.7) and the initial conditions $u(p) = \mathrm{D}u(p) = 0$, one would find that $u \equiv 0$, on any geodesic. Therefore, we would have $u \equiv 0$ on the whole M, against the hypothesis that u is positive in the interior. This proves that the level set $\partial M = \{u = 0\}$ is regular, and in particular we can use formula (1.1.12) to deduce that the second fundamental form of the components of ∂M is zero. It follows that ∂M is a (possibly disconnected) totally geodesic hypersurface in (M, g_0). The connected components of ∂M will be referred to as *horizons*.

- The constant value of $|\mathrm{D}u|$ on a given connected component of the boundary can be interpreted as the *surface gravity* of that horizon. This definition is coherent with the one given for Killing horizons in Subsection 2.1.2, as it will be discussed in the next subsection. In this regard, it is important to notice that on one hand the equations in (2.1.7) are invariant by rescaling of the *static potential*, whereas on the other hand the value of $|\mathrm{D}u|$ heavily depends on such a choice. Hence, in order to deal with meaningful objects, one needs to remove this ambiguity, fixing a normalization of the function u. This is done in different ways, depending on the sign of the cosmological constant as well as on some natural geometric assumptions.

  - In the case where $\Lambda > 0$ and M is compact, which is the one we are more interested in, we will define the *surface gravity* of an horizon $S \subset \partial M$ as the quantity
  $$\kappa(S) \;=\; \frac{|\mathrm{D}u|_{|_S}}{\max_M u}\,.$$
  This definition coincides with the one suggested in [BH96, PK17]. Of course, up to normalize u in such a way that $\max_M u = 1$, the above definition reduces to $\kappa(S) = |\mathrm{D}u|_{|_S}$.

  - In the case where $\Lambda = 0$ and (M, g_0, u) is asymptotically flat with bounded *static potential*, the *surface gravity* of an horizon $S \subset \partial M$ is defined as
  $$\kappa(S) \;=\; \frac{|\mathrm{D}u|_{|_S}}{\sup_M u}\,. \tag{2.1.9}$$

Again, under the usual normalization $\sup_M u = 1$, the above definition reduces to $\kappa(S) = |\mathrm{D}u|_{|S}$.

– In the case where $\Lambda < 0$ and (M, g_0, u) is a conformally compact static solution (see Definition 2.4.1 below), such that the scalar curvature $\mathrm{R}^{\partial_\infty M}$ of the metric induced by (the smooth extension of) $\overline{g} = u^{-2} g$ on the boundary at infinity $\partial_\infty M$ is constant and nonvanishing, the *surface gravity* of an horizon $S \subset \partial M$ is defined as

$$\kappa(S) \;=\; \frac{|\mathrm{D}u|_{|S}}{\sqrt{\left|\dfrac{n\,\mathrm{R}^{\partial_\infty M}}{2(n-2)\Lambda}\right|}}\,.$$

In this case, according to [CS01, Section VII], a natural normalization for the static potential is the one for which, under the above assumptions, one has that the constant value of $\left|\,\mathrm{R}^{\partial_\infty M}\,\right|$ coincides with $-2(n-2)\Lambda/n$. Having fixed this value, the surface gravity can be computed as $\kappa(S) = |\mathrm{D}u|_{|S}$.

2.1.4 Surface gravity on static spacetimes.

In Subsection 2.1.2 we have introduced the notion of surface gravity of a Killing horizon, whereas in Subsection 2.1.3 we have given a different definition in the static case. Here we show that the two definitions are coherent.

Let (M, g_0, u) be a solution to problem (2.1.7) such that $u = 0$ on ∂M. We have already observed that any such triple defines a spacetime

$$(X, \gamma) \;=\; \big(\mathbb{R} \times (M \setminus \partial M), -u^2 dt \otimes dt + g_0\big)\,.$$

In this case, there is a canonical choice of a Killing vector field, that is

$$K \;=\; \frac{\partial}{\partial t}\,.$$

If it were possible to extend the spacetime (X, γ) beyond its boundary $\mathbb{R} \times \partial M$, we would have that the connected components of $\mathbb{R} \times \partial M$ are Killing horizons. In fact, they would be null hypersurfaces invariant under the flow of K, and such that $|K|^2_\gamma = -u^2 = 0$ on them. However, notice that, in our coordinates, the metric γ becomes degenerate on $\mathbb{R} \times \partial M$ and it is not clear in general how to perform a change of coordinates in a neighborhood of $\mathbb{R} \times \partial M$ in such a way that γ remains Lorentzian. This change of coordinates can be made explicit in some specific cases. For instance, the Kruskal coordinates extend the Schwarzschild metric (2.3.3), in such a way that the set $\{|x| = r_0(m)\}$ becomes a bifurcate Killing horizon (see for instance [Chr15, Section 1.2.3] for the details). More generally, similar coordinates can be constructed under the hypothesis that the triple (M, g_0, u) is rotationally symmetric, see [Wal70]. However, to the author's knowledge, there is not a general way to perform such an extension for an arbitrary static spacetime. For this reason, in general, it is not convenient to use identity (2.1.2) to compute the surface gravity of a component of $\mathbb{R} \times \partial M$.

On the other hand, formula (2.1.3) is far easier to apply. In fact, the metric γ and the Killing vector being smooth, we can compute $|\nabla K|^2_\gamma$ on X, and then take the limit as we approach the boundary $\mathbb{R} \times \partial M$ to compute the surface gravity. To this end, we introduce coordinates $(x^1, \ldots, x^n)$ on an open set of M, so that $(x^0 := t, x^1, , \ldots, x^n)$ are coordinates on an open set of $X = \mathbb{R} \times M$. In the following computations, we will use greek letters for indices that vary between 0 and n, and latin letters for indices that vary from 1 to n. With these conventions, one has $K^\alpha = \delta^\alpha_0$, whereas the Christoffel symbols of γ satisfy

$$\Gamma^\beta_{0\alpha} \,=\, \frac{\gamma^{\beta\eta}}{2}\left(\partial_0\gamma_{\alpha\eta} + \partial_\alpha\gamma_{0\eta} - \partial_\eta\gamma_{0\alpha}\right) \,=\, \begin{cases} \dfrac{\partial_\alpha u}{u} & \text{if } \alpha \neq 0,\ \beta = 0\,, \\ u\, g_0^{\beta\eta}\, \partial_\eta u & \text{if } \alpha = 0,\ \beta \neq 0\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Now we compute

$$\nabla_\alpha K^\beta \,=\, \partial_\alpha K^\beta + \Gamma^\beta_{\alpha\eta} K^\eta \,=\, \Gamma^\beta_{0\alpha}\,,$$

hence

$$\begin{aligned} |\nabla K|^2_\gamma \,&=\, \gamma^{\alpha\eta}\,\gamma_{\beta\mu}\,\nabla_\alpha K^\beta\,\nabla_\eta K^\mu \\ &=\, \gamma^{\alpha\eta}\,\gamma_{\beta\mu}\,\Gamma^\beta_{0\alpha}\,\Gamma^\mu_{0\eta} \\ &=\, \left[-\frac{1}{u^2}\,g_{0jq}\,(u\,g_0{}^{jr}\,\partial_r u)\,(u\,g_0{}^{qs}\,\partial_s u) - u^2\,g_0{}^{ip}\,\frac{\partial_i u}{u}\,\frac{\partial_p u}{u}\right] \\ &=\, -2\,|\mathrm{D}u|^2\,. \end{aligned}$$

If S is a connected component of $\mathbb{R} \times \partial M$, taking the limit of formula (2.1.3) as we approach S, we obtain

$$\kappa(S) \,=\, |\mathrm{D}u|_{|_S}\,, \tag{2.1.10}$$

as expected. Formula (2.1.10) justifies in some sense the canonical normalizations introduced in Subsection 4.1.1, that is, $\max_M u = 1$ for $\Lambda > 0$, $\sup_M u = 1$ for $\Lambda = 0$, $\mathrm{R}^{\partial_\infty M} = -2(n-2)\Lambda/n$ if $\Lambda < 0$. In fact, these normalizations are the ones under which the surface gravity of an horizon S, coincides precisely with $|\mathrm{D}u|_{|_S}$.

It is also worth mentioning that in the static case we can give a further physical interpretation of the notion of surface gravity. Roughly speaking the idea is to consider an observer sitting far away from the horizon so that its measurement of the gravitational field must be corrected by a suitable time dilation factor, giving rise to a finite number. Such a number turns out to be related to the mass of the black hole and allows to distinguish between black holes of large mass and black holes with small mass. Thus, it can reasonably be interpreted as the *surface gravity* of the black hole. For further insights about the physical meaning of this concept, we refer the reader to [Wal84, Section 12.5].

Having this in mind, it is not surprising that the behavior of $|\mathrm{D}u|$, either at the horizons or along the geometric ends of a *static solution*, can be put in relation with the mass aspect of the solution itself. We will come back to this point in Section 2.3.2, where we will show that the ADM mass of asymptotically flat static solutions of (2.1.7) can be explicitly computed in terms of the quantity $|Du|$. This will provide our motivation for the definition of the virtual mass of compact static solutions with $\Lambda > 0$, which will be given in Chapter 3.

2.2 Warped product solutions

One of the first ways of finding exact solutions to system (2.1.7) has been to assume that the Riemannian manifold (M, g_0) had a warped product structure. The reason for this is twofold. First of all, these solutions represent the simplest solutions to problem (2.1.7) and they can be computed explicitly. Another reason comes from the following well known characterization of locally conformally flat solutions, proved independently by Kobayashi and Lafontaine.

Theorem 2.2.1 ([Kob82, Laf83])**.** *Let (M, g_0, u) be a locally conformally flat solution to problem (2.1.7). Then (M, g_0, u) is covered by a warped product solution of the form*

$$M = I \times E\,, \qquad g_0 \,=\, d\rho \otimes d\rho \,+\, f^2(\rho)\, g_E\,, \qquad u = v(\rho)\,,$$

where (E, g_E) is a $(n-1)$-dimensional Riemannian manifold, $I \subset \mathbb{R}$ is an interval and $f : I \to \mathbb{R}$ and $v : I \to \mathbb{R}$ are smooth positive functions.

In this section we follow the analysis in [Kob82] and we classify all the possible warped product solutions. We start by studying the easiest case, that is the case of product solutions.

Theorem 2.2.2 (Classification of product solutions)**.** *Let (M, g_0, u) be a solution to the Static Einstein System (2.1.7) such that $u = 0$ on ∂M and $u > 0$ in the interior of M. Suppose that (M, g_0, u) is isometric to a triple of the form*

$$M = I \times E\,, \qquad g_0 \,=\, d\rho \otimes d\rho \,+\, k^2\, g_E\,, \qquad u = v(\rho)\,, \tag{2.2.1}$$

where (E, g_E) is a $(n-1)$-dimensional Riemannian manifold, $I \subset \mathbb{R}$ is an interval, $v : I \to \mathbb{R}$ is a smooth positive function and $k > 0$ is a constant. Then necessarily (E, g_E) is an Einstein manifold and up to rescaling we can suppose $\mathrm{Ric}_{g_E} = 0$ or $\mathrm{Ric}_{g_E} = (n-2)g_E$ or $\mathrm{Ric}_{g_E} = -(n-2)g_E$. Moreover

(i) if $\Lambda = 0$ then $\mathrm{Ric}_{g_E} = 0$ and (M, g_0, u) is isometric to one of the following triples

$$\left(\mathbb{R} \times E\,,\, dr \otimes dr + g_E\,,\, 1\right)\,,$$
$$\left([0, +\infty) \times E\,,\, dr \otimes dr + g_E\,,\, r\right)\,,$$

(ii) if $\Lambda > 0$ then $\mathrm{Ric}_{g_E} = (n-2)g_E$, and (M, g_0, u) is isometric to the following triple

$$\left([0, \pi] \times E\,,\, \frac{n-1}{2\Lambda}\,[dr \otimes dr + (n-2)g_E]\,,\, \sin(r)\right)\,,$$

(iii) *if* $\Lambda < 0$ *then* $\mathrm{Ric}_{g_E} = -(n-2)g_E$*, and* (M, g_0, u) *is isometric to one of the following triples*

$$\left([0, +\infty) \times E\,,\, -\frac{n-1}{2\Lambda}\left[dr \otimes dr + (n-2)g_E\right]\,,\, \sinh(r)\right)\,,$$
$$\left(\mathbb{R} \times E\,,\, -\frac{n-1}{2\Lambda}\left[dr \otimes dr + (n-2)g_E\right]\,,\, \cosh(r)\right)\,,$$
$$\left(\mathbb{R} \times E\,,\, -\frac{n-1}{2\Lambda}\left[dr \otimes dr + (n-2)g_E\right]\,,\, e^r\right)\,,$$

Proof. Since (M, g_0) is a product manifold, it is clear that also the Ricci tensor splits. This means that, for any $p \in M$

$$\mathrm{Ric}(X,Y)_{|_p} = \begin{cases} \mathrm{Ric}_{g_I}(X,Y)_{|_p} & \text{if } X \in T_pI, Y \in T_pI\,, \\ \mathrm{Ric}_{g_E}(X,Y)_{|_p}\,, & \text{if } X \in T_pE, Y \in T_pE\,, \\ 0\,, & \text{if } X \in T_pI, Y \in T_pE\,, \end{cases}$$

where we have denoted by $g_I = dr \otimes dr$ the metric on I, and by T_pI and T_pE the tangent spaces at p to I and E respectively, considered with their natural inclusions inside T_pM. Moreover, since I is 1-dimensional, we have $\mathrm{Ric}_{g_I} \equiv 0$. It follows that the first equation in system (2.1.7) rewrites as

$$\begin{cases} v\,\mathrm{Ric}_{g_E} = \dfrac{2\Lambda}{n-1}\,v\,k^2\,g_E\,, \\ 0 = \ddot{v} + \dfrac{2\Lambda}{n-1}\,v\,, \end{cases}$$

whereas the second equation in (2.1.7) on the laplacian is just redundant. We distinguish three cases depending on the sign of Λ:

Case $\Lambda = 0$. From the equations above we obtain that E is Ricci flat and that v is an affine function, that is, $v = a\rho + b$ with $a, b \in \mathbb{R}$ and at least one between a and b different from zero. Up to rescaling g_E, we can suppose that $k = 1$. If $a \neq 0$ we can consider the new coordinate $r = a\rho + b$, so that we have $v = r$. If instead $a = 0$, then v is constant, and up to rescaling it, we can set $v = 1$. In the first case ($v = r$), since we want $u > 0$ in M and $u = 0$ on ∂M we have $M = [0, +\infty) \times E$. In the second case ($v = 1$) the function u is constant and never zero, hence the manifold M has no boundary and $M = \mathbb{R} \times E$.

Case $\Lambda > 0$. The system above gives

$$\begin{aligned} \mathrm{Ric}_{g_E} &= \frac{2\Lambda}{n-1}\,k^2\,g_E\,, \\ v &= A\,\sin\left(\sqrt{\frac{2\Lambda}{n-1}}\,\rho\right) + B\,\cos\left(\sqrt{\frac{2\Lambda}{n-1}}\,\rho\right)\,, \end{aligned} \tag{2.2.2}$$

where $A, B \in \mathbb{R}$ are constants. Up to rescaling g_E and u, we can set $k^2 = (n-1)(n-2)/(2\Lambda)$, so that $\mathrm{Ric}_{g_E} = (n-2)g_E$, and $\sqrt{A^2 + B^2} = 1$. It follows that there exists an angle $\theta \in [0, 2\pi]$ such that $\cos\theta = A$ and $\sin\theta = -B$, and

$$v = \sin\left(\sqrt{\frac{2\Lambda}{n-1}}\rho + \theta\right).$$

We can now consider the new coordinate $r = \sqrt{2\Lambda/(n-1)}\,\rho$, so that we get $v = \sin(r)$. In particular, we have $M = [0, \pi] \times E$.

<u>*Case* $\Lambda < 0$.</u> From the two equations in the system we get

$$\begin{aligned} \mathrm{Ric}_{g_E} &= \frac{2\Lambda}{n-1} k^2 g_E, \\ v &= A \sinh\left(\sqrt{-\frac{2\Lambda}{n-1}}\rho\right) + B \cosh\left(\sqrt{-\frac{2\Lambda}{n-1}}\rho\right), \end{aligned} \tag{2.2.3}$$

where $A, B \in \mathbb{R}$ are constants. Analogously to the case $\Lambda > 0$, we can rescale g_E so that $k^2 = (n-1)(n-2)/(-2\Lambda)$ and $\mathrm{Ric}_{g_E} = -(n-2)g_E$. In order to find nicer expressions for the function v, we have to distinguish three cases:

- if $|B| > |A|$, we first observe that B cannot be negative. In fact, otherwise we would have $u = v(\rho) < 0$ for all ρ, and we are not interested in these kind of solutions. Therefore, we can find a value $\theta \in \mathbb{R}$ such that $\cosh\theta = B/\sqrt{B^2 - A^2}$ and $\sinh\theta = A/\sqrt{B^2 - A^2}$. With this choice of θ, the function v rewrites as

$$v = \sqrt{B^2 - A^2}\cosh\left(\sqrt{-\frac{2\Lambda}{n-1}}\rho + \theta\right).$$

  Up to a rescaling of v and setting $r = \sqrt{-2\Lambda/(n-1)}\,\rho$, we can then achieve $v = \cosh(r)$. In particular, v is positive everywhere and so $M = \mathbb{R} \times E$.

- if $|A| > |B|$, then we can find a value $\theta \in \mathbb{R}$ such that $\cosh\theta = A/\sqrt{A^2 - B^2}$ and $\sinh\theta = B/\sqrt{A^2 - B^2}$. The function v can then be written as

$$v = \sqrt{A^2 - B^2}\sinh\left(\sqrt{-\frac{2\Lambda}{n-1}}\rho + \theta\right),$$

  and as usual up to a rescaling of v and a change to a new appropriate coordinate r we can make it so that $v = \sinh(r)$. It follows that the manifold M has to be chosen as $M = [0, +\infty) \times E$.

- if $|A| = |B|$, then again we necessarily have $B > 0$, otherwise $v < 0$ for all ρ. Moreover, up to change the sign of ρ, we can also suppose $A > 0$. Finally, rescaling u and changing to a new coordinate r, we can make it so that $v = \sinh(r) + \cosh(r) = e^r$. In particular, v is positive for all $r \in \mathbb{R}$, hence $M = \mathbb{R} \times E$.

This concludes the analysis of all the possible static solutions satisfying the hypotheses of Theorem 2.2.2. □

Now we pass to the classification of the warped product solutions with non-costant warping factor, that is, we want to find all solutions of the form

$$M = I \times E\,, \qquad g_0 \,=\, d\rho \otimes d\rho \,+\, f^2(\rho)\, g_E\,, \qquad u = v(\rho)\,. \tag{2.2.4}$$

Before stating the result, it is convenient to begin with some preliminary computations. Since we are requiring that f is not constant, there exists at least a value of ρ such that $\dot{f} \neq 0$, where $\dot{f}$ is the first derivative of f. In particular, in a neighborhood $J \subset I$ of that value, we can consider the new coordinate $r = f(\rho)$. With respect to r, in $J \times E$ the metric g_0 rewrites as

$$g_0 \,=\, \frac{dr \otimes dr}{V(r)} \,+\, r^2\, g_E\,, \quad \text{where } V(r) \,=\, \dot{f}^2(f^{-1}(\rho))\,. \tag{2.2.5}$$

This new coordinate system will help to simplify the following computations. Notice that formula (2.2.5) ceases to be valid when $V(r) = 0$, which corresponds, in the old coordinate, to $\dot{f}(\rho) = 0$. However, later we will see that u and V are necessarily related by $u^2 = V$. In particular, V is zero if and only if u is zero, which means that $V = 0$ only on ∂M. Therefore, *a posteriori*, $\dot{f}$ will be seen to be different from zero in the interior of M, hence $J = I$, that is, our change of coordinates is defined on the whole M. With respect to a metric of the form (2.2.5), using the formulæ for the Ricci tensor of a warped product shown in [Bes08, Proposition 9.106], system (2.1.7) rewrites as

$$\begin{cases} v\, \mathrm{Ric}_{g_E} - \left[\dfrac{1}{2} r\dot{V} + (n-2)V\right] v\, g_E \,=\, \left(rV\dot{v} \,+\, \dfrac{2\Lambda}{n-1}\, v\, r^2\right) g_E\,, \\ \qquad -\dfrac{n-1}{2}\,\dfrac{v\dot{V}}{r} \,=\, V\ddot{v} \,+\, \dfrac{1}{2}\dot{V}\dot{v} \,+\, \dfrac{2\Lambda}{n-1}\, v\,, \\ V\ddot{v} + \left[(n-1)\dfrac{V}{r} + \dfrac{1}{2}\dot{V}\right] \dot{v} \,=\, -\dfrac{2\Lambda}{n-1} v\,, \end{cases}$$

From the first equation we deduce that Ric_{g_E} is necessarily proportional to g_E, that is, (E, g_E) is Einstein. On the other hand, combining the second and third equalities, we obtain that the functions v, V satisfy

$$\dot{V}\, v \,-\, 2\, V\, \dot{v} \,=\, 0\,, \quad \text{or equivalently} \quad \frac{d}{dr}(\log V) \,=\, \frac{\dot{V}}{V} \,=\, 2\,\frac{\dot{v}}{v} \,=\, \frac{d}{dr}(\log(v^2))\,,$$

from which we deduce that, up to a normalization of v, it holds $v = \sqrt{V}$. Substituting in the second or third equation in the system, we find

$$\ddot{V} \,+\, \frac{n-1}{r}\,\dot{V} \,+\, \frac{4\Lambda}{n-1} \,=\, 0\,,$$

and this differential equation can be explicitly solved, giving

$$V \,=\, k - \frac{2\Lambda}{n(n-1)}\, r^2 \,-\, 2mr^{2-n}\,, \tag{2.2.6}$$

with $k, m \in \mathbb{R}$ constants. Now we turn back to the first equation. Putting all these new informations inside it, we find

$$\mathrm{Ric}_{g_E} = (n-2)\, k\, g_E\,.$$

Of course we can also rescale g_E in such a way that k is equal to 0, $+1$ or -1, depending on its sign. Moreover, notice that our solutions have a singularity when $r = 0$. In fact, the restriction of the metric g_0 to the slice E is $r^2 g_E$, thus the metric becomes singular as we approach $r = 0$. The same holds for u and V, because the term $-2mr^{2-n}$ diverges as $r \to 0$. For this reason, the connected interval of definition of the coordinate r cannot contain zero. Therefore, up to change the sign of r, we can suppose $r > 0$. It is clear that the behavior of V and g_0 depends on the signs of Λ, k, m. We are now ready to state our classification results. In order to simplify the formulæ, it is convenient to introduce the following notation. According to (2.2.6), given three constants $k, \Lambda, m \in \mathbb{R}$ we will denote by $V^{\Lambda}_{k,m}$ the function

$$V^{\Lambda}_{k,m}(r) = k - \frac{2\Lambda}{n(n-1)}\, r^2 - 2mr^{2-n}\,.$$

Since the list of solutions is rather long, we will separate the three cases $\Lambda = 0, \Lambda < 0, \Lambda > 0$. The proof is essentially just an analysis of all the possible combinations of $m(>, =, < 0)$ and $k(= +1, 0, -1)$.

Theorem 2.2.3 (Classification of Warped Product Solutions, case $\Lambda = 0$)**.** *Let (M, g_0, u) be a solution to the Static Einstein System (2.1.7) such that $u = 0$ on ∂M and $u > 0$ in the interior of M. Suppose that (M, g_0, u) is isometric to a triple*

$$M = I \times E\,, \qquad g_0 = d\rho \otimes d\rho + f^2(\rho)\, g_E\,, \qquad u = v(\rho)\,,$$

where (E, g_E) is a $(n-1)$-dimensional Riemannian manifold, $I \subset \mathbb{R}$ is an interval and $f : I \to \mathbb{R}$ and $v : I \to \mathbb{R}$ are smooth positive functions with f not constant. Then necessarily (E, g_E) is an Einstein manifold and up to rescaling we can suppose $\mathrm{Ric}_{g_E} = 0$ or $\mathrm{Ric}_{g_E} = (n-2)g_E$ or $\mathrm{Ric}_{g_E} = -(n-2)g_E$. We distinguish three cases depending on the sign of Ric_{g_E}:

- *if $\mathrm{Ric}_{g_E} = 0$ then (M, g_0, u) is isometric to the triple*

$$\left((0, +\infty) \times E\,,\ \frac{dr \otimes dr}{V^0_{0,m}} + r^2 g_E\,,\ \sqrt{V^0_{0,m}} \right)\,, \quad \textit{with } m < 0\,,$$

- *if $\mathrm{Ric}_{g_E} = (n-2)g_E$ then (M, g_0, u) is isometric to one of the following triples*

$$\left(\left[(2m)^{\frac{1}{n-2}}, +\infty\right) \times E\,,\ \frac{dr \otimes dr}{V^0_{1,m}} + r^2 g_E\,,\ \sqrt{V^0_{1,m}} \right)\,, \quad \textit{with } m > 0\,,$$

$$\left((0, +\infty) \times E\,,\ \frac{dr \otimes dr}{V^0_{1,m}} + r^2 g_E\,,\ \sqrt{V^0_{1,m}} \right)\,, \quad \textit{with } m \leq 0\,,$$

- *if* $\mathrm{Ric}_{g_E} = -(n-2)g_E$ *then* (M, g_0, u) *is isometric to the triple*

$$\left(\left(0,(-2m)^{\frac{1}{n-2}}\right)\times E\,,\,\frac{dr\otimes dr}{V^0_{-1,m}}+r^2 g_E\,,\,\sqrt{V^0_{-1,m}}\right)\,,\quad \textit{with } m<0\,.$$

Proof. With respect to the change of coordinates $r = f(\rho)$ introduced in (2.2.5), the computations above tell us that necessarily

$$g_0 \;=\; \frac{dr\otimes dr}{V(r)}+r^2 g_E\,,\quad u=v(r)=\sqrt{V(r)}\,,\qquad \text{with}\quad V(r)=k-2mr^{2-n}\,.$$

The behavior of the function V changes depending on the sign of k and m, so we distinguish different cases, depending on whether $k = +1, 0, -1$, $m >, =, < 0$.

- If $m = 0$ then, in order for V to be positive, we necessarily have $k = 1$, and we obtain the solution

$$u \;=\; v(r) \;=\; 1,\quad g_0 \;=\; dr\otimes dr \;+\; r^2 g_E\,,$$

  with $\mathrm{Ric}_{g_E} = (n-2)g_E$. This solution is well defined for all $r > 0$, hence $M = (0, +\infty) \times E$.

- If $m > 0$, then the cases $k = 0, -1$ are not admissible, because V would be negative for all $r > 0$. Therefore $k = 1$ and we have the solution $V(r) = 1 - 2mr^{2-n}$, which is only defined on $M = [r_0, +\infty) \times E$, where $r_0 = (2m)^{1/(n-2)}$. It is easily seen that the obtained metric

$$g_0 \;=\; \frac{dr\otimes dr}{V(r)}+r^2 g_E\,,$$

  which is a priori defined only in the interior of M because $V(r_0) = 0$, extends up to the boundary. In fact, one can easily check that $\dot V(r_0) \neq 0$, hence the function V can be rewritten as $V(r) = \chi(r)\cdot(r-r_0)$, where χ is a function such that $\chi(r_0) \neq 0$. Turning back to the original coordinate $\rho = f^{-1}(r)$, recalling the relation $\dot f^2(\rho) = V(r)$, one computes that the function f near $f^{-1}(r_0)$ expands as

$$f(\rho) \;=\; r_0 \;+\; \frac{1}{4}\,\chi(r_0)\,\rho^2 \;+\; o(\rho^2)\,.$$

  In particular, we have $f(0) = r_0$, so the metric g_0 is defined for all $\rho \in [0, +\infty)$, and it behaves near $\{r = r_0\} = \{\rho = 0\}$ as

$$g_0 \;=\; d\rho\otimes d\rho \;+\; r_0^2\, g_E\,.$$

  It is now clear that the metric g_0 is defined on $M = [r_0, +\infty) \times E$, and it induces the metric $r_0^2 g_E$ on the boundary $\partial M = \{r_0\} \times E$.

- If $m < 0$, we distinguish three cases depending on the sign of k
  - if $k = 0$, we have that V is positive for $r > 0$, hence $M = (0, +\infty) \times E$.

- if $k = +1$, then $V = 1 - 2mr^{2-n}$ is positive for all $r > 0$, hence $M = (0, +\infty) \times E$.
- if $k = -1$, then $V = -1 - 2mr^{2-n}$ is positive for $r \in (0, r_0)$, with $r_0 = (-2m)^{1/(n-2)}$, hence $M = (0, r_0] \times E$. Proceeding as in the case $m > 0$ above, one can show that the metric g_0 given by (2.2.5) with respect to these choices of m, k, V, extends smoothly up to the boundary $\partial M = \{r_0\} \times E$.

This concludes the characterization in the case $\Lambda = 0$. □

Theorem 2.2.4 (Classification of Warped Product Solutions, case $\Lambda > 0$). *Let (M, g_0, u) be a solution to the Static Einstein System (2.1.7) such that $u = 0$ on ∂M and $u > 0$ in the interior of M. Suppose that (M, g_0, u) is isometric to a triple of the form*

$$M = I \times E\,, \qquad g_0 \,=\, d\rho \otimes d\rho \,+\, f^2(\rho)\, g_E\,, \qquad u = v(\rho)\,,$$

where (E, g_E) is a $(n-1)$-dimensional Riemannian manifold, $I \subset \mathbb{R}$ is an interval and $f : I \to \mathbb{R}$ and $v : I \to \mathbb{R}$ are smooth positive functions with f not constant. Then necessarily (E, g_E) is an Einstein manifold and up to rescaling we can suppose $\mathrm{Ric}_{g_E} = 0$ or $\mathrm{Ric}_{g_E} = (n-2)g_E$ or $\mathrm{Ric}_{g_E} = -(n-2)g_E$. We distinguish three cases depending on the sign of Ric_{g_E}:

- *if $\mathrm{Ric}_{g_E} = 0$ then (M, g_0, u) is isometric to the triple*

$$\left(\left(0, \left|\frac{n(n-1)m}{\Lambda}\right|^{\frac{1}{n}}\right] \times E\,,\; \frac{dr \otimes dr}{V^{\Lambda}_{0,m}} + r^2 g_E\,,\; \sqrt{V^{\Lambda}_{0,m}} \right)\,, \quad \textit{with } m < 0\,,$$

- *if $\mathrm{Ric}_{g_E} = (n-2)g_E$ then (M, g_0, u) is isometric to one of the following triples*

$$\left([r_-(m), r_+(m)] \times E\,,\; \frac{dr \otimes dr}{V^{\Lambda}_{1,m}} + r^2 g_E\,,\; \sqrt{V^{\Lambda}_{1,m}} \right)\,, \quad \textit{with } 0 < m < \bar{m}\,,$$

$$\left((0, r_0(m)] \times E\,,\; \frac{dr \otimes dr}{V^{\Lambda}_{1,m}} + r^2 g_E\,,\; \sqrt{V^{\Lambda}_{1,m}} \right)\,, \quad \textit{with } m \le 0\,,$$

where, for any $m \le 0$ the value $r_0(m)$ is the unique positive solution to

$$1 - \frac{2\Lambda}{n(n-1)} r_0^2(m) - 2m r_0^{2-n}(m) \,=\, 0\,,$$

whereas for $m > 0$ the numbers $0 < r_-(m) < r_+(m)$ are the two positive solutions to

$$1 - \frac{2\Lambda}{n(n-1)} r_\pm^2(m) - 2m r_\pm^{2-n}(m) \,=\, 0\,,$$

which only exist when

$$m < \bar{m} := \frac{1}{n} \left[\frac{(n-1)(n-2)}{2\Lambda} \right]^{\frac{n-2}{2}}\,,$$

- *if* $\mathrm{Ric}_{g_E} = -(n-2)g_E$ *then* (M, g_0, u) *is isometric to the triple*

$$\left((0, r_1(m)] \times E\,,\, \frac{dr\otimes dr}{V^\Lambda_{-1,m}} + r^2 g_E\,,\, \sqrt{V^\Lambda_{-1,m}} \right)\,, \quad \textit{with } m<0\,,$$

where for any $m<0$ *the value* $r_1(m)$ *is the unique positive solution to*

$$-1 - \frac{2\Lambda}{n(n-1)} r_1^2(m) - 2m r_1^{2-n}(m) \;=\; 0\,.$$

Proof. With respect to the change of coordinates $r = f(\rho)$ discussed above, we have

$$g_0 \;=\; \frac{dr\otimes dr}{V(r)} + r^2 g_E\,, \quad u = \sqrt{V(r)}\,, \qquad \text{with} \quad V(r) = k - \frac{2\Lambda}{n(n-1)}\,r^2 \,-\, 2mr^{2-n}\,,$$

and we have to study the function V depending on the choices of $k = +1, 0, -1$ and $m >, =, < 0$.

- If $k = 0$, then m has to be negative, otherwise V would be nonpositive for all $r > 0$. For $m < 0$, we have that V is nonnegative when $0 < r \leq r_0$, with

$$r_0 = \left[-\frac{n(n-1)m}{\Lambda} \right]^{\frac{1}{n}}.$$

  With these choices, the metric g_0 is defined on the whole $M = (0, r_0] \times E$, and extends smoothly up to the boundary $\{r_0\} \times E$ as usual.

- If $k = -1$, then again we need $m < 0$ in order for V to be positive for some values of r. If $m < 0$, then V is positive in $M = (0, r_0] \times E$, where r_0 is the unique positive solution to

$$-1 - \frac{2\Lambda}{n(n-1)}\,r_0^2 \,-\, 2mr_0^{2-n} \;=\; 0\,.$$

  Again, the metric g_0 obtained through (2.2.5) extends up to the boundary of M.

- If $k = 1$, then we have to distinguish two cases depending on the sign of m.

  - If $m > 0$, then one can compute that the equation

$$1 - \frac{2\Lambda}{n(n-1)}\,r^2 \,-\, 2mr^{2-n} \;=\; 0\,,$$

    has two positive solutions $r_- \leq r_+$ if and only if $m < \bar{m}$, where

$$\bar{m} \;=\; \frac{1}{n}\left[\frac{(n-1)(n-2)}{2\Lambda}\right]^{\frac{n-2}{2}}.$$

    Therefore, for $0 < m < \bar{m}$ we have that V is positive in $M = [r_-, r_+] \times E$, whereas if $m \geq m_+$, the function V is everywhere nonpositive.

– If $m \leq 0$, then the function V is positive in $0 < r \leq r_0$, where r_0 is the unique positive solution of

$$1 - \frac{2\Lambda}{n(n-1)}\, r_0^2 \,-\, 2m r_0^{2-n} \;=\; 0\,.$$

In the case $m = 0$, r_0 can be explicitated as

$$r_0 \;=\; \sqrt{\frac{n(n-1)}{2\Lambda}}\,.$$

$\square$

Theorem 2.2.5 (Classification of Warped Product Solutions, case $\Lambda < 0$)**.** *Let (M, g_0, u) be a solution to the Static Einstein System (2.1.7) such that $u = 0$ on ∂M and $u > 0$ in the interior of M. Suppose that (M, g_0, u) is isometric to a triple of the form*

$$M = I \times E\,, \qquad g_0 \;=\; d\rho \otimes d\rho \,+\, f^2(\rho)\, g_E\,, \qquad u = v(\rho)\,,$$

where (E, g_E) is a $(n-1)$-dimensional Riemannian manifold, $I \subset \mathbb{R}$ is an interval and $f : I \to \mathbb{R}$ and $v : I \to \mathbb{R}$ are smooth positive functions with f not constant. Then necessarily (E, g_E) is an Einstein manifold and up to rescaling we can suppose $\mathrm{Ric}_{g_E} = 0$ or $\mathrm{Ric}_{g_E} = (n-2)g_E$ or $\mathrm{Ric}_{g_E} = -(n-2)g_E$. We distinguish three cases depending on the sign of Ric_{g_E}:

- *if $\mathrm{Ric}_{g_E} = 0$ then (M, g_0, u) is isometric to one of the following triples*

$$\left(\left[\left|\frac{n(n-1)m}{\Lambda}\right|^{\frac{1}{n}}, +\infty\right) \times E\,,\; \frac{dr \otimes dr}{V_{0,m}^{\Lambda}} + r^2 g_E\,,\; \sqrt{V_{0,m}^{\Lambda}}\right)\,, \quad \text{with } m > 0\,,$$

$$\left((0, +\infty) \times E\,,\; \frac{dr \otimes dr}{V_{0,m}^{\Lambda}} + r^2 g_E\,,\; \sqrt{V_{0,m}^{\Lambda}}\right)\,, \quad \text{with } m \leq 0\,,$$

- *if $\mathrm{Ric}_{g_E} = (n-2)g_E$ then (M, g_0, u) is isometric to one of the following triples*

$$\left([r_1(m), +\infty) \times E\,,\; \frac{dr \otimes dr}{V_{1,m}^{\Lambda}} + r^2 g_E\,,\; \sqrt{V_{1,m}^{\Lambda}}\right)\,, \quad \text{with } m > 0\,,$$

$$\left((0, +\infty) \times E\,,\; \frac{dr \otimes dr}{V_{1,m}^{\Lambda}} + r^2 g_E\,,\; \sqrt{V_{1,m}^{\Lambda}}\right)\,, \quad \text{with } m \leq 0\,,$$

where we have denoted by $r_1(m)$, for any $m > 0$, the unique positive solution of

$$1 - \frac{2\Lambda}{n(n-1)} r_1^2(m) - 2m r_1^{2-n}(m) \;=\; 0\,,$$

- *if* $\mathrm{Ric}_{g_E} = -(n-2)g_E$ *then* (M, g_0, u) *is isometric to one of the following triples*

$$\left([r_+(m), +\infty) \times E\,,\, \frac{dr\otimes dr}{V^{\Lambda}_{-1,m}} + r^2 g_E\,,\, \sqrt{V^{\Lambda}_{-1,m}}\right)\,, \quad \textit{with } m > -\bar{m}\,,$$

$$\left((0, r_-(m)] \times E\,,\, \frac{dr\otimes dr}{V^{\Lambda}_{-1,m}} + r^2 g_E\,,\, \sqrt{V^{\Lambda}_{-1,m}}\right)\,, \quad \textit{with } -\bar{m} < m < 0\,,$$

$$\left(\left((n\bar{m})^{\frac{1}{n-2}}, +\infty\right) \times E\,,\, \frac{dr\otimes dr}{V^{\Lambda}_{-1,-\bar{m}}} + r^2 g_E\,,\, \sqrt{V^{\Lambda}_{-1,-\bar{m}}}\right)\,,$$

$$\left(\left(0, (n\bar{m})^{\frac{1}{n-2}}\right) \times E\,,\, \frac{dr\otimes dr}{V^{\Lambda}_{-1,-\bar{m}}} + r^2 g_E\,,\, \sqrt{V^{\Lambda}_{-1,-\bar{m}}}\right)\,,$$

$$\left((0, +\infty) \times E\,,\, \frac{dr\otimes dr}{V^{\Lambda}_{-1,m}} + r^2 g_E\,,\, \sqrt{V^{\Lambda}_{-1,m}}\right)\,, \quad \textit{with } m < -\bar{m}\,,$$

where we have set

$$\bar{m} = \frac{1}{n}\left|\frac{(n-1)(n-2)}{2\Lambda}\right|^{\frac{n-2}{2}}\,,$$

and we have denoted by $r_+(m)$*, for any* $m > -\bar{m}$*, the biggest positive solution of*

$$-1 - \frac{2\Lambda}{n(n-1)} r_+^2(m) - 2m r_+^{2-n}(m) = 0\,.$$

In case there is another positive solution of the equation above (this happens for $-\bar{m} < m < 0$*), we have denoted it by* $r_-(m)$*.*

Proof. Again, with respect to the change of coordinates $r = f(\rho)$, we have shown before that necessarily

$$g_0 = \frac{dr\otimes dr}{V(r)} + r^2 g_E\,, \quad u = \sqrt{V(r)}\,, \qquad \text{with} \quad V(r) = k - \frac{2\Lambda}{n(n-1)}\, r^2 - 2m r^{2-n}\,.$$

We now study the possible combinations of signs of k and m. We first distinguish depending on k.

- If $k = 0$, we have
$$V = -\frac{2\Lambda}{n(n-1)} r^2 - 2m r^{2-n}\,.$$
If $m \leq 0$, this solution is defined on the whole $M = (0, +\infty) \times E$. If $m > 0$, then V is nonnegative only for $r > r_0$, where
$$r_0 = \left[-\frac{n(n-1)m}{\Lambda}\right]^{\frac{1}{n}}\,.$$

- If $k = +1$, then
$$V = 1 - \frac{2\Lambda}{n(n-1)} r^2 - 2m r^{2-n}\,,$$

which is positive on the whole $M = (0, +\infty) \times E$ if $m \leq 0$. If $m > 0$, then V is nonnegative only for $r \geq r_0$, where r_0 is the positive solution of

$$1 - \frac{2\Lambda}{n(n-1)} r_0^2 - 2m r_0^{2-n} \; = \; 0 \, .$$

In this case we are not able to provide an explicit formula for r_0.

- If $k = -1$, then
$$V = -1 - \frac{2\Lambda}{n(n-1)} r^2 - 2m r^{2-n} \, .$$
If $m \geq 0$, then V is positive for $r \geq r_0$, where r_0 is the only positive solution to
$$-1 - \frac{2\Lambda}{n(n-1)} r_0^2 - 2m r_0^{2-n} \; = \; 0 \, .$$
If $m < 0$, we have to distinguish three cases, depending on whether m is less than, equal to or greater than $-\bar{m}$, with
$$\bar{m} \; = \; \frac{1}{n} \left[-\frac{(n-1)(n-2)}{2\Lambda} \right]^{\frac{n-2}{2}} \, .$$
  - If $m < -\bar{m}$, then one can check that V is always positive for $r > 0$, hence our solution is defined on the whole $M = (0, +\infty) \times E$.
  - If $m > -\bar{m}$, then $V(r) = 0$ has two positive solutions $r_-(m) < r_+(m)$ and V is nonnegative when $0 < r \leq r_-$ and when $r > r_+$. Therefore, we have two choices for the manifold M, that is, we can set $M = (0, r_-] \times E$ or $M = [r_+, +\infty) \times E$. In both cases, the metric g_0 defined by (2.2.5) extends up to the boundaries $\{r_-\} \times E$ and $\{r_+\} \times E$ in the usual way.
  - If $m = -\bar{m}$, then V is always nonnegative, but it has a zero at
  $$r_0 = \left[-\frac{n(n-1)(n-2)\bar{m}}{2\Lambda} \right]^{\frac{1}{n}} \; = \; \sqrt{-\frac{(n-1)(n-2)}{2\Lambda}} \, .$$
  Therefore, we have again two choices for the manifold M. We can set $M = (0, r_0) \times E$ or $M = (r_0, +\infty) \times E$. However, notice that, contrarily to the case $m > \bar{m}$, in the case $m = \bar{m}$ the metric does not extend up to the boundary. An easy but indirect way to see this is by observing that, if g_0 could be extended up to the boundary, then that boundary would have surface gravity $|\mathrm{D}u| = 0$, and this is impossible as discussed in Subsection 2.1.4. A more direct way of seeing it is to observe that, since in this case the derivative of V with respect to r annihilates at r_0 (but not its second derivative, as one can check), then V can be rewritten as $V(r) = \chi(r)(r - r_0)^2$, where χ is a function with $\chi(r_0) \neq 0$. Returning to the old coordinate $\rho = f^{-1}(r)$, and recalling that $\dot{f}^2(\rho) = V(r)$, we see that f behaves near $f^{-1}(r_0)$ as
  $$f(\rho) \; = \; r_0 + e^{\rho} \, .$$

In particular, the value $r = r_0$ corresponds to the limit $\rho \to -\infty$. From this one easily deduces that the geodesics of M do not reach $\{r_0\} \times E$ in a finite time, hence $\{r_0\} \times E$ has to be regarded as an end of the manifold and not as a boundary.

This concludes the analysis of all the cases, and the classification of the warped product solutions is now complete. □

Looking at the list of solutions in Theorems 2.2.3, 2.2.5 and 2.2.4, one sees that there are a number of triples with nonzero masses and with a singularity in zero. For all these solutions, one can check that zero is in fact a *naked singularity*. This means that the solution is metrically incomplete, that is, geodesics can reach zero in a finite time, but the metric cannot be extended beyind it. The inextensibility of the metric can be shown by observing that the function $V^{\Lambda}_{k,m}$ behaves near $r = 0$ as r^{2-n}, and using this information to compute that the norm $|\operatorname{Riem}|^2$ of the Riemannian curvature tensor blows up as we approach the singularity. In the rest of the paper we will concentrate on metrically complete solutions, and for this reason we are going to ignore this family of solutions.

Among the listed solutions, there are also three triples with zero mass and with $\operatorname{Ric}_{g_E} = (n-2)g_E$, which we write down here more explicitly

$$\big((0,+\infty) \times E\,,\, dr \otimes dr + r^2 g_E\,,\, 1\big)\,, \quad \text{with } \Lambda = 0\,,$$

$$\left(\left(0, \sqrt{\frac{n(n-1)}{2\Lambda}}\right] \times E\,,\, \frac{dr \otimes dr}{1 - \frac{2\Lambda}{n(n-1)} r^2} + r^2 g_E\,,\, \sqrt{1 - \frac{2\Lambda}{n(n-1)} r^2}\right)\,, \quad \text{with } \Lambda > 0,$$

$$\left((0,+\infty) \times E\,,\, \frac{dr \otimes dr}{1 - \frac{2\Lambda}{n(n-1)} r^2} + r^2 g_E\,,\, \sqrt{1 - \frac{2\Lambda}{n(n-1)} r^2}\right)\,, \quad \text{with } \Lambda < 0\,.$$

These three metrics behave near $r = 0$ as $dr \otimes dr + r^2 g_E$. It follows that the geodesics reach $r = 0$ in a finite time, but the metric on the second factor, $r^2 g_E$, becomes degenerate as we approach $r = 0$. Therefore, the metric above are singular, except for one case: when E is isometric to the round sphere $(\mathbb{S}^{n-1}, g_{\mathbb{S}^{n-1}})$, then we can see the metrics above as polar coordinates and add the point in the center. In this way, the metrics above becomes complete, and we obtain the flat space form when $\Lambda = 0$, the spherical space form when $\Lambda > 0$ and the hyperbolic space form when $\Lambda < 0$.

Finally, in the case $\Lambda < 0$, there are two other solutions with $m = 0$, one with $\operatorname{Ric}_{g_E} = -(n-2)g_E$, which we have already shown being metrically complete during the proof of Theorem 2.2.5, and one with $\operatorname{Ric}_{g_E} = 0$ that we rewrite here

$$\left((0,+\infty) \times E\,,\, \frac{dr \otimes dr}{-\frac{2\Lambda}{n(n-1)} r^2} + r^2 g_E\,,\, \sqrt{-\frac{2\Lambda}{n(n-1)} r^2}\right)\,.$$

One can easily check that the geodesics do not reach $r = 0$ in finite time, which means that this solution does not have a naked singularity in $r = 0$. Hence, this solution is metrically complete.

2.3 Static solutions with $\Lambda = 0$

2.3.1 Statement of the problem and main solutions.

Let us start by rewriting system (2.1.7) in the case where $\Lambda = 0$. For simplicity, we add the natural request that the boundary of the manifold is compact.

$$\begin{cases} u\,\mathrm{Ric} = \mathrm{D}^2 u\,, & \text{in } M\,, \\ \Delta u = 0\,, & \text{in } M\,, \\ u > 0\,, & \text{in } M \setminus \partial M\,, \\ u = 0\,, & \text{on } \partial M\,, \end{cases} \qquad \text{with } \partial M \text{ compact}\,. \tag{2.3.1}$$

The manifold M is usually taken to be noncompact, but let us briefly address the compact case. If M is compact, then $\partial M = \{u = 0\}$ in necessarily empty. In fact, otherwise, since u is harmonic, from the Maximum and Minimum Principle we would deduce that $u \equiv 0$ on the whole M, against the hypothesis that u is positive in the interior. Now that $\partial M = \emptyset$, arguing using the Mean Value Property around a local maximum or minimum point, we obtain that necessarily u is constant. Therefore, up to a normalization, we have $u \equiv 1$ on the whole M, and from the first equation in (2.3.1) we obtain that (M, g_0) is Ricci flat. Ricci flat manifolds are fairly well understood, see for instance [FW74, FW75]. More generally, we will see later (see Theorem 2.3.1) that any (compact or noncompact) solution with empty boundary is Ricci flat.

Recalling the classification in Theorem 2.2.3, we now list all the possible *metrically complete* warped product solutions of problem (2.3.1), and we discuss them in some details.

- Minkowski solution (Flat Space Form).

$$\big(\,\mathbb{R}^n\ ,\ d|x| \otimes d|x| + |x|^2 g_{\mathbb{S}^{n-1}}\ ,\ 1\,\big) \tag{2.3.2}$$

  It is easy to check that the metric g_0, which a priori is well defined only in $M \setminus \{0\}$, extends smoothly through the origin. This model solution has zero mass and can be seen as the limit of the following Schwarzschild solutions, when the parameter $m \to 0^+$.

- Schwarzschild solutions.

$$\left(\,[r_0(m), +\infty) \times E\ ,\ \frac{dr \otimes dr}{1 - 2mr^{2-n}} + r^2 g_E\ ,\ \sqrt{1 - 2mr^{2-n}}\,\right) \tag{2.3.3}$$
$$\text{with } m > 0 \text{ and } (E^{n-1}, g_E) \text{ such that } \mathrm{Ric}_{g_E} = (n-2) g_E\,.$$

  Here, the so called Schwarzschild radius $r_0(m) = (2m)^{1/(n-2)}$ is the only positive solution to $1 - 2mr^{2-n} = 0$. It is not hard to check that both the metric g_0 and the function u, which a priori are well defined only in the interior of M, extend smoothly up to the boundary. The solutions of the form (2.3.3) were found by Schwarzschild [Sch03] just months after the publication of Einstein's theory of General Relativity, and they have an important hystorical meaning. In fact, they were the first nontrivial example of static spacetimes and allowed to compute the exact precession of Mercury, thus providing the first strong proof in support of General Relativity.

- Boost solutions.

$$\begin{gathered}\Big([0,+\infty)\times E\ ,\ dr\otimes dr+g_E\ ,\ r\Big)\\ \text{with } (E^{n-1},g_E) \text{ such that } \mathrm{Ric}_{g_E}=0\,.\end{gathered} \tag{2.3.4}$$

  Usually E is taken to be a torus, so that the boundary $\partial M=\{0\}\times E$ is compact. The Boost triples (2.3.4) are the simplest solutions to problem (2.3.1) that are not asymptotically flat in the sense of Definition 2.3.2 below.

- Cylindrical solutions.

$$\begin{gathered}\Big((-\infty,+\infty)\times E\ ,\ dr\otimes dr+g_E\ ,\ 1\Big)\\ \text{with } (E^{n-1},g_E) \text{ such that } \mathrm{Ric}_{g_E}=0\,.\end{gathered} \tag{2.3.5}$$

  These solutions are probably the less meaningful among this list. In fact, in the 3-dimensional case (which is the most important from a physical point of view) the pair (E,g_E) is necessarily a flat torus, so that this solution is just a quotient of the Minkowski solution (2.3.2).

Aside from the warped solutions described above, another important family of solutions is that of the Myers/Korotkin-Nicolai black holes [Mye87, KN94]. These solutions are obtained by superposing along an axis an infinite number of Schwarzschild black holes, separated by a fixed distance ℓ. This way, one obtains a periodic infinite triple, which can be seen to still be a solution to problem (2.3.1). This periodic infinite solution is then quotiented by a translation of $k\ell$ along the axis (with k positive integer), obtaining solutions that are toroidals, with k holes corresponding to the original Schwarzschild black holes, all separated by a distance ℓ.

2.3.2 Main properties.

The case $\Lambda=0$ is the one where the strongest results are known. We will discuss the main properties here. First of all, we recall the following strong characterization of the case where M has empty boundary.

Theorem 2.3.1 (Uniqueness of Minkowski spacetime)**.** *Let (M,g_0,u) be a solution to problem* (2.3.1)*. If $\partial M=\emptyset$, then $u\equiv 1$ and (M,g_0) is Ricci flat. In particular, in dimension $n=3$, (M,g_0,u) is covered by the Minkowski solution* (2.3.2)*.*

The above theorem was proved by Anderson [And99] in dimension $n=3$, and then generalized by Case [Cas10] to arbitrary dimensions $n\geq 3$. Case's proof will be discussed in Section 3.1.

When the boundary is not empty the analysis is more complicated. In order to obtain interesting uniqueness results, it is usually assumed that the triple

(M, g_0, u) is asymptotically flat in the following sense (see for instance [AM17b, Definition 1]).

Definition 2.3.2 (Asymptotically flat static triples)**.** *A triple (M, g_0, u) that solves problem (2.3.1) is said to be* asymptotically flat *if the following conditions hold*

(i) There exists a compact set $K \subset M$ such that, for each connected component U of $M \setminus K$, there exists a diffeomorphism $x^{(U)} = (x^1, \dots, x^n) : U \to \mathbb{R}^n \setminus \mathbb{B}^n$, where $\mathbb{B}^n$ is the unit ball. The component U is called end *of M, whereas the diffeomorphism $x^{(U)}$ is called* chart at infinity *of U.*

(ii) In each connected component U of $M \setminus K$, the metric g_0 and the potential u can be expressed with respect to the coordinates induced by the diffeomorphism $x^{(U)}$ as

$$\begin{aligned} g^{(0)}_{\alpha\beta} &= \delta_{\alpha\beta} + \eta_{\alpha\beta}\,, \qquad u \,=\, 1 - \frac{2m}{|x|^{n-2}} + w\,, \\ \textit{with } \eta_{\alpha\beta} &= o(|x|^{\frac{2-n}{2}})\,, \ \frac{\partial \eta_{\alpha\beta}}{\partial x^\gamma} = o(|x|^{-\frac{n}{2}})\,, \ \frac{\partial^2 \eta_{\alpha\beta}}{\partial x^\gamma x^\sigma} = o(|x|^{\frac{-2-n}{2}}) \\ \textit{and } \ w &= o(|x|^{2-n})\,, \ \frac{\partial w}{\partial x^\gamma} = o(|x|^{1-n})\,, \ \frac{\partial^2 w}{\partial x^\gamma x^\sigma} = o(|x|^{-n}) \end{aligned}$$

for every $\alpha, \beta, \gamma, \sigma = 1, \dots, n$.

Notice that any asymptotically flat triple (M, g_0, u) is such that the manifold (M, g_0) is asymptotically flat in the sense of Definition 1.2.1. Conversely, it was proved by Beig [Bei80] that, if (M, g_0, u) is a solution to problem (2.3.1) such that (M, g_0) is asymptotically flat in the sense of Definition 1.2.1 and if u is bounded in M, then, up to a normalization of u, (M, g_0, u) is asymptotically flat in the sense of Definition 2.3.2.

