Scuola Normale Superiore
PhD Course in Mathematics

Symbols for Matrix-Sequences: Theory and Application-Driven Structure

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Introduction

Differential equations (DEs) are extensively used in physics, engineering and applied sciences in order to model real-world problems. A closed form for the analytical solution of such DEs is normally not available. It is therefore of fundamental importance to approximate the solution u of a DE by means of some numerical method.

Despite the differences that allow one to distinguish among the various numerical methods, the principle on which most of them are based is essentially the same: they first discretize the continuous DE by introducing a mesh, characterized by some discretization parameter n, and then they compute the corresponding numerical solution u_n, which will (hopefully) converge in some topology to the solution u of the DE as $n \to \infty$, i.e., as the mesh is progressively refined. If we consider a linear DE

$$\mathscr{L}u = f,$$

and a linear numerical method, then the actual computation of the numerical solution reduces to solving a linear system

$$A_n u_n = f_n$$

whose size s_n diverges with n. Solving high-dimensional linear systems in an efficient way is fundamental to compute accurate solutions in a reasonable time. In this direction, it is known that the convergence properties of mainstream iterative solvers, such as multigrid and preconditioned Krylov methods, strongly depend on the spectral features of the matrices to which they are applied. The knowledge of the asymptotic distribution of the sequence $\{A_n\}_n$ is therefore a useful tool we can use to choose or to design the best solver and method of discretization. In support of this claim, we recall that noteworthy estimates on the superlinear convergence of the conjugate gradient method obtained by Beckermann and Kuijlaars in [16] are closely related to the asymptotic spectral distribution of the considered matrices.

In addition to the design and analysis of appropriate solvers, the spectral distribution of DE discretization matrices is important also in itself whenever the eigenvalues of such matrices represent physical quantities of interest. This is the case for a broad class of problems arising in engineering and applied sciences, such as the study of natural vibration frequencies for an elastic material; see the review [47] and the references therein. At the same time, what is often observed in practice is that spectral distribution for the sequence $\{A_n\}_n$ is somehow connected to the spectrum of the differential operator $\mathscr{L}$.

These and many others are motivations to introduce the central focus of the current research: the spectral symbol, or, in general, the symbol. The so-called spectral symbol is an entity associated with a matrix-sequence of increasing size, and it represents the asymptotic distribution of the spectra of the matrices. In case of Toeplitz or Toeplitz-like matrices, this topic has been the subject of several studies and research, starting from Szegő [58], Avram [4], Parter [66], and followed by various other authors [21, 25, 26, 27, 28, 83, 87, 88, 90, 92]. Asymptotic distributions also naturally arise in the theory of orthogonal polynomials, in which case the zeros of the polynomials are seen as the eigenvalues of appropriate Jacobi matrices [45, 56, 59, 60, 89].

The symbol may appear in various forms, as a measurable function, a functional, a measure or even as a matrix-valued function, but all the different representations are able to describe in a compact way the asymptotic behaviour of the eigenvalues of the considered matrix-sequence. For example, when the symbol is a function or a measure, and when the related matrix-sequence is Hermitian (or quasi-Hermitian), we can sample the symbol to obtain an approximation of the spectrum of the matrices with large size n. Regarding this topic, we report the works [15, 43] and the references within, where the authors focus on the approximation of the eigenvalues of Toeplitz and Toeplitz-like matrices, starting with the analysis of the spectral symbol.

Each different representation of the symbol brings new properties to the space of matrix-sequences, for example metric or algebraic structures, and gives us new ways to analyse the matrices, their spectra, and also their singular values. In fact, we will define a concept of convergence on the space of matrix-sequences, and show that it is induced by a complete pseudometric, and that in special cases, it coincides with the natural metrics that we find on the space of symbols. Moreover, we will be able to connect the natural structure of algebra on particular subspaces of matrix-sequences to an analogous structure on the space of symbols.

Using all the previous tools, we can select special spaces of sequences and symbols, that we call generalized locally Toeplitz (GLT) spaces, and that make it easier to find the spectral symbols of several discretization matrix-sequences coming from practical applications. In fact, the GLT spaces are built so that the association between sequences and symbols is an isomorphism of algebras and an isometry of metric spaces, and hence we can compute the symbol of a matrix-sequence $\{A_n\}_n$ from sums, products and limits of simpler sequences for which we already know the spectral symbol.

The theory of GLT sequences stems from Tilli's work on locally Toeplitz (LT) sequences [84] and from the spectral theory of Toeplitz matrices. Nowadays, the main and most comprehensive sources in the literature for theory and applications of GLT spaces are the books [52, 53, 13, 14], in which we can find a careful and complete description of GLT sequences, block GLT sequences, and their respective multivariate versions.

In this regard, it has often been observed that the sequences $\{A_n\}_n$ coming from the discretization of linear DEs belong to one of the GLT spaces, so it has been possible to analyse the spectral behaviour of such sequences, thus justifying the interest in these spaces. Sequences $\{A_n\}_n$ with a spectral behaviour that can be analysed by resorting to GLT spaces have been encountered in the context of finite difference methods (see [52, Section 10.5], [53, Section 7.3], and [24, 75, 76]), finite elements methods (see [52, Section 10.6], [53, Section 7.4], and [17, 24, 42, 76]), finite volumes (see [18, 40]), isogeometric analysis (see [52, Section 10.7], [53, Sections 7.5–7.7], and [36, 48, 49, 50, 51, 69]) , and so on and so forth. In the context of Galerkin and collocation Isogeometric Analysis (IgA) discretizations of elliptic DEs, the spectral distribution computed through the theory of GLT sequences was also exploited in [34, 35, 37] to devise and analyse optimal and robust multigrid solvers for IgA linear systems. We also mention remarkable applications of the theory in the field of fractional differential equations [39, 40], and we refer the reader to [52, Section 10.4] and [1, 70] for a look at the GLT approach for sequences of matrices arising from IE discretizations.

In this thesis, we briefly review the theory of symbols in its generality, showing the steps that lead to define the GLT spaces, and how useful they are in the applications. In particular, we skim through the proofs whenever they can be found in the literature, while we prove in the appendices the results that never appeared before. Most of the arguments proposed herein can be found in one of the works [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 52, 53], but there are also original results. In fact, we formally define and thoroughly analyse the space of reduced GLT, that has previously been hinted by some authors ([17, 63, 75, 76, 77]), but has never been fully explained. Moreover, in the last chapter, we give an overview of open questions and possible future directions of research. Most of the applications in Part 3, and all the results proven in the appendices are also new. The document is structured as follows.

In Part 1, we review the concept of "symbol", we give a number of equivalent definitions and show some analytical and metrical properties of the space of matrix-sequences enjoying a symbol.

In Chapter 1, we start by providing an intuition of what a symbol is and how it is linked with the eigenvalues of the matrix-sequences $\{A_n\}_n$, by means of examples coming from graph theory and linear DEs. From figures and plots, we will learn how to express the symbol through a measurable function, a measure on $\mathbb{C}$ or also a matrix-valued function, and we will show that a single sequence may admit more than one symbol, depending on the ordering considered for the eigenvalues. In the same way, we will determine which kind of perturbation we can apply to the matrix-sequences without losing the symbol, and so we will be ready to formulate a first definition of symbol purely coming from the intuitions gained.

In Chapter 2, we state the classical definition of symbol, and we show that it grasps all forms of symbol considered in the previous chapter (function, measure, matrix-valued function). From the definition, expressed through an ergodic formula, we will be able to prove most of the intuitive properties of the first chapter, and in particular, we analyse the relations between different symbols describing the same spectral distribution, through the concept of rearrangement. We also propose a new equivalent definition of symbol, proving that any symbol can describe the asymptotic rates of eigenvalues for $\{A_n\}_n$ contained in almost all balls of $\mathbb{C}$. Eventually, we notice that the same arguments hold for the singular values of $\{A_n\}_n$, so we adapt the most important definitions and results also for this setting.

In Chapter 3, we closely inspect how a perturbation of $\{A_n\}_n$ reflects in its symbol. In particular, we introduce the concept of approximating classes of sequences (a.c.s.) as a notion of convergence on the sequences, and later we check that it is induced by a complete pseudometric. It will turn out to be the first formal link between the space of matrix-sequences and the space of symbols represented by measurable functions, endowed with the convergence in measure. The connection will be strengthened by closure results and an equivalence relation identifying two sequences whenever their a.c.s. distance is zero. In the last paragraph, we point out the differences between symbols referred to eigenvalues and symbols referred to singular values. In fact, in the case where our sequences and perturbations are not Hermitian, we exhibit examples supporting that the a.c.s. convergence is not enough for describing the behaviour of spectral symbols, and hence more careful results are presented.

Chapter 4 investigates the spectral measures, that are measures on $\mathbb{C}$ representing a spectral symbol for some sequence $\{A_n\}_n$. It turns out that for every symbol there exists a finite measure representing it, with mass 1 or less, thanks to Riesz representation theorem. Through the theory of standard spaces, we can link probability measures and measurable functions, and by generalizing this connection we find for every sequence $\{A_n\}_n$ a function $\kappa : [0,1] \to \mathbb{C} \cup \{\infty\}$ acting as a symbol. When we specialize to probability measures, we can exploit the vague or weak* convergence to come up with a new metric on the space of sequences that identifies two sequences if and only if they enjoy the same spectral symbol. In the last paragraph, we report some of the main results of other chapters for which the use of spectral measures has been essential in the proof, thus affirming the importance of measure theory for the actual research.

In Part 2, we analyse which kind of algebraic structures we can exploit to gain more information about the symbols. This examination will eventually lead to the construction and analysis of the GLT spaces.

In Chapter 5, we start by considering the space $\mathscr{S}$ of pairs $(\{A_n\}_n, \kappa)$ where κ is a symbol for $\{A_n\}_n$. An algebra or a group $\mathscr{A}$ inside $\mathscr{S}$ is a set of sequences and symbols where we can perform algebraic operations simultaneously on both spaces. In such sets, we can compute the symbol for complex matrix sequences as a result of sums, products and limits of known simpler pairs $(\{A_n\}_n, \kappa)$. We first build some simple instances of algebras, such as the space of diagonal sampling sequences, and the circulant algebra, and we exhibit an example where the algebraic and metric structure help in the determination of the spectral symbol. An additional discussion about normal sequences is included, since it is a case where spectral information and singular values are strongly connected. We later formulate the exact definitions of algebras and groups, and state the results we empirically observed. We also endow the space of algebras with different concepts of preorder relations, and we investigate necessary and sufficient conditions apt to identify maximal elements.

In Chapter 6, the GLT spaces are defined, analysed, and also applied for some classical problems. We summarize the standard construction through LT sequences, that we couple with the analogous locally circulant (LC) sequences. Both of them lead to the same space of GLT sequences, and we briefly discuss the pros and cons of the two methodologies. We then proceed to apply the results presented in the previous chapters, obtaining a list of basic properties, that we group in a list of “GLT axioms”. After reviewing the principal properties helping us in deriving the spectral symbol of sequences, we show how it can be applied to banded Toeplitz matrices with both theory and examples. Similar arguments and construction can be repeated to build the other GLT spaces, in particular the Multilevel GLT space, that let us face multidimensional problems, and the Block GLT space, that let us tackle for example systems of DEs and higher order FE discretizations. Lastly, we introduce for the first time the space of Reduced GLT sequences, with the aim of analysing DE discretizations DEs on irregular domains. All the principal results are reported in the section, and proved in the appendix, and some applications are reported in the succeeding part.

In Part 3, we present some recent applications of the developed theory, with a focus on reduced GLT sequences, fractional PDE and g-Toeplitz matrices.

In Chapter 7, we discuss about the convection-diffusion-reaction differential equations defined on a Peano-Jordan measurable set Ω contained in $[0,1]^d$. We analyse the modified FD scheme adopted in the Shortley-Weller approximation, coming up with the spectral symbol of the discretization matrix-sequence through the use of reduced GLT theory. If we specialize to $d = 2$, we can also apply a FE scheme through the P_1 method. Therefore, with the same tools, we examine the cases with square domain, Peano-Jordan measurable sets, and lastly we also consider the case of an irregular grid generated by a regular map.

Chapter 8 deals with fractional and time-dependent PDEs, both in the unidimensional and multidimensional settings. In both cases, we use the shifted Grünwald definition of fractional derivatives in space, and an implicit Euler method in time. In the multidimensional case, we work first on $[0,1]^2$ in space, and then we switch to a more general Peano-Jordan measurable set $\Omega \subseteq [0,1]^2$. Again, the Reduced GLT theory will prove to be essential for obtaining a spectral symbol for the discretization sequences.

In Chapter 9, we deal with particular structured matrices called g-Toeplitz matrices. In the past, their symbols have already been derived and studied, but now the Block GLT theory opens a new point of view on the question, leading to an elegant proof of the same results. The same argument is also applicable to the multidimensional case.

Lastly, Chapter 10 presents an overview of possible future works, by recalling some of the open questions introduced throughout the document and by proposing also new and still unexplored ideas.

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It's still magic even if you know how it's done

Terry Pratchett

Part I

Symbols for matrix-sequences

Chapter 1

The Idea Behind

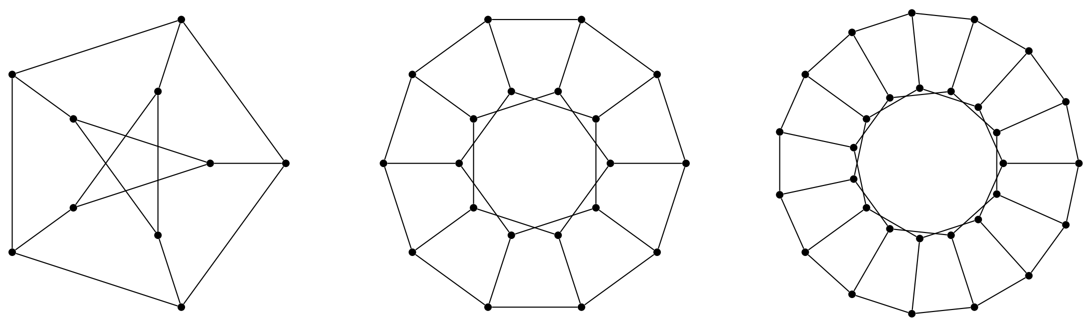
Before giving a formal definition of *symbol*, let us show some example first, to give an informal idea about the main concepts behind the formulae. At the end of this chapter, we sum up the observations and provide an intuitive definition of symbol.

1.1 Examples in Graph Theory

When dealing with graphs and their associated matrices, the knowledge of their spectra is fundamental to derive several combinatorial properties [61], and is useful for various applications, ranging from random walks [80], error correcting codes [82] to network flows [46] and random graphs [91].

1.1.1 Generalized Petersen Graphs

A famous class of graphs are the generalized Petersen graphs $\mathrm{GPG}(n, k)$. They are well-studied in graph theory since they have a lot of interesting properties regarding transitivity of edges, nodes, and are useful structures to keep in mind when looking for counterexamples. More specifically, $\mathrm{GPG}(n, k)$ is an undirected 3-regular graph with $2n$ nodes named $u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n$, and it is defined for $k < n/2$ and $n > 2$. Every u_i is connected to u_{i-1}, u_{i+1} and the corresponding node v_i, whereas v_i is connected to u_i, v_{i+k} and v_{i-k}, where all the indices are to be considered cyclically over n. When we fix k, we thus obtain a sequence of graphs $\{\mathrm{GPG}(n, k)\}_n$ where n ranges over all the natural numbers greater than $2k$. In Figure 1.1 three elements of the sequence with $k = 2$ are shown, where the first one is the original Petersen graph.



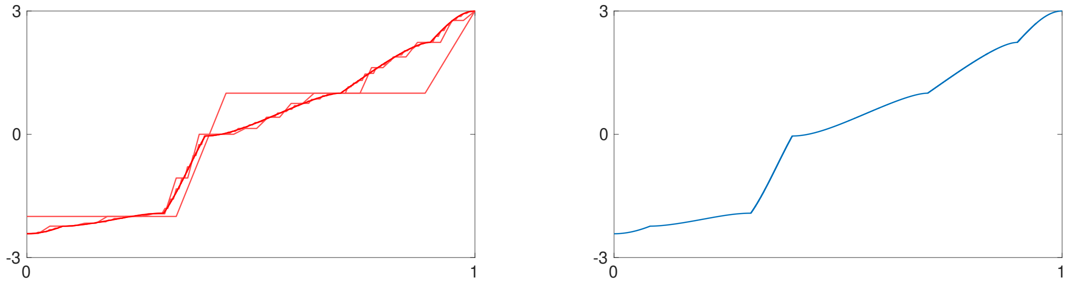
(a) GPG(5,2) *(b) GPG(10,2)* *(c) GPG(15,2)*

Figure 1.1: generalized Petersen graphs GPG(n, k) for $k = 2$ and $n = 5, 10, 15$.

Each graph $\mathrm{GPG}(n, k)$ is naturally associated with an adjacency matrix A_n of size $2n \times 2n$. The matrix A_n is symmetric, sparse and highly structured, and it is easy to compute all its eigenvalues. When we fix k, it is possible to see that all the eigenvalues $\Lambda(A_n) = \{\lambda_1^{(n)}, \ldots, \lambda_{2n}^{(n)}\}$ follow a certain pattern, that gets more evident as n rises. A way to visually show what is happening is to choose a sequence of growing values n, and for each n, plot the eigenvalues of A_n on a regular grid over the interval $[0, 1]$. In particular, for every n in the sequence, we plot a piecewise linear functions connecting the points $\left(\frac{i-1}{2n-1}, \lambda_i^{(n)}\right)$ for $i = 1, \ldots, 2n$, where the spectrum is sorted as in

$$\lambda_1^{(n)} \leq \lambda_2^{(n)} \leq \cdots \leq \lambda_{2n}^{(n)}.$$

The plot (a) in Figure 1.2 shows these function for $n = 5, 20, 80, 320, 1280$, $k = 2$, and one can notice that, except for the first 3 values of n, the plots coincide with a limit function $\kappa(x)$, whose plot is shown in (b). We will study in paragraph 1.3.3 what $\kappa(x)$ is in this case, and that it is actually more interesting to plot the eigenvalues following a different sorting.



(a) Plot of the eigenvalues of A_n.

(b) Asymptotic spectral distribution $\kappa(x)$.

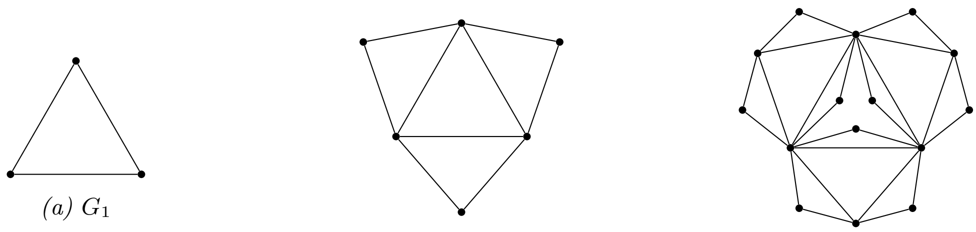
Figure 1.2: Comparison between the eigenvalue plot of A_n for $n = 5, 20, 80, 320, 1280$ and its limit spectral distribution, where A_n is the adjacency matrix of GPG$(n, 2)$.

When this phenomenon happens, we say that the sequence of matrix $\{A_n\}_n$ enjoys an asymptotic spectral distribution, and the function $\kappa(x)$ is called the *spectral symbol* associated to $\{A_n\}_n$. It is common to several kind of sequences $\{A_n\}_n$ coming from very different branch of mathematics and physics. In general, whenever we have a set of rules to build A_n for every n, its spectrum is expected to have an asymptotic behaviour.

Let us see one more example of this phenomenon in graph theory.

1.1.2 Deterministic Growing Graph

Let us define a deterministic growing graph with a simple building rule, as shown in [41]. Let G_1 be a triangle graph, or also said, the 3-complete graph. We want to build a sequence of graphs $\{G_n\}_n$ by applying the following rule: starting from G_n, we build G_{n+1} by adding a new node for each edge in G_n and by connecting it to the two ends of the edge. In Figure 1.3 it is possible to see the first 3 graphs in the sequence.



(a) G_1

(b) G_2

(c) G_3

Figure 1.3: Deterministic growing graphs G_n for $n = 1, 2, 3$.

The number of nodes and edges in G_n grows exponentially, and it is an example, in the limit of large n, of "scale-free" graph with the distribution of degrees following a power rule. The adjacency matrix A_n of G_n is a symmetric matrix with size $3\frac{3^{n-1}+1}{2} \times 3\frac{3^{n-1}+1}{2}$. It is hard to compute the eigenvalues exactly, but when we look at its eigenvalues, we discover again that they seem to converge, as shown in Figure 1.4. Here we plotted again the sorted eigenvalues of A_n on a regular grid over $[0, 1]$ for $n = 2, 3, \ldots, 9$. Notice that the limit function does not seem to be continuous and maybe not even limited, but nonetheless the plots converge.

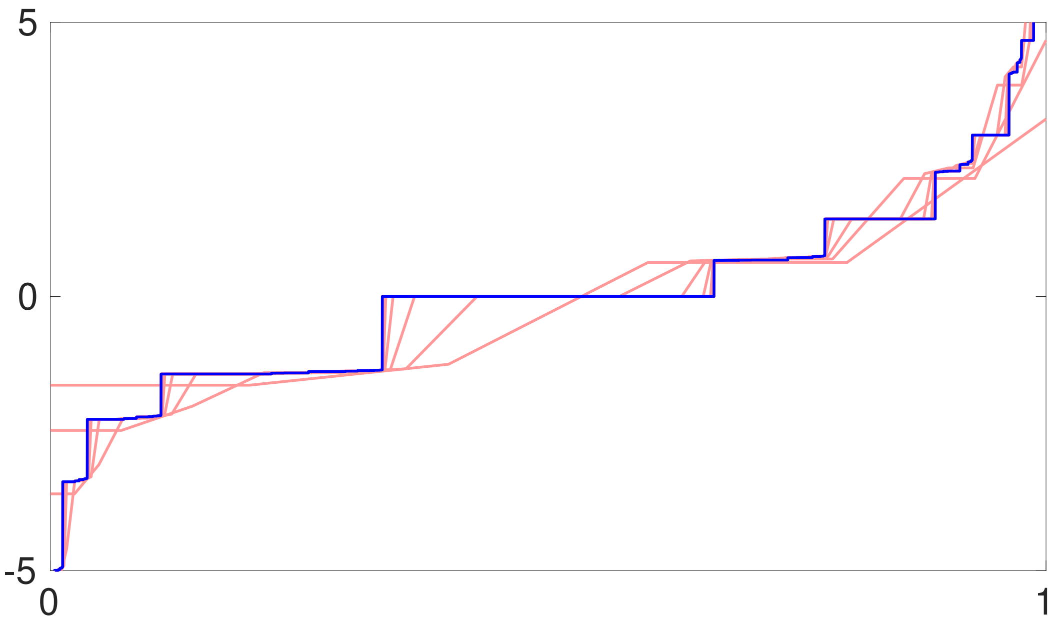


Figure 1.4: Eigenvalue plot of the adjacency matrix for G_n with $n = 2, 3, \ldots, 9$.

1.2 Ordering and Perturbation

From the preceding examples, we have seen that, given a sequence of matrices $\{A_n\}_n$, where $A_n \in \mathbb{C}^{s_n \times s_n}$ and s_n is an increasing sequence, there may happen that the spectra of the matrices converge in some sense to a function $\kappa(x)$, that we call spectral symbol of the sequence, or asymptotic spectral distribution. To symbolize this kind of dependence between a sequence and a function, we write as follows:

$$\{A_n\}_n \sim_\lambda \kappa(x).$$

In this section we address the following questions: do we always have to sort the eigenvalues increasingly? What kind of convergence are we requiring on the plots of the eigenvalues?

1.2.1 Laplacian Discretization

Consider the second-order differential equation with Dirichlet boundary conditions

$$\begin{cases} -u''(x) = f(x), & x \in (0,1), \\ u(0) = \alpha, \qquad u(1) = \beta, \end{cases} \tag{1.1}$$

where $f(x)$ is a continuous function. We employ central second-order finite differences for approximating the given equation. We define the step-size $h = \frac{1}{n+1}$ and the points $x_k = kh$ for k belonging to the interval $[0, n+1]$. Notice that the points x_k form a regular grid on $[0,1]$. For every $j = 1, \ldots, n$ we have

$$-u''(x)|_{x=x_j} \approx -\frac{u'(x_{j+\frac{1}{2}}) - u'(x_{j-\frac{1}{2}})}{h} \approx -\frac{\frac{u(x_{j+1})-u(x_j)}{h} - \frac{u(x_j)-u(x_{j-1})}{h}}{h} = \frac{-u(x_{j+1}) + 2u(x_j) - u(x_{j-1})}{h^2},$$

so we compute approximations u_j of the values $u(x_j)$ for $j = 1, \ldots, n$ by solving the following linear system

$$A_n \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{pmatrix} = h^2 \begin{pmatrix} f_1 + \frac{1}{h^2}\alpha \\ f_2 \\ \vdots \\ f_{n-1} \\ f_n + \frac{1}{h^2}\beta \end{pmatrix},$$

where $f_j := f(x_j)$ and

$$A_n = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}.$$

The accuracy of the approximation depends on the refinement parameter n, so that finer grids and larger n are associated to better solutions. The velocity of convergence for several linear solving algorithms (Krylov subspaces methods, iterative methods, etc.) depends on the spectrum of the matrix A_n, so prior knowledge on the asymptotic distribution of the eigenvalues leads to an estimate on the speed of the method.

In this instance, the eigenvalues of A_n are perfectly known, thanks to the tridiagonal and Toeplitz structure of the matrix.

$$\lambda_j(A_n) = 2 - 2\cos\left(j\frac{\pi}{n+1}\right), \qquad j = 1, \dots, n.$$

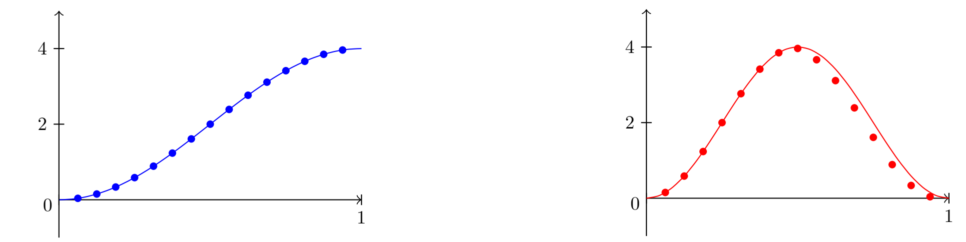
Here it is evident that $\lambda_j(A_n) = 2 - 2\cos(\pi x_j)$ where $x_j = j/(n+1)$ and $\{ x_0, \dots, x_{n+1} \}$ is a regular grid over $[0, 1]$, so the function $\kappa(x) = 2 - 2\cos(\pi x)$ is a spectral symbol for $\{A_n\}_n$. The eigenvalues are already sorted in an increasing way, but what happens if we take another order? For example, consider

$$\lambda_2, \lambda_4, \lambda_6, \dots, \lambda_{2\lfloor n/2\rfloor}, \lambda_{2\lceil n/2\rceil-1}, \lambda_{2\lceil n/2\rceil-3}, \dots, \lambda_3, \lambda_1$$

where we choose first the even numbered λ_i in increasing order, and then the odd numbered λ_i in decreasing order. We can denote this ordering as $\{\widetilde{\lambda}_j\}_j$, where

$$\widetilde{\lambda}_j(A_n) = \begin{cases} \lambda_{2j}(A_n), & 2j \le n, \\ \lambda_{2n+1-2j}(A_n), & 2j > n. \end{cases}$$

When we plot $\widetilde{\lambda}_j(A_n)$ we can see that they are approximatively a sampling of the function $\widetilde{\kappa}(x) = 2-2\cos(2\pi x)$ over $[0, 1]$ and when n goes to infinite, the plot of the eigenvalues gets closer and closer to the plot of $\widetilde{\kappa}(x)$. In Figure 1.5 we can see that when $n = 15$ the values $\widetilde{\lambda}_j(A_{15})$ are already very close to $\widetilde{\kappa}(x)$, whilst $\lambda_j(A_{15})$ coincide perfectly to the sampling of $\kappa(x)$.



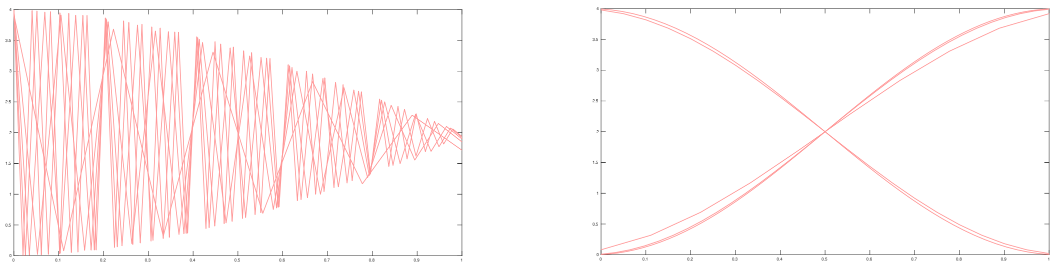
(a) $\lambda_j(A_{15})$ and $\kappa(x) = 2 - 2\cos(\pi x)$. *(b) $\widetilde{\lambda}_j(A_{15})$ and $\widetilde{\kappa}(x) = 2 - 2\cos(2\pi x)$.*

Figure 1.5: Different orderings for the eigenvalues of A_{15}. Depending on the choice of the sorting, we obtain different spectral symbols.

We can thus say that $\widetilde{\kappa}(x) = 2-2\cos(2\pi x)$ is also a spectral symbol for $\{A_n\}_n$. In this case, it is also possible to find specific orderings for the spectrum of A_n so that the eigenvalue plots converge to $2 - 2\cos(s\pi x)$ with s being any non-zero integer.

At the same time, not every ordering of the eigenvalues leads to a spectral symbol, since the plots may wildly oscillate, or in general they may not follow any pattern. Simple examples may be the following orderings:

$$\overline{\lambda}_j(A_n) = \begin{cases} \lambda_s(A_n), & j = 2s, \\ \lambda_{n-s}(A_n), & j = 2s+1, \end{cases} \qquad \widehat{\lambda}_j(A_n) = \begin{cases} \lambda_j(A_n), & n \text{ even}, \\ \lambda_{n-j}(A_n), & n \text{ odd}. \end{cases}$$



(a) Plot of $\overline{\lambda}_j(A_n)$.

(b) Plot of $\widehat{\lambda}_j(A_n)$.

Figure 1.6: Plot of the eigenvalues of A_n following the orderings dictated by $\overline{\lambda}_j(A_n)$ and $\widehat{\lambda}_j(A_n)$. in (a) we considered $n = 10, 20, 30, 40, 50$, whereas in (b) we considered $n = 10, 21, 32, 43, 54$.

In Figure 1.6 we have plotted the eigenvalues $\overline{\lambda}_j$ and $\widehat{\lambda}_j$ for some values of n, and we clearly see that in both cases, for different reasons, we can never achieve convergence.

With this example, we showed that a single sequence may admit multiple (actually, infinite) spectral symbols, depending on the ordering chosen on the eigenvalues of the matrices, but also that not every ordering brings to a spectral symbol.

The argument here may seem a bit nebulous, but will become more clear when we will provide the formal definition of spectral symbol, and it will be easier to explain this property formally and with more details. We will also show that the ordering of the eigenvalues is a fundamental feat to consider when we want to add algebraic structures to specific sets of sequences.

1.2.2 Plot Convergence

In paragraph 1.2.1 we showed an example of matrix-sequence $\{A_n\}_n$ present eigenvalues that are exactly a sampling of the function $2 - 2\cos(\pi x)$ over an uniform grid on $[0, 1]$. We can ask how much we can perturb the sequence $\{A_n\}_n$ without losing or changing its spectral distribution.

Notice that the addition of the identity matrices I_n to A_n uniformly shift all the eigenvalues by one, so

$$\{A_n\}_n \sim_\lambda \kappa(x) \iff \{A_n + I_n\}_n \sim_\lambda \kappa(x) + 1.$$

Similar results hold for every multiple εI_n with arbitrarily small ε. As a consequence, we need a perturbation sequence $\{N_n\}_n$ that has no subsequence with spectral norm bounded below by a constant $c > 0$. In other words, we require that $\|N_n\| \xrightarrow{n\to\infty} 0$. For example, we choose a skew-symmetric tridiagonal Toeplitz sequence $\{N_n\}_n$ defined by

$$N_n = \frac{1}{\log(n)} \begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 0 & -1 \\ & & & 1 & 0 \end{pmatrix}$$

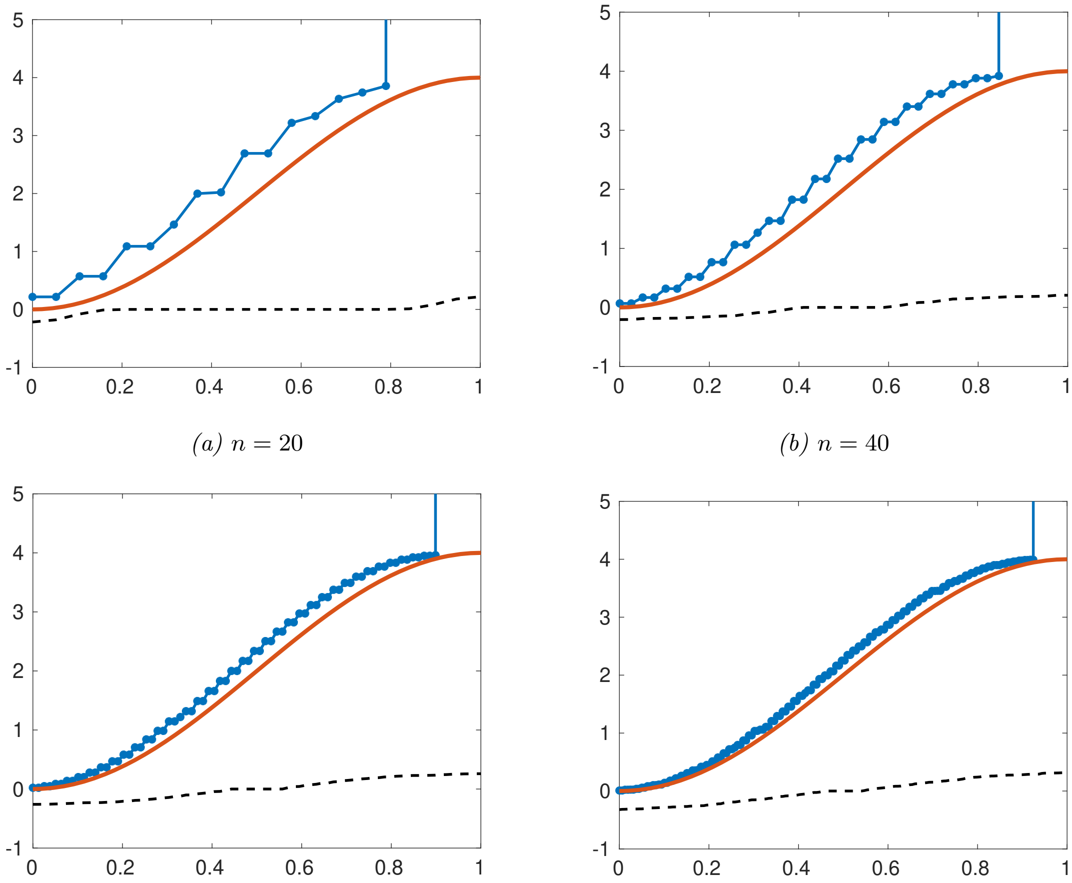
with spectral norm proportional to $1/\log(n)$. The inverse logarithmic spectral converges slowly to zero, so the effect of the perturbation on the spectrum will be noticeable even for large n. The structure of N_n is also recurring in the discretization of PDE, since it represents a first order formula for the first order derivatives.

A full rank perturbation must have small norm to asymptotically conserve the spectral structure of the sequence, but the same is not true for low-rank symmetric perturbation. Let $\boldsymbol{e}^{(n)} \in \mathbb{R}^n$ be the all-ones n-dimensional vector, let $\boldsymbol{e}_i^{(n)} \in \mathbb{R}^n$ be the canonical basis n-dimensional vectors, denote $g(n) = \lfloor \sqrt{n} \rfloor$ and define the matrices R_n as

$$R_n = 2^n \sum_{i=1}^{g(n)} (\boldsymbol{e}^{(n)} - \boldsymbol{e}_i^{(n)})(\boldsymbol{e}^{(n)} - \boldsymbol{e}_i^{(n)})^T.$$

Note that every R_n is symmetric, has rank at most $g(n)$ that is negligible when compared to the matrix size for large values of n, and that the multiplicative constant 2^n is chosen large enough to better show the effect of the perturbation on the spectrum.

Call $B_n = A_n + N_n + R_n$ the perturbed matrices. In Figure 1.7 we show the effects of the perturbation on the spectrum of A_n for $n = 20, 40, 80, 160$. First of all, due to the non-Hermitianity of the matrix N_n, we have complex eigenvalues. We plot separately the real part and the imaginary part of the eigenvalues, and we compare them with the plot of the symbol $\kappa(x) = 2 - 2\cos(\pi x)$ of $\{A_n\}_n$. As we have previously seen, the eigenvalues of $\{A_n\}_n$ fit perfectly $\kappa(x)$, whereas we note a general loss of accuracy of the eigenvalues of B_n caused by N_n and the explosion of a portion of the spectrum given by R_n. Nonetheless, the imaginary part of the eigenvalues have small intensity when compared with the real part, and they will converge to zero for $n \to \infty$. The plots of the real parts also converge almost everywhere to the symbol plot, since the number of huge eigenvalues tends to be negligible when compared with the size of the matrix.



(a) $n = 20$ *(b) $n = 40$*

(c) $n = 80$ *(d) $n = 160$*

Figure 1.7: Comparison between real part (blue dotted line), imaginary part (black dashed line) of the eigenvalues of B_n and the plot of the spectral symbol $\kappa(x) = 2 - 2\cos(\pi x)$ (red continuous line) for $n = 20, 40, 80, 160$.

From the graph we can again conclude that $\{B_n\}_n$ admits a spectral symbol, and $\kappa(x)$ fits the role. We can deduce that, by considering a weak enough convergence for the plots, the spectral symbol is conserved under vanishing norm and negligible ranked perturbations. But what kind of "weak convergence" do we need?

Note that in Figure 1.7 both the real and the imaginary part of the eigenvalues are sorted incrementally, but if we write down the actual eigenvalues, great real parts are not generally associated with great imaginary parts and vice versa. If we consider the actual eigenvalues and we order them according their real parts, then we find that the (complex-valued) piecewise linear functions $\kappa_n(x)$ interpolating the points $\left(\frac{j-1}{n-1}, \lambda_j(B_n)\right)$ do not converge to $\kappa(x)$ punctually almost everywhere. Instead, for every $\varepsilon > 0$,

$$\ell_1 \left\{ x \in [0,1] \mid |\kappa_n(x) - \kappa(x)| > \varepsilon \right\} \xrightarrow{n\to\infty} 0$$

where ℓ_1 is the unidimensional Lebesgue measure. In other words, we have that $\kappa_n \to \kappa$ in measure.

Using all the ideas accumulated from these examples, we are now able to give an intuitive notion of spectral symbol.

1.3 An Informal Definition

In all the previous example we have based our intuition of what is a spectral symbol on the behaviour of the spectral plots for a sequence $\{A_n\}_n$. Let us wrap up the pieces of information we have to finally come up with a definition.

1.3.1 The Intuitive Definition

Definition 1.3.1. *Let $\{A_n\}_n$ be a sequence of matrices, where $A_n \in \mathbb{C}^{s_n\times s_n}$ and s_n is a sequence of natural numbers that diverges. For every n, let $h_n = s_n^{-1}$ and call $\kappa_n(x)$ the continuous function that is linear on every interval $[(j-1)h_n, jh_n]$ for $j = 1,\dots,s_n$ and such that*

$$\kappa_n(0) = 0, \qquad \kappa_n(jh_n) = \lambda_j(A_n) \quad \forall\, j = 1,\dots,s_n,$$

*where $\lambda_j(A_n)$ are the eigenvalues of A_n considered with their algebraic multiplicity. Let $\kappa : [0,1] \to \mathbb{C}$ be a measurable function. We say that $\{A_n\}_n$ admits $\kappa(x)$ as a **spectral symbol**, and we write*

$$\{A_n\}_n \sim_\lambda \kappa(x),$$

when there exists an ordering of the eigenvalues $\Lambda(A_n) = \{\lambda_1(A_n),\dots,\lambda_{s_n}(A_n)\}$ for every n such that the sequence $\{\kappa_n(x)\}_n$ converges in measure to $\kappa(x)$. In this case, we simply say that the eigenvalues of $\{A_n\}_n$ converge to κ.

The above definition can be found in [6] under the name of "piecewise convergence", but the idea was already present in [33]. From the definition, we can state some evident properties.

- The spectral symbol depends exclusively on the spectra of the matrices A_n, so any sequence of invertible matrices $\{M_n\}_n$ of the appropriate size, produces an other sequence with the same symbol by base change. In formula,
$$\{A_n\}_n \sim_\lambda \kappa(x) \iff \{M_nA_nM_n^{-1}\}_n \sim_\lambda \kappa(x).$$
- If the plots of a sequence $\{A_n\}_n$ converge, then any subsequence has the same limit, so
$$\{A_n\}_n \sim_\lambda \kappa(x) \implies \{A_{n_k}\}_k \sim_\lambda \kappa(x).$$
- If we modify a finite number of matrices A_n in the sequence, the limit does not change.
- If $\kappa(x)$ is continuous, an uniform sampling of s_n points on $\kappa(x)$ give an approximation for the eigenvalues of A_n, that gets better as $n \to \infty$.

An other consideration is that when we plot the eigenvalues, we can exchange the grid $\left(\frac{i-1}{n-1}\right)_{i=1,\dots,n}$ over $[0,1]$ with any other regular grid without changing the asymptotic spectral distribution, since in the limit for $n\to\infty$ the plots will result all identical. In particular, we can also choose $\left(\frac{i}{n}\right)_{i=1,\dots,n}$ or $\left(\frac{i}{n+1}\right)_{i=1,\dots,n}$.

Definition 1.3.1 is a natural follow up of the examples showed before, but it comes short in term of usefulness and manageability. In fact, all the different results regarding the spectral symbol become quite technical to prove, differently from the less intuitive but more abstract framework we will be working with, starting from chapter 2.

Sadly, there are cases in which an asymptotic spectral distribution exists, but it is hard (and sometimes nearly impossible) to find a proper ordering on the eigenvalues to make the piecewise linear interpolation functions converge to a function on $[0,1]$. In these cases, a suitable spectral symbol may be represented by a function on a different domain than $[0,1]$, or by a matrix valued function, or even by a finite positive measure on the complex space.

In such cases, we can adopt a more 'philosophical' definition than Definition 1.3.1, and say that a spectral symbol is **a compact way to globally describe the overall spectral distribution of a matrix-sequence**, and it can be expressed in several different forms. A symbol, in fact, can be a measurable function, a finite measure on an euclidean space, a functional on $C_c(\mathbb{C})$, or much more.

In the following paragraphs we report some 'artificial' examples, and in the next chapters we will see how it is possible to formalize such different types of symbols.

1.3.2 Spectral Distribution on a Disk

Let C_n be the most simple circulant matrix of size $n \times n$, defined as

$$[C_n]_{i,j} = \begin{cases} 1, & i-j \equiv 1 \pmod{n}, \\ 0, & \text{otherwise} \end{cases} \implies C_n = \begin{pmatrix} & & & & 1 \\ 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \end{pmatrix}.$$

Let F_n the the Fourier matrix of size $n \times n$,

$$F_n = \frac{1}{\sqrt{n}}(e^{-2\pi \mathrm{i}(j-1)(i-1)/n})_{i,j=1}^n$$

that is an unitary symmetric matrix diagonalising C_n

$$F_n^H C_n F_n = \begin{pmatrix} 1 & & & & \\ & e^{2\pi \mathrm{i}/n} & & & \\ & & e^{2*2\pi \mathrm{i}/n} & & \\ & & & \ddots & \\ & & & & e^{(n-1)*2\pi \mathrm{i}/n} \end{pmatrix}.$$

Let also S_n be the following diagonal matrix

$$S_n = \begin{pmatrix} 1/n & & & & \\ & 2/n & & & \\ & & 3/n & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}.$$

The sequence we want to analyse is $A_n = S_n \otimes C_n$ where A_n has size $n^2 \times n^2$ and it is a block diagonal matrix with circulant block. In particular it is defined as

$$[A_n]_{nk+i,ns+j} = \delta_{k-s}\frac{k+1}{n}[C_n]_{i,j} \qquad \forall k,s = 0,\dots,n-1, \quad \forall i,j = 1,\dots,n.$$

We can compute exactly its eigenvalues through the diagonalizing unitary base change $I_n \otimes F_n$.

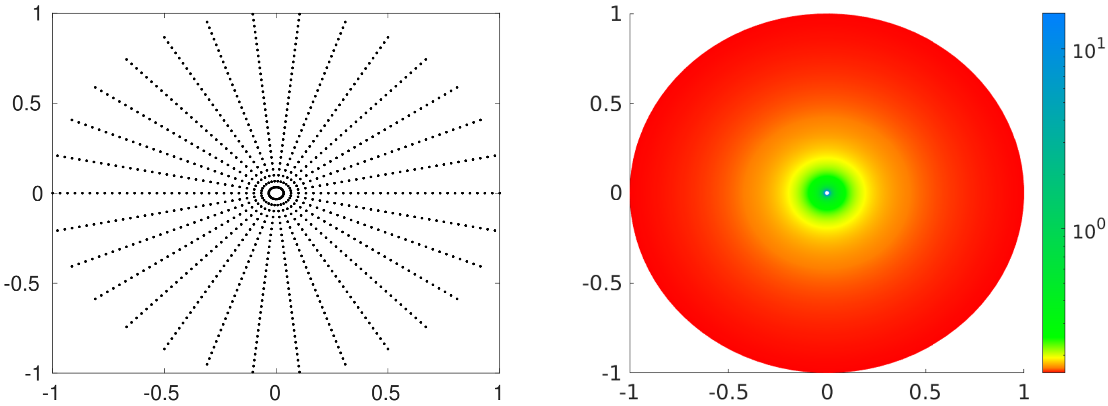
$$\lambda_{a,b} = \frac{a}{n}e^{2b\pi \mathrm{i}/n}, \qquad \forall a = 1,\dots,n, \quad \forall b = 0,\dots,n-1.$$

In Figure 1.8 (a), we have plotted the complex eigenvalues of A_{30}. From the plot, it is clear that the sequence $\{A_n\}_n$ enjoys a spectral asymptotic distribution, but it is unclear which ordering can lead to a spectral symbol as in Definition 1.3.1. A piecewise linear plot would not be useful, since any reasonable ordering of $\lambda_{a,b}$ would make the graph oscillate too much, leading to a non-converging sequence of functions.

Therefore, we start from the analytical formulation for $\lambda_{a,b}$ to deduce a symbol. In fact it is immediate to notice that the eigenvalues fit perfectly an uniform sampling of the function

$$\kappa(x,\theta) = xe^{\mathrm{i}\theta}, \qquad x \in [0,1], \quad \theta \in [0,2\pi].$$

It is evident that $\kappa(x,\theta)$ has the right 'distributional properties' to represent the asymptotic behaviour of the matrices spectra, but due to the change of domain, κ is not a spectral symbol according to Definition 1.3.1. Actually, we can say that the graph of $\kappa(x,\theta)$ is the limit of a sequence of surfaces interpolating the eigenvalues $\lambda_{a,b}$ in a specific order (namely, the lexicographical order on the couples (a,b)), so in a sense we can say that it is a proper spectral symbol of the sequence $\{A_n\}_n$.



(a) Eigenvalues of A_n for $n = 30$. *(b) Colourmap for $f(x)$.*

Figure 1.8: On the left, the plot of the eigenvalues of A_{30}. On the right, the density $f(x)$ associated to the absolutely continuous spectral measure μ representing the spectral asymptotic distribution for $\{A_n\}_n$.

This is the first case we encounter where it may be also convenient to represent the spectral distribution by a positive measure μ on the complex plane $\mathbb{C}$.

In fact, from the plot (a) in Figure 1.8 we note that the eigenvalues span the unitary disk $B(0,1) = \{ z \in \mathbb{C} \mid |z| \leq 1 \}$, and the union of the eigenvalues of A_n for all different n produce a dense set over $B(0,1)$. The distribution, though, is not uniform, since a lot of small eigenvalues are concentrated near the origin, whereas fewer eigenvalues are near the boundary of $B(0,1)$. Since we expect a radial density of the eigenvalues, it is enough to compute the rate of eigenvalues lying inside $B(0,r)$ for every n and $0 < r \leq 1$. A simple calculation shows that the rates of eigenvalues inside $B(0,r)$ converge to r, so we can easily infer a radial asymptotically density function $f(z) = g(|z|)$ for the spectra of A_n.

$$r = \int_{B(0,r)} f(z)\mathrm{d}z = 2\pi \int_0^r g(x)x\mathrm{d}x \implies 2\pi g(x)x = \frac{\partial r}{\partial r} = 1 \implies g(x) = \frac{1}{2\pi x} \implies f(z) = \frac{1}{2\pi |z|}.$$

In Figure 1.8 (b) we can see a map for the density $f(z)$ over the complex disk $B(0,1)$, where it is clear that higher density of eigenvalues are found near the centre. Notice that the mass of $f(x)$ over $B(0,1)$ is unitary, thus making it a probability measure μ that is absolutely continuous with respect to the 2-dimensional Lebesgue measure ℓ_2. From the construction of $f(z)$, it is possible to prove that for any open set $U \subseteq \mathbb{C}$, we have

$$\mu(U) = \int_U f(z)\mathrm{d}z$$

meaning that $\mu(U)$ is the asymptotic rate of eigenvalues in A_n that lie inside U. Note moreover that even in this instance, a sampling of s_n points over $B(0,1)$ weighted by the measure μ gives an approximation of the eigenvalues of A_n, that gets better when n goes to infinity. It is thus clear that μ can be considered a symbol for the sequence $\{A_n\}_n$, since it describes in a compact way the spectral distribution of the sequence. Later on, we will call μ a *spectral measure* for the sequence $\{A_n\}_n$, and we will provide a formal definition.

1.3.3 A Matrix-Valued Symbol

For our last example, we analyse more carefully the adjacency matrices of the generalized Petersen graphs (GPG) introduced in paragraph 1.1.1. Recall that GPG(n,k) is a graph with $2n$ nodes named $u_1, u_2, \dots, u_n$, $v_1, v_2, \dots, v_n$, where every u_i is connected according to the regular polygon to u_{i-1} and u_{i+1} and the corresponding node v_i, whereas v_i is connected to v_{i+k} and v_{i-k}, where all the indices are to be considered cyclically over n. In term of adjacency matrix, it means that we have a 2×2 block structure

$$A_n = A(GPG(n,k)) = \begin{pmatrix} C_n + C_n^T & I_n \\ I_n & C_n^k + (C_n^k)^T \end{pmatrix}$$

where C_n is the circulant matrix defined in paragraph 1.3.2. Note that the Fourier matrix F_n diagonalizes C_n and it is symmetric, so it is also a diagonalizing base change for C_n^k and $(C_n^k)^T$. A simple computation shows that

$$\begin{pmatrix} F_n^H & \\ & F_n^H \end{pmatrix} \begin{pmatrix} C_n + C_n^T & I_n \\ I_n & C_n^k + (C_n^k)^T \end{pmatrix} \begin{pmatrix} F_n & \\ & F_n \end{pmatrix} =$$

$$= \left(\begin{array}{cccc|cccc} 2\cos(0) & & & & 1 & & & \\ & 2\cos(2\pi/n) & & & & 1 & & \\ & & \ddots & & & & \ddots & \\ & & & 2\cos((n-1)2\pi/n) & & & & 1 \\ \hline 1 & & & & 2\cos(0) & & & \\ & 1 & & & & 2\cos(2\pi k/n) & & \\ & & \ddots & & & & & \\ & & & 1 & & & & 2\cos((n-1)2\pi k/n) \end{array}\right)$$

and an opportune permutation of rows and columns leads to a block diagonal matrix with 2×2 blocks, that we call B_n.

$$\left(\begin{array}{ccccc} \boxed{\begin{array}{cc} 2\cos(0) & 1 \\ 1 & 2\cos(0) \end{array}} & & & & \\ & \boxed{\begin{array}{cc} 2\cos(2\pi/n) & 1 \\ 1 & 2\cos(2\pi k/n) \end{array}} & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \boxed{\begin{array}{cc} 2\cos((n-1)2\pi/n) & 1 \\ 1 & 2\cos((n-1)2\pi k/n) \end{array}} \end{array}\right).$$

Up until now we have operated only (unitary) base changes, so the set of eigenvalues has not changed, and also the spectral symbol is the same. In other words, κ is a spectral symbol for $\{A_n\}_n$ if and only if it is a spectral symbol for $\{B_n\}_n$.

Notice that the eigenvalues of B_n are simple to compute, since its spectrum is the union of the spectra of 2×2 matrices. Moreover, we can enumerate the blocks and call them $X_{n,i}$ with $i = 1,\dots,n$ so that $B_n = \operatorname{diag}(X_{n,i})_{i=1}^n$. The $X_{n,i}$ are easily written as

$$X_{n,i} = \begin{pmatrix} 2\cos((i-1)2\pi/n) & 1 \\ 1 & 2\cos((i-1)2\pi k/n) \end{pmatrix}$$

with eigenvalues

$$\begin{aligned} \lambda_{n,i,1} &= \cos((i-1)2\pi/n) + \cos((i-1)2\pi k/n) + \sqrt{[\cos((i-1)2\pi/n) - \cos((i-1)2\pi k/n)]^2 + 1}, \\ \lambda_{n,i,2} &= \cos((i-1)2\pi/n) + \cos((i-1)2\pi k/n) - \sqrt{[\cos((i-1)2\pi/n) - \cos((i-1)2\pi k/n)]^2 + 1}. \end{aligned}$$

Let us now define a matrix-valued function $\Upsilon : [0,1] \to \mathbb{C}^{2\times 2}$, that can also be seen as a 2×2 matrix of functions $\Upsilon_{i,j} : [0,1] \to \mathbb{C}$, as follows

$$\Upsilon(x) = \begin{pmatrix} 2\cos(2\pi x) & 1 \\ 1 & 2\cos(2\pi k x) \end{pmatrix}.$$

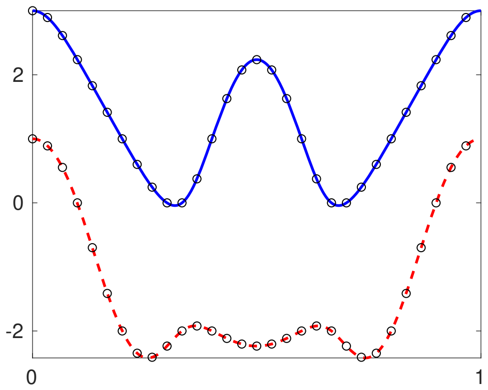
For every x, we can write the two eigenvalues of $\Upsilon(x)$, but they change continuously with respect to x, so we actually obtain two eigenvalue functions $\lambda_1 : [0,1] \to \mathbb{C}$, $\lambda_2 : [0,1] \to \mathbb{C}$ with expressions

$$\begin{aligned} \lambda_1(x) &= \cos(2\pi x) + \cos(2\pi k x) + \sqrt{[\cos(2\pi x) - \cos(2\pi k x)]^2 + 1}, \\ \lambda_2(x) &= \cos(2\pi x) + \cos(2\pi k x) - \sqrt{[\cos(2\pi x) - \cos(2\pi k x)]^2 + 1}. \end{aligned}$$

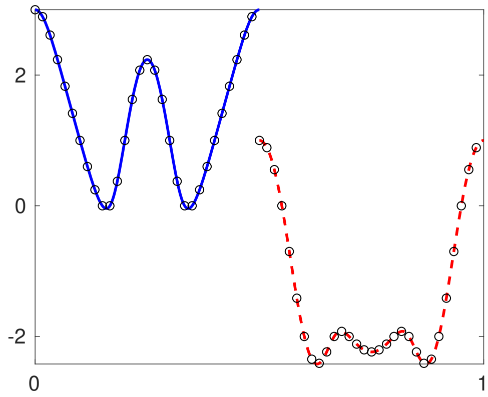
It is now straightforward to see that $X_{n,i}$ are actually an uniform sampling of the function $\Upsilon(x)$, and consequently even $\lambda_{n,i,1}$ and $\lambda_{n,i,2}$ are an exact uniform sampling of $\lambda_1(x)$ and $\lambda_2(x)$. In particular,

$$X_{n,i} = \Upsilon\left(\frac{i-1}{n}\right) \implies \lambda_{n,i,1} = \lambda_1\left(\frac{i-1}{n}\right), \quad \lambda_{n,i,2} = \lambda_2\left(\frac{i-1}{n}\right),$$

as we can also see from Figure 1.9 (a), where we assumed $k = 2$.



(a) Eigenvalues of B_{30} plotted over $\lambda_1(x)$ and $\lambda_2(x)$.



(b) Eigenvalues of B_{30} plotted over $\widetilde{\kappa}(x)$.

Figure 1.9: Scatter plot of the eigenvalues $\lambda_{n,i,1/2}$ of B_n when $k=2$, compared with the plots of $\lambda_{1/2}(x)$ and of $\widetilde{\kappa}(x)$.

This time, we have split the eigenvalues into two sets, converging respectively to $\lambda_1(x)$ and $\lambda_2(x)$, but singularly they are not spectral symbols for $\{A_n\}_n$ (and $\{B_n\}_n$). A possible solution is to order the eigenvalues as follows

$$\lambda_{1,n,1}, \lambda_{2,n,1}, \ldots, \lambda_{n,n,1}, \lambda_{1,n,2}, \lambda_{2,n,2}, \ldots, \lambda_{n,n,2}$$

in order to converge to a spectral symbol $\widetilde{\kappa}(x)$ obtained by joining $\lambda_1(x)$ and $\lambda_2(x)$, as shown in Figure 1.9 (b).

$$\widetilde{\kappa}(x) = \begin{cases} \lambda_1(2x), & x \in [0, 1/2), \\ \lambda_2(2x-1), & x \in [1/2, 1]. \end{cases}$$

$\widetilde{\kappa}(x)$ is thus a spectral symbol for $\{A_n\}_n$, and if we take the increasing rearrangement (that we will define in paragraph 2.2.1) of $\widetilde{\kappa}(x)$ we obtain the function $\kappa(x)$ plotted in Figure 1.2 (b), that automatically becomes a spectral symbol for $\{A_n\}_n$ since it has the same distribution of $\widetilde{\kappa}(x)$.