For asymptotically flat static triples (M, g_0, u), one can introduce the following (Newtonian-like) notion of mass

$$m(M, g_0, u) \;=\; \frac{1}{(n-2)|\mathbb{S}^{n-1}|} \int_\Sigma \langle\, \mathrm{D}u \,|\, \nu \,\rangle \, \mathrm{d}\sigma\,,$$

where Σ is any closed two sided regular hypersurface enclosing the compact boundary of M and ν its exterior unit normal (see for instance [Ced12, Corollary 4.2.4] and the discussion below). Using the Divergence Theorem and the fact that $\Delta u = 0$, one has that the above quantity may also be computed in the following two ways

$$\begin{aligned} m(M, g_0, u) &= \frac{1}{(n-2)|\mathbb{S}^{n-1}|} \int_{\partial M} |\mathrm{D}u| \, \mathrm{d}\sigma \\ &= \frac{1}{(n-2)|\mathbb{S}^{n-1}|} \lim_{R \to +\infty} \int_{S_R} \langle\, \mathrm{D}u \,|\, \nu \,\rangle \, \mathrm{d}\sigma\,, \end{aligned} \tag{2.3.6}$$

where the set S_R that appears in the rightmost hand term is a coordinates sphere of radius R. In other words, if $x^1, \dots, x^n$ are asymptotically flat coordinates, then $S_R = \{|x| = R\}$. In particular, we notice that the last expression makes sense even if M has no boundary and agrees (up to a constant factor) with the more general concept of Komar mass (or KVM mass) as it is defined in [Bei79, Formula (4)]. In the same paper, it is proven that the above definition agrees with the notion of ADM mass of the asymptotically flat manifold (M, g_0), which we have introduced in Definition 1.2.2 and which we recall here

$$m_{ADM}(M, g_0) \;=\; \frac{1}{2(n-1)|\mathbb{S}^{n-1}|} \lim_{R\to+\infty} \int_{S_R} \sum_{i,j} \left[\frac{\partial (g_0)_{ij}}{\partial x^i} - \frac{\partial (g_0)_{ii}}{\partial x^j} \right] \nu^j \, d\sigma \,.$$

On the other hand, when the boundary of M is non-empty, the first expression in (2.3.6) is also of some interest, since it can be employed to prove that, when ∂M is connected, surface gravity and mass are simply proportional to each other. Summarising the above discussion, we have that for a static vacuum asymptotically flat solution (M, g, u) to (2.1.7) with $\Lambda = 0$ and compact connected boundary ∂M, one has that

$$\frac{|\partial M|}{(n-2)|\mathbb{S}^{n-1}|}\, \kappa(\partial M) \;=\; m_{ADM}(M, g_0)\,.$$

This kind of considerations will be extremely important for the discussion in Chapters 3 and 4, where, based on the concept of *surface gravity*, we are going to propose a definition of mass for static solutions with positive cosmological constant. In fact, unlike for the cases $\Lambda = 0$ and $\Lambda < 0$, where natural asymptotic conditions (i.e., asymptotical flatness and asymptotical hyperbolicity) can be imposed to the solutions in order to deal with objects for which efficient notions of mass are available and fairly well understood (see Section 1.2), there is no clear definition of mass in the case where the cosmological constant is taken to be positive, at least to the authors' knowledge.

Going back, to solutions of problem (2.3.1), we have already shown the strong characterization of the boundaryless case given in Theorem 2.3.1. In the case where $\partial M \neq \emptyset$, the classical requirement of asymptotic flatness allows to prove the following well known uniqueness result for the Schwarzschild solution.

Theorem 2.3.3 (Black Hole Uniqueness Theorem for $\Lambda = 0$)**.** *Let (M, g_0, u) be an asymptotically flat solution to problem* (2.3.1)*, in the sense of Definition 2.3.2. Then the triple (M, g_0, u) is isometric to the Schwarzschild solution.*

This very general result was first proved in dimension $n = 3$ thanks to the works of Israel [Isr67], Robinson [Rob77] and Bunding and Masood-ul-Alam [BMu87]. In particular, the argument in the proof of [BMu87] was generalized in [GIS02, Hwa98] to dimensions $n \geq 3$, provided the Positive Mass Theorem 1.2.3 of Schoen and Yau was in force. Since the Positive Mass Theorem has recently been proved for any $n \geq 3$, Theorem 2.3.3 is now known to hold in all dimensions.

An improvement of Theorem 2.3.3 has been proposed by Reiris [Rei15], who relaxed the hypothesis of asymptotic flatness to the notion of *geodesic completeness*.

Theorem 2.3.4 ([Rei15, Theorem 1.2])**.** *Let (M, g_0, u) be a 3-dimensional solution to problem* (2.3.1) *with $\partial M \neq \emptyset$. If the following conditions hold*

- *the spacetime*
$$(X, \gamma) = \left(\mathbb{R} \times M \,,\, \frac{dt \otimes dt}{u^2} + u^2 g_0 \right)$$
is geodesically complete*, that is, every geodesic is either infinite or it stops at the boundary* $\mathbb{R} \times \partial M$*,*

- *there exists a compact set* K *such that a connected component of* $M \setminus K$ *is diffeomorphic to* $\mathbb{R}^n \setminus \mathbb{B}^n$*,*

then (M, g_0, u) *is isometric to the Schwarzschild solution* (2.3.3)

Building on this result, Reiris proposed a classification of 3-dimensional static spacetimes with $\Lambda = 0$ without any condition on the asymptotics. It is stated in [Rei, Theorem 1.0.2] that a 3-dimensional metrically complete static solution to problem (2.3.1) with nonempty boundary, is

(i) either isometric to the Schwarzschild solution (2.3.3), or

(ii) covered by a Boost (2.3.4), or

(iii) it has the same topology and asymptotic behavior of the Myers/Korotkin-Nicolai black holes [Mye87, KN94].

It follows that, in order to complete the study of 3-dimensional static spacetimes with zero cosmological constant, it only remains to classify the static solutions of type (iii).

2.4 Static solutions with $\Lambda < 0$

2.4.1 Statement of the problem and main solutions.

In this section we study static solutions (M, g_0, u) of problem (2.1.7) in the case where the cosmological constant Λ is negative. First of all, let us rescale the metric g_0 in such a way that $\Lambda = -n(n-1)/2$. We recall that the static potential u is taken to be positive in the interior of M and zero at the boundary ∂M. As in the case $\Lambda = 0$, it is also usual to require that ∂M is compact. On the other hand, we observe that M has at least one end. In fact, if M were compact, since from the second equation in (2.1.7) we have $\Delta u = -nu < 0$, arguing exactly as in the case $\Lambda = 0$, we would find that the function u is constant and nonzero on M. But this is in contradiction with the second equation in (2.1.7). Finally, we observe that the static potential of the solutions listed in Theorem 2.2.5 is usually unbounded. From these considerations, we are led to study the following problem

$$\begin{cases} u \,\mathrm{Ric} = \mathrm{D}^2 u - n\, u\, g_0 \,, & \text{in } M \,, \\ \Delta u = n\, u \,, & \text{in } M \,, \\ u > 0 \,, & \text{in } M \setminus \partial M \,, \quad \text{with } \partial M \text{ compact} \,, \\ u = 0 \,, & \text{on } \partial M \,, \\ u(x) \to +\infty \,, & \text{as } x \to \infty \,, \end{cases} \tag{2.4.1}$$

where we recall that the notation $x \to \infty$ means that we are taking the limit as we approach an end of the manifold. More precisely, $u(x) \to +\infty$ as $x \to \infty$ means that, for any number $C > 0$, we can find a compact set $K \subset M$ such that $u > C$ in $M \setminus K$.

Recalling Theorem 2.2.5, we now write down all the metrically complete warped product solutions to problem (2.4.1). For complete solutions with empty boundary, we adopt the notations

$$u_{\min} \; = \; \min_M u \qquad \text{and} \qquad \mathrm{MIN}(u) = \{p \in M \, : \, u(p) = u_{\min}\} \, .$$

It is also convenient to set

$$m_{\max} = \sqrt{\frac{(n-2)^{n-2}}{n^n}} \, . \tag{2.4.2}$$

- Anti de Sitter solution (Hyperbolic Space Form).

$$\left(\mathbb{R}^n \, , \, \frac{d|x| \otimes d|x|}{1+|x|^2} + |x|^2 g_{\mathbb{S}^{n-1}} \, , \, \sqrt{1+|x|^2} \right) . \tag{2.4.3}$$

It is easy to check that the metric, which a priori is well defined only in $M \setminus \{0\}$, extends smoothly through the origin. This model solution has zero mass and can be seen as the limit of the following Schwarzschild–Anti de Sitter solutions (2.4.5), when the parameter $m \to 0^+$. The Anti de Sitter solution has one end, empty boundary, and the set $\mathrm{MIN}(u)$ consists of a single point, the origin. Moreover, both the function u and the quantity $|\mathrm{D}u|$ tend to $+\infty$ as one approaches the end of the manifold and more precisely they obey the following simple relation

$$\lim_{|x| \to \infty} \left(u^2 - u_{\min}^2 - |\mathrm{D}u|^2 \right) \; = \; 0 \, . \tag{2.4.4}$$

This fact will be of some relevance for the classification results presented in Section 3.3.

- Schwarzschild–Anti de Sitter solutions.

$$\left([r_0(m), +\infty) \times E \, , \, \frac{dr \otimes dr}{1+r^2-2mr^{2-n}} + r^2 g_E \, , \, \sqrt{1+r^2-2mr^{2-n}} \right) \tag{2.4.5}$$
with $m > 0$ and (E^{n-1}, g_E) such that $\mathrm{Ric}_{g_E} = (n-2) g_E$.

Here, $r_0(m)$ is the only positive solution to $1 + r^2 - 2mr^{2-n} = 0$. It is not hard to check that both the metric g_0 and the function u, which a priori are well defined only in the interior of M, extend smoothly up to the boundary

$$\partial M \; = \; \{r = r_0(m)\} \, .$$

The triple (2.4.5) is conformally compact in the sense of Definition 2.4.1 below. Moreover, in the rotationally symmetric case, that is, when $E = \mathbb{S}^{n-1}$, the triple (2.4.5) is also asymptotically Anti de Sitter in the sense of

Definition 2.4.4 below. The metric $\bar{g} = u^{-2} g_0$ induces the metric g_E on the conformal boundary $\partial_\infty M$ (for the definition of conformal infinity, see below Definition 1.2.7). It follows that the scalar curvature $\mathrm{R}^{\partial_\infty M}$ of the metric induced by $\bar{g}$ on $\partial_\infty M$ is constant and equal to $(n-1)(n-2) = -2(n-2)\Lambda/n$, hence, according to the normalization suggested in Subsection 2.1.3, the surface gravity of the horizon ∂M can be computed as

$$\kappa(\partial M) \;=\; |\mathrm{D}u|_{|\partial M} \;=\; r_0(m)\,\big[1 + (n-2)\,m\,r_0^{-n}(m)\big]\,.$$

In formal analogy with (2.4.4) one has that the quantities u and $|\mathrm{D}u|$ obey the following relation

$$\lim_{|x|\to\infty}\left(u^2 - \frac{\mathrm{R}^{\partial_\infty M}}{(n-1)(n-2)} - |\mathrm{D}u|^2\right) \;=\; 0\,. \tag{2.4.6}$$

This is due to the fact that the asymptotic behavior of the Schwarzschild–Anti de Sitter solution is very similar to the one of the Anti de Sitter solution (2.4.3). However, an important distinction is that since the boundary of the Schwarzschild–Anti de Sitter solution is non-empty and it coincides with the zero level set of the static potential, one is not allowed to replace the constant $\mathrm{R}^{\partial_\infty M}/(n-1)(n-2)$ with the quantity $u^2_{\min}$ in the above limit.

- Schwarzschild–Anti de Sitter solutions with flat topology.

$$\left([r_0(m), +\infty) \times E\,,\; \frac{dr\otimes dr}{r^2 - 2mr^{2-n}} + r^2 g_E\,,\; \sqrt{r^2 - 2mr^{2-n}} \right) \tag{2.4.7}$$
$$\text{with } m>0 \text{ and } (E^{n-1}, g_E) \text{ such that } \mathrm{Ric}_{g_E} = 0\,.$$

Here $r_0(m) = (2m)^{1/n}$ is the positive solution of $r^2 - 2mr^{2-n} = 0$. As usual, the metric g_0 and the function u extend smoothly up to the boundary, and it holds

$$|\mathrm{D}u| \;=\; (n-1)m^{\frac{1}{n}} \qquad \text{on } \partial M\,.$$

The triple (2.4.5) is conformally compact in the sense of Definition 2.4.1 below, and the metric $\bar{g} = u^{-2} g_0$ induces the metric g_E on the conformal boundary $\partial_\infty M$ (for the definition of conformal boundary see below Definition 1.2.7). In particular, the scalar curvature $\mathrm{R}^{\partial_\infty M}$ of the metric induced by $\bar{g}$ on $\partial_\infty M$ is constant and equal to 0. In this case, according to the discussion in Subsection 2.1.3, we have not a standard way of renormalizing u in order to obtain an unambiguous notion of surface gravity.

Finally, the functions u and $|\mathrm{D}u|$ go to ∞ as we approach the conformal infinity, and more precisely we have the following asymptotic behavior

$$\lim_{r\to\infty}\left(u^2 - \frac{\mathrm{R}^{\partial_\infty M}}{(n-1)(n-2)} - |\mathrm{D}u|^2\right) \;=\; \lim_{r\to\infty}\left(u^2 - |\mathrm{D}u|^2\right) \;=\; 0\,.$$

- Schwarzschild–Anti de Sitter solutions with hyperbolic topology.

$$\left([r_0(m), +\infty) \times E \, , \, \frac{dr \otimes dr}{-1 + r^2 - 2mr^{2-n}} + r^2 g_E \, , \, \sqrt{-1 + r^2 - 2mr^{2-n}} \right) \quad \text{with } m > -m_{\max} \text{ and } (E^{n-1}, g_E) \text{ such that } \mathrm{Ric}_{g_E} = -(n-2) g_E \, . \tag{2.4.8}$$

Here $r_0(m)$ is the greatest positive solution of $-1 + r^2 - 2mr^{2-n} = 0$. We remark that, in order for such an $r_0(m)$ to exits, it is sufficient to set $m > -m_{\max}$, where $m_{\max}$ is defined as in (2.4.2). In particular, negative masses are acceptable. The triple (2.4.8) is conformally compact in the sense of Definition 1.2.7 below, and the metric $\bar{g} = u^{-2} g_0$ induces the metric g_E on the conformal boundary $\partial_\infty M$ (for the definition of conformal boundary see below Definition 1.2.7). In particular, the scalar curvature $\mathrm{R}^{\partial_\infty M}$ of the metric induced by $u^{-2} g_0$ on $\partial_\infty M$ is constant and equal to $-(n-1)(n-2) = 2(n-2)\Lambda/n$, hence, according to the definition given in Subsection 2.1.3, the surface gravity of the horizon ∂M can be computed as

$$|\mathrm{D}u|_{|_{\partial M}} \; = \; r(m) \left[1 + \frac{(n-2)m}{r^n(m)} \right] .$$

Finally, the quantities u and $|\mathrm{D}u|$ obey the following asymptotic behavior

$$\lim_{r \to \infty} \left(u^2 - \frac{\mathrm{R}^{\partial_\infty M}}{(n-1)(n-2)} - |\mathrm{D}u|^2 \right) \; = \; 0 \, .$$

- Anti Nariai solutions (Complete non-compact Cylinders).

$$\left((-\infty, +\infty) \times E \, , \, \frac{1}{n} \left[dr \otimes dr + (n-2) \, g_E \right] \, , \, \cosh(r) \right) \quad \text{with } (E^{n-1}, g_E) \text{ such that } \mathrm{Ric}_{g_E} = -(n-2) g_E \, . \tag{2.4.9}$$

$$\left([0, +\infty) \times E \, , \, \frac{1}{n} \left[dr \otimes dr + (n-2) \, g_E \right] \, , \, \sinh(r) \right) \quad \text{with } (E^{n-1}, g_E) \text{ such that } \mathrm{Ric}_{g_E} = -(n-2) g_E \, . \tag{2.4.10}$$

Solution (2.4.9) has empty boundary and the set

$$\mathrm{MIN}(u) \; = \; \{ p \in M \; : \; u(p) = 1 \} \, ,$$

coincides with the sphere $\{0\} \times \mathbb{H}^{n-1}$. Moreover, this solution has two ends, where the function u goes to infinity, and

$$u^2 - 1 - \frac{1}{n} |\mathrm{D}u|^2 \; = \; 0$$

holds pointwise on M. Concerning solution (2.4.9), it has a boundary component corresponding to $r = 0$, where it holds $|\mathrm{D}u| = \sqrt{n}$. It also has one end and the identity

$$u^2 + 1 - \frac{1}{n} |\mathrm{D}u|^2 \; = \; 0$$

holds pointwise. In particular, in contrast with the (Schwarzschild–)Anti de Sitter solutions, both solutions (2.4.9) and (2.4.10) satisfy

$$\lim_{r\to\infty}(u^2 - |\mathrm{D}u|^2) = -\infty\,.$$

- Solutions with different asymptotics.

$$\left((-\infty,+\infty)\times E\,,\ \frac{1}{n}\left[dr\otimes dr+(n-2)\,g_E\right]\,,\ e^r\right)$$
$$\text{with } (E^{n-1},g_E) \text{ such that } \mathrm{Ric}_{g_E} = -(n-2)g_E\,. \tag{2.4.11}$$

$$\left((0,+\infty)\times E\,,\ \frac{dr\otimes dr}{r^2}+r^2 g_E\,,\ r\right)$$
$$\text{with } (E^{n-1},g_E) \text{ such that } \mathrm{Ric}_{g_E} = 0\,. \tag{2.4.12}$$

$$\left(\left((nm_{\max})^{\frac{1}{n-2}},+\infty\right)\,,\ \frac{dr\otimes dr}{-1+r^2+2m_{\max}r^{2-n}}+r^2 g_E\,,\ \sqrt{-1+r^2+2m_{\max}r^{2-n}}\right)$$
$$\text{with } (E^{n-1},g_E) \text{ such that } \mathrm{Ric}_{g_E} = -(n-2)g_E\,. \tag{2.4.13}$$

These solutions are all metrically complete, but their asymptotic behavior is not the one prescribed in problem (2.4.1). In fact, they have two ends, one at which $u\to+\infty$, and one at which $u\to 0$. The end where $u\to 0$ is usually referred to as a *degenerate horizon*, in the sense that it can be seen as a boundary component of M, that cannot be reached by finite lenght geodesics and where $|\mathrm{D}u|$ vanishes pointwise. The triple (2.4.12) can be interpreted as the limit of the Schwarzschild–Anti de Sitter solutions with flat topology as $m\to 0$, whereas the triple (2.4.13) can be seen as the limit of the Schwarzschild–Anti de Sitter solutions with hyperbolic topology as $m\to -m_{\max}$. In turn, the triple (2.4.11) can be thought as representing the asymptotic behavior of (2.4.13) near $-\infty$, and, in this respect, it is in some sense the natural analogue of the Nariai solution. The existence of these "strange" solutions testifies once again that the realm of solutions with $\Lambda < 0$ is much richer than the cases $\Lambda = 0$ and $\Lambda > 0$. The presence of all these different solutions complicate the analysis and the characterization of the most interesting ones.

2.4.2 Main properties.

Here we collect the main known properties of static spacetimes with negative cosmological constant. To this end, we proceed in analogy with Subsection 1.2.2 and we define the corresponding notions of conformal compactness and asymptotic hyperbolicity (Definitions 1.2.7 and 1.2.11) in the static case.

Definition 2.4.1. *Let (M,g_0,u) be a solution to problem (2.4.1). The triple (M,g_0,u) is said to be* conformally compact *if (M,g_0) is conformally compact in the sense of Definition 1.2.7, and $1/u$ is a defining function of the conformal boundary $\partial_\infty M$.*

As already observed below Definition 1.2.7, in general the conformal boundary $\partial_\infty M$ is not connected. We also notice that static metrics allow for a good control of the scalar curvature, but not of the Ricci tensor, which means that Theorem 1.2.8 is not directly applicable. Nevertheless, the next result shows that the connectedness of the conformal boundary is granted under opportune hypotheses.

Proposition 2.4.2 ([CS01, Theorem I.1], [HM14, Proposition 2]). *Let (M, g_0, u) be a conformally compact solution to problem (2.4.1). If*

- *either $n = 3$,*
- *or the metric $(u^{-2}g_0)_{|\partial_\infty M}$ has nonnegative scalar curvature and $\partial M = \emptyset$,*

then the conformal infinity $\partial_\infty M$ is connected.

Recalling formula (1.2.5), we obtain that if (M, g_0, u) is conformally compact, then the Riemannian tensor of g_0 satisfies

$$\mathrm{R}_{\alpha\beta\gamma\eta} \;=\; -|d(u^{-1})|^2_{\bar{g}} \left[g^{(0)}_{\alpha\gamma}\, g^{(0)}_{\beta\eta} - g^{(0)}_{\alpha\eta}\, g^{(0)}_{\beta\gamma} \right] + \mathcal{O}(u^{-3})\,,$$

as $u \to \infty$. In particular, since the scalar curvature of g_0 is constant and equal to $-n(n-1)$, it follows that $|d(u^{-1})|_{\bar{g}}$ goes to 1 as we approach the infinity. Therefore, the sectional curvature of g converge to -1, and the manifold (M, g_0) is asymptotically locally hyperbolic in the sense of Definition 1.2.9. In particular, Lemma 1.2.10 tells us that there exists a special defining function r with $|dr|_{\bar{g}} \equiv 1$ in a whole collar neighborhood of $\partial_\infty M$. The special defining function allows to prove the following asymptotic formula for the static potential u.

Lemma 2.4.3 ([HM14, Lemma 3]). *Let (M, g_0, u) be an asymptotically locally Anti de Sitter solution of problem (2.4.1), r be the special defining function of the conformal boundary $\partial_\infty M$ with respect to the restriction of the metric $\bar{g} = u^{-2}g_0$ to $\partial_\infty M$. Then it holds*

$$u = \frac{1}{r} + \frac{\varrho}{4}\, r\,,$$

where $\varrho \in \mathscr{C}^{2,\alpha}(M) \cap \mathscr{C}^0(\bar{M})$ is a function such that

$$\varrho_{|\partial_\infty M} = \frac{\mathrm{R}^{\partial_\infty M}_{\bar{g}}}{(n-1)(n-2)}\,, \qquad |\mathrm{D}\varrho| = \mathcal{O}(r^{\frac{\alpha}{2}})\,,$$

for some $\alpha > 0$, where $\mathrm{R}^{\partial_\infty M}_{\bar{g}}$ is the scalar curvature of $(\partial_\infty M, \bar{g}_{|\partial_\infty M})$.

As a consequence, if (M, g_0, u) is asymptotically locally Anti de Sitter, one easily computes the following asymptotic behavior of the potential u as we approach the conformal infinity

$$\lim_{u \to +\infty} \left(u^2 - |\mathrm{D}u|^2\right) \;=\; \frac{\mathrm{R}^{\partial_\infty M}_{\bar{g}}}{(n-1)(n-2)}\,. \tag{2.4.14}$$

Despite these interesting properties, the hypothesis of conformal compactness alone is not strong enough to characterize the Anti de Sitter solution. In fact, it is proven in [ACD02, Theorem 1.1] that, for any Riemannian metric γ on $\mathbb{S}^2$ with the property that the Lorentzian manifold $(\mathbb{R} \times \mathbb{S}^2, -dt \otimes dt + \gamma)$ has positive scalar curvature, there exists a conformally compact 3-dimensional solution (M, g_0, u) of (3.3.1) such that the metric induced by $\bar{g} = u^{-2} g_0$ on the conformal infinity coincides with γ. Such a general result is not available in higher dimensions, however the existence of an infinite family of solutions to (3.3.1) for any $n \geq 4$ has been proven in [ACD05, Theorem 1.1], showing for instance that any small perturbation of the standard metric on $\mathbb{S}^{n-1}$ can be realised as the metric induced on the conformal infinity of a conformally compact static solution through the usual formula.

It follows that, in order to prove a clean uniqueness result for solutions of problem (2.4.1), one needs to require a more precise asymptotic behavior, as well as some topological condition on the conformal infinity, as in the following classical definition, which is the static counterpart of the notion of asymptotic hyperbolicity (Definitions 1.2.11 and 1.2.14).

Definition 2.4.4. *The triple (M, g_0, u) is said to be* asymptotically Anti de Sitter *if it is conformally compact, the conformal boundary $\partial_\infty M$ is diffeomorphic to a sphere $\mathbb{S}^{n-1}$ and the metric $u^{-2} g_0$ extends to the standard spherical metric $g_{\mathbb{S}^{n-1}}$ on $\partial_\infty M$.*

It is easily seen (see for instance [Wan01]) that the Anti de Sitter solution (2.4.3) and all the Schwarzschild–Anti de Sitter solutions (2.4.5), (2.4.7) and (2.4.8) are conformally compact, but of course only the Anti de Sitter triple (2.4.3) and the Schwarzschild–Anti de Sitter triple with spherical topology (2.4.5) are asymptotically Anti de Sitter, because they are the only ones whose conformal infinity is isometric to a sphere.

Notice that, from (2.4.14), it follows that an asymptotically Anti de Sitter triple (M, g_0, u) satisfies

$$\lim_{u \to +\infty} \left(u^2 - 1 - |\mathrm{D}u|^2 \right) = 0 \,.$$

To highlight the connection between the notion of asymptotically Anti de Sitter and asymptotically hyperbolic, we observe that in [Wan05] it is shown, following a computation in [Gra00], that if $\partial_\infty M$ is diffeomorphic to the sphere $\mathbb{S}^{n-1}$ and the metric $(u^{-2} g_0)_{|\partial_\infty M}$ is conformal to the standard spherical metric $g_{\mathbb{S}^{n-1}}$, then the expansion (1.2.7) is always in force. Therefore the following holds.

Proposition 2.4.5. *If (M, g_0, u) is an asymptotically Anti de Sitter solution of problem* (2.4.1)*, then (M, g_0) is asymptotically hyperbolic.*

The following theorem shows an application of the Positive Mass Theorem 1.2.13 to the proof of an uniqueness statement for the Anti de Sitter triple, which generalizes a classical result in [BGH84].

Theorem 2.4.6 ([Wan05, Theorem 1])**.** *Let (M, g_0, u) be an asymptotically Anti de Sitter solution of problem* (2.4.1) *such that M has empty boundary. If M is spin, then (M, g_0, u) is isometric to the Anti de Sitter triple* (2.4.3)*.*

In particular, since every 3-manifold is spin, this proves the uniqueness of the Anti de Sitter solution among asymptotically Anti de Sitter triples for $n = 3$. This result has been further extended by Qing in [Qin04], who was able to drop the spin assumption. The idea of [Qin04] is to glue an asymptotically flat end to the conformally compactified manifold, so that it is possible to use the Positive Mass Theorem of Schoen-Yau, more precisely the version with corners due to Miao (see Theorem 1.2.5). For completeness, we also recall that Chruściel and Herzlich have given a different definition of mass (see Definition 1.2.15), and as a consequence they were able to give an alternative proof of Theorem 2.4.6 (see [CH03, Theorem 4.3]).

Another extension of Theorem 2.4.6, stated below, allows for a different topology of the conformal infinity provided that an opportune spinor field exists on the conformal boundary.

Theorem 2.4.7 ([HM14, Theorem 8]). *Let (M, g_0, u) be a conformally compact solution of problem (2.4.1) and let $\bar{g} = u^{-2} g_0$. If M has empty boundary, is spin and its conformal infinity $(\partial_\infty M, \bar{g}_{|\partial_\infty M})$ admits a nontrivial Killing or parallel spinor field, then (M, g_0, u) is isometric to the Anti de Sitter triple (2.4.3).*

It is important to remark that, when our static triple is not asymptotically Anti de Sitter, in general the mass introduced by Wang [Wan01] and Chruściel-Herzlich [CH03] is not defined. For this reason, one is led to investigate other possible definitions of mass which allow for less rigid behaviours at infinity, while preserving some interesting properties. We cite one of the most important of these alternative masses, which we have already introduced, in a slightly different form, for asymptotically flat manifolds in Subsection 1.2.1. Given a 3-dimensional solution (M, g_0, u) to problem (2.4.1) and a closed compact surface Σ in M, the *Hawking mass* of Σ is defined as

$$m_H(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}} \left[1 - \mathrm{genus}(\Sigma) - \frac{1}{16\pi} \int_\Sigma (\mathrm{H}^2 - 4)\, \mathrm{d}\sigma\right],$$

where H is the mean curvature of Σ. The following result of Chruściel and Simon compares the Hawking mass with another definition of mass, which is very much in the spirit of the virtual mass that we are going to define in Chapter 3 for the case $\Lambda > 0$.

Theorem 2.4.8 ([CS01, Theorem I.5]). *Let (M, g_0, u) be a 3-dimensional solution of system (2.4.1). Suppose that (M, g_0, u) is conformally compact, that the conformal infinity $\partial_\infty M$ (which is connected thanks to Proposition 2.4.2) satisfies $\mathrm{genus}(\partial_\infty M) \geq 2$, and that the scalar curvature $\mathrm{R}^{\partial_\infty M}$ induced by $u^{-2} g_0$ on $\partial_\infty M$ is constant and equal to -6. Suppose further that*

$$0 \leq \kappa := \max_{\partial M} |\mathrm{D}u| \leq 1 .$$

Then there is a unique value $\mu = \mu(M, g_0, u) \leq 0$ such that the boundary of the Schwarzschild–Anti de Sitter triple with hyperbolic topology (2.4.8) has surface gravity equal to κ, and it holds

$$m_H(\{u = t\}) \leq \mu , \tag{2.4.15}$$

for all t. Moreover, if the equality $m_H(\{u = t\}) = \mu$ holds for some t, then the triple (M, g_0, u) is isometric to the Schwarzschild–Anti de Sitter solution with hyperbolic topology (2.4.8).

We emphasize that the parameter μ in the above theorem is the natural analogue of the virtual mass that we will define in Chapter 3 in the case $\Lambda > 0$. Formula (2.4.15) shows a connection between μ and the Hawking mass. Moreover, as it will be discussed below, the parameter μ plays an important role in the proof of an area bound and a Black Hole Uniqueness Theorem for Schwarzschild–Anti de Sitter solutions with hyperbolic topology in dimension $n = 3$. These results are in line with the ones that will be proven for solutions with $\Lambda > 0$ in Chapter 4. We stress that the hypothesis $\mu \leq 0$ (or equivalently $\kappa \leq 1$) in Theorem 2.4.8 above is necessary. In fact, the crucial step of the proof is the application of the Maximum Principle to the differential inequality [BS92, formula (V.4)], which is elliptic only if $\mu \leq 0$.

Under the same hypotheses of Theorem 2.4.8 and with the same notations, Chruściel and Simon also prove that, for any boundary component $S \subset \partial M$ with maximum surface gravity κ, it holds

$$|S| \geq \frac{\mathrm{genus}(S) - 1}{\mathrm{genus}(\partial_\infty M) - 1}\, 4\pi\, r^2(\mu)\,, \tag{2.4.16}$$

where $r(\mu)$ is the largest positive solution of $1 - x^2 - 2\mu/x = 0$. Building on this, Lee and Neves proved the following Black Hole Uniqueness Theorem for Schwarzschild–Anti de Sitter solutions with hyperbolic topology (2.4.8) and with nonpositive virtual mass.

Theorem 2.4.9 ([LN15, Theorem 2.1])**.** *In the same hypotheses and notations of Theorem 2.4.8, suppose also that there exists a component $S \subset \partial M$ with largest surface gravity κ and with*

$$\mathrm{genus}(S) \geq \mathrm{genus}(\partial_\infty M)\,.$$

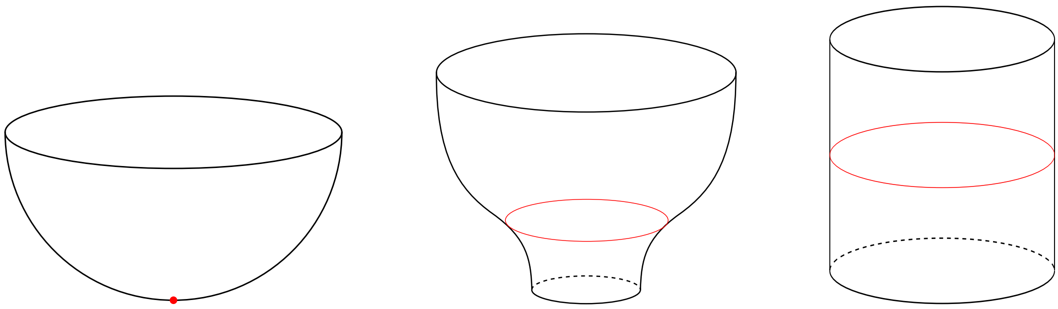
Then (M, g_0, u) is isometric to the Schwarzschild–Anti de Sitter solution with hyperbolic topology (2.4.8) *and with mass μ.*

The proof of this theorem is based on the monotonicity of the Hawking mass under inverse mean curvature flow, in the spirit of the classical work of Huisken-Ilmanen [HI01] in the asymptotically flat case. This monotonicity is used to prove a bound from above on $|S|$, which combined with inequality (2.4.16), recalling from (2.4.15) that the Hawking mass is controlled by μ, gives the thesis.

2.5 Static solutions with $\Lambda > 0$

2.5.1 Statement of the problem and main solutions.

Here we focus on the case $\Lambda > 0$. First of all, it is convenient to normalize g_0 so that $\Lambda = n(n-1)/2$. Contrarily to the case $\Lambda \leq 0$, when $\Lambda > 0$ the most



(a) de Sitter (b) Schwarzschild–de Sitter (c) Nariai

Figure 2.1: Rotationally symmetric solutions to problem (4.1.1). The red dot and red lines represent the set $\mathrm{MAX}(u)$ for the three models.

interesting warped product solutions are compact, as shown in Theorem 2.2.4. For this reason, we are led to consider the following problem

$$\begin{cases} u\,\mathrm{Ric} \,=\, \mathrm{D}^2 u \,+\, n\,u\,g_0\,, & \text{in } M\,,\\ \Delta u \,=\, -\,n\,u\,, & \text{in } M\,,\\ u \,>\, 0\,, & \text{in } M\setminus\partial M\,,\\ u \,=\, 0\,, & \text{on } \partial M\,, \end{cases} \qquad \text{with } M \text{ compact}\,. \tag{2.5.1}$$

According to (2.1.8), these solutions are normalized to have constant scalar curvature equal to $\mathrm{R} \equiv n(n-1)$. We also adopt the notations

$$u_{\max} \,=\, \max_M u \qquad \text{and} \qquad \mathrm{MAX}(u) = \{p \in M \,:\, u(p) = u_{\max}\}\,,$$

for the static potential u. Moreover, from (2.4.2) we recall the notation

$$m_{\max} = \sqrt{\frac{(n-2)^{n-2}}{n^n}}\,.$$

- <u>de Sitter solution (Spherical Space Form), Figure 2.1(a).</u>

$$\left(\overline{B(0,1)} \subset \mathbb{R}^n\,,\ \frac{d|x|\otimes d|x|}{1-|x|^2} + |x|^2 g_{\mathbb{S}^{n-1}}\,,\ \sqrt{1-|x|^2}\right) \tag{2.5.2}$$

It is not hard to check that both the metric g_0 and the static potential u, which a priori are well defined only in the interior of $M\setminus\{0\}$, extend smoothly up to the boundary and through the origin. This model solution can be seen as the limit of the following Schwarzschild–de Sitter solutions (2.5.3), when the parameter $m \to 0^+$. The de Sitter solution is such that the maximum of the potential is $u_{\max} = 1$, achieved at the origin, and it has only one connected horizon with surface gravity

$$|\mathrm{D}u| \,\equiv\, 1 \qquad \text{on} \quad \partial M\,.$$

Hence, according to Definition 3.2.3 below, one has that this horizon is of cosmological type.

- Schwarzschild–de Sitter solutions, Figure 2.1(b).

$$\left([r_-(m), r_+(m)] \times E \,,\; \frac{dr \otimes dr}{1 - r^2 - 2mr^{2-n}} + r^2 g_E \,,\; \sqrt{1 - r^2 - 2mr^{2-n}} \right) \tag{2.5.3}$$
with $0 < m < m_{\max}$ and (E^{n-1}, g_E) such that $\mathrm{Ric}_{g_E} = (n-2) g_E$.

Here $r_-(m)$ and $r_+(m)$ are the two positive solutions to $1 - r^2 - 2mr^{2-n} = 0$. We notice that, for $r_-(m), r_+(m)$ to be real and positive, one needs (2.4.2). It is not hard to check that both the metric g_0 and the function u, which a priori are well defined only in the interior of M, extend smoothly up to the boundary. This latter has two connected components with different character

$$\partial M_+ = \{|x| = r_+(m)\} \qquad \text{and} \qquad \partial M_- = \{|x| = r_-(m)\}\,.$$

In fact, it is easy to check (see formulæ (2.5.6) and (2.5.7)) that the surface gravities satisfy

$$\kappa(\partial M_+) = \frac{|\mathrm{D}u|}{u_{\max}} < \sqrt{n} \qquad \text{on } \partial M_+\,,$$
$$\kappa(\partial M_-) = \frac{|\mathrm{D}u|}{u_{\max}} > \sqrt{n} \qquad \text{on } \partial M_-\,.$$

Hence, according to Definition 3.2.3 below, one has that ∂M_+ is of cosmological type, whereas ∂M_- is of black hole type. We also notice that it holds

$$u_{\max} = \sqrt{1 - \left(\frac{m}{m_{\max}}\right)^{\frac{2}{n}}}\,, \qquad \mathrm{MAX}(u) = \left\{ |x| = \left[(n-2)m\right]^{\frac{1}{n}} \right\}\,, \tag{2.5.4}$$

and $M \setminus \mathrm{MAX}(u)$ has exactly two connected components: M_+ with boundary ∂M_+ and M_- with boundary ∂M_-. According to Definition 3.2.3, we have that M_+ is an outer region, whereas M_- is an inner region.

- Nariai solution (Compact Round Cylinder), Figure 2.1(c).

$$\left([0,\pi] \times E \,,\; \frac{1}{n}\left[dr \otimes dr + (n-2)\, g_E\right] \,,\; \sin(r) \right) \tag{2.5.5}$$
with $m > 0$ and (E^{n-1}, g_E) such that $\mathrm{Ric}_{g_E} = -(n-2) g_E$.

This model solution can be seen as the limit of the previous Schwarzschild–de Sitter solutions, when the parameter m approaches $m_{\max}$, after an appropriate rescaling of the coordinates and of the potential u. This was shown for $n = 3$ in [GP83] and then extended to all dimensions $n \geq 3$ in [CDL04], see also [Bou03, BH96] and Subsection 2.5.3. The Nariai solution has $u_{\max} = 1$ and $\mathrm{MAX}(u) = \{\pi/2\} \times \mathbb{S}^{n-1}$. Moreover, the boundary of M has two connected components with the same constant value of the surface gravity, namely

$$|\mathrm{D}u| \equiv \sqrt{n} \qquad \text{on} \quad \partial M = \{0\} \times \mathbb{S}^{n-1} \cup \{\pi\} \times \mathbb{S}^{n-1}\,.$$

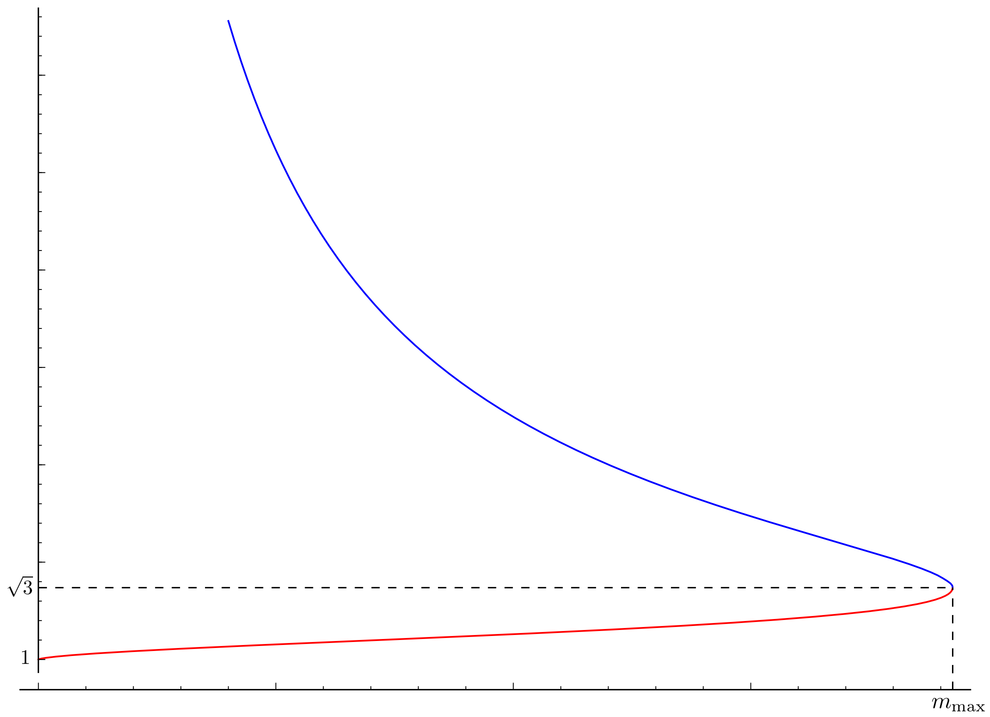


Figure 2.2: Plot of the surface gravities $|\mathrm{D}u|/u_{\max}$ of the two boundaries of the Schwarzschild–de Sitter solution (2.5.3) as functions of the mass m for $n = 3$. The red line represents the surface gravity of the boundary $\partial M_+ = \{r = r_+(m)\}$, whereas the blue line represents the surface gravity of the boundary $\partial M_- = \{r = r_-(m)\}$. Notice that for $m = 0$ we recover the constant value $|\mathrm{D}u| \equiv 1$ of the surface gravity on the (connected) cosmological horizon of the de Sitter solution (2.5.2). The other special situation is when $m = m_{\max}$. In this case the plot assigns to $m_{\max} = 1/(3\sqrt{3})$ the unique value $\sqrt{3}$ achieved by the surface gravity on both the connected components of the boundary of the Nariai solution (2.5.5).

In Subsection 3.2.2, we are going to use the above listed solutions as reference configurations in order to define the concept of *virtual mass* of a solution (M, g_0, u) to (3.2.1). For this reason it is useful to introduce since now the functions k_+ and k_-, whose graphs are plotted, for $n = 3$, in Figure 2.2. They represent the surface gravities of the model solutions as functions of the mass parameter m.

- The *outer* surface gravity function
$$k_+ : [0, m_{\max}) \longrightarrow [1, \sqrt{n}\,) \tag{2.5.6}$$
is defined by
$$\begin{aligned} k_+(0) &= 1\,, && \text{for } m = 0\,, \\ k_+(m) &= \sqrt{\frac{r_+^2(m)\,\left[1-(n-2)mr_+^{-n}(m)\right]^2}{1-(m/m_{\max})^{2/n}}}\,, && \text{if } 0 < m < m_{\max}\,, \end{aligned}$$
where $r_+(m)$ is the largest positive solution to $1 - r^2 - 2mr^{2-n} = 0$. Loosely speaking, $k_+(m)$ is nothing but the constant value of $|Du|/u_{\max}$ at $\{|x| = r_+(m)\}$ for the Schwarzschild–de Sitter solution with mass parameter equal to m. We also observe that k_+ is continuous, strictly increasing and $k_+(m) \to \sqrt{n}$, as $m \to m_{\max}^-$.

- The *inner* surface gravity function
$$k_- : (0, m_{\max}\,] \longrightarrow [\,\sqrt{n}, +\infty\,) \tag{2.5.7}$$
is defined by
$$\begin{aligned} k_-(m_{\max}) &= \sqrt{n}\,, && \text{for } m = m_{\max}\,, \\ k_-(m) &= \sqrt{\frac{r_-^2(m)\,\left[1-(n-2)mr_-^{-n}(m)\right]^2}{1-(m/m_{\max})^{2/n}}}\,, && \text{if } 0 < m < m_{\max}\,, \end{aligned}$$
where $r_-(m)$ is the smallest positive solution to $1 - r^2 - 2mr^{2-n} = 0$. Loosely speaking, $k_-(m)$ is nothing but the constant value of $|Du|/u_{\max}$ at $\{|x| = r_-(m)\}$ for the Schwarzschild–de Sitter solution with mass parameter equal to m. We also observe that k_- is continuous, strictly decreasing and $k_-(m) \to +\infty$, as $m \to 0^+$.

2.5.2 Main properties.

One of the first result concerning the properties of static solutions with positive cosmological constant was the following classical area bound proven by Boucher-Gibbons-Horowitz.

Theorem 2.5.1 (Boucher-Gibbons-Horowitz Inequality [BGH84])**.** *Let (M, g_0, u) be a 3-dimensional solution to problem* (2.5.1)*. If ∂M is connected, then*
$$|\partial M| \,\leq\, 4\pi\,, \tag{2.5.8}$$
and the equality holds if and only if (M, g_0, u) is isometric to the de Sitter solution (2.5.2)*.*

In the same paper, the authors proposed the following strong characterization of the de Sitter solution.

Conjecture (Cosmic No Hair Conjecture). *The only solution to problem* (2.5.1) *with connected boundary ∂M is the de Sitter spacetime.*

This conjecture has been disproven, at least for dimensions $4 \leq n \leq 8$, where an infinite family of solutions with connected boundary diffeomorphic to $\mathbb{S}^n$ has been provided by Gibbons, Hartnoll and Pope in [GHP03]. The construction of this family will be discussed in Subsection 2.5.4. For the moment, let us only observe that, as a consequence, for $n \geq 4$ we cannot expect a topological condition on the boundary to be sufficient to prove a classification result for solutions of problem (2.5.1). Not many results are known for general dimensions, we only cite the following generalization of Theorem 2.5.1, due to Chruściel [Chr] exploiting some early computations by Lindblom [Lin88].

Theorem 2.5.2. *Let (M, g_0, u) be a solution to problem* (2.5.1)*. Then it holds*

$$\int_{\partial M} |\mathrm{D}u| \left[\mathrm{R}^{\partial M} - (n-1)(n-2) \right] \mathrm{d}\sigma \geq 0\,, \tag{2.5.9}$$

where $\mathrm{R}^{\partial M}$ is the scalar curvature of the metric induced by g_0 on ∂M.

We recall that the quantity $|\mathrm{D}u|$ is constant on every component of ∂M. Therefore, if ∂M is connected, inequality (2.5.9) becomes

$$\int_{\partial M} \frac{\mathrm{R}^{\partial M}}{(n-1)(n-2)} \mathrm{d}\sigma \geq |\partial M|\,.$$

In particular, in dimension $n = 3$, using the Gauss-Bonnet Formula we obtain again Theorem 2.5.1. Moreover, notice that, in particular, the equality in (2.5.9) is achieved when ∂M, endowed with the metric induced by g_0, is isometric to a standard round sphere. In particular, we have the following partial version of the Cosmic No Hair Conjecture.

Theorem 2.5.3 ([Chr]). *Let (M, g_0, u) be a solution to problem* (2.5.1)*. Suppose that the boundary ∂M is connected and diffeomorphic to a sphere $\mathbb{S}^{n-1}$, and that the metric induced by g_0 is the standard round metric $g_{\mathbb{S}^{n-1}}$. Then (M, g_0, u) is isometric to the de Sitter triple* (2.5.2)*.*

Concerning the three-dimensional case, a couple of interesting recent results have been shown by Ambrozio in [Amb15]. The following theorem can be seen as a refinement of the Boucher-Gibbons-Horowitz Inequality.

Theorem 2.5.4 ([Amb15, Theorem C]). *Let (M, g_0, u) be a 3-dimensional solution to problem* (2.5.1)*. If (M, g_0, u) is not isometric to the de Sitter triple or to the Nariai triple, then*

$$\sum_{i=1}^{p} \kappa_i |S_i| < \frac{4\pi}{3} \sum_{i=1}^{p} \kappa_i\,, \tag{2.5.10}$$

where $S_1, \ldots, S_p$ are the connected components of ∂M and $\kappa_1, \ldots, \kappa_p$ are their surface gravities.

Notice that, when ∂M is connected, inequality (2.5.10) becomes

$$|\partial M| \,<\, \frac{4\pi}{3}\,,$$

which is much stronger than the analogue result by Boucher, Gibbons and Horowitz. In the same paper, the following strong topological result is also proven.

Theorem 2.5.5 ([Amb15, Theorem B]). *Let (M, g_0, u) be a 3-dimensional orientable solution to problem (2.5.1). Then it holds*

(i) the universal cover of M is compact,

(ii) the connected components of ∂M are all diffeomorphic to spheres,

(iii) there is at most one unstable connected component of ∂M. Moreover, if there is an unstable component of ∂M, then M is simply connected.

2.5.3 Limits of the Schwarzschild–de Sitter solution.

Here we show that the Nariai solution can be obtained as the limit of the Schwarzschild–de Sitter solution when $m \to m_{\max}$. To this end, it is convenient to set

$$r_0 \,=\, \big[(n-2)m\big]^{\frac{1}{n}}\,.$$

The static potential of the Schwarzschild–de Sitter solution with mass m satisfies the following chain of equalities

$$\begin{aligned}
r^{n-2}\,u^2 \,&=\, r^{n-2} - r^n - 2m \\
&=\, (r^{n-2} - r_0^{n-2}) - (r^n - r_0^n) + (r_0^{n-2} - r_0^n - 2m) \\
&=\, (r - r_0)\left(\sum_{i=0}^{n-3} r_0^i r^{n-3-i}\right) - (r-r_0)\left(\sum_{i=0}^{n-1} r_0^i r^{n-1-i}\right) + \big[(n-2)m\big]^{\frac{n-2}{n}} - nm \\
&=\, (r - r_0)\,A(r) \,+\, \big[(n-2)m\big]^{\frac{n-2}{n}} - nm\,.
\end{aligned}$$

The function $A(r)$ can be written as

$$\begin{aligned}
A(r) \,&=\, \sum_{i=0}^{n-3} r_0^i r^{n-3-i} - \sum_{i=0}^{n-1} r_0^i r^{n-1-i} \\
&=\, \sum_{i=0}^{n-3} r_0^i\left(r^{n-3-i} - r_0^{n-3-i}\right) - \sum_{i=0}^{n-1} r_0^i\left(r^{n-1-i} - r_0^{n-1-i}\right) + (n-2)r_0^{n-3} - nr_0^{n-1} \\
&=\, (r-r_0)\left[\sum_{i=0}^{n-3} r_0^i\left(\sum_{j=0}^{n-4-i} r_0^j r^{n-4-i-j}\right) - \sum_{i=0}^{n-1} r_0^i\left(\sum_{j=0}^{n-2-i} r_0^j r^{n-2-i-j}\right)\right] + \\
&\qquad\qquad + (n-2)r_0^{n-3} - nr_0^{n-1} \\
&=\, (r-r_0)B(r) + \frac{n-2}{r_0}\left\{\big[(n-2)m\big]^{\frac{n-2}{n}} - nm\right\}
\end{aligned}$$

where $B(r)$ is a polynomial function in r. Finally, one can compute

$$\begin{aligned}
B(r_0) &= \sum_{i=0}^{n-3} r_0^i \left(\sum_{j=0}^{n-4-i} r_0^j r_0^{n-4-i-j} \right) - \sum_{i=0}^{n-1} r_0^i \left(\sum_{j=0}^{n-2-i} r_0^j r_0^{n-2-i-j} \right) \\
&= \sum_{i=0}^{n-3} (n-3-i) r_0^{n-4} - \sum_{i=0}^{n-1} (n-1-i) r_0^{n-2} \\
&= \frac{(n-2)(n-3)}{2} r_0^{n-4} - \frac{n(n-1)}{2} r_0^{n-2} \\
&= \frac{1}{2r_0^2} \left[(n-2)(n-3)[(n-2)m]^{\frac{n-2}{n}} - n(n-1)(n-2)m \right] \\
&= \frac{n-2}{2r_0^2} \left[(n-3) \left([(n-2)m]^{\frac{n-2}{n}} - nm \right) - 2nm \right] .
\end{aligned}$$

When m approaches $m_{\max}$, we have that $[(n-2)m]^{\frac{n-2}{n}} - nm$ goes to zero, hence for big enough values of m, the quantity $B(r_0)$ is strictly negative. To recap, we have obtained

$$r^{n-2}u^2 = (r-r_0)^2 B(r) + \left\{ \left[(n-2)m\right]^{\frac{n-2}{n}} - nm \right\} \left[1 + (n-2)\frac{r-r_0}{r_0} \right] , \quad (2.5.11)$$

with $B(r)$ polynomial function of r, depending on m, such that for big values of m we have $B(r_0) < 0$. Now we want to study the behavior of the Schwarzschild–de Sitter potential (2.5.11) as $m \to m_{\max}$. This is equivalent to study the behavior of (2.5.11) as $\varepsilon \to 0$, where $\varepsilon > 0$ is related to the mass by

$$-B(r_0)\varepsilon^2 = \left[(n-2)m\right]^{\frac{n-2}{n}} - nm .$$

Notice that the right hand side is negative for all $0 < m < m_{\max}$, therefore the observation that $B(r_0) < 0$ for big values of m is important for ε to be well defined. Consider now the change of coordinates

$$r = r_0 + \varepsilon \cos \chi .$$

The new coordinate χ is defined for the values $[0, \pi]$ such that

$$r_-(m) \le r_0 + \varepsilon \cos \chi \le r_+(m) .$$

However, one can check that, at the limit $\varepsilon \to 0$, the value of $u(\chi \pm \varepsilon)$ goes to zero up to the second order in ε, and as a consequence $r_0 \pm \varepsilon$ converge to $r_\pm(m)$, so that at the limit the change of coordinates will be defined for any $\chi \in [0, \pi]$. In fact, in the new coordinate, from (2.5.11) we find that the potential satisfies the following identity

$$\begin{aligned}
u^2 &= r^{2-n}\varepsilon^2 \left[B(r)\cos^2\chi - B(r_0) - (n-2)\frac{\varepsilon \cos \chi}{r_0} B(r_0) \right] \\
&= -B(r_0) r_0^{2-n} \varepsilon^2 \sin^2 \chi + \mathcal{O}(\varepsilon^3) . \qquad (2.5.12)
\end{aligned}$$

In particular

$$\frac{u^2}{\max_M(u)} = \sin^2 \chi + \mathcal{O}(\varepsilon) ,$$

so that at the limit $u/\max_M(u)$ annihilates at $\chi = 0$ and $\chi = \pi$, as wished. From (2.5.12) it follows that the Schwarzschild–de Sitter metric can be rewritten as

$$\begin{aligned} g_0 &= \frac{dr \otimes dr}{u^2} + r^2 g_{\mathbb{S}^{n-1}} \\ &= \frac{d\chi \otimes d\chi}{-B(r_0) r_0^{2-n} + \mathcal{O}(\varepsilon)} + \left[r_0^2 + \mathcal{O}(\varepsilon)\right] g_{\mathbb{S}^{n-1}} \end{aligned}$$

and at the limit $m \to m_{\max}$, one has

$$\varepsilon \to 0\,, \quad r_0 \to \sqrt{\frac{n-2}{n}}\,, \quad -B(r_0) r_0^{2-n} \to n\,,$$

therefore, at the limit we obtain precisely the Nariai metric. Concerning the static potential, we have already observed that we can normalize it at will. Therefore, dividing formula (2.5.12) by $-B_0(r_0) r_0^{2-n}\varepsilon^2$ we obtain that u converges to $\sin\chi$ when $\varepsilon \to 0$. This proves that the Nariai solution is indeed the limit of the Schwarzschild–de Sitter solution.

Before concluding this subsection, it is worth mentioning that the limit procedure described above is not the only possible one. We take a step back and we recall that, if we do not fix Λ via a normalization of g_0, then the metric of the Schwarzschild–de Sitter spacetime writes as

$$-V dt \otimes dt + \frac{dr \otimes dr}{V} + r^2 g_E\,, \quad \text{with } V = 1 - \frac{2\Lambda}{n(n-1)} - 2mr^{2-n}$$

where $0 < m < \bar{m}$, $\mathrm{Ric}_{g_E} = (n-2) g_E$ and the quantity

$$\bar{m} = \frac{1}{n}\left[\frac{(n-1)(n-2)}{2\Lambda}\right]^{\frac{n-2}{2}},$$

coincides with $m_{\max}$, given by (2.4.2), when $\Lambda = n(n-1)/2$. We also recall that the function V is nonnegative for $r \in [r_-(m), r_+(m)]$, where $r_-(m), r_+(m)$ are the two positive solutions of $V(r) = 0$. In particular, since $m < \bar{m}$, the Schwarzschild–de Sitter solution is defined for all $m > 0$ such that

$$m^2 \Lambda^{n-2} < \frac{1}{n}\left[\frac{(n-1)(n-2)}{2}\right]^{\frac{n-2}{2}}. \tag{2.5.13}$$

Now one can ask more generally what happens at the limit when the left hand side of (2.5.13) converges to the right hand side. The Nariai spacetime represents only one of the possible limits, that is obtained by fixing Λ and varying m, as explained above. However, this is not the only possibility. Another interesting limit is the *extreme Schwarzschild–de Sitter spacetime*, which is obtained by fixing m and varying Λ. However, we will not be interested in this solution, since this limit is not a static spacetime. In fact, one can see that, at the limit, the static part of the solution ($r_-(m) \le r \le r_+(m)$) just collapse, whereas for $r < r_-(m)$ and $r > r_+(m)$ the function V is negative, meaning that r becomes a time coordinate and t becomes a space coordinate. For more details on this subject, we refer the reader to [Gey80, LR77, Pod99].

2.5.4 Counterexamples to the Cosmic No Hair Conjecture.

Here we discuss a family of solutions to problem (2.5.1) described in [GHP03] for dimensions $4 \leq n \leq 8$, which are obtained starting from a family of Einstein metrics on the sphere $\mathbb{S}^{n+1}$, found by Bohm in [Böh99]. The construction of Bohm exploits a foliation of $\mathbb{S}^{n-1}$ that we now describe. First of all, take $2 \leq p \leq n-1$ and consider the following map

$$\begin{aligned}\Theta : [0,2\pi] \times [0,\pi]^{p-1} &\longrightarrow \mathbb{S}^p \subset \mathbb{R}^{p+1}\\ (\theta^1,\dots,\theta^p) &\longmapsto (\sin\theta^1 \dots \sin\theta^p\,,\, \cos\theta^1 \sin\theta^2 \dots \sin\theta^p\,,\\ &\qquad \cos\theta^2 \sin\theta^3 \dots \sin\theta^p\,,\,\dots\,,\, \cos\theta^{p-1}\sin\theta^p\,,\,\cos\theta^p)\,.\end{aligned}$$

The function Θ is a surjective map on $\mathbb{S}^p$. However, Θ is not injective. More precisely, the restriction of Θ to $(0,2\pi) \times (0,\pi^{p-1})$ is injective, whereas concerning the rest of the points we have that for each $j = 2,\dots,p$ the sets $\Theta(\{\theta^j = 0\})$ and $\Theta(\{\theta^j = \pi\})$ are two hemispheres of dimension $p-j$, so that the set

$$\Theta\left(\{\theta^j = 0\} \cup \{\theta^j = \pi\}\right)$$

is homeomorphic to a $(p-j)$-dimensional sphere. Moreover, the set $\Theta(\{\theta^1 = 0\}) = \Theta(\{\theta^1 = 2\pi\})$ is homeomorphic to a sphere of dimension $p-1$, and we have the inclusions

$$\begin{aligned}\Theta\Big(\{\theta^p = 0\} \cup \{\theta^p = \pi\}\Big) \,\subset\, \Theta\Big(\{\theta^{p-1} = 0\} \cup \{\theta^{p-1} = \pi\}\Big) \,\subset\, \cdots\\ \cdots \subset \Theta\Big(\{\theta^2 = 0\} \cup \{\theta^2 = \pi\}\Big) \,\subset\, \Theta\Big(\{\theta^1 = 0\}\Big)\,.\end{aligned}$$

A similar map, with analogous properties, can be constructed for the sphere $\mathbb{S}^q$, $q = n - p$

$$\begin{aligned}\Phi : [0,2\pi] \times [0,\pi]^{q-1} &\longrightarrow \mathbb{S}^q \subset \mathbb{R}^{q+1}\\ (\phi^1,\dots,\phi^q) &\longmapsto (\sin\phi^1 \dots \sin\phi^q\,,\, \cos\phi^1 \sin\phi^2 \dots \sin\phi^q\,,\\ &\qquad \cos\phi^2 \sin\phi^3 \dots \sin\phi^q\,,\,\dots\,,\, \cos\phi^{q-1}\sin\phi^q\,,\,\cos\phi^q)\,.\end{aligned}$$

Let now $c > 0$ and consider the map

$$\begin{aligned}\Psi \,:\, [0,c] \times \Big([0,2\pi] \times [0,\pi]^{p-1}\Big) \times \Big([0,2\pi] \times [0,\pi]^{q-1}\Big) &\longrightarrow \mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}\\ (\rho,\theta^1,\dots,\theta^p,\phi^1,\dots,\phi^q) &\longmapsto \Big(\rho^2 \cdot \Theta(\theta^1,\dots,\theta^p)\,,\,(c^2-\rho^2)\cdot \Phi(\phi^1,\dots,\phi^q)\Big)\,.\end{aligned}$$

Again, this map is surjective, and when restricted to the interior of the domain, it is also injective. Notice that $\{\rho = 0\}$ maps into a sphere of dimension q, whereas $\{\rho = c\}$ maps into a sphere of dimension p. For any other value $x \in (0,c)$, the set $\{\rho = x\}$ is mapped by Ψ into a product $\mathbb{S}^p \times \mathbb{S}^q$. Therefore, the map Ψ can be interpreted as a foliation of $\mathbb{S}^p \times \mathbb{S}^q$ on $\mathbb{S}^{n+1} \setminus (\mathbb{S}^p \sqcup \mathbb{S}^q)$. Using this map, we can define the following ansatz metric on $\mathbb{S}^{n+1}$

$$g \,=\, d\rho \otimes d\rho \,+\, a^2(\rho)\, g_{\mathbb{S}^p} \,+\, b^2(\rho)\, g_{\mathbb{S}^q}\,, \tag{2.5.14}$$

on $[0,c] \times \mathbb{S}^p \times \mathbb{S}^q$, where a and b are functions of the coordinate ρ and $g_{\mathbb{S}^p}$, $g_{\mathbb{S}^q}$ are the standard spherical metrics on $\mathbb{S}^p$ and $\mathbb{S}^q$. In order for g to be a metric on $\mathbb{S}^{n+1}$, we have to require that g behaves nicely at the endpoints of the coordinate ρ. This can be achieved by requiring

$$\begin{aligned} a(0) &= 0\,, & \dot a(0) &= 1\,, & \dot a(c) &= 0\,, \\ b(c) &= 0\,, & \dot b(0) &= 0\,, & \dot b(c) &= -1\,, \end{aligned} \tag{2.5.15}$$

where we have denoted by $\dot a, \dot b$ the derivatives of a and b with respect to ρ. In fact, when $\rho \to 0^+$, the factor $\mathbb{S}^p$ collapses and $a(\rho) \to 0$, so the metric looks like

$$d\rho \otimes d\rho + \rho^2 g_{\mathbb{S}^q}\,.$$

Therefore, the behavior of the metric g near $\rho = 0$ looks like the one at the origin of a spherical coordinate system. The same argument works also when $\rho \to c^-$, and we conclude that, if a, b satisfy the conditions (2.5.15), then the metric g is a metric on $\mathbb{S}^{n+1}$.

We now require the metric g to be Einstein, more precisely we ask $\mathrm{Ric}_g = ng$. Then (see the computations in [GHP03, Subsection 3.1]) the functions a, b must satisfy the differential equations

$$\begin{aligned} \frac{\ddot a}{a} + q\,\frac{\dot a}{a}\,\frac{\dot b}{b} + (p-1)\,\frac{\dot a^2+1}{a^2} + n &= 0\,, \\ \frac{\ddot b}{b} + p\,\frac{\dot a}{a}\,\frac{\dot b}{b} + (q-1)\,\frac{\dot b^2+1}{b^2} + n &= 0\,, \\ p(p-1)\,\frac{\dot a^2+1}{a^2} + q(q-1)\,\frac{\dot b^2+1}{b^2} + 2pq\,\frac{\dot a}{a}\,\frac{\dot b}{b} + n(n-1) &= 0\,. \end{aligned} \tag{2.5.16}$$

where we have denoted by $\dot a, \ddot a, \dot b, \ddot b$ the first and second derivatives of a and b, as usual. Conversely, if we can find functions a, b that satisfy the equations in (2.5.16) and that behave nicely when $\rho \to 0$ and $\rho \to c$, then the metric g defined as in (2.5.14) gives us an Einstein metric on $\mathbb{S}^{n+1}$. The result by Bohm can be stated as follows

Theorem 2.5.6 ([Böh99, Theorem 7.3]). *Let $4 \le n \le 8$, and let $p \ge 2, q \ge 2$ be integers such that $p + q = n$. Then there exists an infinite family of triples (a, b, c), where $a, b : [0, c] \to \mathbb{R}$ are functions that solve (2.5.16) with initial conditions (2.5.15). In particular, for each of these triples, the corresponding metric (2.5.14) is an Einstein metric on the sphere $\mathbb{S}^{n+1}$.*

Remark 2.5.7. *The construction by Böhm is quite more general, and it allows to find families of Einstein metrics on several manifolds. In turn, this allowed Gibbons, Hartnoll and Pope [GHP03] to find other families of static solutions, this time with boundary diffeomorphic to a product of spheres. The construction of these other families is very similar to the one that we describe in this subsection, and we will avoid to give the details.*

The simplest solution to system (2.5.14) is

$$a(\rho) = \sin(\rho)\,, \qquad b(\rho) = \cos(\rho)\,, \qquad \rho \in [0, \pi/2]\,,$$

and the corresponding metric (2.5.14) is just the standard round metric. The other solutions a, b cannot be explicitated, and they all give rise to inhomogeneus Einstein metrics. Some numerical calculations are shown in [GHP03, Appendix A].

We want to use the family of Riemannian Einstein metrics on $\mathbb{S}^{n+1}$ given in Theorem 2.5.6 in order to obtain vacuum spacetimes, that is, Lorentzian Einstein metrics. To this end, recalling the explicit expression of $g_{\mathbb{S}^p}, g_{\mathbb{S}^q}$ with respect to the coordinates $\theta^1, \dots, \theta^p$, $\phi^1, \dots, \phi^q$ (see for instance [AT11, Esempio 6.5.22]), we observe that the metric g can be written as

$$g = d\rho \otimes d\rho + a^2(\rho) \left[\sum_{i=1}^{p} \left(\sin\theta^{i+1} \dots \sin\theta^p \right)^2 d\theta^i \otimes d\theta^i \right] + b^2(\rho) \left[\sum_{i=1}^{q} \left(\sin\phi^{i+1} \dots \sin\phi^p \right)^2 d\phi^i \otimes d\phi^i \right].$$

In particular, the metric g is diagonal and its coefficients do not depend on θ^1. This allows us to use the formal operation of *analytic continuation*, that is, in our chart we substitute the real coordinate θ^1 with the immaginary coordinate $t = i\theta^1$. This way, the term $d\theta^1 \otimes d\theta^1$ becomes $-dt \otimes dt$, whereas the rest of the metric remains unchanged. We obtain the metric

$$\begin{aligned} g_L = & -a^2(\rho) \left(\sin\theta^2 \dots \sin\theta^p \right)^2 dt \otimes dt + d\rho \otimes d\rho \\ & + a^2(\rho) \left[\sum_{i=2}^{p} \left(\sin\theta^{i+1} \dots \sin\theta^p \right)^2 d\theta^i \otimes d\theta^i \right] \\ & + b^2(\rho) \left[\sum_{i=1}^{q} \left(\sin\phi^{i+1} \dots \sin\phi^p \right)^2 d\phi^i \otimes d\phi^i \right]. \end{aligned}$$

The next lemma tells us that the metric g_L is indeed Einstein.

Lemma 2.5.8. *Let g be a Riemannian metric and let $x^1, \dots, x^n$ be a chart in which the metric is diagonal, that is*

$$g = \sum_{i=1}^{n} f_i \, dx^i \otimes dx^i .$$

Suppose also that all the coefficients f_i do not depend on x^1, that is $\partial f_i / \partial x^1 = 0$ for all i, at all points. Then the Lorentzian metric

$$g_L = -f_1 dx^1 \otimes dx^1 + \sum_{i=2}^{n} f_i \, dx^i \otimes dx^i .$$

is such that the coefficients R^L_{ij} of its Ricci tensor Ric_{g_L} are related to the coefficients R_{ij} of the Ricci tensor Ric of g by

$$\mathrm{R}^L_{ij} = \begin{cases} -\mathrm{R}_{ij}, & \text{if } i = j = 1, \\ \mathrm{R}_{ij}, & \text{if } i \neq 1 \text{ or } j \neq 1. \end{cases}$$

In particular, if $\mathrm{Ric} = \lambda g$ for some $\lambda \in \mathbb{R}$, then $\mathrm{Ric}_{g_L} = \lambda g_L$ for the same λ.

Proof. To ease the notation, in this proof we drop the subscript 1 from f_1, and we denote by $\bar{g}$ the metric $\bar{g} = \sum_{i=2}^{n} f_i\, dx^i \otimes dx^i$, so that

$$g \,=\, f\, dx^1 \otimes dx^1 \,+\, \bar{g}\,, \qquad g_L \,=\, -f\, dx^1 \otimes dx^1 \,+\, \bar{g}\,.$$

Let us denote by $\Gamma^{\gamma}_{\alpha\beta}$ and $\bar{\Gamma}^{\gamma}_{\alpha\beta}$ the Christoffel symbols of g and $\bar{g}$, respectively. We easily computes

$$\Gamma^{\gamma}_{\alpha\beta} \,=\, \begin{cases} \bar{\Gamma}^{\gamma}_{\alpha\beta} & \text{if } \alpha > 1, \beta > 1, \gamma > 1\,, \\ 0 & \text{if } \{\alpha,\beta,\gamma\} \,=\, \{1,i,j\} \text{ with } i > 1, j > 1\,, \\ \dfrac{1}{2f}\partial_j f & \text{if } \{\alpha,\beta\} \,=\, \{1,i\}, \gamma = 1 \text{ with } i > 1\,, \\ -\dfrac{\bar{g}^{\gamma k}}{2}\partial_k f & \text{if } \alpha \,=\, \beta \,=\, 1,\, \gamma \neq 1\,, \\ 0 & \text{if } \alpha \,=\, \beta \,=\, \gamma \,=\, 1\,, \end{cases}$$

In particular, since g_L is obtained from g by changing the sign of f, the Christoffel symbols of g_L are the same as the ones of g, with the only exception of Γ^{γ}_{11}, $\gamma > 1$, that changes sign. Therefore, to prove the thesis, it is enough to show that this change of sign only affects the component $1,1$ of the Ricci tensor. We recall that the Ricci tensor of g can be computed from the Christoffel symbols by

$$\mathrm{R}_{\alpha\beta} \,=\, \partial_\eta \Gamma^{\eta}_{\alpha\beta} \,-\, \partial_\beta \Gamma^{\eta}_{\alpha\eta} \,+\, \Gamma^{\eta}_{\eta\mu}\,\Gamma^{\mu}_{\alpha\beta} \,-\, \Gamma^{\mu}_{\alpha\eta}\,\Gamma^{\eta}_{\beta\mu}\,. \tag{2.5.17}$$

If $\alpha > 1$ and $\beta > 1$, the term Γ^{γ}_{11} never appears in the formula above, hence we immediately have $\mathrm{R}_{\alpha\beta} = \mathrm{R}^L_{\alpha\beta}$ for any $\alpha > 1, \beta > 1$. If $\alpha = 1$ and $\beta > 1$ then the Christoffel symbol Γ^{γ}_{11} appears only in the last term $-\Gamma^{\mu}_{\alpha\eta}\,\Gamma^{\eta}_{\beta\mu}$ when $\eta = 1$. In that case one obtains $-\Gamma^{\mu}_{\alpha\eta}\,\Gamma^{\eta}_{\beta\mu} \,=\, -\Gamma^{\mu}_{11}\,\Gamma^{1}_{\beta\mu}$. For Γ^{μ}_{11} to be nonzero, one needs $\mu \neq 1$, but if $\mu \neq 1$ then $\Gamma^{1}_{\beta\mu}$ is zero. Therefore, the term $-\Gamma^{\mu}_{11}\,\Gamma^{1}_{\beta\mu}$ is zero for all μ, and it follows $\mathrm{R}_{1\beta} = \mathrm{R}^L_{1\beta}$. Finally, in the case $\alpha = \beta = 1$, identity (2.5.17) becomes

$$\mathrm{R}_{11} \,=\, \partial_\eta \Gamma^{\eta}_{11} \,-\, \partial_\beta \Gamma^{\eta}_{1\eta} \,+\, \Gamma^{\eta}_{\eta\mu}\,\Gamma^{\mu}_{11} \,-\, \Gamma^{\mu}_{1\eta}\,\Gamma^{\eta}_{1\mu}\,.$$

The first and third term on the right hand side change sign if we change sign of f, while the second one is zero for all η's. Concerning the last term $\Gamma^{\mu}_{1\eta}\,\Gamma^{\eta}_{1\mu}$, for it to be nonzero we need $\eta = 1, \mu \neq 1$ or viceversa, hence it also changes sign. Therefore, the change of sign of f implies the change of sign of R_{11} as wished. $\square$

Let us write g_L in a more confortable form. The coordinate t, who came from the old angular coordinate θ^1, will now be allowed to take values in all the real line. The remaining angular coordinates $(\theta^2, \ldots, \theta^p)$ parametrize, via the map Θ, a $(p-1)$-dimensional hemisphere $\mathbb{S}^{p-1}_{+}$, whose boundary coincides with the set where $\theta^2 = 0$. The metric

$$\sum_{i=2}^{p} \left(\sin\theta^{i+1} \ldots \sin\theta^{p}\right)^2 \, d\theta^i \otimes d\theta^i$$

is the spherical metric on the hemisphere $\mathbb{S}^{p-1}_+$, so we can denote it by $g_{\mathbb{S}^{p-1}_+}$. It follows that the metric g_L can be written as

$$g_L \,=\, -\,a^2(\rho)\,\big(\sin\theta^2\ldots\sin\theta^p\big)^2\,dt\otimes dt\,+\,d\rho\otimes d\rho + a^2(\rho)g_{\mathbb{S}^{p-1}_+} + b^2(\rho)g_{\mathbb{S}^q}\,,$$

for any $t\in\mathbb{R}$, $\rho\in[0,c]$, $(\theta^2,\ldots,\theta^p)\in[0,\pi]^{p-1}$, $(\phi^1,\ldots,\phi^q)\in[0,2\pi]\times[0,\pi]^{q-1}$. Now we notice that the coordinates $(\rho,\theta^2,\ldots,\theta^p,\phi^1,\ldots,\phi^q)$ parametrize, via Ψ, an hemisphere $\mathbb{S}^n_+$, whose boundary coincides again with the points where $\theta^2=0$. Moreover, with a discussion analogue as the one given above for g, we observe that

$$g_0 \,=\, d\rho\otimes d\rho\,+\,a^2(\rho)g_{\mathbb{S}^{p-1}_+} + b^2(\rho)g_{\mathbb{S}^q}$$

is a metric on the whole $\mathbb{S}^n_+$. Summing all up, we have obtained that the pair (X,g_L), with

$$X=\mathbb{R}\times\mathbb{S}^n_+\,,\qquad g_L\,=\,-u^2dt\otimes dt\,+\,g_0\,,\qquad \text{where }\; u\,=\,a(\rho)\sin\theta^2\ldots\sin\theta^p\,,$$

is a Lorentzian manifold with $\mathrm{Ric}_{g_L}\,=\,ng_L$. In particular, (X,g_L) is a static spacetime with positive cosmological constant $\Lambda=n(n-1)/2$, the hemisphere $\mathbb{S}^n_+$ endowed with the metric g_0 is the space manifold and u is its static potential. The function $u=a(\rho)\sin\theta^2\ldots\sin\theta^p$ is everywhere nonnegative, since $\theta^2,\ldots,\theta^p$ all take values in $[0,\pi]$, and it is zero on

$$\{\rho=0\}\cup\{\theta^2=0\}\cup\cdots\cup\{\theta^p=0\}\cup\{\theta^2=\pi\}\cup\cdots\cup\{\theta^p=\pi\}\,.$$

If one looks again at the map Ψ, it is clear that this set simply coincides with $\{\theta^2=0\}$. In particular, u is positive in the interior of $\mathbb{S}^n_+$ and it is zero on its boundary, which is a $(n-1)$-dimensional sphere. Therefore, the triple $(\mathbb{S}^n_+,g_0,u)$ is a static solution with connected horizon diffeomorphic to $\mathbb{S}^{n-1}$. The construction of the triple $(\mathbb{S}^n_+,g_0,u)$ can be done starting from any Einstein metric on $\mathbb{S}^{n+1}$ of the form (2.5.14), and it is clear that two different Einstein metrics give rise to two different static solutions. Therefore, as a consequence of Theorem 2.5.6 we obtain the main result of this subsection.

Theorem 2.5.9. *In dimensions $4\leq n\leq 8$, there exist infinite static solutions of* (2.5.1) *with a connected horizon diffeomorphic to* $\mathbb{S}^{n-1}$.

3

Characterization of massless solutions

Looking at the physical literature, it can be easily checked that there is a general agreement about the fact that the mass parameter m that shows up in the Schwarzschild triple (2.3.3), the Schwarzschild–de Sitter triple (2.5.3) and the Schwarzschild–Anti de Sitter triples (2.4.5), (2.4.7) and (2.4.8), should be physically interpreted as the mass of the solution. In particular, the solutions with $m \neq 0$ represent the basic models for static black holes. These solutions are usually listed among static vacua, since the massive bodies which are responsible for the curvature of the space can be thought as hidden beyond some connected components of the boundary of the manifolds (horizons of black hole type). On the other hand, the solutions with zero mass should be regarded as the true static vacua. Their curvature does not depend on the presence of – possibly hidden – matter but it is only due to the presence of a cosmological constant. For this reasons they represent the most basic solutions to (2.1.7) and correspond to the three fundamental geometric shapes (space forms).

The aim of the present chapter is to propose a possible characterisation of the rotationally symmetric static solutions with zero mass in presence of a cosmological constant, namely the de Sitter triple (2.3.3) and the Anti de Sitter triple (2.4.3). As it will be made precise in the following sections, we are going to prove that these are in fact the only possible solutions to system (2.1.7) which satisfy respectively a natural bound on the *surface gravity*, in the $\Lambda > 0$ case, and a growth condition inspired by (2.4.4), in the $\Lambda < 0$ case. For $\Lambda > 0$, we will also give, in Subsection 3.2.2, an interpretation of the above mentioned result in terms of a Positive Mass Statement (see Theorem 3.2.5 below). The contents of this chapter are based on the work [BM17a], which in turn is inspired by the analysis started in [BM16]. In the whole chapter, the dimension of our manifolds is assumed to be $n \geq 3$.

3.1 Characterization of the Minkowski solution

Before stating the main results, which deal with a nonzero cosmological constant, in this section we retrace the proof of Case's classification of static solutions with zero cosmological constant and with empty boundary. For convenience, we rewrite here the problem that we want to study, that is system (2.3.1), and we add the additional condition of empty boundary

$$\begin{cases} u\,\mathrm{Ric} = \mathrm{D}^2 u\,, & \text{in } M\,,\\ \Delta u = 0\,, & \text{in } M\,,\\ u > 0\,, & \text{in } M\,, \end{cases} \quad \text{with } \partial M = \emptyset\,. \tag{3.1.1}$$

We also write down the statement that we want to prove, that is, Theorem 2.3.1.

Theorem 3.1.1 ([And99, Cas10])**.** *If (M, g_0, u) is a solution to problem (3.1.1), then $u \equiv 1$ and (M, g_0) is Ricci flat.*

When M is compact, we have already proved Theorem 3.1.1 in Subsection 2.3.1, so it remains to discuss the noncompact case. As anticipated, we will follow Case's proof and we will keep the same notations as much as possible. It should be noticed that the analysis and the results presented in [Cas10] are quite more general, as they apply to the family of the Bakry-Émery tensors Ric^m_f, which are defined by

$$\mathrm{Ric}^m_f \,=\, \mathrm{Ric} + \mathrm{D}^2 f \,-\, \frac{1}{m} df \otimes df\,, \tag{3.1.2}$$

for a fixed function $f \in \mathscr{C}^\infty(M)$ and a constant $0 < m \le \infty$. This family of tensors is well studied, see for instance the survey [WW07] and the references therein. One of the reasons for their interest lies in the fact that manifolds with nonnegative Bakry-Émery tensor ($\mathrm{Ric}^m_f \ge 0$) share strong analogies with manifolds with nonnegative Ricci tensor ($\mathrm{Ric} \ge 0$). For instance, later we will need the following *laplacian comparison theorem*, which is a direct analogue of the well known laplacian comparison theorem for manifolds with nonnegative Ricci tensor. The statement given here is enough for our purposes but it can be improved, see [BQ05, Theorem 4.2] or [WW07, Theorem A.1] for a more general version.

Theorem 3.1.2 (Laplacian comparison)**.** *Let (M, g_0) be a n-dimensional Riemannian manifold such that $\mathrm{Ric}^m_f \ge 0$, for some $f \in \mathscr{C}^\infty(M), 0 < m < \infty$. Consider a point $x \in M$ and a geodesically complete neighborhood $x \in U \subset M$, and denote by $r : U \to \mathbb{R}$ the distance function from the point x with respect to the metric g_0. Then, at any point which is not in the cut locus of x, it holds*

$$\Delta_f r \,\le\, \frac{m+n-1}{r}\,.$$

In the theorem above we have denoted by Δ_f the drifted laplacian defined by

$$\Delta_f w \,=\, \Delta w \,-\, \langle \mathrm{D} f \,|\, \mathrm{D} w\rangle\,, \quad \text{for } w \in \mathscr{C}^\infty(M)\,. \tag{3.1.3}$$

We also recall that the cut locus of a point $x \in M$ is the set of the points which are images via the exponential map $\exp_x$ of vectors $v \in T_x M$ such that the geodesic $t \mapsto \exp_x(tv)$ is minimizing for all $t \le 1$ but it is not minimizing for $t > 1$.

Our interest in the Bakry-Émery tensor is due to the fact that it is naturally related to our original problem (3.1.1) by the following lemma, which follows from an immediate computation.

Lemma 3.1.3. *If the triple (M, g_0, u) is a solution to problem (3.1.1) then the function $f = -\log u$ satisfies*

$$\mathrm{Ric}_f^1 = 0 \quad and \quad \Delta_f f = 0 \quad on \ M\,, \tag{3.1.4}$$

where $\mathrm{Ric}_f^1, \Delta_f$ are defined as in (3.1.2), (3.1.3).

Following Case, in the next subsection we are going to prove that a function f satisfying (3.1.4) is necessarily constant. This in turn will immediately imply that u is constant and, recalling the first equation in (3.1.1), that (M, g_0) is Ricci flat, thus concluding the proof of Theorem 3.1.1.

3.1.1 Proof of Theorem 3.1.1.

Let (M, g_0, u) be a solution of problem (3.1.1), and let $f = -\log u$. As observed above, in order to prove Theorem 3.1.1, it is enough to show that f has to be constant. To this end, we start by writing the Bochner formula for f

$$\Delta |\mathrm{D}f|^2 \,=\, 2\,|\mathrm{D}^2 f|^2 \,+\, 2\,\mathrm{Ric}(\mathrm{D}f, \mathrm{D}f) \,+\, 2\,\langle \mathrm{D}\Delta f \,|\, \mathrm{D}f\rangle\,.$$

On the other hand, from (3.1.4) and the definition of Ric_f^1 we compute

$$\begin{aligned} 0 \,&=\, 2\,\mathrm{Ric}_f^1(\mathrm{D}f, \mathrm{D}f) \,=\, 2\,\mathrm{Ric}(\mathrm{D}f, \mathrm{D}f) \,+\, 2\,\mathrm{D}^2 f(\mathrm{D}f, \mathrm{D}f) \,-\, 2\,|\mathrm{D}f|^4 \\ &=\, 2\,\mathrm{Ric}(\mathrm{D}f, \mathrm{D}f) \,+\, \langle \mathrm{D}|\mathrm{D}f|^2 \,|\, \mathrm{D}f\rangle \,-\, 2\,|\mathrm{D}f|^4\,, \end{aligned}$$

and also

$$0 \,=\, \Delta_f f \,=\, \Delta f \,-\, |\mathrm{D}f|^2\,.$$

Moreover, from the Cauchy-Schwartz inequality we have $|\mathrm{D}^2 f|^2 \geq (\Delta f)^2/n = |\mathrm{D}f|^4/n$. Substituting these informations inside the Bochner formula, we obtain

$$\Delta |\mathrm{D}f|^2 \,\geq\, \frac{2(n+1)}{n}\,|\mathrm{D}f|^4 \,-\, \langle \mathrm{D}|\mathrm{D}f|^2 \,|\, \mathrm{D}f\rangle\,. \tag{3.1.5}$$

Now let us fix $x \in M$ and let r be the distance function from the point x with respect to the metric g_0. Let us also suppose for the moment that the point x_0 is not in the cut locus of x, so that r is smooth at x_0. Consider the ball $B(x, a) = \{r < a\} \subset M$ for some radius $a > 0$. On that ball, we define the function $F : B(x, a) \to \mathbb{R}$ as

$$F \,=\, (a^2 - r^2)^2\, |\mathrm{D}f|^2\,.$$

By definition, $F \geq 0$ in the whole ball $B(x, a)$, and $F = 0$ on its boundary. In particular, F admits an interior maximum, that we will call x_0. In particular, at the point x_0 it holds

$$\begin{aligned} 0 \,&=\, \mathrm{D}F \,=\, (a^2 - r^2)^2\, \mathrm{D}|\mathrm{D}f|^2 \,-\, 2\,(a^2 - r^2)\, |\mathrm{D}f|^2\, \mathrm{D}r^2\,, \\ 0 \,\geq\, \Delta F \,&=\, 2\,|\mathrm{D}f|^2\, |\mathrm{D}r^2|^2 \,-\, 4\,(a^2 - r^2)\, \langle \mathrm{D}|\mathrm{D}f|^2 \,|\, \mathrm{D}r^2\rangle \,+ \\ &\qquad\qquad +\, (a^2 - r^2)^2 \Delta |\mathrm{D}f|^2 \,-\, 2\,(a^2 - r^2) |\mathrm{D}f|^2 \Delta r^2\,. \end{aligned}$$

From the first of the two formulæ we obtain that at x_0 it holds

$$2\,\frac{\mathrm{D}r^2}{a^2-r^2}\;=\;\frac{\mathrm{D}|\mathrm{D}f|^2}{|\mathrm{D}f|^2}\,.$$

Using this relation and the fact that $|\mathrm{D}r|\equiv 1$, the second formula simplifies to

$$0\geq \Delta F\;=\;-24\,r^2\,|\mathrm{D}f|^2\,+\,(a^2-r^2)^2\Delta|\mathrm{D}f|^2\,-\,2\,(a^2-r^2)|\mathrm{D}f|^2\Delta r^2\,. \tag{3.1.6}$$

Moreover, at x_0 we have the following estimate

$$\Delta r^2\;=\;2\,r\,\Delta r\,+\,2\,|\mathrm{D}r|^2\;=\;2\,r\,[\Delta_f r\,+\,\langle \mathrm{D}f\,|\,\mathrm{D}r\rangle]\,+\,2\;\leq\;2(n+1)\,+\,\langle \mathrm{D}f\,|\,\mathrm{D}r^2\rangle\,,$$

where in the latter inequality we have used Theorem 3.1.2. Usign this inequality, and recalling also formula (3.1.5), we have that (3.1.6) simplifies to

$$\begin{aligned}0\;&\geq\;-24\,r^2\,|\mathrm{D}f|^2\,-\,4(n+1)\,(a^2-r^2)|\mathrm{D}f|^2\,+\,\frac{2(n+1)}{n}(a^2-r^2)^2|\mathrm{D}f|^4\\&\geq\;\left[-24\,a^2\;-\,4(n+1)\,a^2\,+\,\frac{2(n+1)}{n}\,F\right]|\mathrm{D}f|^2\\&=\;\left[-4(n+7)\,a^2\,+\,\frac{2(n+1)}{n}\,F\right]|\mathrm{D}f|^2\,.\end{aligned} \tag{3.1.7}$$

We recall that the above formula holds at x_0, and that we have used the information that x_0 is not in the cut locus in order to obtain it.

Now we want to prove that formula (3.1.7) holds also when x_0 is in the cut locus of x. To this end, one can use the support functions method as described for instance in the proof of [Pet06, Lemma 42], that we now briefly recap. Let $\sigma:[0,\ell]\to M$ be a unit speed minimizing geodesic with $\sigma(0)=x$ and $\sigma(\ell)=x_0$. It is not hard to see that for any $0<\varepsilon<\ell$, the point x_0 is not in the cut locus of $\sigma(\varepsilon)$. In fact, if this were the case, this would easily imply that the restriction $\sigma_{|}:[\sigma(\varepsilon),\ell]\to M$, which is minimizing by construction, should stop being minimizing when prolonged in any of the two directions. In particular, the whole geodesic $\sigma:[0,\ell]\to M$ wuold not be minimizing, against our hypothesis. Since we have proved that x_0 is not in the cut locus of $\sigma(\varepsilon)$, it follows that Theorem 3.1.2 applies to the function $r_\varepsilon(y)=d(\sigma(\varepsilon),y)$, which is the distance function from the point $\sigma(\varepsilon)$, for any fixed $0<\varepsilon<\ell$. In particular, one can retrace the computations above using r_ε in place of r, and then take the limit as $\varepsilon\to 0$ to recover formula (3.1.7).

This proves that inequality (3.1.7) always holds at the point x_0. Since x_0 is the maximum of F, it follows that on the whole ball $B(x,a)$ it holds

$$(a^2-r^2)^2|\mathrm{D}f|^2\;=\;F\;\leq\;\frac{2n(n+7)}{n+1}a^2\,.$$

Finally, evaluating this inequality in the point x, we have

$$|\mathrm{D}f|^2(x)\;\leq\;\frac{2n(n+7)}{(n+1)a^2}\,.$$

Since our manifold (M,g_0) is unbounded, we can take the value a as big as we wish. In particular, for $a\to\infty$ we obtain $\mathrm{D}f(x)=0$. Since the same argument can be made at any point of M, it follows that f is constant. This concludes the proof.

3.2 Characterization of the de Sitter solution

3.2.1 Statement of the main results.

For static solutions with positive cosmological constant, it is physically reasonable to assume, according to the explicit examples listed in Subsection 2.5.1 above, that M is compact with non-empty boundary. As usual u will be strictly positive in the interior of M and such that $\partial M = \{u = 0\}$. In order to get rid of the scaling invariance of system (2.1.7), we adopt the usual normalization for the static metric g_0, so that we are led to study system (2.5.1), that we rewrite here for reference

$$\begin{cases} u\,\mathrm{Ric} = \mathrm{D}^2 u + n\,u\,g_0, & \text{in } M \\ \Delta u = -n\,u, & \text{in } M \\ u > 0, & \text{in } M \\ u = 0, & \text{on } \partial M \end{cases} \qquad \text{with } \ M \ \text{ compact}. \tag{3.2.1}$$

Our first result is the following characterization of the de Sitter solution in terms of the surface gravity of the boundary. To state the result, we recall the notation $u_{\max} = \max_M u$ introduced in Subsection 2.5.1 and for a given $S \in \pi_0(\partial M)$ (i.e., for a given connected component $S \subset \partial M$ of the boundary) we let

$$\kappa(S) \;=\; \frac{|\mathrm{D}u|_{|_S}}{\max_M u} \;\in \mathbb{R} \tag{3.2.2}$$

be the surface gravity of the horizon S, according to the normalization proposed in Subsection 2.1.3.

Theorem 3.2.1. *Let (M, g_0, u) be a solution to problem (3.2.1). Then*

$$\max_{S \in \pi_0(\partial M)} \kappa(S) \;\geq\; 1\,.$$

Moreover, if the equality holds, then the triple (M, g_0, u) is isometric to the de Sitter solution (2.5.2).

Recalling that the de Sitter solution (2.5.2) satisfies $\kappa(\partial M) = |\mathrm{D}u|/u_{\max} = |\mathrm{D}u| \equiv 1$ on ∂M, the above result implies that the de Sitter triple has the least possible surface gravity among all the solutions to problem (3.2.1) with connected boundary. The proof of the above statement is an elementary argument based on the Maximum Principle and will be presented in Section 3.4. More precisely, what we will prove in Theorem 3.4.2 below is that, if a solution (M, g_0, u) to (3.2.1) satisfies the inequality

$$\frac{|\mathrm{D}u|}{u_{\max}} \leq 1 \qquad \text{on} \quad \partial M\,,$$

then (M, g_0, u) is necessarily isometric to the de Sitter solution (2.5.2). Combining this Maximum Principle argument together with the Monotonicity Formula of Subsection 3.5.2, we obtain a relevant enhancement of Theorem 3.2.1, whose importance will be clarified in a moment. To introduce this result, we let $\mathrm{MAX}(u)$ be the set where the maximum of u is achieved, namely

$$\mathrm{MAX}(u) = \{p \in M \,:\, u(p) = u_{\max}\}\,,$$

and we observe that every connected component of $M \setminus \mathrm{MAX}(u)$ has non-empty intersection with ∂M. This follows easily from the Weak Minimum Principle and it is proven in the No Island Lemma 3.5.1 below. Our main result in the case $\Lambda > 0$ reads:

Theorem 3.2.2. *Let (M, g_0, u) be a solution to problem (3.2.1), let N be a connected component of $M \setminus \mathrm{MAX}(u)$, and let $\partial N = \partial M \cap N$ be the non-empty and possibly disconnected boundary portion of ∂M that lies in N. Then*

$$\max_{S \in \pi_0(\partial N)} \kappa(S) \;\geq\; 1\,.$$

Moreover, if the equality holds, then the triple (M, g_0, u) is isometric to the de Sitter solution (2.5.2).

In other words, Theorem 3.2.2 is a localized version of Theorem 3.2.1. In fact, what we will actually prove (see Theorem 3.5.5 in Section 3.5) is that if on a single connected component N of $M \setminus \mathrm{MAX}(u)$ it holds

$$\frac{|\mathrm{D}u|}{u_{\max}} \;\leq\; 1 \qquad \text{on} \quad \partial N\,,$$

then the entire solution (M, g_0, u) must be isometric to the de Sitter solution, in particular the boundary ∂M and the set $\mathrm{MAX}(u)$ are both connected *a posteriori*.

3.2.2 Surface gravity and mass.

We are now in the position to present an interpretation of both Theorem 3.2.1 and Theorem 3.2.2 in terms of the mass aspect of a static solution (M, g_0, u). As already observed, the main conceptual issue lies in the fact that, unlike for asymptotically flat and asymptotically hyperbolic manifolds, there is no clear notion of mass, when the cosmological constant is positive. To overcome this difficulty, we are going to exploit some very basic relationships between surface gravity and mass in the case of static solutions. In doing this we are motivated by the exemplification given in Subsection 2.3.1 for $\Lambda = 0$ as well as by the explicit role played by the mass parameter m in the model solutions (see Subsection 2.5.1). In particular these latter are used as reference configuration in the following definition of *virtual mass*. As it will be clear from the forthcoming discussion, it is also useful to use them in order to distinguish between the different characters of boundary components. For this reasons we give the following definitions.

Definition 3.2.3. *Let (M, g_0, u) be a solution to problem (3.2.1). A connected component S of ∂M is called an* horizon. *An horizon is said to be:*

- *of* cosmological type *if:* $\quad \kappa(S) < \sqrt{n},$
- *of* black hole type *if:* $\quad \kappa(S) > \sqrt{n},$
- *of* cylindrical type *if:* $\quad \kappa(S) = \sqrt{n},$

where $\kappa(S)$ is the surface gravity *of S defined in (3.2.2). A connected component N of $M \setminus \mathrm{MAX}(u)$ is called:*

- *an* outer region *if all of its horizons are of cosmological type, i.e., if*

$$\max_{S\in\pi_0(\partial N)} \kappa(S) \ < \ \sqrt{n}\,,$$

- *an* inner region *if it has at least one horizon of black hole type, i.e., if*

$$\max_{S\in\pi_0(\partial N)} \kappa(S) \ > \ \sqrt{n}\,,$$

- *a* cylindrical region *if there are no black hole horizons and there is at least one cylindrical horizon, i.e., if*

$$\max_{S\in\pi_0(\partial N)} \kappa(S) \ = \ \sqrt{n}\,.$$

We introduce now the concept of virtual mass of a given connected component of $M \setminus \mathrm{MAX}(u)$.

Definition 3.2.4 (Virtual Mass)**.** *Let (M, g_0, u) be a solution to (3.2.1) and let N be a connected component of $M \setminus \mathrm{MAX}(u)$. The virtual mass of N is denoted by $\mu(N, g_0, u)$ and it is defined in the following way:*

(i) If N is an outer region, then we set

$$\mu(N, g_0, u) \ = \ k_+^{-1}\left(\max_{\partial N} \frac{|\mathrm{D}u|}{u_{\max}}\right), \tag{3.2.3}$$

where k_+ is the outer surface gravity function defined in (2.5.6).

(ii) If N is an inner region, then we set

$$\mu(N, g_0, u) \ = \ k_-^{-1}\left(\max_{\partial N} \frac{|\mathrm{D}u|}{u_{\max}}\right), \tag{3.2.4}$$

where k_- is the inner surface gravity function defined in (2.5.7).

In other words, the virtual mass of a connected component N of $M \setminus \mathrm{MAX}(u)$ can be thought as the mass (parameter) that on a model solution would be responsible for (the maximum of) the surface gravity measured at ∂N. In this sense the rotationally symmetric solutions described in Subsection 2.5.1 are playing here the role of reference configurations. As it is easy to check, if (M, g_0, u) is either the de Sitter, or the Schwarzschild–de Sitter, or the Nariai solution, then the virtual mass coincides with the explicit mass parameter m that appears in Subsection 2.5.1. In dimension $n = 3$ these are the only known solutions to problem (3.2.1). To discuss some other examples, let us consider the 4-dimensional case. We recall that in Subsection 2.5.4 we have discussed other solutions to problem (3.2.1) found by Gibbons, Hartnoll and Pope [GHP03] for $4 \le n \le 8$. Among these solutions, there is a sequence of 4-dimensional triples (M_i, g_i, u_i) with connected horizon diffeomorphic to $\mathbb{S}^3$. These triples are not explicit, but it is easily seen that they converge quickly to a limit singular solution, which corresponds to a double cone singular metric on the sphere $\mathbb{S}^4$, see the discussion and the figures in [GHP03, Appendix]. As a consequence, one can infer that the value κ_i of the surface gravity of the connected boundary ∂M_i rapidly converge to the value $\sqrt{3}$ as $i \to +\infty$. In particular, the virtual masses of this solutions can be computed as

$$\mu_i := \mu(M_i, g_i, u_i) = k_+^{-1}(\kappa_i),$$

and we have $\mu_i \to k_+^{-1}(\sqrt{3}) \approx 0,1136$ as $i \to +\infty$. Since the convergence is very fast, the virtual mass μ_i is actually approximately equal to $0,1136$ for any $i \geq 3$. A similar qualitative analysis can be employed to estimate the virtual masses of the other families of solutions provided in [GHP03].

It is very important to notice that the well–posedness of Definition 3.2.4 for outer regions is not *a priori* guaranteed. In fact, one would have to check that, for any given solution (M, g_0, u) to (3.2.1), the quantity $\max_{\partial N} |\mathrm{D}u|/u_{\max}$ lies in the domain of definition of the function k_+^{-1}, namely in the real interval $[1, \sqrt{n})$. This is the content of the following Positive Mass Statement, whose proof is a direct consequence of Theorem 3.2.2.

Theorem 3.2.5 (Positive Mass Statement for Static Metrics with Positive Cosmological Constant)**.** *Let (M, g_0, u) be a solution to problem* (3.2.1)*. Then, every connected component of $M \setminus \mathrm{MAX}(u)$ has well defined and thus nonnegative virtual mass. Moreover, as soon as the virtual mass of some connected component vanishes, the entire solution (M, g_0, u) is isometric to the de Sitter solution* (2.5.2)*.*

In order to justify the terminology employed, it is useful to put the above result in correspondence with the classical statement of the Positive Mass Theorem for asymptotically flat manifolds with nonnegative scalar curvature. In this perspective it is clear that in our context the connected components of $M \setminus \mathrm{MAX}(u)$ play the same role as the asymptotically flat ends of the classical situation. In fact, the virtual mass is well defined and nonnegative on every single connected component, in perfect analogy with the ADM mass of every single asymptotically flat end. This correspondence holds true also for the rigidity statements. In fact, as soon as the mass (either virtual or ADM) annihilates on one single piece, the whole manifold must be isometric to the model solution with zero mass (either de Sitter or Minkowski).

Another important observation comes from the fact that the above statement should be put in contrast with the so called Min-Oo's conjecture (see Subsection 1.2.3), which, we recall, asserts that a compact Riemannian manifold (M^n, g), whose boundary is isometric to $\mathbb{S}^{n-1}$ and totally geodesic, must be isometric to the standard round hemisphere $(\mathbb{S}^n_+, g_{\mathbb{S}^n})$, provided $\mathrm{R}_g \geq n(n-1)$. For long time, this conjecture has been considered as the natural counterpart of the rigidity statement of the Positive Mass Theorem in the case of positive cosmological constant. However, as already discussed, this conjecture has finally been disproved in a remarkable paper [BMN11] by Brendle, Marques and Neves, see Theorem 1.2.19. In contrast with this, our Positive Mass Statement seems to indicate – at least in the case of static solutions – a different possible approach towards the extension of the classical Positive ADM Mass Theorem to the context of positive cosmological constant. In this perspective, it would be very interesting to see if the above statement could be extended to a broader class of metrics of physical relevance, leading to a more comprehensive definition of mass. The first step in this direction would be to consider the case of stationary solutions to the Einstein Field Equations with Killing horizons, so that the concept of surface gravity is well defined (see Subsection 2.1.2). This will be the object of further investigations.

3.2.3 Area bounds.

Further evidences in favour of the virtual mass will be presented in Chapter 4, where sharp area bounds will be obtained for horizons of black hole and cosmological type, the equality case being characterised by the Schwarzschild–de Sitter solution (2.5.3). In order to anticipate these results, we discuss in this subsection a local version of inequality (2.5.9), which we recall was proved by Chruściel [Chr] (see also [HMR15]) generalizing the approach of Boucher, Gibbons and Horowitz [BGH84].

Theorem 3.2.6. *Let (M, g_0, u) be a solution to problem (3.2.1), let N be a connected component of $M \setminus \mathrm{MAX}(u)$, and let $\partial N = \partial M \cap N$ be the non-empty and possibly disconnected boundary portion of ∂M that lies in N. Then it holds*

$$0 \;\leq\; \int_{\partial N} |\mathrm{D}u| \left[\mathrm{R}^{\partial N} - (n-1)(n-2) \right] \mathrm{d}\sigma \,, \tag{3.2.5}$$

where $\mathrm{R}^{\partial N}$ is the scalar curvature of the metric induced by g_0 on ∂N. Moreover, if the equality holds, then the triple (M, g_0, u) is isometric to the de Sitter solution (2.3.3).

The proof of this result follows closely the one presented in [Chr, Section 6], see Subsection 3.4.3 for the details.

In order to emphasize the analogy with the forthcoming results in the case of conformally compact static solutions with negative cosmological constant (see Corollary 3.3.3 below), it is useful to single out the following straightforward consequence of the above theorem.

Corollary 3.2.7. *Let (M, g_0, u) be a solution to problem (3.2.1), let N be a connected component of $M \setminus \mathrm{MAX}(u)$, and let $\partial N = \partial M \cap N$ be the non-empty and possibly disconnected boundary portion of ∂M that lies in N. Then, if the inequality*

$$\mathrm{R}^{\partial N} \;\leq\; (n-1)(n-2)$$

holds on ∂N, where $\mathrm{R}^{\partial N}$ is the scalar curvature of the metric induced by g_0 on ∂N, then the triple (M, g_0, u) is isometric to the de Sitter solution (2.3.3).

If in Theorem 3.2.6 we also assume that ∂N is connected and orientable, and that $n = 3$, then $|\mathrm{D}u|$ is constant on the whole ∂N, and from the Gauss-Bonnet Theorem we have $\int_{\partial N} \mathrm{R}^{\partial N} \mathrm{d}\sigma = 4\pi\chi(\partial N)$. Therefore, with these additional hypotheses, the thesis of Theorem 3.2.6 translates into

$$|\partial N| \;\leq\; 2\pi\, \chi(\partial N) \,,$$

where $|\partial N|$ is the hypersurface area of ∂N with respect to the metric g_0. In particular, $\chi(\partial N)$ has to be positive, which implies that ∂N is diffeomorphic to a sphere and $\chi(\partial N) = 2$. This proves the following corollary.

Corollary 3.2.8. *Let (M, g_0, u) be a 3-dimensional orientable solution to problem (3.2.1), let N be a connected component of $M \setminus \mathrm{MAX}(u)$, and suppose that $\partial N = \partial M \cap N$ is connected. Suppose also that $|\mathrm{D}u|^2 \leq C(u_{\max} - u)$ on the whole M, for some constant $C \in \mathbb{R}$. Then ∂N is a sphere and it holds*

$$|\partial N| \;\leq\; 4\pi\,. \tag{3.2.6}$$

Moreover, if the equality holds then the triple (M, g_0, u) is isometric to the de Sitter solution (2.3.3).

This corollary is a local version of the well known Boucher-Gibbons-Horowitz inequality, see Theorem 2.5.1.

In Chapter 4, we are going to prove stronger versions of both inequality (3.2.5) and (3.2.6), see Theorems 4.1.1 and 4.1.3. These will then be used to prove a Black Hole Uniqueness Statement, provided e.g. the set $\mathrm{MAX}(u)$ is a two sided regular hypersurface that divides M into an inner region and an outer region, whose virtual mass is controlled by the one of the inner region, see Theorem 4.1.8.

3.3 Characterization of the Anti de Sitter solution

In this section we discuss the case $\Lambda < 0$. The results that we will show in this section, namely Theorems 3.3.1 and 3.3.2, seem to be in line with Theorem 3.1.1 for the case $\Lambda = 0$, shown in Section 3.1. We observe that, while in the case $\Lambda = 0$ no hypothesis on the behavior at infinity of the solutions was required, in the case $\Lambda < 0$ we cannot expect our uniqueness results to remain true without additional assumptions. In fact, the Anti de Sitter triple is not the only solution to (3.3.1). Another one is the Anti Nariai triple (2.4.9) described in Subsection 2.4.1, and we also point out that the existence of an infinite family of conformally compact solutions has been proven in [ACD02, ACD05] (see Subsection 3.3.2 for some more details). To rule these solutions out and obtain uniqueness statements for the Anti de Sitter triple, we suggest the following possibility. Recalling the asymptotic behaviour (2.4.4) that is expected on the model solution (2.4.3)

$$\lim_{|x|\to\infty} \left(u^2 - u_{\min}^2 - |\mathrm{D}u|^2\right) \;=\; 0\,,$$

and looking at this formula as to a necessary condition, we are going to show that it also yields a fairly neat sufficient condition in order to conclude that a complete static triple with $\Lambda < 0$ is isometric to the Anti de Sitter solution. Formally, this translates in the characterisation of the equality case in formulæ (3.3.2) and (3.3.3) below.

3.3.1 Statement of the main results.

In analogy with the properties of the Anti de Sitter triple (2.4.3) described in Subsection 2.4.1, it is natural to restrict our attention to static solutions

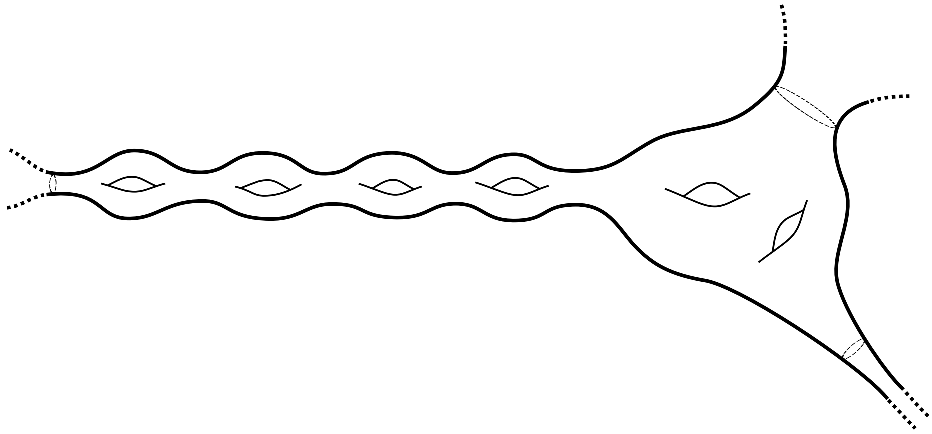


Figure 3.1: The ends of the solutions to (3.3.1) are usually assumed to be diffeomorphic to a product. However, our analysis does not exclude *a priori* more peculiar topologies, like the end represented on the left hand side of the figure.

(M, g_0, u) with negative cosmological constant such that the manifold M is non-compact and with empty boundary. We point out that the latter assumption, which is unavoidable in the present framework, excludes *a priori* the family of the Schwarzschild–Anti de Sitter solutions (2.4.5) from our treatment. For simplicity, we will also suppose that the number of ends of M is finite. We recall (see for instance [Gui16, Section 3.1]) that the ends of M are defined as the sequences $U_1 \supset U_2 \supset \dots$, where, for every $i \in \mathbb{N}$, U_i is an unbounded connected component of $M \setminus K_i$ and $\{K_i\}$ is an exhaustion by compact sets of M. It is easy to see that the definition of end does not really depend on the choice of the exhaustion by compact sets, in the sense that there is a clear one-to-one correspondence between the ends of M defined with respect to two different exhaustions. We emphasize the fact that – in contrast with other characterisations of the Anti de Sitter solution – we are not making any *a priori* assumption on the topology of the ends, as it is explained in Figure 3.1 and the discussion below. Starting from system (2.1.7), and rescaling g_0 as in Subsection 2.4.1, we are led to study problem (2.4.1), that we rewrite here, explicitating also the additional hypothesis of empty boundary

$$\begin{cases} u\,\mathrm{Ric} = \mathrm{D}^2 u - n\,u\,g_0, & \text{in } M \\ \Delta u = n\,u, & \text{in } M \\ u > 0, & \text{in } M \\ u(x) \to +\infty & \text{as } x \to \infty \end{cases} \qquad \text{with} \quad \partial M = \emptyset\,. \tag{3.3.1}$$

We recall that, with the notation $u(x) \to +\infty$ as $x \to \infty$, we mean that, given an exhaustion of M by compact sets $\{K_i\}_{i\in\mathbb{N}}$, we have that for any sequence of points $x_i \in M \setminus K_i$, $i \in \mathbb{N}$, it holds $\lim_{i\to+\infty} u(x_i) = +\infty$. Recalling the notation $u_{\min} = \min_M u$ introduced in Subsection 2.4.1, we are now able to state our first result in the $\Lambda < 0$ case. The proof follows the same line as the one of Theorem 3.2.1 in the $\Lambda > 0$ case.

Theorem 3.3.1. *Let (M, g_0, u) be a solution to problem (3.3.1). Then*

$$\liminf_{x\to\infty} \left(u^2 - u_{\min}^2 - |\mathrm{D}u|^2\right)(x) \leq 0\,. \tag{3.3.2}$$

Moreover, if the equality holds then the triple (M, g_0, u) is isometric to the Anti de Sitter solution (2.4.3).

To avoid ambiguity, we recall that inequality (3.3.2) means that, taken an exhaustion of M by compact sets $\{K_i\}_{i\in\mathbb{N}}$, we have that for any sequence of points $x_i \in M \setminus K_i$, $i \in \mathbb{N}$, it holds $\liminf_{i\to+\infty}(u^2 - u_{\min}^2 - |\mathrm{D}u|^2)(x_i) \leq 0$. We have already observed in (2.4.4) that the Anti de Sitter triple (2.4.3) is such that $u^2 - u_{\min}^2 - |\mathrm{D}u|^2$ goes to zero as one approaches the end of the manifold. Therefore, Theorem 3.3.1 characterizes the Anti de Sitter triple among the solutions to (3.3.1) as the one that maximises the left hand side of (3.3.2). In fact, what we will actually prove (see Theorem 3.4.3 below) is that the only solution to (3.3.1) that satisfies

$$\liminf_{x\to\infty} \left(u^2 - u_{\min}^2 - |\mathrm{D}u|^2\right)(x) \geq 0\,,$$

is the Anti de Sitter triple (2.4.3).

We are now going to state a local version of Theorem 3.3.1. To this end, we recall the notation

$$\mathrm{MIN}(u) = \{p \in M \,:\, u(p) = u_{\min}\}\,,$$

for the set of the minima of u and we notice that any connected component N of $M \setminus \mathrm{MIN}(u)$ must contain at least one of the ends of M by the No Island Lemma 3.6.1. In particular, the $\liminf_{x\to\infty}$ in formula (3.3.3) below is completely justified.

Arguing as in the case $\Lambda > 0$, we obtain through a Maximum Principle and a suitable Monotonicity Formula the following analogue of Theorem 3.2.2.