As a consequence, we have found two spectral symbols for the same sequence, but $\kappa(x)$ is actually hard to compute exactly, and $\widetilde{\kappa}(x)$ is a discontinuous function. A possibility in such cases is to work with the two functions $\lambda_{1/2}(x)$, for which we have explicit formulas, but we will see that a good choice is to consider as spectral symbol the matrix-valued function $\Upsilon(x)$. In fact, it is indeed a compact formulation that contains full information about the asymptotic spectral distribution of $\{A_n\}_n$, so we write

$$\{A_n\}_n \sim_\lambda \Upsilon.$$

As a further parallel, we can also notice that the eigenvalues of A_n are actually the eigenvalues of an uniform sampling of $\Upsilon(x)$ over $[0, 1]$, thus supporting the fact that it is fit to be considered a spectral symbol.

Chapter 2

A Rigorous Definition

Here we will define rigorously what a spectral symbol is, and then we try to link the definition to the intuitions provided in the previous chapter.

2.1 Ergodic Formula

From now on, a matrix-sequence $\{A_n\}_n$ is always a sequence of matrices $A_n \in C^{s_n\times s_n}$ where s_n is a sequence of natural numbers that diverges to $+\infty$. In the most classic settings, (as in paragraph 1.2.1) we have $s_n = n$, but we have already showed examples where $s_n = 2n$ (paragraph 1.1.1), $s_n = n^2$ (paragraph 1.3.3) or $s_n = 3\frac{3^{n-1}+1}{2}$ (paragraph 1.1.2). Obviously, a subsequence of $\{A_n\}_n$ is still a matrix-sequence.

Definition 2.1.1. *Given a matrix-sequence $\{A_n\}_n$, and a functional $\phi : C_c(\mathbb{C}) \to \mathbb{R}$, we say that ϕ is a **spectral symbol** for the sequence $\{A_n\}_n$ if*

$$\lim_{n\to\infty} \frac{1}{s_n}\sum_{j=1}^{s_n} G(\lambda_j(A_n)) = \phi(G), \qquad \forall\, G \in C_c(\mathbb{C}), \tag{2.1}$$

where $\lambda_j(A_n)$ are the eigenvalues of A_n. In this case, we write

$$\{A_n\}_n \sim_\lambda \phi.$$

Equation 2.1 is also referred to as the *ergodic formula* for the spectral symbol, and is the starting point to understand what a symbol really is. Notice that ϕ, when it exists, is univocally determined by $\{A_n\}_n$, since

$$\{A_n\}_n \sim_\lambda \phi, \quad \{A_n\}_n \sim_\lambda \phi' \implies \phi(G) = \phi'(G) \quad \forall G \in C_c(\mathbb{C}), \implies \phi \equiv \phi'.$$

2.1.1 From Functionals To Functions and Measures

To move on, we have to first reconnect to the intuitive spectral symbols presented in chapter 1, so measurable functions $\kappa(x)$, matrix-valued functions $\Upsilon(x)$ and measures μ.

- Let $D \subseteq \mathbb{R}^d$ be a measurable set with positive and finite d-Lebesgue measure, and let $\kappa : D \to \mathbb{C}$ be a measurable function. Consider the functional $\phi_\kappa : C_c(\mathbb{C}) \to \mathbb{R}$ defined as

  $$\phi_\kappa(G) := \frac{1}{\ell_d(D)}\int_D G(\kappa(\mathbf{x}))\mathrm{d}\mathbf{x}, \qquad \forall G \in C_c(\mathbb{C}),$$

  where ℓ_d is the d-dimensional Lebesgue measure. In this case $\{A_n\}_n \sim_\lambda \phi_\kappa$ means

  $$\lim_{n\to\infty} \frac{1}{s_n}\sum_{j=1}^{s_n} G(\lambda_j(A_n)) = \frac{1}{\ell_d(D)}\int_D G(\kappa(\mathbf{x}))\mathrm{d}\mathbf{x}, \qquad \forall\, G \in C_c(\mathbb{C}). \tag{2.2}$$

- Let $D \subseteq \mathbb{R}^d$ be a measurable set with positive and finite d-Lebesgue measure, and let $\Upsilon : D \to \mathbb{C}^{s\times s}$ be a measurable matrix-valued function. Consider the functional $\phi_\Upsilon : C_c(\mathbb{C}) \to \mathbb{R}$ defined as

$$\phi_\Upsilon(G) := \frac{1}{\ell_d(D)} \int_D \frac{\sum_{i=1}^s G(\lambda_i(\Upsilon(\mathbf{x})))}{s} \mathrm{d}\mathbf{x}, \qquad \forall G \in C_c(\mathbb{C}),$$

where ℓ_d is the d-dimensional Lebesgue measure. In this case $\{A_n\}_n \sim_\lambda \phi_\Upsilon$ means

$$\lim_{n\to\infty} \frac{1}{s_n} \sum_{j=1}^{s_n} G(\lambda_j(A_n)) = \frac{1}{\ell_d(D)} \int_D \frac{\sum_{i=1}^s G(\lambda_i(\Upsilon(\mathbf{x})))}{s} \mathrm{d}\mathbf{x}, \qquad \forall\, G \in C_c(\mathbb{C}). \tag{2.3}$$

- Let μ be a positive measure on $\mathbb{C}$ of mass $|\mu| \leq 1$. Consider the functional $\phi_\mu : C_c(\mathbb{C}) \to \mathbb{R}$ defined as

$$\phi_\mu(G) := \int_{\mathbb{C}} G \mathrm{d}\mu, \qquad \forall G \in C_c(\mathbb{C}).$$

In this case $\{A_n\}_n \sim_\lambda \phi_\mu$ means

$$\lim_{n\to\infty} \frac{1}{s_n} \sum_{j=1}^{s_n} G(\lambda_j(A_n)) = \int_{\mathbb{C}} G \mathrm{d}\mu, \qquad \forall\, G \in C_c(\mathbb{C}). \tag{2.4}$$

Notice that ϕ_κ, ϕ_Υ and ϕ_μ are all well-defined since the conditions $\ell_d(D) < \infty$, $|\mu| \leq 1$ and $G \in C_c(\mathbb{C})$ make the right hand sides of Equation 2.2, Equation 2.3 and Equation 2.4 dominated by $\|G\|_\infty$. In chapter 4 we will conduct an analysis of the operators ϕ_μ, that will require some concepts from measure theory. We can see that Equation 2.2 is just a particular case of Equation 2.3 with $s = 1$, but higher values of s are interesting only for practical purposes, so for now we put them aside and focus on ϕ_κ. The first result we present links the expressions $\{A_n\}_n \sim_\lambda \phi_\kappa$ and $\{A_n\}_n \sim_\lambda \kappa$, showing that under the right conditions, they are equivalent.

2.1.2 Link To The Intuition

In order to formally link the ergodic formula with the intuition provided in chapter 1, we need several arguments that we will introduce in the following chapters. In particular, here is a list of the most important tools we still miss.

- A complete pseudometric structure on the space of sequences and symbols.
- Algebraic structures on particular subspaces of sequences and symbols, and how they are linked to the ordering of eigenvalues.
- An analysis of the space of spectral measures.

Using all the listed arguments, we can come up with the following theorem.

Theorem 2.1.2 ([7]). *Let $\{A_n\}_n$ be a matrix-sequence, and let κ be a complex-valued measurable function on* $[0, 1]$.

$$\{A_n\}_n \sim_\lambda \phi_\kappa \iff \{A_n\}_n \sim_\lambda \kappa.$$

In [6], a similar result appears, but considers only real-valued symbol. It is important, though, because its proof does not involve measure theory, and stress the importance of the *increasing rearrangement* of functions that we introduce in the next section.

Theorem 2.1.2 draws a closure for the argument 'spectral symbol', letting us enunciate a general definition.

Definition 2.1.3. *Given a matrix-sequence $\{A_n\}_n$ and a measurable function $\kappa : D \to \mathbb{C}$ on $D \subseteq \mathbb{R}^d$ where $0 < \ell_d(D) < \infty$, we say that κ is a* ***spectral symbol*** *for $\{A_n\}_n$ if $\{A_n\}_n \sim_\lambda \phi_\kappa$, and in this case we write*

$$\{A_n\}_n \sim_\lambda \kappa.$$

Notice that when $D = [0, 1]$, Definition 2.1.3 and Definition 1.3.1 coincide thanks to Theorem 2.1.2. For this reason, from now on we will use mostly the simple notation $\{A_n\}_n \sim_\lambda \kappa$ instead of $\{A_n\}_n \sim_\lambda \phi_\kappa$.

In the next paragraphs and sections we analyse Equation 2.2, leading to an other equivalent characterization of spectral symbol.

2.1.3 Asymptotic Rates of Eigenvalues

Set κ to be a measurable function over $D \subseteq \mathbb{R}^d$ with $0 < \ell_d(D) < \infty$. From the definition, we know that $\{A_n\}_n \sim_\lambda \kappa$ when Equation 2.2 holds,

$$\lim_{n\to\infty} \frac{1}{s_n}\sum_{j=1}^{s_n} G(\lambda_j(A_n)) = \frac{1}{\ell_d(D)}\int_D G(\kappa(\mathbf{x}))\mathrm{d}\mathbf{x}, \qquad \forall\, G \in C_c(\mathbb{C}).$$

Let us now consider a complex disk $B = B(z_0, r) := \{\, z \in \mathbb{C} \mid |z - z_0| \le r \,\}$ and take $G \in C_c(\mathbb{C})$ close to χ_B so that

$$\chi_{B(z_0,r-\varepsilon)}(z) \le G(z) \le \chi_{B(z_0,r+\varepsilon)}(z) \qquad \forall z \in \mathbb{C}$$

for a small enough $\varepsilon > 0$. Analysing both sizes of the ergodic formula, we can see that the left side is almost the asymptotic rate of eigenvalues of A_n lying inside B

$$\lim_{n\to\infty} \frac{1}{s_n}\sum_{j=1}^{s_n} G(\lambda_j(A_n)) \sim \lim_{n\to\infty} \frac{1}{s_n}\sum_{j=1}^{s_n} \chi_B(\lambda_j(A_n)) = \lim_{n\to\infty} \frac{\#\{\, j \mid \lambda_j(A_n) \in B \,\}}{s_n},$$

whereas the right hand side value is roughly the measure of the set $\kappa^{-1}(B)$, divided by $\ell_d(D)$

$$\frac{1}{\ell_d(D)}\int_D G(\kappa(\mathbf{x}))\mathrm{d}\mathbf{x} \sim \frac{1}{\ell_d(D)}\int_D \chi_B(\kappa(\mathbf{x}))\mathrm{d}\mathbf{x} = \frac{\ell_d\{\, \mathbf{x} \in D \mid \kappa(\mathbf{x}) \in B \,\}}{\ell_d(D)}.$$

In other words, the ergodic formula is telling us that the rates of eigenvalues lying in a disk B (or in an interval in case of real eigenvalues) converge approximatively to the normalized measure of the set where $\kappa(x)$ takes values in B. It is evident when $\lambda_i(A_n)$ form an exact sampling of the symbol, like in Figure 1.5 (a), and in general, it holds for almost every disk (or interval). Sadly, there are cases when this correspondence is not perfect, and for some disk the limit

$$\lim_{n\to\infty} \frac{\#\{\, j \mid \lambda_j(A_n) \in B \,\}}{s_n}$$

may not even exists. What can be proved is just that

$$\begin{aligned}\frac{\ell_d\{\, \mathbf{x} \in D \mid \kappa(\mathbf{x}) \in B^\circ \,\}}{\ell_d(D)} \le \liminf_{n\to\infty} \frac{\#\{\, j \mid \lambda_j(A_n) \in B \,\}}{s_n} \le \\ \le \limsup_{n\to\infty} \frac{\#\{\, j \mid \lambda_j(A_n) \in B \,\}}{s_n} \le \frac{\ell_d\{\, \mathbf{x} \in D \mid \kappa(\mathbf{x}) \in \overline{B} \,\}}{\ell_d(D)}.\end{aligned}$$

Here we present a simple example of this behaviour, that is also our first example of spectral symbol according to Definition 2.1.1.

Example 2.1.4

- Let $A_n = \left(1 + (-1)^n \frac{1}{n}\right) I_n$ and let $\kappa(x) \equiv 1$ be a measurable function on $D = [0, 1]$. Given any $G \in \mathbb{C}_c(\mathbb{C})$, we have that

$$\frac{1}{s_n}\sum_{j=1}^{s_n} G(\lambda_j(A_n)) = G\left(1 + (-1)^n \frac{1}{n}\right) \xrightarrow{n\to\infty} G(1),$$
$$\frac{1}{\ell_1(D)}\int_D G(\kappa(\mathbf{x}))\mathrm{d}\mathbf{x} = \frac{1}{\ell_1([0,1])}\int_{[0,1]} G(1)\mathrm{d}\mathbf{x} = G(1),$$

so $\{A_n\}_n \sim_\lambda \kappa$. Consider now the disk $B = B(0, 1)$ and let us compute the rate of eigenvalues inside B.

$$\frac{\#\{\, j \mid \lambda_j(A_n) \in B \,\}}{s_n} = \begin{cases} 1, & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}$$

We can clearly see that the limit does not exists, but

$$\limsup_{n\to\infty} \frac{\#\{\, j \mid \lambda_j(A_n) \in B \,\}}{s_n} = \frac{\ell_1\{\, \mathbf{x} \in D \mid \kappa(\mathbf{x}) \in \overline{B} \,\}}{\ell_1(D)} = 1,$$
$$\liminf_{n\to\infty} \frac{\#\{\, j \mid \lambda_j(A_n) \in B \,\}}{s_n} = \frac{\ell_1\{\, \mathbf{x} \in D \mid \kappa(\mathbf{x}) \in B^\circ \,\}}{\ell_1(D)} = 0.$$

In this situation, χ_B does not satisfy the ergodic formula, but it is easy to prove that if $1 \notin \partial B'$, then the ergodic formula holds for $\chi_{B'}$. In fact,

$$\lim_{n\to\infty} \frac{1}{s_n} \sum_{j=1}^{s_n} \chi_{B'}(\lambda_j(A_n)) = \frac{1}{\ell_1(D)} \int_D \chi_{B'}(\kappa(\mathbf{x}))\mathrm{d}\mathbf{x} = \begin{cases} 1, & 1 \in B'^{\circ}, \\ 0, & 1 \notin \overline{B}'. \end{cases}$$

Consequently, the only 'bad' disks are the ones with the value 1 on the boundary.

In the example, the ergodic formula holds with χ_B for almost any disk B, and in general, one can prove that this property is a characterization of the spectral symbol. It can be seen as a corollary of Theorem 4.1.4 and Lemma 4.1.6.

Theorem 2.1.5. *Let $D \subseteq \mathbb{R}^d$ be a measurable set with positive and finite d-Lebesgue measure, and let $\kappa : D \to \mathbb{C}$ be a measurable function.*

- *If $\{A_n\}_n \sim_\lambda \kappa$, then the set*
$$\{\, r \in \mathbb{R}^+ \mid \chi_{B(z_0,r)} \textit{ does not satisfy (2.2)}\,\}$$
contains at most numerable points for every $z_0 \in \mathbb{C}$.
- *If the set*
$$E_{z_0} := \{\, r \in \mathbb{R}^+ \mid \chi_{B(z_0,r)} \textit{ does not satisfy (2.2)}\,\}$$
has Lebesgue measure zero for every $z_0 \in \mathbb{C}$, then $\{A_n\}_n \sim_\lambda \kappa$.

It is a fundamental property that we will explore in the sext section, since it is useful to tell when different functions can be spectral symbols for the same sequence.

2.2 Rearrangement

In paragraph 1.2.1 we have seen that different functions can represent the spectral distribution of the same sequence. If we rely only on Definition 2.1.1, the spectral symbol is univocally determined by $\{A_n\}_n$, so ϕ_κ and $\phi_{\kappa'}$ are spectral symbols for the same sequence if and only if $\phi_\kappa \equiv \phi_{\kappa'}$. In such case, we say that κ is a *rearrangement* of κ'.

Definition 2.2.1. *Given two functions $\kappa : D \to \mathbb{C}$ and $\kappa' : D' \to \mathbb{C}$ where $D \subseteq \mathbb{R}^d$, $D' \subseteq \mathbb{R}^{d'}$ are measurable sets and have finite non-zero d or d' Lebesgue measure, we say that κ is a **rearrangement** of κ' if for every $G \in C_c(\mathbb{C})$, we have*

$$\frac{1}{\ell_d(D)} \int_D G(\kappa(x))\mathrm{d}x = \frac{1}{\ell_{d'}(D')} \int_{D'} G(\kappa'(x))\mathrm{d}x.$$

The 'rearrangement' is an equivalence relation between functions, and in particular it is transitive. Moreover, from the arguments in paragraph 2.1.3, two different functions κ and κ', even on different domains D, D', produce the same distribution when $\ell(\kappa^{-1}(B))/\ell(D) = \ell(\kappa'^{-1}(B))/\ell(D')$ for almost every disk $B \subseteq \mathbb{C}$, since they are both equal to the asymptotic rate of eigenvalues of A_n inside B. This is not only intuitive, but can be actually proved through some measure theory, and extended to include all disks in $\mathbb{C}$. We can thus wrap up the information we have in the following result.

Lemma 2.2.2. *Let $\kappa : D \to \mathbb{C}$ and $\kappa' : D' \to \mathbb{C}$ be two measurable functions where $D \subseteq \mathbb{R}^d$, $D' \subseteq \mathbb{R}^{d'}$ are measurable sets and $0 < \ell_d(D) < \infty$, $0 < \ell_{d'}(D') < \infty$. Let moreover $\mathscr{B}$ be the set of all closed disks in $\mathbb{C}$. The following propositions are equivalent:*

1. *$\{A_n\}_n \sim_\lambda \kappa \iff \{A_n\}_n \sim_\lambda \kappa'$,*
2. *$\phi_\kappa \equiv \phi_{\kappa'}$,*
3. *κ is a rearrangement of κ',*

4. $\frac{\ell_d\{x|\kappa(x)\in U\}}{\ell_d(D)} = \frac{\ell_{d'}\{x|\kappa'(x)\in U\}}{\ell_{d'}(D')}, \qquad \forall U \in \mathscr{B}.$

Proof. (1. $\implies$ 2.) It comes from the unicity of spectral symbol ϕ.

(2. $\implies$ 1.) Obvious

(2. $\iff$ 3.) It is evident from Definition 2.2.1.

(4. $\implies$ 1.) It follows from Theorem 2.1.5.

(1. $\implies$ 4.) If $G_m \in C_c(\mathbb{C})$ are nonnegative functions that converge punctually to χ_U and are all bounded by an L^1 function, then

$$\frac{1}{\ell_d(D)}\int_D G_m(\kappa(x)) \to \frac{1}{\ell_d(D)}\int_D \chi_U(\kappa(x)), \qquad \frac{1}{\ell_{d'}(D')}\int_{D'} G_m(\kappa'(x)) \to \frac{1}{\ell_{d'}(D')}\int_{D'} \chi_U(\kappa'(x)).$$

Since

$$\lim_{n\to\infty}\frac{1}{s_n}\sum_{i=1}^{s_n} G_m(\lambda_i(A_n)) = \frac{1}{\ell_d(D)}\int_D G_m(\kappa(x)) = \frac{1}{\ell_{d'}(D')}\int_{D'} G_m(\kappa'(x)) \qquad \forall m$$

we conclude that

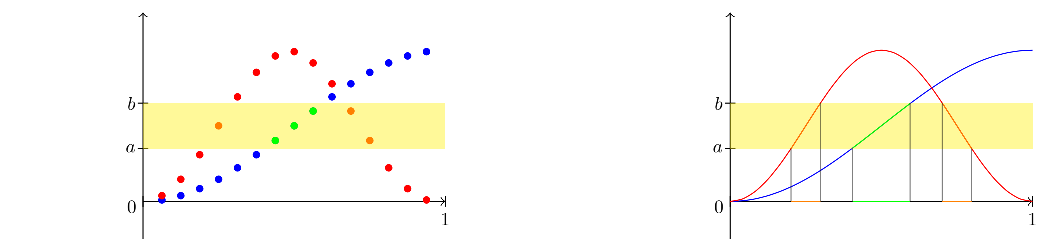
$$\frac{\ell_d\{x|\kappa(x)\in U\}}{\ell_d(D)} = \frac{1}{\ell_d(D)}\int_D \chi_U(\kappa(x)) = \frac{1}{\ell_{d'}(D')}\int_{D'} \chi_U(\kappa'(x)) = \frac{\ell_{d'}\{x|\kappa'(x)\in U\}}{\ell_{d'}(D')}.$$

□

Example 2.2.3

- ► An example of function rearrangement has already been shown in paragraph 1.2.1, where $\kappa_k(x) := 2 - 2\cos(k\pi x)$ where all spectral symbols for the same sequence, and it can be proved that they are all rearrangements of $\kappa_1(x) = 2-2\cos(\pi x)$. In Figure 2.1 we can see that the function κ_2 is a rearrangement of κ_1, since for every interval $[a,b]$, the set $\kappa_1^{-1}[a,b]$ (in green) has the same measure of $\kappa_2^{-1}[a,b]$ (in orange), and their measure is the asymptotic rate of eigenvalues inside $[a,b]$, that does not change after a reordering.

- ► Another example is provided by Figure 1.2 (b), that is the a rearrangement of the function showed in Figure 1.9 (b). This is also a case in which a matrix-valued symbol Υ can be analysed through a scalar symbol κ, and in a sense we can say κ is a rearranged version of Υ, since $\phi_\Upsilon = \phi_\kappa$.



(a) Eigenvalues of A_{15} according two different sorting. *(b) $\kappa_1(x)$ and $\kappa_2(x)$.*

Figure 2.1: Plot of the spectrum of A_{15} according to the orderings that converge to κ_1 and κ_2. Given the interval $[a,b]$ on the abscissae, the sets $\kappa_1^{-1}[a,b]$, $\kappa_2^{-1}[a,b]$ and the eigenvalues inside $[a,b]$ are also indicated.

Recall that Definition 1.3.1 refers only to symbols on the domain $[0,1]$. Later we defined the spectral symbol for more general measurable functions, but in [7] we showed that any function $\kappa : D \to \mathbb{C}$ admits a rearrangement ξ on $[0,1]$, even though κ is complex-valued. Thanks to Lemma 2.2.2 and Theorem 2.1.2, we can thus conclude that id $\{A_n\}_n \sim_\lambda \kappa$, then it admits also a spectral symbol on $[0,1]$, so Definition 1.3.1 is able to identify all the sequences with a symbol represented by a measurable function.

Lemma 2.2.4 ([7]). *$\{A_n\}_n$ enjoys a spectral symbol κ if and only if the eigenvalues of $\{A_n\}_n$ converge to a function ξ on $[0,1]$, and in this case, ξ is a rearrangement for κ.*

In particular, it means that for any sequence with a symbol, we can find an ordering of the eigenvalues that converge to a measurable function in $[0,1]$, even when it seems impossible, like in paragraph 1.3.2. In particular, in the next paragraph, we show that in case the eigenvalues are real, they always converge in increasing order.

2.2.1 Increasing Rearrangement

In this paragraph, we focus on real-valued functions κ, since we will use the natural ordering provided by $\mathbb{R}$. For any measurable function $\kappa : D \to \mathbb{R}$ over a measurable set with finite non-zero d-Lebesgue measure $D \subseteq \mathbb{R}^d$, define the monotone increasing function $\xi : [0,1] \to \mathbb{R}$ as

$$\xi(y) := \inf \left\{ z : \frac{\ell_d\{x : \kappa(x) < z\}}{\ell_d(D)} \geq y \right\}. \tag{2.5}$$

Some of the properties of ξ are contained in the following result.

Lemma 2.2.5 ([57]). *If ξ is defined as in Equation 2.5, then*

- *ξ is increasing, left continuous and its essential range coincides with the essential range of κ,*
- *ξ is a rearrangement of κ,*
- *if ξ' is an increasing rearrangement of κ on $[0,1]$, then $\xi' = \xi$ except for at most numerable points, and they are all discontinuity points for ξ,*
- *ξ is the only left continuous increasing rearrangement of κ on $[0,1]$.*

We have thus a characterization of ξ, that we call *increasing rearrangement* of κ. Notice that if the essential range of κ is an interval, then its increasing rearrangement is continuous, even if κ is discontinuous. We already showed an example in Figure 1.2 (b), that is the increasing rearrangement of the discontinuous function showed in Figure 1.9 (b).

A consequence of Lemma 2.2.5 and the transitivity of the 'rearrangement' relation, is that, given a couple of real valued functions κ and κ', even on different domains, κ is a rearrangement for κ' if and only if their increasing rearrangement coincide. Another way to put it is that every equivalence class of real-valued measurable functions under the 'rearrangement' relation admits an unique left continuous increasing representative. Combining it with Lemma 2.2.2, we obtain the following result.

Lemma 2.2.6. *Let $\kappa : D \to \mathbb{C}$ and $\kappa' : D' \to \mathbb{C}$ be two measurable real-valued functions where $D \subseteq \mathbb{R}^d$, $D' \subseteq \mathbb{R}^{d'}$ are measurable sets and $0 < \ell_d(D) < \infty$, $0 < \ell_{d'}(D') < \infty$. Let ξ and ξ' be the increasing rearrangement of κ and κ' respectively. The following propositions are equivalent:*

1. *$\{A_n\}_n \sim_\lambda \kappa \iff \{A_n\}_n \sim_\lambda \kappa'$,*
2. *$\xi \equiv \xi'$.*

Going back to Definition 1.3.1, the increasing rearrangement plays an important role, since it is often the only symbol for which we know what ordering we must impose on the eigenvalues in order to achieve convergence.

Lemma 2.2.7 ([6]). *Given a sequence $\{A_n\}_n$ where every A_n has only real eigenvalues, and a real-valued measurable function $\kappa : [0,1] \to \mathbb{R}$, consider $\xi : [0,1] \to \mathbb{R}$ the increasing rearrangement of κ. The following holds:*

$$\{A_n\}_n \sim_\lambda \kappa \iff \{A_n\}_n \sim_\lambda \xi,$$

and moreover, the increasing ordering on the eigenvalues of A_n makes the sequence converge to ξ according to Definition 1.3.1.

In other words, if we have only real eigenvalues, then the sequence $\{A_n\}_n$ admits a spectral symbols if and only if its increasingly sorted eigenvalues converge to a measurable function. This result was also hinted by the work [22].

2.3 Singular Value Symbol

Up until now, we have worked with the eigenvalues of matrices A_n. We could do so, because in all our examples our matrices where Hermitian (paragraph 1.2.1, paragraph 1.1.2, paragraph 1.1.1), or normal (paragraph 1.3.2, paragraph 1.3.3) or small perturbation of the above (paragraph 1.2.2). When the sequences do not belong to any of those classes, the eigenvalues may behave in unpredictable ways, and thus it becomes more useful to study the singular values of the matrices.

Recall that any matrix $A \in \mathbb{C}^{n\times n}$ admits a decomposition $A = U\Sigma V$ where U, V are unitary matrices, and Σ is a diagonal, real and entry-wise nonnegative matrix. The diagonal entries of Σ do not depend on the decomposition, they are nonnegative real numbers, and they are called *singular values* of A, that we will denote as $\sigma_1(A), \dots, \sigma_n(A)$. When the singular values of the matrices A_n tend to a distribution for $n \to \infty$, we say that the sequence enjoys a *spectral value symbol*. Its general definition makes again use of functionals.

Definition 2.3.1. *Given a matrix-sequence $\{A_n\}_n$, and given a functional $\phi : C_c(\mathbb{R}) \to \mathbb{R}$, we say that ϕ is a **singular value symbol** for the sequence $\{A_n\}_n$ if*

$$\lim_{n\to\infty} \frac{1}{s_n} \sum_{j=1}^{s_n} G(\sigma_j(A_n)) = \phi(G), \qquad \forall\, G \in C_c(\mathbb{R}), \tag{2.6}$$

where $\sigma_j(A_n)$ are the singular values of A_n. In this case, we write

$$\{A_n\}_n \sim_\sigma \phi.$$

When $\phi = \phi_\kappa$, we can repeat all the arguments of the precedent sections, and come up with analogous results, with the only difference that this time we want information about the asymptotic distribution of the singular values for a sequence $\{A_n\}_n$. In particular, we can always increasingly order the singular values, since they are real numbers, and check if their plot on $[0, 1]$ converge to an increasing function $\kappa : [0, 1] \to \mathbb{R}^+$. In short, both the plot convergence argument and the ergodic formula can be adapted, obtaining two equivalent conditions for a function $\kappa : D \to \mathbb{R}^+$ to be a symbol for $\{A_n\}_n$, where the range must be in $\mathbb{R}^+$ since σ_i are all real and nonnegative.

Here, though, we want to add another degree of freedom on the choice of the symbol. In particular we want a meaningful condition that can be checked on any measurable function $\kappa : D \to \mathbb{C}$, so we define the singular value symbol for a sequence as follows.

Definition 2.3.2. *Given a matrix-sequence $\{A_n\}_n$, and a measurable function $\kappa : D \to \mathbb{C}$ on $D \subseteq \mathbb{R}^d$, where $0 < \ell_d(D) < \infty$, we say that κ is a **singular value symbol** for $\{A_n\}_n$ if*

$$\lim_{n\to\infty} \frac{1}{s_n} \sum_{j=1}^{s_n} G(\sigma_j(A_n)) = \frac{1}{\ell_d(D)} \int_D G(|\kappa(\mathbf{x})|) \mathrm{d}\mathbf{x} = \phi_{|\kappa|}(G), \qquad \forall\, G \in C_c(\mathbb{R}). \tag{2.7}$$

In this case, we write

$$\{A_n\}_n \sim_\sigma \kappa.$$

From the definition, we can make the following observations.

- A singular value symbol for a sequence $\{A_n\}_n$ depends only on the singular values of the matrices, so it does not change when we multiply the matrices A_n by unitary matrices on any side.
- The ergodic formula has to be checked only on $C_c(\mathbb{R})$, opposed to the usual $C_c(\mathbb{C})$ space, since $|\kappa(\mathbf{x})| \in \mathbb{R}$.
- If κ is a singular value symbol for $\{A_n\}_n$, automatically even $|\kappa|$ is a singular value symbol for the same sequence.
- Differently from the spectral symbol, κ is a complex-valued function, so usually it is not a rearrangement of the real-valued function $|\kappa|$. As a consequence, the rearrangement result does not hold for the singular value symbols.

Here we list the results of the preceding sections adapted to the case of singular value symbols.

Theorem 2.3.3. *Let $D \subseteq \mathbb{R}^d$ be a measurable set with positive and finite d-Lebesgue measure, and let $\kappa : D \to \mathbb{C}$ be a measurable function.*

- *If $\{A_n\}_n \sim_\sigma \kappa$, then the set*
$$\{\, r \in \mathbb{R}^+ \mid \chi_{B(x_0,r)} \text{ does not satisfy (2.7)} \,\}$$
contains at most numerable points for every $x_0 \in \mathbb{R}$.
- *If the set*
$$E_{z_0} := \{\, r \in \mathbb{R}^+ \mid \chi_{B(z_0,r)} \text{ does not satisfy (2.7)} \,\}$$
has Lebesgue measure zero for every $z_0 \in \mathbb{C}$, then $\{A_n\}_n \sim_\sigma \kappa$.

Lemma 2.3.4. *Let $\kappa : D \to \mathbb{C}$ and $\kappa' : D' \to \mathbb{C}$ be two measurable functions where $D \subseteq \mathbb{R}^d$, $D' \subseteq \mathbb{R}^{d'}$ are measurable sets and $0 < \ell_d(D) < \infty$, $0 < \ell_{d'}(D') < \infty$. Let moreover $\mathscr{B}$ be the set of all closed intervals in $\mathbb{R}$. The following propositions are equivalent:*

- $\{A_n\}_n \sim_\sigma \kappa \iff \{A_n\}_n \sim_\sigma \kappa'$,
- $\phi_{|\kappa|} \equiv \phi_{|\kappa'|}$,
- $|\kappa|$ *is a rearrangement of* $|\kappa'|$,
- $\frac{\ell_d\{x \mid |\kappa(x)| \in U\}}{\ell_d(D)} = \frac{\ell_{d'}\{x \mid |\kappa'(x)| \in U\}}{\ell_{d'}(D')}, \qquad \forall U \in \mathscr{B}.$

Theorem 2.3.5. *Given a measurable function $\kappa : D \to \mathbb{C}$ on $D \subseteq \mathbb{R}^d$, where $0 < \ell_d(D) < \infty$, let ξ be the increasing rearrangement of $|\kappa|$. Let $\{A_n\}_n$ be a sequence of matrices, where $A_n \in \mathbb{C}^{s_n \times s_n}$ and s_n is a sequence of natural numbers that diverges. For every n, let $h_n = s_n^{-1}$ and call $\xi_n(x)$ the function that is linear on every interval $[(j-1)h_n, jh_n]$ for $j = 1, \dots, s_n$ and such that*

$$\xi_n(0) = 0, \qquad \xi_n(jh_n) = \sigma_j(A_n) \quad \forall\, j = 1, \dots, s_n,$$

where $\sigma_j(A_n)$ are the singular values of A_n considered in non-decreasing order

$$\sigma_1(A_n) \le \sigma_2(A_n) \le \cdots \le \sigma_{s_n}(A_n).$$

In this case, κ is a singular value symbol for $\{A_n\}_n$ if and only if $\{\xi_n(x)\}_n$ converges in measure to $\xi(x)$.

Moreover, for every ξ' rearrangement on $[0,1]$ of $|\kappa|$, there exists an ordering of the singular values for which $\{\xi_n(x)\}_n$ converges in measure to $\xi'(x)$.

The reason for this choice is not easy to explain, but is linked to the fact that for a normal matrix A, we have $\sigma_i(A) = |\lambda_i(A)|$, so for every normal sequence $\{A_n\}_n$, it is easy to check from the ergodic formula that

$$\{A_n\}_n \sim_\lambda \kappa \implies \{A_n\}_n \sim_\sigma \kappa.$$

We will see better how spectral symbol and singular value symbols are connected in section 5.2.

Chapter 3

Perturbation

In the previous sections, we have defined what is a symbol (spectral or referred to singular values) of a matrix-sequence $\{A_n\}_n$. In particular, we also gave an intuitive notion of symbol, in the case it is represented by a measurable function κ over $[0,1]$. Here we want to examine how a perturbation of the matrices A_n affects the symbol of the sequence.

We have already seen some modifications of the sequence that does not change the symbol. In particular, if $\{A_n\}_n \sim_\lambda \kappa$ (or $\{A_n\}_n \sim_\sigma \kappa$), then the symbol stays unchanged if

- we extract a subsequence from $\{A_n\}_n$,
- we modify a finite number of elements in $\{A_n\}_n$,
- we operate a base change on every A_n (or, in the case of singular values, multiply orthogonal matrices to every A_n).

From paragraph 1.2.2 we have also seen that a perturbation of low norm or low rank seems to not affect the symbol, and in the present section we are going to explain formally why.

In order to analyse the perturbation, though, we first need to specify the space we are working on, and fix some notation. After fixing a diverging sequence of positive integers $\{s_n\}_n$, we denote the space of matrix-sequences as

$$\mathscr{E} := \{\, \{A_n\}_n \mid A_n \in \mathbb{C}^{s_n \times s_n} \,\}$$

and the space of Hermitian matrix-sequences as

$$\mathscr{E}_H := \{\, \{A_n\}_n \mid A_n \in \mathbb{C}^{s_n \times s_n}, \quad A_n = A_n^H \ \forall\, n \,\}\,.$$

A sequence of elements in $\mathscr{E}$ or $\mathscr{E}_H$ will always be indexed by m, and denoted as $\{B_{n,m}\}_{n,m}$ where it is intended that $B_{n,m} \in \mathbb{C}^{s_n \times s_n}$ for every m.

On the other side, we also need to denote the space of symbols we consider. We always work with complex-valued functions on $D \subseteq \mathbb{R}^d$ that are measurable according to the d-Lebesgue measure ℓ_d, where $0 < \ell_d(D) < \infty$. We denote this space as

$$\mathscr{M}_D := \{\, \kappa : D \to \mathbb{C} \quad \ell_d\text{-measurable} \,\}\,/ \sim,$$

where we identify two functions κ and κ' through the relation $\sim$ if they are equal ℓ_d almost everywhere.

3.1 Approximating Classes of Sequences

let us return to Definition 1.3.1, where we defined the spectral symbol through the convergence of the plot of eigenvalues to a limit function in measure. Note that if for every matrix A_n we change few eigenvalues, even with a huge gap, or if we globally perturb the eigenvalues by a quantity $\epsilon_n \xrightarrow{n\to\infty} 0$, the convergence in measure of the eigenvalues plot still holds.

On practical terms, though, a perturbation of the matrices A_n is always additive, meaning that the perturbed sequence will be of the form $\{A_n + E_n\}_n$. It is not easy in general to analyse the eigenvalues of a sum of matrices, but there are classical results in linear algebra that can help us.

3.1.1 Definition and Properties

From now on, when we write the eigenvalues of an Hermitian matrix $X \in \mathbb{C}^{m\times m}$ as $\lambda_j(X)$, or the singular values of a generic matrix $X \in \mathbb{C}^{m\times m}$ as $\sigma_j(X)$, we always suppose that they are ordered in a non-increasing way, meaning that

$$\sigma_1(X) \geq \ldots \geq \sigma_m(X), \qquad \lambda_1(X) \geq \ldots \geq \lambda_m(X),$$

and by convention, we impose that when we have σ_j or λ_j with $j \notin [1, m]$, then

$$\lambda_j(X) = \begin{cases} +\infty, & j < 1, \\ -\infty, & j > m, \end{cases} \qquad \sigma_j(X) = \begin{cases} +\infty, & j < 1, \\ 0, & j > m. \end{cases}$$

In what follows, the notation $\|X\|$ stands for the 2-norm or spectral norm of the matrix X, that is namely the biggest singular value of X, and, in case X is normal, it is also the spectral radius of X.
Moreover, the notation $\|X\|_p$ stand for the *p-Schatten norm* of the matrix A, defined as

$$\|A\|_p := (\sigma_1(A)^p + \cdots + \sigma_n(A)^p)^{1/p}$$

for any $p \in [1, \infty)$, and

$$\|A\|_\infty := \sigma_1(A) = \|A\|.$$

In particular, the 2-Schatten norm coincides with the Frobenius norm.

$$\|A\|_2^2 = \sigma_1(A)^2 + \cdots + \sigma_n(A)^2 = \mathrm{tr}(A^H A) = \sum_{i,j=1}^{n} |a_{i,j}|^2 = \|A\|_F^2.$$

Using these notations, we can formulate the Interlacing Theorems and some consequences of Weyl Inequalities for eigenvalues and singular values of matrices.

3.1.1.1 Interlacing Theorems and Weyl's Inequalities

Theorem 3.1.1 (**interlacing theorem for singular values**, [20, Exercise 111.2.4]). *Let $Y = X + E$, where $X, E \in \mathbb{C}^{m\times m}$ and $\mathrm{rk}(E) = k$. Then*

$$\sigma_{j-k}(X) \geq \sigma_j(Y) \geq \sigma_{j+k}(X), \qquad j = 1, \ldots, m.$$

Theorem 3.1.2 (**interlacing theorem for eigenvalues**, [20, Problem III.6.4]). *Let $Y = X + E$, where $X, E \in \mathbb{C}^{m\times m}$ are Hermitian and $\mathrm{rk}(E) = k$. Then*

$$\lambda_{j-k}(X) \geq \lambda_j(Y) \geq \lambda_{j+k}(X), \qquad j = 1, \ldots, m.$$

These results have to be interpreted keeping in mind the informal definition for symbol. In fact, if $\{A_n\}_n \sim_\lambda \kappa$ and A_n are Hermitian matrices, then the eigenvalues of A_n tend to converge to the increasing rearrangement ξ of κ on $[0, 1]$. Remembering that the plot of the eigenvalues associates $\lambda_j(A_n)$ to the ordinate j/n, it is easy to show that given a number $x \in [0, 1]$ and a function $f(n) = o(n)$,

$$\lambda_{\lfloor xn\rfloor - f(n)}(A_n) - \lambda_{\lfloor xn\rfloor + f(n)}(A_n) \xrightarrow{n\to\infty} 0$$

for almost every x, since they both tend to the value $\xi(x)$. It means that if $f(n) = \mathrm{rk}(E_n) = o(n)$, then the equation

$$\lambda_{j-f(n)}(A_n) \geq \lambda_j(A_n + E_n) \geq \lambda_{j+f(n)}(A_n)$$

shows that almost all the eigenvalues of the perturbed sequence are approximated well by the eigenvalues of A_n, except for the first $f(n)$ and the last $f(n)$. In other words, a perturbation $\{E_n\}_n$ with $\mathrm{rk}(E_n) = o(n)$ changes significantly only $o(n)$ eigenvalues for every n, and it is not enough to disrupt the convergence in measure of the eigenvalues plot, and thus the overall spectral symbol. The same argument can be adapted to the case of singular values.

Theorem 3.1.3 (**perturbation theorem for singular values**, [20, Problem 111.6.5]). *Let $Y = X + E$, where $X, E \in \mathbb{C}^{m\times m}$. Then*

$$|\sigma_j(X) - \sigma_j(Y)| \leq \|E\|, \quad j = 1, \ldots, m.$$

Theorem 3.1.4 (**perturbation theorem for eigenvalues**, [20, Corollary III.2.6]). *Let $Y = X + E$, where $X, E \in \mathbb{C}^{m\times m}$ are Hermitian. Then*

$$|\lambda_j(X) - \lambda_j(Y)| \le \|E\|, \quad j = 1, \dots, m.$$

These results show that the perturbation E globally affects the eigenvalues and singular values proportionately to its norm $\|E\|$, so a sequence $\{E_n\}_n$ with $\|E_n\| = o(1)$ cannot change the symbol of $\{A_n\}_n \sim_{\lambda,\sigma} \kappa$. Following this lead, one can define a notion of convergence on $\mathscr{E}$ due to Serra-Capizzano [73], but that was actually inspired by Tilli's pioneering paper on LT sequences [84].

3.1.1.2 Definition

Definition 3.1.5. *Given a sequence of matrix-sequences $\{B_{n,m}\}_{n,m}$, it is said to be an* ***approximating class of sequences*** *(in short a.c.s.) for $\{A_n\}_n$ if there exist $\{N_{n,m}\}_{n,m}$ and $\{R_{n,m}\}_{n,m}$ such that for every m there exists n_m with*

$$A_n = B_{n,m} + N_{n,m} + R_{n,m}, \qquad \|N_{n,m}\| \le \omega(m), \qquad \mathrm{rk}(R_{n,m}) \le s_n c(m)$$

for every $n > n_m$, and

$$\omega(m) \xrightarrow{m\to\infty} 0, \qquad c(m) \xrightarrow{m\to\infty} 0.$$

In this case, we say that $\{B_{n,m}\}_{n,m}$ is a.c.s. convergent to $\{A_n\}_n$, and we use the notation $\{B_{n,m}\}_{n,m} \xrightarrow{a.c.s.} \{A_n\}_n$.

In other words, $\{B_{n,m}\}_{n,m}$ converges to $\{A_n\}_n$ if the difference $\{A_n - B_{n,m}\}_n$ can be decomposed into $\{N_{n,m}\}_{n,m}$ of 'small norm' and $\{R_{n,m}\}_{n,m}$ of 'small rank', according to the intuition given by the classical results in linear algebra.

Example 3.1.6

- Let us modify the sequences in paragraph 1.2.2. In particular, define

$$N_{n,m} = \frac{1}{\log(m)} \begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 0 & -1 \\ & & & 1 & 0 \end{pmatrix}, \qquad R_{n,m} = 2^n \sum_{i=1}^{\lfloor n/m \rfloor} (\boldsymbol{e}^{(n)} - \boldsymbol{e}_i^{(n)})(\boldsymbol{e}^{(n)} - \boldsymbol{e}_i^{(n)})^T,$$

  and

$$A_n = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}, \qquad B_{n,m} = A_n + N_{n,m} + R_{n,m},$$

  where $B_{n,m}, A_n, N_{n,m}, R_{n,m}$ are all $n \times n$ matrices. We have thus built an example of sequences of matrix-sequences where

$$\|N_{n,m}\| \le \frac{2}{\log(m)} \xrightarrow{m\to\infty} 0, \qquad \frac{\mathrm{rk}(R_{n,m})}{n} \le \frac{1}{n}\left\lfloor \frac{n}{m} \right\rfloor \le \frac{1}{m} \xrightarrow{m\to\infty} 0.$$

  and consequently $\{B_{n,m}\}_{n,m} \xrightarrow{a.c.s.} \{A_n\}_n$.

Here we report a result often used in practice when checking for a.c.s. convergence, even if it does not work well with high-norm low-rank perturbations, like in Example 3.1.6.

Theorem 3.1.7 ([52, Corollary 5.3]). *For every $p \in [1, \infty]$, if*

$$\lim_{m\to\infty} \limsup_{n\to\infty} \, (s_n)^{-1/p} \|A_n - B_{n,m}\|_p = 0,$$

then $\{B_{n,m}\}_{n,m} \xrightarrow{a.c.s.} \{A_n\}_n$. Here we are using the notation $1/\infty = 0$, and $\|\cdot\|_p$ stands for the p-Schatten norm introduced at the start of paragraph 3.1.1.

Notice that the a.c.s. convergence is defined for all sequences, but in Theorem 3.1.4 and Theorem 3.1.2 we need Hermitian sequences in order to work with the eigenvalues. In fact, when we deal with general sequences and perturbation, the eigenvalues may behave in a very erratic way. The only cases we are able to spectrally analyse are non-Hermitian perturbations of Hermitian or normal sequences, but the a.c.s. convergence is not enough, and we need more specific results. We will cover the argument in paragraph 3.2.3.

3.1.1.3 Sparsely Unbounded Sequences

To conclude the present paragraph, we provide a result about the algebraic properties of the a.c.s. convergence on sequences, but first we need an additional definition.

Definition 3.1.8. *Let $\{A_n\}_n$ be a matrix-sequence. We say that $\{A_n\}_n$ is **sparsely unbounded** (s.u.) if for every $M > 0$ there exists n_M such that, for $n \geq n_M$,*

$$\frac{\#\{i \in \{1, \dots, s_n\} : \sigma_i(A_n) > M\}}{s_n} \leq r(M),$$

where $\lim_{M\to\infty} r(M) = 0$.
*Moreover, we say that $\{A_n\}_n$ is **spectrally sparsely unbounded** (s.s.u.) if for every $M > 0$ there exists n_M such that, for $n \geq n_M$,*

$$\frac{\#\{i \in \{1, \dots, s_n\} : |\lambda_i(A_n)| > M\}}{s_n} \leq r(M),$$

where $\lim_{M\to\infty} r(M) = 0$.

A sequence $\{A_n\}_n$ is sparsely unbounded when for every $\varepsilon > 0$ we can find a constant M such that the rate of singular values grater then M is definitively less then ε. In particular, if $\{A_n\}_n \sim_\sigma \kappa$, then $\{A_n\}_n$ is s.u., since the rate of the singular values grater than M correspond to the measure of the set $\{\, x \in D \mid |\kappa| > M \,\}$ and it converges to the empty set for $M \to \infty$ ([52, Proposition 5.4]). A similar argument shows us that the same holds for the spectral symbols.

Being s.u. or s.s.u. is thus a property shared by all sequences admitting a symbol κ, but the same cannot be said for sequences admitting symbol ϕ in general. Actually, it turns out that for sequences admitting symbol ϕ, s.u. and s.s.u. are sufficient conditions for the existence of a function κ such that $\phi \equiv \phi_\kappa$. Since the proof of the next result is quite technical, we will report it in Appendix B.

Lemma 3.1.9. *Let $D \subseteq \mathbb{R}^d$ be any measurable set with $0 < \ell_d(D) < \infty$. Given a sequence $\{A_n\}_n \sim_\lambda \phi$ for some functional $\phi : C_C(\mathbb{C}) \to \mathbb{R}$, then*

$$\{A_n\}_n \text{ s.s.u.} \iff \exists \kappa \in \mathscr{M}_D : \phi = \phi_\kappa.$$

Given a sequence $\{A_n\}_n \sim_\sigma \phi$ for some functional $\phi : C_C(\mathbb{R}) \to \mathbb{R}$, then

$$\{A_n\}_n \text{ s.u.} \iff \exists \kappa \in \mathscr{M}_D : \phi = \phi_\kappa.$$

Going back to the a.c.s. convergence, the s.u. property comes into play when dealing with multiplication of sequences, as shown by the next result.

Theorem 3.1.10 ([54, Theorem 2.3]). *If $\{B_{n,m}\}_{n,m} \xrightarrow{a.c.s.} \{A_n\}_n$ and $\{B'_{n,m}\}_{n,m} \xrightarrow{a.c.s.} \{A'_n\}_n$, then*

- $\{B^H_{n,m}\}_{n,m} \xrightarrow{a.c.s.} \{A^H_n\}_n$,
- $\{\alpha B_{n,m} + \beta B'_{n,m}\}_{n,m} \xrightarrow{a.c.s.} \{\alpha A_n + \beta A'_n\}_n$,
- $\{B_{n,m}B'_{n,m}\}_{n,m} \xrightarrow{a.c.s.} \{A_n A'_n\}_n$ *whenever* $\{A_n\}_n$ *and* $\{A'_n\}_n$ *are s.u.,*
- $\{B_{n,m}C_n\}_{n,m} \xrightarrow{a.c.s.} \{A_n C_n\}_n$ *whenever* $\{C_n\}_n$ *is s.u.*

3.1.2 Complete Pseudometric

The concept of a.c.s. convergence on the space of matrix-sequences $\mathscr{E}$ is actually induced by a pseudometric $d : \mathscr{E} \times \mathscr{E} \to \mathbb{R}^+$, that is a symmetric function for which the triangular inequality holds, and the only reason it is not a real distance, is that it may assume the value zero even when evaluated on pairs of distinct elements.

Definition 3.1.11. *Given a matrix $A \in \mathbb{C}^{s_n \times s_n}$, we define the function*

$$p^{(s_n)}_{a.c.s.}(A) := \min_{i=1,\dots,s_n+1} \left\{ \frac{i-1}{s_n} + \sigma_i(A) \right\}.$$

Given now a sequence $\{A_n\}_n \in \mathscr{E}$, we can denote

$$\rho_{a.c.s.}(\{A_n\}_n) := \limsup_{n\to\infty} p^{(s_n)}_{a.c.s.}(A_n),$$

that lets us to introduce the ***a.c.s. distance*** *$d_{a.c.s.}$ on $\mathscr{E}$:*

$$d_{a.c.s.}(\{A_n\}_n, \{B_n\}_n) := \rho_{a.c.s.}(\{A_n - B_n\}_n).$$

The intuition behind this definition is quite straightforward. By Definition 3.1.5, we know that two sequences $\{A_n\}_n$ and $\{B_n\}_n$ are close according to the a.c.s. convergence when their difference $\{A_n - B_n\}_n$ can be written as a sum of a low norm sequence and a low rank sequence. We want thus to choose the decomposition

$$A_n - B_n = N_n + R_n$$

that minimizes the quantity $\|N_n\| + \mathrm{rk}(R_n)/s_n$ for every n. The answer to this problem is provided by the singular value decomposition (SVD) of $A_n - B_n = U\Sigma V$. In fact, if we split the diagonal matrix Σ into $\widetilde{\Sigma}_i + \widehat{\Sigma}_i$ where

$$\Sigma = \begin{pmatrix} \sigma_1 & & & & & \\ & \sigma_2 & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & \sigma_{s_n-1} & \\ & & & & & \sigma_{s_n} \end{pmatrix} = \underbrace{\begin{pmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_{i-1} & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}}_{\widetilde{\Sigma}_i} + \underbrace{\begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & \sigma_i & & \\ & & & & \ddots & \\ & & & & & \sigma_{s_n} \end{pmatrix}}_{\widehat{\Sigma}_i}$$

and we call $R_n^{(i)} = U\widetilde{\Sigma}_i V$, $N_n^{(i)} = U\widehat{\Sigma}_i V$, we have that

$$A_n - B_n = N_n^{(i)} + R_n^{(i)}, \qquad \mathrm{rk}(R_n^{(i)}) = i-1, \quad \|N_n^{(i)}\| = \sigma_i.$$

It can be proved that the decomposition minimizing $\mathrm{rk}(R_n)/s_n + \|N_n\|$ is necessarily one of the pairs $(R_n^{(i)}, N_n^{(i)})$ for some i, so that

$$\inf\{\mathrm{rk}(R_n)/s_n + \|N_n\|\} = \min_{i=1,\dots,s_n+1} \left\{ \frac{i-1}{s_n} + \sigma_i \right\} = p_{a.c.s.}(A_n - B_n).$$

Consequently, we conclude that the a.c.s. convergence is induced by $d_{a.c.s.}$.

Theorem 3.1.12 ([52, Theorem 5.1])**.** *The function $d_{a.c.s.}$ induces the a.c.s. convergence on $\mathscr{E}$, meaning that for every sequences $\{B_{n,m}\}_{n,m}$ and $\{A_n\}_n$ we have*

$$\{B_{n,m}\}_{n,m} \xrightarrow{a.c.s.} \{A_n\}_n \iff d_{a.c.s.}(\{B_{n,m}\}_n, \{A_n\}_n) \xrightarrow{m\to\infty} 0.$$

Let us now analyse the functions $p_{a.c.s.}$, $\rho_{a.c.s.}$ and $d_{a.c.s.}$. Notice that, by convention, $\sigma_{s_n+1}(A_n) = 0$, so

$$p^{(s_n)}_{a.c.s.}(A_n) = \min_{i=1,\dots,s_n+1} \left\{ \frac{i-1}{s_n} + \sigma_i(A_n) \right\} \le \frac{s_n}{s_n} = 1 \qquad \forall n$$

and consequently

$$\rho_{a.c.s.}(\{A_n\}_n) = \limsup_{n\to\infty} p^{(s_n)}_{a.c.s.}(A_n) \le 1,$$

$$d_{a.c.s.}(\{A_n\}_n, \{B_n\}_n) = \rho_{a.c.s.}(\{A_n - B_n\}_n) \le 1.$$

If we analyse the function $p^{(s_n)}_{a.c.s.}$, we can prove that ([52, Lemma 5.1])

- $0 \le p^{(s_n)}_{a.c.s.}(A) \le 1$ for every $A \in \mathbb{C}^{s_n \times s_n}$ and every n,
- $p^{(s_n)}_{a.c.s.}(O_{s_n}) = 0$ for every n, where O_{s_n} is the $s_n \times s_n$ zero matrix,
- $p^{(s_n)}_{a.c.s.}(A) = p^{(s_n)}_{a.c.s.}(-A)$ for every $A \in \mathbb{C}^{s_n \times s_n}$ and every n,
- $p^{(s_n)}_{a.c.s.}(A + B) \le p^{(s_n)}_{a.c.s.}(A) + p^{(s_n)}_{a.c.s.}(B)$ for every $A, B \in \mathbb{C}^{s_n \times s_n}$ and every n,

and consequently

$$d^{(s_n)}_{a.c.s.}(A, B) := p^{(s_n)}_{a.c.s.}(A - B)$$

is a pseudometric on the space of $s_n \times s_n$ complex matrices. The function $d_{a.c.s.}$ is the limit of $d^{(s_n)}_{a.c.s.}$, and we can prove that it is a complete pseudometric.

Theorem 3.1.13 ([5, 7, 11])**.** *Let d_n be pseudometrics on the space of matrices $\mathbb{C}^{n\times n}$ bounded by the same constant $L > 0$ for every n. Then the function*

$$d(\{A_n\}_n, \{B_n\}_n) := \limsup_{n\to\infty} d_{s_n}(A_n, B_n)$$

is a complete pseudometric on the space of matrix-sequences $\mathscr{E}$. In particular, given $\{B_{n,m}\}_{n,m}$ a Cauchy sequence with respect to the pseudometric d, there exists an increasing map $m : \mathbb{N} \to \mathbb{N}$ with $\lim_{n\to\infty} m(n) = \infty$ such that for every increasing map $m' : \mathbb{N} \to \mathbb{N}$ that respects

- $m'(n) \le m(n) \quad \forall n$
- $\lim_{n\to\infty} m'(n) = \infty$

we get

$$\{B_{n,m}\}_{n,m} \xrightarrow{d} \{B_{n,m'(n)}\}_n.$$

The last result holds for every limit of pseudometrics, so $d_{a.c.s.}$ is indeed a complete pseudometric on $\mathscr{E}$. Moreover, it shows that every a.c.s. convergent sequence $\{B_{n,m}\}_{n,m}$ admits an a.c.s. limit that is a sequence built using only matrices chosen among the $B_{n,m}$ themselves.

$d_{a.c.s.}$ is not the only complete pseudometric inducing the a.c.s. convergence. Actually, we can give infinite examples of such distances, one for every concave function with some properties.

Definition 3.1.14. *Let $\varphi : [0, \infty) \to [0, \infty)$ be a concave bounded continuous function such that $\varphi(0) = 0$ and $\varphi > 0$ on $(0, \infty)$. Given a matrix $A \in \mathbb{C}^{s_n \times s_n}$, we can define the function*

$$p^{\varphi,s_n}(A) = \frac{1}{s_n} \sum_{i=1}^{s_n} \varphi(\sigma_i(A)).$$

Given now a sequence $\{A_n\}_n \in \mathscr{E}$, we can denote the ***$\varphi$ distance*** *as*

$$\rho^{\varphi}(\{A_n\}_n) = \limsup_{n\to\infty}\, p^{\varphi,s_n}(A_n),$$

$$d^{\varphi}(\{A_n\}_n, \{B_n\}_n) = \rho^{\varphi}(\{A_n - B_n\}_n).$$

In [11], it has been shown that Theorem 3.1.13 still apply with similar arguments, and we can conclude that d^{φ} is again a complete pseudometric on $\mathscr{E}$, and it also induces the a.c.s. convergence, thus providing us infinite possibilities to check the a.c.s. convergence for sequences.

Theorem 3.1.15 ([11]). *Let $\varphi : [0,\infty) \to [0,\infty)$ be a concave bounded continuous function such that $\varphi(0) = 0$ and $\varphi > 0$ on $(0,\infty)$. Then, the function d^{φ} is a complete pseudometric on $\mathscr{E}$ inducing the a.c.s. convergence.*

The perk of using a concave function induced pseudometric, is that it requires different knowledge about the singular values of the matrices from the ones needed in the computation of $d_{a.c.s.}$.
Example 3.1.16

- $$\varphi_1(x) = \min\{x, 1\}$$

  leads to the distance

  $$d^{\varphi_1}(\{A_n\}_n, \{B_n\}_n) = \limsup_{n\to\infty} \frac{1}{s_n} \sum_{i=1}^{s_n} \min\{\sigma_i(A_n - B_n), 1\}.$$

- $$\varphi_2(x) = \frac{x}{x+1}$$

  leads to the distance

  $$d^{\varphi_2}(\{A_n\}_n, \{B_n\}_n) = \limsup_{n\to\infty} \frac{1}{s_n} \sum_{i=1}^{s_n} \frac{\sigma_i(A_n - B_n)}{\sigma_i(A_n - B_n) + 1}.$$

Now that we have defined the a.c.s. convergence and pseudometric, it is time to see how it affects the symbols of the sequences.