Theorem 3.3.2. *Let (M, g_0, u) be a solution to problem (3.3.1), and let N be a connected component of $M \setminus \mathrm{MIN}(u)$. Then*

$$\liminf_{x\in N,\, x\to\infty} \left(u^2 - u_{\min}^2 - |\mathrm{D}u|^2\right)(x) \leq 0\,. \tag{3.3.3}$$

Moreover, if the equality holds then the triple (M, g_0, u) is isometric to the Anti de Sitter solution (2.4.3).

Theorem 3.3.2 is a stronger version of Theorem 3.3.1, in the sense that the asymptotic behavior of the quantity in (3.3.3) has to be checked only along the ends of N. In this sense, the relation between Theorem 3.3.1 and Theorem 3.3.2 is the same as the one between Theorem 3.2.1 and Theorem 3.2.2. In the next subsection, we are going to compare Theorem 3.3.2 with other known characterisations of the Anti de Sitter solution.

3.3.2 Comparison with other known characterizations.

Classically, the study of static solutions with $\Lambda < 0$ has been tackled by requiring some additional information on the asymptotic behavior of the triple (M, g_0, u). These assumptions, albeit natural, are usually very strong, in the sense that they restrict the topology of the ends as well as the asymptotic behavior of the function u. We have already discussed the main definitions and known results in Subsection 2.4.2. Here, we quickly recall them in order to draw the state of the art and put our results in perspective.

The most widely used assumption is to ask for the triple (M, g_0, u) to be *conformally compact* in the sense of Definition 2.4.1. This hypothesis forces (M, g_0) to be isometric to the interior of a compact manifold $\overline{M}_\infty = M \cup \partial_\infty M$, where $\partial_\infty M$ is the boundary of $\overline{M}_\infty$ and is called the *conformal infinity* of M. It also requires the metric $\bar{g} = u^{-2} g_0$ to extend to the conformal infinity with some regularity. Despite this being a somewhat standard assumption, almost nothing being known without requiring it, it still imposes some strong topological and analytical *a priori* restrictions on a mere solution to (3.3.1). For instance, if $n = 3$, we know from [CS01] (see also Proposition 2.4.2) that the conformal infinity $\partial_\infty M$ is necessarily connected, that is, M has a unique end. Therefore, for 3-dimensional *conformally compact* triples, Theorems 3.3.1 and 3.3.2 are completely equivalent. Nevertheless, as already explained in Subsection 2.4.2, the results in [ACD02, ACD05] show that the conformal compactness *per se* is not strong enough to characterize the Anti de Sitter solution.

This implies that additional assumptions are needed in order to prove a rigidity statement. To introduce our next result, we recall that, for conformally compact solutions, the quantity $u^2 - u_{\min}^2 - |\mathrm{D}u|^2$ extends smoothly to a function on the whole $\overline{M}_\infty = M \cup \partial_\infty M$ and it holds (see formula (2.4.14))

$$u^2 - u_{\min}^2 - |\mathrm{D}u|^2 \;=\; \frac{\mathrm{R}^{\partial_\infty M}}{(n-1)(n-2)} - u_{\min}^2 \qquad \text{on } \partial_\infty M\,, \tag{3.3.4}$$

where $\mathrm{R}^{\partial_\infty M}$ is the scalar curvature of the metric induced by $\bar{g}$ on $\partial_\infty M$.

In order to introduce the next result, we first fix a couple of notations. Given a connected component N of $M \setminus \mathrm{MAX}(u)$, we denote by $\partial_\infty N$ the conformal infinity of N, that is,

$$\partial_\infty N \;=\; \partial_\infty M \cap \overline{N}^{\overline{M}_\infty}\,,$$

where $\overline{N}^{\overline{M}_\infty}$ is the closure of N in $\overline{M}_\infty$. From formula (3.3.4) and Theorem 3.3.2 we deduce the following corollary, that represents the precise analogue of Corollary 3.2.7, proven in the case $\Lambda > 0$.

Corollary 3.3.3. *Let (M, g_0, u) be a conformally compact solution to problem (3.3.1) and let N be a connected component of $M \setminus \mathrm{MIN}(u)$. Suppose that the scalar curvature $\mathrm{R}^{\partial_\infty N}$ of the metric induced by $\bar{g} = u^{-2} g_0$ on the conformal infinity of N satisfies the following inequality*

$$\mathrm{R}^{\partial_\infty N} \;\geq\; (n-1)(n-2)\, u_{\min}^2 \tag{3.3.5}$$

on the whole $\partial_\infty N$. Then the triple (M, g_0, u) is isometric to the Anti de Sitter solution (2.4.3).

Imposing stronger assumptions on the asymptotics of the triple (M, g_0, u) leads to even cleaner statements. The fee for this is that the class of solutions where the uniqueness can be proven is *a priori* much smaller than the ones considered above. For example, if one requires the triple (M, g_0, u) to be *asymptotically Anti de Sitter* in the sense of Definition 2.4.4, then it is possible to conclude uniqueness as in Corollary 3.3.4 below. However, this assumption forces the conformal infinity of M – endowed with the metric induced on it by $\bar{g} = u^{-2}g$ – to be connected and isometric to the standard sphere. In particular the quantity $\mathrm{R}^{\partial_\infty N}$ in Corollary 3.3.3 is equal to $(n-1)(n-2)$ on the whole $\partial_\infty N = \partial_\infty M$.

Corollary 3.3.4. *Let (M, g_0, u) be an asymptotically Anti de Sitter solution to problem (3.3.1) such that M has empty boundary. If $u_{\min} \leq 1$, then the triple (M, g_0, u) is isometric to the Anti de Sitter solution (2.4.3).*

We remark that a stronger version of Corollary 3.3.4 is already known. In fact, from the works of Wang [Wan05] and Qing [Qin04] it follows that the same thesis holds without the need of the assumption $u_{\min} \leq 1$, see Theorem 2.4.6 and the discussion below. It is worth mentioning that the methods employed to obtain these uniqueness results heavily rely on (some kind of) the Positive Mass Theorem. More precisely, Wang's result relies on the Positive Mass Theorem for asymptotically hyperbolic manifolds, proved in [Wan01], whereas Qing's result exploit the Positive Mass Theorem for asymptotically flat manifolds, proved by Schoen-Yau [SY79, SY17].

3.4 Shen's Identity and its consequences

In this section we give the proofs of Theorems 3.2.1 and 3.3.1, which consist on the analysis via the Strong Maximum Principle of Shen's Identity (3.4.1).

3.4.1 Computations via Bochner formula.

In order to prove our theorems, we need the following preparatory result, which is a simple application of the Bochner formula.

Proposition 3.4.1 (Shen's Identity [Amb15, formula (8)], [She97, formula (12)])**.** *Let (M, g_0, u) be a solution of either system (3.2.1) or system (3.3.1). Then it holds*

$$\operatorname{div}\left[\frac{1}{u}\left(\mathrm{D}|\mathrm{D}u|^2 - \frac{2}{n}\Delta u \mathrm{D}u\right)\right] = \frac{2}{u}\left[|\mathrm{D}^2 u|^2 - \frac{(\Delta u)^2}{n}\right] \geq 0\,. \tag{3.4.1}$$

Proof. Since the two cases are very similar, we will do the computations for both solutions of (3.2.1) and (3.3.1) at the same time. We first recall that, from the first and second equation in (3.2.1) and (3.3.1), we have $\Delta u = \mp n u$, and $\mathrm{Ric} = \mathrm{D}^2 u \pm n u g_0$. Using these equalities together with the Bochner formula, we compute

$$\begin{aligned}\Delta|\mathrm{D}u|^2 &= 2\,|\mathrm{D}^2u|^2 + 2\,\mathrm{Ric}(\mathrm{D}u,\mathrm{D}u) + 2\langle \mathrm{D}\Delta u\,|\,\mathrm{D}u\rangle \\ &= 2\,|\mathrm{D}^2u|^2 + 2\left[\frac{1}{u}\,\mathrm{D}^2u(\mathrm{D}u,\mathrm{D}u) \pm n\,|\mathrm{D}u|^2\right] \mp 2n\,|\mathrm{D}u|^2 \\ &= 2\,|\mathrm{D}^2u|^2 + \frac{1}{u}\,\langle \mathrm{D}|\mathrm{D}u|^2\,|\,\mathrm{D}u\rangle\,. \end{aligned} \tag{3.4.2}$$

Letting

$$Y = \mathrm{D}|\mathrm{D}u|^2 - \frac{2}{n}\,\Delta u\,\mathrm{D}u\,,$$

and using (3.4.2), we compute

$$\begin{aligned}\mathrm{div}(Y) &= \Delta|\mathrm{D}u|^2 - \frac{2}{n}\langle \mathrm{D}\Delta u\,|\,\mathrm{D}u\rangle - \frac{2}{n}(\Delta u)^2 \\ &= 2\left[\,|\mathrm{D}^2u|^2 - \frac{(\Delta u)^2}{n}\right] + \frac{1}{u}\,\langle \mathrm{D}|\mathrm{D}u|^2\,|\,\mathrm{D}u\rangle \pm 2\,|\mathrm{D}u|^2\,. \end{aligned}$$

More generally, for every nonzero $\mathscr{C}^1$ function $\alpha = \alpha(u)$, it holds

$$\begin{aligned}\frac{1}{\alpha}\mathrm{div}(\alpha Y) &= \mathrm{div}(Y) + \frac{\dot\alpha}{\alpha}\,\langle Y\,|\,\mathrm{D}u\rangle \\ &= 2\left[\,|\mathrm{D}^2u|^2 - \frac{(\Delta u)^2}{n}\right] + \left(\frac{\dot\alpha}{\alpha} + \frac{1}{u}\right)\left(\langle \mathrm{D}|\mathrm{D}u|^2\,|\,\mathrm{D}u\rangle \pm 2u\,|\mathrm{D}u|^2\right). \end{aligned}$$

where $\dot\alpha$ is the derivative of α with respect to u. The computation above suggests us to choose

$$\alpha(u) = \frac{1}{u}\,.$$

so that $\dot\alpha/\alpha = -1/u$, and we obtain

$$\mathrm{div}\left(\frac{1}{u}\,Y\right) = \frac{2}{u}\left[\,|\mathrm{D}^2u|^2 - \frac{(\Delta u)^2}{n}\right]. \tag{3.4.3}$$

The square root in the right hand side of (3.4.3) coincides with the g_0-norm of the trace-free part of D^2u, in particular it is always positive, and the thesis follows. □

Proposition 3.4.1 is already well known, and it has a number of applications. The most significant one is a proof of the Boucher-Gibbons-Horowitz inequality, see Theorem 2.5.1, for which we refer the reader to the following Subsection 3.4.3. Another interesting application of formula (3.4.1) has appeared recently in [Amb15], where it is used to deduce some relevant topological features of the solutions to system (3.2.1), see Theorem 2.5.5.

3.4.2 Proof of Theorems 3.2.1 and 3.3.1.

In this subsection, we combine Proposition 3.4.1 with the Strong Maximum Principle, in order to recover Theorems 3.2.1 and 3.3.1. Despite the two proofs present some analogies, we prefer to prove each theorem independently. We start with Theorem 3.2.1, that we rewrite here in an alternative – but equivalent – form, for the reader's convenience.

Theorem 3.4.2. *Let (M, g_0, u) be a solution to problem (3.2.1), and suppose that*

$$\frac{|\mathrm{D}u|}{u_{\max}} \leq 1 \qquad \text{on} \quad \partial M\,.$$

Then the triple (M, g_0, u) is isometric to the de Sitter solution (2.5.2). In particular, ∂M is connected.

Proof. Combining the equation $\Delta u = -nu$ with formula (3.4.1) in Proposition 3.4.1, we get

$$0 \leq 2\left[\,|\mathrm{D}^2 u|^2 - \frac{(\Delta u)^2}{n}\,\right] = \Delta\big(|\mathrm{D}u|^2 + u^2\big) - \frac{1}{u}\big\langle \mathrm{D}u \,\big|\, \mathrm{D}\big(|\mathrm{D}u|^2 + u^2\big)\big\rangle\,. \tag{3.4.4}$$

We claim that $\big(|\mathrm{D}u|^2 + u^2\big)$ is constant and its value coincides with $u_{\max}^2$. This follows essentially from the Maximum Principle, however some attention should be payed to the coefficient $1/u$, since it blows up at ∂M. Hence, for the sake of completeness, we prefer to present the details.

As it is pointed out in Subsection 2.1.3, the function u is analytic, and thus its critical level sets as well as its critical value are discrete. On the other hand, one has that $|\mathrm{D}u| > 0$ on ∂M, so that the zero level set of u is regular. Moreover, it is possible to choose a positive number $\eta > 0$ such that each level set $\{u = \varepsilon\}$ is regular (and diffeomorphic to ∂M), provided $0 < \varepsilon \leq \eta$. Setting $M_\varepsilon = \{u \geq \varepsilon\}$, it is immediate to observe that the coefficient $1/u$ is now bounded above by $1/\varepsilon$ in M_ε, moreover we have that

$$\max_{M_\varepsilon}\big(|\mathrm{D}u|^2 + u^2\big) \leq \max_{\partial M_\varepsilon}\big(|\mathrm{D}u|^2 + u^2\big)\,,$$

by the Maximum Principle. In particular, for every $0 < \varepsilon \leq \eta$ it holds

$$\max_{\partial M_\eta}\big(|\mathrm{D}u|^2 + u^2\big) \leq \max_{\partial M_\varepsilon}\big(|\mathrm{D}u|^2 + u^2\big)\,.$$

Moreover, it is easily seen that $\lim_{\varepsilon\to 0^+} \max_{\partial M_\varepsilon}\big(|\mathrm{D}u|^2 + u^2\big) = |\mathrm{D}u|_{|\partial M}$, so that, using the assumption $|\mathrm{D}u| \leq u_{\max}$ on ∂M, one gets

$$\max_{\partial M_\eta}\big(|\mathrm{D}u|^2 + u^2\big) \leq u_{\max}^2\,.$$

On the other hand, it is clear that $\mathrm{MAX}(u) = \{p \in M : u(p) = u_{\max}\} \subset M_\eta$ and that for every $p \in \mathrm{MAX}(u)$ it holds $\big(|\mathrm{D}u|^2 + u^2\big)(p) = u_{\max}^2$. The Strong Maximum Principle implies that $|\mathrm{D}u|^2 + u^2 \equiv u_{\max}^2$ on M_η. Since $\eta > 0$ can be chosen arbitrarily small, we conclude that $|\mathrm{D}u|^2 + u^2 \equiv u_{\max}^2$ on M.

Plugging the latter identity in formula (3.4.4), we easily obtain $|\mathrm{D}^2 u|^2 = (\Delta u)^2/n$, from which it follows $\mathrm{D}^2 u = -ug$ and in turns that $\mathrm{Ric} = (n-1)g$, where in the last step we have used the first equation of system (3.2.1). Now we can conclude by exploiting the results in [Oba62]. To this end, we double the manifold along the totally geodesic boundary, obtaining a closed compact Einstein manifold $(\hat{M}, \hat{g})$ with $\mathrm{Ric}_{\hat{g}} = (n-1)\hat{g}$. On $\hat{M}$ we define the function $\hat{u}$ as $\hat{u} = u$ on one copy of M and as $\hat{u} = -u$ on the other copy. Since $\mathrm{D}^2 u = 0$ on ∂M, after the gluing the function $\hat{u}$ is easily seen to be $\mathscr{C}^2$ on $\hat{M}$. Moreover, $\hat{u}$ is an eigenvalue of the laplacian, and more precisely it holds $-\Delta_{\hat{g}_0}\hat{u} = n\hat{u}$. Therefore [Oba62, Theorem 2] applies and we conclude that $(\hat{M}, \hat{g})$ is isometric to a standard sphere. □

We pass now to the proof of Theorem 3.3.1, that we restate here in an alternative form. Albeit its strict analogy with the above argument, we will present the proof of Theorem 3.3.1 in full details, since this will give us the opportunity to show how the required adjustments are essentially related to the different topology of the manifold M.

Theorem 3.4.3. *Let (M, g_0, u) be a solution to problem (3.3.1), and suppose*

$$\liminf_{x\to\infty} \left(u^2 - u^2_{\min} - |\mathrm{D}u|^2\right)(x) \;\geq\; 0\,.$$

Then the triple (M, g_0, u) is isometric to the Anti de Sitter solution (2.4.3). In particular, M has a unique end.

Proof. Recalling $\Delta u = nu$ and formula (3.4.1) in Proposition 3.4.1, we obtain

$$0 \;\leq\; 2\Big[\,|\mathrm{D}^2u|^2 - \frac{(\Delta u)^2}{n}\,\Big] \;=\; \Delta\big(|\mathrm{D}u|^2 \,-\, u^2\big) \,-\, \frac{1}{u}\big\langle\, \mathrm{D}u \,\big|\, \mathrm{D}\big(|\mathrm{D}u|^2 \,-\, u^2\big)\,\big\rangle\,. \tag{3.4.5}$$

We want to proceed in the same spirit as in the proof of Theorem 3.4.2 above. In this case the boundary is empty and the quantity $1/u$ is bounded from above by $1/u_{\min}$ on the whole M. On the other hand, this time the manifold M is complete and noncompact, so we have to pay some attention to the behavior of our solution along the ends. Let then $\{K_i\}_{i\in\mathbb{N}}$ be an exhaustion by compact sets of M. Without loss of generality, we can assume that the exhaustion is ordered by inclusion, namely $K_i \subset K_j$, whenever $i < j$. Applying the Weak Maxiimum Principle to the differential inequality (3.4.5) one gets

$$\left(|\mathrm{D}u|^2 - u^2 + u^2_{\min}\right)(x) \;\leq\; \max_{\partial K_i}\left(|\mathrm{D}u|^2 - u^2 + u^2_{\min}\right)\,, \tag{3.4.6}$$

for every $x \in K_i$ and every $i \in \mathbb{N}$. On the other hand, the assumption is clearly equivalent to $\limsup_{x\to\infty}(|\mathrm{D}u|^2 - u^2 + u^2_{\min})(x) \leq 0$. This implies that, for any given $\varepsilon > 0$, there exists a large enough $j_\varepsilon \in \mathbb{N}$ so that

$$\max_{\partial K_j}\left(|\mathrm{D}u|^2 - u^2 + u^2_{\min}\right) \;\leq\; \varepsilon\,, \qquad \text{for every } j \geq j_\varepsilon\,. \tag{3.4.7}$$

Combining the last two inequalities, we easily conclude that

$$|\mathrm{D}u|^2 - u^2 + u^2_{\min} \;\leq\; 0\,,$$

on the whole M. In particular, as soon as a compact subset K of M contains $\mathrm{MIN}(u) = \{p \in M \,:\, u(p) = u_{\min}\}$ in its interior, we have that $\max_{\partial K}(|\mathrm{D}u|^2 - u^2 + u^2_{\min}) \leq 0$. Since on $\mathrm{MIN}(u)$ it clearly holds $|\mathrm{D}u|^2 - u^2 + u^2_{\min} = 0$, the Strong Maximum Principle implies that $|\mathrm{D}u|^2 - u^2 + u^2_{\min} \equiv 0$ on K. From the analyticity of u, it follows that $|\mathrm{D}u|^2 - u^2 + u^2_{\min} \equiv 0$ on the whole M. Plugging this information in (3.4.5), we easily obtain $|\mathrm{D}^2u|^2 = (\Delta u)^2/n$, from which we deduce $\mathrm{D}^2u = ug_0$ and we can conclude using [Qin04, Lemma 3.3]. □

3.4.3 Boucher-Gibbons-Horowitz method revisited.

In this subsection we illustrate another consequence of Proposition 3.4.1, namely, we prove a local version of the Boucher-Gibbons-Horowitz inequality. To do that we are going to retrace the approach used in [Chr, Section 6], which essentially consists in integrating identity (3.4.1) on M and using the Divergence Theorem. The main difference is that instead of working on the whole M, we will focus on a single connected component N of $M \setminus \mathrm{MAX}(u)$. This will lead us to the proof of Theorem 3.2.6, which we have restated here for the reader's convenience.

Theorem 3.4.4. *Let (M, g_0, u) be a solution to problem (3.2.1), let N be a connected component of $M \setminus \mathrm{MAX}(u)$, and let $\partial N = \partial M \cap N$ be the non-empty and possibly disconnected boundary portion of ∂M that lies in N. Then it holds*

$$0 \;\leq\; \int_{\partial N} |\mathrm{D}u| \left[\mathrm{R}^{\partial N} - (n-1)(n-2) \right] \mathrm{d}\sigma \,,$$

where $\mathrm{R}^{\partial N}$ is the scalar curvature of the restriction of the metric g_0 to ∂N. Moreover, if the equality holds then (M, g_0, u) is isometric to the de Sitter solution (2.3.3).

Proof. From Proposition 3.4.1 we have

$$\mathrm{div} \left[\frac{1}{u} \left(|\mathrm{D}u|^2 + u^2 \right) \right] \;=\; \frac{2}{u} \left[|\mathrm{D}^2 u|^2 - \frac{(\Delta u)^2}{n} \right] \;\geq\; 0 \,.$$

To simplify the computations, we are going to integrate by parts the inequality

$$\mathrm{div} \left[\frac{1}{u} \left(|\mathrm{D}u|^2 + u^2 \right) \right] \;\geq\; 0 \,. \tag{3.4.8}$$

Proceeding in this way, we are going to prove the validity of the inequality mentioned in the statement of the theorem. In order to deduce the rigidity one has to keep into account also the quadratic term

$$\left[|\mathrm{D}^2 u|^2 - \frac{(\Delta u)^2}{n} \right] .$$

However, since this part of the argument is completely similar to what we have done in the previous subsection, we omit the details, leaving them to the interested reader.

In Subsection 2.1.3 we have shown that $|\mathrm{D}u| \neq 0$ on $\partial M = \{u = 0\}$, and that the critical values of u are always discrete. Therefore, from the compactness of M and the properness of u, it follows that we can choose $\varepsilon > 0$ so that the level sets $\{u = t\}$ are regular for all $0 \leq t \leq \varepsilon$ and for all $u_{\max} - \varepsilon \leq t < u_{\max}$. Integrating inequality (3.4.8) on the domain $\{\varepsilon < u < u_{\max} - \varepsilon\} \cap N$ and using the Divergence Theorem, we obtain

$$\begin{aligned} \int_{\{u = u_{\max} - \varepsilon\} \cap N} \left\langle \frac{\mathrm{D}\big(|\mathrm{D}u|^2 + u^2\big)}{u} \,\middle|\, \nu \right\rangle \mathrm{d}\sigma \;&\geq \\ \geq\; \int_{\{u = \varepsilon\} \cap N} &\left\langle \frac{\mathrm{D}\big(|\mathrm{D}u|^2 + u^2\big)}{u} \,\middle|\, \nu \right\rangle \mathrm{d}\sigma \,, \end{aligned} \tag{3.4.9}$$

where we have used the short hand notation $\nu = \mathrm{D}u/|\mathrm{D}u|$, for the unit normal to the set $\{\varepsilon < u < u_{\max} - \varepsilon\} \cap N$. Using the first equation in (3.2.1), we get

$$\begin{aligned}
\left\langle \frac{\mathrm{D}\big(|\mathrm{D}u|^2 + u^2\big)}{u} \,\middle|\, \nu \right\rangle &= 2\left[\frac{\mathrm{D}^2u(\mathrm{D}u,\mathrm{D}u) + u|\mathrm{D}u|^2}{u\,|\mathrm{D}u|}\right] \\
&= 2\left[\frac{\mathrm{Ric}(\mathrm{D}u,\mathrm{D}u) \,-\, n\,|\mathrm{D}u|^2 + |\mathrm{D}u|^2}{|\mathrm{D}u|}\right] \\
&= 2\,|\mathrm{D}u|\,\big[\,\mathrm{Ric}(\nu,\nu) \,-\, (n-1)\,\big]\,.
\end{aligned}$$

In view of this identity, inequality (3.4.9) becomes

$$\begin{aligned}
\int_{\{u=u_{\max}-\varepsilon\}\cap N} |\mathrm{D}u|\,\big[\,\mathrm{Ric}(\nu,\nu) - (n-1)\,\big]\,\mathrm{d}\sigma \;\geq \\
\geq \int_{\{u=\varepsilon\}\cap N} |\mathrm{D}u|\,\big[\mathrm{Ric}(\nu,\nu) - (n-1)\big]\,\mathrm{d}\sigma\,. \qquad (3.4.10)
\end{aligned}$$

We now claim that the lim inf of the left hand side vanishes when $\varepsilon \to 0$. Since M is compact and g_0 is smooth, the quantity $\mathrm{Ric}(\nu,\nu) - (n-1)$ is continuous, thus bounded, on M. Therefore, to prove the claim, it is sufficient to show that

$$\liminf_{t\to u_{\max}} \int_{\{u=t\}\cap N} |\mathrm{D}u|\,\mathrm{d}\sigma \,=\, 0\,.$$

If this is not the case, then we can suppose that the lim inf in the above formula is equal to some positive constant $\delta > 0$. This means that up to choose a small enough $\alpha > 0$, we could insure that

$$\int_{\{u=t\}\cap N} |\mathrm{D}u|\,\mathrm{d}\sigma \;\geq\; \frac{\delta}{2}\,, \qquad \text{for } u_{\max} - \alpha < t < u_{\max}\,.$$

Combining this fact with the coarea formula, one has that for every $0 < \varepsilon < \alpha$, it holds

$$\begin{aligned}
\int_{\{u_{\max}-\alpha<u<u_{\max}-\varepsilon\}\cap N} \left(\frac{|\mathrm{D}u|^2}{u_{\max}-u}\right)\mathrm{d}\mu \;&=\; \int_{u_{\max}-\alpha}^{u_{\max}-\varepsilon} \frac{\mathrm{d}t}{u_{\max}-t}\int_{\{u=t\}\cap N}|\mathrm{D}u|\,\mathrm{d}\sigma \;\geq \\
\geq\; \frac{\delta}{2}\int_{u_{\max}-\alpha}^{u_{\max}-\varepsilon}\frac{\mathrm{d}t}{u_{\max}-t} \;&=\; \frac{\delta}{2}\int_{\varepsilon}^{\alpha}\frac{\mathrm{d}\tau}{\tau}\,.
\end{aligned}$$

Now we make use of the inequality $|\mathrm{D}u|^2 \leq C(u_{\max} - u)$, which will be proven in the following chapter, see Proposition 4.2.8 and Remark 4.2.9. From this it follows that the leftmost hand side is bounded above by $C\,|N|$. On the other hand, the rightmost hand side tends to $+\infty$, as $\varepsilon \to 0$. Since we have reached a contradiction, the claim is proven. Hence, taking the $\liminf_{\varepsilon\to 0}$ in (3.4.10), we arrive at

$$\int_{\partial N} |\mathrm{D}u|\,\big[-\,\mathrm{Ric}(\nu,\nu) \,+\, (n-1)\big]\,\mathrm{d}\sigma \,\geq\, 0\,. \qquad (3.4.11)$$

To conclude, we observe that, on the totally geodesic boundary ∂N of our connected component, the Gauss equation reads

$$-\,\mathrm{Ric}(\nu,\nu) \;=\; \frac{\mathrm{R}^{\partial N} - \mathrm{R}}{2} \;=\; \frac{\mathrm{R}^{\partial N} \,-\, n(n-1)}{2}\,.$$

Substituting the latter identity in formula (3.4.11) we obtain the thesis. □

3.5 Local lower bound for the surface gravity

In this section we focus on the case $\Lambda > 0$, and we are going to present the complete proof of Theorem 3.2.2. As discussed in Subsection 3.2.2, the local nature of this lower bound for the *surface gravity* is at the basis of our definition of *virtual mass*, as explained in Theorem 3.2.5.

3.5.1 Some preliminary results.

As usual, we denote by N a connected component of $M \setminus \mathrm{MAX}(u)$. The next lemma shows that the set $\partial N = \partial M \cap N$ is always nonempty, and thus it is necessarily given by a disjoint union of *horizons*.

Lemma 3.5.1 (No Islands Lemma, $\Lambda > 0$)**.** *Let (M, g_0, u) be a solution to problem (3.2.1) and let N be a connected component of $M \setminus \mathrm{MAX}(u)$. Then $N \cap \partial M \neq \emptyset$.*

Proof. Let N be a connected component of $M \setminus \mathrm{MAX}(u)$ and assume by contradiction that $N \cap \partial M = \emptyset$. Since $\mathrm{MAX}(u) \cap \partial M = \emptyset$, one has that $\overline{N} \setminus N \subseteq \mathrm{MAX}(u)$, where we have denoted by $\overline{N}$ the closure of N in M. On the other hand, the scalar equation in (3.2.1) implies that $\Delta u \leq 0$ in N and thus, by the Weak Minimum Principle , one can deduce that

$$\min_{\overline{N}} u \;=\; \min_{\overline{N}\setminus N} u \;\geq\; \min_{\mathrm{MAX}(u)} u \;=\; u_{\max}\,.$$

In other words $u \equiv u_{\max}$ on N. Since N has non-empty interior, u must be constant on the whole M, by analyticity. This yields the desired contradiction. □

As an easy application of the Maximum Principle, we obtain the following gradient estimate, which is the first step in the proof of the main result.

Lemma 3.5.2. *Let (M, g_0, u) be a solution to problem (3.2.1), let N be a connected component of $M \setminus \mathrm{MAX}(u)$, and let $\partial N = \partial M \cap N$ be the non-empty and possibly disconnected boundary portion of ∂M that lies in N. If $|\mathrm{D}u| \leq u_{\max}$ on ∂N, then it holds $|\mathrm{D}u|^2 \leq u_{\max}^2 - u^2$ on the whole N.*

Proof. The thesis will essentially follow by the Maximum Principle applied to the equation (3.4.4) on the whole domain N. However, as in the proof of Theorem 3.4.3, we have to pay attention to the coefficient $1/u$, which blows up at the boundary ∂N. We also notice that in general the set $\overline{N} \cap \mathrm{MAX}(u)$ is not necessarily a regular hypersurface. Albeit this does not represent a serious issue for the applicability of the Maximum Principle, we are going to adopt the same treatment for both ∂N and $\overline{N} \cap \mathrm{MAX}(u)$, considering subdomains of the form $N_\varepsilon = N \cap \{\varepsilon \leq u \leq u_{\max} - \varepsilon\}$, for ε sufficiently small. To be more precise, we first recall from Subsection 2.1.3 that the function u is analytic and then the set of its critical values is discrete. Therefore, there exists $\eta > 0$ such that, for every $0 < \varepsilon \leq \eta$ the level sets $\{u = \varepsilon\}$ and $\{u = u_{\max} - \varepsilon\}$ are regular. Applying the Maximum Principle to equation (3.4.4), we get

$$\max_{N_\varepsilon}(|\mathrm{D}u|^2 + u^2) \;\leq\; \max_{\partial N_\varepsilon}(|\mathrm{D}u|^2 + u^2)\,.$$

On the other hand we have that $|\mathrm{D}u|^2 + u^2 = u_{\max}^2$ on $\mathrm{MAX}(u)$, and $|\mathrm{D}u|^2 + u^2 \leq u_{\max}^2$ on ∂N, by our assumption. Hence, letting $\varepsilon \to 0$ in the above inequality, we get the desired conclusion. $\square$

3.5.2 Monotonicity formula.

Let (M, g_0, u) be a solution to problem (3.2.1), and let N be a connected component of $M \setminus \mathrm{MAX}(u)$. Proceeding in analogy with [AM17b, BM16], we introduce the function $U : [0, u_{\max}) \to \mathbb{R}$ given by

$$t \longmapsto U(t) = \left(\frac{1}{u_{\max}^2 - t^2}\right)^{\frac{n}{2}} \int\limits_{\{u=t\}\cap N} |\mathrm{D}u| \, \mathrm{d}\sigma . \tag{3.5.1}$$

We remark that the function $t \mapsto U(t)$ is well defined, since the integrand is globally bounded and the level sets of u have finite hypersurface area. In fact, since u is analytic (see [Chr05, zH70]), the level sets of u have locally finite $\mathscr{H}^{n-1}$–measure by the results in [KP02]. Moreover, they are compact and thus their hypersurface area is finite. To give further insights about the definition of the function $t \mapsto U(t)$, we note that, using the explicit formulæ (2.5.2), one easily realizes that the quantities

$$M \ni x \longmapsto \frac{|\mathrm{D}u|}{\sqrt{u_{\max}^2 - u^2}}(x) \quad \text{and} \quad [0, u_{\max}) \ni t \longmapsto \int\limits_{\{u=t\}} \left(\frac{1}{u_{\max}^2 - u^2}\right)^{\frac{n-1}{2}} \mathrm{d}\sigma \tag{3.5.2}$$

are constant on the de Sitter solution. We notice that the function $t \mapsto U(t)$ can be rewritten in terms of the above quantities as

$$U(t) = \int\limits_{\{u=t\}\cap N} \left(\frac{|\mathrm{D}u|}{\sqrt{u_{\max}^2 - u^2}}\right) \left(\frac{1}{u_{\max}^2 - u^2}\right)^{\frac{n-1}{2}} \mathrm{d}\sigma ,$$

hence the function $t \mapsto U(t)$ is constant on the de Sitter solution. In the next proposition we are going to show that, for a general solution, the function U is monotonically nonincreasing, provided the surface gravity of the connected component of ∂N is bounded above by 1.

Proposition 3.5.3 (Monotonicity, case $\Lambda > 0$)**.** *Let (M, g_0, u) be a solution to problem (3.2.1), let N be a connected component of $M \setminus \mathrm{MAX}(u)$, and let $\partial N = \partial M \cap N$ be the non-empty and possibly disconnected boundary portion of ∂M that lies in N. If $|\mathrm{D}u| \leq u_{\max}$ on ∂N, then the function $U(t)$ defined in (3.5.1) is monotonically nonincreasing.*

Proof. Recalling $\Delta u = -nu$, we easily compute

$$\begin{aligned} \mathrm{div}\left[\frac{\mathrm{D}u}{(u_{\max}^2 - u^2)^{\frac{n}{2}}}\right] &= \frac{\Delta u}{(u_{\max}^2 - u^2)^{\frac{n}{2}}} + n\,u\,\frac{|\mathrm{D}u|^2}{(u_{\max}^2 - u^2)^{\frac{n}{2}+1}} \\ &= -\frac{n\,u}{(u_{\max}^2 - u^2)^{\frac{n}{2}+1}}\left(u_{\max}^2 - u^2 - |\mathrm{D}u|^2\right) \leq 0 , \end{aligned} \tag{3.5.3}$$

where the last inequality follows from Lemma 3.5.2. Integrating by parts inequality (3.5.3) in $\{t_1 \leq u \leq t_2\} \cap N$ for some $t_1 < t_2$, and applying the Divergence Theorem, we deduce

$$\int_{\{u=t_1\}\cap N} \left\langle \frac{\mathrm{D}u}{(u_{\max}^2 - u^2)^{\frac{n}{2}}} \,\middle|\, \mathrm{n} \right\rangle d\sigma + \int_{\{u=t_2\}\cap N} \left\langle \frac{\mathrm{D}u}{(u_{\max}^2 - u^2)^{\frac{n}{2}}} \,\middle|\, \mathrm{n} \right\rangle d\sigma =$$
$$- \int_{\{t_1\leq u\leq t_2\}\cap N} \frac{n\,u}{(u_{\max}^2 - u^2)^{\frac{n}{2}+1}} \left(u_{\max}^2 - u^2 - |\mathrm{D}u|^2\right) \leq 0\,, \quad (3.5.4)$$

where n is the outer g_0-unit normal to the boundary of the set $\{t_1 \leq u \leq t_2\}$. In particular, one has $\mathrm{n} = -\mathrm{D}u/|\mathrm{D}u|$ on $\{u = t_1\}$ and $\mathrm{n} = \mathrm{D}u/|\mathrm{D}u|$ on $\{u = t_2\}$, thus formula (3.5.4) rewrites as

$$\int_{\{u=t_2\}\cap N} \frac{|\mathrm{D}u|}{(u_{\max}^2 - u^2)^{\frac{n}{2}}} d\sigma \;\leq\; \int_{\{u=t_1\}\cap N} \frac{|\mathrm{D}u|}{(u_{\max}^2 - u^2)^{\frac{n}{2}}} d\sigma\,,$$

from which it follows $U(t_2) \leq U(t_1)$, as wished. □

3.5.3 Proof of Theorem 3.2.2.

In the previous subsection, we have shown the monotonicity of the function U. In order to prove Theorem 3.2.2, we also need an estimate of the behavior of $U(t)$ as t approaches $u_{\max}$. This is the content of the following proposition.

Proposition 3.5.4. *Let (M, g_0, u) be a solution to problem (3.2.1). Let N be a connected component of $M \setminus \mathrm{MAX}(u)$ and let U be the function defined as in (3.5.1). If $\mathscr{H}^{n-1}(\mathrm{MAX}(u) \cap \overline{N}) > 0$, then $\lim_{t\to u_{\max}^-} U(t) = +\infty$.*

Proof. From the Łojasiewicz inequality (see [Łoj63, Théorème 4] or [KP94]), we know that for every point $p \in \mathrm{MAX}(u)$ there exists a neighborhood $p \ni V_p \subset M$ and real numbers $c_p > 0$ and $0 < \theta_p < 1$, such that for each $x \in V_p$ it holds

$$|\mathrm{D}u|(x) \;\geq\; c_p \,[u_{\max} - u(x)]^{\theta_p}\,.$$

Up to possibly restricting the neighborhood V_p, we can suppose $u_{\max} - u < 1$ on V_p, so that for every $x \in V_p$ it holds

$$|\mathrm{D}u|(x) \;\geq\; c_p \,[u_{\max} - u(x)]\,.$$

Since $\mathrm{MAX}(u)$ is compact, it is covered by a finite number of sets $V_{p_1}, \ldots, V_{p_k}$. In particular, setting $c = \min\{c_{p_1}, \ldots, c_{p_k}\}$, the set $V = V_{p_1} \cup \cdots \cup V_{p_k}$ is a neighborhood of $\mathrm{MAX}(u)$ and the inequality

$$|\mathrm{D}u| \;\geq\; c\,(u_{\max} - u) \tag{3.5.5}$$

is fulfilled on the whole V. Now we notice that the function $U(t)$ can be rewritten as follows

$$U(t) \;=\; \left(\frac{1}{u_{\max}^2 - t^2}\right)^{\frac{n-2}{2}} \int_{\{u=t\}\cap N} \frac{|\mathrm{D}u|}{(u_{\max} - u)(u_{\max} + u)} \,d\sigma\,.$$

Thanks to the compactness of M and to the properness of u, it follows that for t sufficiently close to $u_{\max}$ we have $\{u = t\} \cap N \subset V$. For these values of t, using inequality (3.5.5) we obtain the following estimate

$$U(t) \;\geq\; \frac{c}{2u_{\max}} \left(\frac{1}{u_{\max}^2 - t^2} \right)^{\frac{n-2}{2}} \cdot \; \big|\{u = t\} \cap N\big| \,.$$

Therefore, in order to prove the thesis, it is sufficient to show that if $\mathscr{H}^{n-1}\big(\mathrm{MAX}(u) \cap \overline{N}\big) > 0$, then

$$\limsup_{t \to u_{\max}^-} \big|\{u = t\} \cap N\big| > 0 \,. \tag{3.5.6}$$

To this end, we recall that, since u is analytic and $\mathscr{H}^{n-1}\big(\mathrm{MAX}(u) \cap \overline{N}\big) > 0$, it follows from [Łoj91] (see also [KP02, Theorem 6.3.3]) that the set $\mathrm{MAX}(u) \cap \overline{N}$ contains a smooth non-empty, relatively open hypersurface Σ such that $\mathscr{H}^{n-1}\big((\mathrm{MAX}(u) \cap \overline{N}) \setminus \Sigma\big) \;=\; 0$. In particular, given a point p on Σ, we are allowed to consider an open neighbourhood Ω of p in M, where the signed distance to Σ

$$r(x) \;=\; \begin{cases} +\,d(x, \Sigma) & \text{if } x \in \Omega \cap N \,, \\ -\,d(x, \Sigma) & \text{if } x \in \Omega \setminus N \,. \end{cases}$$

is a well defined smooth function (see for instance [Foo84, KP81], where this result is discussed in full details in the Euclidean setting, however, as it is observed in [Foo84, Remarks (1) and (2)], the proofs extend with small modifications to the Riemannian setting). In order to prove (3.5.6), we are going to perform a local analysis, in a compact cylindrical neighborhood $C_\delta \subset \Omega$ of p. Let us define such a neighborhood and set up our framework:

- First consider a smooth embedding F_0 of the $(n-1)$-dimensional closed unit ball $\overline{B^{n-1}}$ into M

  $$F_0 \,:\, \overline{B^{n-1}} \;\hookrightarrow\; M \,, \qquad (\theta^1, \ldots, \theta^{n-1}) \;\mapsto\; F_0(\theta^1, \ldots, \theta^{n-1})$$

  such that $\Sigma_0 \;=\; F_0(\overline{B^{n-1}})$ is strictly contained in the interior of $\Sigma \cap \Omega$.

- Given a small enough real number $\delta > 0$, use the flow of $\mathrm{D}r$ to extend the map F_0 to the cartesian product $[-\delta, \delta] \times \overline{B^{n-1}}$, obtaining a new map

  $$F \,:\, [-\delta, \delta] \times \overline{B^{n-1}} \;\hookrightarrow\; M \,, \qquad (\rho, \theta^1, \ldots, \theta^{n-1}) \;\mapsto\; F(\rho, \theta^1, \ldots, \theta^{n-1})$$

  satisfying the initial value problem

  $$\frac{dF}{d\rho} \;=\; \mathrm{D}r \circ F \,, \qquad F(0, \cdot\,) \;=\; F_0(\,\cdot\,) \,.$$

  It is not hard to check that the relation $r\big(F(\rho, \theta^1, \ldots, \theta^{n-1})\big) \;=\; \rho$ must be satisfied, so that for every $\rho \in [-\delta, \delta]$, the image $\Sigma_\rho = F(\rho, \overline{B^{n-1}})$ belongs to the level set $\{r = \rho\}$ of the signed distance.

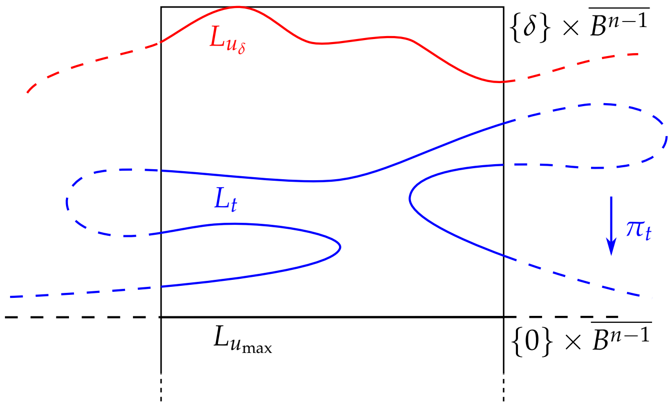


Figure 3.2: A section of the cylinder $[-\delta, \delta] \times \overline{B^{n-1}}$. The arrow shows the action of the function π_t, that sends the points of L_t to their projection on $L_{u_{\max}} = \{0\} \times \overline{B^{n-1}}$.

- Define the cylindrical neighbourhood C_δ of p simply as $F\big([-\delta, \delta] \times \overline{B^{n-1}}\big)$. By construction, the map F is a parametrisation of C_δ. Moreover, still denoting by g the metric pulled-back from M through the map F, we have that
$$g_0 = d\rho \otimes d\rho \, + \, g_{ij}(\rho, \theta^1, \ldots, \theta^{n-1}) d\theta^i \otimes d\theta^j \,,$$
where the g_{ij}'s are smooth functions of the coordinates $(\rho, \theta^1, \ldots, \theta^{n-1})$ of $[-\delta, \delta] \times \overline{B^{n-1}}$. In particular, for any fixed $0 < \varepsilon < 1$, we can suppose that, up to diminishing the value of $\delta > 0$, the following estimates hold true
$$(1 - \varepsilon)^2 \, g_{ij}(0, \theta) \ \leq \ g_{ij}(\rho, \theta) \ \leq \ (1 + \varepsilon)^2 \, g_{ij}(0, \theta) \,, \tag{3.5.7}$$
for every $\theta = (\theta^1, \ldots, \theta^{n-1}) \in \overline{B^{n-1}}$ and every $\rho \in [-\delta, \delta]$.

- Finally, let $u_\delta = \max_{\Sigma_\delta} u$. It follows from the construction that $u_\delta < u_{\max}$. For $u_\delta \leq t \leq u_{\max}$, we are going to consider the (pulled-back) level sets of u given by
$$L_t \ = \ F^{-1}\big(\{u = t\} \cap \overline{N}\big) \ \subset \ [0, \delta] \times \overline{B^{n-1}} \,,$$
together with their natural projection on $L_{u_{\max}} = \{0\} \times \overline{B^{n-1}}$. These are defined by
$$\pi_t \, : \, L_t \longrightarrow \{0\} \times \overline{B^{n-1}} \,, \qquad \pi_t \, : \, (\rho, \theta) \longmapsto (0, \theta) \,.$$
It is not hard to see that for $u_\delta \leq t \leq u_{\max}$, the projection π_t is surjective. This follows from the fact that for any given $\theta \in \overline{B^{n-1}}$ the assignment
$$[0, \delta] \ni \rho \longmapsto (u \circ F)(\rho, \theta)$$
is continuous and its range contains the closed interval $[u_\delta, u_{\max}]$.

With the notations introduced above, we claim that for every $u_\delta \leq t \leq u_{\max}$ and every connected open set $S \subset L_t$, we have
$$\mathrm{diam}_{g_0}(S) \ \geq \ (1 - \varepsilon) \ \mathrm{diam}_{g_0}(\pi_t(S)) \,. \tag{3.5.8}$$

Since we have already shown that π_t is surjective, it follows from the very definition of the Hausdorff measure that the claim implies the inequality

$$\mathscr{H}^{n-1}(L_t) \,\geq\, (1-\varepsilon)^{n-1}\, \mathscr{H}^{n-1}(L_{u_{\max}})\,,$$

which is clearly equivalent to (3.5.6). To prove (3.5.8), let us fix $t \in [u_\delta, u_{\max}]$ and consider a $\mathscr{C}^1$ curve

$$\gamma\,:\, I \longrightarrow L_t\,, \qquad s \longmapsto \gamma(s) \,=\, \big(\rho(s), \theta(s)\big)\,,$$

where $I \subset \mathbb{R}$ is an interval. We want to show that the lenght of γ is controlled from below by the lenght of its projection $\pi_t \circ \gamma$, which is the curve on $L_{u_{\max}}$ defined by $(\pi_t \circ \gamma)(s) = (0, \theta(s))$ for every $s \in I$. Recalling the expression of g_0 with respect to the coordinates (ρ, θ) and the estimate (3.5.7), we compute

$$\begin{aligned}
\left|\frac{d\gamma}{ds}\right|_g^2(s) \,&=\, \left|\frac{d\rho}{ds}\right|_g^2(s) \,+\, g_{ij}\big(\rho(s), \theta(s)\big)\,\frac{d\theta^i}{ds}(s)\,\frac{d\theta^j}{ds}(s)\\
&\geq\, (1-\varepsilon)^2\, g_{ij}\big(0, \theta(s)\big)\,\frac{d\theta^i}{ds}(s)\,\frac{d\theta^j}{ds}(s)\\
&=\, (1-\varepsilon)^2 \left|\frac{d(\pi_t \circ \gamma)}{ds}\right|_g^2(s)\,.
\end{aligned}$$

In particular, the same inequality holds between the lenghts of γ and its projection $\pi_t \circ \gamma$. Claim (3.5.8) follows. □

Combining Propositions 3.5.3 and 3.5.4 we easily obtain Theorem 3.2.2, that we restate here – in an alternative form – for the ease of reference.

Theorem 3.5.5. *Let (M, g_0, u) be a solution to problem (3.2.1), let N be a connected component of $M \setminus \mathrm{MAX}(u)$, and let $\partial N = \partial M \cap N$ be the boundary portion of ∂M that lies in N. Suppose that*

$$\frac{|\mathrm{D}u|}{u_{\max}} \,\leq\, 1 \qquad \textit{on} \quad \partial N\,.$$

Then the triple (M, g_0, u) is isometric to the de Sitter solution (2.5.2). In particular ∂M and $M \setminus \mathrm{MAX}(u)$ are both connected.

Proof. Let us consider the function $t \mapsto U(t)$ defined in (3.5.1). Thanks to the assumption $|\mathrm{D}u| \leq u_{\max}$ on ∂N, we have that Proposition 3.5.3 is in force, and thus $t \mapsto U(t)$ is monotonically nonincreasing. In particular, we get

$$\lim_{t \to u_{\max}^-} U(t) \,\leq\, U(0) \,=\, \int\limits_{\partial N} |\mathrm{D}u|\, \mathrm{d}\sigma \,\leq u_{\max}\, |\partial N| \,<\, \infty\,.$$

In light of Proposition 3.5.4, this fact tells us that $\mathscr{H}^{n-1}(\mathrm{MAX}(u) \cap \overline{N}) = 0$. This means that $\mathrm{MAX}(u) \cap \overline{N}$ cannot disconnect the domain N from the rest of the manifold M. In other words, N is the only connected component of $M \setminus \mathrm{MAX}(u)$. In particular particular, $\partial M \cap N = \partial M$ and Theorem 3.2.1 applies, giving the thesis. □

3.6 A characterization of the Anti de Sitter solution

In this section we focus on the case $\Lambda < 0$ and, proceeding in analogy with Section 3.5, we prove Theorem 3.3.2.

3.6.1 Some preliminary results.

Here we prove the analogues of Lemmata 3.5.1 and 3.5.2. For a connected component N of $M \setminus \mathrm{MIN}(u)$, we will denote by $\overline{N}$ the closure of N in M. Notice that $\overline{N}$ is a manifold with boundary $\partial\overline{N} = \mathrm{MIN}(u) \cap \overline{N}$. Since $\mathrm{MIN}(u)$ might be singular, the boundary $\partial\overline{N}$ is not necessarily smooth in general. Another important feature of $\overline{N}$ is that it must be noncompact, as we are going to show in the following lemma.

Lemma 3.6.1 (No Islands Lemma, $\Lambda < 0$)**.** *Let (M, g_0, u) be a solution to problem (3.3.1) and let N be a connected component of $M \setminus \mathrm{MIN}(u)$. Then $\overline{N}$ has at least one end.*

Proof. Let N be a connected component of $M \setminus \mathrm{MIN}(u)$ and assume by contradiction that $\overline{N}$ has no ends. In particular, $\overline{N}$ is compact, and since also $\mathrm{MIN}(u)$ is compact, one has that $\overline{N} \setminus N \subseteq \mathrm{MIN}(u)$. On the other hand, from (3.3.1) we have $\Delta u \geq 0$ in N, hence, by the Weak Maximum Principle, one obtains

$$\max_{\overline{N}} u \;=\; \max_{\overline{N}\setminus N} u \;\leq\; \max_{\mathrm{MIN}(u)} u \;=\; u_{\min}\,.$$

This implies that $u \equiv u_{\min}$ on N. Since N has non-empty interior, u must be constant on the whole M, by analyticity. This yields the desired contradiction. □

A similar application of the Maximum Principle leads to the following result, which is the analogue of Lemma 3.5.2 in the case $\Lambda < 0$.

Lemma 3.6.2. *Let (M, g_0, u) be a solution to problem (3.3.1), and let N be a connected component of $M \setminus \mathrm{MIN}(u)$. If*

$$\liminf_{x \in N,\, x \to \infty} \left(u^2 - u_{\min}^2 - |\mathrm{D}u|^2\right) \geq 0\,,$$

then it holds $|\mathrm{D}u|^2 \leq u^2 - u_{\min}^2$ on the whole N.

Proof. We recall from Subsection 2.1.3 that the function u is analytic and its critical level sets are discrete. It follows that there exists $\eta > 0$ such that the level sets $\{u = u_{\min} + \varepsilon\}$ and $\{u = 1/\varepsilon\}$ are regular for any $0 < \varepsilon \leq \eta$. For any $0 < \varepsilon \leq \eta$, let $N_\varepsilon = N \cap \{u_{\min} + \varepsilon \leq u \leq 1/\varepsilon\}$. We have $|\mathrm{D}u|^2 - u^2 = -u_{\min}^2$ on $\mathrm{MIN}(u)$ and from the hypothesis

$$\limsup_{x \in N, x \to \infty} \left(|\mathrm{D}u|^2 - u^2 + u_{\min}^2\right) \leq 0\,.$$

In particular, for any $\delta > 0$, there exists $\varepsilon > 0$ small enough so that $|\mathrm{D}u|^2 - u^2 + u_{\min}^2 \leq \delta$ on $\{u = 1/\varepsilon\}$. In fact, if this were not the case, it would exist a sequence $\{\varepsilon_i\}_{i \in \mathbb{N}}$ of positive real numbers converging to zero such that for every

$i \in \mathbb{N}$ there exists $p_i \in \{u = 1/\varepsilon_i\}$ with $(|\mathrm{D}u|^2 - u^2 + u_{\min}^2)(p_i) > \delta$, and the superior limit of this sequence would be greater than δ, in contradiction with the hypothesis. We have thus proved that

$$\lim_{\varepsilon \to 0^+} \max_{\partial N_\varepsilon}(|\mathrm{D}u|^2 - u^2 + u_{\min}^2) \leq 0\,. \tag{3.6.1}$$

On the other hand, we can apply the Maximum Principle to (3.4.5) inside N_ε for an arbitrarily small $\varepsilon > 0$, and using (3.6.1) we find

$$\max_N(|\mathrm{D}u|^2 - u^2) = \lim_{\varepsilon \to 0^+} \max_{N_\varepsilon}(|\mathrm{D}u|^2 - u^2) = \lim_{\varepsilon \to 0^+} \max_{\partial N_\varepsilon}(|\mathrm{D}u|^2 - u^2) \leq -u_{\min}^2\,.$$

The thesis follows. □

3.6.2 Proof of Theorem 3.3.2.

The strategy of the proof of Theorem 3.3.2 is completely analogue to the one employed in Section 3.5 for the proof of Theorem 3.2.2. For this reason, we will avoid to give some details, that can be easily recovered by the interested reader. First of all, we introduce the function $U : (u_{\min}, +\infty) \to \mathbb{R}$ defined as

$$t \longmapsto U(t) = \left(\frac{1}{t^2 - u_{\min}^2}\right)^{\frac{n}{2}} \int_{\{u=t\}\cap N} |\mathrm{D}u| \, \mathrm{d}\sigma. \tag{3.6.2}$$

Reasoning as in Subsection 3.5.3, one sees that the function U is well defined and constant on the Anti de Sitter solution. Furthermore, now we prove that U is always nondecreasing in t.

Proposition 3.6.3 (Monotonicity, case $\Lambda < 0$)**.** *Let (M, g_0, u) be a solution to problem (3.3.1). Let N be a connected component of $M \setminus \mathrm{MIN}(u)$ and let U be the function defined as in (3.6.2). If*

$$\liminf_{x \in N,\, x \to \infty} \left(u^2 - u_{\min}^2 - |\mathrm{D}u|^2\right) \geq 0\,,$$

then the function U is monotonically nondecreasing.