3.1.3 Metric on Measurable Functions

First of all, let us fix a metric on the space of measurable functions $\mathscr{M}_D$.

Definition 3.1.17. *On the space $\mathscr{M}_D$ with $D \subseteq \mathbb{R}^d$, we can define the function*

$$p_{mea}(f) := \inf_{L\ge 0} \left\{ \frac{\ell_d\{x \in D \mid |f| > L\}}{\ell_d(D)} + L \right\}.$$

This function induces the ***distance***

$$d_{mea}(f, g) = p_{mea}(f - g).$$

The distance d_{mea} induces the convergence in measure on $\mathscr{M}_D$ and it is a complete metric. Here we report the definition of $p^{(n)}_{a.c.s.}$.

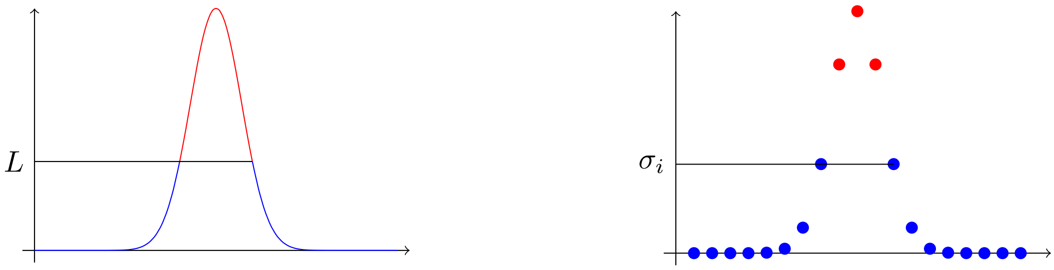
$$p^{(n)}_{a.c.s.}(A) := \min_{i=1,\dots,n+1} \left\{ \frac{i-1}{n} + \sigma_i(A) \right\}.$$

It is immediate to notice that p_{mea} and $p^{(n)}_{a.c.s.}$ have explicit similarities. In fact they are both infimum of a sum with two addends. In $p^{(n)}_{a.c.s.}$, we have σ_i and the rate of singular values greater then σ_i. In p_{mea}, we have L and the rate of the domain where f takes value greater then L. Remembering Lemma A.0.1, we see that if $\{A_n\}_n \sim_\sigma f$ and $\sigma_i = L$, then a limit operation brings one quantity into the other, as seen in Figure 3.1.

Moreover, p_{mea} aims to minimize, over all possible partitions of the domain D into two sets

$$\widetilde{D} = \{x \in D \mid f(x) \le L\}, \quad \widehat{D} = \{x \in D \mid f(x) > L\},$$

the sum of the supremum of $|f|$ over $\widehat{D}$ and the normalized measure of $\widetilde{D}$, that mimics exactly the decomposition $\Sigma = \widetilde{\Sigma}_i + \widehat{\Sigma}_i$ seen in paragraph 3.1.2 that minimizes $\|N_n\| + \mathrm{rk}(R_n)/s_n$.



(a) Partition of the domain D into $\{ f \leq L \}$ (in blue) and $\{ f > L \}$ (in red).

(b) Partition of the singular values into $\{ \sigma \leq \sigma_i \}$ (in blue) and $\{ \sigma > \sigma_i \}$ (in red).

Figure 3.1: Plot of f and the singular values of A_n with $\{A_n\}_n \sim_\sigma f$. The rate of eigenvalues greater than $L = \sigma_i$ converges to the normalized measure of $\{ f > L \}$.

It is thus not surprising that we can find a very close relation between the two metrics, expressed by the following result.

Theorem 3.1.18 ([5])**.** *Given $\{A_n\}_n \in \mathscr{E}$ and $\kappa \in \mathscr{M}_D$, then*

$$\{A_n\}_n \sim_\sigma \kappa \implies \rho_{a.c.s.}(\{A_n\}_n) = p_{mea}(\kappa).$$

Moreover, for every convex function φ we have induced a distance on $\mathscr{E}$, so it is natural to think that there may exist corresponding measures on $\mathscr{M}_D$. That is indeed true, as established by the following definition and result.

Definition 3.1.19. *Let $\varphi : [0, \infty) \to [0, \infty)$ be a concave bounded continuous function such that $\varphi(0) = 0$ and $\varphi > 0$ on $(0, \infty)$. Let*

$$p^\varphi_{mea}(f) = \frac{1}{\mu_k(D)} \int_D \varphi(|f|) \mathrm{d}\mu_k,$$
$$d^\varphi_{mea}(f, g) = p^\varphi_{mea}(f - g),$$

for any $f, g \in \mathscr{M}_D$.

Theorem 3.1.20 ([11])**.** *Let $\varphi : [0, \infty) \to [0, \infty)$ be a bounded continuous function such that $\varphi(0) = 0$. Then, for every $\{A_n\}_n \in \mathscr{E}$ and $\kappa \in \mathscr{M}_D$,*

$$\{A_n\}_n \sim_\sigma \kappa \qquad \implies \qquad \rho^\varphi(\{A_n\}_n) = p^\varphi_{mea}(\kappa).$$

Notice that in Theorem 3.1.20, φ is not necessarily a concave function, and from the proof in [11] we can see that we could weaken the hypothesis. The concavity is essential when proving that d^φ_{mea} and d^φ are distances inducing the convergence in measure on $\mathscr{M}_D$ and the a.c.s. convergence on $\mathscr{E}$ respectively, as showed in Theorem 3.1.15.

3.2 Invariance and Convergence of Symbols

3.2.1 Closure Results

Like we anticipated, the a.c.s. convergence was formulated in order to link the convergence of the sequences to the convergence of their symbol. We have already seen that the similarities between $d_{a.c.s.}$ and d_{mea}, or between $d^\varphi_{a.c.s.}$ and d^φ_{mea}, brought to Theorem 3.1.18 and Theorem 3.1.20, showing that

$$\{A_n\}_n \sim_\sigma \kappa \qquad \implies \qquad \rho^\varphi(\{A_n\}_n) = p^\varphi_{mea}(\kappa), \quad \rho_{a.c.s.}(\{A_n\}_n) = p_{mea}(\kappa).$$

The link between the two spaces is actually much stronger than that. In fact, one can prove that the a.c.s. convergence on the spaces $\mathscr{E}$ and $\mathscr{E}_H$ is connected to the pointwise convergence on the space of functional symbols $C_c(\mathbb{R})'$ or $C_c(\mathbb{R})'$ respectively.

$$\begin{tikzcd}
\{B_{n,m}\}_{n,m} \arrow[r, "a.c.s."] \arrow[d, "\sim"'] & \{A_n\}_n \arrow[d, dashed, "\sim"] \\
\phi_m \arrow[r] & \phi
\end{tikzcd}$$

Theorem 3.2.1 ([52, Theorem 5.4]). *Let $\{A_n\}_n, \{B_{n,m}\}_{n,m} \in \mathscr{E}$ and let $\phi, \phi_m \in C_c'(\mathbb{R})$. Suppose that*

1. *$\{B_{n,m}\}_{n,m} \sim_\sigma \phi_m$ for every m,*
2. *$\{B_{n,m}\}_{n,m} \xrightarrow{a.c.s.} \{A_n\}_n$,*
3. *$\phi_m \to \phi$ pointwise.*

Then $\{A_n\}_n \sim_\sigma \phi$.

Theorem 3.2.2 ([52, Theorem 5.6]). *Let $\{A_n\}_n, \{B_{n,m}\}_{n,m} \in \mathscr{E}_H$ and let $\phi, \phi_m \in C_c'(\mathbb{R})$. Suppose that*

1. *$\{B_{n,m}\}_{n,m} \sim_\lambda \phi_m$ for every m,*
2. *$\{B_{n,m}\}_{n,m} \xrightarrow{a.c.s.} \{A_n\}_n$,*
3. *$\phi_m \to \phi$ pointwise.*

Then $\{A_n\}_n \sim_\lambda \phi$.

Suppose now that $\phi_m = \phi_{\kappa_m}$ and $\phi = \phi_\kappa$ for some measurable functions κ_m, κ on the same domain. If $\kappa_m \to \kappa$ in measure, it is easy to see that $\phi_m \to \phi$ punctually, so we can state some immediate corollaries of the last theorems.

Corollary 3.2.3 ([52, Corollary 5.1]). *Let $\{A_n\}_n, \{B_{n,m}\}_{n,m} \in \mathscr{E}$ and let $\kappa, \kappa_m : D \subseteq \mathbb{R}^k \to \mathbb{C}$ be measurable functions defined on a set D with $0 < \ell_k(D) < \infty$. Suppose that*

1. *$\{B_{n,m}\}_{n,m} \sim_\sigma \kappa_m$ for every m,*
2. *$\{B_{n,m}\}_{n,m} \xrightarrow{a.c.s.} \{A_n\}_n$,*
3. *$\kappa_m \to \kappa$ pointwise.*

Then $\{A_n\}_n \sim_\sigma \kappa$.

Corollary 3.2.4 ([52, Corollary 5.2]). *Let $\{A_n\}_n, \{B_{n,m}\}_{n,m} \in \mathscr{E}_H$ and let $\kappa, \kappa_m : D \subseteq \mathbb{R}^k \to \mathbb{R}$ be measurable functions defined on a set D with $0 < \ell_k(D) < \infty$. Suppose that*

1. *$\{B_{n,m}\}_{n,m} \sim_\lambda \kappa_m$ for every m,*
2. *$\{B_{n,m}\}_{n,m} \xrightarrow{a.c.s.} \{A_n\}_n$,*
3. *$\kappa_m \to \kappa$ pointwise.*

Then $\{A_n\}_n \sim_\lambda \kappa$.

On the practical side, the corollaries gives a way to find the symbol of a sequence $\{A_n\}_n$. In fact, if we can find a convergent sequence $\{B_{n,m}\}_{n,m}$ of simpler matrices for which we already know the symbol, we automatically produce a candidate for the symbol of $\{A_n\}_n$.

On a theoretical point of view, the last results prove that the set of couples $(\{A_n\}_n, \kappa) \in \mathscr{E} \times \mathscr{M}_D$ such that $\{A_n\}_n \sim_{\sigma,\lambda} \kappa$ the metric is closed with respect to $d_{a.c.s.} \times d_{mea}$. Later on, we will use this property to generate a space of sequences by the operation of closure with respect to $d_{a.c.s.} \times d_{mea}$.

Actually, the a.c.s. convergence is much stronger and binding on the singular values than one can think. In fact it is possible to prove that the set $\mathscr{S} \subseteq \mathscr{E}$ of sequences admitting a measurable function as a singular value symbol (or the set $\mathscr{S}_H \subseteq \mathscr{E}_H$ of Hermitian sequences admitting a measurable function as spectral symbol) is closed with respect to the a.c.s. convergence. Note that $\mathscr{S}$ depends on the domain D of the measurable functions, but if we denote $\mathscr{S}(D)$ the space od sequences admitting a singular value symbol on D, then Lemma 3.1.9 shows that for every couple of domains D, D',

$$\exists \kappa \in \mathscr{M}_D : \{A_n\}_n \sim_\sigma \kappa \iff \{A_n\}_n \text{ s.u. } \iff \exists \kappa' \in \mathscr{M}_{D'} : \{A_n\}_n \sim_\sigma \kappa'$$

and as a consequence, $\mathscr{S}(D) = \mathscr{S}(D')$ (respectively, $\mathscr{S}_H(D) = \mathscr{S}_H(D')$). All the spaces $\mathscr{S}(D)$ thus coincide and there is thus no need to specify the domain.

$$\begin{tikzcd}
\{B_{n,m}\}_{n,m} \arrow[r, "a.c.s."] \arrow[d, "\sim"'] & \{A_n\}_n \arrow[d, dashed, "\sim"] \\
\kappa_m & \kappa
\end{tikzcd}$$

Theorem 3.2.5. *Let $\{A_n\}_n, \{B_{n,m}\}_{n,m} \in \mathscr{E}$ and let $\kappa_m : D_m \subseteq \mathbb{R}^{k_m} \to \mathbb{C}$ be measurable functions defined on sets D_m with $0 < \ell_{k_m}(D_m) < \infty$. Suppose that*

1. *$\{B_{n,m}\}_{n,m} \sim_\sigma \kappa_m$ for every m,*
2. *$\{B_{n,m}\}_{n,m} \xrightarrow{a.c.s.} \{A_n\}_n$.*

Then there exists a measurable function $\kappa : [0,1] \to \mathbb{C}$ that is a singular value symbol for $\{A_n\}_n$.

Theorem 3.2.6. *Let $\{A_n\}_n, \{B_{n,m}\}_{n,m} \in \mathscr{E}_H$ and let $\kappa_m : D_m \subseteq \mathbb{R}^{k_m} \to \mathbb{R}$ be measurable functions defined on sets D_m with $0 < \ell_{k_m}(D_m) < \infty$. Suppose that*

1. *$\{B_{n,m}\}_{n,m} \sim_\lambda \kappa_m$ for every m,*
2. *$\{B_{n,m}\}_{n,m} \xrightarrow{a.c.s.} \{A_n\}_n$.*

Then there exists a measurable function $\kappa : [0,1] \to \mathbb{R}$ that is a spectral symbol for $\{A_n\}_n$.

The proof of Theorem 3.2.5 is long and technical, so it is reported in Appendix C, and Theorem 3.2.6 is proved following the same steps.

The results in this paragraph link the a.c.s. convergence with the convergence in measure, and we know that they are induced, respectively, by $d_{a.c.s.}$ and d_{mea}. A difference between d_{mea} and $d_{a.c.s.}$, or any pair of distances induced by convex functions, is that d_{mea} is a distance on $\mathscr{M}_D$, since two functions f, g that coincide ℓ_d-almost everywhere are identified, whereas $d_{a.c.s.}$ induces only a pseudometric on $\mathscr{E}$. In the next paragraph we will study this particular behaviour.

3.2.2 Zero-Distributed Sequences and A.C.S. Equivalence

The function $d_{a.c.s.}$ induces a pseudometric on $\mathscr{E}$, meaning that if $\{A_n\}_n, \{B_n\}_n \in \mathscr{E}$ and

$$d_{a.c.s.}(\{A_n\}_n, \{B_n\}_n) = \rho_{a.c.s.}(\{A_n\}_n - \{B_n\}_n) = 0,$$

then it may be not true that $\{A_n\}_n = \{B_n\}_n$. Equivalently, there exist non-zero sequences $\{Z_n\}_n \in \mathscr{E}$ such that

$$d_{a.c.s.}(\{Z_n\}_n, \{0_n\}_n) = \rho_{a.c.s.}(\{Z_n\}_n) = 0.$$

We call them zero-distributed sequences.

Definition 3.2.7. *Any sequence $\{Z_n\}_n \in \mathscr{E}$ such that $\rho_{a.c.s.}(\{Z_n\}_n) = 0$ is called a **zero-distributed** sequence. Moreover, we denote as*

$$\mathscr{Z} = \{\, \{Z_n\}_n \in \mathscr{E} \mid \rho_{a.c.s.}(\{Z_n\}_n) = 0 \,\}$$

the space of zero-distributed sequences, and with

$$\mathscr{Z}_H = \{\, \{Z_n\}_n \in \mathscr{E}_H \mid \rho_{a.c.s.}(\{Z_n\}_n) = 0 \,\}$$

the space of Hermitian and zero-distributed sequences.

Before going into the details, here are two examples of zero-distributed sequences.

Example 3.2.8

- Consider the matrices

$$Z_n = \begin{pmatrix} & & & & n^{n-1} \\ n^{-1} & & & & \\ & n^{-1} & & & \\ & & \ddots & & \\ & & & n^{-1} & \end{pmatrix}$$

of size $n \times n$. We can split them into $N_n + R_n$ where

$$N_n = \begin{pmatrix} & & & & 0 \\ n^{-1} & & & & \\ & n^{-1} & & & \\ & & \ddots & & \\ & & & n^{-1} & \end{pmatrix} \qquad R_n = \begin{pmatrix} & & & & n^{n-1} \\ 0 & & & & \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \end{pmatrix}$$

so that $\|N_n\| \le n^{-1}$ and $\mathrm{rk}(R_n) = 1$. Notice that all the singular values of N_n are less than $1/n$, and by Theorem 3.1.1, also all the singular values of Z_n, except for the biggest, are still bounded by $1/n$. As a consequence,

$$\rho_{a.c.s.}(\{Z_n\}_n) = \limsup_{n\to\infty} \min_{i=1,\dots,n+1} \left\{ \frac{i-1}{n} + \sigma_i(Z_n) \right\} \le \limsup_{n\to\infty} \frac{1}{n} + \frac{1}{n} = 0.$$

By definition, $\{Z_n\}_n$ is a zero-distributed sequence.

- Consider the Hermitian matrices

$$Z_n = \begin{pmatrix} 1+n^{-1} & 1 & 1 & \dots & 1 \\ 1 & n^{-1} & & & \\ 1 & & n^{-1} & & \\ \vdots & & & \ddots & \\ 1 & & & & n^{-1} \end{pmatrix}$$

of size $n \times n$. We can split them into $N_n + R_n$ where

$$N_n = \begin{pmatrix} n^{-1} & & & & \\ & n^{-1} & & & \\ & & n^{-1} & & \\ & & & \ddots & \\ & & & & n^{-1} \end{pmatrix} \qquad R_n = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & & & & \\ 1 & & & & \\ \vdots & & & & \\ 1 & & & & \end{pmatrix}$$

so that $\rho(N_n) = n^{-1}$ and $\mathrm{rk}(R_n) = 2$. Notice that all the eigenvalues of N_n are $1/n$, and by Theorem 3.1.2, all the eigenvalues of Z_n, except for the biggest and the smallest, are still $1/n$. Since Z_n is a normal matrix, the singular values are the absolute values of the eigenvalues and so

$$\rho_{a.c.s.}(\{Z_n\}_n) = \limsup_{n\to\infty} \min_{i=1,\dots,n+1} \left\{ \frac{i-1}{n} + \sigma_i(Z_n) \right\} \le \limsup_{n\to\infty} \frac{2}{n} + \frac{1}{n} = 0.$$

By definition, $\{Z_n\}_n$ is a Hermitian zero-distributed sequence.

The pseudometric induced by $d_{a.c.s.}$ identifies sequences that differ by a zero-distributed sequence. It naturally induces an equivalence relation on $\mathscr{E}$, that we call the a.c.s. equivalence.

Definition 3.2.9. *We define the* ***a.c.s. equivalence*** $\sim_{a.c.s.}$ *on* $\mathscr{E}$ *as*

$$\{A_n\}_n \sim_{a.c.s.} \{B_n\}_n \iff \{A_n - B_n\}_n \in \mathscr{Z}$$

Notice that if $\{A_n\}_n, \{B_n\}_n \in \mathscr{E}_H$, then also their difference belong to $\mathscr{E}_H$. The a.c.s. equivalence on $\mathscr{E}_H$ is thus the restriction of the a.c.s. equivalence on $\mathscr{E}$, since

$$\{A_n\}_n, \{B_n\}_n \in \mathscr{E}_H, \quad \{A_n\}_n \sim_{a.c.s.} \{B_n\}_n \iff \{A_n - B_n\}_n \in \mathscr{Z}_H.$$

Using this notation, the zero-distributed sequences are exactly the equivalence class of the zero sequence $\{0_n\}_n$.

From the definition, for every zero-distributed sequence $\{Z_n\}_n$, we have

$$0 = \rho_{a.c.s.}(\{Z_n\}_n) = \lim_{n\to\infty} \min_{i=1,\dots,s_n+1} \left\{ \frac{i-1}{s_n} + \sigma_i(Z_n) \right\}$$

and consequently, if we call i_n the index that realizes the minimum for n, then

$$\lim_{n\to\infty} \frac{i_n - 1}{s_n} = 0, \qquad \lim_{n\to\infty} \sigma_{i_n}(Z_n) = 0.$$

In other words, all but a portion of singular values given by $\frac{i_n-1}{s_n} \to 0$, are uniformly converging to zero. If we plot the singular values of Z_n in increasing order, it is immediate to check that they are converging to zero, and since

$$\{Z_n\}_n \sim_\sigma \kappa \implies \rho_{a.c.s.}(\{Z_n\}_n) = p_{mea}(\kappa)$$

we can conclude that $\{Z_n\}_n$ is zero-distributed if and only if $\{Z_n\}_n \sim_\sigma 0$. The next result wraps up all the observations made.

Theorem 3.2.10 ([52, Theorem 3.2, 5.2.]). *Given a sequence $\{Z_n\}_n \in \mathscr{E}$, the following statements are equivalent.*

1. *$\{Z_n\}_n$ is zero-distributed.*
2. *$\{Z_n\}_n \sim_{a.c.s.} \{0_n\}_n$.*
3. *For every n there exists a decomposition $Z_n = N_n + R_n$ such that*

$$\lim_{n\to\infty} \|N_n\| + \frac{\mathrm{rk}(R_n)}{s_n} = 0.$$

4. *$\{Z_n\}_n \sim_\sigma 0$.*

If furthermore $\{Z_n\}_n \in \mathscr{E}_H$, then the statements are also equivalent to

5. *For every n there exists a decomposition $Z_n = N_n + R_n$ such that N_n and R_n are Hermitian matrices and*

$$\lim_{n\to\infty} \|N_n\| + \frac{\mathrm{rk}(R_n)}{s_n} = 0.$$

6. *$\{Z_n\}_n \sim_\lambda 0$.*

$$\begin{tikzcd}
\{A_n\}_n \arrow[r, "a.c.s."] \arrow[d, "\sim"'] & \{A_n + Z_n\}_n \arrow[d, dashed, "\sim"] \\
\kappa \arrow[r] & \kappa
\end{tikzcd}$$

Notice that we could deduce statement 4. using Theorem 3.2.1 and noticing that if $B_{n,m} = 0_n$, then $\{B_{n,m}\}_{n,m} \xrightarrow{a.c.s.} \{Z_n\}_n$. In the same way, given any sequence $\{A_n\}_n \sim_\sigma \kappa$, we can add a zero-distributed sequence and discover that the symbol does not change. In fact,

$$d_{a.c.s.}(\{A_n\}_n, \{A_n\}_n + \{Z_n\}_n) = \rho_{a.c.s.}(\{Z_n\}_n) = 0,$$

and by Theorem 3.1.12 we know that $\{A_n\}_n \xrightarrow{a.c.s.} \{A_n\}_n + \{Z_n\}_n$. If we set $B_{n,m} = A_n$, then we can apply again Theorem 3.2.1 obtaining that $\{A_n\}_n + \{Z_n\}_n \sim_\sigma \kappa$, and if all the matrices were Hermitian, we could say the same about the spectral symbol.

The reason behind this behaviour can be deduced from Statement 3. and 5. of Theorem 3.2.10, since for any $\{A_n\}_n \in \mathscr{E}$, and $\{Z_n\}_n \in \mathscr{Z}$, we can write their sum as

$$\{A_n\}_n + \{Z_n\}_n = \{A_n\}_n + \{N_n\}_n + \{R_n\}_n.$$

In other words, we are perturbing A_n by a matrix N_n with small norm and a matrix R_n with small rank. Theorem 3.1.3 and Theorem 3.1.1 show that we are changing all but $2\,\mathrm{rk}(R_n) = o(n)$ singular values by at most $\|N_n\| = o(1)$, so the singular value symbol of $\{A_n\}_n$ is preserved. Since the difference of any a.c.s. equivalent couple of sequences is a zero-distributed sequence, we conclude the following result.

Theorem 3.2.11 ([52, Corollary 5.1,5.2])**.** *Given two a.c.s. equivalent sequences $\{A_n\}_n, \{B_n\}_n \in \mathscr{E}$,*

$$\{A_n\}_n \sim_\sigma \phi \iff \{B_n\}_n \sim_\sigma \phi.$$

Given two a.c.s. equivalent sequences $\{A_n\}_n, \{B_n\}_n \in \mathscr{E}_H$,

$$\{A_n\}_n \sim_\lambda \phi \iff \{B_n\}_n \sim_\lambda \phi.$$

To conclude the paragraph, we report a way to test if a sequence is zero-distributed, that will be useful in the applications. From Theorem 3.1.7, one can come up with the following result.

Lemma 3.2.12 ([52, Theorem 3.3])**.** *For every $p \in [1, \infty]$, if*

$$\limsup_{n\to\infty} (s_n)^{-1/p} \|Z_n\|_p = 0,$$

then $\{Z_n\}_n$ is zero-distributed. Here we are using the notation $1/\infty = 0$, and $\|\cdot\|_p$ stands for the p-Schatten norm introduced at the start of paragraph 3.1.1.

Like we noticed when we first defined the a.c.s. convergence, all the results involving spectral symbols require Hermitian sequences. In the next paragraph, we deal with general perturbations of sequences, and we see how they affect the spectral symbol.

3.2.3 Invariance of Spectral Symbol

Up until now we observed that the a.c.s. convergence deals well with the singular values symbols of general sequences, but when we turn to the eigenvalues, we have to restrict ourselves to the space of Hermitian sequences. Here is an example showing that a.c.s. convergence behaves poorly with spectral symbols.
Example 3.2.13

► Consider the matrices

$$Z_n = \begin{pmatrix} & & & & n^{n-1} \\ n^{-1} & & & & \\ & n^{-1} & & & \\ & & \ddots & & \\ & & & n^{-1} & \end{pmatrix}.$$

In Example 3.2.8 we showed that $\{Z_n\}_n$ is a zero-distributed sequence, so we can call $B_{n,m} = Z_n$ and notice that $\{B_{n,m}\}_{n,m} \xrightarrow{a.c.s.} \{0_n\}_n$ since

$$d_{a.c.s.}(\{B_{n,m}\}_{n,m}, \{0_n\}_n) = d_{a.c.s.}(\{Z_n\}_n, \{0_n\}_n) = 0 \qquad \forall\, m.$$

At the same time, Z_n is similar to the circulant matrix C_n defined in paragraph 1.3.2 since

$$Z_n = \begin{pmatrix} 1 & & & & \\ & n^{-1} & & & \\ & & n^{-2} & & \\ & & & \ddots & \\ & & & & n^{-n+1} \end{pmatrix} C_n \begin{pmatrix} 1 & & & & \\ & n & & & \\ & & n^2 & & \\ & & & \ddots & \\ & & & & n^{n-1} \end{pmatrix}.$$

The eigenvalues of Z_n are the same as the eigenvalues of C_n, that are $e^{\frac{2\pi k \mathrm{i}}{n}}$ for $k = 0, 1, \dots, n-1$. From this one can deduce that $\{C_n\}_n$ and $\{Z_n\}_n$ have $\kappa(x) = e^{2\pi \mathrm{i} x}$ as spectral symbol over the domain $D = [0, 1]$. As a consequence,

$$\{B_{n,m}\}_{n,m} \sim_\lambda \kappa(x) \quad \forall\, m, \qquad \{B_{n,m}\}_{n,m} \xrightarrow{a.c.s.} \{0_n\}_n, \qquad \kappa(x) \xrightarrow{m\to\infty} \kappa(x)$$

but it is false that $\{0_n\}_n \sim_\lambda \kappa(x)$.

3.2.3.1 Perturbation of Hermitian Sequences

Example 3.2.13 shows that if we perturb an Hermitian sequence (in this instance, $\{0_n\}_n \sim_\lambda 0$) with a non-Hermitian zero-distributed sequence, then the spectral symbol is not preserved in general. Using Hoffman-Wielandt Theorem ([20, Theorem VI.4.1]), one can compare the perturbation of the spectra with the sequence perturbation in some norm. It enables us to formulate suitable hypothesis on the perturbation to ensure that a Hermitian sequence retains its spectral symbol.

Theorem 3.2.14 ([12]). *Let $\{X_n\}_n \in \mathscr{E}_H$ with spectral symbol $\kappa : D \subseteq \mathbb{R}^q \to \mathbb{R}$ where $0 < \ell_q(D) < \infty$. If $\|Y_n\|_2 = o(\sqrt{n})$, then $\{X_n + Y_n\}_n \sim_\lambda \kappa$.*

Corollary 3.2.15 ([12]). *Let $\{X_n\}_n \in \mathscr{E}_H$ with spectral symbol $\kappa : D \subseteq \mathbb{R}^q \to \mathbb{R}$ where $0 < \ell_q(D) < \infty$. Suppose that any of the following conditions is met.*

1. $\|Y_n\|_p = o(\sqrt{n})$ *for some* $1 \le p \le 2$.
2. $\|Y_n\| = o(1)$.

Then $\{X_n + Y_n\}_n \sim_\lambda \kappa$.

Corollary 3.2.16 ([12]). *Let $\{X_n\}_n \in \mathscr{E}_H$ with spectral symbol $\kappa : D \subseteq \mathbb{R}^q \to \mathbb{R}$ where $0 < \ell_q(D) < \infty$. Suppose that both the following conditions are met.*

1. $\|Y_n\|_1 = o(n)$.
2. $\|Y_n\| = O(1)$.

Then $\{X_n + Y_n\}_n \sim_\lambda \kappa$.

Corollary 3.2.16 is a direct generalization of [56, Theorem 3.4], that has been used in practice to compute the spectral symbol for several sequences of linear systems arising from the discretization of PDE. Even though these results are enough to analyse sequences of matrices arising from several applications, there is a much stronger conjecture that is still an open question.

Conjecture 3.2.17. *Let $\{X_n\}_n \in \mathscr{E}_H$ with spectral symbol $\kappa : D \subseteq \mathbb{R}^q \to \mathbb{R}$ where $0 < \ell_q(D) < \infty$. If $\|Y_n\|_1 = o(n)$, then $\{X_n + Y_n\}_n \sim_\lambda \kappa$.*

In [8] it is possible to find a partial analysis of the conjecture, with several equivalent statements. What happens if we start with a non-Hermitian sequence $\{X_n\}_n$ admitting a spectral symbol?

3.2.3.2 Perturbation of Normal Sequences

When dealing with normal matrices instead of Hermitian matrices, we get different results.

Lemma 3.2.18 ([10, 8]). *Let $\{X_n\}_n \in \mathscr{E}$ be a normal sequence, with spectral symbol $\kappa : D \subseteq \mathbb{R}^q \to \mathbb{R}$ where $0 < \ell_q(D) < \infty$. Consider the following statements:*

1. *$\{Y_n\}_n$ zero-distributed and $X_n + Y_n$ normal,*

2. $\|Y_n\|_p = o(1)$ *where* $1 \le p \le 2$,

3. $\|Y_n\|_p = o(n^{\frac{2}{p}-1})$ *where* $2 \le p < \infty$,

4. $\|Y_n\| = o(\frac{1}{n})$.

If any of them holds, then

$$\{X_n\}_n + \{Y_n\}_n \sim_\lambda \kappa.$$

2. 3. and 4. are direct consequences of [20, Problem VI.8.11]. the statement 1. comes instead from a totally different reasoning that can be found in [10]. In few words, we used the fact that a normal sequence $\{A_n\}_n$ has a spectral symbol κ if and only if $\{p(A_n)\}_n \sim_\sigma p \circ \kappa$ for every complex polynomial $p(x) \in \mathbb{C}[x]$ (Lemma 5.2.3).

When the starting sequence $\{X_n\}_n$ is not even a normal sequence, we can say almost nothing about the perturbations that preserve the spectral symbol, so we report just some observation.

3.2.3.3 Perturbation of General Sequences

Example 3.2.13 and Example 3.2.8 show that we have little hope to find a perturbation result that is valid starting from any sequence. Here we can show that there are even worse examples.

Example 3.2.19

- Consider the matrices

$$Z_n = \begin{pmatrix} & & & n^{k(n-1)} \\ n^{-k} & & & \\ & n^{-k} & & \\ & & \ddots & \\ & & & n^{-k} & \end{pmatrix}$$

of size $n \times n$. following the same steps of Example 3.2.8, we find that we can split Z_n into $N_n + R_n$ where

$$N_n = \begin{pmatrix} & & & 0 \\ n^{-k} & & & \\ & n^{-k} & & \\ & & \ddots & \\ & & & n^{-k} & \end{pmatrix} \qquad R_n = \begin{pmatrix} & & & n^{k(n-1)} \\ 0 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 & \end{pmatrix}$$

so that $\|N_n\| \le n^{-k}$ and $\mathrm{rk}(R_n) = 1$. As a consequence, $\{Z_n\}_n$ is zero-distributed, and at the same time, from Example 3.2.13 we can infer that Z_n is similar to the circulant matrix C_n, so $\{Z_n\}_n \sim_\lambda \kappa(x) = e^{2\pi \mathrm{i}x}$. It is evident that all the eigenvalues of N_n and R_n are zero. It means that if we perturb $\{Z_n\}_n$ with a sequence $\{-N_n\}_n$ with $\|N_n\| = O(n^{-k})$, we obtain

$$\{Z_n\}_n \sim_\lambda e^{2\pi \mathrm{i}x}, \qquad \{Z_n\}_n - \{N_n\}_n = \{R_n\}_n \sim_\lambda 0.$$

Since it works for any $k > 0$, we infer that we cannot an uniform bound $\|N_n\| = o(n^{-k})$ that works for all sequences. Moreover, if we perturb $\{Z_n\}_n$ with a sequence $\{-R_n\}_n$ of rank 1, then

$$\{Z_n\}_n \sim_\lambda e^{2\pi \mathrm{i}x}, \qquad \{Z_n\}_n - \{R_n\}_n = \{N_n\}_n \sim_\lambda 0.$$

What we can prove, starting from the Bauer-Fike result and its generalization ([20, 79]), is that there is a class of perturbation of $\{A_n\}_n$, with norm bounded by a function ε_n depending only on $\{A_n\}_n$, that preserves the symbol.

Theorem 3.2.20 ([8])**.** *Let* $\{A_n\}_n \in \mathscr{E}$ *with spectral symbol* $\kappa : D \subseteq \mathbb{R}^q \to \mathbb{R}$ *where* $0 < \ell_q(D) < \infty$. *There exists a sequence of* $\varepsilon_n > 0$ *such that*

$$\|N_n\| \le \varepsilon_n, \quad \forall\, n \implies \{A_n + N_n\}_n \sim_\lambda \kappa.$$

Chapter 4

Spectral Measure

Up until now we have studied the case when a functional ϕ can be expressed through a measurable function κ. In the case that ϕ is a symbol for a sequence $\{A_n\}_n$, Lemma 3.1.9 tells us that $\phi = \phi_\kappa$ for some function κ if and only if $\{A_n\}_n$ is s.u. (or s.s.u. in case of spectral symbol). Even if $\{A_n\}_n$ is not s.u., it can nonetheless enjoy a symbol ϕ, but we cannot express it through a measurable function.

Example 4.0.1

► Let A_n be a $2n \times 2n$ matrix such that

$$A_n = \begin{pmatrix} (-1)^n n I_n & 0 \\ 0 & I_n \end{pmatrix}$$

where I_n is the $n \times n$ identity matrix. A_n has half of its eigenvalues equal to 1, and the other half equal to $(-1)^n n$. If we define ϕ as

$$\phi(G) = \frac{G(1)}{2},$$

then for any $G \in C_c(\mathbb{C})$,

$$\lim_{n\to\infty} \frac{1}{2n} \sum_{i=1}^{2n} G(\lambda_i(A_n)) = \lim_{n\to\infty} \frac{G(1) + G((-1)^n n)}{2} = \frac{G(1)}{2} = \phi(G),$$

thus $\{A_n\}_n \sim_{\lambda,\sigma} \phi$. At the same time, $\{A_n\}_n$ is not s.u. or s.s.u., since

$$\limsup_{n\to\infty} \frac{\#\{i \in \{1, \dots, 2n\} : \sigma_i(A_n) > M\}}{2n} = \limsup_{n\to\infty} \frac{\#\{i \in \{1, \dots, 2n\} : |\lambda_i(A_n)| > M\}}{2n} \geq \frac{1}{2}$$

for every $M > 0$. Consequently, there does not exists a measurable function κ such that $\phi = \phi_\kappa$.

We can prove also prove it directly. In fact, if $\kappa \equiv 1$ on any domain D, then for any $G \in C_c(\mathbb{C})$ such that $G(1) \neq 0$,

$$\lim_{n\to\infty} \frac{1}{2n} \sum_{i=1}^{2n} G(\lambda_i(A_n)) = \frac{G(1)}{2} \neq G(1) = \frac{1}{\ell(D)} \int_D G(\kappa(x))\mathrm{d}x$$

and if $\kappa \not\equiv 1$, then there exists a $G \in C_c(\mathbb{C})$ such that $G(1) = 0$ and

$$\lim_{n\to\infty} \frac{1}{2n} \sum_{i=1}^{2n} G(\lambda_i(A_n)) = \frac{G(1)}{2} = 0 \neq \frac{1}{\ell(D)} \int_D G(\kappa(x))\mathrm{d}x.$$

On the contrary, we can prove that for any $\phi \in \mathbb{C}_c'(\mathbb{C})$ there exists a measure μ on $\mathbb{C}$ such that $\phi = \phi_\mu$.

4.1 Radon Measures

From Definition 2.1.1, we can immediately deduce that any symbol ϕ must be linear, and

$$\|\phi\|_\infty = \sup_{G\in C_c(\mathbb{C})} \frac{|\phi(G)|}{\|G\|_\infty} = \frac{1}{\|G\|_\infty} \lim_{n\to\infty} \frac{1}{s_n} \left| \sum_{j=1}^{s_n} G(\lambda_j(A_n)) \right| \le \frac{1}{\|G\|_\infty} \lim_{n\to\infty} \frac{1}{s_n} \sum_{j=1}^{s_n} \|G\|_\infty = 1.$$

Moreover, it is easy to check that when we evaluate ϕ on any nonnegative function $G \in C_c(\mathbb{C})$ we obtain a nonnegative value $\phi(G) \ge 0$. We just proved that any spectral symbol ϕ is a linear and continuous positive functional on $C_c(\mathbb{C})$ with norm at most 1. With these hypotheses, we can use the Riesz representation theorem.

Theorem 4.1.1 (Riesz, [2]). *Let $\phi : C_c(X) \to \mathbb{R}$ be a positive linear and continuous function, where X is an Hausdorff and locally compact space. There exists an uniquely determined Radon measure μ such that*

$$\phi(G) = \int_X G \mathrm{d}\mu \qquad \forall G \in C_c(X).$$

As a consequence, whenever $\{A_n\}_n$ enjoys ϕ as a symbol, then there exists a Radon measure μ that is also a symbol for $\{A_n\}_n$. One can prove moreover that every measure μ is finite with mass equal to $\|\phi\|_\infty \le 1$. Since all the finite measures over the Borel set are Radon, from now on we will simply say "measure" instead of "Radon measure".

If $\{A_n\}_n \sim_\lambda \phi$ and μ is the only measure representing ϕ given by Theorem 4.1.1, then we say that μ is the **spectral measure** associated to $\{A_n\}_n$.

Definition 4.1.2. *Given $\{A_n\}_n \in \mathscr{E}$, we say that it has a* ***spectral measure*** *μ if*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} G(\lambda_i(A_n)) = \int_{\mathbb{C}} G \mathrm{d}\mu$$

for every $G \in C_c(\mathbb{C})$. In this case, we write

$$\{A_n\}_n \sim_\lambda \mu.$$

Example 4.1.3

- In Example 4.0.1 we showed that the sequence $\{A_n\}_n$ where

$$A_n = \begin{pmatrix} (-1)^n n I_n & 0 \\ 0 & I_n \end{pmatrix}$$

admits a symbol $\phi(G) = G(1)/2$ but it is not representable by a measurable function. It is evident, instead, that $\phi = \phi_\mu$ where $\mu = \frac{1}{2}\delta_1$ and δ_1 is the atomic probability measure supported on the set $\{1\} \subseteq \mathbb{C}$, since

$$\phi(G) = \frac{G(1)}{2} = \int_{\mathbb{C}} G \mathrm{d}\mu \quad \forall G \in C_c(\mathbb{C}).$$

If $\{A_n\}_n \sim_\lambda \mu$, one can exploit the ergodic formula (2.4), and repeat the same reasoning of paragraph 2.1.3. In particular, for every ball $B \subseteq \mathbb{C}$, the quantity $\mu(B)$ represents the asymptotic rate of eigenvalues of A_n inside B, and since the balls are a base for the euclidean topology, this property is actually an other characterization for the spectral measures. We prove it in Appendix A since the proof is long and technical.

Theorem 4.1.4. *Let μ be a positive finite measure on $\mathbb{C}$.*

Let $D \subseteq \mathbb{R}^d$ be a measurable set with positive and finite d-Lebesgue measure, and let $\kappa : D \to \mathbb{C}$ be a measurable function.

- *If $\{A_n\}_n \sim_\lambda \mu$, then the set*

$$E_{z_0} := \{\, r \in \mathbb{R}^+ \mid \chi_{B(z_0,r)} \text{ does not satisfy (2.2)} \}$$

contains at most numerable points for every $z_0 \in \mathbb{C}$.

- *Suppose that the set*

$$R_{z_0} := \{\, r \in \mathbb{R}^+ \mid \chi_{B(z_0,r)} \text{ does not satisfy (2.2)} \}$$

has Lebesgue measure zero for every $z_0 \in \mathbb{C}$. In this case, $\{A_n\}_n \sim_\lambda \mu$.

4.1.1 Link with Measurable Functions

As previously shown, not every symbol ϕ is representable by measurable functions, but it is always induced by a finite measure μ. Notice moreover that μ is uniquely determined by ϕ and vice versa.

Suppose now that $\phi = \phi_\kappa$ for some measurable function κ. A simple computation shows that in such cases, μ must be a probability measure, and moreover every probability measure μ satisfies $\phi_\mu = \phi_\kappa$ for some measurable function κ. We report the precise results in the following lemmas.

Lemma 4.1.5 ([7])**.** *Let $D \subseteq \mathbb{R}^n$ be a measurable set with finite non-zero measure.*
Then, for any $k \in \mathscr{M}_D$ there exists an unique measure μ such that $\phi_\mu = \phi_\kappa$, and it is a probability measure.
Vice versa, for every probability measure μ, there exists a measurable function $k \in \mathscr{M}_D$ such that $\phi_\mu = \phi_\kappa$.

Actually, by the result in the reference [7], we only know that for every probability measure μ there exists a measurable function on $[0,1]$ such that $\phi_\mu = \phi_\kappa$, but in paragraph 3.2.1 we already showed that a sequence admits a symbol on $[0,1]$ if and only if it admits a symbol on D for any D.

Lemma 4.1.6. *Let $D \subseteq \mathbb{R}^n$ be a measurable set with finite non-zero measure. Given a sequence $\{A_n\}_n \in \mathscr{E}$ with spectral symbol ϕ, the following are equivalent.*

- *$\{A_n\}_n$ is s.s.u.,*
- *there exists $\kappa \in \mathscr{M}_D$ such that $\phi = \phi_\kappa$,*
- *if $\phi = \phi_\mu$, then μ is a probability measure.*

Notice that the last result is just the merge of Lemma 4.1.5 and Lemma 3.1.9 and that it still holds if we substitute "spectral symbol" with "singular value symbol" and "s.s.u." with "s.u.".

An interesting feat to explore here is the possibility to stretch the definition of measurable functions and admit also functions that take value ∞. In fact, it is possible to prove that all sequence with a symbol ϕ admits also such a function as symbol, even if not s.s.u. (or s.u.).

Lemma 4.1.7. *Given a measure μ on $\mathbb{C}$ with mass less than or equal to 1, there exists a measurable function $\kappa : [0,1] \to \mathbb{C} \cup \{\infty\}$ such that*

$$\int_{\mathbb{C}} G \mathrm{d}\mu = \int_{[0,1]} G(\kappa(x)) \mathrm{d}x \qquad \forall\, G \in C_c(\mathbb{C})$$

where by convention $G(\infty) = 0$.

Proof. Consider the probability measure $\nu := \mu/\mu(\mathbb{C})$. By Lemma 4.1.5, there exists a function $\kappa' \in \mathscr{M}_{[0,1]}$ such that $\phi_\nu = \phi_{\kappa'}$, and we can produce our candidate $\kappa : [0,1] \to \mathbb{C}$ as

$$\kappa(x) := \begin{cases} \kappa'\left(\frac{x}{\|\phi\|_\infty}\right), & x \le \|\phi\|_\infty, \\ \infty, & x > \|\phi\|_\infty. \end{cases}$$

Given any $G \in C_c(\mathbb{C})$ we have that

$$\int_{\mathbb{C}} G \mathrm{d}\mu = \int_{\mathbb{C}} G \mathrm{d}\mu = \mu(\mathbb{C}) \int_{\mathbb{C}} G \mathrm{d}\nu = \mu(\mathbb{C}) \int_{[0,1]} G(\kappa'(x)) \mathrm{d}x$$

and if we operate a variable change $x = y/\|\phi\|_\infty$, then

$$\int_{\mathbb{C}} G \mathrm{d}\mu = \mu(\mathbb{C}) \int_{[0,1]} G(\kappa'(x)) \mathrm{d}x = \frac{\mu(\mathbb{C})}{\|\phi\|_\infty} \int_{[0,\|\phi\|_\infty]} G\left(\kappa'\left(\frac{y}{\|\phi\|_\infty}\right)\right) \mathrm{d}y = \int_{[0,1]} G(\kappa(x)) \mathrm{d}x.$$

□

Example 4.1.8

- In Example 4.1.3 we showed that the sequence $\{A_n\}_n$ where

$$A_n = \begin{pmatrix} (-1)^n n I_n & 0 \\ 0 & I_n \end{pmatrix}$$

admits a symbol $\mu = \frac{1}{2}\delta_1$ with $\mu(\mathbb{C}) = \frac{1}{2}$, but it is not representable by a standard measurable function. If $\kappa : [0,1] \to \mathbb{C}$ takes value 1 on $[0,1/2]$ and ∞ otherwise, then for every $G \in C_c(\mathbb{C})$,

$$\int_{[0,1]} G(\kappa(x)) \mathrm{d}x = \frac{G(1)}{2} = \int_{\mathbb{C}} G \mathrm{d}\mu.$$

This shows that, in a way, it is always possible to express the symbol in term of measurable functions. The quantity $\|\phi\|_\infty = \mu(\mathbb{C})$ is important, because gives us additional information on the eigenvalues. In fact, the measure of the interval where the function κ takes value ∞ is $1 - \|\phi\|_\infty$, and we can show that it is an estimation of the rate of eigenvalues (or singular values) that goes to infinity in absolute value. It represents, in fact, a measure of how much not-s.u. (or s.s.u.) is a sequence.

Lemma 4.1.9. *If $\{A_n\}_n \sim_\lambda \mu$, then*

$$\lim_{M\to\infty} \limsup_{n\to\infty} \frac{\#\{i \in \{1,\dots,s_n\} : |\lambda_i(A_n)| > M\}}{s_n} = 1 - \mu(\mathbb{C}).$$

If $\{A_n\}_n \sim_\sigma \mu$, then

$$\lim_{M\to\infty} \limsup_{n\to\infty} \frac{\#\{i \in \{1,\dots,s_n\} : \sigma_i(A_n) > M\}}{s_n} = 1 - \mu(\mathbb{C}).$$

Proof. Let $G_M \in C_c(\mathbb{C})$ such that $\chi_{B(0,M)} \le G_M \le \chi_{B(0,2M)}$ for every $M > 0$. We know that

$$\frac{\#\{i \in \{1,\dots,s_n\} : |\lambda_i(A_n)| > M\}}{s_n} = 1 - \frac{1}{s_n}\sum_{i=1}^{s_n} \chi_{B(0,M)}(\lambda_i(A_n)) \ge 1 - \frac{1}{s_n}\sum_{i=1}^{s_n} G_M(\lambda_i(A_n)).$$

If we take the limits on both sides, then

$$\limsup_{n\to\infty} \frac{\#\{i \in \{1,\dots,s_n\} : |\lambda_i(A_n)| > M\}}{s_n} \ge 1 - \int_{\mathbb{C}} G_M \mathrm{d}\mu \ge 1 - \mu(\mathbb{C}),$$

so

$$\liminf_{M\to\infty} \limsup_{n\to\infty} \frac{\#\{i \in \{1,\dots,s_n\} : |\lambda_i(A_n)| > M\}}{s_n} \ge 1 - \mu(\mathbb{C}).$$

Moreover,

$$\frac{\#\{i \in \{1,\dots,s_n\} : |\lambda_i(A_n)| > M\}}{s_n} = 1 - \frac{1}{s_n}\sum_{i=1}^{s_n} \chi_{B(0,M)}(\lambda_i(A_n)) \le 1 - \frac{1}{s_n}\sum_{i=1}^{s_n} G_{M/2}(\lambda_i(A_n)).$$

If we take the limits on both sides, then

$$\limsup_{n\to\infty} \frac{\#\{i \in \{1,\dots,s_n\} : |\lambda_i(A_n)| > M\}}{s_n} \leq 1 - \int_{\mathbb{C}} G_{M/2}\mathrm{d}\mu \leq 1 - \mu(B(0,M/2))$$

and taking again the limits,

$$\limsup_{M\to\infty}\limsup_{n\to\infty} \frac{\#\{i \in \{1,\dots,s_n\} : |\lambda_i(A_n)| > M\}}{s_n} \leq 1 - \mu(\mathbb{C}).$$

This is enough to prove the thesis.
The version for the singular values can be proved with analogous arguments. □

It may seems a far-fetched argument, but considering that every measurable function is a symbol for some sequence in $\mathscr{E}$, as we will show in Theorem 5.1.5, it leads to the following result.

Lemma 4.1.10.
Each measure μ on $\mathbb{C}$ with mass less than or equal to 1 is a spectral symbol for some sequence $\{A_n\}_n \in \mathscr{E}_H$.
Each measure μ on $\mathbb{R}^+$ with mass less than or equal to 1 is a singular value symbol for some sequence $\{A_n\}_n \in \mathscr{E}$.

Proof. Given a measure μ on $\mathbb{C}$ with mass less than or equal to 1, we know by Lemma 4.1.7 that there exists a measurable function $\kappa : [0,1] \to \mathbb{C} \cup \{\infty\}$ such that takes finite values on $[0, \|\phi\|_\infty]$ and

$$\int_{\mathbb{C}} G\mathrm{d}\mu = \int_{[0,\|\phi\|_\infty]} G(\kappa(x))\mathrm{d}x \qquad \forall\, G \in C_c(\mathbb{C}).$$

Consider a function $\kappa' \in \mathscr{M}_{[0,1]}$ such that

$$\kappa'(x) := \kappa\left(\|\phi\|_\infty x\right).$$

From the results in [5], we can deduce that there always exists a sequence of Hermitian matrices B_n of size $n \times n$ such that $\{B_n\}_n \sim_\lambda \kappa'$. Let us produce a Hermitian sequence $\{A_n\}_n \in \mathscr{E}_H$ by the following rule

$$A_n = \begin{pmatrix} nI_{s_n-g_n} & 0 \\ 0 & B_{g_n} \end{pmatrix}$$

where $s_n \times s_n$ is the size of A_n and $g_n = \lfloor \|\phi\|_\infty s_n \rfloor$. Given any $G \in C_c(\mathbb{C})$ we have that

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{s_n}\sum_{i=1}^{s_n} G(\lambda_i(A_n)) &= \lim_{n\to\infty} \frac{1}{s_n}\left((s_n - g_n)G(n) + \sum_{i=1}^{g_n} G(\lambda_i(B_{g_n}))\right) \\
&= \lim_{n\to\infty} \frac{1}{s_n}\sum_{i=1}^{g_n} G(\lambda_i(B_{g_n})) \\
&= \lim_{n\to\infty} \frac{\lfloor \|\phi\|_\infty s_n \rfloor}{s_n}\frac{1}{g_n}\sum_{i=1}^{g_n} G(\lambda_i(B_{g_n})) \\
&= \|\phi\|_\infty \int_{[0,1]} G(\kappa'(x))\mathrm{d}x
\end{aligned}$$

and by a change of variables $y = \|\phi\|_\infty x$ we conclude that

$$\lim_{n\to\infty} \frac{1}{s_n}\sum_{i=1}^{s_n} G(\lambda_i(A_n)) = \int_{[0,\|\phi\|_\infty]} G(\kappa\,(y))\mathrm{d}y = \int_{\mathbb{C}} G\mathrm{d}\mu.$$

Consequentially, $\{A_n\}_n \in \mathscr{E}_H$ is a sequence with spectral symbol μ. The reasoning for the singular value symbol is totally analogous. □

We have just shown that the measures with mass less than or equal to 1 are all the possible spectral measures, and we denote this space with $\mathbb{P}_{\leq 1}$. In the next paragraph, we see how to use the property of this space to come up with information about the convergence of the symbols.

4.1.2 Vague Convergence

In paragraph 3.2.1 we reported how the a.c.s. convergence on sequences links with the pointwise convergence on the symbols ϕ. Here we recall the theorems.

$$\begin{tikzcd} \{B_{n,m}\}_{n,m} \arrow[r, "a.c.s."] \arrow[d, "\sim"'] & \{A_n\}_n \arrow[d, dashed, "\sim"] \\ \phi_m \arrow[r] & \phi \end{tikzcd}$$

Theorem 3.2.1. *Let $\{A_n\}_n, \{B_{n,m}\}_{n,m} \in \mathscr{E}$ and let $\phi, \phi_m \in C_c'(\mathbb{R})$. Suppose that*

1. $\{B_{n,m}\}_{n,m} \sim_\sigma \phi_m$ *for every* m,
2. $\{B_{n,m}\}_{n,m} \xrightarrow{a.c.s.} \{A_n\}_n$,
3. $\phi_m \to \phi$ *pointwise.*

Then $\{A_n\}_n \sim_\sigma \phi$.

Theorem 3.2.2. *Let $\{A_n\}_n, \{B_{n,m}\}_{n,m} \in \mathscr{E}_H$ and let $\phi, \phi_m \in C_c'(\mathbb{R})$. Suppose that*

1. $\{B_{n,m}\}_{n,m} \sim_\lambda \phi_m$ *for every* m,
2. $\{B_{n,m}\}_{n,m} \xrightarrow{a.c.s.} \{A_n\}_n$,
3. $\phi_m \to \phi$ *pointwise.*

Then $\{A_n\}_n \sim_\lambda \phi$.

Moreover, we can also add two analogous theorems showing even more connections between the two spaces.

$$\begin{tikzcd} \{B_{n,m}\}_{n,m} \arrow[r, "a.c.s."] \arrow[d, "\sim"'] & \{A_n\}_n \arrow[d, "\sim"] \\ \phi_m \arrow[r, dashed] & \phi \end{tikzcd}$$

Theorem 4.1.11 ([52, Theorem 5.5])**.** *Let $\{A_n\}_n, \{B_{n,m}\}_{n,m} \in \mathscr{E}$ and let $\phi, \phi_m \in C_c'(\mathbb{R})$. Suppose that*

1. $\{B_{n,m}\}_{n,m} \sim_\sigma \phi_m$ *for every* m,
2. $\{B_{n,m}\}_{n,m} \xrightarrow{a.c.s.} \{A_n\}_n$,
3. $\{A_n\}_n \sim_\sigma \phi$.

Then $\phi_m \to \phi$ *pointwise.*

Theorem 4.1.12. *Let $\{A_n\}_n, \{B_{n,m}\}_{n,m} \in \mathscr{E}_H$ and let $\phi, \phi_m \in C_c'(\mathbb{R})$. Suppose that*

1. $\{B_{n,m}\}_{n,m} \sim_\lambda \phi_m$ *for every* m,
2. $\{B_{n,m}\}_{n,m} \xrightarrow{a.c.s.} \{A_n\}_n$,
3. $\{A_n\}_n \sim_\lambda \phi$.

Then $\phi_m \to \phi$ *pointwise.*

Pointwise limit of positive functionals ϕ_m is still a positive functional ϕ. By Theorem 4.1.1 we know there exist μ_m and μ Radon measures that represents ϕ_m and ϕ respectively, and since ϕ_m are symbols, we also know that μ_m are finite measures, with mass bounded by 1. Notice that

$$\phi_m \to \phi \iff \phi_m(G) \to \phi(G)\ \forall G \in C_c(\mathbb{C}) \iff \int_{\mathbb{C}} G \mathrm{d}\mu_m \to \int_{\mathbb{C}} G \mathrm{d}\mu \quad \forall G \in C_c(\mathbb{C}).$$

It means that the pointwise convergence of ϕ_m coincides with the vague (or weak*) convergence of finite measures. In particular, since $|\mu_m| \leq 1$ for every m, it is possible to prove that $|\mu| \leq 1$.

If we denote the vague convergence simply as $\mu_m \to \mu$, then we can rewrite the previous results with the notation of measures as follows.

$$\begin{tikzcd}
\{B_{n,m}\}_{n,m} \arrow[r, "a.c.s."] \arrow[d, "\sim"'] & \{A_n\}_n \arrow[d, dashed, "\sim"] \\
\mu_m \arrow[r, dashed] & \mu
\end{tikzcd}$$

Theorem 4.1.13. *Let $\{A_n\}_n, \{B_{n,m}\}_{n,m} \in \mathscr{E}$ and let $\mu, \mu_m \in \mathbb{P}_{\leq 1}$. Suppose that*

1. *$\{B_{n,m}\}_{n,m} \sim_\sigma \mu_m$ for every m,*
2. *$\{B_{n,m}\}_{n,m} \xrightarrow{a.c.s.} \{A_n\}_n$.*

Then

$$\{A_n\}_n \sim_\sigma \mu \iff \mu_m \to \mu.$$

Theorem 4.1.14. *Let $\{A_n\}_n, \{B_{n,m}\}_{n,m} \in \mathscr{E}_H$ and let $\mu, \mu_m \in \mathbb{P}_{\leq 1}$. Suppose that*

1. *$\{B_{n,m}\}_{n,m} \sim_\lambda \mu_m$ for every m,*
2. *$\{B_{n,m}\}_{n,m} \xrightarrow{a.c.s.} \{A_n\}_n$.*

Then

$$\{A_n\}_n \sim_\lambda \mu \iff \mu_m \to \mu.$$

Moreover, if $\phi_m = \phi_{\kappa_m}$ and $\phi = \phi_\kappa$, then convergence in measure $\kappa_m \to \kappa$ induces the pointwise convergence $\phi_m \to \phi$ and, as a consequence, the vague convergence $\mu_m \to \mu$.

Notice that a reverse result would be hard even to formulate, since we have seen in paragraph 1.2.1 that for a single symbol ϕ or μ there may be infinite κ such that $\phi = \phi_\kappa$. An exception is when μ_m, μ are probability measures on $\mathbb{R}$, so that the symbols κ_m and κ must be real-valued, and there is a canonical choice for the symbol, represented by the increasing rearrangement (paragraph 2.2.1).

Lemma 4.1.15. *Let μ_m and μ be probability measures, and let κ_m, κ be measurable functions such that $\phi_{\mu_m} = \phi_{\kappa_m}$ and $\phi_\mu = \phi_\kappa$. If ξ_m, ξ are the increasing rearrangements of κ_m, κ, then*

$$\mu_m \to \mu \iff \xi_m \to \xi \ a.e.$$

In this case, ξ_m and ξ are commonly called the quantiles of μ_m and μ. In addition, in [22], the authors showed that a convergence in distribution of the measures leads to an uniform converge on the quantile functions.