Proof. Recalling $\Delta u = nu$, we easily compute

$$\begin{aligned}\mathrm{div}\left[\frac{\mathrm{D}u}{(u^2 - u_{\min}^2)^{\frac{n}{2}}}\right] &= \frac{\Delta u}{(u^2 - u_{\min}^2)^{\frac{n}{2}}} - n\,u\,\frac{|\mathrm{D}u|^2}{(u^2 - u_{\min}^2)^{\frac{n}{2}+1}} \\ &= \frac{n\,u}{(u^2 - u_{\min}^2)^{\frac{n}{2}+1}}\left(u^2 - u_{\min}^2 - |\mathrm{D}u|^2\right) \geq 0\,,\end{aligned} \tag{3.6.3}$$

where the last inequality follows from Lemma 3.6.2. Integrating by parts inequality (3.6.3) in $\{t_1 \leq u \leq t_2\} \cap N$ for some $t_1 < t_2$, and applying the Divergence Theorem, we deduce

$$\begin{aligned}\int_{\{u=t_1\}\cap N} \left\langle \frac{\mathrm{D}u}{(u^2 - u_{\min}^2)^{\frac{n}{2}}} \,\middle|\, \mathrm{n} \right\rangle \mathrm{d}\sigma + \int_{\{u=t_2\}\cap N} \left\langle \frac{\mathrm{D}u}{(u^2 - u_{\min}^2)^{\frac{n}{2}}} \,\middle|\, \mathrm{n} \right\rangle \mathrm{d}\sigma = \\ \int_{\{t_1 \leq u \leq t_2\}\cap N} \frac{n\,u}{(u^2 - u_{\min}^2)^{\frac{n}{2}+1}}\left(u^2 - u_{\min}^2 - |\mathrm{D}u|^2\right) \geq 0\,,\end{aligned} \tag{3.6.4}$$

where n is the outer g_0-unit normal to the set $\{t_1 \le u \le t_2\}$. In particular, one has $\mathrm{n} = -\mathrm{D}u/|\mathrm{D}u|$ on $\{u = t_1\}$ and $\mathrm{n} = \mathrm{D}u/|\mathrm{D}u|$ on $\{u = t_2\}$. Therefore, formula (3.6.4) rewrites as

$$\int\limits_{\{u=t_2\}\cap N} \frac{|\mathrm{D}u|}{(u^2 - u_{\min}^2)^{\frac{n}{2}}}\, \mathrm{d}\sigma \;\ge\; \int\limits_{\{u=t_1\}\cap N} \frac{|\mathrm{D}u|}{(u^2 - u_{\min}^2)^{\frac{n}{2}}}\, \mathrm{d}\sigma\,,$$

which implies $U(t_2) \ge U(t_1)$, as wished. $\square$

Combining Theorem 3.3.2 with some approximations near the extremal points of the static potential u, we are able to characterize the set $\mathrm{MIN}(u)$ and to estimate the behavior of the $U(t)$'s as t approaches $u_{\min}$.

Proposition 3.6.4. *Let (M, g_0, u) be a solution to problem (3.3.1). Let N be a connected component of $M \setminus \mathrm{MIN}(u)$ and let U be the function defined by (3.6.2). If $\mathscr{H}^{n-1}(\mathrm{MIN}(u) \cap \overline{N}) > 0$, then $\lim_{t \to u_{\min}^+} U(t) = +\infty$.*

Proof. The proof is completely analogue to the proof of Propopsition 3.5.4. From the Łojasiewicz inequality one deduces that there is a neighborhood V of $\mathrm{MIN}(u)$ such that the inequality

$$|\mathrm{D}u| \;\ge\; c\,(u - u_{\min}) \tag{3.6.5}$$

holds on the whole V. The second step is to rewrite $U(t)$ as

$$U(t) \;=\; \left(\frac{1}{t^2 - u_{\min}^2}\right)^{\frac{n-2}{2}} \int\limits_{\{u=t\}\cap N} \left(\frac{|\mathrm{D}u|}{(u - u_{\min})(u + u_{\min})}\right) \mathrm{d}\sigma\,.$$

Thanks to the compactness of M and to the properness of u, for t sufficiently close to $u_{\min}$ we have $\{u = t\} \cap N \subset V$. For these values of t, using inequality (3.6.5), we have the following estimate

$$U(t) \;\ge\; \frac{c}{t + u_{\min}} \left(\frac{1}{t^2 - u_{\min}^2}\right)^{\frac{n-2}{2}} \cdot \big|\{u = t\} \cap N\big|\,.$$

Proceeding exactly an in the proof of Proposition 3.5.4, one can show that

$$\lim_{t \to u_{\min}^+} \big|\{u = t\} \cap N\big| > 0\,, \tag{3.6.6}$$

and this concludes the proof. $\square$

We are now in a position to prove Theorem 3.3.2, that we restate here, in an alternative – but equivalent – form, for reference.

Theorem 3.6.5. *Let (M, g_0, u) be a solution to problem (3.3.1), and let N be a connected component of $M \setminus \mathrm{MIN}(u)$. Suppose that*

$$\liminf_{x \in N,\, x \to \infty} \left(u^2 - u_{\min}^2 - |\mathrm{D}u|^2\right)(x) \;\ge\; 0\,.$$

Then the triple (M, g_0, u) is isometric to the Anti de Sitter solution (2.4.3). In particular, $M \setminus \mathrm{MIN}(u)$ is connected and M has a unique end.

Proof. On N, consider the function U defined as in (3.6.2) and fix $t_0 \in (u_{\min}, \infty)$. From Proposition 3.6.3 we know that U is nondecreasing, hence we have

$$\lim_{t \to u_{\min}^+} U(t) \leq U(t_0) = \left(\frac{1}{t_0^2 - u_{\min}^2}\right)^{\frac{n}{2}} \int\limits_{\{u=t_0\}\cap N} |\mathrm{D}u| \, \mathrm{d}\sigma < +\infty ,$$

where in the latter inequality we have used the fact that $|\mathrm{D}u|$ is a continuous function and $\{u = t_0\}$ is compact (because u is proper and $u \to +\infty$ at the infinity of N). Therefore, Proposition 3.6.4 tells us that $\mathscr{H}^{n-1}\big(\mathrm{MIN}(u) \cap \overline{N}\big) = 0$. This means that $\mathrm{MIN}(u) \cap \overline{N}$ cannot disconnect the manifold M, which in turn proves that $M \setminus \mathrm{MIN}(u)$ is connected. Therefore, we can apply Theorem 3.3.1 to deduce the thesis. □

4 Features of the virtual mass

In this chapter we focus on the case of a positive cosmological constant, continuing the study started in Chapter 3 about the notion of *virtual mass* of a static metric. In particular, we will prove that the virtual mass satisfies a Riemannian Penrose–like inequality and allows to prove a Black Hole Uniqueness Theorem. The results in this chapter are based on [BM17b]. As usual, we stress that in the whole chapter it is always tacitly assumed that the dimension of our manifolds is $n \geq 3$.

4.1 Introduction and statement of the main results

4.1.1 Setting of the problem.

As anticipated, in this chapter we are interested in the study of static spacetimes with positive cosmological constant, or equivalently, of solutions to problem (3.2.1). In order to simplify the exposition of some of the results, it is convenient to suppose that the manifold M is orientable. This of course is not restrictive. In fact, if the manifold is not orientable, we can consider its orientable double covering DM, and the results found on DM can be transfered to the original manifold by means of the projection. Notice that an orientation of M induces an orientation on the horizons, which are the connected components of ∂M. Therefore, in particular, if M is orientable so are the horizons. For the reader convenience, let us rewrite here problem (3.2.1), while explicitating the orientability assumption

$$\begin{cases} u\,\mathrm{Ric} = \mathrm{D}^2 u + n\, u\, g_0, & \text{in } M \\ \Delta u = -n\, u, & \text{in } M \\ u > 0, & \text{in } M \setminus \partial M \\ u = 0, & \text{on } \partial M \end{cases} \qquad \text{with } M \text{ compact orientable}\,. \tag{4.1.1}$$

We recall from Definition 3.2.3 that an horizon $S \in \pi_0(\partial M)$ is said to be of *cosmological*, *black hole* or *cylindrical* type depending of whether its surface gravity $\kappa(S)$ is less than, greater than or equal to $\sqrt{n}$, respectively. Analogously, we also recall that a region $N \subset M \setminus \mathrm{MAX}(u)$ is said to be *outer* if all its horizons are cosmological, *inner* if it contains at least one horizon of black hole type and *cylindrical* if it has at least one horizon of cylindrical type and it has no horizon of black hole type. As we will see, the analysis of these three cases will be slightly different, although in the end we will obtain similar results, that are stated in the next subsections.

4.1.2 Area bounds.

An important feature of Theorem 3.2.5, that we have proved in the previous chapter, is that it gives a complete characterisation of the zero mass solutions. Another very interesting and nowadays classical characterisation of the de Sitter solution is given by the Boucher-Gibbons-Horowitz area bound discussed in Theorem 2.5.1. Having at hand Theorem 2.5.1 and Theorem 3.2.5, it is natural to ask if in the case where the virtual mass is strictly positive and the boundary of M is allowed to have several connected components, it is possible to provide a refined version of both statements, whose rigidity case characterises now the Schwarzschild–de Sitter solutions described in (2.5.3) instead of the de Sitter solution. In accomplishing this program, we are inspired by the well known relation between the Positive Mass Theorem and the Riemannian Penrose Inequality as they are stated in the classical setting, where M^3 is an asymptotically flat Riemannian manifold with nonnegative scalar curvature. We have already discussed these result in Subsection 1.2.1. To introduce our first main result, we recall the three-dimensional version of the Riemannian Penrose Inequality and we observe that, using the definition of the Schwarzschild radius given below formula (2.3.3), it can be rephrased as follows

$$|\partial M| \,\leq\, 16\pi m^2 \,=\, 4\pi(2m)^2 \,=\, 4\pi r_0^2(m)\,, \tag{4.1.2}$$

where $m = m_{ADM}(M^3, g)$. Having these considerations in mind, we can now state one of the main results of the present chapter.

Theorem 4.1.1 (Refined Area Bounds). *Let (M^3, g_0, u) be a 3-dimensional solution to problem (4.1.1), and let N be a connected component of $M^3 \setminus \mathrm{MAX}(u)$ with connected smooth compact boundary ∂N. We then let $m \in (0, 1/(3\sqrt{3})]$ be the virtual mass of N, namely*

$$m \,=\, \mu(N, g_0, u)\,.$$

Then, ∂N is diffeomorphic to the sphere $\mathbb{S}^2$. Moreover, the following inequalities hold:

(i) Cosmological Area Bound. If N is an outer region, then

$$|\partial N| \,\leq\, 4\pi r_+^2(m)\,, \tag{4.1.3}$$

and the equality is fulfilled if and only if the triple (M^3, g_0, u) is isometric to the Schwarzschild–de Sitter solution (2.5.3) with mass m.

(ii) *Riemannian Penrose Inequality. If N is an inner region, then*

$$|\partial N| \leq 4\pi r_-^2(m), \tag{4.1.4}$$

and the equality is fulfilled if and only if the triple (M^3, g_0, u) is isometric to the Schwarzschild–de Sitter solution (2.5.3) with mass m.

(iii) *Cylindrical Area Bound. If N is a cylindrical region, then*

$$|\partial N| \leq \frac{4\pi}{3}, \tag{4.1.5}$$

and the equality is fulfilled if and only if the triple (M^3, g_0, u) is covered by the Nariai solution (2.5.5).

Remark 4.1.2. *Concerning the rigidity statement in point (iii) of the above theorem, we observe that there is only one orientable triple which is not isometric to the 3-dimensional Nariai solution but that is covered by it, which is the quotient of the Nariai triple by the involution*

$$\iota : [0, \pi] \times \mathbb{S}^2 \to [0, \pi] \times \mathbb{S}^2, \qquad \iota(t, x) = (\pi - t, -x),$$

where we have denoted by $-x$ the antipodal point of x on $\mathbb{S}^2$. The existence of this solution was pointed out in [Amb15, Section 7].

About the previous statement some comments are in order. First, the fact that ∂N is necessarily diffeomorphic to a sphere is not a new result. In fact, a stronger result is already known from [Amb15, Theorem B], see Theorem 2.5.5, where it is shown that every connected component of the boundary of a static solution to problem (4.1.1) is diffeomorphic to a sphere. Our approach allows to prove the same topological result, but only in the case where the horizons of (M^3, g_0, u) are somehow separated from each other by the locus $\mathrm{MAX}(u)$. Concerning the area bounds, we observe that, conceptually speaking, the inequality (4.1.3) should be compared with the Boucher-Gibbons-Horowitz Area Bound (2.5.8), since it involves the cosmological horizons of the solution, whereas, the inequality (4.1.4) should be compared with (4.1.2) since it is a statement about horizons of black hole type.

An analogous result holds in higher dimension, giving the natural analog of the inequality

$$|\partial M| \leq \int_{\partial M} \frac{\mathrm{R}^{\partial M}}{(n-1)(n-2)} \, d\sigma, \tag{4.1.6}$$

which has been obtained by Chruŝciel in [Chr, Section 6] in the case of connected boundary, extending the Boucher-Gibbons-Horowitz method to every dimension $n \geq 3$. Of course, in the above inequality $\mathrm{R}^{\partial M}$ stands for the scalar curvature of the boundary. Moreover, the equality is fulfilled if and only if (M, g_0, u) coincides with the de Sitter solution.

Theorem 4.1.3. *Let (M, g_0, u) be a solution to problem (4.1.1) of dimension $n \geq 3$, and let N be a connected component of $M \setminus \mathrm{MAX}(u)$ with connected smooth compact boundary ∂N. We then let $m \in (0, m_{\max}]$ be the virtual mass of N, namely*

$$m = \mu(N, g_0, u).$$

Then, the following inequalities hold:

(i) If N is an outer region, then

$$|\partial N| \,\leq\, \left(\int_{\partial N} \frac{\mathrm{R}^{\partial N}}{(n-1)(n-2)}\,\mathrm{d}\sigma\right) r_+^2(m)\,, \tag{4.1.7}$$

and the equality is fulfilled if and only if the triple (M, g_0, u) is isometric to the Schwarzschild–de Sitter solution (2.5.3) with mass m.

(ii) If N is an inner region, then

$$|\partial N| \,\leq\, \left(\int_{\partial N} \frac{\mathrm{R}^{\partial N}}{(n-1)(n-2)}\,\mathrm{d}\sigma\right) r_-^2(m)\,, \tag{4.1.8}$$

and the equality is fulfilled if and only if the triple (M, g_0, u) is isometric to the Schwarzschild–de Sitter solution (2.5.3) with mass m.

(iii) If N is a cylindrical region, then

$$|\partial N| \,\leq\, \int_{\partial N} \frac{\mathrm{R}^{\partial N}}{n(n-1)}\,\mathrm{d}\sigma\,, \tag{4.1.9}$$

and the equality is fulfilled if and only if the triple (M, g_0, u) is covered by the Nariai solution (2.5.5).

The proof of the above statement will be given in Section 4.4, whereas it is clear that Theorem 4.1.1 follows directly from Theorem 4.1.3, applying the Gauss-Bonnet formula. In turns, Theorem 4.1.3 will be deduced by some more general statements (see Corollaries 4.4.2, 4.4.7 and 4.6.8) which correspond to some balancing formulas, in the case where the boundary of N is allowed to have several connected components. To illustrate this fact, we focus on an outer region $N \subset M \setminus \mathrm{MAX}(u)$ and we present a more general version of formulæ (4.1.3) and (4.1.7). If we let $m = \mu(N, g_0, u)$ be the virtual mass of N and we let $A : \partial N \to (0, 1]$ be the locally constant function defined for every $x \in \partial N$ by

$$A(x) \,=\, \frac{|\mathrm{D}u|}{\max_{\partial N} |\mathrm{D}u|}(x)\,, \tag{4.1.10}$$

then it can be proven (see Corollary 4.4.2) that

$$\begin{aligned}\int_{\partial N} A^3\,\mathrm{d}\sigma \,\leq\, \left(\int_{\partial N} A\,\frac{\mathrm{R}^{\partial N}}{(n-1)(n-2)}\,\mathrm{d}\sigma\right) r_+^2(m) \,-\\ -\,\frac{n(n-4)}{(n-1)(n-2)}\left(\int_{\partial N} A\,\left(1-A^2\right)\,\mathrm{d}\sigma\right) r_+^2(m)\,.\end{aligned}$$

It is clear that, when ∂N is connected, then $A \equiv 1$ so that the above inequality reduces to (4.1.7). A second observation is that it holds $A \leq 1$ by definition, hence for $n \geq 4$ the second summand in the right hand side of the above formula is always nonpositive.

A completely analogous computation can be made for a cylindrical region N. In this case, from Corollary 4.6.8 we deduce

$$\int_{\partial N} A^3 \,\mathrm{d}\sigma \leq \left(\int_{\partial N} A \,\frac{\mathrm{R}^{\partial N}}{n(n-1)}\,\mathrm{d}\sigma\right) - \left(\frac{n-4}{n-1}\right)\left(\int_{\partial N} A\,\left(1-A^2\right)\,\mathrm{d}\sigma\right).$$

Again, when ∂N is connected we have $A \equiv 1$ and we reobtain (4.1.9). Moreover, for $n \geq 4$, the second summand in the right hand side is always nonpositive. It follows that, in both formulæ, for $n \geq 4$ the second summand can be neglected and we obtain the following statement.

Theorem 4.1.4. *Let (M, g_0, u) be a solution to problem (4.1.1), and let N be a connected component of $M \setminus \mathrm{MAX}(u)$ with virtual mass $m = \mu(N, g_0, u)$. Let also A be the function defined by (4.1.10), and suppose that the dimension of M is $n \geq 4$. Then the following statements are satisfied*

(i) If N is an outer region, then it holds

$$\int_{\partial N} A^3 \,\mathrm{d}\sigma \leq \left(\int_{\partial N} A \,\frac{\mathrm{R}^{\partial N}}{(n-1)(n-2)}\,\mathrm{d}\sigma\right) r_+^2(m)\,. \tag{4.1.11}$$

Moreover, if the equality holds, then the solution (M, g_0, u) is isometric to the Schwarzschild–de Sitter solution (2.5.3) with mass m.

(ii) If N is a cylindrical region, then it holds

$$\int_{\partial N} A^3 \,\mathrm{d}\sigma \leq \int_{\partial N} A \,\frac{\mathrm{R}^{\partial N}}{n(n-1)}\,\mathrm{d}\sigma\,. \tag{4.1.12}$$

Moreover, if the equality holds, then (M, g_0, u) is covered by the Nariai triple (2.5.5).

In the above theorem, we have excluded the inner regions. In fact, the analogous integral inequality for inner regions is not as clean as in the outer and cylindrical case, since it depends on a parameter α that cannot be computed explicitly. For this reason, we have decided not to include the inner case of Theorem 4.1.4 in the statement, the interested reader can find it in Corollary 4.4.7. We also notice that the Hölder Inequality can be used to estimate the term in brackets in (4.1.11) as

$$\int_{\partial N} A \,\frac{\mathrm{R}^{\partial N}}{(n-1)(n-2)}\,\mathrm{d}\sigma \leq \left(\int_{\partial N} A^3 \mathrm{d}\sigma\right)^{\frac{1}{3}} \left(\int_{\partial N} \left|\frac{\mathrm{R}^{\partial N}}{(n-1)(n-2)}\right|^{\frac{3}{2}} \mathrm{d}\sigma\right)^{\frac{2}{3}}.$$

A similar computation can be made in the cylindrical case, and as a consequence we obtain the following more geometric version of Theorem 4.1.4.

Corollary 4.1.5. *Let (M, g_0, u) be a solution to problem (4.1.1), and let N be a connected component of $M \setminus \mathrm{MAX}(u)$ with virtual mass $m = \mu(N, g_0, u)$. Let also A be the function defined by (4.1.10), and suppose that the dimension of M is $n \geq 4$. Then the following statements are satisfied* *(i) If N is an outer region, then it holds*

$$\int_{\partial N} A^3 \, d\sigma \leq \left(\int_{\partial N} \left| \frac{\mathrm{R}^{\partial N}}{(n-1)(n-2)} \right|^{\frac{3}{2}} d\sigma \right) r_+^3(m) \, .$$

Moreover, if the equality holds, then the solution (M, g_0, u) is isometric to the Schwarzschild–de Sitter solution (2.5.3) *with mass m.*

(ii) If N is a cylindrical region, then it holds

$$\int_{\partial N} A^3 \, d\sigma \leq \int_{\partial N} \left| \frac{\mathrm{R}^{\partial N}}{n(n-1)} \right|^{\frac{3}{2}} d\sigma \, .$$

Moreover, if the equality holds, then (M, g_0, u) is covered by the Nariai triple (2.5.5)*.*

In dimensions $n = 3$ we are not able to provide such nice inequalities. The interested reader can find the results for $n = 3$ and ∂N disconnected in Theorems 4.4.3, 4.4.8 and 4.6.9.

We conclude this subsection with a comparison of our Theorem 4.1.1 with Theorem 2.5.4, recently proved by Ambrozio. Of course however we emphasize that Ambrozio's result and ours are slightly different in nature, as Theorem 2.5.4 does not require any assumption of $\mathrm{MAX}(u)$ but has a more global nature with respect to Theorem 4.1.1. Let us compare the two statements in a couple of special cases. First of all, if our solution (M, g_0, u) is 3-dimensional and has a single horizon, then Theorem 2.5.4 gives

$$|\partial M| \leq \frac{4\pi}{3} \, ,$$

thus greatly improving the classical Boucher-Gibbons-Horowitz inequality (2.5.8). Our Theorem 4.1.1 gives an even stronger inequality when the (unique) horizon of M is of black hole type, whereas if the horizon is outer it gives a worse result. If the horizon is of cylindrical type, it gives the same inequality.

Let us now pass to compare the two statements in the case upon which our result is modelled, that is, suppose that our solution (M, g_0, u) is 3-dimensional and that

$$M \setminus \mathrm{MAX}(u) = M_+ \sqcup M_- \, ,$$

where M_+ is an outer region with connected boundary ∂M_+ and M_- is an inner region with connected boundary ∂M_-. Let us denote by

$$m_+ = \mu(M_+, g_0, u) \, , \quad m_- = \mu(M_-, g_0, u) \, ,$$

the virtual masses of M_+, M_-. In this case, Ambrozio's Theorem 2.5.4 gives

$$k_+(m_+) \, |\partial M_+| + k_-(m_-) \, |\partial M_-| \leq \frac{4\pi}{3} \left[k_+(m_+) + k_-(m_-) \right] \, , \tag{4.1.13}$$

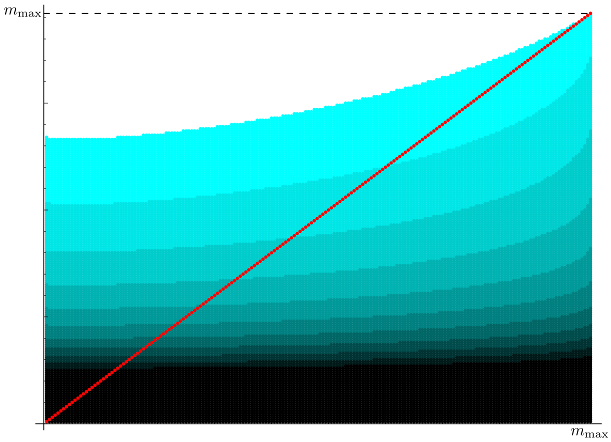


Figure 4.1: In this plot we have numerically analyzed the relation between formulæ (4.1.13) and (4.1.14), in function of the values of m_+ (on the x-axis) and of m_- (on the y-axis). The red line represents the points with $m_+ = m_-$ (which is the case of the Schwarzschild–de Sitter solution). The blue region is the one where (4.1.14) is stronger than (4.1.13). The darker the blue, the better our formula is. To give also a quantitative idea, the black part on the bottom is where the right side of (4.1.13) minus the right side of (4.1.14) is bigger than 3.

whereas from inequalities (4.1.3) and (4.1.4) in Theorem 4.1.1 we get

$$k_+(m_+)\,|\partial M_+| + k_-(m_-)\,|\partial M_-| \leq$$
$$\leq 4\pi\left[k_+(m_+)r_+^2(m_+) + k_-(m_-)r_-^2(m_-)\right]. \quad (4.1.14)$$

The two inequalities (4.1.13), (4.1.14) are compared in Figure 4.1, where we have highlighted the values of m_+, m_- for which formula (4.1.14) improves (4.1.13). This comparison gives us a further proof that our result is particularly strong when our manifold is separated in two regions, one outer and one inner, by the set $\mathrm{MAX}(u)$. It is then not surprising that this is exactly the case in which we are able to make use of our area bounds in order to prove an uniqueness theorem for the Schwarzschild–de Sitter solution. This is discussed in the following subsection.

4.1.3 Uniqueness results.

In this subsection, we discuss a characterization of both the Schwarzschild–de Sitter and the Nariai solution, which is in some ways reminiscent of the well known Black Hole Uniqueness Theorem 2.3.3. We recall that this classical result states that when the cosmological constant is zero, the only asymptotically flat static solutions with nonempty boundary are the Schwarzschild triples described in (2.3.3). In order to clarify what should be expected to hold in the case of positive cosmological constant, let us briefly comment the *asymptotic flatness* hypothesis. We recall (see Definition 2.3.2) that this assumption amounts to both a topological and a geometric requirement. More precisely, each end of the manifold is *a priori* forced to be diffeomorphic to $[0,+\infty) \times \mathbb{S}^{n-1}$ and the metric has to converge to the flat one at a suitable rate, so that, up to a convenient rescaling, the boundary at infinity of the end is isometric to a round sphere. Another important feature of the *asymptotic flatness* assumption is that the static potential approaches its maximum value at infinity.

From this last property, it seems natural to guess that the boundary at infinity of an asymptotically flat static solution with $\Lambda = 0$ should correspond in our framework to the set $\mathrm{MAX}(u)$. The same analogy is also proposed in [BH96, Appendix], where it is used to justify the physical meaning of the normalization (2.1.9) for the surface gravity. We will make use of this correspondence in the discussion that follows Theorem 4.1.8 below, in order to shed some lights on the plausibility of our assumptions. Before presenting the precise statement of this uniqueness result, it is important to underline another feature of the set $\mathrm{MAX}(u)$, that is peculiar of our setting. In fact, in sharp contrast with the $\Lambda = 0$ case, we observe that $\mathrm{MAX}(u)$ may in principle disconnect our manifold. On the other hand, this situation is not only possible but even natural, since it is realized in the model examples given by the Schwarzschild–de Sitter solutions (2.5.3) and the Nariai solutions (2.5.5). Here, the set $\mathrm{MAX}(u)$ separates the manifold into two regions, one of which is outer or cylindrical, while the other is inner or cylindrical. Having this in mind, it is natural to introduce the notion of a 2-sided solution to problem (4.1.1).

Definition 4.1.6 (2-Sided Solution)**.** *A triple (M, g_0, u) is said to be a* 2-sided solution *to problem* (4.1.1) *if*

$$M \setminus \mathrm{MAX}(u) \;=\; M_+ \sqcup M_- \,,$$

where M_+ is either an outer or a cylindrical region, that is

$$\max_{S \in \pi_0(\partial M_+)} \kappa(S) \;=\; \max_{\partial M_+} \frac{|\mathrm{D}u|}{u_{\max}} \;\leq\; \sqrt{n}\,,$$

and M_- is either an inner or a cylindrical region, that is

$$\max_{S \in \pi_0(\partial M_-)} \kappa(S) \;=\; \max_{\partial M_-} \frac{|\mathrm{D}u|}{u_{\max}} \;\geq\; \sqrt{n}\,.$$

Remark 4.1.7. *The definition above is useful for the sake of a clean exposition, as it allows to state our uniqueness result in its clearest form, which is Theorem 4.1.8 below. However, we remark that some of the assumptions made inside Definition 4.1.6 can actually be relaxed. In Subsection 4.5.2 we will discuss this point in further details, showing a couple of generalizations of Theorem 4.1.8 which do not require the full power of Definition 4.1.6. These generalizations come at the cost of a less clean statement.*

The generic shape of a 2-sided solution is shown in Figure 4.2. We recall that, by a classical theorem of Łojasiewicz [Łoj91] (see Theorem 1.1.4), the set $\mathrm{MAX}(u)$ is given *a priori* by a possibly disconnected stratified analytic subvariety of dimensions ranging from 0 to $(n-1)$. In particular, it follows that a 2-sided solution contains a stratified (possibly disconnected) hypersurface $\Sigma \subseteq \mathrm{MAX}(u)$ which separates M_+ and M_-, that is, $\overline{M}_+ \cap \overline{M}_- = \Sigma$.

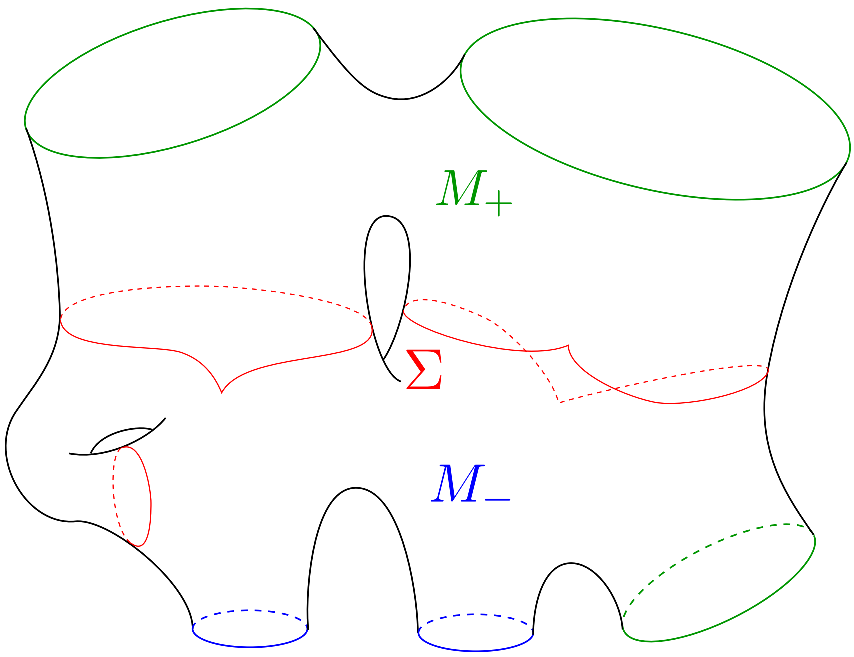


Figure 4.2: The drawing represents the possible structure of a generic 2-sided solution to problem (4.1.1). The red line represents the set $\mathrm{MAX}(u)$, with the separating stratified hypersurface Σ put in evidence. The blue colour of a boundary component indicates a black hole horizon, whereas the green colour indicates a cosmological horizon. Cylindrical horizons are not considered in this figure since they are non generic.

This hypersurface will play an important role in our analysis, as it represents the junction between the regions M_+ and M_-, and for this reason some of the assumptions in the following theorem are about its geometric features. It is also convenient to introduce the continuous function $F : M \to \mathbb{R}$ defined by

$$F(x) \,=\, \begin{cases} \sqrt{\max\limits_M(u) - u(x)} & \text{in } \overline{M}_+\,, \\ -\sqrt{\max\limits_M(u) - u(x)} & \text{in } \overline{M}_-\,. \end{cases} \tag{4.1.15}$$

We are now in the position to state the main result of this subsection.

Theorem 4.1.8 (Black Hole Uniqueness Theorem)**.** *Let (M, g_0, u) be a 2-sided 3-dimensional solution to problem (4.1.1), and let $\Sigma \subseteq \mathrm{MAX}(u)$ be the stratified hypersurface separating M_+ and M_-. Let also*

$$m_+ \,=\, \mu(M_+, g_0, u)\,, \qquad \textit{and} \qquad m_- \,=\, \mu(M_-, g_0, u)$$

be the virtual masses of M_+ and M_-, respectively. Suppose that the following conditions hold

mass compatibility	$m_+ \leq m_-$,
regularity assumption	*F is $\mathscr{C}^2$ in a neighborhood of Σ,*
connected cosmological horizon	*∂M_+ is connected,*

<u>*pinching assumption*</u> $\quad \fint_\Sigma |\mathring{\mathrm{h}}|^2 d\sigma < \mathrm{R}$,

where F is the function introduced in (4.1.15), R is the constant scalar curvature of g_0 and $\mathring{\mathrm{h}}$ is the traceless part of the second fundamental form of Σ. Then, there exists a real number $0 < m \leq m_{\max}$ such that $m_+ = m = m_-$ and the triple (M, g_0, u) is isometric to either the Schwarzschild–de Sitter solution (2.5.3) with mass $0 < m < m_{\max}$ or to the Nariai solution (2.5.5) with mass $m = m_{\max}$.

It is useful to discuss in more details the meaning and the plausibility of the assumptions made in the theorem above.

- Let us start by discussing the *regularity assumption*. Its technical purpose is to ensure that the *global pseudo-radial function*, which will be constructed in Subsection 4.2.1, is also $\mathscr{C}^2$ in a neighborhood of Σ. The pseudo-radial function will then be used to define a *cylindrical ansatz* in the spirit of [AM15, AM16, AM17b], that is, a conformal change of the metric g_0 whose aim is to cylindrify the Riemannian manifold (M, g_0). In the end, the regularity assumption grants us that the new conformal metric is $\mathscr{C}^2$ in a neighborhood of Σ, and this is what we will actually use in the analysis. About the plausibility of this assumption, we point out that the function F is always Lipschitz continuous on the whole manifold M, as it will be shown in Proposition 4.2.8 (see Remark 4.2.9). It may be interesting to see if it is possible to improve our analysis, with the final aim of proving that on a general 2-sided solution to (4.1.1) the map F is always $\mathscr{C}^2$ in a neighborhood of Σ. In fact, something of this sort happens in the case of zero cosmological constant, where it is known from [Bei80] that, for asymptotically flat manifolds, if the static potential is bounded then it satisfies a nice expansion at infinity.

- The assumption of *mass compatibility* will also be of great importance in our argument. In fact, it will allow us to prove a uniform upper bound on the gradient of the static potential on the whole boundary ∂M. In turn, this will be crucial in the proof of Proposition 4.5.1, which will provide us with a uniform upper bound for the gradient of the pseudo-affine function φ (see (4.2.18) for its definition) on the whole manifold. This hypothesis and the regularity assumption can be interpreted as the ones providing a connection between the analysis on the outer region M_+ and the one on the inner region M_-. In particular, these two hypotheses give us some useful information on the separating hypersurface Σ. For instance, in any dimension $n \geq 3$, they allow us to prove (see Propostions 4.5.2 and 4.6.14) that Σ is actually a $\mathscr{C}^2$ hypersurface and that the following formulæ hold

$$\begin{aligned} \mathrm{H} &= (n-1)\sqrt{\frac{n}{n-2}\left[\left(\frac{m_{\max}}{m_+}\right)^{2/n} - 1\right]}\,, \\ [(n-2)m_+]^{2/n}\left(\mathrm{R}^\Sigma + |\mathring{\mathrm{h}}|^2\right) &= (n-1)(n-2)\,. \end{aligned} \tag{4.1.16}$$

Here we have denoted by R^Σ the scalar curvature of the metric induced by g_0 on Σ and by $\mathrm{H}, \mathring{\mathrm{h}}$ the mean curvature and traceless second fundamental form of Σ with respect to g_0 and the unit normal pointing towards the interior of M_+. Another important consequence of the mass compatibility (more precisely a consequence of the global gradient bound for the pseudo-affine function φ) is that it implies the existence of a monotonicity formula, which can be used in combination with (4.1.16) in order to deduce a lower bound for $|\partial M_+|$ (see Propositions 4.5.3 and 4.6.15). More concretely, in every dimension $n \geq 3$, the mass compatibility and the regularity assumption will lead us to the following inequalities

- If $0 < m_+ < m_{\max}$ we have

$$\left[\frac{r_+^n(m_+)}{(n-2)m_+}\right]^{\frac{n-1}{n}} [(n-2)m_+]^{\frac{2}{n}} \int_\Sigma \frac{\mathrm{R}^\Sigma + |\mathring{\mathrm{h}}|^2}{(n-1)(n-2)}\, \mathrm{d}\sigma \;=\; \left[\frac{r_+^n(m_+)}{(n-2)m_+}\right]^{\frac{n-1}{n}} |\Sigma| \;\leq\; |\partial M_+|\,,$$

  and the equality holds if and only if (M, g_0, u) is isometric to a generalized Schwarzschild–de Sitter solution (2.5.3) with mass $m_+ = m_-$.

- If $m_+ = m_{\max}$ we have

$$\int_\Sigma \frac{\mathrm{R}^\Sigma + |\mathring{\mathrm{h}}|^2}{n(n-1)}\, \mathrm{d}\sigma \;=\; |\Sigma| \;\leq\; |\partial M_+|\,,$$

  and the equality holds if and only if (M, g_0, u) is covered by the Nariai solution (2.5.5).

- The *connectedness of* ∂M_+ allows us to apply Theorem 4.1.3, providing a useful upper bound for the quantity $|\partial M_+|$. Combining this with the lower bound discussed above, we obtain the following inequalities in any dimension $n \geq 3$.

  - If $0 < m_+ < m_{\max}$, then

$$\left[\frac{r_+^n(m_+)}{(n-2)m_+}\right]^{\frac{n-3}{n}} \int_\Sigma \left(\mathrm{R}^\Sigma + |\mathring{\mathrm{h}}|^2\right) \mathrm{d}\sigma \;\leq\; \int_{\partial M_+} \mathrm{R}^{\partial M_+}\, \mathrm{d}\sigma\,. \tag{4.1.17}$$

  - If $m_+ = m_{\max}$, then

$$\int_\Sigma \left(\mathrm{R}^\Sigma + |\mathring{\mathrm{h}}|^2\right) \mathrm{d}\sigma \;\leq\; \int_{\partial M_+} \mathrm{R}^{\partial M_+}\, \mathrm{d}\sigma\,. \tag{4.1.18}$$

It is important to point out that, since both the upper and lower bounds on $|\partial M_+|$ come with a rigidity statement, the equality in formula (4.1.17) (respectively (4.1.18)) holds if and only if the solution is isometric to the Schwarzschild–de Sitter solution (respectively if the solution is covered by the Nariai solution).

Another important remark is motivated by the beautiful result in [Amb15, Theorem B], where it is proven that any static solution (M, g_0, u) admits at most one unstable horizon. From a physical perspective, one may expect that the unstable horizons should be the ones of cosmological type, whereas the horizons of black hole type should be stable. This is what happens for the model solutions, as one can easily check. This observation leads us to formulate the following conjecture, which, if proven to be true, would allow to remove the assumption of connected cosmological horizon from Theorem 4.1.8.

Conjecture. *An horizon of cosmological type is necessarily unstable. In particular, every static solution to problem* (4.1.1) *has at most one horizon of cosmological type.*

- As already discussed, the mass compatibility, the regularity assumption and the hypothesis of connected cosmological horizon allow us to prove upper and lower bounds for $|\partial M_+|$ in any dimension $n \geq 3$. In a general dimension, this is the best result that we are able to obtain, whereas for $n = 3$ one can go further by means of the Gauss-Bonnet formula. In particular, it is possible to show that if the *pinching assumption* is satisfied, then the equality is achieved in either (4.1.17) (if $m_+ < m_{\max}$) or (4.1.18) (if $m_+ = m_{\max}$). As we have already pointed out, this implies the desired rigidity and thus concludes the argument.

  It is important to observe that the pinching assumption is written down in its more general and scaling invariant form. Of course, under the chosen normalization for the metric g_0, we know that R is just equal to 6. More than that, a perusal of our proof shows that the pinching assumption could be replaced by the inequality

$$\fint_{\Sigma} |\mathring{\mathrm{h}}|^2 \mathrm{d}\sigma \, < \, \frac{2}{m_+^{2/3}}\,, \tag{4.1.19}$$

  see Theorems 4.5.4 and 4.6.16. Since $m_+^{2/3} \leq m_{\max}^{2/3} = 1/3$, the bound (4.1.19) is more loose, and thus slighly better than the one in Theorem 4.1.8. Another way to replace the pinching assumption is to make the following topological requirement

$$\sum_{i=1}^{k} \chi(\Sigma_i) \, \geq \, 2\,,$$

  where $\Sigma_1, \ldots, \Sigma_k$ are the connected components of Σ, see once again Theorems 4.5.4 and 4.6.16. In particular, it would be enough to assume that Σ is (connected and) diffeomorphic to a sphere, in order to recover the uniqueness result.

  To put the pinching assumption as well as all of its possible variants in perspective, it is sufficient to recall the mentioned analogy between the set $\mathrm{MAX}(u)$ in our setting and the boundary at infinity of an asymptotically flat manifold in the case where $\Lambda = 0$. In fact, we have already noticed that in the second situation the boundary at infinity is forced to have spherical topology. On the other hand, the asymptotic flatness implies that the quantity $\fint_{S_r} |\mathring{\mathrm{h}}|^2 \mathrm{d}\sigma$ goes to zero as the radius r of the coordinate sphere S_r goes to infinity. Having these consideration in mind, our pinching assumption seems to be perfectly justified.

4.1.4 Summary.

In the remainder of the paper we will prove the results stated in this introduction. We will first focus on outer and inner regions, since the analysis of these two cases is similar. Our study is based on the so called *cylindrical ansatz*, introduced in [AM16, AM17b], which consists is finding an appropriate conformal change of the original metric g_0 in terms of the static potential u.

In Section 4.2 we will describe this method, we will set up the formalism and we will provide some preliminary lemmata and computations that will be used throughout the paper.

Building on this, we will prove in Section 4.3 a couple of integral identities in the conformal setting, and in Section 4.4 we will translate these results in terms of the original metric g_0. As a consequence, we will prove the results stated in Subsection 4.1.2, for both the cases of outer and inner regions.

In Section 4.5 we will show that our analysis can be improved under the assumption that the solution is 2-sided, and this will lead us to the proof of Theorem 4.1.8 stated in Subsection 4.1.3, in the case where $m_+ < m_{\max}$.

Finally, in Section 4.6 we will focus on the cylindrical regions. The analysis of the cylindrical case is slightly different, as our model solution will be the Nariai triple instead of the Schwarzschild–de Sitter triple, however the ideas behind our analysis are completely analogous. In this section we will establish the results stated in Subsection 4.1.2 for cylindrical regions and we will complete the proof of Theorem 4.1.8 by studying the case $m_+ = m_{\max}$.

4.2 The cylindrical ansatz

This section is devoted to the setup of the *cylindrical ansatz*, which will be the starting point of the proofs of our main results. We will work on a single region N of our manifold M, and we will always suppose that N is not cylindrical, that is

$$\max_{S\in\pi_0(\partial N)} \kappa(S) \neq \sqrt{n}\,,$$

where $\kappa(S)$ is the surface gravity of S, defined as in (2.1.9). The case of equality requires a different analysis, and will be studied separately in Section 4.6.

The cylindrical ansatz is inspired by the analogue technique used in [AM16, AM17b], and consists in a cylindrification of our triple by means of a conformal change. The idea comes from the observation that the Schwarzschild–de Sitter metric can be cylindrified via a division by $|x|^2$. In fact, the metric

$$\frac{1}{|x|^2}\left(\frac{d|x|\otimes d|x|}{1-|x|^2-2m|x|^{2-n}} + |x|^2 g_{\mathbb{S}^{n-1}}\right) = \frac{d|x|\otimes d|x|}{|x|^2(1-|x|^2-2m|x|^{2-n})} + g_{\mathbb{S}^{n-1}}\,,$$

after a suitable rescaling of the coordinate $|x|$, is just the standard metric of the cylinder $\mathbb{R} \times \mathbb{S}^{n-1}$. We would like to perform a similar change of coordinates on a general solution (M, g_0, u).

To this end, in Subsection 4.2.1 we are going to define on a region N of a general triple (M, g_0, u) a *pseudo-radial function* $\Psi : N \to \mathbb{R}$. The function Ψ will be constructed starting from the static potential u, and in the case where u is as in the Schwarzschild–de Sitter solution (2.5.3), it will simply coincide with $|x|$.

Subsection 4.2.2 is devoted to the proof of the relevant properties of the pseudo-radial function. Most of the results in this subsection are quite technical, and the reader is advised to simply ignore this part of the work and to come back only when needed. However, there is one result that deserves to be mentioned. In Lemma 4.2.3 we will observe that static potentials satisfy the reverse Łojasiewicz inequality (see Theorem 1.1.6), and this will be used to find an estimate on the gradient of the pseudo-radial function near $\mathrm{MAX}(u)$. This estimate will be crucial in the Minimum Principle argument that leads to Proposition 4.2.8. It is interesting to notice that Proposition 4.2.8, in turn, will allow us to improve the reverse Łojasiewicz inequality, as explained in Remark 4.2.9. However, since the proof of Proposition 4.2.8 exploits the equations in (4.1.1), we do not know if the improved Łojasiewicz inequality still holds outside the realm of static potentials.

In Subsection 4.2.3 we will finally use the pseudo-radial function Ψ to set up our cylindrical ansatz. More precisely, on a region N of our initial manifold, we will consider the new metric

$$g = \frac{g_0}{\Psi^2},$$

and we will also define a *pseudo-affine function* φ. The definitions are chosen in such a way that, if (M, g_0, u) is isometric to the Schwarzschild–de Sitter solution, then the metric g is just the standard cylindrical metric and φ is an affine function, that is, the norm of $\nabla\varphi$ with respect to the metric g is constant on M. Conversely, the general idea of the future proofs will be to find opportune conditions that force φ to be affine and g to be cylindrical, thus proving the isometry with the Schwarzschild–de Sitter solution. The highlight of this subsection is Proposition 4.2.6, where we will translate the equations in problem (4.1.1) in terms of g and φ.

In Subsection 4.2.4 we will analyze the level sets of φ, and in particular we will write down the relations between the mean curvature and second fundamental form of the level sets with respect to g_0 and g. Finally, in Subsection 4.2.5 we will apply the Bochner formula and the equations of the conformal reformulation of problem (4.1.1) written down in Proposition 4.2.6, in order to deduce an elliptic inequality for the quantity

$$w = \beta \left(1 - |\nabla\varphi|_g^2\right),$$

where β is a suitably chosen positive function. A Minimum Principle argument, together with an estimate on the behavior of $|\nabla\varphi|_g$ near $\mathrm{MAX}(u)$ (which is provided by the reverse Łojasiewicz inequality proved in Subsection 4.2.2) will allow us to prove that w is positive on our region N. This will give us an important bound from above on the gradient of φ, which will be of great importance in the next sections.

4.2.1 The pseudo-radial function.

Let (M, g_0, u) be a solution to problem (4.1.1), and let N be a connected component of $M \setminus \mathrm{MAX}(u)$. As already discussed above, in this subsection we focus on inner and outer regions. In other words, the quantity

$$\max_{S \in \pi_0(\partial N)} \kappa(S) = \max_{\partial N} \frac{|\mathrm{D}u|}{u_{\max}}$$

will always be supposed to be different from $\sqrt{n}$. In particular, the virtual mass

$$m = \mu(N, g_0, u),$$

is strictly less than $m_{\max}$. The special case $m = m_{\max}$ will be discussed later, in Section 4.6.

The aim of this subsection is that of defining a pseudo-radial function, that is, a function that mimic the behavior of the radial coordinate $|x|$ in the Schwarzschild–de Sitter solution. First of all, we recall that our problem is invariant under a normalization of u, hence we first rescale u in such a way that its maximum is the same as the maximum of the Schwarzschild–de Sitter solution with mass m.

Notation 1. *We will make use of the notations $m_{\max}$ and $u_{\max}$ introduced in* (2.4.2) *and* (2.5.4). *We recall their definitions here*

$$m_{\max} = \sqrt{\frac{(n-2)^{n-2}}{n^n}}, \qquad u_{\max}(m) = \sqrt{1 - \left(\frac{m}{m_{\max}}\right)^{2/n}}.$$

We emphasize that $u_{\max} = u_{\max}(m)$ is a function of the virtual mass m of N. We will explicitate that dependence only when it will be significative.

Normalization 1. *We normalize u in such a way that its maximum is $u_{\max}(m)$, where m is the virtual mass of N and $u_{\max}(m)$ is defined as in Notation 1.*

As usual, we let $r_+(m) > r_-(m) \geq 0$ be the two positive roots of the polynomial $P_m(x) = x^{n-2} - x^n - 2m$, and we define the function

$$\begin{aligned} F_m : [0, u_{\max}(m)] \times [r_-(m), r_+(m)] &\longrightarrow \mathbb{R} \\ (u, \psi) &\longmapsto F_m(u, \psi) = u^2 - 1 + \psi^2 + 2m\psi^{2-n}. \end{aligned}$$

It is a simple computation to show that $\partial F_m / \partial \psi = 0$ if and only if $\psi = 0$ or $\psi = [(n-2)m]^{1/n}$. Therefore, as a consequence of the Implicit Function Theorem we have the following.

Proposition 4.2.1. *Let u be a positive function and let $u_{\max}$ be its maximum value. Then there exist functions*

$$\psi_- : [0, u_{\max}] \longrightarrow \left[r_-(m), [(n-2)m]^{1/n}\right],$$
$$\psi_+ : [0, u_{\max}] \longrightarrow \left[[(n-2)m]^{1/n}, r_+(m)\right],$$

such that $F_m(u, \psi_-(u)) = F_m(u, \psi_+(u)) = 0$ for all $u \in [0, u_{\max}(m)]$.

Let us make a list of the main properties of ψ_+ and ψ_-, that can be derived easily from their definition.

- First of all, we can compute ψ_+, ψ_- and their derivatives using the following formulæ

$$u^2 = 1 - \psi_\pm^2 - 2m\psi_\pm^{2-n}. \tag{4.2.1}$$

$$\dot{\psi}_\pm = -\frac{u}{\psi_\pm\left[1-(n-2)m\psi_\pm^{-n}\right]}, \qquad \ddot{\psi}_\pm = n\frac{\dot{\psi}_\pm^3}{u} + (n-1)\frac{\dot{\psi}_\pm^2}{\psi_\pm} + \frac{\dot{\psi}_\pm}{u}. \tag{4.2.2}$$

- The function ψ_- takes values in $[r_-(m), [(n-2)m]^{1/n}]$, hence $\psi_-^n \leq (n-2)m$ and from the first formula in (4.2.2) we deduce

$$\dot{\psi}_- \geq 0, \qquad \ddot{\psi}_- \geq 0, \qquad \lim_{u\to u_{\max}^-} \dot{\psi}_- = +\infty.$$

- The function ψ_+ takes values in $[[(n-2)m]^{1/n}, r_+(m)]$, hence $\psi_+^n \geq (n-2)m$ and from the first formula in (4.2.2) we deduce that $\dot{\psi}_+$ is nonpositive and diverges as u approaches $u_{\max}$. Moreover, the second formula in (4.2.2) can be rewritten as

$$\ddot{\psi}_+ = \frac{\dot{\psi}_+}{u}\left\{1 + \left[1+(n-1)(n-2)m\psi_+^{2-n}\right]\dot{\psi}_+^2\right\},$$

from which it follows $\ddot{\psi}_+ \leq 0$. Summing up, we have

$$\dot{\psi}_+ \leq 0, \qquad \ddot{\psi}_+ \leq 0, \qquad \lim_{u\to u_{\max}^-} \dot{\psi}_+ = -\infty.$$

Let us now come back to our case of interest, that is, let us consider a region $N \subset M \setminus \mathrm{MAX}(u)$. We want to use the functions $\psi_\pm$ in order to define a *pseudo-radial function* on N. To this end, we distinguish between the case where N is an outer or an inner region, according to Definition 3.2.3.

- If N is an outer region, then our reference model will be the outer region of the Schwarzschild–de Sitter solution (2.5.3). Accordingly, we define the pseudo-radial function Ψ_+ as

$$\begin{aligned} \Psi_+ : N &\longrightarrow \left[[(n-2)m]^{1/n}, r_+(m)\right] \\ p &\longmapsto \Psi_+(p) := \psi_+(u(p)). \end{aligned} \tag{4.2.3}$$

  Notice that, if N is the outer region of the Schwarzschild–de Sitter solution (2.5.3) with mass m, for every $p \in N$ the value of $\Psi_+(p)$ is equal to the value of the radial coordinate $|x|$ at p.

- If N is an inner region, then our reference model will be the inner region of the Schwarzschild–de Sitter solution (2.5.3). Accordingly, we define the pseudo-radial function Ψ_- as

$$\begin{aligned} \Psi_- : N &\longrightarrow \left[r_-(m), [(n-2)m]^{1/n}\right] \\ p &\longmapsto \Psi_-(p) := \psi_-(u(p)). \end{aligned} \tag{4.2.4}$$

Notice that, if N is the inner region of the Schwarzschild–de Sitter solution (2.5.3) with mass m, for every $p \in N$ the value of $\Psi_-(p)$ is equal to the value of the radial coordinate $|x|$ at p.

In the case of 2-sided solutions we will need a global version of the definition above.

- If (M, g_0, u) is a 2-sided solution in the sense of Definition 4.1.6, then we define the *global pseudo-radial function* as

$$\begin{aligned} \Psi : M &\longrightarrow [r_-(m), r_+(m)] \\ p &\longmapsto \Psi(p) := \begin{cases} \psi_+(u(p)) & \text{if } p \in M_+ , \\ \psi_-(u(p)) & \text{if } p \in M_- , \\ [(n-2)m]^{1/n} & \text{if } p \in \mathrm{MAX}(u) . \end{cases} \end{aligned} \tag{4.2.5}$$

If (M, g_0, u) is isometric to the Schwarzschild–de Sitter solution (2.5.3) with mass m, then Ψ coincides with the radial coordinate $|x|$. The function Ψ is continuous by construction, but *a priori* we have no more informations on its regularity near the set $\mathrm{MAX}(u)$. However, in Subsection 4.2.2 we will prove that Ψ is always Lipschitz. Moreover, under suitable hypotheses, we will show that Ψ is also $\mathscr{C}^2$ along the hypersurface $\Sigma \subset \mathrm{MAX}(u)$ that separates M_+ and M_-.

By definition, we have the following relation between the derivatives of the pseudo-radial function Ψ and the potential u.

$$\mathrm{D}\Psi_\pm \,=\, (\dot\psi_\pm \circ u)\, \mathrm{D}u \,, \qquad \mathrm{D}^2\Psi_\pm \,=\, (\dot\psi_\pm \circ u)\, \mathrm{D}^2 u \,+\, (\ddot\psi_\pm \circ u)\, du \otimes du \,. \tag{4.2.6}$$

Notation 2. *In the following sections, we will perform several formal computations. In order to simplify the notations, we will avoid to indicate the subscript $\pm$, and we will simply denote by $\Psi = \psi \circ u$ the pseudo-radial function on a region N of $M \setminus \mathrm{MAX}(u)$, where we understand that Ψ is defined by* (4.2.3) *if we are in an outer region and by* (4.2.4) *if we are in an inner region. When there is no risks of confusion, we will also avoid to explicitate the composition with u, namely, we will write ψ instead of $\psi \circ u$. For instance, the formulæ in* (4.2.6) *will be simply written as*

$$\mathrm{D}\Psi \,=\, \dot\psi\, \mathrm{D}u \,, \qquad \mathrm{D}^2\Psi \,=\, \dot\psi\, \mathrm{D}^2 u \,+\, \ddot\psi\, du \otimes du \,,$$

4.2.2 Preparatory estimates.

Here we collect some lemmata that will be useful in the following. The first one shows an important connection between the value of the pseudo-radial function at the boundary and the surface gravity.

Lemma 4.2.2. *Let (M, g_0, u) be a solution to problem* (4.1.1)*, let N be a connected component of $M \setminus \mathrm{MAX}(u)$ with virtual mass $m = \mu(N, go, u)$, and let $\Psi = \psi \circ u$ be the pseudo-radial function, defined by* (4.2.3) *or* (4.2.4) *depending on whether N is an outer or inner region, respectively. Then it holds*

$$\max_{\partial N} \left| \frac{\mathrm{D}u}{\psi\left[1 - (n-2)m\psi^{-n}\right]} \right| \,=\, 1 \,.$$

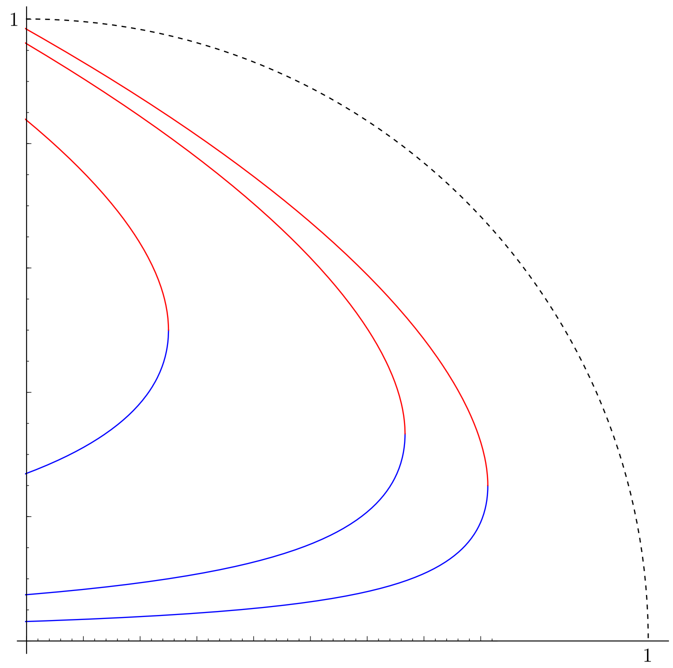


Figure 4.3: Relation between u^2 (on the y-axis) and the pseudo-radial functions (on the x-axis). The blue line represents the relation with Ψ_- whereas the red line represents the relation with Ψ_+.

Proof. The proof is an easy computation. We recall from the definition of the virtual mass m of N, that $\max_{\partial N} |\mathrm{D}u|/u_{\max} = k_\pm(m)$, where $k_\pm$ are the surface gravity functions defined by (2.5.6) and (2.5.7), and the sign $\pm$ depends on whether $\max_{\partial N} |\mathrm{D}u|/u_{\max}$ is less or greater than $\sqrt{n}$.

From the definition of ψ, we have that $\psi(x) = r_\pm(m)$ for all $x \in \partial N$, where $r_+(m) > r_-(m)$ are the two positive solutions of $1 - x^2 - 2mx^{-n} = 0$ (again, the sign $\pm$ depends on the value of $\max_{\partial N} |\mathrm{D}u|/u_{\max}$). Therefore, we have

$$
\begin{aligned}
\max_{\partial N} \left| \frac{\mathrm{D}u}{\psi\left[1-(n-2)m\psi^{-n}\right]} \right| &= \frac{u_{\max}}{r_\pm(m)\left|1-(n-2)mr_\pm(m)^{-n}\right|} \max_{\partial N} \frac{|\mathrm{D}u|}{u_{\max}} \\
&= \frac{u_{\max}}{r_\pm(m)\left|1-(n-2)mr_\pm(m)^{-n}\right|} k_\pm(m) \\
&= 1\,,
\end{aligned}
$$

where the last equality follows from the definition of k_+ and k_-. $\square$

We now pass to discuss an estimate for the gradient of the potential u near the maximum points. To this end, we will make use of the reverse Łojasiewicz inequality proved in Theorem 1.1.7, and more precisely the improved version given in Corollary 1.1.8.

Lemma 4.2.3. *Let (M, g_0, u) be a solution to problem (4.1.1) and let ψ be defined by (4.2.1) with respect to a parameter $m \in (0, m_{\max})$. Then, for every $p \in \mathrm{MAX}(u)$,* *it holds*

$$\lim_{x\to p} \dot\psi^{2\alpha}|\mathrm{D}u|^2(x) \,=\, 0\,,$$

for all $0<\alpha<1$.

Proof. First, we compute

$$\begin{aligned}\frac{u_{\max}-u}{[1-(n-2)m\psi^{-n}]^2} &= \frac{1}{u_{\max}+u}\,\frac{u_{\max}^2-u^2}{[1-(n-2)m\psi^{-n}]^2}\\ &= \frac{1}{u_{\max}+u}\,\frac{1-(m/m_{\max})^{2/n}-1+\psi^2+2m\psi^{2-n}}{[1-(n-2)m\psi^{-n}]^2}\\ &= \frac{1}{(u_{\max}+u)[1-(n-2)m\psi^{-n}]}\left[\psi^2-\frac{n\,m^{\frac{2}{n}}}{(n-2)^{\frac{n-2}{n}}}\,\frac{1-(n-2)^{\frac{n-2}{n}}m^{\frac{n-2}{n}}\psi^{2-n}}{1-(n-2)m\psi^{-n}}\right].\end{aligned}$$

We want to show that the quantity above has a finite nonzero limit as we approach $\mathrm{MAX}(u)$. If we denote $z=[(n-2)m\psi^{-n}]^{1/n}$, the equation above can be rewritten as

$$\begin{aligned}\frac{u_{\max}-u}{[1-(n-2)m\psi^{-n}]^2} &= \frac{u_{\max}-u}{(1-z^n)^2}\\ &= \frac{(n-2)^{\frac{2}{n}}m^{\frac{2}{n}}}{(u_{\max}+u)(1-z^n)}\left[z^{-2}-\frac{n}{n-2}\,\frac{1-z^{n-2}}{1-z^n}\right]\\ &= \frac{(n-2)^{\frac{2}{n}}m^{\frac{2}{n}}}{(u_{\max}+u)(1-z^n)}\left[z^{-2}-\frac{n}{n-2}\,\frac{1+z+\cdots+z^{n-3}}{1+z^{+}\cdots+z^{n-1}}\right]\\ &= \frac{(n-2)^{\frac{2}{n}}m^{\frac{2}{n}}z^{-2}}{(u_{\max}+u)(1+z+\cdots+z^{n-1})^2}\cdot\frac{1+z-\frac{2}{n-2}z^2\left(1+z+\cdots+z^{n-3}\right)}{1-z}.\end{aligned}$$

It is clear that the first factor above has a finite nonzero limit as we approach $\mathrm{MAX}(u)$, that is, when z goes to 1. Concerning the second factor, one easily computes

$$\begin{aligned}&\frac{1+z-\frac{2}{n-2}z^2\left(1+z+\cdots+z^{n-3}\right)}{1-z} =\\ &\qquad\qquad = n-\frac{2}{n-2}(1-z)\sum_{k=1}^{n-2}(n-k-1)(1+z+\cdots+z^{k-1})\,,\end{aligned}$$

and with some easy estimates, when $z\to 1$ we obtain

$$\begin{aligned}&\frac{u_{\max}-u}{[1-(n-2)m\psi^{-n}]^2} =\\ &\qquad\qquad = \frac{[(n-2)m]^{2/n}}{2nu_{\max}}\left[1-\frac{1}{3}n(n-1)(1-z)+\mathcal{O}\big((1-z)^2\big)\right].\end{aligned}\tag{4.2.7}$$

In particular, recalling formula (4.2.2), we have proved that $(u_{\max}-u)\dot\psi^2$ has a finite limit as we approach the set $\mathrm{MAX}(u)$. Therefore, for any $p\in\mathrm{MAX}(u)$ and $0<\alpha<1$, we compute

$$\lim_{x\to p}\dot\psi^{2\alpha}|\mathrm{D}u|^2(x) \,=\, \lim_{x\to p}\left[(u_{\max}-u)\dot\psi^2\right]^{\alpha}\frac{|\mathrm{D}u|^2}{(u_{\max}-u)^{\alpha}}(x)\,,$$

and, since $(u_{\max} - u)\dot{\psi}^2$ has a finite limit on $\mathrm{MAX}(u)$, from Corollary 1.1.8 we conclude. □

Lemma 4.2.3 will be crucial later in the proof of Proposition 4.2.8, where a Minimum Principle argument will be used to prove a stronger result, namely, that the quantity $|\mathrm{D}u|/\dot{\psi}$ is bounded near $\mathrm{MAX}(u)$, see Remark 4.2.9. In particular, since $(u_{\max} - u)/\dot{\psi}^2$ is also bounded near $\mathrm{MAX}(u)$, as shown in the proof of Lemma 4.2.3, it follows that the quantity

$$4\left|\mathrm{D}\big(\sqrt{u_{\max} - u}\big)\right|^2 = \frac{|\mathrm{D}u|^2}{u_{\max} - u}.$$

is bounded near $\mathrm{MAX}(u)$. In other words, the function $\sqrt{u_{\max} - u}$ is always Lipschitz continuous on M.

However, in order to prove the Black Hole Uniqess result for 2-sided solutions stated in Subsection 4.1.3, this regularity is not sufficient and we will need to assume more, namely that the function

$$F(x) = \begin{cases} \sqrt{\max\limits_M(u) - u(x)} & \text{in } \overline{M}_+, \\ -\sqrt{\max\limits_M(u) - u(x)} & \text{in } \overline{M}_-. \end{cases}$$

introduced in (4.1.15) is $\mathscr{C}^2$ in a neighborhood of our separating hypersurface Σ. More precisely, what we will really need is the regularity of Ψ, and the following lemma shows that the regularity of F and the regularity of Ψ are related. Before stating it, it is convenient to introduce the following terminology.

Definition 4.2.4. *Let (M, g_0, u) be a 2-sided solution and let $\Sigma \subseteq \mathrm{MAX}(u)$ be the hypersurface separating M_+ and M_-. We will say that* a function f is $\mathscr{C}^k$ along Σ *if $f \in \mathscr{C}^k(U)$, where $U \supset \Sigma$ is some open neighborhood of Σ in M.*

This definition will be helpful, as we will repeatedly use it in the next lemma, whose proof is based on an analysis of the regularity of F, Ψ and their derivatives near Σ.

Lemma 4.2.5. *Let (M, g_0, u) be a 2-sided solution to problem (4.1.1), let $\Psi = \psi \circ u$ be the global pseudo-radial function defined by (4.2.5) with respect to a parameter $m \in (0, m_{\max})$ and let F be defined as in (4.1.15). Then the function $\sqrt{u_{\max} - u}$ is $\mathscr{C}^2$ along Σ if and only if Ψ is $\mathscr{C}^2$ along Σ.*

Proof. It is clear that Ψ and F are always $\mathscr{C}^0$ along Σ. Using formulæ (4.2.2), we also find that Ψ is $\mathscr{C}^1$ along Σ if and only if

$$\mathrm{D}\Psi = \dot{\psi}\,\mathrm{D}u = -\frac{u}{\psi}\,\frac{\mathrm{D}u}{1-(n-2)m\psi^{-n}} = \frac{1}{2}\,\frac{u}{\psi}\,\frac{\sqrt{u_{\max} - u}}{1-(n-2)m\psi^{-n}}\,\mathrm{D}\big(\sqrt{u_{\max} - u}\big).$$

is $\mathscr{C}^0$ along Σ. On the other hand, since the quantity

$$\frac{\sqrt{u_{\max} - u}}{[1-(n-2)m\psi^{-n}]}$$

changes sign as we pass through Σ and we know from formula (4.2.7) that its absolute value is continuous along Σ, we deduce that Ψ is $\mathscr{C}^1$ along Σ if and only if F is $\mathscr{C}^1$ along Σ. To pass to the $\mathscr{C}^2$ regularity, we first compute

$$\mathrm{D}^2 F \;=\; \begin{cases} -\dfrac{1}{2\sqrt{u_{\max}-u}}\left[\mathrm{D}^2 u + \dfrac{1}{2}\,\dfrac{du\otimes du}{u_{\max}-u}\right] & \text{in } \overline{M}_+\,, \\ \dfrac{1}{2\sqrt{u_{\max}-u}}\left[\mathrm{D}^2 u + \dfrac{1}{2}\,\dfrac{du\otimes du}{u_{\max}-u}\right] & \text{in } \overline{M}_-\,, \end{cases}$$

$$\mathrm{D}^2\Psi \;=\; \dot\psi\,\mathrm{D}^2 u + \left[n\frac{\dot\psi^3}{u} + (n-1)\frac{\dot\psi^2}{\psi} + \frac{\dot\psi}{u}\right] du\otimes du\,.$$

Since we have already observed that

$$\Psi \ \text{is}\ \mathscr{C}^1 \ \text{along}\ \Sigma \ \Leftrightarrow\ F \ \text{is}\ \mathscr{C}^1 \ \text{along}\ \Sigma \ \Leftrightarrow\ \dot\psi\,\mathrm{D}u \ \text{is}\ \mathscr{C}^0 \ \text{along}\ \Sigma\,,$$

we have that $\mathrm{D}^2\Psi$ is $\mathscr{C}^0$ along Σ if and only if

$$\dot\psi\left[\mathrm{D}^2 u + n\frac{\dot\psi^2}{u}\,du\otimes du\right]$$

is $\mathscr{C}^0$ along Σ. Recalling (4.2.2) and formula (4.2.7), this condition is equivalent to requiring that

$$\frac{1}{F}\left[\mathrm{D}^2 u + n\frac{\dot\psi^2}{u}\,du\otimes du\right] \tag{4.2.8}$$

is $\mathscr{C}^0$ along Σ. Using again formulæ (4.2.2) and (4.2.7), near $\mathrm{MAX}(u)$ we compute

$$\begin{aligned} n\,\frac{\dot\psi^2}{u} \;&=\; \frac{n\,u}{\psi^2\,[1-(n-2)m\psi^{-n}]^2} \\ &=\; \frac{n\,u}{\psi^2(u_{\max}-u)}\,\frac{[(n-2)m]^{2/n}}{2nu_{\max}}\left[1-\frac{1}{3}n(n-1)(1-z)+\sigma\right] \\ &=\; \frac{1}{2(u_{\max}-u)}\,z^2\left[1-\frac{1}{3}n(n-1)(1-z)+\sigma\right] \\ &=\; \frac{1}{2(u_{\max}-u)}\left[1-\frac{1}{3}(n^2-n+6)(1-z)+\sigma\right]\,, \end{aligned}$$

where $z=[(n-2)m\psi^{-n}]^{1/n}$ and $\sigma=\mathcal{O}\left((1-z)^2\right)$. Therefore, the quantity (4.2.8) is $\mathscr{C}^0$ along Σ if and only if the quantity

$$\frac{1}{F}\left[\mathrm{D}^2 u + \frac{1}{2}\,\frac{du\otimes du}{u_{\max}-u}\right] + \frac{du\otimes du}{2F^3}\left[-\frac{n^2-n+6}{3}\,(1-z)+\sigma\right] \tag{4.2.9}$$

is $\mathscr{C}^0$ along Σ. If Ψ (or equivalently F) is $\mathscr{C}^1$ along Σ, we have already proved above that $\mathrm{D}u/F$ is $\mathscr{C}^0$ along Σ. Moreover from formula (4.2.7) it follows that also $(1-z)/F$ is $\mathscr{C}^0$ along Σ. Therefore, the second summand in formula (4.2.9) is always $\mathscr{C}^0$ along Σ. On the other hand, the first summand is equal to $-2\,\mathrm{D}^2F$, hence the continuity of (4.2.9) is equivalent to the continuity of D^2F, as wished. $\square$

4.2.3 Conformal reformulation of the problem.

Let (M, g_0, u) be a solution to problem (4.1.1), and let N be a connected component of $M \setminus \mathrm{MAX}(u)$. As already observed, when (M, g_0, u) is the Schwarzschild–de Sitter solution, the pseudo-radial function $\Psi = \psi \circ u$, defined by (4.2.3) or by (4.2.4) depending on whether N is outer or inner, coincides with the radial coordinate $|x|$. As anticipated, we want to proceed via a cylindrical ansatz. For this reason, we are led to consider the following conformal change

$$g = \frac{g_0}{\Psi^2}, \tag{4.2.10}$$

on N and we want to reformulate problem (4.1.1) in terms of g. We fix local coordinates in N and we compute the relation between the Christoffel symbols $\Gamma_{\alpha\beta}^{\gamma}, G_{\alpha\beta}^{\gamma}$ of g, g_0

$$\Gamma_{\alpha\beta}^{\gamma} = G_{\alpha\beta}^{\gamma} - \frac{1}{\psi}\left(\delta_{\alpha}^{\gamma}\partial_\beta\Psi + \delta_{\beta}^{\gamma}\partial_\alpha\Psi - (g_0)_{\alpha\beta}(g_0)^{\gamma\eta}\partial_\eta\Psi\right). \tag{4.2.11}$$

Denote by ∇, Δ_g the Levi-Civita connection and the Laplace-Beltrami operator of g. For every $z \in \mathscr{C}^\infty$, we compute

$$\nabla^2_{\alpha\beta}z = \mathrm{D}^2_{\alpha\beta}z + \frac{1}{\psi}\left(\partial_\alpha z\partial_\beta\Psi + \partial_\alpha\Psi\partial_\beta z - \langle \mathrm{D}z\,|\mathrm{D}\Psi\,\rangle\, g^{(0)}_{\alpha\beta}\right) \tag{4.2.12}$$

$$\Delta_g z = \psi^2\Delta z - (n-2)\psi\langle \mathrm{D}z\,|\mathrm{D}\Psi\,\rangle \tag{4.2.13}$$

Substituting $z = \Psi$ in formulæ (4.2.12) and (4.2.13), and using the equations in (4.1.1) we compute

$$\begin{aligned}
\nabla^2\Psi &= \mathrm{D}^2\Psi + \frac{1}{\psi}\left(2d\Psi\otimes d\Psi - |\mathrm{D}\Psi|^2\, g_0\right) \\
&= \dot\psi\mathrm{D}^2 u + \left(\frac{1}{u\dot\psi} + \frac{n+1}{\psi} + n\frac{\dot\psi}{u}\right) d\Psi\otimes d\Psi - \frac{|\mathrm{D}\Psi|^2}{\psi} g_0 \\
&= u\dot\psi\,\mathrm{Ric} + \left(\frac{1}{u\dot\psi} + \frac{n+1}{\psi} + n\frac{\dot\psi}{u}\right) d\Psi\otimes d\Psi - \frac{|\nabla\Psi|_g^2}{\psi} g - nu\psi^2\dot\psi\, g, \qquad (4.2.14)\\
\Delta_g\Psi &= \psi^2\Delta\Psi - (n-2)\psi|\mathrm{D}\Psi|^2 \\
&= \psi^2\dot\psi\Delta u + \left(\frac{\psi^2}{u\dot\psi} + \psi + n\frac{\psi^2\dot\psi}{u}\right)|\mathrm{D}\Psi|^2 \\
&= -nu\psi^2\dot\psi + \left(\frac{1}{u\dot\psi} + \frac{1}{\psi} + n\frac{\dot\psi}{u}\right)|\nabla\Psi|_g^2. \qquad (4.2.15)
\end{aligned}$$

On the other hand, we know from Theorem 1.1.11 that the Ricci tensors of g_0 and g are related by the formula

$$\begin{aligned}
\mathrm{Ric} &= \mathrm{Ric}_g - \frac{n-2}{\psi}\nabla^2\Psi + \frac{2(n-2)}{\psi^2} d\Psi\otimes d\Psi - \left(\frac{1}{\psi}\Delta_g\Psi + \frac{n-3}{\psi^2}|\nabla\Psi|_g^2\right) g \\
&= \mathrm{Ric}_g - \frac{n-2}{\psi}\nabla^2\Psi + \frac{2(n-2)}{\psi^2} d\Psi\otimes d\Psi + \\
&\qquad + \left[nu\psi\dot\psi - \left(n-2+n\frac{\psi\dot\psi}{u} + \frac{\psi}{u\dot\psi}\right)\frac{|\nabla\Psi|_g^2}{\psi^2}\right] g. \qquad (4.2.16)
\end{aligned}$$

Substituting (4.2.16) in (4.2.14) we obtain

$$\begin{aligned}\mathrm{Ric}_g = \left[\frac{n-2}{\psi} + \frac{1}{u\dot\psi}\right]\nabla^2\Psi - \left[\frac{2(n-2)}{\psi^2} + \frac{n}{u^2} + \frac{1}{u^2\dot\psi^2} + \frac{n+1}{u\psi\dot\psi}\right] d\Psi\otimes d\Psi \\ + \left[n\psi\Big(\psi - u\dot\psi\Big) + \left(n-2+n\frac{\psi\dot\psi}{u} + \frac{2\psi}{u\dot\psi}\right)\frac{|\nabla\Psi|_g^2}{\psi^2}\right] g\,. \quad (4.2.17)\end{aligned}$$

In order to simplify the above expressions, we notice that, *a posteriori*, in the rotationally symmetric case we expect the equality $|\mathrm{D}u|^2 = (u/\dot\psi)^2$, or equivalently $|\nabla\Psi|_g^2 = (u\psi)^2$, to hold pointwise on N. For this reason, it is convenient to introduce a function $\varphi \in \mathscr{C}^\infty(N)$ which satisfies $|\nabla\varphi|_g^2 = |\nabla\Psi|_g^2/(u\psi)^2$, so that, *a posteriori*, we expect $|\nabla\varphi|_g = 1$ pointwise on N, that is, we expect φ to be an affine function. Such a function φ can be defined in several ways. In fact, if φ is affine, so are $c\pm\varphi$, with $c\in\mathbb{R}$. However, all these choices are actually equivalent for our analysis, hence we will fix φ now, once and for all. We define the *pseudo-affine function* φ as

$$\varphi(p) \;=\; \int_{\Psi(p)}^{r_+(m)} \frac{dt}{t\sqrt{1-t^2-2mt^{2-n}}}\,. \quad (4.2.18)$$

Despite the integrand has a singularity for $t = r_\pm(m)$, the integral in (4.2.18) is finite. In fact, setting $s = 1-t^2-2mt^{2-n}$, fixed $\eta > [(n-2)m]^{1/n}$, we have

$$\begin{aligned}\int_\eta^{r_+(m)} \frac{dt}{t\sqrt{s}} &= \int_{1-\eta^2-2m\eta^{2-n}}^{0} \frac{-ds}{2t^2[1-(n-2)mt^{-n}]\sqrt{s}} \\ &\le \frac{1}{2\eta^2[1-(n-2)m\eta^{-n}]}\int_0^{1-\eta^2-2m\eta^{2-n}} \frac{ds}{\sqrt{s}} \\ &= \frac{\sqrt{1-\eta^2-2m\eta^{2-n}}}{\eta^2[1-(n-2)m\eta^{-n}]} \;<\; \infty\,.\end{aligned}$$

The singularity of the integrand when $t = r_-(m)$ can be handled in the same way. It follows that φ is well defined and smooth on N. However, *a priori* we do not know if the gradient of φ is bounded when we approach $\mathrm{MAX}(u)$, because both the numerator and the denominator of formula (4.2.19) below go to zero. This point will be addressed in Proposition 4.2.8, where we will show that $|\nabla\varphi|_g$ is bounded above by 1 on the whole N. Notice that the definition of φ is chosen in such a way that, when N is outer and $p\in\partial N$, we have $\varphi = 0$ on ∂N. Instead, when N is inner and $p\in\partial N$, that is, $\Psi(p) = r_-(m)$, the function φ assumes its maximum value.

For future convenience, we also write down some formulæ for the gradient and the hessian of φ

$$|\nabla\varphi|_g^2 = \frac{|\nabla\Psi|_g^2}{u^2\psi^2} = \frac{\dot\psi^2}{u^2}\,|\mathrm{D}u|^2 = \frac{|\mathrm{D}u|^2}{\psi^2\big[1-(n-2)m\psi^{-n}\big]^2}\,, \tag{4.2.19}$$

$$\nabla\varphi = -\frac{\dot\psi}{u\psi}\mathrm{D}u = \frac{\mathrm{D}u}{\psi^2\big[1-(n-2)m\psi^{-n}\big]}\,, \tag{4.2.20}$$

$$\nabla^2\varphi = -\frac{\nabla^2\Psi}{\psi u} + \frac{1}{\psi^2u^2}\left(u+\frac{\psi}{\dot\psi}\right)d\Psi\otimes d\Psi\,. \tag{4.2.21}$$

Combining equations (4.2.19), (4.2.21) with (4.2.15), (4.2.17), we arrive with some computations to a conformal reformulation of system (4.1.1).

Proposition 4.2.6. *Let (M, g_0, u) be a solution to problem (4.1.1), and let N be an outer or inner region, in the sense of Definition 3.2.3. Let also $\Psi = \psi\circ u$ be the pseudo-radial function defined by (4.2.3) or (4.2.4), depending on whether N is an outer or inner region, respectively. Then the metric $g = g_0/\Psi^2$ and the pseudo-affine function φ defined in (4.2.18) satisfy the following system of differential equations*

$$\begin{cases} \mathrm{Ric}_g = -\Big[(n-2)u+\dfrac{\psi}{\dot\psi}\Big]\nabla^2\varphi - (n-2)d\varphi\otimes d\varphi + \\ \qquad\qquad + \Big\{(n-2)|\nabla\varphi|_g^2 - \Big(u-\dfrac{\psi}{\dot\psi}\Big)\Delta_g\varphi\Big\}g, & \text{in } N, \\ \Delta_g\varphi = n\psi\dot\psi\left(1-|\nabla\varphi|_g^2\right), & \text{in } N, \end{cases} \tag{4.2.22}$$

with initial conditions

$$\begin{cases} \varphi = 0 & \text{on } \partial N, \\ \varphi = \varphi_0 := \displaystyle\int_{[(n-2)m]^{1/n}}^{r_+(m)} \frac{dt}{t\sqrt{1-t^2-2mt^{2-n}}} & \text{on } \overline{N}\cap\mathrm{MAX}(u), \end{cases} \qquad \text{if } N \text{ outer},$$

$$\begin{cases} \varphi = \varphi_{\max} := \displaystyle\int_{r_-(m)}^{r_+(m)} \frac{dt}{t\sqrt{1-t^2-2mt^{2-n}}} & \text{on } \partial N, \\ \varphi = \varphi_0 := \displaystyle\int_{[(n-2)m]^{1/n}}^{r_+(m)} \frac{dt}{t\sqrt{1-t^2-2mt^{2-n}}} & \text{on } \overline{N}\cap\mathrm{MAX}(u), \end{cases} \qquad \text{if } N \text{ inner}.$$

Tracing the first equation of (4.2.22), one obtains

$$\frac{\mathrm{R}_g}{n-1} = (n-2) - (2nu\psi\dot\psi - n\psi^2 + n - 2)\left(1-|\nabla\varphi|_g^2\right) \tag{4.2.23}$$

where R_g is the scalar curvature of g. In the cylindrical situation, which is the conformal counterpart of the Schwarzschild–de Sitter solution, R_g has to be constant. In this case, the above formula implies that also $|\nabla\varphi|_g$ has to be constant and equal to 1, as already anticipated. For these reasons, also in the situation, where we do not know a priori if g is cylindrical, it is natural to think of $\nabla\varphi$ as to a candidate splitting direction and to investigate under which conditions this is actually the case. A first important observation is that the splitting is in force when φ is an affine function, that is, when its hessian $\nabla^2\varphi$ vanishes everywhere in our region.

Proposition 4.2.7. *Let (M, g_0, u) be a solution to problem (4.1.1), and let N be an outer or inner region with virtual mass*

$$m = \mu(N, g_0, u).$$

Let $\Psi = \psi \circ u$ be the pseudo-radial function defined by (4.2.3) or (4.2.4), depending on whether N is an outer or inner region, respectively. Finally, let $g = g_0/\Psi^2$ and let φ be the pseudo-affine function defined by (4.2.18).

If $\nabla^2\varphi \equiv 0$ and $|\nabla\varphi|_g \equiv 1$ on N, then (M, g_0, u) is isometric to the Schwarzschild–de Sitter solution (2.5.3) with mass m.

Proof. Let us suppose that N is an outer region, the inner case being completely equivalent. Proceeding as in the proof of [AM15, Theorem 4.1], we obtain that $(\{0 \le \varphi < \varphi_0\}, g)$ is isometric to the product

$$\big([0, \varphi_0) \times \partial N\,,\, d\varphi \otimes d\varphi + g^{\partial N}\big)\,,$$

where $g^{\partial N}$ is the metric induced by g on ∂N. From the first equation in (4.2.22) we deduce that $\mathrm{Ric}_{g^{\partial N}} = (n-2)g^{\partial N}$. Recalling the definition of φ and the relation between g and g_0, we deduce that g_0 is isometric to

$$\frac{d\Psi \otimes d\Psi}{u^2} + \psi^2 g^{\partial N}.$$

This proves that (N, g_0, u) is isometric to the outer region of the Schwarzschild–de Sitter solution (M^s, g_0^s, u^s) defined by (2.5.3), where Ψ is the radial coordinate. It remains to prove that this isometry between the outer regions extends to an isometry between the whole (M, g_0, u) and the whole (M^s, g_0^s, u^s). To this end, we distinguish two cases, depending on whether the (possibly stratified) hypersurface $\Sigma = \overline{N} \cap \mathrm{MAX}(u)$ is orientable or not.

- Let us start by considering the case in which Σ is an orientable hypersurface. Since M is orientable by hypothesis, the hypersurface Σ is orientable if and only if it is two sided, meaning that any neighborhood of Σ contains both points of N and points outside N. Considering the corresponding chart in (M^s, g_0^s, u^s), by analytic continuation we can extend the isometry between (N, g_0, u) and the outer region of (M^s, g_0^s, u^s) to all the points in the chart. That way, the isometry pass through Σ, and we can continue to argue chart by chart until we finally cover all the manifold M, thus proving the global isometry of (M, g_0, u) and (M^s, g_0^s, u^s).

- If Σ is not orientable, this means that it is one sided, that is, every point of Σ has a neighborhood that is entirely contained inside $\overline{N}$. Therefore, it easily follows that $(M, g_0, u) = (\overline{N}, g_0, u)$ is isometric to $(\overline{M}^s_+, g_0^s, u^s)/\sim$, where $\sim$ is a relation on the points of

$$\mathrm{MAX}^s(u) = \{p \in M^s \,:\, u^s(p) = u_{\max}\} \subset \partial\overline{M}^s_+.$$

  We first observe that this relation is induced by an involution

$$\iota_\sim : \mathrm{MAX}^s(u) \to \mathrm{MAX}^s(u).$$

In fact, the neighborhood of a point $x \in \mathrm{MAX}^s(u)$ inside $(\overline{M}^s_+, g_0)$ is isometric to an half space $\mathbb{R}^n_+$ endowed with a metric such that the boundary $\partial\mathbb{R}^n_+$ is smooth. In order for the manifold $(\overline{M}^s_+, g_0)/\sim$ to be smooth at x it is necessary that there exists exactly one point $\iota_\sim(x) \in \mathrm{MAX}^s(u)$, $\iota_\sim(x) \neq x$, such that $x \sim \iota_\sim(x)$. It is also clear that $\iota_\sim^2$ is the identity, so that $\iota_\sim$ is indeed an involution. Moreover, $\iota_\sim$ has to reverse the orientation.

We now notice that the mean curvature vector $\vec{\mathrm{H}}$ of the hypersurface $\mathrm{MAX}^s(u)$ has constant nonzero norm and it always points outside M^s_+ on the whole $\mathrm{MAX}^s(u)$, hence the same holds on $\Sigma = \mathrm{MAX}^s(u)/\sim$. In particular, at the points x and $\iota_\sim(x)$ the vector $\vec{\mathrm{H}}$ points outside M^s_+, but this is in contrast with the fact that $\iota_\sim$ reverse the orientation.

This concludes the proof. □

4.2.4 The geometry of the level sets.

In the forthcoming analysis a crucial role is played by the study of the geometry of the level sets of φ, which coincide with the level sets of u in N, by definition. Hence, we pass now to describe the second fundamental form and the mean curvature of the regular level sets of φ (or equivalently of u) in both the original Riemannian context (N, g_0) and the conformally related one (N, g). To this aim, we fix a regular level set $\{\varphi = s_0\}$ and we construct a suitable set of coordinates in a neighborhood of it. Note that $\{\varphi = s_0\}$ must be compact, by the properness of φ, thus there exists a real number $\delta > 0$ such that in the tubular neighborhood $\mathcal{U}_\delta = \{s_0 - \delta < \varphi < s_0 + \delta\}$ we have $|\nabla\varphi|_g > 0$ so that $\mathcal{U}_\delta$ is foliated by regular level sets of φ. This means that the analysis in Subsection 1.1.2 apply. In particular, we observe that $\mathcal{U}_\delta$ is diffeomorphic to $(s_0 - \delta, s_0 + \delta) \times \{\varphi = s_0\}$ and the function φ can be regarded as a coordinate in $\mathcal{U}_\delta$. Thus, one can choose a local system of coordinates $\{\varphi, \vartheta^1, ..., \vartheta^{n-1}\}$, where $\{\vartheta^1, ..., \vartheta^{n-1}\}$ are local coordinates on $\{\varphi = s_0\}$. In such a system, the metric g can be written as

$$g = \frac{d\varphi \otimes d\varphi}{|\nabla\varphi|_g^2} + g_{ij}(\varphi, \vartheta^1, ..., \vartheta^{n-1})\, d\vartheta^i \otimes d\vartheta^j\,,$$

where the latin indices vary between 1 and $n-1$. We now fix in $\mathcal{U}_\delta$ the g_0-unit vector field $\nu = \mathrm{D}u/|\mathrm{D}u| = \mathrm{D}\varphi/|\mathrm{D}\varphi|$ and the g-unit vector field $\nu_g = \nabla u/|\nabla u|_g = \nabla\varphi/|\nabla\varphi|_g$. Accordingly, the second fundamental forms of the regular level sets of u or φ with respect to ambient metric g_0 and the conformally-related ambient metric g are respectively given by

$$\mathrm{h}^{(0)}_{ij} = \frac{\mathrm{D}^2_{ij}u}{|\mathrm{D}u|} = \frac{\mathrm{D}^2_{ij}\varphi}{|\mathrm{D}\varphi|} \qquad \text{and} \qquad \mathrm{h}^{(g)}_{ij} = \frac{\nabla^2_{ij}u}{|\nabla u|_g} = \frac{\nabla^2_{ij}\varphi}{|\nabla\varphi|_g}\,, \tag{4.2.24}$$

for $i, j = 1, ..., n-1$. Taking the traces of the above expressions with respect to the induced metrics we obtain the following expressions for the mean curvatures in the two ambients

$$\mathrm{H} = \frac{\Delta u}{|\mathrm{D}u|} - \frac{\mathrm{D}^2u(\mathrm{D}u, \mathrm{D}u)}{|\mathrm{D}u|^3}\,, \qquad \mathrm{H}_g = \frac{\Delta_g\varphi}{|\nabla\varphi|_g} - \frac{\nabla^2\varphi(\nabla\varphi, \nabla\varphi)}{|\nabla\varphi|_g^3}\,. \tag{4.2.25}$$

Taking into account expressions (4.2.19), (4.2.21), one can show that the second fundamental forms are related by

$$|\nabla\varphi|_g\,\mathrm{h}^{(g)}_{ij} \;=\; \frac{\dot\psi}{u\psi}\left[-|\mathrm{D}u|\mathrm{h}^{(0)}_{ij} \,+\, \frac{\dot\psi}{\psi}\,|\mathrm{D}u|^2 g^{(0)}_{ij}\right]. \tag{4.2.26}$$

The analogous formula for the mean curvatures reads

$$|\nabla\varphi|_g\,\mathrm{H}_g \;=\; \frac{\dot\psi}{u}\left[-\psi|\mathrm{D}u|\,\mathrm{H} \,+\, (n-1)\,\dot\psi|\mathrm{D}u|^2\right] \tag{4.2.27}$$

Concerning the nonregular level sets of φ, we first observe that, since u is analytic on M (see [Chr05, zH70]), then φ is analytic on the whole N. As anticipated in Subsection 4.1.1, it follows from the results in [Łoj91] (see also [KP02, Theorem 6.3.3]) that there exists an hypersurface $\Sigma \subseteq \mathrm{Crit}(\varphi)$ such that $\mathscr{H}^{n-1}(\mathrm{Crit}(\varphi)\setminus\Sigma) = 0$. In particular, the $(n-1)$-dimensional Hausdorff measure of the level sets of φ is locally finite. Moreover, the unit normal to a level set is well-defined $\mathscr{H}^{n-1}$-almost everywhere, and so are the second fundamental form h_g and the mean curvature H_g. We will prove now that formulæ (4.2.26) and (4.2.27) hold also at any point $y_0 \in \Sigma$, and therefore they hold $\mathscr{H}^{n-1}$-almost everywhere on any level set. Let ν, ν_g be the unit normal vector fields to Σ at y_0 with respect to g_0, g respectively. Since $|\nu_g|^2_g = 1 = |\nu|^2 = \psi^2\,|\nu|^2_g$, we deduce that $\nu_g = \psi\,\nu$. Let $(\partial/\partial x^1, \ldots, \partial/\partial x^{n-1})$ be a basis of $T_{y_0}\Sigma$, so that in particular $(\partial/\partial x^1, \ldots, \partial/\partial x^{n-1}, \nu_g)$ is a basis of $T_{y_0}M$. Recalling (4.2.11) and observing that the derivatives of u in y_0 are all zero since $y_0 \in \mathrm{Crit}(\varphi) = \mathrm{Crit}(u)$, we have

$$\mathrm{h}^{(g)}_{ij} \;=\; \left\langle \nabla_i \frac{\partial}{\partial x^j}\,\middle|\, \nu_g\right\rangle_g \;=\; \Gamma^n_{ij} \;=\; \mathrm{G}^n_{ij} \;=\; \left\langle \mathrm{D}_i \frac{\partial}{\partial x^j}\,\middle|\, \nu_g\right\rangle_g \;=\; \frac{1}{\psi}\left\langle \mathrm{D}_i \frac{\partial}{\partial x^j}\,\middle|\, \nu\right\rangle \;=\; \frac{1}{\psi}\,\mathrm{h}^{(0)}_{ij}\,.$$

From this we immediately deduce that formula (4.2.26) holds also on Σ, and taking its trace we prove that also (4.2.27) is verified.

4.2.5 Consequences of the Bochner formula.

Starting from the Bochner formula and using the equations in (4.2.22), we find

$$\begin{aligned}\Delta_g|\nabla\varphi|^2_g \;&=\; 2|\nabla^2\varphi|^2_g + 2\mathrm{Ric}_g(\nabla\varphi,\nabla\varphi) + 2\langle\nabla\Delta_g\varphi\,|\,\nabla\varphi\rangle\\ &=\; 2|\nabla^2\varphi|^2_g - \left[(n-2)u + \frac{\psi}{\dot\psi} + 2n\psi\dot\psi\right]\langle\nabla|\nabla\varphi|^2_g\,|\,\nabla\varphi\rangle_g -\\ &\qquad\qquad - 2\left[(n+1)u + n\psi\dot\psi\right]|\nabla\varphi|^2_g\,\Delta_g\varphi\,.\end{aligned} \tag{4.2.28}$$

We will use (4.2.28) to compute the laplacian of the function

$$w = \beta\left(1 - |\nabla\varphi|^2_g\right), \quad \text{where } \beta = \psi^2\left|1-(n-2)m\psi^{-n}\right| = \psi^2\left|\frac{u}{\psi\dot\psi}\right|.$$

The function β is smooth in N. We will denote by β' the derivative of β with respect to φ, more precisely, $\beta' \in \mathscr{C}^\infty(N)$ is the function that satisfies $\nabla\beta = \beta'\nabla\varphi$.

One computes

$$\frac{\beta'}{\beta} = n\psi\dot\psi + (n-2)u\,, \tag{4.2.29}$$

$$\nabla w \;=\; -\beta\nabla|\nabla\varphi|^2_g + \frac{\beta'}{\beta}w\nabla\varphi\,. \tag{4.2.30}$$

In order to compute the laplacian of w, we take the divergence of (4.2.30)

$$\begin{aligned}\Delta_g w \;=\; & -\frac{\beta'}{\beta}\big\langle\beta\nabla|\nabla\varphi|^2_g\,\big|\,\nabla\varphi\big\rangle_g - \beta\Delta_g|\nabla\varphi|^2_g + \\ & + \Big(\frac{\beta'}{\beta}\Big)'w|\nabla\varphi|^2_g + \frac{\beta'}{\beta}\langle\nabla w\,|\,\nabla\varphi\rangle_g + \frac{\beta'}{\beta}w\Delta_g\varphi\,,\end{aligned}$$

and using formula (4.2.28) we obtain

$$\begin{aligned}\Delta_g w \;=\; & -2\beta\left[|\nabla^2\varphi|^2_g - \frac{(\Delta_g\varphi)^2}{n}\right] - \left[(n-2)u + \frac{\psi}{\dot\psi} + 2n\psi\dot\psi - 2\frac{\beta'}{\beta}\right]\langle\nabla w\,|\,\nabla\varphi\rangle_g + \\ & + n\psi\dot\psi\Big(\frac{\beta'}{\beta} - 2\psi\dot\psi\Big)w + \bigg[\Big(\frac{\beta'}{\beta}\Big)' - \Big(\frac{\beta'}{\beta}\Big)^2 + \Big((n-2)u + \frac{\psi}{\dot\psi} + n\psi\dot\psi\Big)\frac{\beta'}{\beta} + \\ & + 2n(n+1)u\psi\dot\psi + 2n(n+1)\psi^2\dot\psi^2\bigg]|\nabla\varphi|^2_g w \\ \leq\; & \left[(n-2)u - \frac{\psi}{\dot\psi}\right]\langle\nabla w\,|\,\nabla\varphi\rangle_g + n(n-2)m\psi^{2-n}\dot\psi^2\big[(n-2) + (n+2)|\nabla\varphi|^2_g\big]w\,.\end{aligned} \tag{4.2.31}$$

In particular, w satisfies an elliptic inequality on our connected component N and, as a consequence of the Minimum Principle, we obtain the following relevant bound on the gradient of the pseudo-affine function φ.

Proposition 4.2.8. *Let (M, g_0, u) be a solution to problem (4.1.1) and let $N \subset M \setminus \mathrm{MAX}(u)$ be an outer or inner region with virtual mass $m = \mu(N, g_0, u)$. Let also g, φ be defined by (4.2.10), (4.2.18). Then it holds*

$$|\nabla\varphi|_g \leq 1\,,$$

on the whole N.

Proof. We recall from (4.2.19) that it holds

$$|\nabla\varphi|_g \;=\; \frac{|\mathrm{D}u|}{\psi\,|1-(n-2)m\psi^{-n}|}\,,$$

therefore, from Lemma 4.2.2 we deduce that $w \geq 0$ on ∂N. On the other hand, from the definition of w, we compute

$$w \;=\; \beta(1-|\nabla\varphi|^2_g) \;=\; \psi^2\left|\frac{u}{\psi\dot\psi}\right| - \left|\frac{\psi\dot\psi}{u}\right||\mathrm{D}u|^2\,.$$

Since β goes to 0 as we approach $\mathrm{MAX}(u)$, and also $\dot{\psi}|\mathrm{D}u|^2$ goes to 0 thanks to Lemma 4.2.3, we have $w \to 0$ as we approach $\mathrm{MAX}(u)$. We also recall that φ, g, thus also w, are analytic in the interior of N. As observed in Subsection 4.1.1, this implies that the critical level sets of w are discrete. Therefore there exists $\eta > 0$ such that any $0 < \varepsilon < \eta$ is a regular values for w. In particular, the set

$$N_\varepsilon \;=\; \{|w| \geq \varepsilon\} \cap \overline{N}$$

has a smooth boundary. Since we have already observed that $w \to 0$ as we approach ∂N and $\mathrm{MAX}(u)$, we have that N_ε is a compact domain contained in the interior of N. In particular, the coefficients of the elliptic inequality (4.2.31) are bounded in N_ε and we can apply the Weak Minimum Principle (see for instance [GT83, Corollary 3.2]) to deduce that

$$\min_{N_\varepsilon} w \;\geq\; \min_{\partial N_\varepsilon} w \;\geq\; -\varepsilon\,, \tag{4.2.32}$$

where in the latter inequality we have used the fact that the boundary of N_ε is contained in $(\{w = \varepsilon\} \cup \{w = -\varepsilon\}) \cap N$. Since inequality (4.2.32) holds for all $0 < \varepsilon < \eta$, taking the limit as $\varepsilon \to 0$ we obtain the thesis. □

Remark 4.2.9. *Translating the thesis of Proposition 4.2.8 back in terms of u, g_0, we have that $|\mathrm{D}u|/\dot{\psi}$ is bounded in N, and recalling formula (4.2.7), we deduce that the quantity $|\mathrm{D}u|^2/(u_{\max} - u)$ is bounded on N. In particular, for any $p \in \overline{N} \cap \mathrm{MAX}(u)$ there exists a collar neighborhood $p \in U_p \subset \overline{N}$ and a constant c_p such that*

$$|\mathrm{D}u|^2(x) \;\leq\; c_p\,[u_{\max} - u(x)] \tag{4.2.33}$$

for any $x \in U_p$. The same proof can be repeated on each inner and outer region, and a similar result will also be shown for cylindrical regions, see Proposition 4.6.3. It follows from these considerations that the inequality (4.2.33) is always in force in a neighborhood of any $p \in \mathrm{MAX}(u)$. This is an improvement of the reverse Łojasiewicz inequality 1.1.7 proved in Chapter 1.

4.2.6 Area lower bound.

In this subsection, we will study the function

$$\Phi(s) = \int_{\{\varphi = s\}} |\nabla\varphi|_g \,\mathrm{d}\sigma_g\,. \tag{4.2.34}$$

which is defined on $s \in [0, \varphi_0)$ or $s \in (\varphi_0, \varphi_{\max}]$ depending on whether the quantity

$$\max_{S \in \pi_0(\partial N)} \kappa(S) \;=\; \max_{\partial N} \frac{|\mathrm{D}u|}{u_{\max}}$$

is less than or greater than $\sqrt{n}$, respectively. As an application of Proposition 4.2.8, one can prove the following monotonicity result for Φ.

Proposition 4.2.10. *Let (M, g_0, u) be a solution to problem (4.1.1), let N be a connected component of $M \setminus \mathrm{MAX}(u)$ with virtual mass $m < m_{\max}$, and let $\Phi(s)$ be the function defined by (4.2.34), with respect to the metric g and the pseudo-affine function φ defined by (4.2.10), (4.2.18).*

(i) If N is an outer region, then the function $\Phi(s)$, defined for $s \in [0, \varphi_0)$, is monotonically nonincreasing. Moreover, if $\Phi(s_1) = \Phi(s_2)$ for two different values $0 \leq s_1 < s_2 < \varphi_0$, then the triple (M, g_0, u) is isometric to the Schwarzschild–de Sitter triple (2.5.3) with mass m.

(ii) If N is an inner region, then the function $\Phi(s)$, defined for $s \in (\varphi_0, \varphi_{\max}]$, is monotonically nondecreasing. Moreover, if $\Phi(s_1) = \Phi(s_2)$ for two different values $\varphi_0 < s_1 < s_2 \leq \varphi_{\max}$, then the solution (M, g_0, u) is isometric to the Schwarzschild–de Sitter triple (2.5.3) with mass m.

Proof. Consider the case N outer, that is, $\max_{\partial N} |\mathrm{D}u|/u_{\max} < \sqrt{n}$. In particular, the determination of ψ is (4.2.3), $\varphi \in [0, \varphi_0)$, $\dot{\psi} \leq 0$ and $\Delta_g \varphi \leq 0$ (this last inequality is a consequence of the second equation of system (4.2.22) and of Proposition 4.2.8). Integrating $\Delta_g \varphi \leq 0$ in $\{s_1 \leq \varphi \leq s_2\}$ for any $0 \leq s_1 < s_2 < \varphi_0$, we get

$$\int\limits_{\{s_1 \leq \varphi \leq s_2\}} \Delta_g \varphi \, \mathrm{d}\sigma_g \leq 0 \,. \tag{4.2.35}$$

Applying the Divergence Theorem to inequality (4.2.35), we easily obtain $\Phi(s_2) \leq \Phi(s_1)$, therefore Φ is nonincreasing. To prove the rigidity statement, we observe that, if the equality $\Phi(s_1) = \Phi(s_2)$ holds for some $0 \leq s_1 < s_2 < \varphi_0$, then $\Delta_g \varphi \equiv 0$ on $\{s_1 \leq \varphi \leq s_2\}$, hence by the analyticity of φ we deduce $\Delta_g \varphi \equiv 0$ on N. Recalling the definition of $\Delta_g \varphi$, this in turn implies $|\nabla \varphi|_g \equiv 1$ on N. Substituting this information in the Bochner formula (4.2.28) we obtain $|\nabla^2 \varphi|_g \equiv 0$, hence we can apply Proposition 4.2.7 to conclude.

If instead N is an inner region, that is, $\max_{\partial N} |\mathrm{D}u|/u_{\max} > \sqrt{n}$, then ψ is as in (4.2.4), $\varphi \in (\varphi_0, \varphi_{\max}]$ and $\Delta_g \varphi \geq 0$. Proceeding in the same way as above, we obtain the opposite monotonicity for Φ. The rigidity statement is proved in the same way as in the preceding case. □

From the monotonicity and the continuity of $\Phi(s)$, it follows that its limit as $s \to \varphi_0$ exists, and since $|\nabla \varphi|_g$ is bounded (this is a consequence of Proposition 4.2.8) and the level sets are compact, this limit is finite. Therefore, from the monotonicity of Φ we can deduce the following global monotonicity property.

Lemma 4.2.11. *Let (M, g_0, u) be a solution to problem (4.1.1), let N be a connected component of $M \setminus \mathrm{MAX}(u)$ with virtual mass $m < m_{\max}$, and let g and φ be defined by (4.2.10), (4.2.18). Then*

$$|\partial N|_g \geq \lim_{s \to \varphi_0} \int_{\{\varphi = s\}} |\nabla \varphi|_g \, \mathrm{d}\sigma_g \,.$$

Moreover, if the equality holds, then (M, g_0, u) is isometric to the Schwarzschild–de Sitter triple (2.5.3) with mass m.

Proof. For $0 \leq s < \varphi_0$ the function $\Phi(s)$ is monotonically nonincreasing by Proposition 4.2.10, therefore $\lim_{s \to \varphi_0} \Phi(s) \leq \Phi(0)$. From Lemma 4.2.2, we know that $|\nabla \varphi|_g \leq 1$ on ∂N, hence $\Phi(0) \leq \int_{\partial N} \mathrm{d}\sigma_g = |\partial N|_g$. This proves the thesis. The case $\varphi_0 < s < \varphi_{\max}$ is proved in the exact same way. □

In order to make use of Lemma 4.2.11, we need some information on the set $\mathrm{MAX}(u)$ and on the behavior of $\nabla\varphi$ at the limit $\varphi \to \varphi_0$. In Section 4.5 we will see how to recover some more explicit information from Lemma 4.2.11 in the case where our solution is 2-sided according to Definition 4.1.6.

4.3 Integral identities

In this section we use the pseudo-affine function φ in order to construct a vector field with a nonnegative divergence. As an application of the Divergence Theorem, we will then deduce a couple of important integral identities. In particular, in Propositions 4.3.1 and 4.3.2 we show two functions that give a nonnegative value when integrated along any level set of φ. Moreover, the integral on a level set is zero only in the case of the Schwarzschild–de Sitter solution.

The analysis of the case where N is an outer region and the case where N is an inner region are slightly different. The outer case will be studied in Subsection 4.3.1 and the inner case will be studied in Subsection 4.3.2.

4.3.1 Integral identities in the outer regions.

We start by considering the case when N is an outer region, that is, in this subsection we will suppose

$$\max_{S\in\pi_0(\partial N)} \kappa(S) \,<\, \sqrt{n}\,,$$

and the pseudo-radial function $\Psi = \psi \circ u$ is chosen as in (4.2.3). Consider the vector field

$$Y \,=\, \nabla|\nabla\varphi|_g^2 \,+\, \Delta_g\varphi\nabla\varphi\,.$$

From the Bochner formula (4.2.28) and the equations in (4.2.22) we compute

$$\begin{aligned}
\mathrm{div}_g(Y) \,&=\, \Delta_g|\nabla\varphi|_g^2 + \mathrm{div}_g(\Delta_g\varphi\nabla\varphi) \\
&=\, -\left[(n-2)u + \frac{\psi}{\dot\psi} + 3n\psi\dot\psi\right]\langle\nabla\varphi\,|\,Y\rangle_g + 2|\nabla^2\varphi|_g^2 + (\Delta_g\varphi)^2 - \\
&\qquad\qquad - 2n(n+2)u\psi\dot\psi|\nabla\varphi|_g^2(1-|\nabla\varphi|_g^2)\,.
\end{aligned}$$

Since

$$u\psi\dot\psi = \psi^2\dot\psi^2\left(\frac{u}{\psi\dot\psi}\right) = -\psi^2\dot\psi^2[1-(n-2)m\psi^{-n}]$$

is negative when the chosen determination of ψ is (4.2.3), we have

$$\begin{aligned}
\mathrm{div}_g(Y) \,+\, &\left[(n-2)u + \frac{\psi}{\dot\psi} + 3n\psi\dot\psi\right]\langle\nabla\varphi\,|\,Y\rangle_g \,= \\
&= 2|\nabla^2\varphi|_g^2 + (\Delta_g\varphi)^2 - 2n(n+2)u\psi\dot\psi|\nabla\varphi|_g^2(1-|\nabla\varphi|_g^2) \geq 0\,. \qquad (4.3.1)
\end{aligned}$$

Now consider the function

$$\gamma = -\frac{u^2\psi^{2n-1}}{\dot\psi^3} \,=\, \frac{\psi^{2(n+1)}}{u}\left[1-(n-2)m\psi^{-n}\right]^3 \qquad (4.3.2)$$

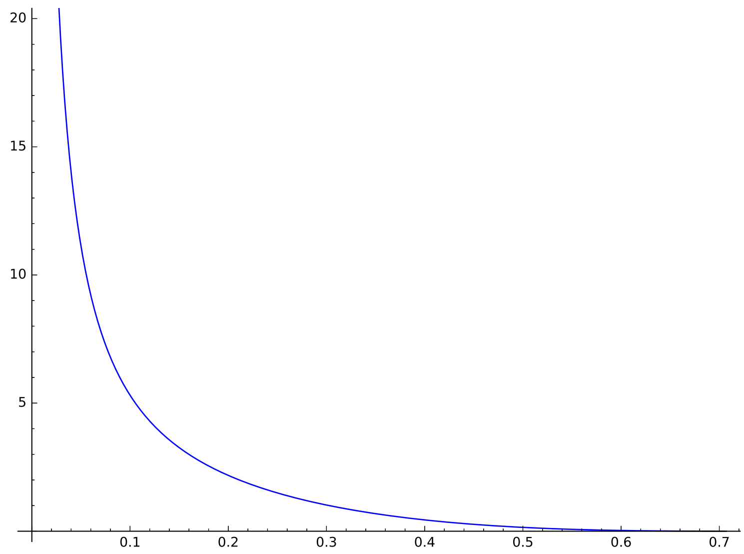


Figure 4.4: Relation between γ (on the y-axis) and u (on the x-axis), when N is an outer region, for $n = 4$ and $m = 1/32$.

(notice that $\gamma \geq 0$ when $\Psi = \psi \circ u$ is as in (4.2.3)). The relation between γ and u is shown in Figure 4.4. We compute

$$\begin{aligned}
\frac{\gamma'}{\gamma} &= \frac{\dot\psi^3}{u^2\psi^{2n-1}} \cdot \frac{du}{d\varphi} \cdot \left(2\frac{u\psi^{2n-1}}{\dot\psi^3} + (2n-1)\frac{u^2\psi^{2n-2}\dot\psi}{\dot\psi^3} - 3\frac{u^2\psi^{2n-1}\ddot\psi}{\dot\psi^4}\right) \\
&= -\frac{\dot\psi^2}{u\psi^{2n-2}}\left(2\frac{u\psi^{2n-1}}{\dot\psi^3} + (2n-1)\frac{u^2\psi^{2n-2}\dot\psi}{\dot\psi^3} - 3\frac{u^2\psi^{2n-1}\ddot\psi}{\dot\psi^4}\right) \\
&= -2\frac{\psi}{\dot\psi} - (2n-1)u + 3\frac{u\psi\ddot\psi}{\dot\psi^2} \\
&= -2\frac{\psi}{\dot\psi} - (2n-1)u + 3n\psi\dot\psi + 3(n-1)u + 3\frac{\psi}{\dot\psi} \\
&= (n-2)u + \frac{\psi}{\dot\psi} + 3n\psi\dot\psi\,.
\end{aligned}$$

Therefore, formula (4.3.1) rewrites as

$$\begin{aligned}
\mathrm{div}_g(\gamma Y) &= \gamma\left[2|\nabla^2\varphi|_g^2 + (\Delta_g\varphi)^2 - 2n(n+2)u\psi\dot\psi|\nabla\varphi|_g^2\left(1 - |\nabla\varphi|_g^2\right)\right] \\
&\geq 0\,.
\end{aligned} \tag{4.3.3}$$

Integrating identity (4.3.3) in $\{\varphi \geq s\}$ for some value s, we have the following integral identity.

Proposition 4.3.1. *Let (M, g_0, u) be a solution to problem (4.1.1), let $N \subset M \setminus \mathrm{MAX}(u)$ be an outer region and let $m = \mu(N, g_0, u)$ be its virtual mass. Let $\Psi = \psi \circ u$, g and φ be defined by (4.2.3), (4.2.10) and (4.2.18). Then for any*

$0 \leq s < \varphi_0$ it holds

$$\int\limits_{\{\varphi=s\}\cap N} |\nabla\varphi|_g \left[|\nabla\varphi|_g \mathrm{H}_g - \frac{3}{2}\Delta_g\varphi\right] \mathrm{d}\sigma_g =$$

$$= \frac{1}{\gamma(s)} \int\limits_{\{s\leq\varphi<\varphi_0\}\cap N} \gamma \left[|\nabla^2\varphi|_g^2 + \frac{1}{2}(\Delta_g\varphi)^2 - n(n+2)u\psi\dot{\psi}|\nabla\varphi|_g^2(1-|\nabla\varphi|_g^2)\right] \mathrm{d}\sigma_g$$

$$\geq 0\,. \tag{4.3.4}$$

where γ is the function defined by (4.3.2). Moreover, if the equality

$$\int_{\{\varphi=s_0\}\cap N} |\nabla\varphi|_g \left[|\nabla\varphi|_g \mathrm{H}_g - \frac{3}{2}\Delta_g\varphi\right] \mathrm{d}\sigma_g = 0\,, \tag{4.3.5}$$

holds for some $0 < s_0 < \varphi_0$, then the solution (M, g_0, u) is isometric to the Schwarzschild–de Sitter triple (2.5.3) with mass m.