Notice that even if μ_m were all probability measure, their vague limit μ might not be a probability measure any more. On this occasion, the quantiles ξ_m do not converge to a function.

Example 4.1.16

► Take the measures

$$\mu_m = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{(-1)^m m}.$$

with their quantiles

$$\xi_m(x) = \begin{cases} -\chi_{[0,1/2]}(x)m + \chi_{(1/2,1]}(x), & m \text{ odd}, \\ \chi_{[0,1/2]}(x) + \chi_{(1/2,1]}(x)m, & m \text{ even}. \end{cases}$$

We can observe that ξ_m do not converge, since they oscillate, whereas $\mu_m \to \mu = \frac{1}{2}\delta_1$.

Notice that the measures in the last example actually mimic the behaviour of the eigenvalues in Example 4.0.1. In fact, we can associate to every matrix A of size $s_n \times s_n$ a probability measure defined as

$$\mu_A := \frac{1}{s_n}\sum_{i=1}^{s_n} \delta_{\lambda_i(A)}$$

and rewrite the ergodic formula as

$$\lim_{n\to\infty} \frac{1}{s_n}\sum_{j=1}^{s_n} G(\lambda_j(A_n)) = \lim_{n\to\infty} \int_{\mathbb{C}} G \mathrm{d}\mu_{A_n} = \int_{\mathbb{C}} G \mathrm{d}\mu, \qquad \forall\, G \in C_c(\mathbb{C}),$$

that is equivalent to

$$\{A_n\}_n \sim_\lambda \mu \iff \mu_{A_n} \to \mu.$$

4.2 Complete Metrics

We have seen that all the symbols are identified with a finite measure with mass at most 1. The space $\mathbb{P}_{\leq 1}$ though, comes short in properties when compared with the space of probability measures $\mathbb{P}$, where every spectral measure can be represented by a measurable function. Consequently, here we focus mainly on $\mathbb{P}$, and, unless it is not clearly specified, we will always be talking about probability measures over the complex plane $\mathbb{C}$

4.2.1 Optimal Matching and Prokhorov Distance

In paragraph 3.2.3 we focused on the perturbations that do not change the spectral symbol of the sequences. Here we can derive new result on the same theme, but using tools that comes from measure theory.

First of all, we observed that the a.c.s. pseudometric was not enough to analyse the perturbations to the spectral symbol, so here we define a different distance on the sequences.

Definition 4.2.1. *Given $v, w \in \mathbb{C}^n$, the* ***modified optimal matching distance*** *is defined as*

$$d'(v, w) := \min_{\sigma \in S_n} \min_{i=1,\dots,n+1} \left\{ \frac{i-1}{n} + |v - w_\sigma|_i^\downarrow \right\},$$

where

$$|v - w_\sigma| = [|v_1 - w_{\sigma(1)}|, |v_2 - w_{\sigma(2)}|, \dots, |v_n - w_{\sigma(n)}|]$$

and $|v - w_\sigma|_i^\downarrow$ is the i-th greatest element in $|v - w_\sigma|$, with the convention $|v - w_\sigma|_{n+1}^\downarrow := 0$.

Given $A, B \in \mathbb{C}^{n\times n}$, we define

$$d'(A, B) := d'(\Lambda(A), \Lambda(B))$$

and if $\{A_n\}_n, \{B_n\}_n \in \mathscr{E}$, we can also define

$$d'(\{A_n\}_n, \{B_n\}_n) := \limsup_{n\to\infty} d'(A_n, B_n).$$

This is actually a modified version of the optimal matching distance, that can be found in [20]. d' is actually not a distance, but like $d_{a.c.s.}$, it induces a complete pseudometric over $\mathscr{E}$ thanks to Theorem 3.1.13, and $d'(\{A_n\}_n, \{B_n\}_n)$ is always bounded by 1. The most important property of d' is that it identifies two sequences if and only if they have the same symbol.

Theorem 4.2.2 ([7])**.** *If $\{A_n\}_n \sim_\lambda \mu$ where $\mu \in \mathbb{P}$, then*

$$\{B_n\}_n \sim_\lambda \mu \iff d'(\{A_n\}_n, \{B_n\}_n) = 0.$$

Notice that the last result works even if we put a measurable function κ instead of μ. This theorem has been used extensively in [12] and [8] to prove all the results reported in paragraph 3.2.3. For this reason, the rest of the section is focused on explaining briefly how to prove it.

First of all, we need to find an analogous distance on the symbols. Since we are working in $\mathbb{P}$ instead of $\mathbb{P}_{\leq 1}$, we can use the Lévy-Prokhorov distance, that induces the vague convergence and it is complete ([67]).

Definition 4.2.3. *the* ***Lévy-Prokhorov Metric*** *on $\mathbb{P}$ is defined as*

$$\pi(\mu, \nu) = \inf \{ \varepsilon > 0 \mid \mu(A) \leq \nu(A^\varepsilon) + \varepsilon,\ \nu(A) \leq \mu(A^\varepsilon) + \varepsilon \ \forall A \in \mathscr{B}(\mathbb{C}) \}$$

where

$$A^\varepsilon := \{ x \in \mathbb{C} \mid dist(x, A) < \varepsilon \} = \{ x + y \mid x \in A, |y| < \varepsilon \}.$$

Since we have seen how to associate a measure to every matrix, it is possible to consider π as a distance on the sequences, so we can write

$$\pi(\{A_n\}_n, \{B_n\}_n) := \limsup_{n\to\infty} \pi(\mu_{A_n}, \mu_{B_n}).$$

Using again Theorem 3.1.13, it is possible to prove that the Lévy-Prokhorov metric is a complete pseudometric on $\mathscr{E}$. The distance π on $\mathscr{E}$ can be transferred naturally on the symbols, as the next result shows.

Lemma 4.2.4 ([7])**.** *If $\{A_n\}_n \sim_\lambda \mu$ and $\{B_n\}_n \sim_\lambda \nu$, with $\{A_n\}_n, \{B_n\}_n \in \mathscr{E}$ and $\mu, \nu \in \mathbb{P}$, then*

$$\pi(\{A_n\}_n, \{B_n\}_n) = \pi(\mu, \nu).$$

In the next paragraph, we show how π and d' are linked.

4.2.2 Closure Results

The most important result of the paragraph is that π and d' are actually equivalent distances on $\mathscr{E}$.

Lemma 4.2.5 ([7])**.** *If $\{A_n\}_n, \{B_n\}_n \in \mathscr{E}$, then*

$$\pi(\{A_n\}_n, \{B_n\}_n) \le d'(\{A_n\}_n, \{B_n\}_n) \le 2\pi(\{A_n\}_n, \{B_n\}_n).$$

Actually, an open question is whether π and d' coincide or not.

Conjecture 4.2.6. *If $\{A_n\}_n, \{B_n\}_n \in \mathscr{E}$, then*

$$\pi(\{A_n\}_n, \{B_n\}_n) = d'(\{A_n\}_n, \{B_n\}_n).$$

Using the equivalence of the distances, one can prove a powerful closure result.

Lemma 4.2.7 ([7])**.** *Let $\{B_{n,m}\}_{n,m} \sim_\lambda \mu_m$, where $\{B_{n,m}\}_{n,m} \in \mathscr{E}$ and $\mu_m \in \mathbb{P}$ for every m. If we consider the statements below*

1. $\pi(\mu_m, \mu) \xrightarrow{m\to\infty} 0$,
2. $\{A_n\}_n \sim_\lambda \mu$,
3. $d'(\{B_{n,m}\}_{n,m}, \{A_n\}_n) \xrightarrow{m\to\infty} 0$,

where $\{A_n\}_n \in \mathscr{E}$ and $\mu \in \mathbb{P}$, then any two of them are true if and only if all of them are true.

$$\begin{tikzcd}
\{B_{n,m}\}_{n,m} \arrow[r, dashed, "d'"] \arrow[d, "\sim_\lambda"'] & \{A_n\}_n \arrow[d, dashed, "\sim_\lambda"] \\
\mu_m \arrow[r, dashed, "\pi"] & \mu
\end{tikzcd}$$

The difference with the closure results in paragraph 4.1.2 is that the a.c.s. distance is affected by base-changes of the matrices, whereas π and d' are not, since they work directly on the spectra. This is the central result to exploit in order to prove Theorem 4.2.2 and many other results that we report in the next paragraph.

4.2.3 Corollaries

All the machinery defined and analysed in the previous sections lets us discover new and unexpected results.

For example, in [9] we focus on symbols of sequences that have eigenvalues supported on a circle of $\mathbb{C}$. Using theorems that characterize a measure through its moments, and transferring them on the eigenvalues of the sequences, we can come up with a characterization of bounded Hermitian matrices that admit spectral symbol.

Corollary 4.2.8 ([9])**.** *Given a sequence $\{A_n\}_n$ of Hermitian matrices with $\|A_n\|$ uniformly bounded by M, the sequence $\{A_n\}_n$ admits a spectral symbol if and only if*

$$\lim_{n\to\infty} \frac{1}{n} Tr(A_n^k) \in \mathbb{R} \qquad \forall k \in \mathbb{N}.$$

Along this document, we have already seen several important results that was possible to prove only using measures. A list of the most crucial is: Theorem 2.1.2, Theorem 2.1.5, Lemma 2.2.2, Lemma 2.2.4 and Lemma 3.1.9. Moreover, the theory developed in this section will be used extensively in all the next results, even if it isn't said explicitly.

Part II

Sequences Algebras and Groups

Chapter 5

Algebraic Structures

In the first part, we introduced the concept of symbol referred to matrix-sequences. In particular, we focused on the cases when the symbol may be represented by a measurable function, or equivalently by a probability measure. In fact, we could analyse the notion of convergence in the space of measurable functions or in the space of probability measures and link them to different pseudometrics on the space of sequences $\mathscr{E}$. With these tools, we could find closure and perturbation results on the spaces $\mathscr{E}, \mathscr{E}_H$ and much more.

Up until now, though, we analysed only the metrical aspect of the spaces, but $\mathscr{E}$ and $\mathscr{E}_H$ have also a natural structure as $\mathbb{R}$-vectorial spaces, and $\mathscr{E}$ is a $\mathbb{C}$-algebra closed for the conjugation operator. The spaces $\mathbb{P}$ or $\mathbb{P}_{\leq 1}$ of finite measures have not such structure, so in this chapter we will focus on the space of measurable functions $\mathscr{M}_D$ in order to draw a link between its algebraic structure and the ones of $\mathscr{E}, \mathscr{E}_H$.

In this same chapter we report some first applications of simple algebra structures in the determination of the symbol for certain matrix-sequences.

5.1 Diagonal Sequences

Maybe the most simple example of a subalgebra in $\mathscr{E}$ is the space of diagonal sequences, since it is immediate to analyse their eigenvalues and singular values. Moreover, diagonal matrices are always normal matrices, so a spectral symbol for a diagonal sequence is also a singular value symbol.

5.1.1 Diagonal Sampling Sequences

Definition 5.1.1. *Given any function $a : [0,1] \to \mathbb{C}$, its associated **diagonal sampling sequence** in $\mathscr{E}$ is $\{D_n(a)\}_n$, where*

$$D_n(a) = \operatorname*{diag}_{i=1,\dots,s_n} a\left(\frac{i}{s_n}\right).$$

The definition for diagonal sampling sequence comes directly from Definition 1.3.1, since we are taking, in order, an exact sampling of the function $a(x)$ over $[0,1]$. Consequently, it is not surprising that $\{D_n(a)\}_n$ has as $a(x)$ as symbol whenever it is regular enough.

Lemma 5.1.2 ([6, 52])**.** *Given any almost everywhere continuous function $a : [0,1] \to \mathbb{C}$,*

$$\{D_n(a)\}_n \sim_{\lambda,\sigma} a(x).$$

Notice that sums and products of a.e. continuous functions are still a.e. continuous functions, and

- $D_n(a) + D_n(b) = D_n(a+b)$,
- $D_n(a)D_n(b) = D_n(ab)$,
- $D_n(\overline{a}) = D_n(a)^H$,
- $zD_n(a) = D_n(za)$,

for every a, b a.e. continuous function and every scalar $z \in \mathbb{C}$. Call $\mathscr{D}'$ the set of diagonal sampling sequences referred to a.e. continuous function. This space is nice because the algebraic structure of the space reflects into the structure of the symbols, building an isomorphism between the two spaces. The space we are interested to analyse, though, is the closure of $\mathscr{D}'$ with respect to the a.c.s. distance,

$$\mathscr{D} := \overline{\mathscr{D}'}^{a.c.s.}.$$

Notice that every sequence in $\mathscr{D}'$ has a measurable function as symbol, so they are all s.u. and s.s.u. sequences by Lemma 3.1.9. As a consequence, Theorem 3.1.10 tells us that $\mathscr{D}$ is still a $\mathbb{C}$-algebra. In the next paragraph we show that every sequence in $\mathscr{D}$ admits a symbol, and that $d_{a.c.s.}$ on $\mathscr{D}$ correspond to the distance d_{mea} on $\mathscr{M}_{[0,1]}$.

5.1.2 Equivalence with Measurable Functions

Consider an element $\{A_n\}_n \in \mathscr{D}$. By definition, there exists a sequence of sequences $\{D_n(a_m)\}_{n,m} \subseteq \mathscr{D}'$ that converges a.c.s. to $\{A_n\}_n$. In particular, $\{D_n(a_m)\}_{n,m}$ is a Cauchy sequence for the a.c.s. distance, and thanks to Theorem 3.1.18 we can show that also $\{a_m\}_m$ is a Cauchy sequence with respect to the distance d_{mea} on $\mathscr{M}_{[0,1]}$ as follows.

$$\begin{aligned} d_{a.c.s.}(\{D_n(a_s)\}_n, \{D_n(a_t)\}_n) &= \rho_{a.c.s.}(\{D_n(a_s) - D_n(a_t)\}_n) \\ &= \rho_{a.c.s.}(\{D_n(a_s - a_t)\}_n) \\ &= p_{mea}(a_s - a_t) \\ &= d_{mea}(a_s, a_t). \end{aligned} \tag{5.1}$$

The distance d_{mea}, though, is complete, so $a_m \to a$ in measure, and by Corollary 3.2.3 we conclude that $\{A_n\}_n \sim_\sigma a(x)$. Moreover, if we suppose that $\{D_n(b_m)\}_{n,m}$ is another sequence in $\mathscr{D}'$ converging to $\{A_n\}_n$, then we can repeat the same reasoning and find a function $b \in \mathscr{M}_{[0,1]}$ such that $b_m \to b$ in measure and $\{A_n\}_n \sim_\sigma b(x)$. As a consequence,

$$\{D_n(a_m)\}_{n,m} \to \{A_n\}_n, \quad \{D_n(b_m)\}_{n,m} \to \{A_n\}_n \implies \{D_n(a_m) - D_n(b_m)\}_{n,m} \to \{0_n\}_n \implies a_m - b_m \to 0$$

and

$$a_m \to a, \quad b_m \to b \implies a_m - b_m \to a - b = 0.$$

In conclusion, we find that we can assign to each sequence $\{A_n\}_n \in \mathscr{D}$ an uniquely determined symbol $a(x)$. If we denote this choice as $\{A_n\}_n \sim_{\mathscr{D}} a(x)$, we can prove that

- $\{D_n(a)\}_n \sim_{\mathscr{D}} a$ for every $\{D_n(a)\}_n \in \mathscr{D}'$,
- $\{A_n\}_n \sim_{\mathscr{D}} a, \quad \{B_n\}_n \sim_{\mathscr{D}} b \implies \{A_n\}_n + \{B_n\}_n \sim_{\mathscr{D}} a + b$,
- $\{A_n\}_n \sim_{\mathscr{D}} a, \quad \{B_n\}_n \sim_{\mathscr{D}} b \implies \{A_n\}_n\{B_n\}_n \sim_{\mathscr{D}} ab$,
- $\{A_n\}_n \sim_{\mathscr{D}} a \implies \{A_n^H\}_n \sim_{\mathscr{D}} \overline{a}$,
- $\{A_n\}_n \sim_{\mathscr{D}} a \implies \{zA_n\}_n \sim_{\mathscr{D}} za$ for every $z \in \mathbb{C}$,
- $\{B_{n,m}\}_{n,m} \sim_{\mathscr{D}} b_m, \quad \{B_{n,m}\}_{n,m} \xrightarrow{a.c.s.} \{A_n\}_n, \quad b_m \to a \implies \{A_n\}_n \sim_{\mathscr{D}} a$,
- $\{A_n\}_n \sim_{\mathscr{D}} a, \quad \{B_n\}_n \sim_{\mathscr{D}} b \implies d_{a.c.s.}(\{A_n\}_n, \{B_n\}_n) = d_{mea}(a, b)$.

Every function $a \in \mathscr{M}_{[0,1]}$ can be expressed by limit in measure of a sequence of continuous functions $\{a_m\}_m \subseteq \mathscr{M}_{[0,1]}$, that is in particular a Cauchy sequence. If we now take the sequence of matrix-sequences $\{D_n(a_m)\}_{n,m} \subseteq \mathscr{D}'$, we know by Equation 5.1 that it is a Cauchy sequence with respect to $d_{a.c.s.}$ and by Theorem 3.1.13 and Corollary 3.2.3, we conclude that there exists a limit sequence $\{D_n\}_n \in \mathscr{D}$ with symbol $a(x)$. In conclusion, we can summarize all the arguments and results of the paragraph in the following theorem.

Theorem 5.1.3 ([6]). *$\mathscr{D}$ is a $\mathbb{C}$-algebra closed for a.c.s. convergence and for the conjugation operator. Each sequence $\{A_n\}_n \in \mathscr{D}$ enjoys a canonical symbol, and the map $\mathfrak{s} : \mathscr{D} \to \mathscr{M}_{[0,1]}$ defined as*

$$\mathfrak{s}(\{A_n\}_n) = a(x) \iff \{A_n\}_n \sim_{\mathscr{D}} a$$

is a surjective homomorphism of $\mathbb{C}$*-algebras that preserves the metrics* $(\mathscr{D}, d_{a.c.s.}) \sim (\mathscr{M}_{[0,1]}, d_{mea})$*. Moreover,*

$$\{A_n\}_n \sim_{\mathscr{D}} a \implies \{A_n\}_n \sim_\sigma a$$

and if A_n *are normal matrices,*

$$\{A_n\}_n \sim_{\mathscr{D}} a \implies \{A_n\}_n \sim_\lambda a$$

Notice that the a.c.s. limit of diagonal sequences is always a.c.s. equivalent to a diagonal sequence thanks to Theorem 3.1.13, so we can conclude that for any function $a \in \mathscr{M}_{[0,1]}$ there exists a diagonal sequence $\{D_n\}_n$ such that $\{D_n\}_n \sim_\sigma a(x)$.

$\{D_n\}_n$ is an a.c.s. limit of diagonal sampling sequence, but it is not true in general that $\{D_n\}_n \sim_{a.c.s.} \{D_n(a)\}_n$.

Example 5.1.4

- Take $a(x) = \chi_{\mathbb{Q}\cap[0,1]}$. It is easy to see that $\{D_n(a)\}_n = \{I_n\}_n$, but $a(x)$ is zero a.e., so

$$\{D_n(a)\}_n \sim_\sigma a(x) \implies \{I_n\}_n \sim_\sigma 0$$

  that is impossible.

A corollary of the previous result is that every measurable function in $\mathscr{M}_{[0,1]}$ is a symbol for some sequence.

Theorem 5.1.5 ([5, 6])**.** *For every function* $a \in \mathscr{M}_{[0,1]}$ *there exists a sequence* $\{A_n\}_n \in \mathscr{E}$ *with*

$$\{A_n\}_n \sim_{\sigma,\lambda} a(x).$$

The set $\mathscr{D}$ is probably the most simple example of sub-algebra in $\mathscr{E}$ where it is possible to choose for each sequence $\{A_n\}_n \subseteq \mathscr{D}$ a canonical symbol such that the algebra structure and the metric of $\mathscr{M}_{[0,1]}$ coincide with the ones of $\mathscr{D}$. Notice that it also admits non-diagonal sequences as elements, and we can denote the space of couple sequences-symbol as

$$\mathcal{D} := \{\, (\{A_n\}_n, \kappa) \mid \{A_n\}_n \sim_{\mathscr{D}} \kappa_m \,\}\,.$$

This closed algebra is important because shows that choosing a symbol in an algebra coincides with choosing an ordering of the eigenvalues.

Theorem 5.1.6 ([6])**.** *Given* $\{D_n\}_n$ *a sequence of diagonal matrices, and* $a : [0,1] \to \mathbb{C}$ *any measurable function, the following are equivalent*

- *the entries of* D_n*, in order, converge to* $a(x)$ *as in Definition 1.3.1*
- $D_n \sim_{\mathscr{D}} a(x)$*.*

Moreover, the two definition of spectral symbols Definition 1.3.1 and Definition 2.1.1 coincide on measurable functions $a : [0,1] \to \mathbb{C}$, so for any diagonal sequence with symbol $a(x)$ we can find an ordering of the entries such that they approximate $a(x)$. In other words, there exists an ordering of the eigenvalues that bring any diagonal sequence with spectral symbol into $\mathcal{D}$.

Theorem 5.1.7 ([7])**.** *Given a measurable function* $a : [0,1] \to \mathbb{C}$*, and a diagonal sequence* $\{D_n\}_n$ *with spectral symbol* $a(x)$*, there exists a sequence* $\{P_n\}_n$ *of permutation matrices such that* $(\{P_n D_n P_n^T\}_n, a(x)) \in \mathcal{D}$*.*

Now let us turn our attention on more general sequences of matrices.

5.2 Normal Sequences

Normal matrices are characterized by the property to be diagonalizable through unitary base-change. It means that an easy way to extract an algebra from the space of normal sequences is to fix a sequence of unitary matrices $\{U_n\}_n$ and define the space

$$\mathscr{D}_U := \{\, \{U_n D_n U_n^H\}_n \mid \{D_n\}_n \in \mathscr{D} \,\} .$$

Notice that all algebraic operations on $\mathscr{D}_U$ reduces to the same operations on $\mathscr{D}$, and since the distance $d_{a.c.s.}$ is invariant by unitary base-change, Theorem 5.1.3 holds also for $\mathscr{D}_U$. In particular, there's a canonical choice of a symbol for each element of $\mathscr{D}_U$ such that sums, products and convergence of sequences reflects on the symbols. A classical example of such algebra can be built starting from the τ-algebra defined in [21], or in general, from ξ-circulant or Hartley-type algebras [23].

5.2.1 Circulant Algebra

In the present paragraph, we explore the algebra of circulant matrices. Recall that we defined C_n as the $s_n \times s_n$ matrix

$$[C_n]_{i,j} = \begin{cases} 1, & i-j \equiv 1 \pmod{s_n}, \\ 0, & \text{otherwise} \end{cases} \implies C_n = \begin{pmatrix} & & & 1 \\ 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

and F_n as the Fourier matrix of size $s_n \times s_n$,

$$F_n = \frac{1}{\sqrt{s_n}} (e^{-2\pi \mathrm{i}(j-1)(i-1)/s_n})_{i,j=1}^{s_n}$$

that is an unitary symmetric matrix diagonalising C_n

$$D_n = F_n^H C_n F_n = \begin{pmatrix} 1 & & & & \\ & e^{2\pi \mathrm{i}/s_n} & & & \\ & & e^{2*2\pi \mathrm{i}/s_n} & & \\ & & & \ddots & \\ & & & & e^{(s_n-1)*2\pi \mathrm{i}/s_n} \end{pmatrix} .$$

If we consider now the sequence $\{D_n(e^{2\pi x \mathrm{i}})\}_n$, we can observe that it is a.c.s. equivalent to $\{D_n\}_n$ as

$$D_n(e^{2\pi x \mathrm{i}}) - D_n = \mathrm{diag}_{i=1,\dots,s_n} e^{2\pi i \mathrm{i}/s_n} - e^{2\pi(i-1)\mathrm{i}/s_n}$$

and

$$|e^{2\pi i \mathrm{i}/s_n} - e^{2\pi(i-1)\mathrm{i}/s_n}| = 2 - 2\cos\left(\frac{2\pi}{s_n}\right) = o(1) \quad \forall\, i = 1,\dots,s_n \implies \|D_n(e^{2\pi x \mathrm{i}}) - D_n\| = o(1).$$

Consequently, we find that $\{D_n\}_n \in \mathscr{D}$ and $\{C_n\}_n \in \mathscr{D}_F$, where the canonical symbol associated to $\{C_n\}_n$ is $e^{2\pi x \mathrm{i}}$. Moreover, $\mathscr{D}_F$ is a $\mathbb{C}$-algebra, so $\{C_n\}_n$ generate a $\mathbb{C}$-subalgebra as follows

$$\mathscr{C} := \{\, \{p(C_n)\}_n \mid p(x) \in \mathbb{C}[x, \overline{x}] \,\} \subseteq \mathscr{D}_F$$

and from the identification between sequences and symbols (Theorem 5.1.3), we conclude that

$$\{p(C_n)\}_n \sim_{\mathscr{D}_F} p(e^{2\pi x \mathrm{i}}) \implies \{p(C_n)\}_n \sim_{\lambda,\sigma} p(e^{2\pi x \mathrm{i}}).$$

Notice moreover that the set of functions $\{\, p(e^{2\pi x \mathrm{i}}) \mid p(x) \in \mathbb{C}[x, \overline{x}] \,\}$ contains all the trigonometric function on the interval $[0, 1]$, so it is a dense set in $\mathscr{M}_{[0,1]}$ with respect to the convergence in measure. It means that if $\{A_n\}_n \sim_{\mathscr{D}_F} a(x)$, then there exists $p_m(e^{2\pi x \mathrm{i}})$ converging to $a(x)$ in measure, and consequently

$$d_{a.c.s.}(\{p_m(C_n)\}_{m,n}, \{A_n\}_n) = d_{mea}(p_m(e^{2\pi x \mathrm{i}}), a(x)) \to 0 \implies \{p_m(C_n)\}_{m,n} \xrightarrow{a.c.s.} \{A_n\}_n.$$

In particular, $\mathscr{C}$ is dense in $\mathscr{D}_F$. This example shows how the algebra structure can help us find symbols for specific matrix-sequences. In paragraph 5.2.3 we use circulant matrices and their symbols to infer spectral properties of sequences of linear systems coming from the discretization of differential equations.

5.2.2 Other Results

When we consider the whole set of normal sequences, we lose most of the nice properties found in the previous paragraph. Nonetheless, some algebraic results can still be proved.

Lemma 5.2.1 ([10]). *Let $\{N_n\}_n$ be a normal sequence, and let $\{A_n\}_n$ be a generic sequence.*

- $\{N_n\}_n \sim_\sigma k \implies \{N_n^s\}_n \sim_\sigma k^s, \qquad \forall s \in \mathbb{N},$
- $\{A_n\}_n \sim_\lambda k \implies \{f(A_n)\}_n \sim_\lambda f \circ k, \qquad \forall f \in C(\mathbb{C}).$

Notice that here we must distinguish spectral symbol from singular values symbols, even when dealing with normal sequences. In the next example, we show a counterexample to the statement

$$\{N_n\}_n \sim_\sigma k \implies \{f(N_n)\}_n \sim_\sigma f \circ k, \qquad \forall f \in C(\mathbb{C})$$

with normal matrices N_n.
Example 5.2.2

► Consider the $2n \times 2n$ normal matrices N_n, where

$$N_n = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}.$$

All the singular values of N_n are 1, so $N_n \sim_\sigma \kappa(x) \equiv 1$, where $\kappa \in \mathscr{M}_{[0,1]}$. Let us now consider the affine function $f(z) = z + 1$, and notice that

$$f(N_n) = \begin{pmatrix} 2I_n & 0 \\ 0 & 0 \end{pmatrix}, \qquad f(\kappa(x)) \equiv 2.$$

If we take now a function $G \in C_c(\mathbb{C})$ such that $G(2) \neq 0$ and $G(0) = 0$, we obtain that

$$\lim_{n\to\infty} \frac{1}{2n} \sum_{i=1}^{2n} G(\lambda_i(f(N_n))) = \frac{G(2)}{2} \neq G(2) = \int_{[0,1]} G(\kappa(x) + 1)\mathrm{d}x.$$

This is enough to say that $\{f(N_n)\}_n \not\sim_\sigma f \circ k$.

Notice that in the example we had to choose a singular value symbol κ that was not a spectral symbol, otherwise Lemma 5.2.1 would have applied. Here an affine function $f(z)$ was able to prove that κ was not a spectral symbol. Actually, it can be proved that affine functions are always enough to distinguish singular values symbols from spectral symbols.

Lemma 5.2.3. *[10] Given $\{N_n\}_n$ a normal sequence such that*

$$\{N_n - cI_n\}_n \sim_\sigma k(z) - c \qquad \forall c \in \mathbb{C}$$

then $\{N_n\}_n \sim_\lambda k$.

This result has been used, for example, to prove statement 1. in Lemma 3.2.18 and has nice consequences on the algebras of sequences that we will introduce in the next section. Actually, an open question that links the algebraic structure of the sequences to their symbol is the following.

Conjecture 5.2.4. *Consider a sequence $\{A_n\}_n$ such that*

$$\{p(A_n)\} \sim_\sigma p \circ \kappa$$

for every polynomial $p(x) \in \mathbb{C}[x, \overline{x}]$. In this case, there exists a normal sequence $\{N_n\}_n \sim_{a.c.s.} \{A_n\}_n$.

In fact, thanks to Lemma 5.2.3, that would mean that any sequence $\{A_n\}_n$ in an algebra with a canonical singular value symbol κ, admits an a.c.s. equivalent "normal form" that has κ as spectral and singular value symbol.

Example 5.2.5

- If $A_n = J_n$ the Jordan block of size $n \times n$ relative to the eigenvalue 0, and C_n is the classical circulant matrix of paragraph 5.2.1, then $A_n = C_n - R_n$ where $\mathrm{rk}(R_n) = 1$. This is enough to prove that $\{A_n\}_n \sim_{a.c.s.} \{C_n\}_n$, so
$$\{A_n\}_n \sim_\sigma \kappa = e^{2\pi x \mathrm{i}}.$$
Thanks to the property of a.c.s. convergence reported in Theorem 3.1.10 and of the zero sequences, one can prove that for every $p(x) \in \mathbb{C}[x, \overline{x}]$ the sequence $p(\{A_n\}_n)$ is a.c.s. equivalent to $p(\{C_n\}_n)$, so
$$\{p(A_n)\} \sim_\sigma p \circ \kappa.$$

Notice that even if the conjecture is true, we cannot conclude that A_n are normal matrices or that $\{A_n\}_n \sim_\lambda \kappa$, as one can infer from the last example. This is the reason why looking for an equivalent normal sequence is important.

5.2.3 A First Application

In the present section, we would like to investigate what happens if we approximate by finite differences an higher-order differential equation. We will focus on the following simple fourth-order problem with homogeneous Dirichlet-Neumann boundary conditions:
$$\begin{cases} u^{(4)}(x) - u^{(2)} = f(x), & x \in (0,1), \\ u(0) = u'(0) = 0, \quad u(1) = u'(1) = 0, & \end{cases} . \tag{5.2}$$

To approximate the fourth derivative $u^{(4)}(x)$, we use the second-order central FD scheme $(1, -4, 6, -4, 1)$, which yields the approximation
$$u^{(4)}(x)|_{x=x_j} \approx \frac{u(x_{j+2}) - 4u(x_{j+1}) + 6u(x_j) - 4u(x_{j-1}) + u(x_{j-2})}{h^4}, \qquad j = 2, \dots, n+1$$
and to approximate the second derivative $u^{(2)}(x)$, we use the second-order central FD scheme $(1, -2, 1)$, which yields the approximation
$$-u''(x)|_{x=x_j} \approx \frac{-u(x_{j+1}) + 2u(x_j) - u(x_{j-1})}{h^2}, \qquad j = 1, \dots, n+2.$$
Here, $x_j = jh$, $j = 0, \dots, n+3$, and $h = \frac{1}{n+3}$. Taking into account the homogeneous boundary conditions, we approximate the solution of (5.2) by the piecewise linear function that takes the value u_j in x_j for $j = 0, \dots, n+3$, where $u_0 = u_1 = u_{n+2} = u_{n+3} = 0$ and $\boldsymbol{u} = (u_2, \dots, u_{n+1})^T$ is the solution of the linear system
$$\big(u_{j+2} - 4u_{j+1} + 6u_j - 4u_{j-1} + u_{j-2}\big) + h^2\big(-u_{j+1} + 2u_j - u_{j-1}\big) = h^4 f(x_j), \qquad j = 2, \dots, n+1.$$
The matrix A_n of this linear system is given by
$$A_n = B_n + N_n = \begin{bmatrix} 6 & -4 & 1 & & \\ -4 & 6 & \ddots & \ddots & \\ 1 & \ddots & \ddots & \ddots & 1 \\ & \ddots & \ddots & 6 & -4 \\ & & 1 & -4 & 6 \end{bmatrix} + h^2 \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}.$$
Notice that B_n can be expressed by a circulant matrix plus a small rank correction.
$$B_n = S_n + R_n = \begin{bmatrix} 6 & -4 & 1 & & 1 & -4 \\ -4 & 6 & \ddots & \ddots & & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & & \ddots & \ddots & 6 & -4 \\ -4 & 1 & & 1 & -4 & 6 \end{bmatrix} - \begin{bmatrix} & & & & 1 & -4 \\ & & & & & 1 \\ & & & & & \\ & & & & & \\ 1 & & & & & \\ -4 & 1 & & & & \end{bmatrix}$$

Since $\|N_n\| \le 4h^2 = o(1)$, $\mathrm{rk}(R_n) = 4 = o(n)$ and all the matrices are Hermitian, we conclude that $\{A_n\}_n$ and $\{S_n\}_n$ are a.c.s. equivalent, and thanks to Theorem 3.2.11 a symbol of $\{S_n\}_n$ is also a symbol for $\{A_n\}_n$. From paragraph 5.2.1, we obtain

$$\{S_n\}_n \sim_\lambda \kappa(x) = e^{-4\pi x \mathrm{i}} - 4e^{-2\pi x \mathrm{i}} + 6 - 4e^{2\pi x \mathrm{i}} + e^{4\pi x \mathrm{i}} = 6 - 8\cos(2\pi x) + 2\cos(4\pi x)$$

where $\kappa : [0,1] \to \mathbb{R}$. We can thus conclude that

$$\{A_n\}_n \sim_\lambda 6 - 8\cos(2\pi x) + 2\cos(4\pi x).$$

This is an example where we were able to prove that the wanted sequence is a.c.s. equivalent to a circulant matrix for which we already know the symbol.

5.3 Groups and Algebras

In the previous section, we showed examples of simple $\mathbb{C}$-algebras of sequences contained in $\mathscr{E}$ for which we could assign a canonical symbol so that operations on the sequences induce the same operations on the symbols. Here we formalize and analyse such structures.

We start by setting the notation.

- With symbols like $\mathscr{N}, \mathscr{A}, \mathscr{G}, \dots$ we will always denote subsets of $\mathscr{E}$ or $\mathscr{M}_D$. For example, we have already defined $\mathscr{D}', \mathscr{D}$ that are sets of diagonal sequences, $\mathscr{C}$ that is the $\mathbb{C}$-algebra generated by $\{C_n\}_n$, and $\mathscr{D}_U$ for every sequence of unitary matrices $\{U_n\}_n$. In general, we will be mainly interested in sets of sequences enjoying a symbol.
- With the calligraphic font $\mathcal{C}, \mathcal{A}, \mathcal{G}, \dots$ we denote subsets of $\mathscr{E} \times \mathscr{M}_D$. For each set $\mathcal{A}$, if $(\{A_n\}_n, \kappa) \in \mathcal{A}$, then we will write $\{A_n\}_n \sim_{\mathcal{A}} \kappa$ and usually it means that κ is a symbol for $\{A_n\}_n$.
- We will typically use $\mathfrak{s}$ or variants to indicate maps $\mathfrak{s} : \mathscr{E} \to \mathscr{M}_D$ or between subsets of $\mathscr{E}$ and $\mathscr{M}_D$. If $\mathfrak{s}(\{A_n\}_n) = \kappa$, then usually it means that κ is a symbol for $\{A_n\}_n$.

Notice that a map $\mathfrak{s} : \mathscr{A} \to \mathscr{M}_D$ induces a set $\mathcal{A} = \{\, (\{A_n\}_n, \kappa) \mid \mathfrak{s}(\{A_n\}_n) = \kappa \,\}$, and in such case, we have three equivalent ways to say that κ is a symbol for some sequence in $\mathcal{A}$.

$$\{A_n\}_n \sim_{\mathcal{A}} \kappa \iff (\{A_n\}_n, \kappa) \in \mathcal{A} \iff \mathfrak{s}(\{A_n\}_n) = \kappa.$$

The reverse is in general not true, since a single sequence $\{A_n\}_n$ may appear in several couples inside $\mathcal{A}$.

In this section, we will consider only structures on sets of sequences in $\mathscr{E}$ admitting a singular value symbol. The same results can be naturally generalized to sets of sequences admitting a spectral symbol, under the condition that all the sequences considered are Hermitian, so they are contained in $\mathscr{E}_H$ by definition, and all the symbols are real-valued functions.

5.3.1 Definitions

Let us start by denoting with $\mathcal{S}_D$ the set of couples sequence-singular value symbol inside $\mathscr{E} \times \mathscr{M}_D$, where $D \subseteq \mathbb{R}^d$ is a set with $0 < \ell_d(D) < \infty$.

Definition 5.3.1.

$$\mathcal{S}_D := \{\, (\{A_n\}_n, \kappa) \in \mathscr{E} \times \mathscr{M}_D \mid \{A_n\}_n \sim_\sigma \kappa \,\}.$$

In paragraph 3.2.1, we showed that the set of sequences with symbol on a domain D does not depend from D, hence we can call it simply $\mathscr{S}$. Moreover, we also denote the union of all $\mathcal{S}_D$ as

$$\mathcal{S} := \sqcup_D \mathcal{S}_D \subseteq \mathscr{S} \times (\sqcup_D \mathscr{M}_D)$$

where the union ranges all possible $D \subseteq \mathbb{R}^d$ for all $d \in \mathbb{N}$.

The set $\mathscr{S}$ has no algebraic structure at all, as we can infer from the following example.

Example 5.3.2

▶ Take A_n and B_n matrices of size $2n \times 2n$ such that

$$A_n = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}, \qquad B_n = \begin{cases} \begin{pmatrix} -I_n & 0 \\ 0 & 0 \end{pmatrix} & n \text{ even} \\ \begin{pmatrix} 0 & 0 \\ 0 & -I_n \end{pmatrix} & n \text{ odd} \end{cases}.$$

It is easy to verify that $\{A_n\}_n \sim_{\sigma,\lambda} \chi_{[0,1/2]}$ and $\{B_n\}_n \sim_{\sigma,\lambda} -\chi_{[0,1/2]}$, so they both belong to $\mathscr{S}$, but their sum

$$A_n + B_n = \begin{cases} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & n \text{ even} \\ \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix} & n \text{ odd} \end{cases}$$

do not possess a singular value symbol or a eigenvalue symbol. Notice that all the matrices are diagonal, and thus normal.

5.3.1.1 Sequences Groups

We can only hope to find algebraic structures on subsets of $\mathscr{S}$, but we want also a corresponding structure on a linked subset of symbol. For this reason, we define a sequences group as a subset of $\mathcal{S}$.

Definition 5.3.3. *We say that $\mathcal{A}$ is a **sequences group** (s.g.) if it is a group inside $\mathcal{S}$.*

Notice that if $\mathcal{A}$ is a s.g., then there always exists a set $D \subseteq \mathbb{R}^d$ such that $\mathcal{A} \subseteq \mathcal{S}_D$. The notation $\{A_n\}_n \sim_{\mathcal{A}} \kappa$ stands for $(\{A_n\}_n, \kappa) \in \mathcal{A}$ and since $\mathcal{A} \subseteq \mathcal{S}_D$, we can infer that $\{A_n\}_n \sim_\sigma \kappa$. Moreover, we can prove that there exists a function $\mathfrak{s}$ inducing $\mathcal{A}$.

Lemma 5.3.4 ([10])**.** *Given $\mathcal{A} \subseteq \mathcal{C}_D$, the following are equivalent.*

1. *$\mathcal{A}$ is a group,*
2. *There exists a subgroup $\mathscr{A} \subseteq \mathscr{E}$ and an homomorphism of group $\mathfrak{s} : \mathscr{A} \to \mathscr{M}_D$ such that*

$$(\{A_n\}_n, \kappa) \in \mathcal{A} \iff \{A_n\}_n \in \mathscr{A}, \quad \mathfrak{s}(\{A_n\}_n) = \kappa.$$

Notice that given $\mathcal{A}$, the function $\mathfrak{s}$ and its domain $\mathscr{A}$ are uniquely determined.

Lemma 5.3.4 shows that building a group in $\mathcal{S}_D$ requires the choice an unique symbol for every sequence $\{A_n\}_n \in \mathscr{A}$. Recall that if a sequence $\{A_n\}_n$ admits a measurable function κ as a symbol, then every rearrangement of κ is also a symbol for $\{A_n\}_n$, and the number of rearrangements of a function is infinite, with the only exception when κ is constant. This was in contrast with the case of symbols expressed with measures μ or functionals ϕ, since they were uniquely determined by the sequence. In the spaces $\mathbb{P}_{\leq 1}$, $\mathbb{P}$, or in the space of functional symbols ϕ, though, there is not a natural algebraic structure that links with the respective sequences. In $\mathscr{M}_D$, instead, we are able to exploit the degree of freedom for the choice of the symbol, in order to add an algebraic structure to a subset of symbols, which in turn induces a respective structure on the matrix-sequences.

We have already shown example of s.g. with diagonal or circulant sequences, and we have observed that in all these cases the distance a.c.s. of the sequences coincide with the distance of their symbols, even when considering d^φ and d^φ_{mea} defined in section 3.1. Following the same steps as in Equation 5.1, it is possible to prove that it holds for every s.g.

Lemma 5.3.5 ([5])**.** *If $\mathcal{A}$ is a s.g. induced by $\mathfrak{s}$, then*

$$\mathfrak{s}(\{A_n\}_n) = \kappa, \quad \mathfrak{s}(\{B_n\}_n) = \kappa' \implies d_{a.c.s.}(\{A_n\}_n, \{B_n\}_n) = d_{mea}(\kappa, \kappa').$$

Moreover, if $\varphi : [0, \infty) \to [0, \infty)$ is a bounded continuous function such that $\varphi(0) = 0$, then

$$\mathfrak{s}(\{A_n\}_n) = \kappa, \quad \mathfrak{s}(\{B_n\}_n) = \kappa' \implies d^\varphi(\{A_n\}_n\{B_n\}_n) = d^\varphi_{mea}(\kappa, \kappa').$$

Any of the results proved until now on $\mathcal{A}$ shows that whenever κ is the symbol in $\mathcal{A}$ for two different sequences, then the two sequences are a.c.s. equivalent, or equivalently, their difference is a zero-distributed sequence. A natural follow up would be to quotient $\mathscr{A}$ by the zero-distributed sequences, so that the map induced by $\mathfrak{s}$ becomes an embedding of the quotient space into $\mathscr{M}_D$.

Lemma 5.3.6 ([5]). *Given a s.g. $\mathcal{A}$ induced by $\mathfrak{s} : \mathscr{A} \to \mathscr{M}_D$, then the map*

$$\widetilde{\mathfrak{s}} : \frac{\mathscr{A}}{\mathscr{A} \cap \mathscr{Z}} \hookrightarrow \mathscr{M}_D$$

is an embedding and preserves the distances $d_{a.c.s.}$ and d_{mea}.

An other property we have observed in the examples is that if we take the closure of the groups with respect to the distance $d_{a.c.s.} \times d_{mea}$ on $\mathscr{E} \times \mathscr{M}_D$, we still obtain a group. Actually, since the two metrics coincide, if a sequence of symbols in $\mathcal{A}$ converge, then also the respective sequence of matrix-sequences converge, and vice versa. We can denote the closure of $\mathcal{A}$ as

$$\overline{\mathcal{A}} := \{ (\{A_n\}_n, \kappa) \in \mathcal{S}_D \mid \exists \{(\{B_{n,m}\}_{n,m}, \kappa_m)\}_m \subseteq \mathcal{A}, \quad \{B_{n,m}\}_{n,m} \xrightarrow{a.c.s.} \{A_n\}_n, \quad k_m \to k \} .$$

Lemma 5.3.7 ([5]). *Let $\mathcal{A}$ be a s.g. induced by $\mathfrak{s} : \mathscr{A} \to \mathscr{M}_D$. Then*

$$\overline{\mathcal{A}} = \mathcal{A} \iff \overline{\mathscr{A}}^{a.c.s.} = \mathscr{A} \implies \overline{\mathfrak{s}(\mathscr{A})} = \mathfrak{s}(\mathscr{A}).$$

Notice that the last implication is not a double one. For example, we could extract the diagonal sequences in $\mathscr{D}$ and still obtain a group that admits all the measurable functions as symbols, but that is not closed.

If the map $\mathfrak{s} : \mathscr{A} \to \mathscr{M}_D$ induces the group $\mathcal{A}$, its closure is induced by $\overline{\mathfrak{s}} : \overline{\mathscr{A}}^{a.c.s.} \to \mathscr{M}_D$ and the image of $\overline{\mathfrak{s}}$ is the closure of the image of $\mathfrak{s}$ with respect to d_{mea}. In particular, since $\{0_n\}_n \in \mathscr{A}$ from the definition of group, then $\mathscr{Z} \subseteq \overline{\mathscr{A}}^{a.c.s.}$ is the kernel of $\overline{\mathfrak{s}}$. Moreover, if the starting symbols were dense in $\mathscr{M}_D$, then all the functions are symbols for $\overline{\mathcal{A}}$. The same argument can be applied to all s.g., leading us to the following result, that will come in handy later, when we will study how the groups con be embedded in each other.

Theorem 5.3.8 ([5]). *Given a s.g. $\mathcal{A}$, induced by $\mathfrak{s} : \mathscr{A} \to \mathscr{M}_D$, its closure with respect to $d_{a.c.s.} \times d_{mea}$ is still an s.g. that contains $\mathcal{A}$ and $\mathscr{Z} \times \{0\}$ and is induced by $\overline{\mathfrak{s}} : \overline{\mathscr{A}}^{a.c.s.} \to \mathscr{M}_D$ where $\overline{\mathfrak{s}}(\overline{\mathscr{A}}^{a.c.s.}) = \overline{\mathfrak{s}(\mathscr{A})}$.*
Moreover, if $\mathfrak{s}(\mathscr{A})$ is dense in $\mathscr{M}_D$, then $\mathfrak{s}$ induces an isomorphism of groups and isometry $\widetilde{\mathfrak{s}}$ on the quotient space of the closure

$$\widetilde{\mathfrak{s}} : \frac{\overline{\mathscr{A}}^{a.c.s.}}{\mathscr{Z}} \cong \mathscr{M}_D.$$

5.3.1.2 Sequences Algebras

If we know consider algebras instead of groups, we obtain totally analogous results, so we list here a selection of the most important ones.

Definition 5.3.9. *We say that $\mathcal{A}$ is a **sequences algebra** (s.a.) if it is a $\mathbb{C}$-algebra inside $\mathcal{S}_D$.*

One of the most noticeable difference is that the map $\mathfrak{s}$ inducing the algebra is now an homomorphism of $\mathbb{C}$-algebras.

Lemma 5.3.10 ([10]). *Given $\mathcal{A} \subseteq \mathcal{C}_D$, the following are equivalent.*

1. *$\mathcal{A}$ is a $\mathbb{C}$-algebra,*

2. *There exists a $\mathbb{C}$-algebra $\mathscr{A} \subseteq \mathscr{E}$ and an homomorphism of $\mathbb{C}$-algebras $\mathfrak{s} : \mathscr{A} \to \mathscr{M}_D$ such that*

$$(\{A_n\}_n, \kappa) \in \mathcal{A} \iff \{A_n\}_n \in \mathscr{A}, \quad \mathfrak{s}(\{A_n\}_n) = \kappa.$$

Since algebras are in particular groups, we find that Lemma 5.3.5, Lemma 5.3.6 and Lemma 5.3.7 still hold for s.a. The last result changes slightly.

Theorem 5.3.11 ([5])**.** *Given a s.a. $\mathcal{A}$, induced by $\mathfrak{s} : \mathscr{A} \to \mathscr{M}_D$, its closure with respect to $d_{a.c.s.} \times d_{mea}$ is still a s.a. that contains $\mathcal{A}$ and $\mathscr{Z} \times \{0\}$ and is induced by $\overline{\mathfrak{s}} : \overline{\mathscr{A}}^{a.c.s.} \to \mathscr{M}_D$ where $\overline{\mathfrak{s}}(\overline{\mathscr{A}}^{a.c.s.}) = \overline{\mathfrak{s}(\mathscr{A})}$. Moreover, if $\mathfrak{s}(\mathscr{A})$ is dense in $\mathscr{M}_D$, then $\mathfrak{s}$ induces an isomorphism of $\mathbb{C}$-algebras and isometry $\widetilde{\mathfrak{s}}$ on the quotient space of the closure*

$$\widetilde{\mathfrak{s}} : \frac{\overline{\mathscr{A}}^{a.c.s.}}{\mathscr{Z}} \cong \mathscr{M}_D.$$

In the next paragraph, we add an order relation to the set of s.a. contained in $\mathcal{S}$.

5.3.2 Order Relation

Call

$$\mathfrak{A}_D := \{ \mathcal{A} \text{ s.a.} \mid \mathcal{A} \subseteq \mathcal{S}_D \}, \qquad \mathfrak{A} := \sqcup_D \mathfrak{A}_D$$

the set of all s.a. respectively in $\mathcal{S}_D$ and in $\mathcal{S}$, and notice that they have a natural partial ordering given by the inclusion. We have already shown examples of this ordering in the previous paragraphs.

- The algebra of circulant sequences $\mathscr{C}$ embeds into the algebra $\mathscr{D}_F$, and the symbols coincide, so it is an inclusion of s.a.
- Every s.a. $\mathcal{A}$ is included into its closure $\overline{\mathcal{A}}$.

Recall that $\mathscr{D}$ is closed, and is symbol map $\mathfrak{s} : \mathscr{D} \to \mathscr{M}_{[0,1]}$ induces a s.a. $\mathcal{D}$, and the same argument can be used to see that also $\mathscr{D}_U$ induces a s.a. $\mathcal{D}_U$ through a map $\mathfrak{s}_U$. The two spaces $\mathcal{D}$ and $\mathcal{D}_U$ enjoy the same set of symbols $\mathscr{M}_{[0,1]}$ and it is easy to prove that

$$\varphi : \mathcal{D} \to \mathcal{D}_U : (\{A_n\}_n, \kappa) \to (\{U_n A_n U_n^H, \kappa\})$$

is an isomorphism of $\mathbb{C}$-algebras and an isometry. Even though $\mathcal{D}$ and $\mathcal{D}_U$ are isomorphic and have the same set of symbols, they are not related through the natural order relation. We thus propose a new concept of embedding for s.a., that identifies $\mathcal{D}$ and $\mathcal{D}_U$.

Definition 5.3.12. *Given $\mathcal{A}, \mathcal{B}$ in $\mathfrak{A}$ induced respectively by $\mathfrak{s}_A : \mathscr{A} \to \mathscr{M}_D$ and $\mathfrak{s}_B : \mathscr{B} \to \mathscr{M}_{D'}$, we say that $\mathcal{A}$ **embeds** into $\mathcal{B}$ if $\mathfrak{s}_A(\mathscr{A}) \subseteq \mathfrak{s}_B(\mathscr{B})$ and there exists a sequence of unitary matrices $\{U_n\}_n$ such that the map*

$$\varphi : \mathcal{A} \hookrightarrow \mathcal{B} : (\{A_n\}_n, \kappa) \to (\{U_n A_n U_n^H, \kappa\})$$

is well-defined. In this case, we write

$$\mathcal{A} \leq \mathcal{B}.$$

*In the case $\mathcal{A} \leq \mathcal{B}$ and $\mathcal{B} \leq \mathcal{A}$, we say that the two s.a. are **equivalent**.*

The order relation $\mathcal{A} \subseteq \mathcal{B}$ is a particular case of $\mathcal{A} \leq \mathcal{B}$ when $\{U_n\}_n = \{I_n\}_n$. Moreover, if $\mathcal{A} \leq \mathcal{B}$ with inclusion map φ, then ([10])

- there exists an unique D such that $\mathcal{A}, \mathcal{B} \in \mathfrak{A}_D$,
- φ is an embedding of $\mathbb{C}$-algebras and preserves the metrics,
- φ brings zero-distributed sequences into zero-distributed sequences,
- the image of φ is still a s.a.

- φ brings a.c.s. equivalent sequences into a.c.s. equivalent sequences.

Similarly as what we have done with $\mathcal{D}$ and $\mathcal{D}_U$, given any s.a. $\mathcal{A}$ and a sequence of unitary matrices $\{U_n\}_n$, we can generate an other s.a. $\mathcal{A}_U$ through the isomorphism

$$\varphi : \mathcal{A} \cong \mathcal{A}_U : (\{A_n\}_n, \kappa) \rightleftarrows (\{U_n A_n U_n^H\}_n, \kappa).$$

Moreover is easy to see that

- $\mathcal{A} \subseteq \mathcal{B} \iff \mathcal{A}_U \subseteq \mathcal{B}_U$,
- $(\mathcal{A}_U)_V = \mathcal{A}_{VU}$.

This is enough to prove the following result.

Lemma 5.3.13.

$$\mathcal{A} \leq \mathcal{B} \iff \exists \{U_n\}_n : \mathcal{A}_U \subseteq \mathcal{B}, \quad \mathcal{A} \subseteq \mathcal{B}_{U^H}.$$

It is evident that φ and φ^{-1} induce the relations $\mathcal{A} \leq \mathcal{A}_U$ and $\mathcal{A}_U \leq \mathcal{A}$, so the two spaces are always equivalent, but in general are very different. In fact, $\leq$ is not an order, but only a preorder, and can identify s.a. that have no containment relations between them, but with the same set of symbols.

Lemma 5.3.14. *The relation $\leq$ is a preorder. If $\mathcal{A} \leq \mathcal{B}$ and $\mathcal{B} \leq \mathcal{A}$, then $\mathfrak{s}_A(\mathscr{A}) = \mathfrak{s}_B(\mathscr{B})$.*

Proof. Since $\mathcal{A} \subseteq \mathcal{A}$, then $\mathcal{A} \leq \mathcal{A}$. If we have $\mathcal{A} \leq \mathcal{B} \leq \mathcal{C}$, from Lemma 5.3.13, there exist $\{U_n\}_n$ and $\{V_n\}_n$ such that

$$\mathcal{A}_U \subseteq \mathcal{B}, \quad \mathcal{B}_V \subseteq \mathcal{C} \implies \mathcal{A}_{VU} \subseteq \mathcal{B}_V \subseteq \mathcal{C} \implies \mathcal{A} \leq \mathcal{C}.$$

We have thus proved that $\leq$ is a preorder. Given now $\mathcal{A} \leq \mathcal{B}$ and $\mathcal{B} \leq \mathcal{A}$, we know by definition that $\mathfrak{s}_A(\mathscr{A}) \subseteq \mathfrak{s}_B(\mathscr{B}) \subseteq \mathfrak{s}_A(\mathscr{A})$ thus proving the thesis. ☐

The map φ inducing the relation $\mathcal{A} \leq \mathcal{B}$ maps a.c.s. equivalent sequences into a.c.s. equivalent sequences, so it induces a map between the quotient spaces

$$\widetilde{\varphi} : \frac{\mathcal{A}}{(\mathscr{Z} \times \{0\}) \cap \mathcal{A}} \hookrightarrow \frac{\mathcal{B}}{(\mathscr{Z} \times \{0\}) \cap \mathcal{B}}$$

and thanks to Lemma 5.3.6, we can state the following result.

Theorem 5.3.15 ([10]). *If $\mathcal{A} \leq \mathcal{B}$ with inclusion map φ, and $\mathfrak{s}_A(\mathscr{A}) = \mathfrak{s}_B(\mathscr{B})$, then the quotient map*

$$\widetilde{\varphi} : \frac{\mathcal{A}}{(\mathscr{Z} \times \{0\}) \cap \mathcal{A}} \cong \frac{\mathcal{B}}{(\mathscr{Z} \times \{0\}) \cap \mathcal{B}}$$

is an isomorphism. If moreover $\mathscr{Z} \times \{0\} \subseteq \mathcal{A}$, then $\mathscr{Z} \times \{0\} \subseteq \mathcal{B}$, φ is an isomorphism and as a consequence $\mathcal{B} \leq \mathcal{A}$.

Notice that when we write $\mathcal{A} \leq \mathcal{B}$, then the two s.a. both belong to the same set $\mathfrak{A}_D$. It is actually possible to generalize the order relation to compare s.a. with symbols with different domains. In fact, in Appendix B we showed that two different domains $D \subseteq \mathbb{R}^d$ and $D' \subseteq \mathbb{R}^{d'}$ are isomorphic 'modulo zero', meaning that there exists a bijective measurable and measure preserving function T

$$T : D \setminus A_1 \to D' \setminus A_2$$

such that φ^{-1} is still measurable and measure preserving, where $A_1 \subseteq D$, $\ell_d(A_1) = 0$ and $A_2 \subseteq D'$, $\ell_{d'}(A_2) = 0$. In particular, T induces an isomorphism and isometry of spaces

$$\psi : \mathscr{M}_{D'} \to \mathscr{M}_D : \kappa \mapsto \kappa \circ T.$$

This map is well defined since measurable functions are always defined almost everywhere. Using such measure preserving map, we are able to rearrange the symbols of a s.a. and change their domain. This let us generalize the order relation with the following definition.

Definition 5.3.16. *Given $\mathcal{A},\mathcal{B}$ in $\mathfrak{A}$ induced respectively by $\mathfrak{s}_A : \mathscr{A} \to \mathscr{M}_D$ and $\mathfrak{s}_B : \mathscr{B} \to \mathscr{M}_{D'}$, we say that $\mathcal{A}$ **embeds** into $\mathcal{B}$ if there exists an isomorphism modulo zero T from D' to D and a sequence of unitary matrices $\{U_n\}_n$ such that the map*

$$\varphi : \mathcal{A} \hookrightarrow \mathcal{B} : (\{A_n\}_n, \kappa) \to (\{U_n A_n U_n^H, \kappa \circ T\})$$

is well-defined. In this case, we write

$$\mathcal{A} \preceq \mathcal{B}.$$

*In the case $\mathcal{A} \le \mathcal{B}$ and $\mathcal{B} \le \mathcal{A}$, we say that the two s.a. are **equivalent**.*

When $\mathcal{A} \subseteq \mathcal{B}$ or $\mathcal{A} \le \mathcal{B}$, then $D = D'$ and $\mathcal{A} \preceq \mathcal{B}$ with T being the identity function. This is again a preorder on $\mathfrak{A}$, and if we call $\mathcal{A}_{U,T}$ the set induced by the isomorphism

$$\varphi : \mathcal{A} \cong \mathcal{A}_{U,T} : (\{A_n\}_n, \kappa) \rightleftarrows (\{U_n A_n U_n^H\}_n, \kappa \circ T).$$

it is possible to prove that

- $\mathcal{A} \subseteq \mathcal{B} \iff \mathcal{A}_{U,T} \subseteq \mathcal{B}_{U,T}$,
- $(\mathcal{A}_{U,T})_{V,S} = \mathcal{A}_{VU,T\circ S}$,
- $\mathcal{A} \le \mathcal{B} \iff \exists \{U_n\}_n, T \,:\, \mathcal{A}_{U,T} \subseteq \mathcal{B}, \quad \mathcal{A} \subseteq \mathcal{B}_{U^H,T^{-1}}$.