Proof. Let us recall that u is an analytic function. In particular, also φ is analytic in the interior of N, hence its critical level sets are discrete. It follows that we can choose $0 < s < S < \varphi_0$, with s arbitrarily close to zero and S arbitrarily close to φ_0 such that both s and S are regular values for φ. Integrating (4.3.3) on $\{s \leq \varphi \leq S\}$ and using the Divergence Theorem 1.1.9, we obtain

$$\int\limits_{\{s\leq\varphi\leq S\}\cap N} \mathrm{div}_g(\gamma Y)\mathrm{d}\sigma_g =$$

$$= \int\limits_{\{\varphi=S\}\cap N} \gamma(S)\left\langle Y \,\Big|\, \frac{\nabla\varphi}{|\nabla\varphi|_g}\right\rangle_g \mathrm{d}\sigma_g - \int\limits_{\{\varphi=s\}\cap N} \gamma(s)\left\langle Y \,\Big|\, \frac{\nabla\varphi}{|\nabla\varphi|_g}\right\rangle_g \mathrm{d}\sigma_g\,. \tag{4.3.6}$$

First of all, we notice that it holds

$$\lim_{S\to\varphi_0} \gamma(S) \int_{\{\varphi=S\}\cap N} \left\langle Y \,\Big|\, \frac{\nabla\varphi}{|\nabla\varphi|_g}\right\rangle_g \mathrm{d}\sigma_g = 0\,. \tag{4.3.7}$$

In fact, using formulæ (4.2.19), (4.2.21) and (4.2.22) to translate the integrand in terms of u, g_0, we find

$$\begin{aligned}
\gamma\left\langle Y \,\Big|\, \frac{\nabla\varphi}{|\nabla\varphi|_g}\right\rangle_g &= \gamma\left(\frac{\langle\nabla|\nabla\varphi|_g^2 \,|\, \nabla\varphi\rangle_g}{|\nabla\varphi|_g} + \Delta_g\varphi\,|\nabla\varphi|_g\right)\\
&= \gamma\,|\nabla\varphi|_g\left[2\,\nabla^2\varphi(\nu_g,\nu_g) + \Delta_g\varphi\right]\\
&= \frac{\psi^{2n}}{\dot{\psi}}|\mathrm{D}u|\left[-2\,\mathrm{D}^2u(\nu,\nu) - 2\frac{\dot{\psi}}{u\psi}\left(n-1+n\frac{\psi\dot{\psi}}{u}\right)|\mathrm{D}u|^2 + \right.\\
&\qquad\left. + \,n\left(1-\frac{\dot{\psi}^2}{u^2}\,|\mathrm{D}u|^2\right)\right],
\end{aligned}$$

where $\nu = \mathrm{D}u/|\mathrm{D}u|, \nu_g = \nabla\varphi/|\nabla\varphi|_g = \psi\,\nu$ are the unit normals to $\{\varphi = S\}$, which exist everywhere because $\{\varphi = S\}$ is a regular level set. Since $|\nabla\varphi|_g^2 =$

$(\dot\psi^2/u^2)|\mathrm{D}u|^2 \leq 1$ by Proposition 4.2.8, we deduce that the limit of the term in square bracket as $S \to \varphi_0$ (or equivalently $u \to u_{\max}$) is bounded from above. Therefore, in order to prove (4.3.7), it is enough to show that

$$\lim_{u\to 1} \int_{\{u=t\}} \frac{1}{\dot\psi} |\mathrm{D}u| \, \mathrm{d}\sigma \, = \, 0 \, .$$

But this can be done proceeding exactly as in the proof of Theorem 3.4.4, via a simple argument using the coarea formula and the facts that $(\dot\psi^2/u^2)|\mathrm{D}u|^2 \leq 1$ and $\dot\psi \to +\infty$ as $u \to u_{\max}$. Therefore, taking the limit as $S \to \varphi_0$ of (4.3.6), we deduce

$$\int\limits_{\{\varphi=s\}\cap N} \gamma(s) \Big\langle Y \,\Big|\, \frac{\nabla\varphi}{|\nabla\varphi|_g} \Big\rangle_g \mathrm{d}\sigma_g \, = \, - \int\limits_{\{s\leq\varphi<\varphi_0\}\cap N} \mathrm{div}_g(\gamma Y) \mathrm{d}\sigma_g \, \leq \, 0 \, , \tag{4.3.8}$$

where in the last inequality we have used (4.3.1). Recalling (4.2.25) we get

$$\begin{aligned} \Big\langle Y \,\Big|\, \frac{\nabla\varphi}{|\nabla\varphi|_g} \Big\rangle_g \, &= \, 2\frac{\nabla^2\varphi(\nabla\varphi,\nabla\varphi)}{|\nabla\varphi|_g} + \Delta_g\varphi|\nabla\varphi|_g \\ &= \, |\nabla\varphi|_g \, \big(-2|\nabla\varphi|_g \mathrm{H}_g + 3\Delta_g\varphi\big) \, . \end{aligned} \tag{4.3.9}$$

Combining (4.3.8) with (4.3.9) we obtain (4.3.4).

To prove the rigidity statement, we start by observing that, if the equality (4.3.5) holds, then necessarily the right-hand side of (4.3.4) is null. In particular, $|\nabla\varphi|_g \equiv 1$ on N. Substituting this information in the Bochner formula (4.2.28) we obtain $|\nabla^2\varphi|_g \equiv 0$, hence we can apply Proposition 4.2.7 to conclude. □

4.3.2 Integral identities in the inner regions.

In this subsection, we deal with the case in which N is an inner region, that is,

$$\max_{S\in\pi_0(\partial N)} \kappa(S) \, > \, \sqrt{n} \, ,$$

and the pseudo-radial function $\Psi = \psi \circ u$ is defined by (4.2.4). This case is slightly more complicated than the outer one, and requires a generalization of the computations of the previous subsection. Let

$$Y_\alpha \, = \, \nabla|\nabla\varphi|_g^2 + \alpha\Delta_g\varphi\nabla\varphi \, ,$$

where $\alpha \in \mathbb{R}$. From the Bochner formula (4.2.28) and the equations in (4.2.22) we compute

$$\begin{aligned} \mathrm{div}_g(Y_\alpha) + \left[(n-2)u + \frac{\psi}{\dot\psi} + 3n\psi\dot\psi\right] \langle \nabla\varphi \,|\, Y_\alpha \rangle_g \, &= \, 2|\nabla^2\varphi|_g^2 + \alpha(\Delta_g\varphi)^2 + \\ + n\psi^2\dot\psi^2 \left[n(\alpha-1)(\alpha+2) - 2(n+\alpha+1)\frac{u}{\psi\dot\psi}\right] &|\nabla\varphi|_g^2(1-|\nabla\varphi|_g^2) \, . \end{aligned}$$

In order for the term $2|\nabla^2\varphi|^2_g + \alpha(\Delta_g\varphi)^2$ to be positive, we want $\alpha \geq -2/n$. Recalling

$$\frac{u}{\psi\dot\psi} = -[1-(n-2)m\psi^{-n}]\,,$$

we have

$$\begin{aligned}\mathrm{div}_g(Y_\alpha) + \left[(n-2)u + \frac{\psi}{\dot\psi} + (\alpha+2)n\psi\dot\psi\right]\langle\nabla\varphi\,|Y_\alpha\rangle_g &= 2|\nabla^2\varphi|^2_g + \alpha(\Delta_g\varphi)^2+\\ +n\psi^2\dot\psi^2\left[n(n\alpha+2)(\alpha+1) - 2(n+\alpha+1)(n-2)\frac{m}{\psi^n}\right]&|\nabla\varphi|^2_g(1-|\nabla\varphi|^2_g)\,.\end{aligned} \tag{4.3.10}$$

The term in square brackets is positive if and only if

$$\frac{\psi^n}{m} \geq 2(n-2)\frac{n+\alpha+1}{(n\alpha+2)(\alpha+1)}\,. \tag{4.3.11}$$

Since the term on the right hand side goes to zero as $\alpha \to \infty$, there exists an α big enough so that

$$\frac{r_-^n(m)}{m} = 2(n-2)\frac{n+\alpha+1}{(n\alpha+2)(\alpha+1)}\,. \tag{4.3.12}$$

Notice that the value of α that satisfies (4.3.12) is greater than or equal to 1 (in fact, if we set $\alpha = 1$ in (4.3.11) we have $\psi^n \geq (n-2)m$, which is never satisfied on M_-). If we choose α as in (4.3.12), we have that the square bracket above is positive for any $\psi \in [r_-(m), \big((n-2)m\big)^{1/n}]$. In particular, for that α we have

$$\mathrm{div}_g(Y_\alpha) + \left[(n-2)u + \frac{\psi}{\dot\psi} + (\alpha+2)n\psi\dot\psi\right]\langle\nabla\varphi\,|Y_\alpha\rangle_g \geq 0\,. \tag{4.3.13}$$

on the whole M_-. Now we choose

$$\gamma = \frac{u^{\alpha+1}\psi^{n\alpha+n-\alpha}}{\dot\psi^{\alpha+2}} = \frac{\psi^{n\alpha+n+2}}{u}\left[1-(n-2)m\psi^{-n}\right]^{\alpha+2} \geq 0\,, \tag{4.3.14}$$

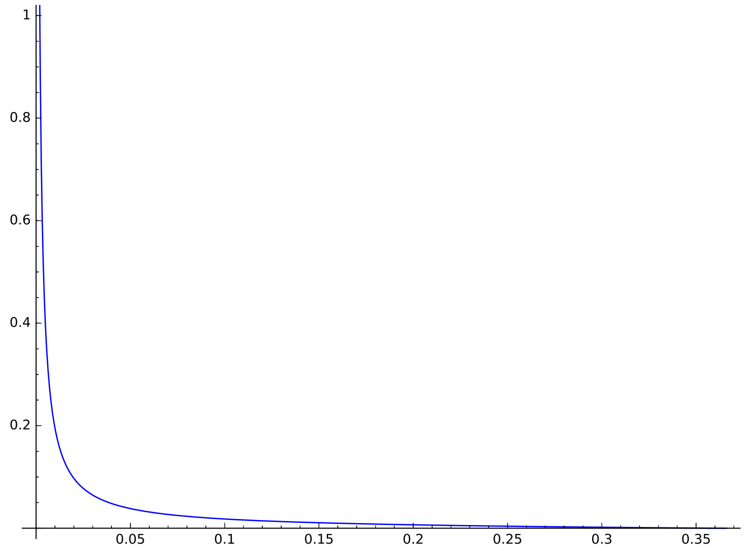


Figure 4.5: Relation between γ (on the y-axis) and u (on the x-axis), when N is an inner region, for $n=4$ and $m=3/32$ (which corresponds to $\alpha=2$).

(notice that $\gamma \geq 0$ when $\Psi = \psi \circ u$ is as in (4.2.4)). The relation between γ and u is shown in Figure 4.5. We compute

$$\begin{aligned}
\frac{\gamma'}{\gamma} &= \frac{\dot{\psi}^{\alpha+2}}{u^{\alpha+1}\psi^{n\alpha+n-\alpha}} \cdot \frac{du}{d\varphi} \cdot \Bigg[(\alpha+1)\frac{u^{\alpha}\psi^{n\alpha+n-\alpha}}{\dot{\psi}^{\alpha+2}} + (n\alpha+n-\alpha)\frac{u^{\alpha+1}\psi^{n\alpha+n-\alpha-1}\dot{\psi}}{\dot{\psi}^{\alpha+2}} \\
&\qquad\qquad - (\alpha+2)\frac{u^2\psi^{n\alpha+n-\alpha}\ddot{\psi}}{\dot{\psi}^{\alpha+3}}\Bigg] \\
&= -\frac{\dot{\psi}^{\alpha+1}}{u^{\alpha}\psi^{n\alpha+n-\alpha-1}}\Bigg[(\alpha+1)\frac{u^{\alpha}\psi^{n\alpha+n-\alpha}}{\dot{\psi}^{\alpha+2}} + (n\alpha+n-\alpha)\frac{u^{\alpha+1}\psi^{n\alpha+n-\alpha-1}\dot{\psi}}{\dot{\psi}^{\alpha+2}} \\
&\qquad\qquad - (\alpha+2)\frac{u^2\psi^{n\alpha+n-\alpha}\ddot{\psi}}{\dot{\psi}^{\alpha+3}}\Bigg] \\
&= -(\alpha+1)\frac{\psi}{\dot{\psi}} - (n\alpha+n-\alpha)u + (\alpha+2)\frac{u\psi\ddot{\psi}}{\dot{\psi}^2} \\
&= -(\alpha+1)\frac{\psi}{\dot{\psi}} - (n\alpha+n-\alpha)u + (\alpha+2)n\psi\dot{\psi} + (\alpha+2)(n-1)u + (\alpha+2)\frac{\psi}{\dot{\psi}} \\
&= (n-2)u + \frac{\psi}{\dot{\psi}} + (\alpha+2)n\psi\dot{\psi}\,.
\end{aligned}$$

From formulæ (4.3.10), (4.3.13) we deduce

$$\begin{aligned}
\operatorname{div}_g(\gamma Y_\alpha) &= 2|\nabla^2\varphi|_g^2 + \alpha(\Delta_g\varphi)^2 + \\
&\quad + n\psi^2\dot{\psi}^2\left[n(n\alpha+2)(\alpha+1) - 2(n+\alpha+1)(n-2)\frac{m}{\psi^n}\right]|\nabla\varphi|_g^2(1-|\nabla\varphi|_g^2) \\
&\geq 0\,. \qquad (4.3.15)
\end{aligned}$$

Integrating (4.3.15) on $\{\varphi \leq s\}$ for some value s, we obtain the following statement.

Proposition 4.3.2. *Let (M, g_0, u) be a solution to problem (4.1.1), let $N \subset M \setminus \mathrm{MAX}(u)$ be an inner region with virtual mass $m = \mu(N, g_0, u)$, and let Ψ, g and φ be defined by (4.2.4), (4.2.10) and (4.2.18). Then for any $\varphi_0 < s < \varphi_{\max}$ it holds*

$$\int\limits_{\{\varphi=s\}\cap N} |\nabla\varphi|_g \left[|\nabla\varphi|_g \mathrm{H}_g - \frac{\alpha+2}{2}\Delta_g\varphi \right] d\sigma_g = $$

$$= -\frac{1}{\gamma(s)} \int\limits_{\{\varphi_0<\varphi\le s\}\cap N} \gamma \left[\psi\dot\psi \left(\frac{1}{2}n(n\alpha+2)(\alpha+1) - (n+\alpha+1)(n-2)\frac{m}{\psi^n} \right) |\nabla\varphi|_g^2 \Delta_g\varphi \right.$$

$$\left. |\nabla^2\varphi|_g^2 + \frac{\alpha}{2}(\Delta_g\varphi)^2 \right] d\sigma_g \le 0\,, \tag{4.3.16}$$

where $\alpha > 1$ is the real number satisfying equation (4.3.12) and γ is the function defined by (4.3.14). Moreover, if the equality

$$\int\limits_{\{\varphi=s_0\}\cap N} |\nabla\varphi|_g \left[|\nabla\varphi|_g \mathrm{H}_g - \frac{\alpha+2}{2}\Delta_g\varphi \right] d\sigma_g = 0\,, \tag{4.3.17}$$

holds for some $\varphi_0 < s_0 \le \varphi_{\max}$, then the solution (M, g_0, u) is isometric to the Schwarzschild–de Sitter triple (2.5.3) with mass m.

Proof. Let us recall that u is an analytic function. In particular, also φ is analytic in the interior of N, hence its critical level sets are discrete. It follows that we can choose $0 < S < s < \varphi_0$, with S arbitrarily close to φ_0 and s arbitrarily close to $\varphi_{\max}$ such that both s and S are regular values for φ. Integrating (4.3.3) on $\{S \le \varphi \le s\}$ and using the Divergence Theorem 1.1.9 we obtain

$$\int\limits_{\{S\le\varphi\le s\}\cap N} \mathrm{div}_g(\gamma Y_\alpha) d\sigma_g =$$

$$= \int\limits_{\{\varphi=s\}\cap N} \gamma(s) \left\langle Y_\alpha \,\middle|\, \frac{\nabla\varphi}{|\nabla\varphi|_g} \right\rangle_g d\sigma_g - \int\limits_{\{\varphi=S\}\cap N} \gamma(S) \left\langle Y_\alpha \,\middle|\, \frac{\nabla\varphi}{|\nabla\varphi|_g} \right\rangle_g d\sigma_g\,. \tag{4.3.18}$$

First of all, with analogous computations to the ones used in Proposition 4.3.1, we obtain

$$\lim_{S\to\varphi_0} \gamma(S) \int_{\{\varphi=S\}\cap N} \left\langle Y_\alpha \,\middle|\, \frac{\nabla\varphi}{|\nabla\varphi|_g} \right\rangle_g d\sigma_g = 0\,.$$

Therefore, taking the limit as $S \to \varphi_0$ of (4.3.18) we obtain

$$\int\limits_{\{\varphi=s\}\cap N} \gamma(s) \left\langle Y_\alpha \,\middle|\, \frac{\nabla\varphi}{|\nabla\varphi|_g} \right\rangle_g d\sigma_g = \int\limits_{\{\varphi_0<\varphi\le s\}\cap N} \mathrm{div}_g(\gamma Y_\alpha)\, d\sigma_g \ge 0\,, \tag{4.3.19}$$

where in the last inequality we have used (4.3.15). Now we compute the integral on the left hand side. Recalling (4.2.25), we obtain

$$\left\langle Y_\alpha \,\middle|\, \frac{\nabla\varphi}{|\nabla\varphi|_g} \right\rangle_g = 2\frac{\nabla^2\varphi(\nabla\varphi, \nabla\varphi)}{|\nabla\varphi|_g} + \alpha\Delta_g\varphi|\nabla\varphi|_g =$$

$$= |\nabla\varphi|_g \left[-2|\nabla\varphi|_g \mathrm{H}_g + (\alpha+2)\Delta_g\varphi \right]\,. \tag{4.3.20}$$

Combining (4.3.19) with (4.3.20), we obtain (4.3.16).

The rigidity statement is proved in the same way as in Proposition 4.3.1. If the equality in (4.3.17) holds, then necessarily the right hand side of (4.3.16) is null. In particular, $|\nabla\varphi|_g \equiv 1$ on N. Substituting this information in the Bochner formula (4.2.28) we obtain $|\nabla^2\varphi|_g \equiv 0$, hence we can apply Proposition 4.2.7 to conclude. □

4.4 Proof of the area bounds

Here we translate the integral identities obtained in Section 4.3 in terms of u and g_0. Some computations will lead to the proof of Theorem 4.1.3 in the case where N is outer (Theorem 4.4.4) and inner (Theorem 4.4.9). As a consequence of the Gauss-Bonnet Formula we will then deduce Theorem 4.1.1 (see Theorems 4.4.3 and 4.4.8). We will also prove some more general statements, in the cases where N has more than one horizon.

4.4.1 The outer case.

Here we concentrate on the case where our region N is outer and translate Proposition 4.3.1, proved in Subsection 4.3.1, in terms of u, g_0.

Proposition 4.4.1. *Let (M, g_0, u) be a solution to problem (4.1.1), let $N \subset M \setminus \mathrm{MAX}(u)$ be an outer region with virtual mass $m = \mu(N, g_0, u)$, and let $\Psi = \psi \circ u$ be the pseudo-radial function defined by (4.2.3). Then, for any $0 < t < u_{\max}$ it holds*

$$\int\limits_{\{u=t\}\cap N} \frac{\psi^n}{|\dot\psi|}|\mathrm{D}u|\left[\psi|\mathrm{D}u|\mathrm{H} - (n-1)\frac{u^2}{\dot\psi} + \left(\frac{3}{2}nu\psi + (n-1)\frac{u^2}{\dot\psi}\right)\left(1 - \frac{\dot\psi^2}{u^2}|\mathrm{D}u|^2\right)\right]\mathrm{d}\sigma$$

$$= \int\limits_{\{u\geq t\}\cap N} \frac{u^2\psi^{n-1}}{|\dot\psi|^3}\left\{2\frac{\psi^2\dot\psi^2}{u^2}|\mathrm{D}^2u|^2 + 4n\frac{\psi\dot\psi^3}{u^2}\left(1+\frac{\psi\dot\psi}{u}\right)\mathrm{D}^2u(\mathrm{D}u,\mathrm{D}u) + n^2\psi^2\dot\psi^2 + \right.$$

$$\left. - 2n^2\left(\frac{\psi^2\dot\psi^2}{u^2} + \frac{\psi\dot\psi}{u}\right)\dot\psi^2|\mathrm{D}u|^2 + \left[3n^2\frac{\psi^2\dot\psi^2}{u^2} + 6n^2\frac{\psi\dot\psi}{u} + 2n(n-1)\right]\frac{\dot\psi^4}{u^2}|\mathrm{D}u|^4\right\}\mathrm{d}\sigma$$

$$\geq 0\,,$$

Moreover, if the equality holds for some $0 < t < u_{\max}$, then the solution (M, g_0, u) is isometric to the Schwarzschild–de Sitter triple (2.5.3) with mass m.

Proof. Writing inequality (4.3.4) in terms of u, g_0, recalling (4.2.27), we find

$$\int_{\{u=t\}\cap N} \frac{\psi^n}{\dot\psi}|\mathrm{D}u|\left[-\psi\,|\mathrm{D}u|\mathrm{H} + (n-1)\,\dot\psi|\mathrm{D}u|^2 - \frac{3}{2}n\psi\dot\psi\left(1 - \frac{\dot\psi^2}{u^2}|\mathrm{D}u|^2\right)\right]\mathrm{d}\sigma =$$

$$= \int\limits_{\{u\geq t\}\cap N} -\frac{u^2\psi^{n-1}}{\dot\psi^3}\left\{2\frac{\psi^2\dot\psi^2}{u^2}|\mathrm{D}^2u|^2 + 4n\frac{\psi\dot\psi^3}{u^2}\left(1+\frac{\psi\dot\psi}{u}\right)\mathrm{D}^2u(\mathrm{D}u,\mathrm{D}u) + n^2\psi^2\dot\psi^2 + \right.$$

$$\left. - 2n^2\left(\frac{\psi^2\dot\psi^2}{u^2} + \frac{\psi\dot\psi}{u}\right)\dot\psi^2|\mathrm{D}u|^2 + \left[3n^2\frac{\psi^2\dot\psi^2}{u^2} + 6n^2\frac{\psi\dot\psi}{u} + 2n(n-1)\right]\frac{\dot\psi^4}{u^2}|\mathrm{D}u|^4\right\}\mathrm{d}\sigma$$

$$\geq 0\,.$$

Since $u, \psi, \dot\psi$ are constant on $\{u = t\}$, and $\dot\psi$ is negative on the whole N, with some easy algebra we obtain the thesis. □

A special case of the above result, which is the one we are more interested in, is when the integral is done over the level set $\{u = 0\} \cap N = \partial N$. We obtain the following integral inequality.

Corollary 4.4.2. *Let (M, g_0, u) be a solution to problem (4.1.1), let $N \subset M \setminus \mathrm{MAX}(u)$ be an outer region with virtual mass $m = \mu(N, g_0, u)$. Then it holds*

$$\int_{\partial N} |\mathrm{D}u| \Big[\mathrm{R}^{\partial N} - (n-1)(n-2) r_+^{-2}(m) \\ + \big((n+2) + 2(n-1)(n-2) m r_+^{-n}(m)\big) \left(1 - \frac{|\mathrm{D}u|^2}{\max_{\partial N} |\mathrm{D}u|^2}\right) \Big] \mathrm{d}\sigma \geq 0 \,.$$

Moreover, if the equality holds for some $0 < t < u_{\max}$, then the solution (M, g_0, u) is isometric to the Schwarzschild–de Sitter triple (2.5.3) with mass m.

Proof. Recalling that $u \,\mathrm{Ric}(\nu,\nu) = -\mathrm{H}|\mathrm{D}u|$, the equation in Proposition 4.4.1 above can be written as

$$\int\limits_{\{u=t\}\cap N} \frac{u\psi^{n+1}}{|\dot\psi|} |\mathrm{D}u| \Big[-\mathrm{Ric}(\nu,\nu) - (n-1)\frac{u}{\psi\dot\psi} + \\ \left(\frac{3}{2}n + (n-1)\frac{u}{\psi\dot\psi}\right)\left(1 - \frac{\dot\psi^2}{u^2}|\mathrm{D}u|^2\right)\Big] \mathrm{d}\sigma = \\ = \int\limits_{\{u\geq t\}\cap N} -\frac{u^2\psi^{n-1}}{\dot\psi^3} \Big\{ 2\frac{\psi^2\dot\psi^2}{u^2}|\mathrm{D}^2 u|^2 + 4n\frac{\psi\dot\psi^3}{u^2}\left(1 + \frac{\psi\dot\psi}{u}\right)\mathrm{D}^2 u(\mathrm{D}u, \mathrm{D}u) + n^2\psi^2\dot\psi^2 - \\ - 2n^2\left[\frac{\psi^2\dot\psi^2}{u^2} + \frac{\psi\dot\psi}{u}\right]\dot\psi^2|\mathrm{D}u|^2 + \left[3n^2\frac{\psi^2\dot\psi^2}{u^2} + 6n^2\frac{\psi\dot\psi}{u} + 2n(n-1)\right]\frac{\dot\psi^4}{u^2}|\mathrm{D}u|^4\Big\}\mathrm{d}\sigma \\ \geq 0 \,. \tag{4.4.1}$$

Now we take the limit as $t \to 0^+$. Recall that, from (4.2.2) it follows that $u/\dot\psi$ has a finite limit as $t \to 0^+$, hence we find

$$\lim_{t\to 0^+} \int\limits_{\{u=t\}\cap N} |\mathrm{D}u| \Big[-\mathrm{Ric}(\nu,\nu) - (n-1)\frac{u}{\psi\dot\psi} + \\ + \left(\frac{3}{2}n + (n-1)\frac{u}{\psi\dot\psi}\right)\left(1 - \frac{\dot\psi^2}{u^2}|\mathrm{D}u|^2\right)\Big] \mathrm{d}\sigma \geq 0 \,,$$

where we remark that the equality holds if and only if the integrand in the right hand side of (4.4.1) is null everywhere on N (and this happens only when the solution is isometric to the Schwarzschild–de Sitter solution).

Using the Gauss-Codazzi equation we have $2\,\mathrm{Ric}(\nu,\nu) = \mathrm{R} - \mathrm{R}^{\partial N} = n(n-1) - \mathrm{R}^{\partial N}$, where $\mathrm{R}^{\partial N}$ is the scalar curvature of the metric induced by g_0 on

∂N. Using again (4.2.2) to explicitate the quantity $u/(\psi\dot{\psi})$, and recalling that $\{u=0\}\cap N=\partial N$, the inequality above rewrites as

$$\int_{\partial N}|\mathrm{D}u|\Big[\mathrm{R}^{\partial N}-(n-1)(n-2)(1+2m\psi^{-n})+ \\ +\big(n+2+2(n-1)(n-2)m\psi^{-n}\big)\left(1-\frac{\dot{\psi}^2}{u^2}|\mathrm{D}u|^2\right)\Big]\,d\sigma\,\geq\,0\,.$$

Now we recall that $\psi\equiv r_+(m)$ on ∂N (because N is an outer region), hence on ∂N it holds

$$1+2m\psi^{-n}=1+2mr_+^{-n}(m)=r_+^{-2}(m)$$

where in the last equality we have used $1-r_+^2(m)-2mr_+^{2-n}(m)=0$ by definition. Moreover, recalling the definition of the virtual mass m, we have

$$\frac{\dot{\psi}^2}{u^2}\max_{\partial N}|\mathrm{D}u|^2\,=\,1\,,$$

Substituting in the integral inequality above we have the thesis. □

In dimension $n=3$, the above result can be made more explicit using the Gauss-Bonnet formula.

Theorem 4.4.3. *Let (M,g_0,u) be a 3-dimensional solution to problem (4.1.1), and let $N\subset M\setminus\mathrm{MAX}(u)$ be an outer region with virtual mass $m=\mu(N,g_0,u)$. Then*

$$\frac{\sum_{i=1}^p\left[\left(\frac{\kappa_i}{\kappa_1}\right)^2-\frac{3}{2}r_+^2(m)\left(1-\left(\frac{\kappa_i}{\kappa_1}\right)^2\right)\right]\kappa_i|S_i|}{\sum_{i=1}^p\kappa_i}\leq 4\pi r_+^2(m)$$

where $\partial N=S_1\sqcup\cdots\sqcup S_p$ and $\kappa_1\geq\cdots\geq\kappa_p$ are the surface gravities of $S_1,\ldots,S_p$. Moreover, if the equality holds then ∂N is connected and (M,g_0,u) is isometric to the Schwarzschild–de Sitter solution with mass m.

Proof. For $n=3$, the formula in Corollary 4.4.2 rewrites as

$$\sum_{i=1}^p\int_{S_i}\kappa_i\left[\mathrm{R}^{S_i}-2r_+^{-2}(m)+\left[5+4mr_+^{-3}(m)\right]\left(1-\frac{\kappa_i^2}{\kappa_1^2}\right)\right]\,d\sigma\,\geq\,0\,.$$

Since $1-r_+^2(m)-2mr_+^{-1}(m)=0$ by definition, we compute $5+4mr_+^{-3}(m)=3+2r_+^{-2}(m)$. Moreover, from the Gauss-Bonnet formula, we have $\int_{S_i}\mathrm{R}^{S_i}d\sigma=4\pi\chi(S_i)$ for all $i=1,\ldots,p$. From [Amb15], we also know that each S_i is diffeomorphic to a sphere, hence $\chi(S_i)=2$. Substituting these informations inside formula above, with some manipulations we arrive to the thesis. □

In the case when ∂N is connected, the constancy of the quantity $|\mathrm{D}u|$ on the whole boundary allows to obtain the following stronger results.

Corollary 4.4.4. *Let (M, g_0, u) be a solution to problem (4.1.1), let N be a connected component of $M \setminus \mathrm{MAX}(u)$ with $\max_{\partial N} |\mathrm{D}u|/u_{\max} < \sqrt{n}$ and virtual mass m. If ∂N is connected, then it holds*

$$\int_{\partial N} \mathrm{R}^{\partial N}\, \mathrm{d}\sigma \,\geq\, (n-1)(n-2) r_+^{-2}(m) |\partial N| \,.$$

Moreover, if the equality holds for some $0 < t < u_{\max}$, then the solution (M, g_0, u) is isometric to the Schwarzschild–de Sitter triple (2.5.3) with mass m.

Proof. The result is an immediate consequence of Corollary 4.4.2 and the fact that $|\mathrm{D}u|$ is constant on ∂N. □

Theorem 4.4.5. *Let (M, g_0, u) be a 3-dimensional solution to problem (4.1.1), and let N be a connected component of $M \setminus \mathrm{MAX}(u)$ with $\max_{\partial N} |\mathrm{D}u|/u_{\max} < \sqrt{3}$ and virtual mass m. If ∂N is connected, then ∂N is diffeomorphic to $\mathbb{S}^2$ and it holds*

$$|\partial N| \,\leq\, 4\pi r_+^2(m) \,.$$

Moreover, if the equality holds, then the solution (M, g_0, u) is isometric to the Schwarzschild–de Sitter triple (2.5.3) with mass m.

Proof. Substituting $n = 3$ in Corollary 4.4.4 and using the Gauss-Bonnet formula, we immediately obtain

$$4\pi\chi(\partial N) \,\geq\, 2 r_+^{-2}(m)\, |\partial N| \,.$$

In particular, $\chi(\partial N)$ has to be positive, hence ∂N is necessarily a sphere and we obtain the thesis. □

4.4.2 The inner case.

Here we proceed as in Subsection 4.4.1 to prove analogous integral identities when N is an inner region.

Proposition 4.4.6. *Let (M, g_0, u) be a solution to problem (4.1.1), let $N \subset M \setminus \mathrm{MAX}(u)$ be an inner region with virtual mass $m = \mu(N, g_0, u)$, and let $\Psi = \psi \circ u$ be the pseudo-radial function defined by (4.2.4). For any $0 < t < u_{\max}$ it holds*

$$\begin{aligned}
&\int\limits_{\{u=t\}\cap N} \frac{u^{\alpha-1}\psi^{n\alpha-\alpha+1}}{\dot\psi^{\alpha}} |\mathrm{D}u| \bigg[\psi |\mathrm{D}u| \mathrm{H} - (n-1)\frac{u^2}{\dot\psi} + \\
&\qquad\qquad + u\psi\left(\frac{\alpha+2}{2} n + (n-1)\frac{u}{\psi\dot\psi}\right)\left(1 - \frac{\dot\psi^2}{u^2}|\mathrm{D}u|^2\right)\bigg]\, \mathrm{d}\sigma \,= \\
&= \int\limits_{\{u\geq t\}\cap N} \frac{u^{\alpha+1}\psi^{n\alpha-\alpha}}{\dot\psi^{\alpha+2}} \bigg\{ 2\frac{\psi^2\dot\psi^2}{u^2} |\mathrm{D}^2 u|^2 + 4n\frac{\psi\dot\psi^3}{u^2}\left(1 + \frac{\psi\dot\psi}{u}\right) \mathrm{D}^2 u(\mathrm{D}u, \mathrm{D}u) + \\
&\qquad + \alpha n^2 \psi^2 \dot\psi^2 + \left[n^2(\alpha+1)(\alpha-2)\frac{\psi^2\dot\psi^2}{u^2} - 2n(n+\alpha-1)\frac{\psi\dot\psi}{u} \right] \dot\psi^2 |\mathrm{D}u|^2 + \\
&+ \left[-n^2(\alpha-2)(\alpha+2)\frac{\psi^2\dot\psi^2}{u^2} + 2n(3n+\alpha-1)\frac{\psi\dot\psi}{u} + 2n(n-1) \right] \frac{\dot\psi^4}{u^2} |\mathrm{D}u|^4 \bigg\}\, \mathrm{d}\sigma \\
&\geq 0 \,,
\end{aligned}$$

where $\alpha > 1$ is the real number satisfying equation (4.3.12). Moreover, if the equality holds for some $0 < t < u_{\max}$, then the solution (M, g_0, u) is isometric to the Schwarzschild–de Sitter triple (2.5.3) with mass m.

Proof. Writing inequality (4.3.16) in terms of u, g_0, recalling (4.2.27), we find

$$\begin{aligned}
&\int\limits_{\{u=t\}\cap N} \frac{u^{\alpha-1}\psi^{n\alpha-\alpha+1}}{\dot\psi^{\alpha}}|\mathrm{D}u|\bigg[-\psi|\mathrm{D}u|\mathrm{H}+(n-1)\dot\psi|\mathrm{D}u|^2-\\
&\qquad\qquad\qquad -\frac{\alpha+2}{2}n\psi\dot\psi\left(1-\frac{\dot\psi^2}{u^2}|\mathrm{D}u|^2\right)\bigg]\mathrm{d}\sigma =\\
&= -\int\limits_{\{u\ge t\}\cap N} \frac{u^{\alpha+1}\psi^{n\alpha-\alpha}}{\dot\psi^{\alpha+2}}\bigg\{2\frac{\psi^2\dot\psi^2}{u^2}|\mathrm{D}^2u|^2+4n\frac{\psi\dot\psi^3}{u^2}\Big(1+\frac{\psi\dot\psi}{u}\Big)\mathrm{D}^2u(\mathrm{D}u,\mathrm{D}u)+\\
&\qquad +\alpha n^2\psi^2\dot\psi^2+\bigg[n^2(\alpha+1)(\alpha-2)\frac{\psi^2\dot\psi^2}{u^2}-2n(n+\alpha-1)\frac{\psi\dot\psi}{u}\bigg]\dot\psi^2|\mathrm{D}u|^2+\\
&+\bigg[-n^2(\alpha-2)(\alpha+2)\frac{\psi^2\dot\psi^2}{u^2}+2n(3n+\alpha-1)\frac{\psi\dot\psi}{u}+2n(n-1)\bigg]\frac{\dot\psi^4}{u^2}|\mathrm{D}u|^4\bigg\}\mathrm{d}\sigma\\
&\le 0\,,
\end{aligned}$$

Since $u, \psi, \dot\psi$ are constant on $\{u = t\}$, and $\dot\psi$ is positive on the whole N, with some easy algebra we obtain the thesis. ☐

As for the outer case, we are particularly interested to the above result applied to the level set $\{u = 0\} \cap N = \partial N$. We obtain the following.

Corollary 4.4.7. *Let (M, g_0, u) be a solution to problem (4.1.1), let $N \subset M \setminus \mathrm{MAX}(u)$ be an inner region with virtual mass $m = \mu(N, g_0, u)$. Then it holds*

$$\begin{aligned}
\int_{\partial N}|\mathrm{D}u|\Big[&\mathrm{R}^{\partial N}-(n-1)(n-2)r_-^{-2}(m)+\\
&+\big[(\alpha n+2)+2(n-1)(n-2)mr_-^{-n}(m)\big]\left(1-\frac{|\mathrm{D}u|^2}{\max_{\partial N}|\mathrm{D}u|^2}\right)\Big]\,\mathrm{d}\sigma \ge 0\,.
\end{aligned}$$

Moreover, if the equality holds for some $0 < t < u_{\max}$, then the solution (M, g_0, u) is isometric to the Schwarzschild–de Sitter triple (2.5.3) with mass m.

Proof. Recalling that $u\,\mathrm{Ric}(\nu,\nu) = -\mathrm{H}|\mathrm{D}u|$, the equation in Proposition 4.4.1

above can be written as

$$\int\limits_{\{u=t\}\cap N} \frac{u^{\alpha}\psi^{n\alpha-\alpha+2}}{\dot\psi^{\alpha}}|\mathrm{D}u|\bigg[-\mathrm{Ric}(\nu,\nu)-(n-1)\frac{u}{\psi\dot\psi}+ + \left(\frac{\alpha+2}{2}n+(n-1)\frac{u}{\psi\dot\psi}\right)\left(1-\frac{\dot\psi^2}{u^2}|\mathrm{D}u|^2\right)\bigg]\,\mathrm{d}\sigma =$$

$$= \int\limits_{\{u\geq t\}\cap N} \frac{u^{\alpha+1}\psi^{n\alpha-\alpha}}{\dot\psi^{\alpha+2}}\bigg\{2\frac{\psi^2\dot\psi^2}{u^2}|\mathrm{D}^2u|^2+4n\frac{\psi\dot\psi^3}{u^2}\left(1+\frac{\psi\dot\psi}{u}\right)\mathrm{D}^2u(\mathrm{D}u,\mathrm{D}u)+ + \alpha n^2\psi^2\dot\psi^2+\left[n^2(\alpha+1)(\alpha-2)\frac{\psi^2\dot\psi^2}{u^2}-2n(n+\alpha-1)\frac{\psi\dot\psi}{u}\right]\dot\psi^2|\mathrm{D}u|^2+ + \left[-n^2(\alpha-2)(\alpha+2)\frac{\psi^2\dot\psi^2}{u^2}+2n(3n+\alpha-1)\frac{\psi\dot\psi}{u}+2n(n-1)\right]\frac{\dot\psi^4}{u^2}|\mathrm{D}u|^4\bigg\}\mathrm{d}\sigma \geq 0\,. \tag{4.4.2}$$

Now we take the limit as $t\to 0^+$. Recall that, from (4.2.2), it follows that $u/\dot\psi$ has a finite limit as $t\to 0^+$, hence we find

$$\lim_{t\to 0^+}\int\limits_{\{u=t\}\cap N}|\mathrm{D}u|\bigg[-\mathrm{Ric}(\nu,\nu)-(n-1)\frac{u}{\psi\dot\psi} + \left(\frac{\alpha+2}{2}n+(n-1)\frac{u}{\psi\dot\psi}\right)\left(1-\frac{\dot\psi^2}{u^2}|\mathrm{D}u|^2\right)\bigg]\,\mathrm{d}\sigma \geq 0\,,$$

where we remark that the equality holds if and only if the integrand in the right hand side of (4.4.2) is null everywhere on N (and this happens only when the solution is isometric to the Schwarzschild–de Sitter solution).

Using the Gauss-Codazzi equation we have $2\,\mathrm{Ric}(\nu,\nu)=\mathrm{R}-\mathrm{R}^{\partial N}=n(n-1)-\mathrm{R}^{\partial N}$, where $\mathrm{R}^{\partial N}$ is the scalar curvature of the metric induced by g_0 on ∂N. Using again (4.2.2) to explicitate the quantity $\frac{u}{\psi\dot\psi}$, and recalling that $\{u=0\}=\partial N$, the inequality above rewrites as

$$\int_{\partial N}|\mathrm{D}u|\bigg[\mathrm{R}^{\partial N}-(n-1)(n-2)(1+2m\psi^{-n})+ + \left(\alpha n+2+2(n-1)(n-2)m\psi^{-n}\right)\left(1-\frac{\dot\psi^2}{u^2}|\mathrm{D}u|^2\right)\bigg]\,\mathrm{d}\sigma \geq 0\,.$$

Now we recall that $\psi\equiv r_-(m)$ on ∂N (because N is an inner region), hence on ∂N it holds

$$1+2m\psi^{-n}=1+2mr_-^{-n}(m)=r_-^{-2}(m)$$

where in the last equality we have used $1-r_-^2(m)-2mr_-^{2-n}(m)=0$ by definition. Moreover, by definition of virtual mass m, we have

$$\frac{\dot\psi^2}{u^2}\max_{\partial N}|\mathrm{D}u|^2 = 1\,,$$

Substituting in the integral inequality above we have the thesis. □

In dimension $n = 3$, the above formula can be made more explicit using the Gauss-Bonnet formula.

Theorem 4.4.8. *Let (M, g_0, u) be a 3-dimensional solution to problem (4.1.1), and let $N \subset M \setminus \mathrm{MAX}(u)$ be an inner region with virtual mass $m = \mu(N, g_0, u)$. Then*

$$\frac{\sum_{i=1}^{p}\left[\left(\frac{\kappa_i}{\kappa_1}\right)^2 - \frac{3}{2}\alpha r_-^2(m)\left(1 - \left(\frac{\kappa_i}{\kappa_1}\right)^2\right)\right]\kappa_i|S_i|}{\sum_{i=1}^{p}\kappa_i} \leq 4\pi r_-^2(m)$$

where $\partial N = S_1 \sqcup \cdots \sqcup S_p$ and $\kappa_1 \geq \cdots \geq \kappa_p$ are the surface gravities of $S_1, \ldots, S_p$. Moreover, if the equality holds then ∂N is connected and (M, g_0, u) is isometric to the Schwarzschild–de Sitter triple (2.5.3) with mass m.

Proof. For $n = 3$, the formula in Corollary 4.4.7 rewrites as

$$\sum_{i=1}^{p}\int_{S_i}\kappa_i\left[\mathrm{R}^{S_i} - 2r_-^{-2}(m) + \left[3\alpha + 2 + 4mr_-^{-3}(m)\right]\left(1 - \frac{\kappa_i^2}{\kappa_1^2}\right)\right]\mathrm{d}\sigma \geq 0\,.$$

Since $1 - r_-^2(m) - 2mr_-^{-1}(m) = 0$ by definition, we compute $3\alpha + 2 + 4mr_-^{-3}(m) = 3\alpha + 2r_-^{-2}(m)$. Moreover, from the Gauss-Bonnet formula, we have $\int_{S_i}\mathrm{R}^{S_i}\mathrm{d}\sigma = 4\pi\chi(S_i)$ for all $i = 1, \ldots, p$. From [Amb15], we also know that each S_i is diffeomorphic to a sphere, hence $\chi(S_i) = 2$. Substituting these informations inside formula above, with some manipulations we arrive to the thesis. □

In the case when ∂N is connected, the constancy of the quantity $|\mathrm{D}u|$ on the whole boundary allows to obtain the following stronger results.

Corollary 4.4.9. *Let (M, g_0, u) be a solution to problem (4.1.1), let N be a connected component of $M \setminus \mathrm{MAX}(u)$ with $\max_{\partial N}|\mathrm{D}u|/u_{\max} > \sqrt{n}$ and virtual mass m. If ∂N is connected, then it holds*

$$\int_{\partial N}\mathrm{R}^{\partial N}\,\mathrm{d}\sigma \geq (n-1)(n-2)r_-^{-2}(m)|\partial N|\,.$$

Moreover, if the equality holds for some $0 < t < u_{\max}$, then the solution (M, g_0, u) is isometric to the Schwarzschild–de Sitter triple (2.5.3) with mass m.

Proof. The result is an immediate consequence of Corollary 4.4.2 and the fact that $|\mathrm{D}u|$ is constant on ∂N. □

Theorem 4.4.10. *Let (M, g_0, u) be a 3-dimensional solution to problem (4.1.1), and let N be a connected component of $M \setminus \mathrm{MAX}(u)$ with $\max_{\partial N}|\mathrm{D}u|/u_{\max} > \sqrt{3}$ and virtual mass m. If ∂N is connected, then ∂N is diffeomorphic to $\mathbb{S}^2$ and it holds*

$$|\partial N| \leq 4\pi r_-^2(m)\,.$$

Moreover, if the equality holds, then the solution (M, g_0, u) is isometric to the Schwarzschild–de Sitter triple (2.5.3) with mass m.

Proof. Substituting $n = 3$ in Corollary 4.4.4 and using the Gauss-Bonnet formula, we immediately obtain

$$4\pi\chi(\partial N) \geq 2r_-^{-2}(m)\,|\partial N|\,.$$

In particular, $\chi(\partial N)$ has to be positive, hence ∂N is necessarily a sphere and we obtain the thesis. $\square$

4.5 Black Hole Uniqueness Theorem

This section is dedicated to the proof of the Black Hole Uniqueness results stated in Subsection 4.1.3. Again, we avoid to study the cylindrical case, that will be dealt with in Section 4.6.

In Lemma 4.2.11, we have proven a bound from below on the area of the boundary of N, in terms of the limit of $\Phi(s)$ as $s \to \varphi_0$. We recall that $\varphi = \varphi_0$ corresponds to the set $\overline{N} \cap \mathrm{MAX}(u)$, hence, in general, since we have no information on the behavior of φ near $\overline{N} \cap \mathrm{MAX}(u)$, we are not able to make use of this lower bound.

However, under the assumption that our triple is 2-sided, in Subsection 4.5.1 we will show that the additional information on the shape of our solution near the separating hypersurface $\Sigma \subseteq \mathrm{MAX}(u)$ will allow us to deduce a more explicit lower bound. Comparing it with the upper bounds already proved in Corollaries 4.4.4 and 4.4.9, we will deduce our uniqueness results. Furthermore, in Subsection 4.5.2 we will discuss possible improvements, and we will show some possible weakenings of the assumptions.

4.5.1 Analysis of 2-sided solutions.

In this subsection we will prove the Black Hole Uniqueness Theorem 4.1.8 stated in Subsection 4.1.3, in the case where $m_+ < m_{\max}$. In fact, the hypothesis $m_+ < m_{\max}$ will be necessary in all the results in this section, since it allows to use the functions Ψ, φ defined in the previous sections by formulæ (4.2.1), (4.2.18). The case $m_+ = m_{\max}$ requires a different conformal change and a different analysis, as the model solution will be the Nariai triple (2.5.5), and it will be studied in Section 4.6. We start by proving the following global estimate of the gradient.

Proposition 4.5.1. *Let (M, g_0, u) be a 2-sided solution to problem (4.1.1), and let*

$$m_+ = \mu(M_+, g_0, u)\,, \qquad m_- = \mu(M_-, g_0, u)$$

be the virtual masses of M_+ and M_-. Suppose $m_+ < m_{\max}$ and $m_+ \leq m_-$. Let $\Psi = \psi \circ u$ be the global pseudo-radial function (4.2.5), and let g and φ be defined by, (4.2.10) and (4.2.18), with respect to the parameter $m \in [m_+, m_-]$, $m \neq m_{\max}$. Then $|\nabla\varphi|_g \leq 1$ on the whole $M \setminus \mathrm{MAX}(u)$.

Proof. The proof is an easy adjustment of the proof of Proposition 4.2.8. Following the proof of Lemma 4.2.2, it is easily seen that, since $m \in [m_+, m_-]$, it holds

$$|\nabla \varphi|_g = \left| \frac{\mathrm{D}u}{\psi\left[1-(n-2)m\psi^{-n}\right]} \right| \leq 1$$

on the whole boundary $\partial M = \partial M_+ \sqcup \partial M_-$. The thesis follows applying the Minimum Principle to the elliptic inequality (4.2.31) on each connected component of M_+ and M_-. □

Proposition 4.5.1, and also the most part of the following results, work using as a parameter any number $m \in [m_+, m_-] \cap (0, m_{\max})$. However, the cleanest and strongest results are obtained when $m = m_+$, so in the following we limit to this case, for simplicity. As an easy consequence of Proposition 4.5.1, using the regularity assumption, that implies the regularity of the global pseudo-radial function Ψ, we obtain the following.

Proposition 4.5.2. *Let (M, g_0, u) be a 2-sided solution to problem (4.1.1), and let $\Sigma \subseteq \mathrm{MAX}(u)$ be the stratified hypersurface separating M_+ and M_-. Let also*

$$m_+ = \mu(M_+, g_0, u)\,, \qquad m_- = \mu(M_-, g_0, u)$$

be the virtual masses of M_+ and M_-, and suppose $m_+ < m_{\max}$. Suppose that the following conditions hold

mass compatibility	$m_+ \leq m_-$,
regularity assumption	F *is* $\mathscr{C}^2$ *along* Σ,

where F is the function defined as in (4.1.15). Then Σ is a $\mathscr{C}^2$ hypersurface and it holds

$$\begin{aligned} \mathrm{H} &= (n-1)\sqrt{\frac{n}{n-2}\left[\left(\frac{m_{\max}}{m_+}\right)^{\frac{2}{n}} - 1\right]}\,, \\ [(n-2)m_+]^{2/n}\left(\mathrm{R}^\Sigma + |\mathring{\mathrm{h}}|^2\right) &= (n-1)(n-2)\,, \end{aligned} \tag{4.5.1}$$

where R^Σ is the scalar curvature of the metric induced by g_0 on Σ and $\mathrm{H}, \mathring{\mathrm{h}}$ are the mean curvature and traceless second fundamental form of Σ with respect to g_0 and the unit normal pointing towards the interior of M_+.

Proof. Let us recall from Definition 4.2.4 that a function F is said to be $\mathscr{C}^k$ *along* Σ if $F \in \mathscr{C}^k(U)$, where $U \supset \Sigma$ is some open neighborhood of Σ in M. Let $\Psi = \psi \circ u$, g and φ be defined by (4.2.5), (4.2.10) and (4.2.18) with respect to the parameter $m = m_+$. From the assumptions we have that F is $\mathscr{C}^2$ along Σ, hence from Lemma 4.2.5 it follows that also the global pseudo-radial function $\Psi = \psi \circ u$ introduced in (4.2.5) is $\mathscr{C}^2$ along Σ. In turn, also the pseudo-affine function φ defined by (4.2.18) is $\mathscr{C}^2$ along Σ, and so is the metric g. In particular, the scalar curvature R_g is continuous along Σ, and from formula (4.2.23) we deduce that also $\dot\psi(1 - |\nabla\varphi|_g^2)$ has to be continuous along Σ. We also notice that

$|\nabla\varphi|_g \leq 1$ everywhere by Proposition 4.5.1, while $\dot\psi$ has positive sign on M_- and negative sign on M_+. Therefore, $\dot\psi(1-|\nabla\varphi|_g^2)$ has to change sign when passing through Σ, hence $\dot\psi(1-|\nabla\varphi|_g^2) = 0$ on Σ. In particular, $\Delta_g\varphi = 0$ and $|\nabla\varphi|_g = 1$ on Σ. It follows that $\Sigma \subseteq \{\varphi = \varphi_0\}$ is a $\mathscr{C}^2$ hypersurface. Moreover, $|\nabla\varphi|_g$ has a maximum on Σ, hence $\nabla|\nabla\varphi|_g^2 = 0$ on Σ. In particular, $\nabla^2\varphi(\nu_g,\nu_g) = \langle\nabla\varphi \,|\, \nabla|\nabla\varphi|_g^2\rangle_g / |\nabla\varphi|_g^2 = 0$.

Since φ has no critical points on Σ as proved above, the second fundamental form h_g and the mean curvature H_g of Σ with respect to g can be computed using formulæ (4.2.24) and (4.2.25). From the Gauss-Codazzi equation we find

$$\begin{aligned}\mathrm{R}_g^\Sigma &= \mathrm{R}_g - 2\mathrm{Ric}_g(\nu_g,\nu_g) - |\mathrm{h}_g|_g^2 + \mathrm{H}_g^2 \\ &= \mathrm{R}_g + 2\Big[(n-2)u + \frac{\psi}{\dot\psi}\Big]\nabla^2\varphi(\nu_g,\nu_g) + 2\Big[u - \frac{\psi}{\dot\psi}\Big]\Delta_g\varphi - |\mathrm{h}_g|_g^2 + \mathrm{H}_g^2,\end{aligned} \tag{4.5.2}$$

where $\nu_g = \nabla\varphi/|\nabla\varphi|_g = \nabla\varphi$ is the g-unit normal vector field to Σ, and in the second inequality we have used the first equation in (4.2.22).

Since we have proved above that $\Delta_g\varphi = \nabla^2\varphi(\nu_g,\nu_g) = 0$ on Σ, then from formula (4.2.25) we deduce

$$\mathrm{H}_g = 0, \tag{4.5.3}$$

on Σ, and from (4.2.23) we have $\mathrm{R}_g = (n-1)(n-2)$ on Σ. Plugging all these pieces of information inside (4.5.2), we obtain

$$\mathrm{R}_g^\Sigma = (n-1)(n-2) - |\mathrm{h}_g|_g^2, \tag{4.5.4}$$

Translating (4.5.3) in terms of g_0 recalling (4.2.27), and using the fact that $|\nabla\varphi|_g = |\dot\psi/u|\,|\mathrm{D}u| = 1$ on Σ, we obtain

$$\mathrm{H} = -\left[\frac{(n-1)u}{\psi}\right]_{|\Sigma} = (n-1)\sqrt{\frac{1}{[(n-2)m_+]^{2/n}} - \frac{n}{n-2}}\,.$$

Substituting this information in (4.2.26), we also find $|\mathrm{h}_g|_g^2 = [(n-2)m_+]^{2/n}|\mathring{\mathrm{h}}|^2$. Finally, noticing that $\mathrm{R}_g^\Sigma = [(n-2)m_+]^{2/n}\mathrm{R}^\Sigma$, where R^Σ is the scalar curvature of the metric induced by g_0 on Σ, from identity (4.5.4) we obtain

$$[(n-2)m_+]^{2/n}\left(\mathrm{R}^\Sigma + |\mathring{\mathrm{h}}|^2\right) = \mathrm{R}_g^\Sigma + |\mathrm{h}_g|_g^2 = (n-1)(n-2)\,.$$

This concludes the proof. □

The above result concludes the proof of formula (4.1.16) stated in Subsection 4.1.3, at least in the case $m_+ < m_{\max}$. As anticipated, the case $m_+ = m_{\max}$ will be discussed in Section 4.6.

The next result is obtained combining Proposition 4.5.2 with the technical Lemma 4.2.11, in order to obtain a lower bound on $|\partial M_+|$.

Proposition 4.5.3. *Let (M, g_0, u) be a 2-sided solution to problem (4.1.1), and let $\Sigma \subseteq \mathrm{MAX}(u)$ be the stratified hypersurface separating M_+ and M_-. Let also*

$$m_+ = \mu(M_+, g_0, u)\,, \qquad m_- = \mu(M_-, g_0, u)$$

be the virtual masses of M_+ and M_-, and suppose $m_+ < m_{\max}$. If the following conditions are satisfied *mass compatibility* $\qquad m_+ \leq m_-$,

regularity assumption $\qquad F$ *is* $\mathscr{C}^2$ *along* Σ,

where F *is the function defined in* (4.1.15)*, then it holds*

$$\frac{1}{[(n-2)m_+]^{\frac{n-3}{n}}}\int_\Sigma \frac{(\mathrm{R}^\Sigma + |\mathring{\mathrm{h}}|^2)\,\mathrm{d}\sigma}{(n-1)(n-2)} \,=\, \frac{|\Sigma|}{[(n-2)m_+]^{\frac{n-1}{n}}} \,\leq\, \frac{|\partial M_+|}{r_+^{n-1}(m_+)}\,.$$

Moreover, if the equality holds in the latter inequality, then (M, g_0, u) *is isometric to the Schwarzschild–de Sitter solution* (2.5.3) *with mass* $m_+ = m_-$.