Most of the other results also generalize to $\preceq$ with similar arguments.

In the next paragraph we want to study the maximal elements for these order relations.

5.3.3 Maximality

First of all, notice that for the inclusion partial order, we always are sure that every s.a. $\mathcal{A}$ is contained in a maximal element in $(\mathfrak{A}, \subseteq)$ thanks to the Zorn's Lemma. One can easily prove that the maximal elements of $(\mathfrak{A}, \subseteq)$ coincide with the maximal elements for $(\mathfrak{A}, \le)$, so the same result holds also for the partial order $\le$.

Lemma 5.3.17. *The maximal elements for $(\mathfrak{A}, \subseteq)$ coincide with the maximal elements for $(\mathfrak{A}, \le)$. In particular, each s.a. $\mathcal{A}$ is contained in a maximal element for $(\mathfrak{A}, \le)$.*

Proof. Given $\mathcal{A} \in \mathfrak{A}$, consider the set

$$\mathfrak{B} := \{\, \mathcal{B} \in \mathfrak{A} \mid \mathcal{A} \subseteq \mathcal{B} \,\}.$$

$\mathfrak{B}$ is non-empty since it contains $\mathcal{A}$, and every chain in $\mathfrak{B}$ has an upper bound in $\mathfrak{B}$ given by the union of the elements in the chain. Thanks to Zorn's Lemma, one can conclude that $\mathfrak{B}$ admits at least one maximal element $\mathcal{M}$ in $(\mathfrak{B}, \subseteq)$ and it contains $\mathcal{A}$. It is also a maximal element for $(\mathfrak{A}, \subseteq)$ since

$$\mathcal{M} \subseteq \mathcal{C} \implies \mathcal{A} \subseteq \mathcal{C} \implies \mathcal{C} \in \mathfrak{B} \implies \mathcal{C} \subseteq \mathcal{M} \implies \mathcal{C} = \mathcal{M}.$$

Suppose now that $\mathcal{M}$ is a maximal element for $(\mathfrak{A}, \subseteq)$. If $\mathcal{M} \le \mathcal{C}$, then for Lemma 5.3.13 there exists an unitary sequence $\{U_n\}_n$ such that

$$\mathcal{M} \subseteq \mathcal{C}_{U^H},$$

but $\mathcal{M}$ is maximal for the inclusion, so $\mathcal{M} = \mathcal{C}_{U^H}$. In particular, again for Lemma 5.3.13,

$$\mathcal{C}_{U^H} \subseteq \mathcal{M} \implies \mathcal{C} \le \mathcal{M}$$

and as a consequence $\mathcal{M}$ and $\mathcal{C}$ are equivalent for $\le$. This means that $\mathcal{M}$ is a maximal element also for $(\mathfrak{A}, \le)$.

Suppose now that $\mathcal{M}$ is a maximal element for $(\mathfrak{A}, \le)$. We know there exists $\mathcal{N}$ maximal element for $(\mathfrak{A}, \subseteq)$, that contains $\mathcal{M}$, but from the maximality of both s.a. we obtain

$$\mathcal{M} \subseteq \mathcal{N} \implies \mathcal{M} \le \mathcal{N} \implies \mathcal{N} \le \mathcal{M} \implies \mathcal{N} \subseteq \mathcal{M}_{U^H} \implies \mathcal{N} = \mathcal{M}_{U^H} \implies \mathcal{N}_U = \mathcal{M}.$$

If now $\mathcal{M} \subseteq \mathcal{A}$, then

$$\mathcal{N}_U \subseteq \mathcal{A} \implies \mathcal{N} \subseteq \mathcal{A}_{U^H} \implies \mathcal{N} = \mathcal{A}_{U^H} \implies \mathcal{M} = \mathcal{N}_U = \mathcal{A},$$

and this is enough to prove that $\mathcal{M}$ is a maximal element for $(\mathfrak{A}, \subseteq)$. $\square$

We already noticed that every s.a. is included in its closure, so a maximal element for $(\mathfrak{A}, \subseteq)$ or for $(\mathfrak{A}, \leq)$ is always closed.
The last result shows that the maximal elements in $(\mathfrak{A}, \subseteq)$ are also maximal in $(\mathfrak{A}, \leq)$ and vice versa, but the preorder $\leq$ identifies different maximal elements. It is possible to show that two maximal elements identified are isomorphic through an unitary sequence $\{U_n\}_n$.

Lemma 5.3.18. *Every maximal element in $(\mathfrak{A}, \leq)$ is closed. Moreover, if two maximal elements $\mathcal{A}$ and $\mathcal{B}$ for $(\mathfrak{A}, \leq)$ are equivalent, then they are isomorphic, and there exists an unitary sequence $\{U_n\}_n$ such that $\mathcal{A} = \mathcal{B}_U$.*

Proof. Consider $\mathcal{A}$ and $\mathcal{B}$ maximal elements and suppose they are equivalent. From $\mathcal{A} \leq \mathcal{B}$ and the maximality of the s.a. we infer

$$\mathcal{A} \subseteq \mathcal{B}_U \implies \mathcal{A} = \mathcal{B}_U \cong \mathcal{B}.$$

□

It is quite a feat to find a maximal element, and also to prove that a specific s.a. is maximal. The last result says that being closed is a necessary condition, but it is not sufficient. For example, the set $\mathscr{Z} \times \{0\}$ is a closed s.a., but it is contained in any other closed s.a., so it is very far from being maximal. A sufficient condition is given by the following result.

Theorem 5.3.19 ([5])**.** *If $\mathcal{A}$ is a closed s.a. induced by $\mathfrak{s}_A : \mathscr{A} \to \mathscr{M}_D$, and $\mathfrak{s}_A$ is surjective, then $\mathcal{A}$ is maximal in $(\mathfrak{A}, \subseteq)$.*

In a sense, if a closed s.a. admits all measurable functions as symbols, then we cannot add other sequences, since its symbol is already inside. For example, both $\mathcal{D}$ and $\mathcal{D}_U$ are e closed s.a. with set of symbol $\mathscr{M}_{[0,1]}$, so they are maximal elements in $(\mathfrak{A}, \subseteq)$, and they are also equivalent in $(\mathfrak{A}, \leq)$.

An open question is whether the inverse of Theorem 5.3.19 holds or not. In other words, is it true that all the maximal elements $\mathcal{A}$ are induced by a surjective map $\mathfrak{s}_A$? In particular it would mean that every s.a. $\mathcal{A}$ could be expanded by adding sequences and symbols until we obtain all measurable functions as symbols.

Here, though, we want to formulate an even stronger conjecture. Is it true that all the maximal elements inside $\mathfrak{A}_D$ are equivalent?

Conjecture 5.3.20. *For every D, $(\mathfrak{A}_D, \leq)$ admits a maximum element.*

When we turn to examine $(\mathfrak{A}, \preceq)$, we find that similar results hold. Since the proofs follow the same arguments, we do not report them.

Lemma 5.3.21. *The maximal elements for $(\mathfrak{A}, \subseteq)$ and for $(\mathfrak{A}, \preceq)$ coincide. In particular, each s.a. $\mathcal{A}$ is contained in a maximal element for $(\mathfrak{A}, \preceq)$.*

Lemma 5.3.22. *Every maximal element in $(\mathfrak{A}, \preceq)$ is closed. Moreover, if two maximal elements $\mathcal{A}$ and $\mathcal{B}$ for $(\mathfrak{A}, \preceq)$ are equivalent, then $\mathcal{A} = \mathcal{B}_{U,T}$.*

In what follows, we will find also examples of s.a. equivalent for $\preceq$ but that are not on the same domain. In particular, every s.a. $\mathcal{A}$ we consider is always embeddable into $\mathcal{D}$ defined at the end of section 5.1. We do not know whether all the s.a. embed into $\mathcal{D}$ or not, but it is a consequence of Conjecture 5.3.20. The only inclusion result we are able to prove is the following.

Theorem 5.3.23 ([10])**.** *Suppose $\mathcal{A}$ is a s.a. with the following properties.*

- *$\mathcal{A}$ is a closed s.a. containing the element $(\{I_n\}_n, 1)$.*

- *$\mathcal{A}$ is induced by a surjective map $\mathfrak{s}_A : \mathscr{A} \to \mathscr{M}_D$.*
- *There exists an unitary sequence $\{U_n\}_n$ such that for every $\kappa \in \mathscr{M}_D$ there exists an element $(\{N_n\}_n, \kappa) \in \mathcal{A}$ where $\{N_n\}_n = \{U_n D_{N,n} U_n^H\}_n$ with $D_{N,n}$ diagonal matrices.*

If $T : [0,1] \to D$ is an isomorphism of measurable spaces, then there exists an unitary sequence $\{V_n\}_n$ such that $\mathcal{A}_{V,T} \equiv \mathcal{D}$. In particular, $\mathcal{A}$ and $\mathcal{D}$ are isomorphic.

Actually, if Conjecture 5.2.4 were true, then we could drop the third hypothesis, leaving that every closed s.a. with identity and all the measurable functions as symbols is isomorphic to $\mathcal{D}$.

Chapter 6

GLT World

This entire section is dedicated to build and analyse the generalized locally Toeplitz (GLT) sequences algebra, a maximal s.a. that has been used to determine the symbols for sequences of linear system coming from the discretization of integro-differential linear equations. Similar structures carrying the GLT name arising from practical purposes will also be shortly presented, but for a detailed presentation, we suggest to consult the books [52, 53, 13, 14].

6.1 Generalized Locally Toeplitz Sequences

In the present section, we deal with sequences in $\mathscr{E}$ with size $s_n = n$. All the results naturally generalize to a diverging sequence s_n, but for clarity of exposition, we keep the convention that the matrices A_n are of size $n \times n$.

6.1.1 Locally Circulant and Locally Toeplitz Sequences

6.1.1.1 Locally Circulant Sequences

In paragraph 5.2.1, we showed how to build the algebra $\mathbb{C}$ of circulant matrices. In particular, recall that all the sequences $\{p(C_n)\}_n$, where $p(x) \in \mathbb{C}[x, \overline{x}]$, can be diagonalized through the sequence of Fourier matrices $\{F_n\}_n$, and

$$\{p(C_n)\}_n \sim_\lambda p(e^{2\pi x \mathrm{i}}).$$

The symbol is considered over $x \in [0,1]$, but a simple transformation of domain $T : [0,1] \to [-\pi, \pi]$ shows that we can consider the symbol $e^{\mathrm{i}\theta}$ instead of $e^{2\pi x\mathrm{i}}$, where $\theta \in [-\pi, \pi]$. Starting from this setting, we want to build the s.a. of locally circulant (LC) sequences.

Definition 6.1.1. *Given any $a \in C[0,1]$ and $f(\theta) = p(e^{2\pi x\mathrm{i}})$ for $p \in \mathbb{C}[x, \overline{x}]$, let*

$$\begin{aligned}
LC_n^m(a,f) &= [D_m(a) \otimes p(C_{\lfloor n/m \rfloor})] \oplus 0_{n \ mod \ m} \\
&= \mathrm{diag}_{i=1,\dots,m}\left[a\left(\frac{i}{m}\right) p(C_{\lfloor n/m \rfloor})\right] \oplus 0_{n \ mod \ m} \\
&= \begin{pmatrix}
\boxed{a(1/m)p(C_{\lfloor n/m \rfloor})} & & & & \\
& \boxed{a(2/m)p(C_{\lfloor n/m \rfloor})} & & & \\
& & \ddots & & \\
& & & \boxed{a(1)p(C_{\lfloor n/m \rfloor})} & \\
& & & & \boxed{0_{n \ mod \ m}}
\end{pmatrix}.
\end{aligned}$$

*We define the **locally circulant** sequence referred to a, f as*

$$LC_n(a,f) := LC_n^{\lfloor \sqrt{n} \rfloor}(a,f).$$

From the definition, it is easy to see that the operator LC_n is bilinear for every n in a, f, and

$$LC_n(a_1a_2, f_1f_2) = LC_n(a_1, f_1)LC_n(a_2, f_2).$$

This is enough to conclude that the sequences $\{LC_n(a, f)\}_n$ form a $\mathbb{C}$-algebra inside $\mathscr{E}$. Moreover, the LC operator produces normal matrices, that are also easily diagonalizable. In fact,

$$LC_n(a, f) = Q_nD_n(a, f)Q_n^H,$$

where $m = \lfloor\sqrt{n}\rfloor$ and

$$Q_n = \begin{pmatrix} \boxed{F_{\lfloor n/m\rfloor}} & & & & \\ & \boxed{F_{\lfloor n/m\rfloor}} & & & \\ & & \ddots & & \\ & & & \boxed{F_{\lfloor n/m\rfloor}} & \\ & & & & \boxed{I_{n \bmod m}} \end{pmatrix}.$$

The matrices Q_n are unitary and do not depend on a or f. Moreover, the entries of the diagonal matrices $D_n(a, f)$, excluding the last zero block, are a uniform sampling of the function $a(x)p(e^{\mathrm{i}\theta})$ over $D = [0, 1]\times[-\pi, \pi]$. We can thus conclude that the LC sequences induce a s.a. when associated to the respective symbols $a(x)f(\theta)$.

Definition 6.1.2.

$$\mathcal{LC} := \{\, (\{LC_n(a, f)\}_n, a(x)f(\theta)) \mid a \in C[0, 1], \quad f(\theta) = p(e^{2\pi x\mathrm{i}}), \quad p \in \mathbb{C}[x, \overline{x}] \,\}\,.$$

6.1.1.2 Locally Toeplitz Sequences

Like we have already showed in paragraph 5.2.3, in applications it may happen to find Toeplitz matrices instead of circulant ones. When dealing with sequences of uniformly banded Toeplitz matrices, though, we can show that they are a.c.s. equivalent to circulant sequences. In particular, we can generate a space of locally Toeplitz (LT) sequences following the same steps for the LC sequences, and showing that they are also a.c.s. equivalent. Let us start by defining what a Toeplitz sequence relative to a function $f(\theta)$ is.

Definition 6.1.3. *Given a function $f \in L^1[-\pi, \pi]$, its* ***associated Toeplitz sequence*** *is $\{T_n(f)\}_n$, where*

$$T_n(f) = [f_{i-j}]_{i,j=1}^n, \qquad f_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-ik\theta}\mathrm{d}\theta.$$

From this, we can build the space of LT sequences.

Definition 6.1.4. *Given any $a \in C[0, 1]$ and $f \in L^1[-\pi, \pi]$, let*

$$\begin{aligned} LT_n^m(a, f) &= [D_m(a) \otimes T_{\lfloor n/m\rfloor}(f)] \oplus 0_{n \bmod m} \\ &= \mathrm{diag}_{i=1,\dots,m}\left[a\left(\frac{i}{m}\right)T_{\lfloor n/m\rfloor}(f)\right] \oplus 0_{n \bmod m} \\ &= \begin{pmatrix} \boxed{a(1/m)T_{\lfloor n/m\rfloor}(f)} & & & & \\ & \boxed{a(2/m)T_{\lfloor n/m\rfloor}(f)} & & & \\ & & \ddots & & \\ & & & \boxed{a(1)T_{\lfloor n/m\rfloor}(f)} & \\ & & & & \boxed{0_{n \bmod m}} \end{pmatrix}. \end{aligned}$$

We define the ***locally Toeplitz*** *sequence referred to a, f as*

$$LT_n(a, f) := LT_n^{\lfloor\sqrt{n}\rfloor}(a, f).$$

When $f(\theta)$ is a trigonometric polynomial $p(e^{\mathrm{i}\theta})$, there is only a finite number of non-zero Fourier coefficients f_i. In particular,

$$p(x) = \sum_{i=-s}^{s} a_i x^i, \quad f(\theta) = p(e^{\mathrm{i}\theta}) \implies f_i = a_i \quad \forall\, i \in \mathbb{Z}.$$

It is fairly easy to see that $\mathrm{rk}(T_n(f) - p(C_n)) \leq 2s^2 = o(n)$, so the difference $\{T_n(f) - C_n(f)\}_n$ is a zero-distributed sequence and $\{T_n(f)\}_n \sim_{a.c.s.} \{C_n(f)\}_n$. As a consequence, even LT and LC sequences are a.c.s. equivalent, since

$$\mathrm{rk}(C_{\lfloor n/m \rfloor}(f) - T_{\lfloor n/m \rfloor}(f)) \leq 2s^2 \implies \mathrm{rk}(LC_n(a,f) - LT_n(a,f)) \leq 2s^2 \lfloor \sqrt{n} \rfloor = o(n).$$

We can thus associate to each LT sequence $\{LT_n(a,f)\}_n$ the symbol $a(x)f(\theta)$, and induce the following set of sequences-symbol.

Definition 6.1.5.

$$\mathcal{LT} := \{\, (\{LT_n(a,f)\}_n, a(x)f(\theta)) \mid a \in C[0,1], \quad f(\theta) = p(e^{2\pi x \mathrm{i}}), \quad p \in \mathbb{C}[x, \overline{x}] \,\}\,.$$

Notice that $\mathcal{LT}$ is not an s.a., since product of Toeplitz matrices is not in general a Toeplitz matrix. Nonetheless, the couples $(\{LT_n(a,f)\}_n, a(x)f(\theta))$ belong to the closure of $\mathcal{LC}$, and for every LC sequence, there exists the corresponding a.c.s. equivalent LT sequence. As a consequence, the two closures coincide, and Theorem 5.3.11 tells us that the closure is still an s.a.

Theorem 6.1.6. *The two sets $\overline{\mathcal{LT}}$ and $\overline{\mathcal{LC}}$ coincide, and the resulting set is a s.a.*

6.1.1.3 LT or LC?

We have thus two equivalent ways to build the same space, but which is the more convenient? Notice that by using the LC sequences, we have the following properties.

- The LC sequences are explicitly built using normal matrices whose eigenvalues, when diagonalized with a known unitary base change, are sorted according to the sampling of the symbols over a regular grid. Moreover, the base change is the same for every sequence.
- Every sequence in $\overline{\mathcal{LC}}$ can be expressed by normal matrices, with the same diagonalizing base change, that sorts the eigenvalues according to an approximated sampling of the associated symbol.
- The diagonalization lets us draw a connection between LC sequences and more simple diagonal sequences with one dimensional symbols.

The downsides of the LC sequences are described in Remark 0.1 of [76], where it is discussed the reasons why to use LT sequences instead.

- For every $f \in L^1$, we can prove that $(\{T_n(f)\}_n, f(\theta))$ is an element of $\overline{\mathcal{LT}}$ ([58]).
- It is easy to describe a Toeplitz sequence $\{T_n(f)\}_n$ associated to $f(\theta) \in L^1[-\pi, \pi]$ through its Fourier coefficients, whereas a circulant sequence $\{C_n(f)\}_n$ associated to f can only be generated through a.c.s. convergence of banded circulant sequences, or through the Frobenius optimal circulant approximation, that requires a more involved analysis [31, 72].
- Toeplitz and Toeplitz-like sequences appear often in discretization of linear PDEs.

In other words, the circulant approach helps us on a theoretical point of view, whereas the Toeplitz approach is more fitted for applications. For these and other reason, we call the generated s.a. the algebra of generalized locally Toeplitz (GLT) sequences.

Definition 6.1.7. *The space of **generalized locally Toeplitz** (GLT) sequences is*

$$\mathcal{G} := \overline{\mathcal{LT}} = \overline{\mathcal{LC}}$$

and we indicate its elements as

$$(\{A_n\}_n, \kappa(x,\theta)) \in \mathcal{G} \iff \{A_n\}_n \sim_{GLT} \kappa(x,\theta).$$

In the next paragraph, we study the fundamental properties of this s.a.

6.1.2 Properties of GLT Sequences

Let us start by noticing that

- the diagonal sampling sequence $\{D_n(a)\}_n$ defined in paragraph 5.1.1 is a.c.s. equivalent to $\{LT_n(a,1)\}_n$ for every $a \in C^\infty[0,1]$,
- the sequences $\{T_n(f)\}_n$ and $\{LT_n(1,f)\}_n$ are a.c.s. equivalent for $f(\theta) = e^{k\mathrm{i}\theta}$ and $k \in \mathbb{Z}$.

As a remark, we can notice that $\{T_n(f)\}_n \sim_{a.c.s.} \{LT_n(1,f)\}_n$ for $f(\theta) = e^{k\mathrm{i}\theta}$, because all the matrices $T_n(f)$ are k-semiseparable, where k does not depend from n, and $LT_n(1,f)$ deletes $o(n)$ submatrices out of the lower (or upper) triangular part of $T_n(f)$. As a consequence, the difference between the two matrices has rank $o(n)$, and so it is zero-distributed. This means that the same argument can be repeated with algebras of semiseparable sequences substituting Toeplitz and circulant matrices, and all the results still hold.

Since $C^\infty[0,1]$ is dense in $C[0,1]$, and we can generate any trigonometrical polynomial with the basis functions $e^{k\mathrm{i}\theta}$, the diagonal sampling sequences $\{D_n(a)\}_n$ with $a \in C^\infty[0,1]$ and the Toeplitz sequences $\{T_n(e^{k\mathrm{i}\theta})\}_n$ with $k \in \mathbb{Z}$ are sufficient to generate the GLT space. Notice that all the symbols $a(x)$ and $e^{k\mathrm{i}\theta}$ have to be considered as functions on $[0,1] \times [-\pi,\pi]$.

Theorem 6.1.8 ([52])**.** *Consider the set*

$$\mathcal{F} := \{\, (\{D_n(a)\}_n, a) \mid a \in C[0,1] \,\} \cup \{\, (\{T_n(e^{k\mathrm{i}\theta})\}_n, e^{k\mathrm{i}\theta}) \mid k \in \mathbb{Z} \,\}.$$

The algebra generated by $\mathcal{F}$ is a s.a. and its closure coincides with $\mathcal{G}$.

In particular, we find that if $a \in C^\infty[0,1]$ and $f(\theta) = e^{k\mathrm{i}\theta}$, then

$$\{D_n(a)\}_n\{T_n(f)\}_n \sim_{GLT} a(x)f(\theta).$$

Moreover, the set

$$\{\, a(x)f(\theta) \mid a \in C[0,1], \quad f(\theta) = p(e^{2\pi x\mathrm{i}}), \quad p \in \mathbb{C}[x,\overline{x}] \,\}$$

is dense in $\mathscr{M}_{[0,1]\times[-\pi,\pi]}$, so we can use Lemma 5.3.7, Theorem 5.3.11 and Theorem 5.3.19 to conclude the following result.

Theorem 6.1.9 ([5])**.** *$\mathcal{G}$ is a closed and maximal s.a. admitting all functions in $\mathscr{M}_{[0,1]\times[-\pi,\pi]}$ as symbols. Moreover, if we quotient the space by the zero-distributed s.a. $\mathscr{Z} \times \{0\}$, it becomes isomorphic and isometrically equivalent to $\mathscr{M}_{[0,1]\times[-\pi,\pi]}$.*

As we said in the last paragraph, the connection between GLT and measurable function is quite strong, since for every sequence $\{A_n\}_n \in \mathcal{G}$ we can find an a.c.s. equivalent normal sequence whose eigenvalues converge to the symbol κ. As ulterior proof of the connections between the two spaces, we give in Appendix D a way to produce some "derivation" operator T_x and T_θ on the GLT sequences, so that for several sequences $\{A_n\}_n \sim_{GLT} k$ we have

$$T_x(\{A_n\}_n) \sim_{GLT} {}^{\partial k}\!/_{\partial x}, \qquad T_\theta(\{A_n\}_n) \sim_{GLT} {}^{\partial k}\!/_{\partial \theta}.$$

The link highlighted by Theorem 6.1.9 (and more in general Theorem 5.3.8) is so deep that they may lead to a "bridge", in the precise mathematical sense established in [30], between measure theory and the asymptotic linear algebra theory underlying the notion of a.c.s.; a bridge that could be exploited to obtain matrix theory results from measure theory results and vice versa.

Turning to study the maximality of this s.a., we notice that Theorem 5.3.23 applies to $\mathcal{G}$, since it is a closed s.a., it contains $(\{D_n(1)\}_n, 1) = (\{I_n\}_n, 1)$, admits all functions $\mathscr{M}_{[0,1]\times[-\pi,\pi]}$ as symbols, and the construction through the LC sequences assures us that there exists an unitary sequence $\{V_n\}_n$ such that any GLT sequence is a.c.s. equivalent to a normal sequence diagonalized by $\{V_n\}_n$. Consequentially, $\mathcal{G}$ and $\mathcal{D}$ are isomorphic.

Theorem 6.1.10 ([10]). *If $T : [0,1] \to [0,1] \times [-\pi, \pi]$ is an isomorphism of measurable spaces, then there exists an unitary sequence $\{U_n\}_n$ such that the map*

$$\phi : \mathcal{G} \to \mathcal{D}, \qquad \varphi(\{A_n\}_n, \kappa) = (\{U_n A_n U_n^H\}_n, \kappa(T)).$$

is an isomorphism of $\mathbb{C}$-algebras and an isometry.

Notice that if Conjecture 5.3.20 were true, then any s.a. would embed into $\mathcal{G}$, thus transforming it into a maximum element for $\mathfrak{A}_{[0,1]\times[-\pi,\pi]}$ and $\mathfrak{A}$ in general. When we consider the simple inclusion order, though, it is easy to find algebras not contained in $\mathcal{G}$, even simple ones built only with diagonal sequences.

Example 6.1.11

- Let $\{A_n\}_n = \{D_n((-1)^i)_i\}_n$. Notice that $\{A_n\}_n$ is diagonal, and thus normal, and admits $\chi_{[0,1/2]} - \chi_{[1/2,1]}$ as spectral symbol. If $\{A_n\}_n \sim_{GLT} \kappa$, and $B_n = A_n T_n(e^{i\theta}) - T_n(e^{i\theta}) A_n$, then

$$\{B_n\}_n \sim_{GLT} 0 \implies \{B_n B_n^T\} \sim_{GLT} 0$$

but

$$B_n = A_n T_n(e^{i\theta}) - T_n(e^{i\theta}) A_n = \begin{pmatrix} 0 & 1 & 0 & & & \\ 1 & 0 & -1 & & & \\ & -1 & 0 & 1 & & \\ & & 1 & \ddots & \ddots & \\ & & & \ddots & \ddots & \pm 1 \\ & & & & \pm 1 & 0 \end{pmatrix}$$

and

$$B_n^T B_n = \begin{pmatrix} 1 & 0 & -1 & & & & \\ 0 & 2 & 0 & -1 & & & \\ -1 & 0 & 2 & 0 & -1 & & \\ & -1 & 0 & \ddots & \ddots & \ddots & \\ & & -1 & \ddots & 2 & 0 & -1 \\ & & & \ddots & 0 & 2 & 0 \\ & & & & -1 & 0 & 1 \end{pmatrix} \sim_{a.c.s.} T_n(2 - 2cos(2\theta)) \sim_{GLT} 2 - 2cos(2\theta) \neq 0.$$

So $\{A_n\}_n$ is not a GLT sequence, but it is normal and admits a spectral symbol, so it generates an algebra that is not contained in $\mathcal{G}$.

6.1.3 Axioms of GLT Sequences

In [52] we can find all the main properties of the GLT space summarized in 9 points. Here we report them, and discuss or generalize some of them.

GLT 1. If $\{A_n\}_n \sim_{\mathrm{GLT}} \kappa$ then $\{A_n\}_n \sim_\sigma \kappa$. If $\{A_n\}_n \sim_{\mathrm{GLT}} \kappa$ and each A_n is Hermitian then $\{A_n\}_n \sim_\lambda \kappa$.

GLT 2. If $\{A_n\}_n \sim_{\mathrm{GLT}} \kappa$ and $A_n = X_n + Y_n$, where

- every X_n is Hermitian,
- $n^{-1/2}\|Y_n\|_2 \to 0$,

then $\{A_n\}_n \sim_\lambda \kappa$.

GLT 3. Here we list three fundamental examples of GLT sequences.

- Given a function f in $L^1([-\pi,\pi])$, its associated Toeplitz sequence is $\{T_n(f)\}_n$, where

$$T_n(f) = [f_{i-j}]_{i,j=1}^n, \qquad f_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-\mathrm{i}k\theta}\mathrm{d}\theta.$$

  $\{T_n(f)\}_n$ is a GLT sequence with symbol $\kappa(x,\theta) = f(\theta)$.
- Given any a.e. continuous function $a : [0,1] \to \mathbb{C}$, its associated diagonal sampling sequence is $\{D_n(a)\}_n$, where

$$D_n(a) = \operatorname*{diag}_{i=1,\dots,n} a\Big(\frac{i}{n}\Big).$$

  $\{D_n(a)\}_n$ is a GLT sequence with symbol $\kappa(x,\theta) = a(x)$.
- A zero-distributed sequence is a matrix-sequence such that $\{Z_n\}_n \sim_\sigma 0$, i.e.,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} G(\sigma_i(A_n)) = G(0)$$

  for every continuous function $G:\mathbb{R}\to\mathbb{C}$ with compact support. Any zero-distributed sequence is a GLT sequence with symbol $\kappa(x,\theta) = 0$.

GLT 4. If $\{A_n\}_n \sim_{\mathrm{GLT}} \kappa$ and $\{B_n\}_n \sim_{\mathrm{GLT}} \xi$, then

- $\{A_n^H\}_n \sim_{\mathrm{GLT}} \overline{\kappa}$, where A_n^H is the conjugate transpose of A_n,
- $\{\alpha A_n + \beta B_n\}_n \sim_{\mathrm{GLT}} \alpha\kappa + \beta\xi$ for all $\alpha,\beta \in \mathbb{C}$,
- $\{A_n B_n\}_n \sim_{\mathrm{GLT}} \kappa\xi$.

GLT 5. If $\{A_n\}_n \sim_{\mathrm{GLT}} \kappa$ and $\kappa \neq 0$ a.e., then $\{A_n^\dagger\}_n \sim_{\mathrm{GLT}} \kappa^{-1}$, where $A_n^\dagger$ is the Moore–Penrose pseudoinverse of A_n.

GLT 6. If $\{A_n\}_n \sim_{\mathrm{GLT}} \kappa$ and each A_n is Hermitian, then $\{f(A_n)\}_n \sim_{\mathrm{GLT}} f(\kappa)$ for all continuous functions $f:\mathbb{C}\to\mathbb{C}$.

GLT 7. $\{A_n\}_n \sim_{\mathrm{GLT}} \kappa$ if and only if there exist GLT sequences $\{B_{n,m}\}_n \sim_{\mathrm{GLT}} \kappa_m$ such that κ_m converges to κ in measure and $\{B_{n,m}\}_n \xrightarrow{\text{a.c.s.}} \{A_n\}_n$ as $m\to\infty$.

GLT 8. Suppose $\{A_n\}_n \sim_{\mathrm{GLT}} \kappa$ and $\{B_{n,m}\}_n \sim_{\mathrm{GLT}} \kappa_m$, where both A_n and $B_{n,m}$ have the same size. Then, $\{B_{n,m}\}_n \xrightarrow{\text{a.c.s.}} \{A_n\}_n$ as $m\to\infty$ if and only if κ_m converges to κ in measure.

GLT 9. If $\{A_n\}_n \sim_{\mathrm{GLT}} \kappa$ then there exist functions $a_{i,m}, f_{i,m},\ i = 1,\dots,N_m$, such that

- $a_{i,m} \in C^\infty([0,1])$ and $f_{i,m}$ is a trigonometric polynomial,
- $\sum_{i=1}^{N_m} a_{i,m}(x)f_{i,m}(\theta)$ converges to $\kappa(x,\theta)$ a.e.,
- $\big\{\sum_{i=1}^{N_m} D_n(a_{i,m})T_n(f_{i,m})\big\}_n \xrightarrow{\text{a.c.s.}} \{A_n\}_n$ as $m\to\infty$.

Most of the properties listed are easy corollaries of Theorem 6.1.8 and Theorem 6.1.9, so we can explain individually the few remaining.

GLT 1 says that $\{A_n\}_n \sim_{GLT} \kappa \implies \{A_n\}_n \sim_\lambda \kappa$, and it is an easy corollary of Lemma 5.2.3. Actually, the same results shows that it holds even for normal sequences.

Lemma 6.1.12 ([10]). *If A_n are normal matrices,*

$$\{A_n\}_n \sim_{GLT} \kappa \implies \{A_n\}_n \sim_\lambda \kappa.$$

GLT 2 comes directly from Theorem 3.2.14. Notice that we can also use Lemma 3.2.18 paired with Lemma 6.1.12, to come up with similar results.

Lemma 6.1.13 ([10, 8])**.** *Consider $\{A_n\}_n \sim_{\mathrm{GLT}} \kappa$ and $A_n = X_n + Y_n$, where every X_n is normal. Then $\{A_n\}_n \sim_\lambda \kappa$ if one of the following holds.*

- $\|Y_n\|_p = o(1)$ *where* $1 \le p \le 2$,
- $\|Y_n\|_p = o(n^{\frac{2}{p}-1})$ *where* $2 \le p < \infty$,
- $\|Y_n\| = o(\frac{1}{n})$.

GLT 5 comes from the observation that if $\{B_n\}_n \sim_{GLT} \kappa^{-1}$ then both $\{A_n^\dagger A_n\}_n$ and $\{A_n B_n\}_n$ are a.c.s. equivalent to $\{I_n\}_n$, so

$$\{A_n^\dagger\}_n \sim_{a.c.s.} \{A_n^\dagger A_n B_n\}_n \sim_{a.c.s.} \{B_n\}_n.$$

GLT 6 can be proved by approximating f with a sequence of polynomials p_m converging to it. Actually, from Lemma 6.1.12 and the fact that A_n normal implies $f(A_n)$ normal, we can generalize the result to normal GLT sequences.

Lemma 6.1.14. *If $\{A_n\}_n \sim_{\mathrm{GLT}} \kappa$ and each A_n is normal, then $\{f(A_n)\}_n \sim_{\mathrm{GLT}} f(\kappa)$ for all continuous functions $f : \mathbb{C} \to \mathbb{C}$.*

GLT 7 and **GLT 8** descend directly from the isometry between $\mathcal{G}$ and $\mathscr{M}_{[0,1]\times[-\pi,\pi]}$ stated in Theorem 6.1.9, and can be generalized into the following result.

Lemma 6.1.15 ([5])**.** *Given the conditions*

1. $\{B_{n,m}\}_{n,m} \sim_{GLT} \kappa_m$,
2. $\{A_n\}_n \sim_{GLT} \kappa$,
3. $\kappa_m \to \kappa$ *in measure,*
4. $\{B_{n,m}\}_{n,m} \xrightarrow{a.c.s.} \{A_n\}_n$,

the following statements hold:

- $(1),(2) \implies ((3) \iff (4))$,
- $(1),(3) \implies \exists\{A_n\}_n : (4),(2)$,
- $(1),(4) \implies \exists\kappa : (2),(3)$,
- $(2),(3) \implies \exists\{B_{n,m}\}_{n,m} : (1),(4)$.

$$\begin{array}{ccc}
\{B_{n,m}\}_{n,m} & \xrightarrow{a.c.s.} & \{A_n\}_n \\
{\scriptstyle \sim_{GLT}}\big\downarrow & & \big\downarrow{\scriptstyle \sim_{GLT}} \\
\kappa_m & \longrightarrow & \kappa
\end{array}$$

6.1.4 Spectral Symbols and Banded Matrices

The GLT axioms are very useful to determine singular value and spectral symbols of matrix-sequences derived from practical applications. In fact we can find the GLT symbol of a sequence through the algebra operations in **GLT 4**, convergence results **GLT 7,8,9** or applying specific functions **GLT 5,6** starting from simple sequences **GLT 3**, and the GLT symbol is always a singular value symbol thanks to **GLT 1**.

When we want to find the spectral symbol of a sequence, then it is automatic if we have a normal sequence thanks to Lemma 6.1.12, otherwise we have to prove that our sequence is close to an Hermitian one, like in **GLT 2**. The last case is actually the one that happens most frequently in applications, so here we report some additional results that usually help in practice. The proofs for these results are reported in Appendix E.

Lemma 6.1.16. *Given a sequence of Hermitian sequences $\{B_{n,m}\}_n$ with GLT symbol k_m, suppose that there exists a sequence $\{B_n\}_n$ with*

$$\lim_{m\to\infty} \limsup_{n\to\infty} \frac{1}{n}\|B_{n,m} - B_n\|_2^2 = 0.$$

In this case, there exists a limit function $k_m \to k$ and $\{B_n\}_n \sim_\lambda k$.

This lemma descends naturally from the perturbation result Theorem 3.2.14, but it is an important piece that we need to prove the last result of this section on banded matrix-sequences.

Theorem 6.1.17. *Given $m+1$ Riemann Integrable function $a_0, a_1, \ldots, a_m$, we have*

$$\sum_{k=0}^{m} \{D_n(a_k)T_n(e^{\mathrm{i}k\theta}) + D_n(a_k)^H T_n(e^{\mathrm{i}k\theta})^H\}_n \sim_\lambda \sum_{k=0}^{m} 2\Re(a_k(x)e^{\mathrm{i}k\theta}).$$

6.1.5 An Example of Application

We consider the second-order differential equation with Dirichlet boundary conditions

$$\begin{cases} -(a(x)u'(x))' + b(x)u'(x) + c(x)u(x) = f(x), & x \in (0,1), \\ u(0) = \alpha, \qquad u(1) = \beta, \end{cases} \tag{6.1}$$

where

- $a(x), c(x)$ are continuous and real-valued functions, defined in $[0,1]$,
- $b(x)$ is a real-valued function on $[0,1]$, such that $|b(x)x^\alpha|$ is bounded for some $\alpha < 3/2$,

while $f(x)$ is a general function. For the GLT analysis presented in the present section we do not need further assumptions. We employ central second-order finite differences for approximating the given equation. We define the stepsize $h = \frac{1}{n+1}$ and the points $x_k = kh$ for k belonging to the interval $[0, n+1]$. For every $j = 1, \ldots, n$ we have

$$\begin{aligned}
-(a(x)u'(x))'|_{x=x_j} &\approx -\frac{a(x_{j+\frac{1}{2}})u'(x_{j+\frac{1}{2}}) - a(x_{j-\frac{1}{2}})u'(x_{j-\frac{1}{2}})}{h} \\
&\approx -\frac{a(x_{j+\frac{1}{2}})\frac{u(x_{j+1})-u(x_j)}{h} - a(x_{j-\frac{1}{2}})\frac{u(x_j)-u(x_{j-1})}{h}}{h} \\
&= \frac{-a(x_{j+\frac{1}{2}})u(x_{j+1}) + (a(x_{j+\frac{1}{2}}) + a(x_{j-\frac{1}{2}}))u(x_j) - a(x_{j-\frac{1}{2}})u(x_{j-1})}{h^2} \\
b(x)u'(x)|_{x=x_j} &\approx b(x_j)\frac{u(x_{j+1}) - u(x_{j-1})}{2h} \\
c(x)u(x)|_{x=x_j} &= c(x_j)u(x_j).
\end{aligned}$$

Let $a_k := a(x_{\frac{k}{2}})$ for any $k \in [0, 2n+2]$ and set $b_j := b(x_j)$, $c_j := c(x_j)$, $f_j := f(x_j)$ for every $j = 0, \ldots, n+1$. We compute approximations u_j of the values $u(x_j)$ for $j = 1, \ldots, n$ by solving the following linear system

$$A_n \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{pmatrix} + B_n \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{pmatrix} + C_n \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{pmatrix} = h^2 \begin{pmatrix} f_1 + \frac{1}{h^2}a_1\alpha + \frac{1}{2h}b_1\alpha \\ f_2 \\ \vdots \\ f_{n-1} \\ f_n + \frac{1}{h^2}a_{2n+1}\beta - \frac{1}{2h}b_n\beta \end{pmatrix},$$

where

$$A_n = \begin{pmatrix} a_1 + a_3 & -a_3 & & & \\ -a_3 & a_3 + a_5 & -a_5 & & \\ & \ddots & \ddots & \ddots & \\ & & -a_{2n-3} & a_{2n-3} + a_{2n-1} & -a_{2n-1} \\ & & & -a_{2n-1} & a_{2n-1} + a_{2n+1} \end{pmatrix},$$

$$B_n = \frac{h}{2} \begin{pmatrix} 0 & b_1 & & & \\ -b_2 & 0 & b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & -b_{n-1} & 0 & b_{n-1} \\ & & & -b_n & 0 \end{pmatrix}, \qquad C_n = h^2 \operatorname{diag}(c_1, \ldots, c_n).$$

In the case where $a(x) \equiv 1$, in accordance with the notation of axiom **GLT 3**, we obtain the basic Toeplitz structures

$$T_n(2-2\cos(\theta)) = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}. \tag{6.2}$$

Recall that the diagonal sampling sequence $\{D_n(a)\}_n$ associated to $a(x)$ is written as

$$\{D_n(a)\}_n = \mathrm{diag}_{i=1,\dots,n}\, a\left(\frac{i}{n}\right)$$

Notice that

$$\begin{aligned}\|D_n(a)T_n(2-2\cos(\theta)) - A_n\|_\infty &\le \\ &\le \max_{i=1,\dots,n}\left\{\left|-a\left(\frac{i}{n}\right)+a_{2i-1}\right| + \left|2a\left(\frac{i}{n}\right) - a_{2i-1} - a_{2i+1}\right| + \left|-a\left(\frac{i}{n}\right)+a_{2i+1}\right|\right\}\end{aligned}$$

where $a_k = a(k/(2n+2))$, and if we call $\omega_a(x)$ the modulus of continuity for a, we have

$$\begin{aligned}\left|-a\left(\frac{i}{n}\right)+a_{2i-1}\right| &= \left|-a\left(\frac{i}{n}\right)+a\left(\frac{2i-1}{2n+2}\right)\right| \le \omega_a\left(\frac{3}{2n+2}\right) = o(1),\\ \left|-a\left(\frac{i}{n}\right)+a_{2i-1}\right| &= \left|-a\left(\frac{i}{n}\right)+a\left(\frac{2i+1}{2n+2}\right)\right| \le \omega_a\left(\frac{1}{2n+2}\right) = o(1),\end{aligned}$$

so

$$\|D_n(a)T_n(2-2\cos(\theta)) - A_n\|_\infty = o(1).$$

As a consequence, Lemma 3.2.12 proves that $\{D_n(a)\}_n\{T_n(2-2\cos(\theta))\}_n \sim_{a.c.s.} \{A_n\}_n$ and thanks to **GLT 3** and **GLT 4**, we obtain

$$\{A_n\}_n \sim_{GLT} a(x)(2-2\cos(\theta)).$$

Following a similar reasoning, we discover that $n^2\{C_n\}_n \sim_{a.c.s.} \{D_n(c)\}_n \sim_{GLT} c(x)$, so

$$\{n^{-2}I_n\}_n \sim_{GLT} 0 \implies \{C_n\}_n \sim_{a.c.s.} \{n^{-2}D_n(c)\}_n \sim_{GLT} 0,$$

which, again by axioms **GLT 3** and **GLT 4**, implies that $\{A_n\}_n + \{C_n\}_n \sim_{GLT} a(x)(2-2\cos(\theta))$. By exploiting the real symmetry of all the considered matrices and in view of axiom **GLT 1**, we obtain

$$\{A_n\}_n + \{C_n\}_n \sim_\lambda a(x)(2-2\cos(\theta)).$$

For the matrices B_n, taking into account **GLT 2**, we are interested in estimating their Schatten 2-norm. Suppose that there exists a constant $C > 0$ such that $|b(x)x^\alpha| < C$. If $\alpha < 1$, then $|b(x)x| \le |b(x)x^\alpha| < C$, so we analyse only the case $\alpha \ge 1$. We have

$$\|B_n\|_2^2 \le \frac{h^2}{2}\sum_{i=1}^n b_i^2 = \frac{h^2}{2}\sum_{i=1}^n b(ih)^2 \le \frac{C^2h^2}{2}\sum_{i=1}^n (ih)^{-2\alpha} = \frac{C^2h^{2-2\alpha}}{2}\sum_{i=1}^n i^{-2\alpha}. \tag{6.3}$$

Since $-2\alpha \le -2$, we can estimate the last sum with the integral of $x^{2\alpha+1}$, in the following way

$$\sum_{i=1}^n i^{-2\alpha} \le 1 + \int_1^n x^{-2\alpha}\mathrm{d}x = \frac{2\alpha - n^{-2\alpha+1}}{2\alpha-1}. \tag{6.4}$$

Substituting (6.4) into (6.3), we obtain

$$\|B_n\|_2^2 \le \frac{C^2h^{2-2\alpha}}{2}\sum_{i=1}^n i^{-2\alpha} \le \frac{C^2\alpha}{2\alpha-1}h^{2-2\alpha} - \frac{C^2}{4\alpha-2}h,$$

which implies

$$\|B_n\|_2 = O(n^{\alpha-1}). \tag{6.5}$$

Observe that $\frac{3}{2} > \alpha$ leads to $\|B_n\|_2 = o(\sqrt{n})$. By invoking **GLT 2**, we simply conclude

$$\{A_n\}_n + \{B_n\}_n + \{C_n\}_n \sim_\lambda a(x)(2-2\cos(\theta)),$$

with $\{X_n\}_n = \{A_n\}_n + \{C_n\}_n$, $\{Y_n\}_n = \{B_n\}_n$.

6.2 GLT Variants

Here we present some of the variants of the original GLT space, built in order to analyse sequences coming from the discretization of multidimensional PDEs or from

6.2.1 Multilevel GLT

Similarly to the one-dimensional case, a multilevel GLT sequence $\{A_{\boldsymbol{n}}\}_n$ is a sequence of matrices with increasing size, equipped with one of its singular values symbols κ, which is referred to as the *GLT Symbol* and is defined over a domain D of the form $[0,1]^q \times [-\pi,\pi]^q$, $q \geq 1$. For a complete analysis of the multilevel GLT sequences, the reader can refer to [53]. Here we report only the principal results, omitting all the proofs.

A point of $D = [0,1]^q \times [-\pi,\pi]^q$ is usually denoted by $(\mathbf{x}, \boldsymbol{\theta})$, where $\mathbf{x} = (x_1, \ldots, x_q)$ and $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_q)$ are vectors of variables. When dealing with multilevel sequences, matrices and vectors, we will use the multi-index notation. A multi-index $\boldsymbol{i} \in \mathbb{Z}^q$, also called a q-index, is simply a vector in $\mathbb{Z}^q$; its components are denoted by $i_1, \ldots, i_q$.

- $\mathbf{0}$, $\mathbf{1}$, $\mathbf{2}$, ... are the vectors of all zeros, all ones, all twos, ... (their size will be clear from the context).
- For any q-index $\boldsymbol{m}$, $N(\boldsymbol{m}) = \prod_{j=1}^q m_j$ and $\boldsymbol{m} \to \infty$ means that $\min(\boldsymbol{m}) = \min_{j=1,\ldots,q} m_j \to \infty$.
- If $\boldsymbol{h}, \boldsymbol{k}$ are q-indices, $\boldsymbol{h} \leq \boldsymbol{k}$ means that $h_r \leq k_r$ for all $r = 1, \ldots, q$, while $\boldsymbol{h} \not\leq \boldsymbol{k}$ means that $h_r > k_r$ for at least one $r \in \{1, \ldots, q\}$.
- If $\boldsymbol{h}, \boldsymbol{k}$ are q-indices such that $\boldsymbol{h} \leq \boldsymbol{k}$, the multi-index range $\boldsymbol{h}, \ldots, \boldsymbol{k}$ is the set $\{\boldsymbol{j} \in \mathbb{Z}^q : \boldsymbol{h} \leq \boldsymbol{j} \leq \boldsymbol{k}\}$. We assume for the multi-index range $\boldsymbol{h}, \ldots, \boldsymbol{k}$ the standard lexicographic ordering:
$$\Big[\ldots \Big[\, [\, (j_1, \ldots, j_q) \,]_{j_q = h_q, \ldots, k_q} \Big]_{j_{q-1} = h_{q-1}, \ldots, k_{q-1}} \ldots \Big]_{j_1 = h_1, \ldots, k_1}. \tag{6.6}$$
For instance, in the case $q = 2$ the ordering is
$$(h_1, h_2),\ (h_1, h_2+1),\ \ldots,\ (h_1, k_2),\ (h_1+1, h_2),\ (h_1+1, h_2+1),\ \ldots,\ (h_1+1, k_2),$$
$$\ldots\ldots,\ (k_1, h_2),\ (k_1, h_2+1),\ \ldots,\ (k_1, k_2).$$
- When a q-index $\boldsymbol{j}$ varies over a multi-index range $\boldsymbol{h}, \ldots, \boldsymbol{k}$ (this is sometimes written as $\boldsymbol{j} = \boldsymbol{h}, \ldots, \boldsymbol{k}$), it is understood that $\boldsymbol{j}$ varies from $\boldsymbol{h}$ to $\boldsymbol{k}$ following the specific ordering (6.6). For instance, if $\boldsymbol{m} \in \mathbb{N}^d$ and if we write $\mathbf{x} = [x_{\boldsymbol{i}}]_{\boldsymbol{i}=\mathbf{1}}^{\boldsymbol{m}}$, then $\mathbf{x}$ is a vector of size $N(\boldsymbol{m})$ whose components $x_{\boldsymbol{i}}$, $\boldsymbol{i} = \mathbf{1}, \ldots, \boldsymbol{m}$, are ordered in accordance with (6.6): the first component is $x_{\mathbf{1}} = x_{(1,\ldots,1,1)}$, the second component is $x_{(1,\ldots,1,2)}$, and so on until the last component, which is $x_{\boldsymbol{m}} = x_{(m_1,\ldots,m_q)}$. Similarly, if $X = [x_{\boldsymbol{ij}}]_{\boldsymbol{i},\boldsymbol{j}=\mathbf{1}}^{\boldsymbol{m}}$, then X is a $N(\boldsymbol{m}) \times N(\boldsymbol{m})$ matrix whose components are indexed by two d-indices $\boldsymbol{i}, \boldsymbol{j}$, both varying from $\mathbf{1}$ to $\boldsymbol{m}$ according to the lexicographic ordering (6.6).
- Operations involving q-indices that have no meaning in the vector space $\mathbb{Z}^q$ must always be interpreted in the componentwise sense. For instance, $\boldsymbol{ij} = (i_1 j_1, \ldots, i_q j_q)$, $\boldsymbol{i}/\boldsymbol{j} = (i_1/j_1, \ldots, i_q/j_q)$, etc.

In this context, by a sequence of matrices (or matrix-sequence) we mean a sequence of the form $\{A_{\boldsymbol{n}}\}_n$, where $\boldsymbol{n} = (n_1, \ldots, n_d)$ depends on n and $\boldsymbol{n} \to \infty$ as $n \to \infty$. In many cases, it is natural to assume that $\boldsymbol{n} = n\mathbf{c}$, where $\mathbf{c}$ is a vector of rational constants and n diverges to infinity. It is always understood that a matrix $A_{\boldsymbol{n}}$ parametrized by a q-index $\boldsymbol{n}$ has dimension $N(\boldsymbol{n}) = n_1 \cdot \cdots \cdot n_q$; its entries will be indexed by two q-indices $\boldsymbol{i}, \boldsymbol{j}$.

The main theoretical properties of one-dimensional GLT sequences **GLT 1**−**GLT 9** still hold in the multidimensional context, upon substituting the sequences $\{A_n\}_n$ with the multilevel sequences $\{A_{\boldsymbol{n}}\}_n$. The only exceptions are **GLT 3** and **GLT 9**, that have to be rewritten in order to include q-level Toeplitz matrices generated by an L^1 q-variate function and q-level diagonal sampling matrices associated with an almost everywhere continuous q-variate function.

GLT 3. Here we list three important examples of GLT sequences.

- Given a function f in $L^1([-\pi,\pi]^q)$, its associated Toeplitz sequence is $\{T_{\boldsymbol{n}}(f)\}_n$, where the elements are multidimensional Fourier coefficients of f:
$$T_{\boldsymbol{n}}(f) = [f_{\boldsymbol{i}-\boldsymbol{j}}]_{\boldsymbol{i},\boldsymbol{j}=\mathbf{1}}^{\boldsymbol{n}}, \qquad f_{\boldsymbol{k}} = \frac{1}{(2\pi)^q} \int_{-\pi}^{\pi} f(\boldsymbol{\theta}) e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\theta}} \mathrm{d}\theta.$$
$\{T_{\boldsymbol{n}}(f)\}_n$ is a GLT sequence with symbol $\kappa(\mathbf{x}, \boldsymbol{\theta}) = f(\boldsymbol{\theta})$.

- Given an almost everywhere continuous function, $a : [0,1]^q \to \mathbb{C}$, its associated diagonal sampling sequence $\{D_{\boldsymbol{n}}(a)\}_n$ is defined as

$$D_{\boldsymbol{n}}(a) = \operatorname{diag}\left(\left\{a\left(\frac{\boldsymbol{i}}{\boldsymbol{n}}\right)\right\}_{\boldsymbol{i}=1}^{\boldsymbol{n}}\right).$$

  $\{D_{\boldsymbol{n}}(a)\}_n$ is a GLT sequence with symbol $\kappa(\mathbf{x},\boldsymbol{\theta}) = a(\mathbf{x})$.
- Any zero-distributed sequence $\{Z_{\boldsymbol{n}}\}_n \sim_\sigma 0$ is a GLT sequence with symbol $\kappa(\mathbf{x},\boldsymbol{\theta}) = 0$.

GLT 9. If $\{A_{\boldsymbol{n}}\}_n \sim_{\mathrm{GLT}} \kappa$ then there exist functions $a_{i,m}, f_{i,m},\ i = 1,\dots,N_m$, such that

- $a_{i,m} \in C^\infty([0,1]^q)$ and $f_{i,m}$ is a trigonometric polynomial in q variables,
- $\sum_{i=1}^{N_m} a_{i,m}(\mathbf{x}) f_{i,m}(\boldsymbol{\theta})$ converges to $\kappa(\mathbf{x}, b\theta)$ a.e.,
- $\left\{\sum_{i=1}^{N_m} D_{\boldsymbol{n}}(a_{i,m}) T_{\boldsymbol{n}}(f_{i,m})\right\}_n \xrightarrow{\text{a.c.s.}} \{A_{\boldsymbol{n}}\}_n$ as $m \to \infty$.

The construction of the s.a. of q-level GLT $\mathcal{G}_q$ is totally analogous to the unidimensional case, but we start with q-level diagonal sampling sequences $\{D_{\boldsymbol{n}}(a)\}_n$ and q-level Toeplitz sequences $\{T_{\boldsymbol{n}}(f)\}_n$. In particular, in the case of $\{D_{\boldsymbol{n}}(a)\}_n$ for $a \in C([0,1]^q)$, it is intuitive to associate the symbol $a(\mathbf{x})$ to the sequence, since it contains an uniform sampling of the function over a regular grid on $[0,1]^q$. In the case of separable functions $f(\boldsymbol{\theta})$ (for example, when they are trigonometric polynomials), it is possible to express the multilevel Toeplitz sequences as a Kronecker product of simpler Toeplitz sequences

$$T_{n_1}(f_1) \otimes \cdots \otimes T_{n_q}(f_q) = T_{\boldsymbol{n}}(f_1 \cdot \cdots \cdot f_q).$$

The properties of the Kronecker product let us associate to each such multilevel Toeplitz sequence the symbol $f_1 \cdot \cdots \cdot f_q$. With these basilar sequences, it is possible to build the set of multilevel locally Toeplitz sequences, and its closure will result in $\mathcal{G}_q$.

All the other results from the unidimensional GLT settings can also be generalized to the multidimensional case. In particular, if we call $\mathcal{G}_q$ the space of couples $(\{A_{\boldsymbol{n}}\}_n, \kappa)$ such that $\{A_{\boldsymbol{n}}\}_n \sim_{GLT} \kappa$, with $\kappa \in \mathscr{M}_{[0,1]^q \times [-\pi,\pi]^q}$, then we can state the following propositions.

Theorem 6.2.1. *$\mathcal{G}_q$ is a closed and maximal s.a. admitting all functions in $\mathscr{M}_{[0,1]^d \times [-\pi,\pi]^q}$ as symbols. Moreover, if we quotient the space by the zero-distributed s.a. $\mathscr{Z} \times \{0\}$, it becomes isomorphic and isometrically equivalent to $\mathscr{M}_{[0,1]^d \times [-\pi,\pi]^q}$.*

Theorem 6.2.2. *If $T : [0,1] \to [0,1]^q \times [-\pi,\pi]^q$ is an isomorphism of measurable spaces, then there exists an unitary sequence $\{U_{\boldsymbol{n}}\}_n$ such that the map*

$$\phi : \mathcal{G}_q \to \mathcal{D}, \qquad \varphi(\{A_{\boldsymbol{n}}\}_n, \kappa) = (\{U_{\boldsymbol{n}} A_{\boldsymbol{n}} U_{\boldsymbol{n}}^H\}_n, \kappa(T)).$$

is an isomorphism of $\mathbb{C}$-algebras and an isometry.

Lemma 6.2.3. *If $A_{\boldsymbol{n}}$ are normal matrices,*

$$\{A_{\boldsymbol{n}}\}_n \sim_{GLT} \kappa \implies \{A_{\boldsymbol{n}}\}_n \sim_\lambda \kappa.$$

Lemma 6.2.4. *If $\{A_{\boldsymbol{n}}\}_n \sim_{\mathrm{GLT}} \kappa$ and each $A_{\boldsymbol{n}}$ is normal, then $\{f(A_{\boldsymbol{n}})\}_n \sim_{\mathrm{GLT}} f(\kappa)$ for all continuous functions $f : \mathbb{C} \to \mathbb{C}$.*

6.2.2 Multidimensional Diffusion Problem

$$\begin{cases} -\nabla \cdot \mathrm{A}\nabla u = f, & \text{in } (0,1)^2, \\ u = 0, & \text{on } \partial((0,1)^2), \end{cases} \tag{6.7}$$

where $A : [0,1]^2 \to \mathbb{R}^{2\times 2}$ is a symmetric matrix of functions a_{hk}. As in the 1-dimensional case, we should assume at least $a_{hk} \in C^1([0,1]^2)$ to ensure the well posedness of problem (6.7), but for the analysis we only need the weaker condition that a_{hk} are continuous a.e. on $[0,1]^2$.

We observe that (6.7) is equivalent to

$$\begin{cases} -\displaystyle\sum_{h,k=1}^{2} \frac{\partial}{\partial x_h}\Big(a_{hk}\frac{\partial u}{\partial x_k}\Big) = f, & \text{in } (0,1)^2, \\ u = 0, & \text{on } \partial((0,1)^2). \end{cases} \tag{6.8}$$

We consider the classical central FD discretizations of (6.8). We choose $\boldsymbol{n} \in \mathbb{N}^2$ and we set $\boldsymbol{h} = \frac{\boldsymbol{1}}{\boldsymbol{n}+\boldsymbol{1}}$ and $x_{\boldsymbol{j}} = \boldsymbol{j}\boldsymbol{h}$ for $\boldsymbol{j} = \boldsymbol{0}, \dots, \boldsymbol{n}+\boldsymbol{1}$. [1] For simplicity, we also suppose that $n = n_1 = n_2$, meaning that $\boldsymbol{n} = (n,n)$ and $h = h_1 = h_2$.

We refer the reader to [8], where this application is treated in more generality (generic dimension q instead of 2, and non-uniform parameter $\boldsymbol{n} = (n_1, n_2, \dots, n_q)$), and more details. Here we report a simplified sketch of the analysis.

Let $\mathbf{e}_k$ be the kth vector of the canonical basis of $\mathbb{R}^2$ and notice that $x_{\boldsymbol{j}} + sh_k\mathbf{e}_k = x_{\boldsymbol{j}+s\mathbf{e}_k}$. Then, for $\boldsymbol{j} = \boldsymbol{1}, \dots, \boldsymbol{n}$, we can approximate the terms appearing in (6.8) as follows:

$$\begin{aligned} \frac{\partial}{\partial x_k}\left(a_{kk}\frac{\partial u}{\partial x_k}\right)\Big|_{x=x_{\boldsymbol{j}}} &\approx \frac{a_{kk}\frac{\partial u}{\partial x_k}(x_{\boldsymbol{j}+\mathbf{e}_k/2}) - a_{kk}\frac{\partial u}{\partial x_k}(x_{\boldsymbol{j}-\mathbf{e}_k/2})}{h} \\ &\approx a_{kk}(x_{\boldsymbol{j}+\mathbf{e}_k/2})\frac{u(x_{\boldsymbol{j}+\mathbf{e}_k}) - u(x_{\boldsymbol{j}})}{h^2} - a_{kk}(x_{\boldsymbol{j}-\mathbf{e}_k/2})\frac{u(x_{\boldsymbol{j}}) - u(x_{\boldsymbol{j}-\mathbf{e}_k})}{h^2} \end{aligned} \tag{6.9}$$

$$\begin{aligned} \frac{\partial}{\partial x_h}\left(a_{hk}\frac{\partial u}{\partial x_k}\right)\Big|_{x=x_{\boldsymbol{j}}} &\approx \frac{a_{hk}\frac{\partial u}{\partial x_k}(x_{\boldsymbol{j}+\mathbf{e}_h}) - a_{hk}\frac{\partial u}{\partial x_k}(x_{\boldsymbol{j}-\mathbf{e}_h})}{2h} \\ &\approx a_{hk}(x_{\boldsymbol{j}+\mathbf{e}_h})\frac{u(x_{\boldsymbol{j}+\mathbf{e}_h+\mathbf{e}_k}) - u(x_{\boldsymbol{j}+\mathbf{e}_h-\mathbf{e}_k})}{4h^2} - a_{hk}(x_{\boldsymbol{j}-e_h})\frac{u(x_{\boldsymbol{j}-\mathbf{e}_h+\mathbf{e}_k}) - u(x_{\boldsymbol{j}-\mathbf{e}_h-\mathbf{e}_k})}{4h^2}, \end{aligned} \tag{6.10}$$

for $h, k = 1, 2,\quad h \neq k$. The evaluations $u(x_{\boldsymbol{j}})$ of the solution of (7.2) at the grid points $x_{\boldsymbol{j}}$ are approximated by the values $u_{\boldsymbol{j}}$, where $u_{\boldsymbol{j}} = 0$ for $\boldsymbol{j} \in \{\boldsymbol{0}, \dots, \boldsymbol{n}+\boldsymbol{1}\}\backslash\{\boldsymbol{1}, \dots, \boldsymbol{n}\}$, and the vector $\mathbf{u} = (u_{\boldsymbol{1}}, \dots, u_{\boldsymbol{n}})^T$ is the solution of the linear system

$$\begin{aligned} &-\sum_{k=1}^{2} a_{kk}(x_{\boldsymbol{j}+\mathbf{e}_k/2})\frac{u_{\boldsymbol{j}+\mathbf{e}_k} - u_{\boldsymbol{j}}}{h^2} - a_{kk}(x_{\boldsymbol{j}-\mathbf{e}_k/2})\frac{u_{\boldsymbol{j}} - u_{\boldsymbol{j}-\mathbf{e}_k}}{h^2} \\ &-\sum_{\substack{h,k=1\\h\neq k}}^{2} a_{hk}(x_{\boldsymbol{j}+e_h})\frac{u_{\boldsymbol{j}+\mathbf{e}_h+\mathbf{e}_k} - u_{\boldsymbol{j}+\mathbf{e}_h-\mathbf{e}_k}}{4h^2} - a_{hk}(x_{\boldsymbol{j}-e_h})\frac{u_{\boldsymbol{j}-\mathbf{e}_h+\mathbf{e}_k} - u_{\boldsymbol{j}-\mathbf{e}_h-\mathbf{e}_k}}{4h^2} \\ &= f(x_{\boldsymbol{j}}), \qquad \boldsymbol{j} = \boldsymbol{1}, \dots, \boldsymbol{n}. \end{aligned} \tag{6.11}$$

We now want to understand the structure of the matrix $A_{\boldsymbol{n}}$ associated with the linear system (6.11). Luckily, the multi-index language allows us to provide a compact and easy-to-manage expression of this matrix. For example, the matrix representing the addendum

$$-a_{11}(x_{\boldsymbol{j}+\mathbf{e}_1/2})\frac{u_{\boldsymbol{j}+\mathbf{e}_1} - u_{\boldsymbol{j}}}{h^2}, \qquad \boldsymbol{j} = \boldsymbol{1}, \dots, \boldsymbol{n}$$

can be expressed as

$$\frac{1}{h^2}\Big(\operatorname{diag}_{\boldsymbol{j}=\boldsymbol{1},\dots,\boldsymbol{n}} a_{11}(x_{\boldsymbol{j}+\mathbf{e}_1/2})\Big)\begin{pmatrix} \boxed{I_n} & \boxed{-I_n} & & \\ & \boxed{I_n} & \ddots & \\ & & \ddots & \boxed{-I_n} \\ & & & \boxed{I_n} \end{pmatrix}$$

[1] Operations involving q-indices in $\mathbb{Z}^q$ must be interpreted in the componentwise sense. In the present case, given $\boldsymbol{n} = (n_1, \dots, n_q)$, the vector of discretization steps $\boldsymbol{h} = \frac{\boldsymbol{1}}{\boldsymbol{n}+\boldsymbol{1}}$ and the grid points $x_{\boldsymbol{j}} = \boldsymbol{j}\boldsymbol{h}$ are given by $\boldsymbol{h} = \left(\frac{1}{n_1+1}, \dots, \frac{1}{n_q+1}\right) = (h_1, \dots, h_q)$ and $x_{\boldsymbol{j}} = (j_1h_1, \dots, j_qh_q)$.

so

$$\begin{aligned} & \frac{1}{h^2}\Big(\operatorname{diag}_{\boldsymbol{j}=\mathbf{1},\dots,\boldsymbol{n}} a_{11}(x_{\boldsymbol{j}+\mathbf{e}_1/2})\Big)\big(T_n(1-e^{\mathrm{i}\theta})\otimes I_n\big) \\ = & \quad \frac{1}{h^2}\Big(\operatorname{diag}_{\boldsymbol{j}=\mathbf{1},\dots,\boldsymbol{n}} a_{11}(x_{\boldsymbol{j}+\mathbf{e}_1/2})\Big)T_{\boldsymbol{n}}(1-e^{\mathrm{i}\theta_1}). \end{aligned}$$

With analogous reasoning for the other addends, we note that $A_{\boldsymbol{n}}$ admits the following natural decomposition:

$$\begin{aligned} A_{\boldsymbol{n}} \;=\; & \sum_{k=1}^{2}\frac{1}{h^2}\Big(\operatorname{diag}_{\boldsymbol{j}=\mathbf{1},\dots,\boldsymbol{n}} a_{kk}(x_{\boldsymbol{j}+\mathbf{e}_k/2})\Big)T_{\boldsymbol{n}}(1-e^{\mathrm{i}\theta_k}) \\ & +\sum_{k=1}^{2}\frac{1}{h^2}\Big(\operatorname{diag}_{\boldsymbol{j}=\mathbf{1},\dots,\boldsymbol{n}} a_{kk}(x_{\boldsymbol{j}-\mathbf{e}_k/2})\Big)T_{\boldsymbol{n}}(1-e^{-\mathrm{i}\theta_k}) \\ & +\sum_{\substack{h,k=1\\ h\neq k}}^{2} -\frac{1}{h^2}\Big(\operatorname{diag}_{\boldsymbol{j}=\mathbf{1},\dots,\boldsymbol{n}} a_{hk}(x_{\boldsymbol{j}+\mathbf{e}_h})\Big)\frac{\mathrm{i}}{2}\,T_{\boldsymbol{n}}(e^{\mathrm{i}\theta_h}\sin\theta_k) \\ & +\sum_{\substack{h,k=1\\ h\neq k}}^{2} \frac{1}{h^2}\Big(\operatorname{diag}_{\boldsymbol{j}=\mathbf{1},\dots,\boldsymbol{n}} a_{hk}(x_{\boldsymbol{j}-\mathbf{e}_h})\Big)\frac{\mathrm{i}}{2}\,T_{\boldsymbol{n}}(e^{-\mathrm{i}\theta_h}\sin\theta_k). \end{aligned} \tag{6.12}$$

In view of axiom **GLT 3**, all the diagonal matrix-sequences that appear in (6.12) belong to the class of GLT sequences. More precisely, we observe

$$\{\operatorname{diag}_{\boldsymbol{j}=\mathbf{1},\dots,\boldsymbol{n}} a_{kk}(x_{\boldsymbol{j}\pm\mathbf{e}_k/2})\}_n \sim_{a.c.s.} \{D_{\boldsymbol{n}}(a_{kk})\}_n \sim_{GLT} a_{kk}(\mathbf{x}), \tag{6.13}$$

$$\{\operatorname{diag}_{\boldsymbol{j}=\mathbf{1},\dots,\boldsymbol{n}} a_{hk}(x_{\boldsymbol{j}\pm\mathbf{e}_h})\}_n \sim_{a.c.s.} \{D_{\boldsymbol{n}}(a_{hk})\}_n \sim_{GLT} a_{hk}(\mathbf{x}). \tag{6.14}$$

Using axiom **GLT 4**, we thus obtain

$$\{n^{-2}A_{\boldsymbol{n}}\}_n \quad \sim_{GLT} \quad \sum_{k=1}^{2} a_{kk}(\mathbf{x})(2-2\cos\theta_k) + \sum_{\substack{h,k=1\\ h\neq k}}^{2} a_{hk}(\mathbf{x})\sin\theta_h\sin\theta_k, \tag{6.15}$$

that can be rewritten as

$$\{n^{-2}A_{\boldsymbol{n}}\}_n \sim_{GLT} \mathbf{1}(\mathrm{A}(\mathbf{x})\circ H(\boldsymbol{\theta}))\mathbf{1}^T,$$

where $H:[0,1]^2\to\mathbb{R}^{2\times 2}$ is the symmetric matrix of continuous functions defined by

$$\begin{aligned} (H(\boldsymbol{\theta}))_{kk} &= 2-2\cos\theta_k, && k=1,2, \\ (H(\boldsymbol{\theta}))_{hk} &= \sin\theta_h\sin\theta_k, && 1\le h\neq k\le 2. \end{aligned}$$

Now $n^{-2}A_{\boldsymbol{n}}$ are Hermitian matrices, so by axioms **GLT 1** we infer

$$\{n^{-2}A_{\boldsymbol{n}}\}_n \sim_{\mathrm{GLT},\sigma,\lambda} \mathbf{1}(\mathrm{A}(\mathbf{x})\circ H(\boldsymbol{\theta}))\mathbf{1}^T.$$

6.2.3 Block GLT

Notice that the sequences we deal with in paragraph 6.2.1 are block matrices, with blocks of increasing sizes. If we fix the size of the block to a constant, we obtain a different, yet similar, family of sequences, called the Block GLT sequences. The main difference with all the other sequences we encountered until now is that they admit a matrix-valued function as symbols. In other words, the algebra of block GLT sequences will consist of couples $(\{A_n\}_n, \Upsilon)$ where

$$\Upsilon : [0,1]\times[-\pi,\pi]\to\mathbb{C}^{s\times s},$$

and the matrix A is an s-block matrix

$$A_n = (a_{i,j})_{i,j=1}^{n}, \qquad a_{i,j}\in\mathbb{C}^{s\times s} \quad \forall\, i,j=1,\dots,n.$$

Here we use n for simplicity of exposition, but we could use a general diverging sequence s_n and all the results still hold. In particular, A_n has size $sn\times sn$. When $s=1$ we recover the original GLT sequences, but otherwise the considered algebra is not even a s.a. in sense of Definition 5.3.9, because its symbols do not belong to

$\mathscr{M}_D$. Nonetheless, one can prove that all the theory regarding s.a. and s.g. can be generalized to spaces with matrix-valued symbols, and we find again most of the properties of the GLT sequences. For a complete analysis of the block GLT sequences, the reader can refer to [13]. Here we report only the principal results, omitting all the proofs.

In what follows, unless expressly indicated, all the sequences $\{A_n\}_n$ are s-block matrix-sequences, and all the symbols are in $\mathscr{M}^{(s)}$, the space of measurable functions with values in $\mathbb{C}^{s\times s}$. In particular, $L^1([-\pi,\pi],s)$ are the matrix valued L^1 functions with domain $[-\pi,\pi]$ and $C([0,1,s])$ are the continuous matrix valued functions with domain $[0,1]$.

GLT 1. If $\{A_n\}_n \sim_{\mathrm{GLT}} \Upsilon$ then $\{A_n\}_n \sim_\sigma \Upsilon$. If $\{A_n\}_n \sim_{\mathrm{GLT}} \Upsilon$ and each A_n is Hermitian then Υ is Hermitian a.e. and $\{A_n\}_n \sim_\lambda \Upsilon$.

GLT 2. If $\{A_n\}_n \sim_{\mathrm{GLT}} \Upsilon$ and $A_n = X_n + Y_n$, where

- every X_n is Hermitian,
- $n^{-1/2}\|Y_n\|_2 \to 0$,

then $\{P_n^H A_n P_n\}_n \sim_{\sigma,\lambda} \Upsilon$ for every sequence $\{P_n\}_n$ such that $P_n^H P_n = I_{\delta_n}$, $\delta_n \le sn$, and $\delta_n/sn \to 1$.

GLT 3. Here we list three fundamental examples of GLT sequences.

- Given a function f in $L^1([-\pi,\pi],s)$, its associated Toeplitz sequence is $\{T_n(f)\}_n$, where

$$T_n(f) = [f_{i-j}]_{i,j=1}^n, \qquad f_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-\mathrm{i}k\theta}\mathrm{d}\theta.$$

  $\{T_n(f)\}_n$ is a block GLT sequence with symbol $\Upsilon(x,\theta) = f(\theta)$.
- Given any a.e. continuous function $a : [0,1] \to \mathbb{C}^{s\times s}$, its associated diagonal sampling sequence is $\{D_n(a)\}_n$, where

$$D_n(a) = \operatorname*{diag}_{i=1,\dots,n} a\Big(\frac{i}{n}\Big).$$

  $\{D_n(a)\}_n$ is a block GLT sequence with symbol $\Upsilon(x,\theta) = a(x)$.
- A zero-distributed sequence is a matrix-sequence such that $\{Z_n\}_n \sim_\sigma 0$, i.e.,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{sn} G(\sigma_i(A_n)) = G(0)$$

  for every continuous function $G : \mathbb{R} \to \mathbb{C}$ with compact support. Any zero-distributed sequence is a block GLT sequence with symbol $\Upsilon(x,\theta) = 0_s$.

GLT 4. If $\{A_n\}_n \sim_{\mathrm{GLT}} \Upsilon$ and $\{B_n\}_n \sim_{\mathrm{GLT}} \Xi$, then

- $\{A_n^H\}_n \sim_{\mathrm{GLT}} \Upsilon^H$, where A_n^H is the conjugate transpose of A_n,
- $\{\alpha A_n + \beta B_n\}_n \sim_{\mathrm{GLT}} \alpha\Upsilon + \beta\Xi$ for all $\alpha,\beta \in \mathbb{C}$,
- $\{A_n B_n\}_n \sim_{\mathrm{GLT}} \Upsilon\Xi$,
- $\{A_n^\dagger\}_n \sim_{GLT} \Upsilon^{-1}$ if Υ is invertible a.e.

GLT 5. If $\{A_n\}_n \sim_{\mathrm{GLT}} \Upsilon$ and each A_n is Hermitian, then $\{f(A_n)\}_n \sim_{\mathrm{GLT}} f(\Upsilon)$ for all continuous functions $f : \mathbb{C} \to \mathbb{C}$.

GLT 6. If $\{A_{n,ij}\}_n \sim_{\mathrm{GLT}} \Upsilon_{ij} \in \mathscr{M}^{(r)}$ for $i,j = 1,\dots,s$ and we set $\Upsilon = [\Upsilon]_{i,j=1}^s \in \mathscr{M}^{(rs)}$ and $A_n = [A_{n,ij}]_{i,j=1}^s$, then there exists a permutation matrix Π_n such that $\{\Pi_n^T A_n \Pi_n\}_n \sim_{GLT} \Upsilon$.

GLT 7. $\{A_n\}_n \sim_{\mathrm{GLT}} \Upsilon$ if and only if there exist GLT sequences $\{B_{n,m}\}_n \sim_{\mathrm{GLT}} \Upsilon_m$ such that Υ_m converges to Υ in measure and $\{B_{n,m}\}_n \xrightarrow{\text{a.c.s.}} \{A_n\}_n$ as $m \to \infty$.

GLT 8. If $\{A_n\}_n \sim_{GLT} \Upsilon$ and $\{B_n\}_n \sim_{GLT} \Xi$, then $d_{a.c.s.}(\{A_n\}_n, \{B_n\}_n) = d_{mea}(\Upsilon,\Xi)$, and for every gauge function φ, $d^{\varphi}_{a.c.s.}(\{A_n\}_n, \{B_n\}_n) = d^{\varphi}_{mea}(\Upsilon,\Xi)$.

GLT 9. If $\{A_n\}_n \sim_{\mathrm{GLT}} \Upsilon$ then there exist functions $a_{i,m}, f_{i,m},\ i = 1,\dots,N_m$, such that

- $a_{i,m} \in C^\infty([0,1])$ and $f_{i,m}$ is a trigonometric monomial for every entry,
- $\sum_{i=1}^{N_m} a_{i,m}(x) f_{i,m}(\theta)$ converges to $\Upsilon(x,\theta)$ a.e.,
- $\left\{\sum_{i=1}^{N_m} D_n(a_{i,m} I_s) T_n(f_{i,m})\right\}_n \xrightarrow{\text{a.c.s.}} \{A_n\}_n$ as $m \to \infty$.

Notice that in $T_n(f)$ all the Fourier coefficient are $s \times s$ matrices, and in $D_n(a)$ all the diagonal entries are also $s \times s$ matrices. Moreover, the distances reported in **GLT 8** are the usual $d_{a.c.s.}$ distance, the distance d_{mea} between matrix-valued functions defined as

$$d_{mea}(\Upsilon, \Xi) = \inf_{y \geq 0} \left\{ x + \frac{1}{s} \sum_{i=1}^{s} \frac{\ell_2 \{ (x,\theta) \in [0,1] \times [-\pi,\pi] \mid \sigma_i(\Upsilon(x,\theta) - \Xi(x,\theta)) > y \}}{2\pi} \right\}$$

and for every function φ as in Theorem 3.1.15,

$$d^{\varphi}_{a.c.s.}(\{A_n\}_n, \{B_n\}_n) = \limsup_{n\to\infty} \frac{1}{s_n} \sum_{i=1}^{s_n} \varphi(\sigma_i(A_n - B_n)),$$

$$d^{\varphi}_{mea}(\Upsilon, \Xi) = \frac{1}{2\pi} \int_{[0,1]\times[-\pi,\pi]} \frac{\sum_{i=1}^{s} \varphi(\sigma_i(\Upsilon(x,\theta) - \Xi(x,\theta)))}{s} \mathrm{d}(x,\theta).$$

For a proof of the isometry between $d_{a.c.s.}$ and d_{mea} refer to Appendix G.

The construction of the algebra of s-block GLT $\mathcal{G}^s$ is totally analogous to the unidimensional case, but we start with s-block diagonal sampling sequences $\{D_n(a)\}_n$ and s-block Toeplitz sequences $\{T_n(f)\}_n$. In particular, in the case of $\{D_n(a)\}_n$ for $a \in C([0,1], s)$, it is intuitive to associate the symbol $a(x)$ to the sequence, since it contains an uniform sampling of the function over a regular grid on $[0,1]$.

In the case of entry-wise L^1 functions $f(\theta)$ where only one matrix entry $f_{i,j}(\theta)$ is non-zero (and in this instance we write $f(\theta) = f_{i,j}(\theta) E_{i,j}$) it is possible to permute the rows and columns through the permutation matrix in **GLT 6** and obtain a block matrix with $s \times s$ blocks of size n, and where only the i,j-th block is non-zero and equal to $T_n(f_{i,j}(x))$. It is thus natural to associate to each such s-block Toeplitz sequence the symbol $f_{i,j}(\theta) E_{i,j}$.

With these basilar sequences, it is possible to build the set of block locally Toeplitz, and its closure will result in $\mathcal{G}^s$.

An ulterior extension of the theory is provided in [14], where one can consider multivariate matrix-valued functions $\Upsilon : [0,1]^d \times [-\pi,\pi]^d \to \mathbb{C}^{s\times s}$ and the associated d-multilevel s-block GLT sequences. The analysis follows the same lines as paragraph 6.2.1 and this paragraph argument, and similar results to **GLT 1-9** are shown.

6.2.4 Petersen Graphs

In paragraph 1.3.3 we encountered for the first time an example of matrix-valued symbol Υ associated to the sequence of adjacency matrices associated to generalized Petersen graphs. Here we can show how that it is actually a block GLT symbol.

Recall that the adjacency matrices of GPG(n,k) present a 2×2 block structure

$$A_n = A(GPG(n,k)) = \begin{pmatrix} C_n + C_n^T & I_n \\ I_n & C_n^k + (C_n^k)^T \end{pmatrix},$$

where C_n is the circulant matrix defined in paragraph 1.3.2. As we have shown in paragraph 6.1.1, the circulant sequences are actually a.c.s. equivalent to the Toeplitz sequences, so it is easy to show that $\{A_n\}_n \sim_{a.c.s.} \{B_n\}_n$ where

$$B_n = \begin{pmatrix} T_n(2\cos(\theta)) & T_n(1) \\ T_n(1) & T_n(2\cos(k\theta)) \end{pmatrix}.$$

Now we can apply **GLT 6** with $r = 1$ and $s = 2$, and easily conclude that there exist a permutation sequence $\{\Pi_n\}_n$ such that

$$\{\Pi_n^T B_n \Pi_n\}_n \sim_{GLT} \Upsilon = \begin{pmatrix} 2\cos(\theta) & 1 \\ 1 & 2\cos(k\theta) \end{pmatrix}.$$

Notice that Υ coincides, up to a transformation of domain $[0,1] \to [-\pi,\pi]$, with the symbol found in paragraph 1.3.3. Since A_n and B_n are both Hermitian, and since a permutation does not change eigenvalues and singular values, we conclude that

$$\{A_n\}_n \sim_{\sigma,\lambda} \Upsilon.$$

6.2.5 High Order FE Discretization

Consider the diffusion problem

$$\begin{cases} -u''(x) = f(x), & x \in (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{6.16}$$

The weak form of (6.16) reads as follows [29, Chapter 8]: find $u \in H_0^1([0,1])$ such that

$$\int_0^1 u'(x)w'(x)\mathrm{d}x = \int_0^1 f(x)w(x)\mathrm{d}x, \qquad \forall\, w \in H_0^1([0,1]).$$

We use a finite elements method [68, Chapter 4], with of basis functions $B_{1,[p,k]}, \ldots, B_{n(p-k)+k+1,[p,k]} : \mathbb{R} \to \mathbb{R}$ the B-splines of degree p and smoothness C^k, for $p, n \geq 1$ and $0 \leq k \leq p-1$, defined on the knot sequence

$$\begin{aligned} &\{\tau_1, \ldots, \tau_{n(p-k)+p+k+2}\} \\ &= \Big\{\underbrace{0, \ldots, 0}_{p+1}, \underbrace{\frac{1}{n}, \ldots, \frac{1}{n}}_{p-k}, \underbrace{\frac{2}{n}, \ldots, \frac{2}{n}}_{p-k}, \ldots, \underbrace{\frac{n-1}{n}, \ldots, \frac{n-1}{n}}_{p-k}, \underbrace{1, \ldots, 1}_{p+1}\Big\}. \end{aligned} \tag{6.17}$$

For simplicity, we denote the basis functions as

$$\varphi_i = B_{i+1,[p,k]}, \qquad i = 1, \ldots, n(p-k)+k-1, \tag{6.18}$$

and we look for an approximation of the exact solution in the space $\mathscr{W} = \mathrm{span}(\varphi_1, \ldots, \varphi_N)$ by solving the following discrete problem: find $u_{\mathscr{W}} \in \mathscr{W}$ such that

$$\int_0^1 u_{\mathscr{W}}'(x)w'(x)\mathrm{d}x = \int_0^1 f(x)w(x)\mathrm{d}x, \qquad \forall\, w \in \mathscr{W}.$$

Since $\{\varphi_1, \ldots, \varphi_N\}$ is a basis of $\mathscr{W}$, we can write $u_{\mathscr{W}} = \sum_{j=1}^N u_j\varphi_j$ for a unique vector $\mathbf{u} = (u_1, \ldots, u_N)^T$. By linearity, the computation of $u_{\mathscr{W}}$ (i.e., of $\mathbf{u}$) reduces to solving the linear system

$$A\mathbf{u} = \mathbf{f},$$

where $\mathbf{f} = \left(\int_0^1 f(x)\varphi_1(x)\mathrm{d}x, \ldots, \int_0^1 f(x)\varphi_N(x)\mathrm{d}x\right)^T$ and A is the stiffness matrix,

$$A = A_{n,[p,k]} = \left[\int_0^1 B_{j+1,[p,k]}'(x)B_{i+1,[p,k]}'(x)\mathrm{d}x\right]_{i,j=1}^{n(p-k)+k-1}. \tag{6.19}$$

The main properties of $B_{1,[p,k]}, \ldots, B_{n(p-k)+k+1,[p,k]}$ that we need to analyse the linear system are listed below ([32, 71]).

- The support of the ith B-spline is given by

$$\mathrm{supp}(B_{i,[p,k]}) = [\tau_i, \tau_{i+p+1}], \qquad i = 1, \ldots, n(p-k)+k+1. \tag{6.20}$$

- The derivatives of the B-splines satisfy

$$\sum_{i=1}^{n(p-k)+k+1} |B_{i,[p,k]}'| \leq C_p n \quad \text{over} \quad \mathbb{R}, \tag{6.21}$$

  where C_p is a constant depending only on p. Note that the derivatives $B_{i,[p,k]}'$ may not be defined at some of the grid points $0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1$ in the case of C^0 smoothness ($k = 0$). In (6.21) it is assumed that the undefined values are excluded from the summation.

- All the B-splines, except for the first $k+1$ and the last $k+1$, are uniformly shifted-scaled versions of $p-k$ fixed reference functions $\beta_{1,[p,k]}, \ldots, \beta_{p-k,[p,k]}$, namely the first $p-k$ B-splines defined on the reference knot sequence

$$\underbrace{0, \ldots, 0}_{p-k}, \underbrace{1, \ldots, 1}_{p-k}, \ldots, \underbrace{\eta, \ldots, \eta}_{p-k}, \qquad \eta = \left\lceil \frac{p+1}{p-k} \right\rceil.$$

The precise formula we shall need later on is the following: setting

$$\nu = \left\lceil \frac{k+1}{p-k} \right\rceil, \tag{6.22}$$

for the B-splines $B_{k+2,[p,k]}, \ldots, B_{k+1+(n-\nu)(p-k),[p,k]}$ we have

$$\begin{aligned} &B_{k+1+(p-k)(r-1)+q,[p,k]}(x) = \beta_{q,[p,k]}(nx - r + 1), \\ &r = 1, \ldots, n-\nu, \qquad q = 1, \ldots, p-k. \end{aligned} \tag{6.23}$$

We point out that the supports of the reference B-splines $\beta_{q,[p,k]}$ satisfy

$$\mathrm{supp}(\beta_{1,[p,k]}) \subseteq \mathrm{supp}(\beta_{2,[p,k]}) \subseteq \ldots \subseteq \mathrm{supp}(\beta_{p-k,[p,k]}) = [0, \eta]. \tag{6.24}$$

The last property, in particular, shows that all the B-splines, except for the first $k+1$ and the last $k+1$, can be partitioned into subsets of $p-k$ functions, and each subset is a normalized and shifted version of the set reference functions $\beta_{1,[p,k]}, \ldots, \beta_{p-k,[p,k]}$. This naturally leads, up to a constant number of rows and columns, to a Toeplitz $(p-k)$-block structure for the matrix $A_{n,[p,k]}$.

In fact, consider the entry i,j of $A_{n,[p,k]}$ and compare it with entry $i+s(p-k), j+s(p-k)$, where $s>0$ and all the indices $i, j, i+s(p-k), j+s(p-k)$ are in the range $[k+2, n(p-k)]$. In other words, we are comparing the same element in different blocks to check if it is the same. Suppose moreover that

$$i = k+1+(p-k)(r-1)+i', \qquad j = k+1+(p-k)(R-1)+j',$$

with $i', j' \in [1, p-k]$, so that

$$B_i(x) = \beta_{i',[p,k]}(nx - r + 1), \qquad B_j(x) = \beta_{j',[p,k]}(nx - R + 1).$$

We find that

$$\begin{aligned} &(A_{n,[p,k]})_{i,j} \\ &= \int_0^1 B_i'(x) B_j'(x) \mathrm{d}x \\ &= n^2 \int_{\mathbb{R}} \beta'_{i',[p,k]}(nx - r + 1) \beta'_{j',[p,k]}(nx - R + 1) \mathrm{d}x \\ &= n \int_{\mathbb{R}} \beta'_{i',[p,k]}(y) \beta'_{j',[p,k]}(y - R + r) \mathrm{d}y \end{aligned}$$

and

$$\begin{aligned} &(A_{n,[p,k]})_{i+s(p-k),j+s(p-k)} \\ &= \int_0^1 B'_{i+s(p-k)}(x) B'_{j+s(p-k)}(x) \mathrm{d}x \\ &= n^2 \int_{\mathbb{R}} \beta'_{i',[p,k]}(nx - r - s + 1) \beta'_{j',[p,k]}(nx - R - s + 1) \mathrm{d}x \\ &= n \int_{\mathbb{R}} \beta'_{i',[p,k]}(y) \beta'_{j',[p,k]}(y - R + r) \mathrm{d}y. \end{aligned}$$

As a consequence, we can conclude that $(A_{n,[p,k]})_{i,j} = (A_{n,[p,k]})_{i+s(p-k),j+s(p-k)}$ for every i, j, s such that the indices $i, j, i+s(p-k), j+s(p-k)$ are between $k+2$ and $n(p-k)$. It is thus possible to prove that the sequence $\{n^{-1}A_{n,[p,k]}\}_n$ has the same eigenvalues and singular values as a block GLT matrix, that is in particular a banded and symmetric block Toeplitz sequence with symbol

$$\kappa_{[p,k]}(\theta) = \sum_{\ell \in \mathbb{Z}} K^{[\ell]}_{[p,k]} \mathrm{e}^{\mathrm{i}\ell\theta} = K^{[0]}_{[p,k]} + \sum_{\ell > 0} \Big(K^{[\ell]}_{[p,k]} \mathrm{e}^{\mathrm{i}\ell\theta} + (K^{[\ell]}_{[p,k]})^T \mathrm{e}^{-\mathrm{i}\ell\theta} \Big), \tag{6.25}$$

where

$$K^{[\ell]}_{[p,k]} = \left[\int_{\mathbb{R}} \beta'_{j,[p,k]}(t) \beta'_{i,[p,k]}(t-\ell) \mathrm{d}t \right]_{i,j=1}^{p-k}, \qquad \ell \in \mathbb{Z}. \tag{6.26}$$

We conclude that

$$\{n^{-1}A_{n,[p,k]}\}_n \sim_{\sigma,\lambda} \kappa_{[p,k]}(\theta).$$

We refer to [13, 14] for a complete analysis of this applications and all the detailed proofs of the results, altogether with generalizations to convection-diffusion-reaction equations, multidimensional setting, and much more.

6.2.5.1 Periodic Table of Finite Elements

The method used to derive the symbol in paragraph 6.2.5 can be generalized to a number of other FE methods. Consider in fact the following steps.

- Consider a set of g fixed reference functions $\mathscr{B}$, and fix a point P on their common domain.
- Take a regular grid Ξ_n on the domain of definition of the PDE.
- For almost each point $x \in \Xi_n$, consider the composition of a shift $P \mapsto x$ and a dilation with ratio $r(n) = o(1)$, and associate to x the set $\mathscr{B}_x$ of the shifted-scaled versions of $\mathscr{B}$.
- For each n, take $\mathscr{B}_n = \mathscr{F}_n \cup (\cup_{x\in\Xi_n} \mathscr{B}_x)$ as the basis function for the FE method, where $\mathscr{F}_n$ are few additional functions, whose cardinality is at most $\#\mathscr{F}_n = o(\#\mathscr{B}_n)$.

Any FE method that uses a basis function built with these steps, generally leads to discretization matrices admitting a (matrix-valued) symbol.

Noticeable examples of such FE methods are contained in the Periodic Table of the Finite Elements, by Douglas Arnold and Anders Logg ([3]). In the table, it is presented a compact way to enumerate different types of finite elements methods using the notation of homological algebra.

Each cell in the table is uniquely determined by 4 parameters: the general polynomial spaces where we find our finite elements, the dimension of the space (n), the form order considered (k) and the degree of the polynomials (r). From these parameter is possible to come up with a well defined base of functions for the finite elements space.

In particular, when we use a method of the table, every element in the triangulation considered $\mathscr{T}_h$ presents the same degrees of freedom, so a "regular" mesh for the domain lead to basis of functions composed by a finite number of elements $\varphi_1, \dots, \varphi_g$ and their translations, plus a small number of "border" basis function. This naturally leads to a structured matrix, where it is possible to identify as a $g \times g$ blocks Toeplitz structure, plus few additional rows and columns. Moreover, the system turns out to be Hermitian, so we fall in the class of Multilevel Block GLT sequences, for which we can easily compute a spectral and singular value symbol.

6.3 Reduced GLT

When dealing with Linear PDE such as

$$\begin{cases} \mathscr{L}(u)(x) = f(x) & x \in [0,1]^d \\ B.C. & x \in \partial([0,1]^d) \end{cases}$$

the discretization methods often lead to sequences of linear systems admitting a GLT symbol with domain $([0,1] \times [-\pi,\pi])^d$, like we have shown in paragraph 6.1.5, paragraph 6.2.2 and paragraph 6.2.5. Interestingly enough, it has been observed that when we consider a different domain Ω of u that is regular enough, a similar analysis can be conducted.

First of all, we know that there are already well-known cases of linear PDE on non-rectangular domains. For example, in the context of Finite Elements methods with constant coefficients, the domains of the basis functions can be arbitrary since the main focus is on the values of the bilinear form evaluated on couples of basis functions, so the resulting symbols have domain $[-\pi,\pi]^d$. The cases of FE or collocation methods with variable coefficients have been studied on the condition that the physical domain Ω can be described by a global geometry function $G : [0,1] \to \Omega$, which is invertible and satisfies $G(\partial([0,1]^d)) = \partial\Omega$.

Now we want to explore a more general case, starting from a domain Ω with few properties. We require that Ω is contained in $[0,1]^d$, and we work in the restricted euclidean topology and Lebesgue measure ℓ_d of $[0,1]^d$, unless specified differently. We will show that the same arguments that let us build the GLT sequences, also apply here with slight modifications, and lead to a s.a. where the symbol are measurable functions defined on $\Omega \times [-\pi,\pi]^d$. We will call it the algebra of reduced GLT sequences. We will explain briefly all the principal steps that let us define the space, postponing the proofs of the results we present in Appendix F

6.3.1 Characteristic Sequences

We know that every measurable function with support in $([0,1]\times[-\pi,\pi])^d$ is a multilevel GLT symbol for a sequence of matrices. Using this connection, we can associate to each Ω a diagonal sequence $\{D_{\boldsymbol{n}}\}_n$ such that $\{D_{\boldsymbol{n}}\}_n \sim_{GLT} \chi_\Omega$ and $\{D_{\boldsymbol{n}}\}_n$ is a diagonal sequences with binary entries. This is easy to see in the case the characteristic function χ_Ω is continuous almost everywhere, since we know that

$$\{D_{\boldsymbol{n}}(\chi_\Omega)\}_n \sim_{GLT} \chi_\Omega.$$

Let us focus on the case χ_Ω is continuous a.e., that is a condition common to almost every domain used in linear PDE. Given a set Ω, the following assertions are equivalent:

- the function χ_Ω is continuous a.e.,
- the function χ_Ω is Riemann integrable,
- $\mu(\partial\Omega) = 0$,
- the set Ω is Peano-Jordan measurable.

Moreover, every measurable set Ω respecting the condition, is equal, up to a negligible set, to its interior Ω° and to its closure $\overline{\Omega}$. The matrices $D_{\boldsymbol{n}}(\chi_\Omega)$ give us a natural way to link its rows and columns to the points of type $z_{\boldsymbol{i}} := \frac{\boldsymbol{i}}{\boldsymbol{n}}$ with $\boldsymbol{1} \le \boldsymbol{i} \le \boldsymbol{n}$ inside and outside of Ω. The points $z_{\boldsymbol{i}}$ forms an uniform grid on $[0,1]^d$, but in applications the most used grid, denoted as Ξ_n, is composed by points of the form

$$\frac{\boldsymbol{i}}{\boldsymbol{n}+\boldsymbol{1}} = \left(\frac{i_1}{n_1+1}, \frac{i_2}{n_2+1}, \dots, \frac{i_d}{n_d+1}\right), \qquad i_j = 0,1,2,\dots,n_j,n_j+1, \quad j = 1,2,\dots,d.$$

Consequentially we define a new diagonal matrix associated to Ω

$$I_{\boldsymbol{n}}(\chi_\Omega) := \operatorname{diag}\left(\chi_\Omega\left(\frac{\boldsymbol{i}}{\boldsymbol{n}+\boldsymbol{1}}\right)\right)_{\boldsymbol{i}=\boldsymbol{1},\dots,\boldsymbol{n}}$$

that has dimension $N(\boldsymbol{n}) \times N(\boldsymbol{n})$, the same as $D_{\boldsymbol{n}}(\chi_\Omega)$. More in general, for any continuous a.e. function $a : [0,1]^d \to \mathbb{C}$ we denote

$$I_{\boldsymbol{n}}(a) := \operatorname{diag}\left(a\left(\frac{\boldsymbol{i}}{\boldsymbol{n}+\boldsymbol{1}}\right)\right)_{\boldsymbol{i}=\boldsymbol{1},\dots,\boldsymbol{n}}$$

so that $I_{\boldsymbol{n}}(a)$ and $D_{\boldsymbol{n}}(a)$ have the same dimension, and can actually be proved that they are a.c.s. equivalent. The two diagonal sequences $\{I_{\boldsymbol{n}}(\chi_\Omega)\}_n$ and $\{D_{\boldsymbol{n}}(\chi_\Omega)\}_n$ are thus both multilevel GLT sequences and hold essentially the same information about the domain Ω. We adopt the second one since later it will be fundamental to operate on the grid Ξ_n through diagonal matrices. In particular, the quantity

$$d_n^\Omega := \operatorname{rk}(I_{\boldsymbol{n}}(\chi_\Omega))$$

is important since it counts the number of grid points inside Ω, and it converges, asymptotically, to the measure of Ω.

Lemma 6.3.1. *If Ω is a Peano-Jordan measurable set, then*

$$\lim_{n\to\infty} \frac{d_n^\Omega}{N(\boldsymbol{n})} = \ell_d(\Omega).$$

Moreover, we also need that the number of grid points in Ξ_n that are arbitrarily near the border of Ω are negligible when compared with the ones inside Ω. This is useful in the applications, since it lets us ignore the conditions that arise from grid points that are close enough to the boundary.

Lemma 6.3.2. *Call*

$$K_c = \{\, p \in [0,1]^d \mid d(p,\partial\Omega) \le c \,\}$$

the set of points whose distance from $\partial\Omega$ is at most $c \geq 0$. Given a sequence h_n of real nonnegative numbers converging to zero, and a Peano-Jordan measurable set Ω, then

$$\lim_{n\to\infty} \frac{d_n^{K_{h_n}}}{N(\boldsymbol{n})} = 0.$$

In particular, if $\ell_d(\Omega) > 0$, then

$$\lim_{n\to\infty} \frac{d_n^{K_{h_n}}}{d_n^{\Omega}} = 0.$$

One can show that $\partial\Omega$ is also Peano-Jordan measurable and a null-set, so $d_n^{\partial\Omega} = o(d_n^{\Omega}) = o(N(\boldsymbol{n}))$. As a corollary, we can also derive the limits of $d_n^{\overline{\Omega}}(N(\boldsymbol{n}))^{-1}$ and $d_n^{\Omega^\circ}(N(\boldsymbol{n}))^{-1}$, since we know that $\overline{\Omega}$ and Ω° differ from Ω for a negligible set inside $\partial\Omega$.

$$\Omega \cup \partial\Omega = \overline{\Omega} \supseteq \Omega \implies d_n^{\Omega} + d_n^{\partial\Omega} \geq d_n^{\overline{\Omega}} \geq d_n^{\Omega} \implies \lim_{n\to\infty} \frac{d_n^{\overline{\Omega}}}{N(\boldsymbol{n})} = \ell_d(\Omega),$$

$$\Omega \setminus \partial\Omega = \Omega^\circ \subseteq \Omega \implies d_n^{\Omega} - d_n^{\partial\Omega} \leq d_n^{\Omega^\circ} \leq d_n^{\Omega} \implies \lim_{n\to\infty} \frac{d_n^{\Omega^\circ}}{N(\boldsymbol{n})} = \ell_d(\Omega).$$

Notice that Lemma 6.3.1 shows $\lim_{n\to\infty} d_n^{\Omega} = +\infty$ whenever the measure of Ω is not zero, so from now on, we suppose that $\ell_d(\Omega) > 0$.

6.3.2 Restriction and Expansion Operators

In the context of differential equation, we want to zero out, or directly delete, the rows and columns of the resulting matrix linked to points of the grid outside Ω. For this reason, here we define restriction and expansion operators, and we analyse their impact on sequences and symbols.

If we fix a Peano-Jordan measurable set Ω, then we can build the map

$$Z_\Omega : \{A_{\boldsymbol{n}}\}_n \mapsto \{I_{\boldsymbol{n}}(\chi_\Omega) A_{\boldsymbol{n}} I_{\boldsymbol{n}}(\chi_\Omega)\}_n.$$

From now on, we abuse the notation and write $Z_\Omega(A_{\boldsymbol{n}})$ for the matrix $I_{\boldsymbol{n}}(\chi_\Omega) A_{\boldsymbol{n}} I_{\boldsymbol{n}}(\chi_\Omega)$. If we associate each multi-index $\boldsymbol{i}$ in the matrix $A_{\boldsymbol{n}}$ to the point $\frac{\boldsymbol{i}}{\boldsymbol{n}+\boldsymbol{1}} \in \Xi_n$, then Z_Ω sets to zero every row and column corresponding to a point not in Ω. We can delete the zero rows and columns in the matrices, and obtain a matrix with size $d_n^{\Omega} \times d_n^{\Omega}$, through a rectangular matrix $\Pi_{\boldsymbol{n},\Omega}$, so we define the restriction operator

$$R_\Omega : \{A_{\boldsymbol{n}}\}_n \mapsto \{\Pi_{\boldsymbol{n},\Omega} A_{\boldsymbol{n}} (\Pi_{\boldsymbol{n},\Omega})^T\}_n.$$

If we want to invert the map, and add zero rows and columns corresponding to points not belonging to Ω, then we can use the expansion operator

$$E_\Omega : \{S_{\boldsymbol{n}}^{\Omega}\}_n \mapsto \{(\Pi_{\boldsymbol{n},\Omega})^T S_{\boldsymbol{n}}^{\Omega} \Pi_{\boldsymbol{n},\Omega}\}_n.$$

We will use the notation $R_\Omega(A_{\boldsymbol{n}})$ for $\Pi_{\boldsymbol{n},\Omega} A_{\boldsymbol{n}} (\Pi_{\boldsymbol{n},\Omega})^T$ and the notation $E_\Omega(S_{\boldsymbol{n}}^{\Omega})$ for $(\Pi_{\boldsymbol{n},\Omega})^T S_{\boldsymbol{n}}^{\Omega} \Pi_{\boldsymbol{n},\Omega}$. Moreover, unless differently specified, we use the exponent Ω to distinguish the sequences $\{S_{\boldsymbol{n}}^{\Omega}\}_n$ of size $d_n^{\Omega} \times d_n^{\Omega}$ from classical sequences $\{A_{\boldsymbol{n}}\}_n$ of size $N(\boldsymbol{n}) \times N(\boldsymbol{n})$. Note that the operators $E_\Omega, R_\Omega, Z_\Omega$, the matrices $\Pi_{\boldsymbol{n},\Omega}, I_{\boldsymbol{n}}(\chi_\Omega)$ and the quantity d_n^{Ω} can be defined for any measurable set Ω, even if not Peano-Jordan measurable.

The operator $R_\Omega(A)$ always extracts a principal submatrix of A. If we apply it to $Z_\Omega(A)$, though, it deletes only rows and columns that are already zero, so we can easily tell the behaviour of their singular values and eigenvalues. The same argument also apply to S^Ω and $E_\Omega(S^\Omega)$.

Lemma 6.3.3. *There exists a permutation matrix P of size $N(\boldsymbol{n}) \times N(\boldsymbol{n})$ such that for every matrix $A_{\boldsymbol{n}}$ of size $N(\boldsymbol{n}) \times N(\boldsymbol{n})$,*

$$P Z_\Omega(A_{\boldsymbol{n}}) P^T = \begin{pmatrix} R_\Omega(A_{\boldsymbol{n}}) & 0 \\ 0 & 0 \end{pmatrix}.$$

In particular, $Z_\Omega(A_{\boldsymbol{n}})$ has the same eigenvalues and singular values of the matrix $R_\Omega(A_{\boldsymbol{n}})$ except for $N(\boldsymbol{n}) - d_n^{\Omega}$ null eigenvalues and singular values.

Corollary 6.3.4. *There exists a permutation matrix P of size $N(\boldsymbol{n}) \times N(\boldsymbol{n})$ such that for every matrix $S^\Omega_{\boldsymbol{n}}$ of size $d^\Omega_n \times d^\Omega_n$,*

$$PE_\Omega(S^\Omega_{\boldsymbol{n}})P^T = \begin{pmatrix} S^\Omega_{\boldsymbol{n}} & 0 \\ 0 & 0 \end{pmatrix}.$$

In particular, $E_\Omega(S^\Omega_{\boldsymbol{n}})$ has the same eigenvalues and singular values of the matrix $S^\Omega_{\boldsymbol{n}}$ except for $N(\boldsymbol{n}) - d^\Omega_n$ null eigenvalues and singular values.

The operator E_Ω only adds zero eigenvalues and singular values, so it is easy to analyse how it affects the symbol of a sequence. On the other hand, we know how R_Ω acts on the singular values and eigenvalues only for sequences that are in the image of Z_Ω. What we can show is that if the symbol κ takes value zero on a space of (normalized) measure at least $\ell_d(\Omega)$, then the restriction operator acts as a restriction map also on the symbol, leaving out from the domain a space where $\kappa \equiv 0$.

Lemma 6.3.5. *Let $\{A_{\boldsymbol{n}}\}_n$ be a sequence with $A_{\boldsymbol{n}}$ of size $N(\boldsymbol{n}) \times N(\boldsymbol{n})$ that is a fixed point for the operator Z_Ω, and let $\kappa : [0,1]^d \times [-\pi, \pi]^d \to \mathbb{C}$ be a measurable function with $\kappa(x,\theta)|_{x \notin \Omega} = 0$. If $\{A_{\boldsymbol{n}}\}_n \sim_\sigma k$, then*

$$R_\Omega(\{A_{\boldsymbol{n}}\}_n) \sim_\sigma \kappa(x,\theta)|_{x\in\Omega}.$$

If $\{A_{\boldsymbol{n}}\}_n \sim_\lambda \kappa$, then

$$R_\Omega(\{A_{\boldsymbol{n}}\}_n) \sim_\lambda \kappa(x,\theta)|_{x\in\Omega}.$$

Lemma 6.3.6. *Let $\{S^\Omega_{\boldsymbol{n}}\}_n$ be a sequence with $S^\Omega_{\boldsymbol{n}}$ of size $d^\Omega_n \times d^\Omega_n$, let $\kappa : \Omega \times [-\pi, \pi]^d \to \mathbb{C}$ be a measurable function, and define*

$$\kappa'(x,\theta) = \begin{cases} \kappa(x,\theta), & x \in \Omega, \\ 0, & x \in [0,1]^d \setminus \Omega. \end{cases}$$

If $\{S^\Omega_{\boldsymbol{n}}\}_n \sim_\sigma k$, then

$$E_\Omega(\{S^\Omega_{\boldsymbol{n}}\}_n) \sim_\sigma \kappa'(x,\theta).$$

If $\{S^\Omega_{\boldsymbol{n}}\}_n \sim_\lambda \kappa$, then

$$E_\Omega(\{S^\Omega_{\boldsymbol{n}}\}_n) \sim_\lambda \kappa'(x,\theta).$$

When we apply the operator E_Ω, we are just adding zero columns and rows, so it is not surprising to discover that the extension operator preserves the a.c.s. convergence. On the other hand, R_Ω extracts a principal submatrix, so it is not clear whether it complies with the a.c.s. distance, but the Cauchy interlacing theorem for singular values Theorem 3.1.1 is enough to prove that it happens anyway.

Lemma 6.3.7. *Given two sequences $\{A_{\boldsymbol{n}}\}_n$ and $\{B_{\boldsymbol{n}}\}_n$ with matrices of size $N(\boldsymbol{n}) \times N(\boldsymbol{n})$,*

$$d_{a.c.s.}\left(\{A_{\boldsymbol{n}}\}_n, \{B_{\boldsymbol{n}}\}_n\right) \geq \ell_d(\Omega) d_{a.c.s.}\left(R_\Omega(\{A_{\boldsymbol{n}}\}_n), R_\Omega(\{B_{\boldsymbol{n}}\}_n)\right).$$

In particular,

$$\{B_{\boldsymbol{n},m}\}_n \xrightarrow{a.c.s.} \{A_{\boldsymbol{n}}\}_n \implies R_\Omega(\{B_{\boldsymbol{n},m}\}_n) \xrightarrow{a.c.s.} R_\Omega(\{A_{\boldsymbol{n}}\}_n).$$

Lemma 6.3.8. *Given two sequences $\{A^\Omega_{\boldsymbol{n}}\}_n$ and $\{B^\Omega_{\boldsymbol{n}}\}_n$ with matrices of size $d^\Omega_n \times d^\Omega_n$,*

$$d_{a.c.s.}\left(\{A^\Omega_{\boldsymbol{n}}\}_n, \{B^\Omega_{\boldsymbol{n}}\}_n\right) \geq d_{a.c.s.}\left(E_\Omega(\{A^\Omega_{\boldsymbol{n}}\}_n), E_\Omega(\{B^\Omega_{\boldsymbol{n}}\}_n)\right) \geq \ell_d(\Omega) d_{a.c.s.}\left(\{A^\Omega_{\boldsymbol{n}}\}_n, \{B^\Omega_{\boldsymbol{n}}\}_n\right).$$

In particular,

$$\{B^\Omega_{\boldsymbol{n},m}\}_n \xrightarrow{a.c.s.} \{A^\Omega_{\boldsymbol{n}}\}_n \iff E_\Omega(\{B^\Omega_{\boldsymbol{n},m}\}_n) \xrightarrow{a.c.s.} E_\Omega(\{A^\Omega_{\boldsymbol{n}}\}_n).$$

To conclude the paragraph, we report two more results that comes in handy for real applications.

The first proposition, makes use again of the Cauchy interlacing theorem for singular values Theorem 3.1.1 to show that the restriction operator has norm less then one, since it extracts a principal minor from the matrices.

Lemma 6.3.9. *For every $1 \le p \le \infty$,*

$$\|R_\Omega(A)\|_p \le \|A\|_p.$$

As a last result, we can show that if we pick a grid that is slightly different from Ξ_n, we still recover similar results. Remember that the symmetric difference $\triangle$ between two sets is the set of elements belonging to only one of the two sets. In symbols, $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

Lemma 6.3.10. *Let Γ_n be a measurable set in $[0,1]^d$ (not necessarily Peano-Jordan measurable) and let Ω be a Peano-Jordan measurable set with positive measure in $[0,1]^d$. Suppose that*

$$d_n^{\Omega \triangle \Gamma_n} = o(N(\boldsymbol{n})).$$

Given a sequence $\{A_{\boldsymbol{n}}\}_n$ with $A_{\boldsymbol{n}}$ of size $N(\boldsymbol{n}) \times N(\boldsymbol{n})$, and a measurable function k, we have that

$$R_\Omega(\{A_{\boldsymbol{n}}\}_n) \sim_\sigma k \iff \{R_{\Gamma_n}(A_{\boldsymbol{n}})\}_n \sim_\sigma k.$$

Moreover, if $A_{\boldsymbol{n}}$ are Hermitian, then

$$R_\Omega(\{A_{\boldsymbol{n}}\}_n) \sim_\lambda k \iff \{R_{\Gamma_n}(A_{\boldsymbol{n}})\}_n \sim_\lambda k.$$

This result is quite powerful since it tells us that we can add and remove a small number of rows and columns without changing the symbol of the sequence. It will be useful in applications when dealing with near-boundary conditions.

6.3.3 Definition and Properties of Reduced GLT Sequences

In the following propositions, we denote the image of R_Ω when applied to GLT sequences as $\mathscr{G}_d^\Omega := R_\Omega(\mathscr{G}_d)$, and we call it the space of *Reduced GLT* with respect to Ω.

Lemma 6.3.11. *Given a GLT sequence $\{A_{\boldsymbol{n}}\}_n \sim_{GLT} k(x,\theta)$ with $k : [0,1]^d \times [-\pi,\pi]^d \to \mathbb{C}$, then*

$$R_\Omega(\{A_{\boldsymbol{n}}\}_n) \sim_\sigma k(x,\theta)|_{x\in\Omega}.$$

If $A_{\boldsymbol{n}}$ are also Hermitian matrices, then

$$R_\Omega(\{A_{\boldsymbol{n}}\}_n) \sim_\lambda k(x,\theta)|_{x\in\Omega}.$$

Notice that the map R_Ω is not injective, but one can prove that all the GLT sequences with the same image have symbols that coincide on $\Omega \times [-\pi,\pi]$.

Lemma 6.3.12. *Given $\{A_{\boldsymbol{n}}\}_n \sim_{GLT} k$, $\{B_{\boldsymbol{n}}\}_n \sim_{GLT} h$ such that $R_\Omega(\{A_{\boldsymbol{n}}\}_n) = R_\Omega(\{B_{\boldsymbol{n}}\}_n) = \{S_{\boldsymbol{n}}^\Omega\}_n \in \mathscr{G}_d^\Omega$, the symbols k, h coincide on $\Omega \times [-\pi,\pi]^d$.*

As a corollary, every GLT sequence mapped into $\{S_{\boldsymbol{n}}^\Omega\}_n$ possesses a symbol with a fixed value on $\Omega \times [-\pi,\pi]^d$, so we can associate to each reduced GLT sequence $\{S_{\boldsymbol{n}}^\Omega\}_n$ an unique symbol, called *Reduced GLT Symbol*, obtained as the restriction of any GLT symbol of the sequences in the counter-image $R_\Omega^{-1}(\{S_{\boldsymbol{n}}^\Omega\}_n) \cap \mathscr{G}_d$. From now on, we will use the notation $\{S_{\boldsymbol{n}}^\Omega\}_n \sim_{GLT}^\Omega s$ to indicate that $s : \Omega \times [-\pi,\pi]^d \to \mathbb{C}$ is the restriction of a symbol $k : [0,1]^d \times [-\pi,\pi]^d \to \mathbb{C}$ such that $\{A_{\boldsymbol{n}}\}_n \sim_{GLT} k$ and $R_\Omega(\{A_{\boldsymbol{n}}\}_n) = \{S_{\boldsymbol{n}}^\Omega\}_n$.

Given any reduced GLT sequence $\{S_{\boldsymbol{n}}^\Omega\}_n$, it is easy to produce a GLT sequence $\{A_{\boldsymbol{n}}\}_n$ such that $R_\Omega(\{A_{\boldsymbol{n}}\}_n) = \{S_{\boldsymbol{n}}^\Omega\}_n$ using the operator E_Ω. We can thus reverse Lemma 6.3.11.

Lemma 6.3.13. *If $\{S_{\boldsymbol{n}}^\Omega\}_n \sim_{GLT}^\Omega \kappa$, then $E_\Omega(\{S_{\boldsymbol{n}}^\Omega\}_n) \sim_{GLT} k(x,\theta) = \begin{cases} \kappa(x,\theta) & x \in \Omega, \\ 0 & x \notin \Omega. \end{cases}$ and $R_\Omega(E_\Omega(\{S_{\boldsymbol{n}}^\Omega\}_n)) = \{S_{\boldsymbol{n}}^\Omega\}_n$.*

Using the connection between $\mathscr{G}_d$ and $\mathscr{G}_d^\Omega$, we can prove that many properties of the first space transfer to the second.