Proof. The proof is just a collection of the previous results. From Lemma 4.2.11 and the second identity in (4.5.1), we get

$$[(n-2)m_+]^{\frac{2}{n}}\int_\Sigma \frac{\mathrm{R}^\Sigma + |\mathring{\mathrm{h}}|^2}{(n-1)(n-2)}\mathrm{d}\sigma \,=\, |\Sigma|\,,$$

We also recall that $|\nabla\varphi|_g \to 1$ as we approach Σ, as proven in Proposition 4.5.2 above. Therefore, from Lemma 4.2.11 we deduce

$$|\Sigma|_g \,\leq\, |\partial M_+|_g\,,$$

where we recall that the metric g is defined by $g = g_0/\Psi^2$. In particular, it holds

$$|\partial M_+|_g \,=\, \frac{|\partial M_+|}{r_+^{n-1}(m_+)}\,, \qquad |\Sigma|_g \,=\, \frac{|\Sigma|}{[(n-2)m_+]^{\frac{n-1}{n}}}\,.$$

Putting together these formulæ we easily obtain the thesis. □

If we also assume the hypothesis that ∂M_+ is connected, we can use Corollary 4.4.4 to obtain a bound from above on ∂M_+. Combining this bound with the bound from below given by Proposition 4.5.3, we obtain the chain of inequalities

$$\begin{aligned}\frac{1}{[(n-2)m_+]^{\frac{n-3}{n}}}\int_\Sigma \frac{(\mathrm{R}^\Sigma + |\mathring{\mathrm{h}}|^2)\,\mathrm{d}\sigma}{(n-1)(n-2)} \,\leq\, \frac{|\partial M_+|}{r_+^{n-1}(m_+)} \,\leq\, \\ \leq\, \frac{1}{r_+^{n-3}(m_+)}\int_{\partial M_+} \frac{\mathrm{R}^{\partial M_+}\,\mathrm{d}\sigma}{(n-1)(n-2)}\,.\end{aligned} \tag{4.5.5}$$

In dimensions $n \geq 4$, we are not able to improve on this. Instead, in the 3-dimensional case, we can obtain stronger results by combining the inequalities above with the Gauss-Bonnet formula. This leads to our first uniqueness theorem.

Theorem 4.5.4. *Let* (M, g_0, u) *be a* 3*-dimensional* 2*-sided solution to problem* (4.1.1)*, and let* $\Sigma \subseteq \mathrm{MAX}(u)$ *be the stratified hypersurface separating* M_+ *and* M_-*. Let also*

$$m_+ \,=\, \mu(M_+, g_0, u)\,, \qquad m_- \,=\, \mu(M_-, g_0, u)$$

be the virtual masses of M_+ *and* M_-*, and suppose* $m_+ < m_{\max}$*. If the following conditions are satisfied* *mass compatibility* $m_+ \leq m_-$,

regularity assumption F *is* $\mathscr{C}^2$ *along* Σ,

connected cosmological horizon ∂M_+ *is connected,*

where F *is the function defined as in* (4.1.15)*, then*

$$\sum_{i=1}^{k} \chi(\Sigma_i) \leq 2,$$

where $\Sigma_1, \ldots, \Sigma_k$ *are the connected components of* Σ. *Moreover, if the equality holds, then* (M, g_0, u) *is isometric to the Schwarzschild–de Sitter triple* (2.5.3) *with mass* $m_+ = m_-$.

If in addition to the above hypotheses, we also suppose the following

pinching assumption $\fint_\Sigma |\mathring{\mathrm{h}}|^2 \mathrm{d}\sigma < 2/m_+^{2/3}$,

where $\mathring{\mathrm{h}}$ *is the traceless part of the second fundamental form of* Σ *with respect to the metric* g_0*, then* (M, g_0, u) *is isometric to the Schwarzschild–de Sitter solution* (2.5.3) *with mass* $m_+ = m_-$.

Proof. In dimension $n = 3$, the chain of inequalities (4.5.5) tells us that

$$\int_\Sigma \mathrm{R}^\Sigma \, \mathrm{d}\sigma \leq \int_{\partial M_+} \mathrm{R}^{\partial M_+} \, \mathrm{d}\sigma,$$

and the equality holds if and only if (M, g_0, u) is isometric to the Schwarzschild–de Sitter solution (2.5.3). Recalling that Σ has no conical singularities as proved in Proposition 4.5.2, applying the Gauss-Bonnet formula to both sides of the above inequality, we obtain

$$4\pi \sum_{i=1}^{k} \chi(\Sigma_i) \leq 4\pi \chi(\partial M_+).$$

We recall from Theorem 4.4.5 that if ∂M_+ is connected then ∂M_+ is diffeomorphic to a sphere, hence the first part of the thesis follows.

We now move to prove the second part, namely we will show that, if the pinching assumption holds, then $\sum_{i=1}^{k} \chi(\Sigma_i) \geq 2$. Thanks to what we have just proven above, this will imply that (M, g_0, u) is isometric to the Schwarzschild–de Sitter solution. We start by noticing that, in dimension $n = 3$, the second formula in (4.5.1) rewrites as

$$m_+^{2/3} \mathrm{R}^\Sigma = 2 - m_+^{2/3} |\mathring{\mathrm{h}}|^2.$$

Integrating this identity on Σ and using our pinching condition, we get

$$\int_\Sigma \mathrm{R}^\Sigma \, \mathrm{d}\sigma = \frac{1}{m_+^{2/3}} \int_\Sigma \left(2 - m_+^{2/3} |\mathring{\mathrm{h}}|^2\right) \mathrm{d}\sigma > 0.$$

In particular, again from the Gauss-Bonnet formula it follows

$$\sum_{i=1}^{k} \chi(\Sigma_i) = \frac{1}{4\pi} \int_\Sigma \mathrm{R}^\Sigma \mathrm{d}\sigma > 0,$$

but $\sum_{i=1}^{k} \chi(\Sigma_i)$ can only assume even integer values, hence $\sum_{i=1}^{k} \chi(\Sigma_i) \geq 2$, as wished. □

The above theorem is the main result of this subsection. We conclude by noticing that some of the hypotheses can be relaxed if one has some informations on the critical points.

Corollary 4.5.5. *Let (M, g_0, u) be a 3-dimensional 2-sided solution to problem (4.1.1), and let $\Sigma \subseteq \mathrm{MAX}(u)$ be the stratified hypersurface separating M_+ and M_-. Let also*

$$m_+ = \mu(M_+, g_0, u), \qquad m_- = \mu(M_-, g_0, u)$$

be the virtual masses of M_+ and M_-, and suppose $m_+ < m_{\max}$. Suppose also that the hypothesis of mass compatibility ($m_+ \leq m_-$) and the regularity assumption (F is $\mathscr{C}^2$ along Σ) are in force, and that ∂M_+ is connected. If there are no critical points of u in the interior of M_+, then (M, g_0, u) is isometric to the Schwarzschild–de Sitter triple (2.5.3) with mass $m_+ = m_-$.

Proof. If ∂M_+ is connected, then we have proved in Theorem 4.4.5 that ∂M_+ is diffeomorphic to a sphere. Since there are no critical points inside M_+, this means that Σ is diffeomorphic to ∂M_+, hence $\chi(\Sigma) = 2$ and we conclude using Theorem 4.5.4. □

4.5.2 Generalizations of the Black Hole Uniqueness Theorem.

As anticipated in Remark 4.1.7, although the definition of 2-sided solution helps clarifying the exposition and the ideas behind the proof of our uniqueness result, all the power of that definition is not really necessary. In this subsection we are going to show two generalizations of Theorem 4.5.4 which only require relaxed versions of the 2-sided hypothesis.

Theorem 4.5.6. *Let (M, g_0, u) be a 3-dimensional solution to problem (4.1.1). Let*

$$M \setminus \mathrm{MAX}(u) = N_0 \sqcup N_1 \sqcup \cdots \sqcup N_K,$$

where $N_0, \ldots, N_K$ are connected submanifolds. Suppose that N_0 is outer or cylindrical with virtual mass

$$m_+ = \mu(N_0, g_0, u).$$

Denote by $\Sigma \subseteq \mathrm{MAX}(u)$ the (possibly disconnected) stratified hypersurface separating N_0 from $N_1 \sqcup \cdots \sqcup N_K$. If the following conditions are satisfied

mass compatibility	$m_+ \leq \liminf\{\mu(N_i, g_0, u) \,:\, N_i \text{ inner region}\}$,
regularity assumption	F *is* $\mathscr{C}^2$ *along* Σ,
connected cosmological horizon	∂N_0 *is connected,*

where F is the function defined by (4.1.15), then

$$\sum_{i=1}^{k} \chi(\Sigma_i) \leq 2,$$

where $\Sigma_1, \ldots, \Sigma_k$ are the connected components of Σ. Moreover, if the equality holds, then the triple (M, g_0, u) is isometric to either the Schwarzschild–de Sitter solution (2.5.3) with mass $m_+ < m_{\max}$ or to the Nariai solution (2.5.5).

If in addition to the above hypotheses, we also suppose the following

- *pinching assumption* $\qquad \fint_\Sigma |\mathring{\mathrm{h}}|^2 d\sigma < 2/[\mu(N_0, g_0, u)]^{2/3}$,

where $\mathring{\mathrm{h}}$ is the traceless part of the second fundamental form of Σ with respect to the metric g_0, then the triple (M, g_0, u) is isometric to either the Schwarzschild–de Sitter solution (2.5.3) with mass $m_+ < m_{\max}$ or to the Nariai solution (2.5.5).

Proof. If $m < m_{\max}$, the proof is analogous to the one of Theorem 4.5.4. To emphasize this analogy, let us set

$$M_+ := N_0\,, \qquad M_- := \overline{(N_1 \sqcup \cdots \sqcup N_K)} \setminus \Sigma\,,$$

and

$$m_+ := \mu(N_0, g_0, u)\,, \qquad m_- := \liminf\{\mu(N_i, g_0, u) \,:\, N_i \text{ inner region}\}\,.$$

We will focus only on the case $m_+ < m_{\max}$. The case $m_+ = m_{\max}$ can be proven in a similar fashion proceeding as discussed in Section 4.6.

Defining $\Psi = \psi \circ u$, g and φ as usual with respect to the parameter m_+, proceeding as in Proposition 4.5.1 one obtains again $|\nabla\varphi|_g \leq 1$ on the whole M. Now one can follow exactly the same steps discussed in Subsection 4.5.1 to obtain the thesis. □

It is worth remarking that in the above theorem it is not necessary to assume that the regions $N_1, \dots, N_K$ are inner. It is even possible for $N_1, \dots, N_K$ to be all outer, in which case $\liminf\{\mu(N_i, g_0, u) \,:\, N_i \text{ inner region}\} = +\infty$ and the mass compatibility assumption is automatically satisfied. A similar reasoning can be repeated focusing the analysis on the inner region. One obtains the following.

Theorem 4.5.7. *Let (M, g_0, u) be a 3-dimensional solution to problem (4.1.1). Let*

$$M \setminus \mathrm{MAX}(u) = N_0 \sqcup N_1 \sqcup \cdots \sqcup N_K\,,$$

where $N_0, \dots, N_K$ are connected submanifolds. Suppose that N_0 is inner or cylindrical with virtual mass

$$m_- = \mu(N_0, g_0, u)\,,$$

and that $N_1, \dots, N_K$ are outer or cylindrical. Denote by $\Sigma \subseteq \mathrm{MAX}(u)$ the (possibly disconnected) stratified hypersurface separating N_0 from $N_1 \sqcup \cdots \sqcup N_K$. If the following conditions are satisfied

mass compatibility $\qquad m_- \geq \limsup\{\mu(N_i, g_0, u) \,:\, i = 1, \dots, K\}$,

regularity assumption $\qquad F$ *is* $\mathscr{C}^2$ *along* Σ,

connected cosmological horizon $\qquad \partial N_0$ *is connected,*

where F is the function defined by (4.1.15), then

$$\sum_{i=1}^{k} \chi(\Sigma_i) \leq 2\,,$$

where $\Sigma_1, \dots, \Sigma_k$ are the connected components of Σ. Moreover, if the equality holds, then the triple (M, g_0, u) is isometric to either the Schwarzschild–de Sitter solution (2.5.3) with mass $m_- < m_{\max}$ or to the Nariai solution (2.5.5).

If in addition to the above hypotheses, we also suppose the following *pinching assumption* $\quad \fint_\Sigma |\mathring{\mathrm{h}}|^2 d\sigma < 2/m_-^{2/3}$,

where $\mathring{\mathrm{h}}$ is the traceless part of the second fundamental form of Σ with respect to the metric g_0, then the triple (M, g_0, u) is isometric to either the Schwarzschild–de Sitter solution (2.5.3) with mass $m_- < m_{\max}$ or to the Nariai solution (2.5.5).

Proof. Again, the proof is analogous to the one of Theorem 4.5.4. To emphasize this analogy, let us set

$$M_+ := \overline{(N_1 \sqcup \cdots \sqcup N_K)} \setminus \Sigma\,, \qquad M_- := N_0\,,$$

and

$$m_+ := \limsup\{\mu(N_i, g_0, u) \,:\, i = 1, \ldots, K\}\,, \qquad m_- := \mu(N_0, g_0, u)\,.$$

As above, we only consider the case $m_- < m_{\max}$, as the case $m_- = m_{\max}$ is handled as discussed in Section 4.6.

We define $\Psi = \psi \circ u$, g and φ as usual, this time with respect to the parameter m_-. Proceeding as in Proposition 4.5.1 one obtains again $|\nabla\varphi|_g \leq 1$ on the whole M. To obtain the thesis, one can follow the same steps discussed in Subsection 4.5.1, but working inside the domain M_- instead of M_+. We leave the details to the interested reader. □

4.6 The cylindrical case

In this section we deal with the case where the virtual mass of a region N is equal to $m_{\max}$. We notice that the metric and the static potential of the Schwarzschild–de Sitter solution (2.5.3) collapse as the mass m approaches $m_{\max}$. Nevertheless, it is well known (see for instance Subsection 2.5.3) that, if one rescales the static potential and the coordinates during the limit process in order to avoid singularities, then the limit of the Schwarzschild–de Sitter solution as the mass m approaches $m_{\max}$ can be seen to be the Nariai triple (2.5.5). Therefore, in this section, the Nariai triple will play the role of the reference model. While the following computations are different from the ones shown in the preceding sections, the ideas and the conclusions will be analogue.

Normalization 2. *According to the Nariai solution, throughout all this section, the static potential u is normalized in such a way that $u_{\max} := \max_M(u) = 1$.*

4.6.1 Conformal reformulation.

Let (M, g_0, u) be a solution to system (4.1.1), and let N be a connected component of $M \setminus \mathrm{MAX}(u)$ such that $\max_{\partial N} |\mathrm{D}u| = \sqrt{n}$. On N, consider the metric

$$g = \frac{n}{n-2}\, g_0\,. \tag{4.6.1}$$

We want to reformulate problem (4.1.1) in terms of g.

Remark 4.6.1. *We notice that this conformal change is analogue to the conformal change (4.2.10) (in fact, the value of the pseudo-radial function Ψ defined in Section 4.2.1 goes to $\sqrt{(n-2)/n}$ as $m \to m_{\max}$). In this case, the conformal change (4.6.1) is just a rescaling of the metric, hence it is not really necessary for the following analysis. However, we have preferred to introduce it, since it allows for an easier comparison between the following computations and the ones shown in the previous sections for $m \neq m_{\max}$.*

We fix local coordinates in M and we denote by $\Gamma^{\gamma}_{\alpha\beta}, G^{\gamma}_{\alpha\beta}$ the Christoffel symbols of g, g_0. It is clear that $\Gamma^{\gamma}_{\alpha\beta} = G^{\gamma}_{\alpha\beta}$. Denote by ∇, Δ_g the Levi-Civita connection and the Laplace-Beltrami operator of g. For every $z \in \mathscr{C}^\infty$, we compute

$$\nabla^2_{\alpha\beta} z = \mathrm{D}^2_{\alpha\beta} z \tag{4.6.2}$$

$$\Delta_g z = \frac{n-2}{n}\,\Delta z \tag{4.6.3}$$

Moreover, since the Ricci tensor is invariant under rescaling, we have $\mathrm{Ric}_g = \mathrm{Ric}$. Consider now the function

$$\varphi \,=\, \frac{\arcsin(u)}{\sqrt{n-2}}\,. \tag{4.6.4}$$

Since u is normalized in such a way that $u_{\max} = 1$, the function φ is well defined, is zero on ∂N and goes to $\pi/(2\sqrt{n-2})$ when we approach $\mathrm{MAX}(u)$. Moreover, the gradient and hessian of φ satisfy the following identities

$$|\nabla \varphi|^2_g \,=\, \frac{1}{n}\,\frac{|\mathrm{D}u|^2}{1-u^2}\,, \tag{4.6.5}$$

$$\nabla^2 \varphi \,=\, \frac{1}{\sqrt{n-2}\sqrt{1-u^2}}\left[\mathrm{D}^2 u \,+\, \frac{u}{1-u^2}\, du \otimes du\right]. \tag{4.6.6}$$

Some more calculations show that, with respect to (φ, g), the equations in (4.1.1) rewrites in N as

$$\begin{cases} \mathrm{Ric}_g = \dfrac{\sqrt{n-2}}{\tan(\sqrt{n-2}\,\varphi)}\,\nabla^2\varphi - (n-2)d\varphi\otimes d\varphi + (n-2)\,g, & \text{in } N \\ \Delta_g \varphi = -\sqrt{n-2}\,\tan(\sqrt{n-2}\varphi)\left(1 - |\nabla\varphi|^2_g\right), & \text{in } N \\ \varphi = 0, & \text{on } \partial N \\ \varphi = \varphi_0 := \dfrac{\pi}{2\sqrt{n-2}} & \text{on } \overline{N} \cap \mathrm{MAX}(u). \end{cases} \tag{4.6.7}$$

We observe that, since g is just a rescaling of g_0, we have $\mathrm{Ric}_g = \mathrm{Ric}$. In particular the scalar curvature of g is constant and more precisely

$$\mathrm{R}_g = (n-1)(n-2)\,. \tag{4.6.8}$$

We can also prove the analogue of Proposition 4.2.7.

Proposition 4.6.2. *Let (M, g_0, u) be a solution to problem (4.1.1), and let N be a cylindrical region. Let $g = [n/(n-2)]g_0$ and let φ be the pseudo-affine function defined by (4.6.4).*

If $\nabla^2\varphi \equiv 0$ and $|\nabla\varphi|_g \equiv 1$ on N, then (M, g_0, u) is covered by the Nariai solution (2.5.5).

Proof. Proceeding as in the proof of Proposition 4.2.7 one shows that (N, g_0, u) is isometric to a region (M^N_+, g^N_0, u^N) of a Nariai solution (M^N, g^N_0, u^N). We then distinguish two cases, depending on whether the hypersurface $\Sigma = \overline{N} \cap \mathrm{MAX}(u)$ is two-sided or one-sided.

- If Σ is two sided, then one can proceed exactly as in Proposition 4.2.7 to prove that the isometry extends beyond Σ. Therefore, (M, g_0, u) is isometric to the Nariai solution (2.5.5).

- If Σ is one sided then, reasoning as in Proposition 4.3.1, we have that $(M, g_0, u) = (\overline{N}, g_0, u)$ is isometric to $(\overline{M}^N_+, g^N_0, u^N)/\sim$, where $\sim$ is a relation on the points of
$$\mathrm{MAX}^N(u) = \{p \in M^N : u^N(p) = u_{\max}\} \subset \partial\overline{M}^N_+ .$$
induced by an involution of $\mathrm{MAX}^N(u)$. But, up to a rescaling, $\mathrm{MAX}^N(u)$, with the metric induced by g^N_0 is isometric to an Einstein manifold (E, g_E), hence the relation $\sim$ gives rise to an isometric involution $\iota_\sim : E \to E$. But then one can check that
$$(\overline{M}^N_+, g^N_0, u^N)/\sim = (M^N, g^N_0, u^N)/\iota$$
where $\iota : M^N \to M^N$ is the involution defined, for any $(r, x) \in [0, \pi] \times E = M^N$, by
$$\iota(r, x) = (\pi - r, \iota_\sim(x)) .$$
In particular, $(\overline{M}^N_+, g^N_0, u^N)/\sim$ is covered by the Nariai solution (2.5.5) with fiber E, and so the same holds for our initial manifold (M, g_0, u).

This concludes the proof. $\square$

Proceeding as in Subsection 4.2.4, from identity (4.6.6) one can also prove the following formulæ for the second fundamental form and mean curvature of a level set $\{\varphi = s\}$

$$\mathrm{h}_g\,|\nabla\varphi|_g = \frac{1}{\sqrt{n-2}}\,\frac{|\mathrm{D}u|}{\sqrt{1-u^2}}\,\mathrm{h}, \qquad \mathrm{H}_g\,|\nabla\varphi|_g = \frac{\sqrt{n-2}}{n}\,\frac{|\mathrm{D}u|}{\sqrt{1-u^2}}\,\mathrm{H}. \tag{4.6.9}$$

Furthermore, starting from the Bochner formula and using the equations in (4.6.7), we find

$$\begin{aligned}\Delta_g|\nabla\varphi|^2_g - \sqrt{n-2}\left[\frac{1+2\tan^2(\sqrt{n-2}\,\varphi)}{\tan(\sqrt{n-2}\,\varphi)}\right]\langle\nabla|\nabla\varphi|^2_g\,|\,\nabla\varphi\rangle_g &= \\ = 2|\nabla^2\varphi|^2_g - 2(n-2)\tan^2(\sqrt{n-2}\,\varphi)\,|\nabla\varphi|^2_g\,(1-|\nabla\varphi|^2_g)\end{aligned} \tag{4.6.10}$$

Let $w = \beta(1 - |\nabla\varphi|_g^2)$, where $\beta = |\cos(\sqrt{n-2}\,\varphi)|$. With computations analogue to the ones shown in Subsection 4.2.5, we arrive to the following equation

$$\begin{aligned} \Delta_g w \,-\, \sqrt{n-2}\,\frac{1}{\tan(\sqrt{n-2}\,\varphi)}\langle\nabla\varphi\,|\,\nabla w\rangle \,-\\ -\,(n-2)\,\tan^2(\sqrt{n-2}\,\varphi)\,\Big[(n+2)|\nabla\varphi|_g^2 + (n-2)\Big]\,w \,=\\ =\,-2\,|\cos(\sqrt{n-2}\,\varphi)|\,\left[|\nabla^2\varphi|_g^2 - \frac{(\Delta_g\varphi)^2}{n}\right] \,\leq\, 0\,. \end{aligned} \tag{4.6.11}$$

In particular, we can apply a Minimum Principle and find the following analogue of Proposition 4.2.8.

Proposition 4.6.3. *Let (M, g_0, u) be a solution to problem (4.1.1), let N be a connected component of $M \setminus \mathrm{MAX}(u)$ with virtual mass $m = m_{\max}$, and let g, φ be defined by, (4.6.1), (4.6.4). Then $|\nabla\varphi|_g \leq 1$ on the whole N.*

Proof. The proof is completely analogue to the proof of Proposition 4.2.8 for the case $m \neq m_{\max}$, so we will not give all the details.

Since $\max_{\partial N}|Du| = \sqrt{n}$, from (4.6.5) we deduce $w \geq 0$ on ∂N. Moreover, again from (4.6.5), and Lemma 4.2.3, we have that w goes to zero as we approach $\mathrm{MAX}(u)$. In particular, since $\cos(\sqrt{n-2}\varphi) \to 0$ as $\varphi \to \varphi_0$, we have $w \to 0$ as we approach $\mathrm{MAX}(u)$. In particular, for any $\varepsilon > 0$ we can find a small neighborhood U_ε of $\mathrm{MAX}(u)$ such that $w \geq -\varepsilon$ on $N \setminus U_\varepsilon$. The thesis follows applying the Minimum Principle in $N \setminus U_\varepsilon$, and then letting ε and the volume of U_ε go to zero. □

Now we consider the function

$$\Phi(s) = \int_{\{\varphi = s\}} |\nabla\varphi|_g \, \mathrm{d}\sigma_g\,. \tag{4.6.12}$$

which is defined on $s \in [0, \varphi_0]$, where we recall that we have set $\varphi_0 = \pi/(2\sqrt{n-2})$. Proceeding as in the proof of Proposition 4.2.10, as an application of Proposition 4.6.3 one can prove the following monotonicity result for Φ.

Proposition 4.6.4. *Let (M, g_0, u) be a solution to problem (4.1.1), let $N \subseteq M \setminus \mathrm{MAX}(u)$ be a cylindrical region, and let $\Phi(s)$ be the function defined by (4.6.12), with respect to the metric g and the pseudo-affine function φ defined by (4.6.1) and (4.6.4). Then the function $\Phi(s)$ is monotonically nonincreasing. Moreover, if $\Phi(s_1) = \Phi(s_2)$ for two different values $0 \leq s_1 < s_2 < \varphi_0$, then the solution (M, g_0, u) is covered by the Nariai triple (2.5.5).*

If the limit of $\Phi(s)$ exists as $s \to \varphi_0$, then this limit is finite since $|\nabla\varphi|_g$ is bounded (this is a consequence of Proposition 4.2.8) and the level sets are compact. Therefore, from the monotonicity of Φ we can deduce the following global monotonicity property, which is the analogue of Proposition 4.6.5 and is proved in the same way.

Lemma 4.6.5. *Let (M, g_0, u) be a solution to problem (4.1.1), let $N \subseteq M \setminus \mathrm{MAX}(u)$ be a cylindrical region, and let g and φ be defined by (4.6.1), (4.6.4). Then*

$$|\partial N|_g \geq \lim_{s \to \varphi_0} \int_{\{\varphi = s\}} |\nabla \varphi|_g \, d\sigma_g \, .$$

Moreover, if the equality holds, then (M, g_0, u) is covered by the Nariai solution (2.5.5).

In order to make use of Lemma 4.6.5, we need some information on the set $\mathrm{MAX}(u)$ and on the behavior of $\nabla\varphi$ at the limit $\varphi \to \varphi_0$. In Section 4.5 we will see how to recover some more explicit information from Lemma 4.2.11 in the case where our solution is 2-sided according to Definition 4.1.6.

4.6.2 Integral identities.

Consider the vector field $Y = \nabla|\nabla\varphi|_g^2 + \Delta_g\varphi\nabla\varphi$. Starting from the Bochner formula (4.6.10), we easily compute

$$\mathrm{div}_g(Y) - \sqrt{n-2}\left[\frac{1 + 3\tan(\sqrt{n-2}\varphi)}{\tan(\sqrt{n-2}\varphi)}\right]\langle \nabla\varphi \,|\, Y\rangle_g = 2|\nabla^2\varphi|_g^2 + (\Delta_g\varphi)^2 \geq 0 \, .$$

If we introduce the function

$$\gamma = \frac{\cos^3(\sqrt{n-2}\,\varphi)}{\sin(\sqrt{n-2}\,\varphi)} \, , \tag{4.6.13}$$

the identity above can be rewritten as

$$\mathrm{div}_g(\gamma Y) = \gamma\left[2\,|\nabla^2\varphi|_g^2 + (\Delta_g\varphi)^2\right] \geq 0 \, . \tag{4.6.14}$$

As an application of the Divergence Theorem, we obtain the following result, which is the analogue of Propositions 4.3.1 and 4.3.2.

Proposition 4.6.6. *Let (M, g_0, u) be a solution to problem (4.1.1), let N be a connected component of $M \setminus \mathrm{MAX}(u)$ with virtual mass $m = m_{\max}$, and let φ be defined by (4.6.4). For any $0 \leq s < \varphi_0$ it holds*

$$\begin{aligned} &\int_{\{\varphi = s\}} |\nabla\varphi|_g \left[|\nabla\varphi|_g \mathrm{H}_g - \frac{3}{2}\Delta_g\varphi\right] d\sigma_g = \\ &\qquad = \frac{1}{\gamma(s)} \int_{\{s \leq \varphi < \varphi_0\}} \gamma\left[|\nabla^2\varphi|_g^2 + \frac{1}{2}(\Delta_g\varphi)^2\right] d\sigma_g \geq 0 \, . \end{aligned} \tag{4.6.15}$$

where γ is the function defined by (4.6.13). Moreover, if the equality

$$\int_{\{\varphi = s_0\}} |\nabla\varphi|_g \left[|\nabla\varphi|_g \mathrm{H}_g - \frac{3}{2}\Delta_g\varphi\right] d\sigma_g = 0 \, , \tag{4.6.16}$$

holds for some $0 < s_0 < \varphi_0$, then the solution (M, g_0, u) is covered by the Nariai triple (2.5.5).

Proof. Let us recall that u is an analytic function. In particular, also φ is analytic in the interior of N, hence its critical level sets are discrete. It follows that we can choose $0 < s < S < \varphi_0$, with s arbitrarily close to 0 and S arbitrarily close to φ_0 such that both s and S are regular values for φ. Integrating (4.6.14) on $\{s \leq \varphi \leq S\}$ and using the Divergence Theorem 1.1.9 we obtain

$$\begin{aligned}\int\limits_{\{s\leq\varphi\leq S\}\cap N} \mathrm{div}_g(\gamma Y)\mathrm{d}\sigma_g &= \\ &= \int\limits_{\{\varphi=S\}\cap N} \gamma(S)\Big\langle Y\,\Big|\,\frac{\nabla\varphi}{|\nabla\varphi|_g}\Big\rangle_g \mathrm{d}\sigma_g - \int\limits_{\{\varphi=s\}\cap N} \gamma(s)\Big\langle Y\,\Big|\,\frac{\nabla\varphi}{|\nabla\varphi|_g}\Big\rangle_g \mathrm{d}\sigma_g\,. \qquad (4.6.17)\end{aligned}$$

First of all, we notice that, proceeding as in Proposition 4.3.1, we get

$$\lim_{S\to\varphi_0} \gamma(S)\int_{\{\varphi=S\}\cap N} \Big\langle Y\,\Big|\,\frac{\nabla\varphi}{|\nabla\varphi|_g}\Big\rangle_g \mathrm{d}\sigma_g = 0\,. \qquad (4.6.18)$$

Therefore, taking the limit as $S \to \varphi_0$ of (4.6.17), we deduce

$$\int_{\{\varphi=s\}\cap N} \gamma(s)\Big\langle Y\,\Big|\,\frac{\nabla\varphi}{|\nabla\varphi|_g}\Big\rangle_g \mathrm{d}\sigma_g = -\int_{\{s\leq\varphi<\varphi_0\}\cap N} \mathrm{div}_g(\gamma Y)\mathrm{d}\sigma_g \leq 0\,, \qquad (4.6.19)$$

where in the last inequality we have used (4.6.14). Now we compute the integral on the left hand side. From (4.6.9) we find

$$\begin{aligned}\Big\langle Y\,\Big|\,\frac{\nabla\varphi}{|\nabla\varphi|_g}\Big\rangle_g &= \\ &= 2\frac{\nabla^2\varphi(\nabla\varphi,\nabla\varphi)}{|\nabla\varphi|_g} + \Delta_g\varphi|\nabla\varphi|_g = |\nabla\varphi|_g\,\big(-2|\nabla\varphi|_g\mathrm{H}_g + 3\Delta_g\varphi\big)\,. \qquad (4.6.20)\end{aligned}$$

Combining (4.6.19) with (4.6.20) we obtain (4.6.15), as wished.

Concerning the rigidity statement, if the equality in (4.6.16) holds, then necessarily the right hand side of (4.6.15) is null. In particular, $|\nabla\varphi|_g \equiv 1$ on N. Substituting this information in the Bochner formula (4.6.10) we obtain $|\nabla^2\varphi|_g \equiv 0$, hence we can apply Proposition 4.6.2 to conclude. $\square$

4.6.3 Consequences.

Here we discuss the consequences of Proposition 4.6.6 proved above. First of all, translating it in terms of u, g_0, we obtain the following result.

Proposition 4.6.7. *Let (M, g_0, u) be a solution to problem (4.1.1) and let N be a connected component of $M \setminus \mathrm{MAX}(u)$ with virtual mass $m_{\max}$. For any $0 < t <$*

$u_{\max}$ *it holds*

$$\int\limits_{\{u=t\}\cap N} \frac{\sqrt{1-u^2}}{u}|\mathrm{D}u|\left[\frac{1}{n}|\mathrm{D}u|\mathrm{H}+\frac{3}{2}u\left(1-\frac{1}{n}\frac{|\mathrm{D}u|^2}{1-u^2}\right)\right]\,\mathrm{d}\sigma =$$

$$= \sqrt{\frac{n-2}{n}}\int\limits_{\{u\geq t\}\cap N}\frac{\sqrt{1-u^2}}{u}\left\{\frac{1}{n}|\mathrm{D}^2u|^2+\frac{2}{n}\frac{u}{1-u^2}\mathrm{D}^2u(\mathrm{D}u,\mathrm{D}u)+\right.$$

$$\left.+\frac{n}{2}u^2-\frac{|\mathrm{D}u|^2}{2(1-u^2)}+\frac{3}{2n}\frac{u^2|\mathrm{D}u|^4}{(1-u^2)^2}\right\}\mathrm{d}\sigma \geq 0\,.$$

Moreover, if the equality holds for some $0<t<u_{\max}$*, then the solution* (M, g_0, u) *is isometric to the Nariai triple* (2.5.5)*.*

As a consequence, we obtain the following result, that should be compared with the analogue Corollaries 4.4.2, 4.4.7.

Corollary 4.6.8. *Let* (M, g_0, u) *be a solution to problem* (4.1.1)*, let* N *be a connected component of* $M\setminus \mathrm{MAX}(u)$ *with* $\max_{\partial N}|\mathrm{D}u|/u_{\max}<\sqrt{n}$ *and virtual mass* m*. Then it holds*

$$\int_{\partial N}|\mathrm{D}u|\left[\mathrm{R}^{\partial N}-n(n-1)+3n\left(1-\frac{|\mathrm{D}u|^2}{\max_{\partial N}|\mathrm{D}u|^2}\right)\right]\,\mathrm{d}\sigma \geq 0\,.$$

Moreover, if the equality holds, then the solution (M, g_0, u) *is isometric to the Nariai triple* (2.5.5)*.*

Proof. Recalling that $u\,\mathrm{Ric}(\nu,\nu)=-\mathrm{H}|\mathrm{D}u|$, the equation in Proposition 4.6.7 above can be written as

$$\int\limits_{\{u=t\}\cap N}\sqrt{1-u^2}\,|\mathrm{D}u|\left[-\frac{1}{n}\mathrm{Ric}(\nu,\nu)+\frac{3}{2}\left(1-\frac{1}{n}\frac{|\mathrm{D}u|^2}{1-u^2}\right)\right]\,\mathrm{d}\sigma =$$

$$= \sqrt{\frac{n-2}{n}}\int\limits_{\{u\geq t\}\cap N}\frac{\sqrt{1-u^2}}{u}\left\{\frac{1}{n}|\mathrm{D}^2u|^2+\frac{2}{n}\frac{u}{1-u^2}\mathrm{D}^2u(\mathrm{D}u,\mathrm{D}u)+\right.$$

$$\left.+\frac{n}{2}u^2-\frac{|\mathrm{D}u|^2}{2(1-u^2)}+\frac{3}{2n}\frac{u^2|\mathrm{D}u|^4}{(1-u^2)^2}\right\}\mathrm{d}\sigma \geq 0\,. \quad (4.6.21)$$

Now, taking the limit as $t\to 0^+$, we find

$$\lim_{t\to 0^+}\int\limits_{\{u=t\}\cap N}|\mathrm{D}u|\left[-\frac{1}{n}\mathrm{Ric}(\nu,\nu)+\frac{3}{2}\left(1-\frac{1}{n}|\mathrm{D}u|^2\right)\right]\,\mathrm{d}\sigma \geq 0\,,$$

where we remark that the equality holds if and only if the integrand in the right hand side of (4.6.21) is null everywhere on N (and this happens only when the solution is isometric to the Nariai solution.

Using the Gauss-Codazzi equation we have $2\,\mathrm{Ric}(\nu,\nu)=\mathrm{R}-\mathrm{R}^{\partial N}=n(n-1)-\mathrm{R}^{\partial N}$, where $\mathrm{R}^{\partial N}$ is the scalar curvature of the metric induced by g_0 on ∂N, hence the inequality above rewrites as

$$\int_{\partial N}|\mathrm{D}u|\left[\mathrm{R}^{\partial N}-n(n-1)+3n\left(1-\frac{1}{n}|\mathrm{D}u|^2\right)\right]\,\mathrm{d}\sigma \geq 0\,.$$

Since $\max_{\partial N} |\mathrm{D}u|^2 = n$ (because N has virtual mass $m_{\max}$ and we recall that u is normalized so that $u_{\max} = 1$), we have proved the thesis. □

In dimension $n = 3$, the above formula can be made more explicit using the Gauss-Bonnet formula.

Theorem 4.6.9. *Let (M, g_0, u) be a 3-dimensional solution to problem (4.1.1), and let N be a connected component of $M \setminus \mathrm{MAX}(u)$ with virtual mass $m_{\max}$. Then*

$$\frac{\sum_{i=1}^{p}\left[\left(\frac{\kappa_i}{\kappa_1}\right)^2 - \frac{1}{2}\left(1 - \left(\frac{\kappa_i}{\kappa_1}\right)^2\right)\right]\kappa_i |S_i|}{\sum_{i=1}^{p}\kappa_i} \le \frac{4\pi}{3}$$

where $\partial N = S_1 \sqcup \cdots \sqcup S_p$ and $\kappa_1 \ge \cdots \ge \kappa_p$ are the surface gravities of $S_1, \ldots, S_p$. Moreover, if the equality holds then ∂N is connected and (M, g_0, u) is isometric to the Nariai triple (2.5.5).

Proof. For $n = 3$, the formula in Corollary 4.6.8 rewrites as

$$\sum_{i=1}^{p} \int_{S_i} \kappa_i \left[\mathrm{R}^{S_i} - 6 + 9\left(1 - \frac{\kappa_i^2}{\kappa_1^2}\right)\right] \mathrm{d}\sigma \ge 0\,.$$

From the Gauss-Bonnet formula, we have $\int_{S_i} \mathrm{R}^{S_i} \mathrm{d}\sigma = 4\pi\chi(S_i)$ for all $i = 1, \ldots, p$. From [Amb15], we also know that each S_i is diffeomorphic to a sphere, hence $\chi(S_i) = 2$. Substituting these informations inside formula above, with some manipulations we arrive to the thesis. □

In the case when ∂N is connected, the constancy of the quantity $|\mathrm{D}u|$ on the whole boundary allows to obtain the following stronger results.

Corollary 4.6.10. *Let (M, g_0, u) be a solution to problem (4.1.1), let N be a connected component of $M \setminus \mathrm{MAX}(u)$ with virtual mass $m_{\max}$. If ∂N is connected, then it holds*

$$\int_{\partial N} \mathrm{R}^{\partial N} \mathrm{d}\sigma \ge n(n-1)|\partial N|\,.$$

Moreover, if the equality holds, then the solution (M, g_0, u) is isometric to the Nariai triple (2.5.5).

Proof. The result is an immediate consequence of Corollary 4.6.8 and the fact that $|\mathrm{D}u|$ is constant on ∂N. □

Theorem 4.6.11. *Let (M, g_0, u) be a 3-dimensional solution to problem (4.1.1), and let N be a connected component of $M \setminus \mathrm{MAX}(u)$ with virtual mass $m_{\max}$. If ∂N is connected, then ∂N is diffeomorphic to $\mathbb{S}^2$ and it holds*

$$|\partial N| \le \frac{4\pi}{3}\,.$$

Moreover, if the equality holds, then the solution (M, g_0, u) is isometric to the Nariai triple (2.5.5).

Proof. Substituting $n = 3$ in Corollary 4.6.10 and using the Gauss-Bonnet formula, we immediately obtain

$$4\pi\chi(\partial N) \geq 6\,|\partial N|\,.$$

In particular, $\chi(\partial N)$ has to be positive, hence ∂N is necessarily a sphere and we obtain the thesis. ☐

4.6.4 Black Hole Uniqueness.

In this section we will complete the proof of Theorem 4.1.8, started in Theorem 4.5.4, by discussing the missing case $m_+ = m_{\max}$. To this end, on $M = \overline{M}_+ \cup \overline{M}_-$ we define the metric g as in (4.6.1), and the function φ as follows

$$\varphi = \begin{cases} \dfrac{\arcsin(u)}{\sqrt{n-2}}\,, & \text{on } M_+\,, \\ \dfrac{\pi - \arcsin(u)}{\sqrt{n-2}}\,, & \text{on } M_-\,. \end{cases} \tag{4.6.22}$$

The function φ defined here is equal to 0 on ∂M_+, it is equal to $\varphi_0 = \pi/(2\sqrt{n-2})$ on $\Sigma = \overline{M}_+ \cap \overline{M}_-$ and is equal to $\varphi_{\max} = \pi/\sqrt{n-2}$ on ∂M_-. Moreover, it is easily checked that φ, g satisfy the equations in (4.6.7) on M_+ and M_-. In particular, the elliptic inequality (4.6.11) is in charge on every connected component of M_+ and M_-, and this leads to the following global estimate for the gradient of φ (which is defined a priori only on $M_- \cup M_+$ and not on Σ).

Proposition 4.6.12. *Let (M, g_0, u) be a 2-sided solution to problem (4.1.1) such that the virtual masses $m_+ = \mu(M_+, g_0, u)$, $m_- = \mu(M_-, g_0, u)$ satisfy $m_+ = m_- = m_{\max}$, and let g, φ be defined by (4.6.1), (4.6.22). Then $|\nabla\varphi|_g \leq 1$ on the whole $M \setminus \mathrm{MAX}(u)$.*

Proof. The proof is an easy adjustment of the proof of Proposition 4.2.8. First of all, we notice that our function φ satisfies formula (4.6.5) hence, thanks to the assumption, we have

$$|\nabla\varphi|_g = \frac{1}{n}|\mathrm{D}u|^2 \leq 1$$

on the whole boundary $\partial M = \partial M_+ \sqcup \partial M_-$. The thesis follows applying the Minimum Principle to the elliptic inequality (4.6.11) on each connected component of M_+ and M_-. ☐

A second important remark is that the regularity of the function

$$F(x) = \begin{cases} \sqrt{1-u(x)} & \text{in } \overline{M}_+\,, \\ -\sqrt{1-u(x)} & \text{in } \overline{M}_-\,. \end{cases} \tag{4.6.23}$$

implies the regularity of φ.

Lemma 4.6.13. *Let (M, g_0, u) be a 2-sided solution to problem (4.1.1), and let φ be defined by (4.6.22). Then the function F defined by (4.6.23) is $\mathscr{C}^2$ along Σ if and only if φ is $\mathscr{C}^2$ along Σ.*

Proof. From the definition of φ, it is clear that it is enough to show that F is $\mathscr{C}^2$ if and only if $\arcsin(u)$ is $\mathscr{C}^2$. This is an easy exercise of analysis. □

As an easy consequence of the above results, we obtain the following.

Proposition 4.6.14. *Let (M, g_0, u) be a 2-sided solution to problem (4.1.1), let $\Sigma \subseteq \mathrm{MAX}(u)$ be the stratified hypersurface separating M_+ and M_-. Let also*

$$m_+ = \mu(M_+, g_0, u), \qquad m_- = \mu(M_-, g_0, u)$$

be the virtual masses of M_+ and M_-. Suppose that the following conditions hold

<u>*mass compatibility*</u> $\qquad m_+ = m_- = m_{\max}$,

<u>*regularity assumption*</u> $\qquad F$ *is $\mathscr{C}^2$ along Σ,*

where F is the function defined by (4.6.23). Then Σ is a $\mathscr{C}^2$ hypersurface and it holds

$$\begin{aligned} \mathrm{H} &= 0\,, \\ [(n-2)m_{\max}]^{2/n}\left(\mathrm{R}^\Sigma + |\mathring{\mathrm{h}}|^2\right) &= \frac{n-2}{n}\left(\mathrm{R}^\Sigma + |\mathring{\mathrm{h}}|^2\right) = (n-1)(n-2)\,, \end{aligned} \tag{4.6.24}$$

where R^Σ is the scalar curvature of the metric induced by g_0 on Σ and $\mathrm{H}, \mathring{\mathrm{h}}$ are the mean curvature and the traceless second fundamental form of Σ with respect to g_0 and the unit normal pointing towards M_+.

Proof. This proof follows the scheme of the proof of Proposition 4.5.2. Let us recall from Definition 4.2.4 that a function F is said to be $\mathscr{C}^k$ *along* Σ if $F \in \mathscr{C}^k(U)$, where $U \supset \Sigma$ is some open neighborhood of Σ in M. Let g, φ be defined by (4.6.1), (4.6.22). Since F is $\mathscr{C}^2$ along Σ by hypothesis, so is φ thanks to Lemma 4.6.13. Therefore $\Delta_g \varphi$ is continuous along Σ, thus from the second formula in (4.6.7) we deduce that also $\tan(\sqrt{n-2}\varphi)(1 - |\nabla\varphi|_g^2)$ can be extended to a continuous function along Σ. We also notice that $|\nabla\varphi|_g \leq 1$ everywhere by Proposition 4.6.12, whereas $\tan(\sqrt{n-2}\varphi)$ has positive sign on M_+ and negative sign on M_-. Therefore, $\tan(\sqrt{n-2}\varphi)(1 - |\nabla\varphi|_g^2)$ has to change sign when passing through Σ, hence $\tan(\sqrt{n-2}\varphi)(1 - |\nabla\varphi|_g^2) = 0$ on Σ. In particular, $\Delta_g \varphi = 0$ and $|\nabla\varphi|_g = 1$ on Σ. Since φ is $\mathscr{C}^2$ and $\nabla\varphi$ is nonzero on $\Sigma \subseteq \{\varphi = \varphi_0\}$, we have that Σ is a $\mathscr{C}^2$ hypersurface. Furthermore, $|\nabla\varphi|_g$ has a maximum on Σ, hence $\nabla|\nabla\varphi|_g^2 = 0$ on Σ. In particular, $\nabla^2\varphi(\nu_g, \nu_g) = \langle \nabla\varphi \,|\, \nabla|\nabla\varphi|_g^2\rangle_g / |\nabla\varphi|_g^2 = 0$, where $\nu_g = \nabla\varphi/|\nabla\varphi|_g = \nabla\varphi$ is the g-unit normal vector field to Σ, and substituting in the first formula in (4.6.7), we obtain $\mathrm{Ric}_g(\nu_g, \nu_g) = 0$ on Σ.

Since φ has no critical points on Σ as proved above, the second fundamental form h_g and the mean curvature H_g of Σ can be computed using formulæ (4.6.9). Moreover, since $\Delta_g \varphi = \nabla^2\varphi(\nu_g, \nu_g) = 0$ on Σ, from (4.6.9) we deduce

$$\mathrm{H}_g = 0\,, \tag{4.6.25}$$

on Σ. Moreover, from the Gauss-Codazzi equation we find

$$\begin{aligned} \mathrm{R}_g^\Sigma &= \mathrm{R}_g - 2\mathrm{Ric}_g(\nu_g, \nu_g) - |\mathrm{h}_g|_g^2 + \mathrm{H}_g^2 \\ &= \mathrm{R}_g - |\mathrm{h}_g|_g^2 \\ &= (n-1)(n-2) - |\mathrm{h}_g|_g^2\,, \end{aligned} \tag{4.6.26}$$

where in the last equality we have used from (4.6.8).

Translating (4.6.25) in terms of g_0 recalling (4.6.9), and using the fact that $|\nabla\varphi|_g^2 = (1/n)\,|Du|^2/(1-u^2) = 1$ on Σ, we obtain

$$\mathrm{H} = 0\,, \qquad |\mathring{\mathrm{h}}|^2 = |\mathrm{h}|^2 = \frac{n-2}{n}\,|\mathrm{h}_g|_g^2\,.$$

Finally, noticing that $\mathrm{R}_g^\Sigma = [(n-2)/n]\mathrm{R}^\Sigma$, where R^Σ is the scalar curvature of the metric induced by g_0 on Σ, from identity (4.6.26) we obtain

$$\begin{aligned}
[(n-2)m_{\max}]^{2/n}\left(\mathrm{R}^\Sigma + |\mathring{\mathrm{h}}|^2\right) &= \frac{n-2}{n}\left(\mathrm{R}^\Sigma + |\mathring{\mathrm{h}}|^2\right)\\
&= \mathrm{R}_g^\Sigma + |\mathrm{h}_g|_g^2\\
&= (n-1)(n-2)\,.
\end{aligned}$$

Thiis concludes the proof. □

The above proposition concludes the proof of the missing case $m_+ = m_{\max}$ of formula (4.1.16) stated in Subsection 4.1.3.

The next result follows combining Propositions 4.6.14, 4.6.4 and the results in Subsection 4.6.3.

Proposition 4.6.15. *Let (M, g_0, u) be a 2-sided solution to problem (4.1.1), and let $\Sigma \subseteq \mathrm{MAX}(u)$ be the stratified hypersurface separating M_+ and M_-. Let also*

$$m_+ = \mu(M_+, g_0, u)\,, \qquad m_- = \mu(M_-, g_0, u)$$

be the virtual masses of M_+ and M_-. If the following conditions are satisfied

mass compatibility $\qquad m_+ = m_- = m_{\max}$,

regularity assumption $\qquad F$ *is* $\mathscr{C}^2$ *along* Σ,

where F is the function defined by (4.6.23), then it holds

$$\int_\Sigma \frac{\mathrm{R}^\Sigma + |\mathring{\mathrm{h}}|^2}{n(n-1)}\,\mathrm{d}\sigma = |\Sigma| \le |\partial M_+|\,.$$

Moreover, if the equality holds, then (M, g_0, u) is isometric to the Nariai solution (2.5.5).

Proof. The proof is just a collection of the previous results. From the second identity in (4.6.24), we immediately get

$$\int_\Sigma \frac{\mathrm{R}^\Sigma + |\mathring{\mathrm{h}}|^2}{n(n-1)}\mathrm{d}\sigma = |\Sigma|\,.$$

Since $m_+ = m_{\max}$, we have $|\nabla\varphi|_g^2 = (1/n)|Du|^2 \le 1$ on ∂M_+. Moreover, we recall from Proposition 4.6.14 that $|\nabla\varphi|_g$, where g and φ are defined by (4.6.1) and (4.6.4) as usual, goes to 1 as we approach Σ. Therefore, from Lemma 4.6.5 we obtain

$$\left(\frac{n}{n-2}\right)^{\frac{n-1}{2}}|\Sigma| = |\Sigma|_g \le |\partial M_+|_g = \left(\frac{n}{n-2}\right)^{\frac{n-1}{2}}|\partial M_+|$$

This concludes the proof of the inequality. The rigidity statement follows from the corresponding rigidity statements in Proposition 4.6.4. □

If we also assume that ∂M_+ is connected, then we can combine Proposition 4.6.15 with Corollary 4.6.10 and we obtain the following inequality

$$\int_\Sigma \frac{\mathrm{R}^\Sigma + |\mathring{\mathrm{h}}|^2}{n(n-1)}\,\mathrm{d}\sigma \,\leq\, \int_{\partial M_+} \frac{\mathrm{R}^{\partial M_+}}{n(n-1)}\,\mathrm{d}\sigma\,. \tag{4.6.27}$$

In dimensions $n \geq 4$, we are not able to improve on this. Instead, in the 3-dimensional case, we can obtain stronger results by combining the inequality above with the Gauss-Bonnet formula. The first uniqueness theorem that we obtain is the following.

Theorem 4.6.16. *Let (M, g_0, u) be a 3-dimensional 2-sided solution to problem (4.1.1), and let $\Sigma \subseteq \mathrm{MAX}(u)$ be the stratified hypersurface separating M_+ and M_-. Let also*

$$m_+ \,=\, \mu(M_+, g_0, u)\,, \qquad m_- \,=\, \mu(M_-, g_0, u)$$

be the virtual masses of M_+ and M_-. If the following conditions are satisfied

mass compatibility $\qquad m_+ = m_- = m_{\max}$,

regularity assumption $\qquad F$ *is* $\mathscr{C}^2$ *along* Σ,

connected cylindrical horizon $\qquad \partial M_+$ *is connected,*

where F is the function defined by (4.6.23), then

$$\sum_{i=1}^k \chi(\Sigma_i) \,\leq\, 2\,,$$

where $\Sigma_1, \dots, \Sigma_k$ are the connected components of Σ. Moreover, if the equality holds, then (M, g_0, u) is isometric to the Nariai triple (2.5.5).

If in addition to the above hypotheses, we also suppose the following

pinching assumption $\qquad \fint_\Sigma \left|\mathring{\mathrm{h}}\right|^2 \mathrm{d}\sigma < 6$,

where $\mathring{\mathrm{h}}$ is the traceless part of the second fundamental form of Σ with respect to the metric g_0, then the triple (M, g_0, u) is isometric to the Nariai solution (2.5.5).

Proof. Inequality (4.6.27) tells us that

$$\int_\Sigma \mathrm{R}^\Sigma\,\mathrm{d}\sigma \,\leq\, \int_{\partial M_+} \mathrm{R}^{\partial M_+}\,\mathrm{d}\sigma\,,$$

and the equality holds if and only if (M, g_0, u) is isometric to the Nariai solution (2.5.5). Recalling that Σ has no conical singularities as proved in Proposition 4.5.2, applying the Gauss-Bonnet formula to both sides of the above inequality, we obtain

$$4\pi \sum_{i=1}^k \chi(\Sigma_i) \,\leq\, 4\pi\chi(\partial M_+)\,.$$

We recall from Theorem 4.6.11 that if ∂M_+ is connected then ∂M_+ is diffeomorphic to a sphere, hence the first part of the thesis follows.

We now pass to the proof of the second part, namely we will show that, if the pinching assumption holds, then $\sum_{i=1}^{k}\chi(\Sigma_i) \geq 2$, so that the thesis will follow from what we have just proven above. To this end, we start by observing that, in dimension $n = 3$, the second identity in (4.6.24) gives

$$\mathrm{R}^{\Sigma} = 6 - |\mathring{\mathrm{h}}|^2 .$$

Integrating this identity on Σ and using our pinching condition, we get

$$\int_{\Sigma} \mathrm{R}^{\Sigma}\, d\sigma = \int_{\Sigma} (6 - |\mathring{\mathrm{h}}|^2) d\sigma > 0 .$$

In particular, again from the Gauss-Bonnet formula it follows

$$\sum_{i=1}^{k} \chi(\Sigma_i) = \frac{1}{4\pi} \int_{\Sigma} \mathrm{R}^{\Sigma} d\sigma > 0 ,$$

but $\sum_{i=1}^{k}\chi(\Sigma_i)$ can only assume even integer values, hence $\sum_{i=1}^{k}\chi(\Sigma_i) \geq 2$, as wished. □

We also mention that some of the hypotheses of the above theorem can be relplaced with some informations on the critical points. For instance, proceeding in the same way as in Corollary 4.5.5 one proves the following.

Corollary 4.6.17. *Let (M, g_0, u) be a 3-dimensional 2-sided solution to problem (4.1.1), and let $\Sigma \subseteq \mathrm{MAX}(u)$ be the stratified hypersurface separating M_+ and M_-. Let also*

$$m_+ = \mu(M_+, g_0, u) , \qquad m_- = \mu(M_-, g_0, u)$$

be the virtual masses of M_+ and M_-. Suppose that the hypothesis of mass compatibility ($m_+ = m_- = m_{\max}$), and the regularity assumption (F is $\mathscr{C}^2$ along Σ) are in force, and suppose also that ∂M_+ is connected. If there are no critical points of u in the interior of M_+, then the solution is isometric to the Nariai solution (2.5.5).

We conclude by mentioning that the 2-sided hypothesis in Theorem 4.6.16 can be relaxed, as already discussed in more details in Subsection 4.5.2.

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