Theorem 6.3.14. *Suppose $\{A_{\boldsymbol{n}}^\Omega\}_n$, $\{B_{\boldsymbol{n}}^\Omega\}_n$ are reduced GLT sequences and $\{X_{\boldsymbol{n}}^\Omega\}_n, \{Y_{\boldsymbol{n}}^\Omega\}_n$ are sequences with $X_{\boldsymbol{n}}^\Omega, Y_{\boldsymbol{n}}^\Omega \in \mathbb{C}^{d_n^\Omega \times d_n^\Omega}$.*

GLT 1. *If $\{A_{\boldsymbol{n}}^\Omega\}_n \sim_{GLT}^\Omega \kappa$ then $\{A_{\boldsymbol{n}}^\Omega\}_n \sim_\sigma \kappa$. If $\{A_{\boldsymbol{n}}^\Omega\}_n \sim_{GLT}^\Omega \kappa$ and each $A_{\boldsymbol{n}}^\Omega$ is Hermitian then $\{A_{\boldsymbol{n}}^\Omega\}_n \sim_\lambda \kappa$.*

GLT 2. *If $\{A_{\boldsymbol{n}}^\Omega\}_n \sim_{GLT}^\Omega \kappa$ and $\{A_{\boldsymbol{n}}^\Omega\}_n = \{X_{\boldsymbol{n}}^\Omega\}_n + \{Y_{\boldsymbol{n}}^\Omega\}_n$, where*

- *every $X_{\boldsymbol{n}}^\Omega$ is Hermitian,*
- $(d_n^\Omega)^{-1}\|Y_{\boldsymbol{n}}^\Omega\|_2^2 \to 0$,

then $\{A_{\boldsymbol{n}}^\Omega\}_n \sim_\lambda \kappa$.

GLT 3. *Here we list three important examples of reduced GLT sequences.*

- *Given a function $f \in L^1([-\pi,\pi]^d)$, its associated Toeplitz sequence is $\{T_{\boldsymbol{n}}^\Omega(f)\}_n = R_\Omega(\{T_{\boldsymbol{n}}(f)\}_n)$, where the elements are multidimensional Fourier coefficients of f:*

$$T_{\boldsymbol{n}}(f) = [f_{\boldsymbol{i}-\boldsymbol{j}}]_{\boldsymbol{i},\boldsymbol{j}=\boldsymbol{1}}^{\boldsymbol{n}}, \qquad f_{\boldsymbol{k}} = \frac{1}{(2\pi)^d}\int_{-\pi}^{\pi} f(\theta)e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\theta}}\mathrm{d}\theta.$$

  $\{T_{\boldsymbol{n}}^\Omega(f)\}_n$ is a reduced GLT sequence with symbol $\kappa(x,\theta) = f(\theta)$.

- *Given an almost everywhere continuous function, $\widetilde{a} : [0,1]^d \to \mathbb{C}$ and its restriction $a = \widetilde{a}|_\Omega$, its associated diagonal sampling sequence $\{D_{\boldsymbol{n}}^\Omega(a)\}_n$ is defined as*

$$D_{\boldsymbol{n}}^\Omega(a) = \mathrm{diag}\left(\left\{a\left(\frac{\phi(i)}{\boldsymbol{n}+\boldsymbol{1}}\right)\right\}_{i=1}^{d_n^\Omega}\right).$$

  $\{D_{\boldsymbol{n}}^\Omega(a)\}_n$ is a reduced GLT sequence with symbol $\kappa(x,\theta) = a(x)$.

- *Any zero-distributed sequence $\{Y_{\boldsymbol{n}}^\Omega\}_n \sim_\sigma 0$ is a reduced GLT sequence with symbol $\kappa(x,\theta) = 0$.*

GLT 4. *If $\{A_{\boldsymbol{n}}^\Omega\}_n \sim_{GLT}^\Omega \kappa$ and $\{B_{\boldsymbol{n}}^\Omega\}_n \sim_{GLT}^\Omega \xi$, then*

- *$\{(A_{\boldsymbol{n}}^\Omega)^H\}_n \sim_{GLT}^\Omega \overline{\kappa}$, where $(A_{\boldsymbol{n}}^\Omega)^H$ is the conjugate transpose of $A_{\boldsymbol{n}}^\Omega$,*
- *$\{\alpha A_{\boldsymbol{n}}^\Omega + \beta B_{\boldsymbol{n}}^\Omega\}_n \sim_{GLT}^\Omega \alpha\kappa + \beta\xi$ for all $\alpha, \beta \in \mathbb{C}$,*
- *$\{A_{\boldsymbol{n}}^\Omega B_{\boldsymbol{n}}^\Omega\}_n \sim_{GLT}^\Omega \kappa\xi$.*

GLT 5. *If $\{A_{\boldsymbol{n}}^\Omega\}_n \sim_{GLT}^\Omega \kappa$ and $\kappa \neq 0$ a.e., then $\{(A_{\boldsymbol{n}}^\Omega)^\dagger\}_n \sim_{GLT}^\Omega \kappa^{-1}$, where $(A_{\boldsymbol{n}}^\Omega)^\dagger$ is the Moore–Penrose pseudoinverse of $A_{\boldsymbol{n}}^\Omega$.*

GLT 6. *If $\{A_{\boldsymbol{n}}^\Omega\}_n \sim_{GLT}^\Omega \kappa$ and each $A_{\boldsymbol{n}}^\Omega$ is Hermitian, then $\{f(A_{\boldsymbol{n}}^\Omega)\}_n \sim_{GLT}^\Omega f(\kappa)$ for all continuous functions $f : \mathbb{C} \to \mathbb{C}$.*

GLT 7. *$\{A_{\boldsymbol{n}}^\Omega\}_n \sim_{GLT}^\Omega \kappa$ if and only if there exist GLT sequences $\{B_{\boldsymbol{n},m}\}_n \sim_{GLT}^\Omega \kappa_m$ such that κ_m converges to κ in measure and $\{B_{\boldsymbol{n},m}\}_n \xrightarrow{\text{a.c.s.}} \{A_{\boldsymbol{n}}^\Omega\}_n$ as $m \to \infty$.*

GLT 8. *Suppose $\{A_{\boldsymbol{n}}^\Omega\}_n \sim_{GLT}^\Omega \kappa$ and $\{B_{\boldsymbol{n},m}^\Omega\}_n \sim_{GLT}^\Omega \kappa_m$, where both $A_{\boldsymbol{n}}^\Omega$ and $B_{\boldsymbol{n},m}^\Omega$ have the same size $d_n^\Omega \times d_n^\Omega$. Then, $\{B_{\boldsymbol{n},m}^\Omega\}_n \xrightarrow{\text{a.c.s.}} \{A_{\boldsymbol{n}}^\Omega\}_n$ as $m \to \infty$ if and only if κ_m converges to κ in measure.*

GLT 9. *If $\{A_{\boldsymbol{n}}^\Omega\}_n \sim_{GLT}^\Omega \kappa$ then there exist functions $a_{i,m}, f_{i,m}$, $i = 1,\dots,N_m$, such that*

- *$a_{i,m} \in C^\infty(\Omega)$ and $f_{i,m}$ is a trigonometric polynomial,*
- *$\sum_{i=1}^{N_m} a_{i,m}(x)f_{i,m}(\theta)$ converges to $\kappa(x,\theta)$ a.e.,*
- *$\left\{\sum_{i=1}^{N_m} D_{\boldsymbol{n}}^\Omega(a_{i,m})T_{\boldsymbol{n}}^\Omega(f_{i,m})\right\}_n \xrightarrow{\text{a.c.s.}} \{A_{\boldsymbol{n}}^\Omega\}_n$ as $m \to \infty$.*

Notice moreover that $\mathscr{G}_d^\Omega$ defined as the set

$$\{\, (\{A_{\boldsymbol{n}}^\Omega\}_n, \kappa) \mid \{A_{\boldsymbol{n}}^\Omega\}_n \sim_{GLT}^\Omega \kappa \,\}$$

is an s.a., it is closed, and every function in $\mathscr{M}_{\Omega\times[-\pi,\pi]^d}$ is a symbol for it. As a consequence, all the results in section 5.3 still hold.

Theorem 6.3.15. *$\mathscr{G}_d^\Omega$ is a closed and maximal s.a., and if we quotient the space by the zero-distributed s.a. $\mathscr{Z}\times\{0\}$, it is isomorphic and isometrically equivalent to $\mathscr{M}_{\Omega\times[-\pi,\pi]^d}$.*

Part III

Applications

Chapter 7

Convection-Diffusion-Reaction Differential Equation on Different Domains

The theory regarding symbols, and especially the theory of GLT sequences, was devised in order to solve the problem of analysing the spectral distribution of matrices arising from the numerical discretization of Differential Equations. A final goal of this spectral analysis is the design of efficient numerical methods for computing the related numerical solutions.

The convection-diffusion-reaction (cdr) differential equation with variable coefficients is maybe the problem that has been analysed the most with spectral techniques. Its most generic version read as follows.

$$\begin{cases} -\displaystyle\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\Big(a_{j,i}\frac{\partial u}{\partial x_i}\Big) + \sum_{i=1}^{d} b_i \frac{\partial u}{\partial x_i} + cu = f, & \text{in } \Omega^\circ, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{7.1}$$

Here $a_{i,j}, b_i, c$ are given functions, and $\Omega \subseteq [0,1]^d \subseteq \mathbb{R}^d$ is a Peano-Jordan set. In (7.1), we can actually change the boundary condition to a generic Dirichlet condition $u = g$ or to a Neumann type condition $\partial u/\partial n = g$, without affecting significantly the analysis we will perform, so from now on we will omit it.

In paragraph 1.2.1, paragraph 6.1.5, paragraph 6.2.2 and paragraph 6.2.5 we have already analysed several particular cases of the problem (7.1), each with a different discretization method. In [52, 53, 13, 14] and references within, it is possible to find countless other analyses referred to the same problem, but to other discretization methods (Galerkin, collocation, IgA, etc.). Almost all the examples cited, though, suppose that $\Omega = [0,1]^d$, so here we will deal with more general domains, and use principally the machinery of reduced GLT sequences introduced in section 6.3.

7.1 Shortley-Weller Approximation

Consider a linear partial differential equation

$$\mathscr{L}(u)(x) = b(x) \qquad x \in \Omega^\circ$$

equipped with some boundary conditions (Dirichlet, Neumann, etc.) when $x \in \partial\Omega$. Suppose that $\Omega \subseteq [0,1]^d$ is a closed Peano-Jordan measurable set and b is a function defined over Ω.

We can try to discretize the equation by considering the d-dimensional grid Ξ_n over $[0,1]^d$ and by applying a Finite Difference method only on the points of the grid inside Ω. Notice that the union of Ξ_n for every n is the set $\mathbb{Q}^d \cap [0,1]^d$, that is dense in $[0,1]^d$, and consequently even the set

$$\bigcup_{n\in\mathbb{N}} (\Xi_n \cap \Omega^\circ) = \mathbb{Q}^d \cap [0,1]^d \cap \Omega^\circ$$

is dense in Ω°. The grids are hence bound to describe well the interior of Ω, but the same cannot be said about the border. In fact, it may happen that

$$\mathbb{Q}^d \cap \delta\Omega = \emptyset,$$

meaning that no point from Ξ_n belongs to $\partial\Omega$, hence the discretization does not take in account the boundary conditions of the problem. When one cannot build regular grids whose points on the border are dense, he has

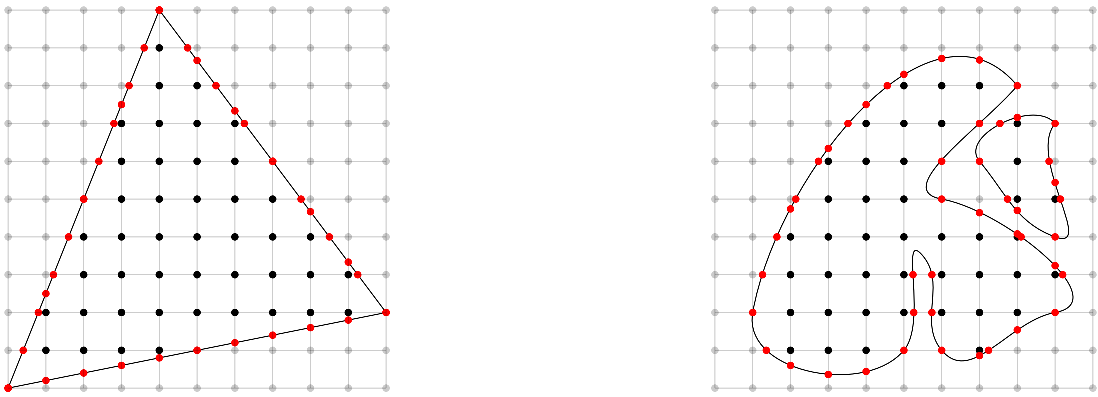


Figure 7.1: Points of the grid Ξ_n over two different domains Ω. The points in Ξ_n' are black, and their neighbours on the boundary are red.

to use non-regular grids shaped accordingly to the boundary, like the ones that arise from the *Shortley-Weller Approximation* for a convection-diffusion-reaction linear PDE.

7.1.1 CDR PDE

Let us consider the problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{d} \frac{\partial}{\partial x_i}\left(a_i \frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{d} b_i \frac{\partial u}{\partial x_i} + cu = f, & \text{in } \Omega^\circ, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{7.2}$$

where a_i, b_i, c and f are given real-valued continuous functions defined on Ω and $a_i \in C^1(\Omega)$. Moreover, suppose that Ω is a closed Peano-Jordan measurable set inside $[0,1]^d$ with positive measure. We set $\boldsymbol{h} = \frac{\mathbf{1}}{\boldsymbol{n}+\mathbf{1}}$, so that $x_{\boldsymbol{j}} = \boldsymbol{jh}$ for $\boldsymbol{j} = \mathbf{0}, \dots, \boldsymbol{n}+\mathbf{1}$ are the points of the grid Ξ_n. It is also natural to assume that $\boldsymbol{n}+\mathbf{1} = n\boldsymbol{c}$, where $\boldsymbol{c}$ is a vector of rational constants. Let $\mathbf{e}_i$ be the vectors of the canonical basis of $\mathbb{R}^d$ and notice that $x_{\boldsymbol{j}} + sh_i\boldsymbol{e}_i = x_{\boldsymbol{j}+s\boldsymbol{e}_i}$. Then, for $\boldsymbol{j} = \mathbf{1}, \dots, \boldsymbol{n}$, we try to approximate the terms appearing in (7.2) according to the classical central FD discretizations on $[0,1]^d$ as follows:

$$\begin{aligned} \frac{\partial}{\partial x_i}\left(a_i \frac{\partial u}{\partial x_i}\right)\Big|_{x=x_{\boldsymbol{j}}} &\approx \frac{a_i \frac{\partial u}{\partial x_i}(x_{\boldsymbol{j}+\mathbf{e}_i/2}) - a_i \frac{\partial u}{\partial x_i}(x_{\boldsymbol{j}-\mathbf{e}_i/2})}{h_i} \\ &\approx a_i(x_{\boldsymbol{j}+\mathbf{e}_i/2}) \frac{u(x_{\boldsymbol{j}+\mathbf{e}_i}) - u(x_{\boldsymbol{j}})}{h_i^2} - a_i(x_{\boldsymbol{j}-\mathbf{e}_i/2}) \frac{u(x_{\boldsymbol{j}}) - u(x_{\boldsymbol{j}-\mathbf{e}_i})}{h_i^2} \end{aligned} \tag{7.3}$$

$$b_i \frac{\partial u}{\partial x_i}\Big|_{x=x_{\boldsymbol{j}}} \approx b_i(x_{\boldsymbol{j}}) \frac{u(x_{\boldsymbol{j}+\mathbf{e}_i}) - u(x_{\boldsymbol{j}-\mathbf{e}_i})}{2h_i}, \tag{7.4}$$

$$cu|_{x=x_{\boldsymbol{j}}} = c(x_{\boldsymbol{j}})u(x_{\boldsymbol{j}}), \tag{7.5}$$

for $i = 1, \dots, d$. This approach requires that all the segments connecting the points $x_{\boldsymbol{j}}$ with $\boldsymbol{j} = \mathbf{1}, \dots, \boldsymbol{n}$, to their neighbours $x_{\boldsymbol{j}\pm\boldsymbol{e}_i}$ still lie inside the domain of the problem. It always happens if the domain is $[0,1]^d$, but when we consider Ω, we need to modify the scheme by adding some points. In particular, we define a new set of neighbours for every point in $\Xi_n' := \Omega^\circ \cap \Xi_n$. Given $x_{\boldsymbol{j}} \in \Xi_n'$ and a direction $\boldsymbol{e}_i$, we can set the numbers $s_i^+(\boldsymbol{j}), s_i^-(\boldsymbol{j})$ as

$$s_i^{\pm}(\boldsymbol{j}) = \sup\{\, t \in [0,1] \mid x_{\boldsymbol{j}} \pm rh_i\boldsymbol{e}_i \in \Omega^\circ \quad \forall\, 0 \le r \le t \,\}$$

that is the size of the biggest connected line contained in the segment connecting $x_{\boldsymbol{j}}$ to $x_{\boldsymbol{j}+\boldsymbol{e}_i}$ and containing $x_{\boldsymbol{j}}$. We can thus call $x_{\boldsymbol{j}} + s_i^{\pm}(\boldsymbol{j})h_i\boldsymbol{e}_i = x_{\boldsymbol{j}+s_i^{\pm}(\boldsymbol{j})\boldsymbol{e}_i}$ the right/left neighbour of $x_{\boldsymbol{j}}$ along the direction $\boldsymbol{e}_i$. The values $s_i^{\pm}(\boldsymbol{j})$ depend on the point $x_{\boldsymbol{j}}$, but when it is evident, we can omit the index and write simply $s_i^{\pm}$.

As we can see in Figure 7.1, even if $x_{\boldsymbol{j}}$ and $x_{\boldsymbol{j}+\boldsymbol{e}_i}$ belong to Ξ_n', it doesn't mean that $s_i^+(\boldsymbol{j}) = 1$, because the segment connecting $x_{\boldsymbol{j}}$ to $x_{\boldsymbol{j}+\boldsymbol{e}_i}$ may not be contained entirely in Ω° (this happens often, for example, when Ω is not convex).

Notice that every neighbour is a point of Ω, so when one of the neighbours is not included in Ξ'_n, it surely belongs the boundary $\partial\Omega$, and in any case we have $s_i^\pm > 0$. Adding these boundary points to Ξ'_n, we obtain the discretization grid Ξ_n^Ω over Ω, and we can rewrite the formulas (7.3)-(7.5) for $x_{\boldsymbol{j}} \in \Xi'_n$ as

$$\begin{aligned}
\frac{\partial}{\partial x_i}\left(a_i\frac{\partial u}{\partial x_i}\right)\Big|_{x=x_{\boldsymbol{j}}} &\approx \frac{a_i\frac{\partial u}{\partial x_i}(x_{\boldsymbol{j}+s_i^+\mathbf{e}_i/2}) - a_i\frac{\partial u}{\partial x_i}(x_{\boldsymbol{j}-s_i^-\mathbf{e}_i/2})}{\frac{1}{2}(s_i^+ + s_i^-)h_i} \\
&\approx a_i(x_{\boldsymbol{j}+s_i^+\mathbf{e}_i/2})\frac{u(x_{\boldsymbol{j}+s_i^+\mathbf{e}_i}) - u(x_{\boldsymbol{j}})}{\frac{1}{2}s_i^+(s_i^+ + s_i^-)h_i^2} - a_i(x_{\boldsymbol{j}-s_i^-\mathbf{e}_i/2})\frac{u(x_{\boldsymbol{j}}) - u(x_{\boldsymbol{j}-s_i^-\mathbf{e}_i})}{\frac{1}{2}s_i^-(s_i^+ + s_i^-)h_i^2} &(7.6)\\
b_i\frac{\partial u}{\partial x_i}\Big|_{x=x_{\boldsymbol{j}}} &\approx b_i(x_{\boldsymbol{j}})\frac{u(x_{\boldsymbol{j}+s_i^+\mathbf{e}_i}) - u(x_{\boldsymbol{j}-s_i^-\mathbf{e}_i})}{(s_i^+ + s_i^-)h_i}, &(7.7)\\
cu|_{x=x_{\boldsymbol{j}}} &= c(x_{\boldsymbol{j}})u(x_{\boldsymbol{j}}), &(7.8)
\end{aligned}$$

called the difference scheme of *Shortley and Weller* [81]. Notice that when $s_j^\pm = 1$ for every j and sign $\pm$, we fall again in the classical scheme of central differences.

The evaluations $u(x_{\boldsymbol{j}})$ of the solution at the grid points $x_{\boldsymbol{j}} \in \Xi_n^\Omega$ are approximated by the values $u_{\boldsymbol{j}}$, where $u_{\boldsymbol{j}} = 0$ for $x_{\boldsymbol{j}} \in \partial\Omega$, and the vector $\boldsymbol{u} = (u_{\boldsymbol{j}})^T_{x_{\boldsymbol{j}}\in\Omega^\circ}$ is the solution of the linear system

$$\begin{aligned}
&-\sum_{i=1}^d a_i(x_{\boldsymbol{j}+s_i^+\mathbf{e}_i/2})\frac{u_{\boldsymbol{j}+s_i^+\mathbf{e}_i} - u_{\boldsymbol{j}}}{\frac{1}{2}s_i^+(s_i^+ + s_i^-)h_i^2} - a_i(x_{\boldsymbol{j}-s_i^-\mathbf{e}_i/2})\frac{u_{\boldsymbol{j}} - u_{\boldsymbol{j}-s_i^-\mathbf{e}_i}}{\frac{1}{2}s_i^-(s_i^+ + s_i^-)h_i^2} \\
&+\sum_{i=1}^d b_i(x_{\boldsymbol{j}})\frac{u_{\boldsymbol{j}+s_i^+\mathbf{e}_i} - u_{\boldsymbol{j}-s_i^-\mathbf{e}_i}}{(s_i^+ + s_i^-)h_i} + c(x_{\boldsymbol{j}})u_{\boldsymbol{j}} = f(x_{\boldsymbol{j}}), \qquad \boldsymbol{j} : x_{\boldsymbol{j}} \in \Omega^\circ. \qquad (7.9)
\end{aligned}$$

If we order the indices $\boldsymbol{j}$ in Ξ'_n by lexicographic order, then we can write the system in compact form as

$$A_{\boldsymbol{n}}^{\Omega^\circ}\boldsymbol{u} = \boldsymbol{f},$$

where $A_{\boldsymbol{n}}^{\Omega^\circ} \in \mathbb{C}^{s_n^{\Omega^\circ}\times s_n^{\Omega^\circ}}$ and $\boldsymbol{f} \in \mathbb{C}^{s_n^{\Omega^\circ}}$. The coefficients are

$$(A_{\boldsymbol{n}}^{\Omega^\circ})_{\boldsymbol{j},\boldsymbol{i}} = \begin{cases} \sum_{i=1}^d \left[\frac{a_i(x_{\boldsymbol{j}+s_i^+\mathbf{e}_i/2})}{\frac{1}{2}s_i^+(s_i^++s_i^-)h_i^2} + \frac{a_i(x_{\boldsymbol{j}-s_i^-\mathbf{e}_i/2})}{\frac{1}{2}s_i^-(s_i^++s_i^-)h_i^2}\right] + c(x_{\boldsymbol{j}}), & \boldsymbol{i} = \boldsymbol{j}, \\ \frac{-a_i(x_{\boldsymbol{j}\pm\mathbf{e}_i/2})}{\frac{1}{2}(s_i^++s_i^-)h_i^2} + \frac{\pm b_i(x_{\boldsymbol{j}})}{(s_i^++s_i^-)h_i}, & \boldsymbol{i} = \boldsymbol{j} \pm \boldsymbol{e}_i, s_i^\pm = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Notice that one can rewrite the non-zero off-diagonal coefficients as

$$(A_{\boldsymbol{n}}^{\Omega^\circ})_{\boldsymbol{j},\boldsymbol{j}\pm e_i} = \frac{-a_i(x_{\boldsymbol{j}\pm\mathbf{e}_i/2})}{\frac{1}{2}(s_i^+ + s_i^-)h_i^2} + \frac{\pm b_i(x_{\boldsymbol{j}})}{(s_i^+ + s_i^-)h_i} = \frac{2}{s_i^+ + s_i^-}\left(\frac{-a_i(x_{\boldsymbol{j}\pm\mathbf{e}_i/2})}{h_i^2} + \frac{\pm b_i(x_{\boldsymbol{j}})}{2h_i}\right).$$

7.1.2 Spectral Analysis

As already noted, if all $s_i^\pm$ are equal to 1, then the relations (7.6)-(7.8) reduces to the classic finite difference scheme (7.3)-(7.5), so we may ask how many are the points $x_{\boldsymbol{j}} \in \Omega^\circ$ such that one of the $s_i^\pm$ is not equal to 1. By the definition of $s_i^\pm$, this is equivalent to say that the segment $(x_{\boldsymbol{j}} - h_i\boldsymbol{e}_i, x_{\boldsymbol{j}} + h_i\boldsymbol{e}_i)$ does not lie completely inside Ω°. In the next result, we will prove that given any positive integer number k, the number of points $x_{\boldsymbol{j}} \in \Xi'_n$ for which there exists a direction $\boldsymbol{e}_i$ such that $(x_{\boldsymbol{j}} - kh_i\boldsymbol{e}_i, x_{\boldsymbol{j}} + kh_i\boldsymbol{e}_i)$ does not lie completely inside Ω° is negligible when compared with the number of points in Ξ'_n.

Lemma 7.1.1. *Let*

$$D(n,k) := \{\, x_{\boldsymbol{j}} \in \Xi'_n \mid \exists i, t \in (-k,k) : x_{\boldsymbol{j}} + th_i\boldsymbol{e}_i \notin \Omega^\circ \,\}.$$

For every $k > 0$, we have

$$\#D(n,k) = o(N(\boldsymbol{n})).$$

Proof. Notice that if $x_{\boldsymbol{j}} \in D(n,k)$, then there exists a direction $\boldsymbol{e}_i$ and a value $t \in (-k,k)$ such that $x_{\boldsymbol{j}} + th_i\boldsymbol{e}_i \in \partial\Omega$ and $t \neq 0$. In particular, we infer that $d(x_{\boldsymbol{j}}, \partial\Omega) < kh_i$ and if we denote $h = \max_i h_i$, then $d(x_{\boldsymbol{j}}, \partial\Omega) < kh$. Using notations and results of Lemma 6.3.2, we know that $x_{\boldsymbol{j}} \in K_{kh} \cap \Xi'_n$, but $kh \to 0$ as n goes to infinity, so

$$\#D(n,k) \leq s_n^{K_{kh}} = o(s_n^{\Omega^\circ}) \implies \#D(n,k) = o(N(\boldsymbol{n})).$$

□

We just proved that, except for few relations, the system (7.9) mimics a classical FD scheme. We can thus consider the extended problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{d} \frac{\partial}{\partial x_i}\left(a'_i \frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{d} b'_i \frac{\partial u}{\partial x_i} + c'u = f', & \text{in } (0,1)^d, \\ u = 0, & \text{on } \partial([0,1]^d). \end{cases} \tag{7.10}$$

where a'_i, b'_i, c', f' are functions that extend a_i, b_i, c, f

$$a'_i(x) = \begin{cases} a_i(x), & x \in \Omega, \\ 0, & x \notin \Omega, \end{cases} \qquad b'_i(x) = \begin{cases} b_i(x), & x \in \Omega, \\ 0, & x \notin \Omega, \end{cases}$$

$$c'(x) = \begin{cases} c(x), & x \in \Omega, \\ 0, & x \notin \Omega, \end{cases} \qquad f'(x) = \begin{cases} f(x), & x \in \Omega, \\ 0, & x \notin \Omega. \end{cases}$$

Notice that b'_i, c' are bounded functions since Ω is a compact set, and moreover a'_i are bounded and continuous a.e. functions. In [8], it is showed that these conditions on the coefficients are enough to prove that the matrices $A_{\boldsymbol{n}}$ induced by the relations (7.3)-(7.5) build a GLT sequence with symbol

$$\{n^{-2}A_{\boldsymbol{n}}\}_n \sim_{GLT} k(x,\theta) = \sum_{i=1}^{d} \nu_i^2 a'_i(x)(2 - 2\cos(\theta_i)),$$

where $\boldsymbol{n} + \boldsymbol{1} = n\boldsymbol{\nu}$. This is also enough to let us conclude that $\{n^{-2}A_{\boldsymbol{n}}^{\Omega^\circ}\}_n$ is actually a reduced GLT sequence.

Theorem 7.1.2.

$$\{n^{-2}A_{\boldsymbol{n}}^{\Omega^\circ}\}_n \sim_{GLT}^{\Omega^\circ} \kappa(x,\theta) = \sum_{i=1}^{d} \nu_i^2 a_i(x)(2 - 2\cos(\theta_i)).$$

Proof. Denote with $B_{\boldsymbol{n}}^{\Omega^\circ}$ and $Z_{\boldsymbol{n}}^{\Omega^\circ}$ the matrices

$$B_{\boldsymbol{n}}^{\Omega^\circ} = R_{\Omega^\circ}(A_{\boldsymbol{n}}), \qquad Z_{\boldsymbol{n}}^{\Omega^\circ} = B_{\boldsymbol{n}}^{\Omega^\circ} - A_{\boldsymbol{n}}^{\Omega^\circ},$$

where the rows and columns are associated to the points $x_{\boldsymbol{j}} \in \Xi'_n$. If $x_{\boldsymbol{j}} \in \Xi'_n \setminus D(n,2)$, then $x_{\boldsymbol{j}}$ is a point of the grid Ξ_n inside Ω° such that all its neighbours still belong to Ω°. In this case,

$$(A_{\boldsymbol{n}}^{\Omega^\circ})_{\boldsymbol{j},\boldsymbol{i}} = (B_{\boldsymbol{n}}^{\Omega^\circ})_{\boldsymbol{j},\boldsymbol{i}} = (A_{\boldsymbol{n}})_{\boldsymbol{j},\boldsymbol{i}} = \begin{cases} c(x_{\boldsymbol{j}}) + \sum_{i=1}^{d} \frac{a_i(x_{\boldsymbol{j}+\boldsymbol{e}_i/2}) + a_i(x_{\boldsymbol{j}-\boldsymbol{e}_i/2})}{h_i^2} & \boldsymbol{i} = \boldsymbol{j}, \\ -\frac{a_i(x_{\boldsymbol{j}\pm\boldsymbol{e}_i/2})}{h_i^2} \pm \frac{b_i(x_{\boldsymbol{j}})}{2h_i} & \boldsymbol{i} = \boldsymbol{j} \pm \boldsymbol{e}_i, \\ 0, & \text{otherwise}, \end{cases}$$

hence the row corresponding to $x_{\boldsymbol{j}}$ in $Z_{\boldsymbol{n}}^{\Omega^\circ}$ is zero. Using the result in Lemma 7.1.1, we conclude that the number of non-zero rows in $Z_{\boldsymbol{n}}^{\Omega^\circ}$ is $o(N(\boldsymbol{n}))$, so $\{Z_{\boldsymbol{n}}^{\Omega^\circ}\}_n$ is a zero-distributed sequence, since Lemma 6.3.1 assures us that

$$\operatorname{rk}(Z_{\boldsymbol{n}}^{\Omega^\circ}) = o(N(\boldsymbol{n})) \implies \operatorname{rk}(Z_{\boldsymbol{n}}^{\Omega^\circ}) = o(s_n^{\Omega^\circ}).$$

From **GLT 3** and **GLT 4**, we conclude that

$$\{n^{-2}Z_{\boldsymbol{n}}\}_n \sim_{GLT}^{\Omega^\circ} 0, \qquad \{n^{-2}B_{\boldsymbol{n}}^{\Omega^\circ}\}_n = R_{\Omega^\circ}(\{n^{-2}A_{\boldsymbol{n}}\}_n) \sim_{GLT}^{\Omega^\circ} \kappa$$

$$\implies \{n^{-2}A_{\boldsymbol{n}}^{\Omega^\circ}\}_n = \{n^{-2}Z_{\boldsymbol{n}}\}_n + \{n^{-2}B_{\boldsymbol{n}}^{\Omega^\circ}\}_n \sim_{GLT}^{\Omega^\circ} \kappa.$$

□

A more involved analysis is needed to conclude that $\{n^{-2}A_{\boldsymbol{n}}^{\Omega^\circ}\}_n \sim_\lambda \kappa$. If $A_{\boldsymbol{n}}^{\Omega^\circ}$ were Hermitian matrices, the result would follow from **GLT 1**, but it is almost never the case. Notice that κ is a real valued function, so we can decompose $A_{\boldsymbol{n}}^{\Omega^\circ}$ into its Hermitian and skew-Hermitian part. Using **GLT 1, 4**, we have

$$\Re(n^{-2}A_{\boldsymbol{n}}^{\Omega^\circ}) = \frac{n^{-2}A_{\boldsymbol{n}}^{\Omega^\circ} + n^{-2}(A_{\boldsymbol{n}}^{\Omega^\circ})^H}{2} \implies \{\Re(n^{-2}A_{\boldsymbol{n}}^{\Omega^\circ})\}_n \sim_{GLT}^{\Omega^\circ} \kappa, \qquad \{\Re(n^{-2}A_{\boldsymbol{n}}^{\Omega^\circ})\}_n \sim_\lambda \kappa.$$

On the other hand, the skew-Hermitian part is zero-distributed, but in order to write the expression for its coefficients, we need to remind that the values $s_i^\pm$ depend on the point $x_{\boldsymbol{j}}$. To avoid confusion, we will denote them by $s_i^\pm(\boldsymbol{j})$.

$$\Im(n^{-2}A_{\boldsymbol{n}}^{\Omega^\circ}) = \frac{n^{-2}A_{\boldsymbol{n}}^{\Omega^\circ} - (n^{-2}A_{\boldsymbol{n}}^{\Omega^\circ})^H}{2} \implies \{n^{-2}\Im(A_{\boldsymbol{n}}^{\Omega^\circ})\}_n \sim_{GLT}^{\Omega^\circ} 0$$

$$(\Im(n^{-2}A_{\boldsymbol{n}}^{\Omega^\circ}))_{\boldsymbol{j},\boldsymbol{i}} = \begin{cases} \frac{n^{-2}}{1+s_i^-(\boldsymbol{j})}\left(\frac{-a_i(x_{\boldsymbol{j}+\mathbf{e}_i/2})}{h_i^2} + \frac{b_i(x_{\boldsymbol{j}})}{2h_i}\right) - \frac{n^{-2}}{s_i^+(\boldsymbol{i})+1}\left(\frac{-a_i(x_{\boldsymbol{i}-\mathbf{e}_i/2})}{h_i^2} + \frac{-b_i(x_{\boldsymbol{i}})}{2h_i}\right), & \boldsymbol{i} = \boldsymbol{j} + \boldsymbol{e}_i, \\ \frac{n^{-2}}{s_i^+(\boldsymbol{j})+1}\left(\frac{-a_i(x_{\boldsymbol{j}-\mathbf{e}_i/2})}{h_i^2} + \frac{-b_i(x_{\boldsymbol{j}})}{2h_i}\right) - \frac{n^{-2}}{1+s_i^-(\boldsymbol{i})}\left(\frac{-a_i(x_{\boldsymbol{i}+\mathbf{e}_i/2})}{h_i^2} + \frac{b_i(x_{\boldsymbol{i}})}{2h_i}\right), & \boldsymbol{i} = \boldsymbol{j} - \boldsymbol{e}_i, \\ 0, & \text{otherwise.} \end{cases}$$

Notice that $s_i^\pm \in (0,1]$, so we can bound every entry by

$$\left|(\Im(n^{-2}A_{\boldsymbol{n}}^{\Omega^\circ}))_{\boldsymbol{j},\boldsymbol{i}}\right| \leq \nu(2\nu\|a\|_\infty + n^{-1}\|b\|_\infty), \tag{7.11}$$

where $\nu = \max_i \nu_i$. Moreover, suppose $x_{\boldsymbol{j}}$ is a grid point in $\Xi_n' \setminus D(n,3)$. In particular, we have $s_i^\pm(\boldsymbol{j}) = s_i^\pm(\boldsymbol{j}+e_i) = s_i^\pm(\boldsymbol{j}-e_i) = 1$ for every i. The row $\boldsymbol{j}$ is easier to write as

$$(\Im(n^{-2}A_{\boldsymbol{n}}^{\Omega^\circ}))_{\boldsymbol{j},\boldsymbol{i}} = \begin{cases} \frac{n^{-2}}{h_i}\left(\frac{b_i(x_{\boldsymbol{j}})+b_i(x_{\boldsymbol{i}})}{4}\right), & \boldsymbol{i} = \boldsymbol{j} + \boldsymbol{e}_i, \\ -\frac{n^{-2}}{h_i}\left(\frac{b_i(x_{\boldsymbol{j}})+b_i(x_{\boldsymbol{i}})}{4}\right), & \boldsymbol{i} = \boldsymbol{j} - \boldsymbol{e}_i, \\ 0, & \text{otherwise,} \end{cases}$$

and we can bound the entries by

$$\left|(\Im(n^{-2}A_{\boldsymbol{n}}^{\Omega^\circ}))_{\boldsymbol{j},\boldsymbol{i}}\right| \leq 2\nu n^{-1}\|b\|_\infty. \tag{7.12}$$

Lemma 7.1.1 assures us that almost all points in Ξ_n' respect these conditions. Now we are ready to prove that $\{n^{-2}A_{\boldsymbol{n}}^{\Omega^\circ}\}_n \sim_\lambda \kappa$.

Theorem 7.1.3.

$$\{n^{-2}A_{\boldsymbol{n}}^{\Omega^\circ}\}_n \sim_\lambda \kappa(x,\theta) = \sum_{i=1}^d \nu_i^2 a_i(x)(2 - 2\cos(\theta_i)).$$

Proof. Using the decomposition into Hermitian and skew-Hermitian part, we write

$$n^{-2}A_{\boldsymbol{n}}^{\Omega^\circ} = \Re(n^{-2}A_{\boldsymbol{n}}^{\Omega^\circ}) + \Im(n^{-2}A_{\boldsymbol{n}}^{\Omega^\circ})$$

where $\Re(n^{-2}A_{\boldsymbol{n}}^{\Omega^\circ})$ are Hermitian and $\{\Re(n^{-2}A_{\boldsymbol{n}}^{\Omega^\circ})\}_n \sim_\lambda \kappa$. Notice that every row of $\Im(A_{\boldsymbol{n}})$ has at most $2d$ non-zero elements. Using Lemma 7.1.1 and the relations (7.11,7.12) we obtain

$$\begin{aligned} \|\Im(n^{-2}A_{\boldsymbol{n}}^{\Omega^\circ})\|_2^2 &= \sum_{\boldsymbol{j}}\sum_{\boldsymbol{i}} |(\Im(n^{-2}A_{\boldsymbol{n}}^{\Omega^\circ}))_{\boldsymbol{j},\boldsymbol{i}}|^2 \\ &= \sum_{\boldsymbol{j}:x_{\boldsymbol{j}}\in\Xi_n'\setminus D(n,3)} \sum_{\boldsymbol{i}} |(\Im(n^{-2}A_{\boldsymbol{n}}^{\Omega^\circ}))_{\boldsymbol{j},\boldsymbol{i}}|^2 + \sum_{\boldsymbol{j}:x_{\boldsymbol{j}}\in D(n,3)} \sum_{\boldsymbol{i}} |(\Im(n^{-2}A_{\boldsymbol{n}}^{\Omega^\circ}))_{\boldsymbol{j},\boldsymbol{i}}|^2 \\ &\leq \sum_{\boldsymbol{j}:x_{\boldsymbol{j}}\in\Xi_n'\setminus D(n,3)} \sum_{\boldsymbol{i}} 4\nu^2 n^{-2}\|b\|_\infty^2 + \sum_{\boldsymbol{j}:x_{\boldsymbol{j}}\in D(n,3)} \sum_{\boldsymbol{i}} \nu^2(2\nu\|a\|_\infty + n^{-1}\|b\|_\infty)^2 \\ &\leq \sum_{\boldsymbol{j}:x_{\boldsymbol{j}}\in\Xi_n'\setminus D(n,3)} 8d\nu^2 n^{-2}\|b\|_\infty^2 + \sum_{\boldsymbol{j}:x_{\boldsymbol{j}}\in D(n,3)} 2d\nu^2(2\nu\|a\|_\infty + n^{-1}\|b\|_\infty)^2 \\ &\leq 8d\nu^2 n^{-2}\|b\|_\infty^2 s_n^{\Omega^\circ} + 2d\nu^2(2\nu\|a\|_\infty + n^{-1}\|b\|_\infty)^2 o(s_n^{\Omega^\circ}) = o(s_n^{\Omega^\circ}). \end{aligned}$$

GLT 2 let us conclude that

$$\{n^{-2}A_{\boldsymbol{n}}^{\Omega^\circ}\}_n \sim_\lambda \kappa.$$

□

The Shortley-Weller approximation just described is actually very general, and it comprehends classical finite differences methods used on regular domains. For example, in 2 dimensions, every triangular domain can be transformed by affine maps into the isosceles right triangle T described by the vertices with coordinates $(0,0),(0,1),(1,0)$. If we superimpose the regular grid Ξ_n onto the triangle, we find that the union of the points on the border for every n is a dense set in δT.

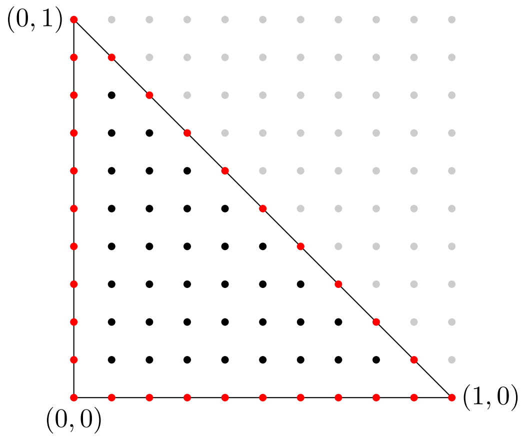


Figure 7.2: Superimposition of Ξ_n onto the triangle T.

Operating a classical second order method to discretize Problem 7.2 in 2 dimensions, we fall again in the Shortley-Weller method, so we already know the symbol of the resulting linear system.

7.2 P_1 Method

Consider a linear partial differential equation

$$\mathscr{L}(u)(x) = f(x) \qquad x \in \Omega^\circ$$

equipped with some boundary conditions (Dirichlet, Neumann, etc.) when $x \in \partial\Omega$, where $\Omega \subseteq [0,1]^d$ is a closed Peano-Jordan measurable set with positive measure and f is a function defined over Ω.

A common way to discretize the problem is to use a finite elements method, that is based on the choice of a basis for the functions on the domain Ω. The basis does not necessarily depend on a grid of points inside Ω, but usually they do, so on a generic Ω there's again the problem to describe the boundary. For this reason, usually the domains are polyhedral or with a regular enough boundary. When we deal with more general shapes, we may need to map the domain into a regular one, or to modify the grids of discretization, and a more involved analysis is required.

Let us consider the problem

$$\begin{cases} -\displaystyle\sum_{i,j=1}^{2} \frac{\partial}{\partial x_j}\left(a_{i,j}\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{2} b_i \frac{\partial u}{\partial x_i} + cu = f, & \text{in } \Omega^\circ, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{7.13}$$

where Ω is a closed set inside $[0,1]^2$ with negligible boundary and positive measure. Moreover $a_{i,j}$, b_i, c and f are given complex-valued continuous functions defined on Ω and $a_{i,j} \in C^1(\Omega)$. If $A = (a_{i,j})_{i,j=1}^2$ is a matrix of functions and $\boldsymbol{b} = (b_1, b_2)^T$, then the equivalent weak form of (7.13) reads as

$$\int_{\Omega^\circ} (\nabla u)^T A \nabla w + (\nabla u)^T \boldsymbol{b} w + cuw = \int_{\Omega^\circ} fw, \qquad \forall w \in H_0^1(\Omega). \tag{7.14}$$

The space $[0,1]^2$ is divided into triangles as shown in Figure 7.5, whose vertices are the nodes of Ξ_n. The P_1 finite elements method, studied in [17, 63], uses base functions supported on the grid triangles that fall inside Ω. We say that the adjacent nodes of a point $p \in \Xi_n$ are its neighbours, and we call $N(p)$ the set composed of p and its neighbours. Each point p is a vertex for at most 6 triangles, that we call $T_{i,p}$ as shown in Figure 7.3, and we denote their union as T_p (notice that they depend also on n, but for brevity we omit the index). The collection of all the triangles in the scheme associated to the grid Ξ_n is

$$\mathscr{T}_n = \{\, T_{i,p} \mid p \in \Xi_n,\ i = 1, \dots, 6 \,\}\,.$$

For every point $p \in \Xi_n$ such that $T_p \subseteq [0,1]^2$, we define a function $\psi_{p,n}$ that is linear on each triangle, whose value is 1 at p and 0 on every other point of Ξ_n.

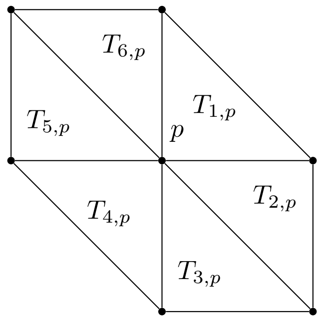


Figure 7.3: triangles and neighbours associated to the point p

We can explicitly write $\psi_{p,n}$ and its partial derivatives. If $p = (x_p, y_p)$ and $\widetilde{x} = x - x_p$, $\widetilde{y} = y - y_p$, then

$$\psi_{p,n}(x,y) = \begin{cases} 1 - \frac{\widetilde{x}+\widetilde{y}}{h}, & (x,y) \in T_{1,p}, \\ 1 - \frac{\widetilde{x}}{h}, & (x,y) \in T_{2,p}, \\ 1 + \frac{\widetilde{y}}{h}, & (x,y) \in T_{3,p}, \\ 1 + \frac{\widetilde{x}+\widetilde{y}}{h}, & (x,y) \in T_{4,p}, \\ 1 + \frac{\widetilde{x}}{h}, & (x,y) \in T_{5,p}, \\ 1 - \frac{\widetilde{y}}{h}, & (x,y) \in T_{6,p}, \\ 0, & \text{otherwise,} \end{cases}$$

$$\frac{\partial}{\partial x}\psi_{p,n}(x,y) = \begin{cases} -\frac{1}{h}, & (x,y) \in T_{1,p}, \\ -\frac{1}{h}, & (x,y) \in T_{2,p}, \\ 0, & (x,y) \in T_{3,p}, \\ \frac{1}{h}, & (x,y) \in T_{4,p}, \\ \frac{1}{h}, & (x,y) \in T_{5,p}, \\ 0, & (x,y) \in T_{6,p}, \\ 0, & \text{otherwise,} \end{cases} \qquad \frac{\partial}{\partial y}\psi_{p,n}(x,y) = \begin{cases} -\frac{1}{h}, & (x,y) \in T_{1,p}, \\ 0, & (x,y) \in T_{2,p}, \\ \frac{1}{h}, & (x,y) \in T_{3,p}, \\ \frac{1}{h}, & (x,y) \in T_{4,p}, \\ 0, & (x,y) \in T_{5,p}, \\ -\frac{1}{h}, & (x,y) \in T_{6,p}, \\ 0, & \text{otherwise,} \end{cases}$$

where $h = 1/(n+1)$. P_1 methods usually arises when the domain is not a square, but it is polyhedral or regular enough. For example, as we can see in Figure 7.4, the subdivision scheme adopted has the property to describe also the boundary of the triangle, in opposition to the classical tensor-product hat-functions considered in [53, Section 7.4].

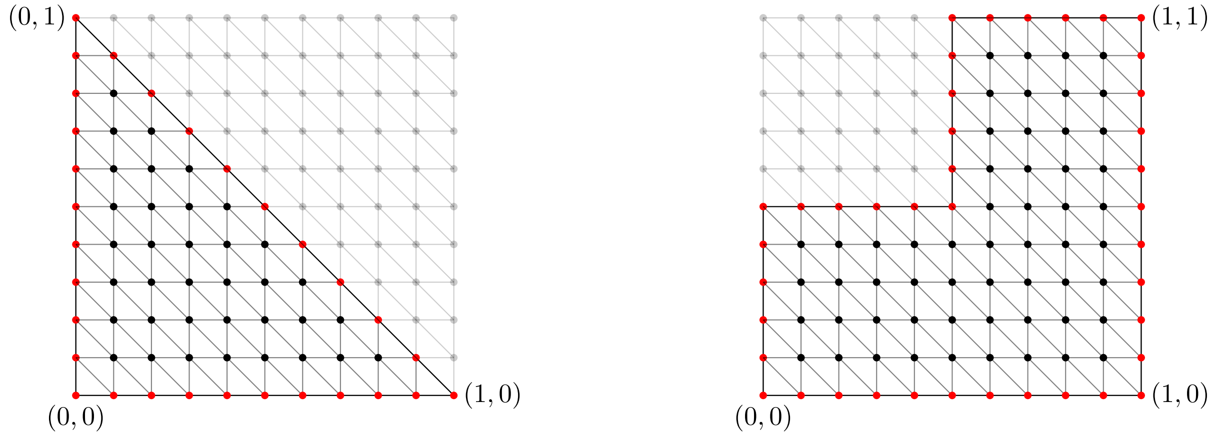


Figure 7.4: Superimposition of Ξ_n onto the triangle T and an L shape. the subdivision mesh is referred to the P_1 finite elements method.

This does not happen when dealing with more complicated domains Ω, as shown in Figure 7.5. In fact we can see that, for example, on a curvilinear shape, the points of Ξ_n are not enough to approximate the boundary $\partial\Omega$.

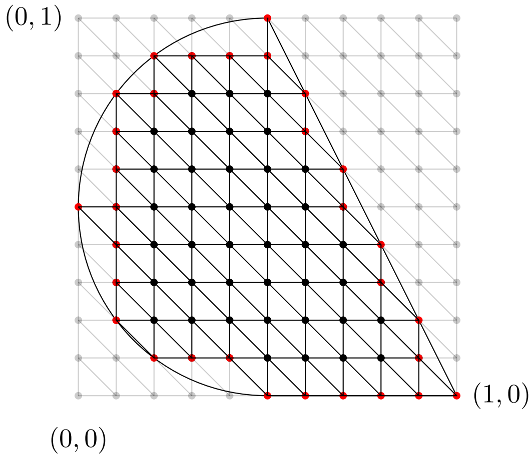


Figure 7.5: Example of a general domain Ω and induced mesh.

When we work on a closed set $\Omega \subseteq [0,1]^2$ with $\mu(\partial\Omega) = 0$, we focus on the points p such that T_p is contained in Ω, so we call

$$\Xi_n(\Omega) := \{\, p \in \Xi_n \mid T_p \subseteq \Omega \,\} \,.$$

We look for a function u that is a linear combination of the $\psi_{p,n}$ such that (7.14) is satisfied for every $w = \psi_{p,n}$. If we substitute $u = \sum_{p\in\Xi_n(\Omega)} u_p \psi_{p,n}$ and $w = \psi_{q,n}$ into (7.14), then we obtain the system

$$\sum_{p\in\Xi_n(\Omega)} s_{q,p} u_p = f_q, \qquad s_{q,p} = \int_{\Omega^\circ} (\nabla\psi_{p,n})^T A \nabla\psi_{q,n} + (\nabla\psi_{p,n})^T \boldsymbol{b} \psi_{q,n} + c\psi_{p,n}\psi_{q,n}, \qquad f_q = \int_{\Omega^\circ} f\psi_{q,n} \quad (7.15)$$

for every $q \in \Xi_n(\Omega)$. We call $S_{\boldsymbol{n}}$ the resulting matrix with entries $s_{p,q}$ for every $p, q \in \Xi_n(\Omega)$, where the nodes are sorted in lexicographic order. We can notice that $p \in \Xi_n(\Omega) \implies p \in \Omega^\circ \cap \Xi_n$, even if the converse is not always true, so

$$|\Xi_n(\Omega)| \le s_n^{\Omega^\circ} = O(n^2)$$

where $|\Xi_n(\Omega)|$ is the size of the matrix $S_{\boldsymbol{n}}$. It leads to solve the system

$$S_{\boldsymbol{n}} \boldsymbol{u} = \boldsymbol{f}.$$

A different boundary condition does not change the stiffness matrix, so the analysis is the same if we impose, for example, $u = g$ on T_D and $\partial u/\partial n = h$ on T_N where $\partial T = T_D \coprod T_N$.

7.2.1 Case on the Square

When $\Omega = [0,1]^2$, we already know that, under suitable hypotheses on the regularity of the coefficients, the sequence of stiffness matrices $\{S_n\}_n$ described in (7.15) is actually a multilevel GLT sequence, for which we can compute GLT and spectral symbol. Here we prove that the same holds when $A, \boldsymbol{b}, c$ are just L^1 functions.

Theorem 7.2.1. *We call B the 3×2 matrix*

$$B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and we indicate with B_1, B_2, B_3 its rows. Given L^1 functions $A : (0,1)^2 \to \mathbb{C}^{2\times 2}$, $\boldsymbol{b} : (0,1)^2 \to \mathbb{C}^2$ and $c : (0,1)^2 \to \mathbb{C}$, we have that the sequence $\{S_{\boldsymbol{n}}\}_n$ is a multilevel GLT sequence with symbol $k(x,\theta)$, where

$$\begin{aligned} k(x,\theta) = & r_{0,0}(x) + r_{0,1}(x)\exp(-i\theta_2) + r_{1,0}(x)\exp(-i\theta_1) + r_{-1,0}(x)\exp(i\theta_1) \\ & + r_{0,-1}(x)\exp(i\theta_2) + r_{1,-1}(x)\exp(-i\theta_1 + i\theta_2) + r_{-1,1}(x)\exp(i\theta_1 - i\theta_2), \end{aligned} \tag{7.16}$$

$$\begin{aligned} & r_{0,0} = B_1A(B_1)^T + B_2A(B_2)^T + B_3A(B_3)^T, \\ r_{0,1} &= -\frac{1}{2}B_3A(B_1)^T - \frac{1}{2}B_1A(B_3)^T, \qquad r_{0,-1} = -\frac{1}{2}B_3A(B_1)^T - \frac{1}{2}B_1A(B_3)^T, \\ r_{1,-1} &= \frac{1}{2}B_2A(B_3)^T + \frac{1}{2}B_3A(B_2)^T, \qquad r_{-1,1} = \frac{1}{2}B_2A(B_3)^T + \frac{1}{2}B_3A(B_2)^T, \\ r_{-1,0} &= -\frac{1}{2}B_1A(B_2)^T - \frac{1}{2}B_2A(B_1)^T, \qquad r_{1,0} = -\frac{1}{2}B_1A(B_2)^T - \frac{1}{2}B_2A(B_1)^T. \end{aligned} \tag{7.17}$$

If A is also Hermitian for every $x \in (0,1)^2$, then the sequence $\{S_{\boldsymbol{n}}\}_n$ has $k(x,\theta)$ as spectral symbol.

Proof. We split the matrix $S_{\boldsymbol{n}}$ into $P_{\boldsymbol{n}} + Z_{\boldsymbol{n}}$, where

$$(P_{\boldsymbol{n}})_{p,q} = \int_{(0,1)^2} (\nabla\psi_{p,n})^T A \nabla\psi_{q,n}, \qquad (Z_{\boldsymbol{n}})_{p,q} = \int_{(0,1)^2} (\nabla\psi_{p,n})^T \boldsymbol{b}\psi_{q,n} + c\psi_{p,n}\psi_{q,n}$$

and we prove that $\{P_{\boldsymbol{n}}\}_n \sim_{GLT} k(x,\theta)$ and $\{Z_{\boldsymbol{n}}\}_n$ is zero-distributed.

Notice that ψ_p is supported on T_p, so $(S_{\boldsymbol{n}})_{p,q}, (P_{\boldsymbol{n}})_{p,q}, (Z_{\boldsymbol{n}})_{p,q}$ are different from zero only when q is one of the 6 neighbours of p or p itself, that is $q \in N(p)$. Moreover, every $\psi_{p,n}$ is nonnegative and less than 1, and each component of $\nabla\psi_{p,n}$ is bounded by $1/h$ in absolute value.

Notice that the area of T_p is $3h^2$ for every p. Moreover, the functions b_1, b_2, c are L^1, so for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\mu(U) \le \delta \implies \int_U |b_1| + |b_2| + |c| \le \varepsilon.$$

Notice that every triangle $T_{(i)}$ of the triangulation $\mathscr{T}_n$ is inside T_p for at most 3 different points p, that are its vertices, and if $3h^2 \leq \delta$, we get

$$
\begin{aligned}
\|Z_{\boldsymbol{n}}\|_2^2 &= \sum_{p,q\in(0,1)^2\cap\Xi_n} |(Z_{\boldsymbol{n}})_{p,q}|^2 = \sum_{p,q\in(0,1)^2\cap\Xi_n} \left| \int_{(0,1)^2} (\nabla\psi_{p,n})^T \boldsymbol{b}\psi_{q,n} + c\psi_{p,n}\psi_{q,n} \right|^2 \\
&\leq \sum_{p\in(0,1)^2\cap\Xi_n} \sum_{q\in N(p)} \left[\int_{T_p} |(\nabla\psi_{p,n})^T \boldsymbol{b}|\psi_{q,n} + |c|\psi_{p,n}\psi_{q,n} \right]^2 \\
&\leq \sum_{p\in(0,1)^2\cap\Xi_n} \sum_{q\in N(p)} \left[\int_{T_p} \frac{|b_1|+|b_2|}{h} + |c| \right]^2 \leq \frac{1}{h^2} \sum_{p\in(0,1)^2\cap\Xi_n} \sum_{q\in N(p)} \left[\int_{T_p} |b_1|+|b_2|+|c| \right]^2 \\
&\leq \frac{7}{h^2} \sum_{p\in(0,1)^2\cap\Xi_n} \left[\int_{T_p} |b_1|+|b_2|+|c| \right] \varepsilon \\
&\leq \frac{7}{h^2}\varepsilon \sum_{T_{(i)}\in\mathscr{T}_n} 3 \left[\int_{T_{(i)}} |b_1|+|b_2|+|c| \right] \\
&\leq 21\frac{\varepsilon}{h^2}(\|b_1\|_1 + \|b_2\|_1 + \|c\|_1).
\end{aligned} \tag{7.18}
$$

Since we can take ε arbitrarily small as n tends to infinity, we infer that $n^{-1}\|Z_{\boldsymbol{n}}\|_2 \to 0$, so we can use **Z2** from [53] and conclude that $\{Z_{\boldsymbol{n}}\}_n$ is zero-distributed.

Let us analyse now the matrix $P_{\boldsymbol{n}}$. The elements of $P_{\boldsymbol{n}}$ on the row associated to $p = x_{\boldsymbol{j}}$ are different from zero only when $q \in N(p)$. Call $t_{p,a,b} = (P_{\boldsymbol{n}})_{p,p+a\boldsymbol{e}_1+b\boldsymbol{e}_2}$, and a computation shows that

$$
\begin{aligned}
h^2 t_{p,0,0} &= \int_{T_{1,p}\cup T_{4,p}} B_1 A(B_1)^T + \int_{T_{2,p}\cup T_{5,p}} B_2 A(B_2)^T + \int_{T_{3,p}\cup T_{6,p}} B_3 A(B_3)^T, \\
h^2 t_{p,0,1} &= -\int_{T_{6,p}} B_3 A(B_1)^T - \int_{T_{1,p}} B_1 A(B_3)^T, \\
h^2 t_{p,1,0} &= -\int_{T_{1,p}} B_1 A(B_2)^T - \int_{T_{2,p}} B_2 A(B_1)^T, \\
h^2 t_{p,1,-1} &= \int_{T_{2,p}} B_2 A(B_3)^T + \int_{T_{3,p}} B_3 A(B_2)^T, \\
h^2 t_{p,0,-1} &= -\int_{T_{3,p}} B_3 A(B_1)^T - \int_{T_{4,p}} B_1 A(B_3)^T, \\
h^2 t_{p,-1,0} &= -\int_{T_{4,p}} B_1 A(B_2)^T - \int_{T_{5,p}} B_2 A(B_1)^T, \\
h^2 t_{p,-1,1} &= \int_{T_{5,p}} B_2 A(B_3)^T + \int_{T_{6,p}} B_3 A(B_2)^T,
\end{aligned}
$$

and $t_{p,a,b} = 0$ for every other a, b.

Assume that A is a continuous function, so that there exists a modulus of continuity ω_A defined as

$$
\omega_A(\delta) = \max_{i,j} \sup_{p,q:|p-q|\leq\delta} |(A(p) - A(q))_{i,j}|
$$

and such that $\lim_{\delta\to 0}\omega_A(\delta) = 0$. Let us define a 2-level GLT sequence $\{G_n\}_n$ as

$$
\begin{aligned}
G_{\boldsymbol{n}} = &D_{\boldsymbol{n}}(r_{0,0})T_{\boldsymbol{n}}(1) + D_{\boldsymbol{n}}(r_{0,1})T_{\boldsymbol{n}}(\exp(-i\theta_2)) + D_{\boldsymbol{n}}(r_{1,0})T_{\boldsymbol{n}}(\exp(-i\theta_1)) + D_{\boldsymbol{n}}(r_{-1,0})T_{\boldsymbol{n}}(\exp(i\theta_1)) \\
&+ D_{\boldsymbol{n}}(r_{0,-1})T_{\boldsymbol{n}}(\exp(i\theta_2)) + D_{\boldsymbol{n}}(r_{1,-1})T_{\boldsymbol{n}}(\exp(-i\theta_1 + i\theta_2)) + D_{\boldsymbol{n}}(r_{-1,1})T_{\boldsymbol{n}}(\exp(i\theta_1 - i\theta_2)),
\end{aligned}
$$

with symbol $k(x,\theta)$. The elements of $P_{\boldsymbol{n}} - G_{\boldsymbol{n}}$ on the row associated to $p = x_{\boldsymbol{j}}$ are different from zero only when $q \in N(p)$. If we call $z_{p,a,b} = (P_{\boldsymbol{n}})_{p,p+\boldsymbol{e}_1+b\boldsymbol{e}_2} - (Q_{\boldsymbol{n}})_{p,p+\boldsymbol{e}_1+b\boldsymbol{e}_2}$, then

$$\begin{aligned}
|z_{p,0,0}| &\le \left| B_1A(p)(B_1)^T - \frac{1}{h^2}\int_{T_{1,p}\cup T_{4,p}} B_1A(B_1)^T \right| + \left| B_2A(p)(B_2)^T - \frac{1}{h^2}\int_{T_{2,p}\cup T_{5,p}} B_2A(B_2)^T \right| \\
&\quad + \left| B_3A(p)(B_3)^T - \frac{1}{h^2}\int_{T_{3,p}\cup T_{6,p}} B_3A(B_3)^T \right| \\
&= \left| \frac{1}{h^2}\int_{T_{1,p}\cup T_{4,p}} B_1(A(p)-A(x))(B_1)^T \mathrm{d}x \right| + \left| \frac{1}{h^2}\int_{T_{2,p}\cup T_{5,p}} B_2(A(p)-A(x))(B_2)^T \mathrm{d}x \right| \\
&\quad + \left| \frac{1}{h^2}\int_{T_{3,p}\cup T_{6,p}} B_3(A(p)-A(x))(B_3)^T \mathrm{d}x \right| \\
&\le 4\omega_A(h\sqrt{2}) + \omega_A(h\sqrt{2}) + \omega_A(h\sqrt{2}) = 6\omega_A(h\sqrt{2}),
\end{aligned}$$

$$\begin{aligned}
|z_{p,0,1}| &\le \left| \frac{1}{2}B_3A(p)(B_1)^T - \frac{1}{h^2}\int_{T_{6,p}} B_3A(B_1)^T \right| + \left| \frac{1}{2}B_1A(p)(B_3)^T - \frac{1}{h^2}\int_{T_{1,p}} B_1A(B_3)^T \right| \\
&= \left| \frac{1}{h^2}\int_{T_{6,p}} B_3(A(p)-A(x))(B_1)^T \mathrm{d}x \right| + \left| \frac{1}{h^2}\int_{T_{1,p}} B_1(A(p)-A(x))(B_3)^T \mathrm{d}x \right| \\
&\le \omega_A(h\sqrt{2}) + \omega_A(h\sqrt{2}) = 2\omega_A(h\sqrt{2}),
\end{aligned}$$

and analogous computations show that $|z_{p,1,0}|, |z_{p,0,-1}|, |z_{p,-1,0}|$ are also bounded by $2\omega_A(h\sqrt{2})$. Moreover,

$$\begin{aligned}
|z_{p,1,-1}| &\le \left| \frac{1}{2}B_2A(p)(B_3)^T - \frac{1}{h^2}\int_{T_{2,p}} B_2A(B_3)^T \right| + \left| \frac{1}{2}B_3A(p)(B_2)^T - \frac{1}{h^2}\int_{T_{3,p}} B_3A(B_2)^T \right| \\
&= \left| \frac{1}{h^2}\int_{T_{2,p}} B_2(A(p)-A(x))(B_3)^T \mathrm{d}x \right| + \left| \frac{1}{h^2}\int_{T_{3,p}} B_3(A(p)-A(x))(B_2)^T \mathrm{d}x \right| \\
&\le \frac{1}{2}\omega_A(h\sqrt{2}) + \frac{1}{2}\omega_A(h\sqrt{2}) = \omega_A(h\sqrt{2}),
\end{aligned}$$

and a similar argument shows that $|z_{p,-1,1}|$ is also bounded by $\omega_A(h\sqrt{2})$. Since every row of $P_{\boldsymbol{n}} - G_{\boldsymbol{n}}$ has at most 7 non-zero elements and they are all bounded in absolute value by $6\omega_A(h\sqrt{2})$, we conclude that

$$\|P_{\boldsymbol{n}} - G_{\boldsymbol{n}}\|_2 \le \sqrt{7n^2 \cdot 36\omega_A(h\sqrt{2})^2} \le 18n\omega_A(h\sqrt{2})^2 = o(n)$$

and using again **Z2** from [53], we obtain that $P_{\boldsymbol{n}} - G_{\boldsymbol{n}}$ is zero-distributed. Since $\{G_{\boldsymbol{n}}\}_n$ has GLT symbol $k(x,\theta)$, we conclude that

$$\{S_{\boldsymbol{n}}\}_n = \{G_{\boldsymbol{n}}\}_n + \{P_{\boldsymbol{n}} - G_{\boldsymbol{n}}\}_n + \{Z_{\boldsymbol{n}}\}_n \sim_{GLT} k(x,\theta).$$

If we now assume that A is an L^1 function, then we can find a sequence A_m of continuous functions such that $\|A - A_m\|_1 \le 2^{-m}$, where

$$\|C\|_1 = \sum_{i,j} \|c_{i,j}\|_1 = \int_{(0,1)^2} B_1|C|(B_1)^T.$$

If we define $r_{a,b,m}$ like in (7.17) with A_m instead of A, and $k_m(x,\theta)$ like in (7.16) with $r_{a,b,m}$ instead of $r_{a,b}$, then we get $k_m \to k$ in L^1. Moreover, if $\{S^{(m)}_{\boldsymbol{n}}\}_n$ is defined as above, but with A_m instead of A, then from the previous analysis, we know that $\{S^{(m)}_{\boldsymbol{n}}\}_n \sim_{GLT} k_m$. The difference

$$\{S^{(m)}_{\boldsymbol{n}}\}_n - \{S_{\boldsymbol{n}}\}_n = \{P^{(m)}_{\boldsymbol{n}}\}_n - \{P_{\boldsymbol{n}}\}_n + \{Z^{(m)}_{\boldsymbol{n}}\}_n - \{Z_{\boldsymbol{n}}\}_n$$

presents two zero-distributed sequences $\{Z^{(m)}_{\boldsymbol{n}}\}_n$ and $\{Z_{\boldsymbol{n}}\}_n$, so we need to analyse the other two sequences. Notice that for every measurable set $U \subseteq [0,1]^2$ and every indices i,j we know that

$$\left| \int_U B_iA(B_j)^T - \int_U B_iA_m(B_j)^T \right| \le B_1 \left[\int_U |A - A_m| \right] (B_1)^T,$$

but $A - A_m$ is also L^1, so given ε there exists a δ such that $\mu(U) < \delta$ implies that

$$B_1 \int_U |A - A_m| (B_1)^T \leq \varepsilon.$$

If $\mu(T_p) = 3h^2 \leq \delta$, then we can bound the 1 Schatten norm of $P_{\boldsymbol{n}}^{(m)} - P_{\boldsymbol{n}}$ by the sum of the absolute values of their elements, so

$$\begin{aligned}
\|P_{\boldsymbol{n}}^{(m)} - P_{\boldsymbol{n}}\|_1 &\leq \sum_{p,q \in (0,1)^2 \cap \Xi_n} |(P_{\boldsymbol{n}}^{(m)} - P_{\boldsymbol{n}})_{p,q}| \\
&\leq \sum_{p \in (0,1)^2 \cap \Xi_n} \sum_{q \in N(p)} |(P_{\boldsymbol{n}}^{(m)} - P_{\boldsymbol{n}})_{p,q}| \\
&\leq \frac{1}{h^2} \sum_{p \in (0,1)^2 \cap \Xi_n} \sum_{q \in N(p)} 6B_1 \int_{T_p} |A - A_m| (B_1)^T \\
&\leq \frac{42}{h^2} \sum_{p \in (0,1)^2 \cap \Xi_n} B_1 \int_{T_p} |A - A_m| (B_1)^T \\
&\leq 3\frac{42}{h^2} \|A - A_m\|_1.
\end{aligned}$$

Using Theorem 3.1.7, we obtain that $\{P_{\boldsymbol{n}}^{(m)}\}_n \xrightarrow{a.c.s.} \{P_{\boldsymbol{n}}\}_n$ and $\{S_{\boldsymbol{n}}^{(m)}\}_n \xrightarrow{a.c.s.} \{S_{\boldsymbol{n}}\}_n$. We conclude that $\{S_{\boldsymbol{n}}\}_n \sim_{GLT} k$.

When A is Hermitian, we can prove that $P_{\boldsymbol{n}}$ is Hermitian. In fact

$$(P_{\boldsymbol{n}})_{p,q} = \int_{(0,1)^2} (\nabla\psi_{p,n})^T A \nabla\psi_{q,n} = \overline{\int_{(0,1)^2} (\nabla\psi_{p,n})^T \overline{A} \nabla\psi_{q,n}} = \overline{\int_{(0,1)^2} (\nabla\psi_{q,n})^T \overline{A} \nabla\psi_{p,n}} = \overline{(P_{\boldsymbol{n}})_{q,p}}.$$

Since $\{S_{\boldsymbol{n}}\}_n = \{P_{\boldsymbol{n}}\}_n + \{Z_{\boldsymbol{n}}\}_n$ and from (7.18), we know that $\|\widetilde{Z}_{\boldsymbol{n}}\|_2 = o(n)$, we can apply **GLT 2** and conclude that $\{S_{\boldsymbol{n}}\}_n \sim_\lambda k$. ☐

7.2.2 Problem on Sub-domains

Let us now consider a closed Peano-Jordan measurable set $\Omega \subseteq [0,1]^2$ with positive measure. Consider the problem (7.14) on Ω, where now $A, \boldsymbol{b}, c$ are L^1 functions defined on Ω. When we apply a P_1 discretization. The resulting matrices form a sequence equivalent to a reduced GLT sequence that descends from the square case. In particular, we can prove the following theorem.

Theorem 7.2.2. *Given a closed Peano-Jordan measurable set $\Omega \subseteq [0,1]^2$ with positive measure. Let $\widetilde{A}$, $\widetilde{\boldsymbol{b}}$ and $\widetilde{c}$ be extensions of A, $\boldsymbol{b}$ and c to $(0,1)^2$, obtained by setting $\widetilde{a}_{i,j}(z) = \widetilde{b}_j(z) = \widetilde{c}(z) = 0$ outside Ω for every i, j. Moreover, let $\widetilde{k}$ be the symbol described in Theorem 7.2.1 referred to the problem with coefficients $\widetilde{A}$, $\widetilde{\boldsymbol{b}}$, $\widetilde{c}$, and denote $k = \widetilde{k}|_{\Omega^\circ}$. If $S_{\boldsymbol{n}}^\Omega$ is the matrix resulting from the P_1 discretization using the grid $\Xi_n(\Omega)$, then*

$$\{S_{\boldsymbol{n}}^\Omega\}_n \sim_\sigma k$$

and if A is Hermitian for every $x \in \Omega$, then k is also a spectral symbol for $\{S_{\boldsymbol{n}}^\Omega\}_n$.

Proof. Let $S_{\boldsymbol{n}}$ be the matrix resulting from the P_1 discretization of the problem with coefficients $\widetilde{A}$, $\widetilde{\boldsymbol{b}}$, $\widetilde{c}$ on the square $[0,1]^2$ using the grid Ξ_n. We want to show that $R_{\Xi_n(\Omega)}(S_{\boldsymbol{n}}) = S_{\boldsymbol{n}}^\Omega$, that is, for every pair of points (p,q) in $\Xi_n(\Omega)$, we prove $(S_{\boldsymbol{n}})_{p,q} = (S_{\boldsymbol{n}}^\Omega)_{p,q}$. From (7.15), the equations for the two quantities are

$$\begin{aligned}
(S_{\boldsymbol{n}})_{p,q} &= \int_{(0,1)^2} (\nabla\psi_{p,n})^T \widetilde{A} \nabla\psi_{q,n} + (\nabla\psi_{p,n})^T \widetilde{\boldsymbol{b}} \psi_{q,n} + \widetilde{c}\psi_{p,n}\psi_{q,n}, \\
(S_{\boldsymbol{n}}^\Omega)_{p,q} &= \int_{\Omega^\circ} (\nabla\psi_{p,n})^T A \nabla\psi_{q,n} + (\nabla\psi_{p,n})^T \boldsymbol{b} \psi_{q,n} + c\psi_{p,n}\psi_{q,n},
\end{aligned}$$

but $p \in \Xi_n(\Omega)$ so $T_p^\circ \subseteq \Omega^\circ$ and therefore the two quantities are the same since $A, \boldsymbol{b}$ and c coincide with $\widetilde{A}$, $\widetilde{\boldsymbol{b}}$ and $\widetilde{c}$ on Ω. In this case, it may happen that $\Xi_n(\Omega) \subsetneq \Xi_n \cap \Omega^\circ$ since Ω may not be convex, but the two sets are actually almost the same. In fact,

$$E_n := (\Xi_n \cap \Omega^\circ) \setminus \Xi_n(\Omega) = \{\, p \in \Xi_n \cap \Omega^\circ \mid T_p \not\subseteq \Omega \,\},$$

so any point $p \in E_n$ is at distance at most $h_n = 1/(n+1)$ from the boundary $\partial\Omega$, and using Lemma 6.3.2, we conclude

$$E_n \subseteq \{\, p \in \Xi_n \mid d(p, \partial\Omega) \le h_n \,\} \implies |E_n| \le s_n^{K_{h_n}} = o(N(\boldsymbol{n})).$$

As a consequence,

$$s_n^{\Omega^\circ \triangle \Xi_n(\Omega)} = |\{\, p \in \Xi_n \cap \Omega^\circ \mid p \notin \Xi_n(\Omega) \,\}| = |E_n| = o(N(\boldsymbol{n}))$$

and Lemma 6.3.10 assures us that it is sufficient to prove the thesis for $R_{\Omega^\circ}(S_{\boldsymbol{n}})$.

Using the definition of reduced GLT, we can affirm that

$$\{R_{\Omega^\circ}(S_{\boldsymbol{n}})\}_n \sim_{GLT}^{\Omega^\circ} k \implies \{R_{\Omega^\circ}(S_{\boldsymbol{n}})\}_n \sim_\sigma k.$$

If we now assume that A is an Hermitian matrix for every $x \in \Omega$, then automatically also $\widetilde{A}$ is Hermitian for every x, since it is equal to A or it is the zero matrix. From the proof of Theorem 7.2.1, we know that $S_{\boldsymbol{n}} = P_{\boldsymbol{n}} + Z_{\boldsymbol{n}}$, where $P_{\boldsymbol{n}}$ is Hermitian and $\|Z_{\boldsymbol{n}}\|_2 = o(n)$. If we call $P_{\boldsymbol{n}}^{\Omega^\circ} = R_{\Omega^\circ}(P_{\boldsymbol{n}})$ and $Z_{\boldsymbol{n}}^{\Omega^\circ} = R_{\Omega^\circ}(Z_{\boldsymbol{n}})$ then we find that $R_{\Omega^\circ}(S_{\boldsymbol{n}}) = P_{\boldsymbol{n}}^{\Omega^\circ} + Z_{\boldsymbol{n}}^{\Omega^\circ}$, $P_{\boldsymbol{n}}^{\Omega^\circ}$ is Hermitian and for Lemmas F.2.3, 6.3.9 and 6.3.1,

$$\|Z_{\boldsymbol{n}}^{\Omega^\circ}\|_2 = \|R_{\Omega^\circ}(Z_{\boldsymbol{n}})\|_2 \le \|Z_{\boldsymbol{n}}\|_2 = o(n) \implies \|Z_{\boldsymbol{n}}^{\Omega^\circ}\|_2 = o\left(\sqrt{s_n^\Omega}\right).$$

Notice that $\{P_{\boldsymbol{n}}^{\Omega^\circ}\}_k \sim_\lambda k$, so we can use **GLT 2**, and conclude that

$$\{R_{\Omega^\circ}(S_{\boldsymbol{n}})\}_k \sim_\lambda k.$$

□

Notice that $k(x, \theta)$ has the same form described in (7.16), (7.17), where A is now defined only on Ω.

7.2.3 P1 on Non-Regular Grids

When the domain Ω is compact, but presents an irregular boundary, or when we want to focus the discretization to particular points in the domain, the adopted grids are usually adapted to the problem geometry. We can find examples of such grids and relative spectral analyses already in [53] for $\Omega = [0,1]^d$ and in [17] for more general domains. In both cases, the grids taken into account were produced starting from a regular grid and by applying an invertible function ϕ. For clarity sake, we start from a smooth (C^1) embedding φ that maps Ω into $[0,1]^d$, and if $D = \varphi(\Omega)$, then we call the inverse $\phi := \varphi^{-1} : D \to \Omega$. Notice that φ is in particular a closed locally Lipschitz map, so D is a compact set in $[0,1]^d$ and it is still Peano-Jordan measurable. We can thus induce a discretization grid on Ω given by $\phi(D \cap \Xi_n)$ for every n.

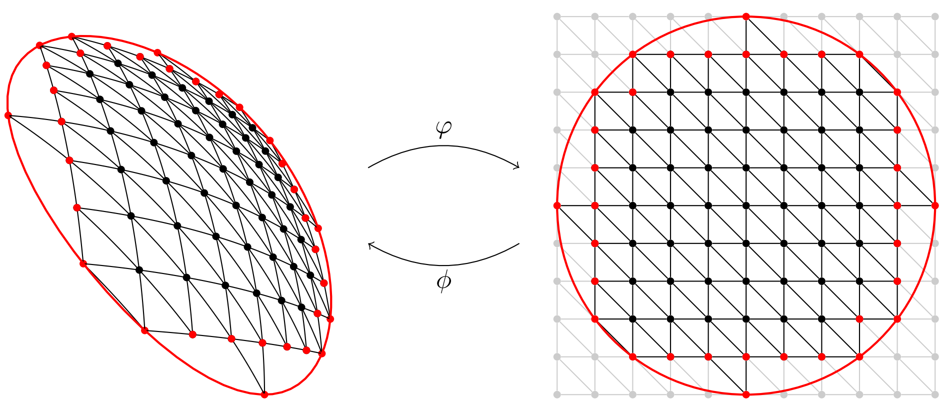


Figure 7.6

We now discretize the diffusion problem (7.13) using modified P_1 finite elements on a compact domain $\Omega \subseteq \mathbb{R}^2$ with positive measure, $\mu(\partial\Omega) = 0$ and grids described by the function ϕ.

$$\begin{cases} -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_j}\Big(a_{i,j}\frac{\partial u}{\partial x_i}\Big) + \sum_{i=1}^{2} b_i \frac{\partial u}{\partial x_i} + cu = f, & \text{in } \Omega^\circ, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{7.13}$$

where $a_{i,j}$, b_i, c and f are given complex-valued L^1 functions defined on Ω.

The basis function we consider on Ω are produced from the classical P_1 elements by composition with the map φ. In fact, if $p \in \Xi_n(D) \subseteq \Xi_n \cap D$ and $p' = \phi(p)$ we can define the basis function associated to p' as

$$\xi_{p',n} := \psi_{p,n} \circ \varphi.$$

Note that the support of $\xi_{p',n}$ is $T_{p'} := \phi(T_p)$ and $T_p \subseteq D \iff T_{p'} \subseteq \Omega$. In the classical P_1 setting, we consider a basis function for each point in $\Xi(D)$, so here we will produce a function $\xi_{p',n}$ only for the points $p' \in \phi(\Xi(D))$, and we call the set of such points

$$\Xi(\Omega) := \phi(\Xi(D)) = \phi\left(\{\, p \in \Xi_n \mid T_p \subseteq D \,\}\right) = \{\, p' \in \phi(\Xi_n \cap D) \mid T_{p'} \subseteq \Omega \,\}.$$

The weak form of the problem (7.14) leads us to a linear system similar to the ones already considered. In fact, if we substitute $u = \sum_{p' \in \Xi_n(\Omega)} u_{p'}\xi_{p',n}$ and $w = \xi_{q',n}$ into problem (7.14), then we obtain the relation

$$\sum_{p' \in \Xi_n(\Omega)} s^{\Omega}_{q',p'} u_{p'} = f_{q'}, \qquad s^{\Omega}_{q',p'} = \int_{\Omega^\circ} (\nabla\xi_{p',n})^T A \nabla\xi_{q',n} + (\nabla\xi_{p',n})^T \boldsymbol{b}\xi_{q',n} + c\xi_{p',n}\xi_{q',n}, \qquad f_{q'} = \int_{\Omega^\circ} f\xi_{q',n} \tag{7.19}$$

for every q' in $\Xi_n(\Omega)$. Sorting the relations in lexicographical order with respect to the appearance of $\varphi(q')$ in the grid Ξ_n, we obtain a linear system $S_n^{\Omega}\boldsymbol{u}_n = \boldsymbol{f}_n$ of size $|\Xi_n(\Omega)| = |\Xi_n(D)|$.

The analysis of this particular instance descends from the fact that we can find opportune coefficients for the problem (7.14) on the domain D so that the linear system arising from the P_1 method applied to the regular grid $\Xi_n(D)$ coincides with S_n^{Ω}. Consider in fact the problem

$$\begin{cases} -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_j}\Big(\widetilde{a}_{i,j}\frac{\partial u}{\partial x_i}\Big) + \sum_{i=1}^{2} \widetilde{b}_i \frac{\partial u}{\partial x_i} + \widetilde{c}u = f, & \text{in } D^\circ, \\ u = 0, & \text{on } \partial D, \end{cases} \tag{7.20}$$

and its weak form

$$\int_{D^\circ} (\nabla u)^T \widetilde{A} \nabla w + (\nabla u)^T \widetilde{\boldsymbol{b}} w + u\widetilde{c}w = \int_{D^\circ} fw, \qquad \forall w \in H^1_0(D). \tag{7.21}$$

where

$$\widetilde{A}(\boldsymbol{x}) := J_\phi^{-1}(\boldsymbol{x}) A(\phi(\boldsymbol{x})) J_\phi^{-T}(\boldsymbol{x}) |\det J_\phi(\boldsymbol{x})|,$$
$$\widetilde{\boldsymbol{b}}(\boldsymbol{x}) := J_\phi^{-1}(\boldsymbol{x})\boldsymbol{b}(\phi(\boldsymbol{x}))|\det J_\phi(\boldsymbol{x})|, \qquad \widetilde{c}(\boldsymbol{x}) := c(\phi(\boldsymbol{x}))|\det J_\phi(\boldsymbol{x})|$$

are L^1 functions on D. Applying the P_1 method to this problem we obtain the relations

$$\sum_{p \in \Xi_n(D)} s^D_{q,p}\widetilde{u}_p = \widetilde{f}_q, \qquad s^D_{q,p} = \int_{D^\circ} (\nabla\psi_{p,n})^T \widetilde{A}\nabla\psi_{q,n} + (\nabla\psi_{p,n})^T \widetilde{\boldsymbol{b}}\psi_{q,n} + \psi_{p,n}\widetilde{c}\psi_{q,n}, \qquad \widetilde{f}_q = \int_{D^\circ} f\psi_{q,n} \tag{7.22}$$

for every $q \in \Xi_n(D)$, that give rise to the system $S_n^D\widetilde{\boldsymbol{u}} = \widetilde{\boldsymbol{f}}_n$ of size $|\Xi_n(D)|$. Notice that if $p', q' \in \Xi_n(\Omega)$ such that $p' = \phi(p)$ and $q' = \phi(q)$, then

$$\begin{aligned} \int_{\Omega^\circ} (\nabla\xi_{p',n})^T A \nabla\xi_{q',n} &= \int_{D^\circ} (\nabla_{\boldsymbol{x}}\psi_p(\boldsymbol{x}))^T J_\phi(\boldsymbol{x})^{-1} A(\phi(\boldsymbol{x})) J_\phi(\boldsymbol{x})^{-T} \nabla_{\boldsymbol{x}}\psi_q(\boldsymbol{x}) |\det J_\phi(\boldsymbol{x})| \mathrm{d}\boldsymbol{x} \\ &= \int_{D^\circ} (\nabla_{\boldsymbol{x}}\psi_p(\boldsymbol{x}))^T \widetilde{A}(\boldsymbol{x}) \nabla_{\boldsymbol{x}}\psi_q(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}, \\ \int_{\Omega^\circ} (\nabla\xi_{p',n})^T \boldsymbol{b}\xi_{q',n} &= \int_{D^\circ} (\nabla_{\boldsymbol{x}}\psi_p(\boldsymbol{x}))^T J_\phi(\boldsymbol{x})^{-1} \boldsymbol{b}(\phi(\boldsymbol{x}))\psi_q(\boldsymbol{x}) |\det J_\phi(\boldsymbol{x})| \mathrm{d}\boldsymbol{x} \\ &= \int_{D^\circ} (\nabla_{\boldsymbol{x}}\psi_p(\boldsymbol{x}))^T \widetilde{\boldsymbol{b}}(\boldsymbol{x})\psi_q(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}, \\ \int_{\Omega^\circ} \xi_{p',n} c \xi_{q',n} &= \int_{D^\circ} \psi_p(\boldsymbol{x}) c(\phi(\boldsymbol{x}))\psi_q(\boldsymbol{x}) |\det J_\phi(\boldsymbol{x})| \mathrm{d}\boldsymbol{x} \\ &= \int_{D^\circ} \psi_p(\boldsymbol{x})\widetilde{c}(\boldsymbol{x})\psi_q(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}, \end{aligned}$$

so comparing Equation 7.22 and Equation 7.19 we conclude that $s^D_{p,q} = s^\Omega_{p',q'}$ for every $p, q \in \Xi(D)$ and therefore $S^\Omega_n = S^D_n$. The symbols of the sequence can be easily computed from Theorem 7.2.2.

Chapter 8

Fractional PDE

8.1 Unidimensional Setting

In [39] it has been studied an initial-boundary value problem presenting factional derivation operators, and the authors derived the spectral symbol of the sequence of systems arising from the discretization of the equations. Using the spectral information, they designed an effective class of preconditioners and tested the performances of the resulting numerical methods. In particular, they considered the following fractional diffusion equations

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = d_+(x,t)\frac{\partial^\alpha u(x,t)}{\partial_+ x^\alpha} + d_-(x,t)\frac{\partial^\alpha u(x,t)}{\partial_- x^\alpha} + f(x,t), & (x,t) \in (L,R)\times(0,T], \\ u(L,t) = u(R,t) = 0, & t \in (0,T], \\ u(x,0) = u_0(x), & x \in [L,R], \end{cases} \tag{8.1}$$

where $\alpha \in (1,2)$, $d_\pm(x,t)$ are nonnegative functions, and we use the shifted Grünwald definition of fractional derivatives given by

$$\begin{aligned} \frac{\partial^\alpha u(x,t)}{\partial_+ x^\alpha} &= \lim_{\Delta x\to 0^+} \frac{1}{\Delta x^\alpha} \sum_{k=0}^{\lfloor (x-L)/\Delta x \rfloor} g_k^{(\alpha)} u(x-(k-1)\Delta x, t), \\ \frac{\partial^\alpha u(x,t)}{\partial_- x^\alpha} &= \lim_{\Delta x\to 0^+} \frac{1}{\Delta x^\alpha} \sum_{k=0}^{\lfloor (R-x)/\Delta x \rfloor} g_k^{(\alpha)} u(x+(k-1)\Delta x, t), \end{aligned} \tag{8.2}$$

with $g_k^{(\alpha)}$ being the alternating fractional binomial coefficients defined as

$$g_k^{(\alpha)} = (-1)^k \binom{\alpha}{k} = \frac{(-1)^k}{k!}\alpha(\alpha-1)\dots(\alpha-k+1) \qquad k = 0,1,\dots.$$

After fixing two positive integers N, M, consider a regular partition on the intervals $[L,R]$ and $[0,T]$ given by the points

$$x_i = L + i\Delta x, \quad \Delta x = \frac{R-L}{N+1}, \quad i = 0,\dots,N+1,$$

$$t_m = m\Delta t, \quad \Delta t = \frac{T}{M}, \quad m = 0,\dots,M.$$

The discretizations used for the problem are an implicit Euler method in time and a shifted Grünwald estimate in space, leading to the following finite difference approximation scheme

$$\frac{u_i^{(m)} - u_i^{(m-1)}}{\Delta t} = \frac{d_{+,i}^{(m)}}{\Delta x^\alpha}\sum_{k=0}^{i} g_k^{(\alpha)} u_{i-k+1}^{(m)} + \frac{d_{-,i}^{(m)}}{\Delta x^\alpha}\sum_{k=0}^{N-i+1} g_k^{(\alpha)} u_{i+k-1}^{(m)} + f_i^{(m)}, \tag{8.3}$$

where $d_{\pm,i}^{(m)} := d_\pm(x_i,t_m)$, $f_i^{(m)} := f(x_i,t_m)$ and $u_i^{(m)}$ is a numerical approximation of $u(x_i,t_m)$. It is possible to summarize all the relations into the expression

$$\mathcal{M}_{\alpha,N}^{(m)}\boldsymbol{u}^{(m)} = \left(\nu_{M,N} I + D_+^{(m)} T_{\alpha,N} + D_-^{(m)} T_{\alpha,N}^T\right)\boldsymbol{u}^{(m)} = \nu_{M,N}\boldsymbol{u}^{(m-1)} + \Delta x^\alpha \boldsymbol{f}^{(m)} \tag{8.4}$$

where $D_{\pm}^{(m)} = \mathrm{diag}(d_{\pm,1}^{(m)}, \dots, d_{\pm,N}^{(m)})$, I is the identity matrix of order N, $\nu_{M,N} = \Delta x^\alpha/\Delta t$ and

$$T_{\alpha,N} = -\begin{pmatrix} g_1^{(\alpha)} & g_0^{(\alpha)} & 0 & \dots & 0 \\ g_2^{(\alpha)} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ g_{N-1}^{(\alpha)} & & \ddots & \ddots & g_0^{(\alpha)} \\ g_N^{(\alpha)} & g_{N-1}^{(\alpha)} & \dots & g_2^{(\alpha)} & g_1^{(\alpha)} \end{pmatrix}, \qquad \boldsymbol{u}^{(m)} = \begin{pmatrix} u_1^{(m)} \\ \vdots \\ u_N^{(m)} \end{pmatrix}, \qquad \boldsymbol{f}^{(m)} = \begin{pmatrix} f_1^{(m)} \\ \vdots \\ f_N^{(m)} \end{pmatrix}$$

If we fix the index m, one can consider $d_{\pm}(x) := d_{\pm}(x, t_m)$ and define the functions

$$f_\alpha(\theta) := -\sum_{k=-1}^{\infty} g_{k+1}^{(\alpha)} e^{\mathrm{i}k\theta} = e^{-\mathrm{i}\theta}\left(1 + e^{\mathrm{i}(\theta+\pi)}\right)^\alpha,$$

$$h_\alpha(x,\theta) := d_+(x) f_\alpha(\theta) + d_-(x) f_\alpha(-\theta), \qquad \hat{h}_\alpha(\hat{x},\theta) := h_\alpha(L + (R-L)\hat{x}, \theta),$$

where $\theta \in [-\pi, \pi]$, $x \in [L, R]$, $\hat{x} \in [0,1]$. It has been proved [39] that whenever $d_{\pm}(x)$ are both Riemann integrable over $[L, R]$, then the sequence of linear systems in Equation 8.4 admits a GLT symbol

$$\{\mathcal{M}_{\alpha,N}^{(m)}\}_N \sim_{GLT} \hat{h}_\alpha$$

and if $d_+ = d_-$, then it admits also a spectral symbol

$$\{\mathcal{M}_{\alpha,N}^{(m)}\}_N \sim_\lambda \hat{h}_\alpha.$$

8.2 Multidimensional Fractional Diffusion Equations

8.2.1 Case on the Square

We can now try to extend the theory to a multidimensional setting. For the sake of simplicity, we only explore the bidimensional case, but the same analysis can be repeated with more dimensions. In particular, let us consider the problem

$$\begin{cases} \frac{\partial u(\boldsymbol{x},t)}{\partial t} = d_+(\boldsymbol{x},t)\frac{\partial^\alpha u(\boldsymbol{x},t)}{\partial_+ x^\alpha} + d_-(\boldsymbol{x},t)\frac{\partial^\alpha u(\boldsymbol{x},t)}{\partial_- x^\alpha} + & \\ \qquad c_+(\boldsymbol{x},t)\frac{\partial^\alpha u(\boldsymbol{x},t)}{\partial_+ y^\alpha} + c_-(\boldsymbol{x},t)\frac{\partial^\alpha u(\boldsymbol{x},t)}{\partial_- y^\alpha} + f(\boldsymbol{x},t), & (\boldsymbol{x},t) \in (0,1)^2 \times (0,T], \\ u(\boldsymbol{x},t) = 0, & (\boldsymbol{x},t) \in \partial[0,1]^2 \times (0,T], \\ u(\boldsymbol{x},0) = u_0(\boldsymbol{x}), & \boldsymbol{x} \in [0,1]^2, \end{cases} \tag{8.5}$$

where $\boldsymbol{x} = (x,y)$, $\alpha \in (1,2)$ and $c_{\pm}(\boldsymbol{x},t), d_{\pm}(\boldsymbol{x},t)$ are nonnegative functions. If we imagine to discretize time as before and space with a 2D regular grid

$$y_i = x_i = i\Delta x, \quad \Delta x = \frac{1}{N+1}, \quad i = 0, \dots, N+1,$$

then the resulting relations are

$$\begin{aligned} \frac{u_{i,j}^{(m)} - u_{i,j}^{(m-1)}}{\Delta t} =& \frac{d_{+,i,j}^{(m)}}{\Delta x^\alpha}\sum_{k=0}^{i} g_k^{(\alpha)} u_{i-k+1,j}^{(m)} + \frac{d_{-,i,j}^{(m)}}{\Delta x^\alpha}\sum_{k=0}^{N-i+1} g_k^{(\alpha)} u_{i+k-1,j}^{(m)} + \\ & \frac{c_{+,i,j}^{(m)}}{\Delta x^\alpha}\sum_{k=0}^{i} g_k^{(\alpha)} u_{i,j-k+1}^{(m)} + \frac{c_{-,i,j}^{(m)}}{\Delta x^\alpha}\sum_{k=0}^{N-i+1} g_k^{(\alpha)} u_{i,j+k-1}^{(m)} + f_{i,j}^{(m)}, \end{aligned} \tag{8.6}$$

where $d_{\pm,i,j}^{(m)} := d_{\pm}(x_i, y_j, t_m)$, $c_{\pm,i,j}^{(m)} := c_{\pm}(x_i, y_j, t_m)$, $f_{i,j}^{(m)} := f(x_i, y_j, t_m)$ and $u_{i,j}^{(m)}$ is a numerical approximation of $u(x_i, y_j, t_m)$.

It is possible to summarize all the relations into the expression

$$\begin{aligned} \mathcal{M}_{\alpha,\boldsymbol{N}}^{(m)}\boldsymbol{u}^{(m)} =& \nu_{M,N}\boldsymbol{u}^{(m)} + \left(D_+^{(m)}(I \otimes T_{\alpha,N}) + D_-^{(m)}(I \otimes T_{\alpha,N}^T)\right)\boldsymbol{u}^{(m)} + \\ & \left(C_+^{(m)}(T_{\alpha,N} \otimes I) + C_-^{(m)}(T_{\alpha,N}^T \otimes I)\right)\boldsymbol{u}^{(m)} = \nu_{M,N}\boldsymbol{u}^{(m-1)} + \Delta x^\alpha \boldsymbol{f}^{(m)} \end{aligned} \tag{8.7}$$

where $D_\pm^{(m)} = \mathrm{diag}(d_{\pm,1,1}^{(m)}, d_{\pm,2,1}^{(m)}, \ldots, d_{\pm,N,N}^{(m)})$, $C_\pm^{(m)} = \mathrm{diag}(c_{\pm,1,1}^{(m)}, c_{\pm,2,1}^{(m)}, \ldots, c_{\pm,N,N}^{(m)})$, I is the identity matrix of order N, $\nu_{M,N} = \Delta x^\alpha/\Delta t$ and

$$T_{\alpha,N} = -\begin{pmatrix} g_1^{(\alpha)} & g_0^{(\alpha)} & 0 & \ldots & 0 \\ g_2^{(\alpha)} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ g_{N-1}^{(\alpha)} & & \ddots & \ddots & g_0^{(\alpha)} \\ g_N^{(\alpha)} & g_{N-1}^{(\alpha)} & \ldots & g_2^{(\alpha)} & g_1^{(\alpha)} \end{pmatrix}, \qquad \boldsymbol{u}^{(m)} = \begin{pmatrix} u_{1,1}^{(m)} \\ \vdots \\ u_{N,N}^{(m)} \end{pmatrix}, \qquad \boldsymbol{f}^{(m)} = \begin{pmatrix} f_{1,1}^{(m)} \\ \vdots \\ f_{N,N}^{(m)} \end{pmatrix}$$

If we fix the index m, one can consider $d_\pm(\boldsymbol{x}) := d_\pm(\boldsymbol{x}, t_m)$, $c_\pm(\boldsymbol{x}) := c_\pm(\boldsymbol{x}, t_m)$ and define the functions

$$f_\alpha(\theta) := -\sum_{k=-1}^{\infty} g_{k+1}^{(\alpha)} e^{\mathrm{i}k\theta} = e^{-\mathrm{i}\theta}\left(1 + e^{\mathrm{i}(\theta+\pi)}\right)^\alpha,$$

$$h_\alpha(\boldsymbol{x}, \boldsymbol{\theta}) := d_+(\boldsymbol{x}) f_\alpha(\theta_1) + d_-(\boldsymbol{x}) f_\alpha(-\theta_1) + c_+(\boldsymbol{x}) f_\alpha(\theta_2) + c_-(\boldsymbol{x}) f_\alpha(-\theta_2),$$

where $\boldsymbol{\theta} = (\theta_1, \theta_2) \in [-\pi, \pi]^2$ and $\boldsymbol{x} \in [0,1]^2$. Using the same arguments of [39], it is possible to conclude that whenever $d_\pm(\boldsymbol{x})$, $c_\pm(\boldsymbol{x})$ are all Riemann integrable over $[0,1]^2$, then the sequence of linear systems in Equation 8.7 admits a GLT symbol

$$\{\mathcal{M}_{\alpha,\boldsymbol{N}}^{(m)}\}_N \sim_{GLT} h_\alpha$$

and if $d_+ = d_- = c_+ = c_-$, then it admits also a spectral symbol

$$\{\mathcal{M}_{\alpha,\boldsymbol{N}}^{(m)}\}_N \sim_\lambda h_\alpha.$$

8.2.2 Case on Other Spatial Domains

In this section, we consider the same problem, but change the spatial domain from $[0,1]^2$ to a closed Peano-Jordan measurable set $\Omega \subseteq [0,1]^2$.

$$\begin{cases} \frac{\partial u(\boldsymbol{x},t)}{\partial t} = d_+(\boldsymbol{x},t)\frac{\partial^\alpha u(\boldsymbol{x},t)}{\partial_+ x^\alpha} + d_-(\boldsymbol{x},t)\frac{\partial^\alpha u(\boldsymbol{x},t)}{\partial_- x^\alpha} + & \\ \qquad c_+(\boldsymbol{x},t)\frac{\partial^\alpha u(\boldsymbol{x},t)}{\partial_+ y^\alpha} + c_-(\boldsymbol{x},t)\frac{\partial^\alpha u(\boldsymbol{x},t)}{\partial_- y^\alpha} + f(\boldsymbol{x},t), & (\boldsymbol{x},t) \in \Omega^\circ \times (0,T], \\ u(\boldsymbol{x},t) = 0, & (\boldsymbol{x},t) \in \partial\Omega \times (0,T], \\ u(\boldsymbol{x},0) = u_0(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \end{cases} \tag{8.8}$$

where $\boldsymbol{x} = (x,y)$, $\alpha \in (1,2)$ and $c_\pm(\boldsymbol{x},t), d_\pm(\boldsymbol{x},t)$ are nonnegative functions.

Note that the fractional derivative defined in Riemann-Liouville form is not a local operator, and it only works for functions defined on connected segments of the real line. For this reason, we need that for every point $p \in \Omega$, the sets

$$\{\, s \in \mathbb{R} \mid p + s(1,0) \in \Omega \,\}, \qquad \{\, s \in \mathbb{R} \mid p + s(0,1) \in \Omega \,\}$$

are just segments. All convex shapes belong to this class of domains, but also non-convex sets such as the L-shapes.

We use the same grid, the same notations and the same approximation rules of section 8.2, except that we only consider points (x_i, y_j) inside Ω°. We call the set of such points Ξ_N, and when a point (x_i, y_j) belongs to Ξ_N, we write $(i,j) \in \Xi_N$ for brevity. The resulting relation for each $(i,j) \in \Xi_N$ is

$$\begin{aligned} \frac{u_{i,j}^{(m)} - u_{i,j}^{(m-1)}}{\Delta t} = & \frac{d_{+,i,j}^{(m)}}{\Delta x^\alpha} \sum_{\substack{k=0,\ldots,i \\ (i-k+1,j)\in\Xi_N}} g_k^{(\alpha)} u_{i-k+1,j}^{(m)} + \frac{d_{-,i,j}^{(m)}}{\Delta x^\alpha} \sum_{\substack{k=0,\ldots,N-i+1 \\ (i+k-1,j)\in\Xi_N}} g_k^{(\alpha)} u_{i+k-1,j}^{(m)} + \\ & \frac{c_{+,i,j}^{(m)}}{\Delta x^\alpha} \sum_{\substack{k=0,\ldots,i \\ (i-k+1,j)\in\Xi_N}} g_k^{(\alpha)} u_{i,j-k+1}^{(m)} + \frac{c_{-,i,j}^{(m)}}{\Delta x^\alpha} \sum_{\substack{k=0,\ldots,N-i+1 \\ (i+k-1,j)\in\Xi_N}} g_k^{(\alpha)} u_{i,j+k-1}^{(m)} + f_{i,j}^{(m)}. \end{aligned} \tag{8.9}$$

It is possible to summarize all the relations into the expression

$$\mathcal{M}_{\alpha,\boldsymbol{N}}^{(m),\Omega} \boldsymbol{u}^{(m)} = \nu_{M,N} \boldsymbol{u}^{(m-1)} + \Delta x^\alpha \boldsymbol{f}^{(m)},$$

where

$$\boldsymbol{u}^{(m)} = \left(u_{i,j}^{(m)}\right)_{(i,j)\in\Xi_n}, \qquad \boldsymbol{f}^{(m)} = \left(f_{i,j}^{(m)}\right)_{(i,j)\in\Xi_n}.$$

Abusing the notation, we can call $d_\pm, c_\pm, f$ the extensions of the functions in Equation 8.8 to $[0,1]^2$, where $d_\pm(\boldsymbol{x}) = c_\pm(\boldsymbol{x}) = f(\boldsymbol{x}) = 1$ whenever $\boldsymbol{x} \notin \Omega^\circ$, and If we fix the index m, then we can call $M_{\alpha,\boldsymbol{N}}^{(m)}$ the matrix arising from the discretization of Equation 8.5 with such functions, along with its symbol $h_\alpha(\boldsymbol{x},\boldsymbol{\theta})$. One can also define the reduced functions

$$h_\alpha^\Omega(\boldsymbol{x},\boldsymbol{\theta}) := h_\alpha(\boldsymbol{x},\boldsymbol{\theta})|_{\boldsymbol{x}\in\Omega^\circ}$$

that will be the symbol of our reduced sequence.

Lemma 8.2.1.

$$\mathcal{M}_{\alpha,\boldsymbol{N}}^{(m),\Omega} = R_{\Omega^\circ}(\mathcal{M}_{\alpha,\boldsymbol{N}}^{(m)})$$

and if $d_\pm, c_\pm$ are all Riemann integrable over Ω at time t_m, then $\{\mathcal{M}_{\alpha,\boldsymbol{N}}^{(m),\Omega}\}_N \sim_{GLT}^{\Omega} h_\alpha^\Omega$.

Proof. Note that if $\boldsymbol{p},\boldsymbol{q} \in \Xi_N$, then a quick comparison of Equation 8.6 and Equation 8.9 shows that $[\mathcal{M}_{\alpha,\boldsymbol{N}}^{(m)}]_{\boldsymbol{p},\boldsymbol{q}}$ and $[\mathcal{M}_{\alpha,\boldsymbol{N}}^{(m),\Omega}]_{\boldsymbol{p},\boldsymbol{q}}$ are equal, since the equations are identical. When $d_\pm, c_\pm$ are all Riemann integrable over Ω, we know that their extension to $[0,1]^2$ are still Riemann integrable, so

$$\{\mathcal{M}_{\alpha,\boldsymbol{N}}^{(m)}\}_N \sim_{GLT} h_\alpha.$$

By definition of reduced GLT sequences and symbols, we conclude that

$$\{\mathcal{M}_{\alpha,\boldsymbol{N}}^{(m),\Omega}\}_N \sim_{GLT}^{\Omega} h_\alpha(\boldsymbol{x},\boldsymbol{\theta})|_{\boldsymbol{x}\in\Omega^\circ} = h_\alpha^\Omega(\boldsymbol{x},\boldsymbol{\theta}).$$

□

Lemma 8.2.2. *If $d_\pm = c_\pm$ and they are all Riemann integrable over Ω at time t_m, then $\{\mathcal{M}_{\alpha,\boldsymbol{N}}^{(m),\Omega}\}_N \sim_\lambda h_\alpha^\Omega$.*

Proof. Let us rewrite here the definition of $\mathcal{M}_{\alpha,\boldsymbol{N}}^{(m)}$

$$\mathcal{M}_{\alpha,\boldsymbol{N}}^{(m)} = \nu_{M,N} I + D_+^{(m)}(I \otimes T_{\alpha,N}) + D_-^{(m)}(I \otimes T_{\alpha,N}^T) + C_+^{(m)}(T_{\alpha,N} \otimes I) + C_-^{(m)}(T_{\alpha,N}^T \otimes I).$$

When $d := d_\pm = c_\pm$, we have that $D^{(m)} := D_+^{(m)} = D_-^{(m)} = C_+^{(m)} = C_-^{(m)}$ so we can rewrite the expression as

$$\mathcal{M}_{\alpha,\boldsymbol{N}}^{(m)} = \nu_{M,N} I + D^{(m)}(I \otimes (T_{\alpha,N} + T_{\alpha,N}^T) + (T_{\alpha,N} + T_{\alpha,N}^T) \otimes I).$$

Consider now the matrices $(D^{(m)})^{-1/2} Z_{\Omega^\circ}(\mathcal{M}_{\alpha,\boldsymbol{N}}^{(m)})(D^{(m)})^{1/2}$. We know that it is a GLT sequence since

$$\{(D^{(m)})^{1/2}\}_N \sim_{GLT} d^{1/2}, \qquad \{(D^{(m)})^{-1/2}\}_N \sim_{GLT} d^{-1/2},$$

$$\{Z_{\Omega^\circ}(\mathcal{M}_{\alpha,\boldsymbol{N}}^{(m)})\}_N \sim_{GLT} \widetilde{h}_\alpha(\boldsymbol{x},\boldsymbol{\theta}) := \begin{cases} h_\alpha(\boldsymbol{x},\boldsymbol{\theta}) & \boldsymbol{x} \in \Omega^\circ \\ 0 & \boldsymbol{x} \notin \Omega^\circ \end{cases}$$

$$\implies \{(D^{(m)})^{-1/2} Z_{\Omega^\circ}(\mathcal{M}_{\alpha,\boldsymbol{N}}^{(m)})(D^{(m)})^{1/2}\}_N \sim_{GLT} \widetilde{h}_\alpha(\boldsymbol{x},\boldsymbol{\theta}).$$

Moreover, all the matrices are Hermitian, since $D^{(m)}$ are diagonal matrices and by definition of Z_{Ω°,

$$\begin{aligned}(D^{(m)})^{-1/2} Z_{\Omega^\circ}(\mathcal{M}_{\alpha,\boldsymbol{N}}^{(m)})(D^{(m)})^{1/2} =& (D^{(m)})^{-1/2} I_{\boldsymbol{n}}(\chi_{\Omega^\circ}) M_{\alpha,\boldsymbol{N}}^{(m)} I_{\boldsymbol{n}}(\chi_{\Omega^\circ})(D^{(m)})^{1/2} \\ =& I_{\boldsymbol{n}}(\chi_{\Omega^\circ})(D^{(m)})^{-1/2} M_{\alpha,\boldsymbol{N}}^{(m)}(D^{(m)})^{1/2} I_{\boldsymbol{n}}(\chi_{\Omega^\circ}) \\ =& \nu_{M,N} I_{\boldsymbol{n}}(\chi_{\Omega^\circ}) + I_{\boldsymbol{n}}(\chi_{\Omega^\circ})(D^{(m)})^{1/2}(I \otimes (T_{\alpha,N} + T_{\alpha,N}^T) \\ &+ (T_{\alpha,N} + T_{\alpha,N}^T) \otimes I)(D^{(m)})^{1/2} I_{\boldsymbol{n}}(\chi_{\Omega^\circ}).\end{aligned}$$

Since it is an Hermitian GLT sequence, we know its spectral symbol

$$\{(D^{(m)})^{-1/2} Z_{\Omega^\circ}(\mathcal{M}^{(m)}_{\alpha,\boldsymbol{N}})(D^{(m)})^{1/2}\}_N \sim_\lambda \widetilde{h}_\alpha(\boldsymbol{x},\boldsymbol{\theta})$$

and as a consequence, we also know the spectral symbol of

$$\{Z_{\Omega^\circ}(\mathcal{M}^{(m)}_{\alpha,\boldsymbol{N}})\}_N \sim_\lambda \widetilde{h}_\alpha(\boldsymbol{x},\boldsymbol{\theta})$$

since we have applied a similitude that does not change the eigenvalues. The sequence $\{Z_{\Omega^\circ}(\mathcal{M}^{(m)}_{\alpha,\boldsymbol{N}})\}_N$ is obviously a fixed point for the operator Z_{Ω° and $\widetilde{h}_\alpha(\boldsymbol{x},\boldsymbol{\theta})|_{\boldsymbol{x}\notin\Omega^\circ} = 0$, so we have all the hypothesis to apply Lemma 6.3.5 and conclude that

$$\{R_{\Omega^\circ} \circ Z_{\Omega^\circ}(\mathcal{M}^{(m)}_{\alpha,\boldsymbol{N}})\}_N = \{\mathcal{M}^{(m),\Omega}_{\alpha,\boldsymbol{N}}\}_N \sim_\lambda \widetilde{h}_\alpha(\boldsymbol{x},\boldsymbol{\theta})|_{\boldsymbol{x}\in\Omega^\circ} = h^\Omega_\alpha(\boldsymbol{x},\boldsymbol{\theta}).$$

□

Chapter 9

g-Toeplitz

9.1 Unidimensional Case

In this section we derive the asymptotic singular value distribution for the sequence of g-Toeplitz matrices $\{T_{n,g}(f)\}_n$. The extension to the multidimensional case uses the same arguments, but is more convoluted due to the notation, so it is advised to read the present section first. Here we provide a fast lookout on the steps we go through to obtain the symbol:

- notice that every matrix $T_{n,g}$ is a non-principal minor of the Toeplitz matrix $T_{ng}(f)$, for which we know a symbol,
- produce two sequences of block diagonal matrices $P_{n,g}$ and $P'_{n,g}$ both enjoying a symbol, such that $P_{n,g}T_{ng}(f)P'_{n,g}$ zeroes out the entries not belonging to $T_{n,g}$,
- deduce a sequence of permutation matrices $R_{n,g}$ such that $T_{n,g}$ is a principal minor of A_n, defined as $P_{n,g}T_{ng}(f)P'_{n,g}R_{n,g}$,
- derive the symbol of $\{A_n\}_n$ and restrict it to obtain the symbol of $\{T_{n,g}(f)\}_n$.

9.1.1 Symbol Derivation

Consider the g-Toeplitz matrix of dimensions n associated to the L^1 functions $f : [-\pi, \pi] \to \mathbb{C}$

$$T_{n,g}(f) = [f_{i-1-g(j-1)}]_{i,j=1}^{n}.$$

We can note that it is a submatrix of the Toeplitz matrix $T_{ng}(f)$, since

$$[T_{n,g}(f)]_{i,j} = f_{i-1-g(j-1)} = [T_{ng}(f)]_{i,1+g(j-1)}$$

so we can define $P_{n,g}$ and $P'_{n,g}$ diagonal binary matrix of dimension ng that zeros out all elements of $T_{ng}(f)$ not belonging to $T_{n,g}(f)$. In particular

$$[P_{n,g}]_{i,j} = \begin{cases} 1 & i = j \le n, \\ 0 & \text{otherwise,} \end{cases} \qquad [P'_{n,g}]_{i,j} = \begin{cases} 1 & i = j \equiv 1 \pmod g, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 9.1.1.

$$[P_{n,g}T_{ng}(f)P'_{n,g}]_{i,j} = \begin{cases} [T_{n,g}(f)]_{i,t} & i \le n, \quad j = 1 + g(t-1) \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. If $i > n$, then

$$[P_{n,g}T_{ng}(f)P'_{n,g}]_{i,j} = \sum_{k=1}^{ng} [P_{n,g}]_{i,k}[T_{ng}(f)P'_{n,g}]_{k,j} = 0$$

and similarly, if $j \not\equiv 1 \pmod g$, then

$$[P_{n,g}T_{ng}(f)P'_{n,g}]_{i,j} = \sum_{k=1}^{ng}[P_{n,g}T_{ng}(f)]_{i,k}[P'_{n,g}]_{k,j} = 0.$$

The only remaining case is $i \leq n$ and $j \equiv 1 \pmod g$, that can be rewritten as $j = 1 + g(t-1)$ for some $t = 1, \dots, n$.

$$\begin{aligned}
[P_{n,g}T_{ng}(f)P'_{n,g}]_{i,j} &= \sum_{k=1}^{ng}[P_{n,g}]_{i,k}[T_{ng}(f)P'_{n,g}]_{k,j} \\
&= [T_{ng}(f)P'_{n,g}]_{i,j} \\
&= \sum_{k=1}^{ng}[T_{ng}(f)]_{i,k}[P'_{n,g}]_{k,j} \\
&= [T_{ng}(f)]_{i,j} = [T_{n,g}(f)]_{i,t}.
\end{aligned}$$

$\square$

Example 9.1.2

- From now on we provide a visual of the various steps with parameters $g = 3, n = 4$.

$$T_{4,3}(f) = \begin{bmatrix} f_0 & f_{-3} & f_{-6} & f_{-9} \\ f_1 & f_{-2} & f_{-5} & f_{-8} \\ f_2 & f_{-1} & f_{-4} & f_{-7} \\ f_3 & f_0 & f_{-3} & f_{-6} \end{bmatrix}$$

$$T_{12}(f) = \left[\begin{array}{ccc|ccc|ccc|ccc}
f_0 & f_{-1} & f_{-2} & f_{-3} & f_{-4} & f_{-5} & f_{-6} & f_{-7} & f_{-8} & f_{-9} & f_{-10} & f_{-11} \\
f_1 & f_0 & f_{-1} & f_{-2} & f_{-3} & f_{-4} & f_{-5} & f_{-6} & f_{-7} & f_{-8} & f_{-9} & f_{-10} \\
f_2 & f_1 & f_0 & f_{-1} & f_{-2} & f_{-3} & f_{-4} & f_{-5} & f_{-6} & f_{-7} & f_{-8} & f_{-9} \\
\hline
f_3 & f_2 & f_1 & f_0 & f_{-1} & f_{-2} & f_{-3} & f_{-4} & f_{-5} & f_{-6} & f_{-7} & f_{-8} \\
f_4 & f_3 & f_2 & f_1 & f_0 & f_{-1} & f_{-2} & f_{-3} & f_{-4} & f_{-5} & f_{-6} & f_{-7} \\
f_5 & f_4 & f_3 & f_2 & f_1 & f_0 & f_{-1} & f_{-2} & f_{-3} & f_{-4} & f_{-5} & f_{-6} \\
\hline
f_6 & f_5 & f_4 & f_3 & f_2 & f_1 & f_0 & f_{-1} & f_{-2} & f_{-3} & f_{-4} & f_{-5} \\
f_7 & f_6 & f_5 & f_4 & f_3 & f_2 & f_1 & f_0 & f_{-1} & f_{-2} & f_{-3} & f_{-4} \\
f_8 & f_7 & f_6 & f_5 & f_4 & f_3 & f_2 & f_1 & f_0 & f_{-1} & f_{-2} & f_{-3} \\
\hline
f_9 & f_8 & f_7 & f_6 & f_5 & f_4 & f_3 & f_2 & f_1 & f_0 & f_{-1} & f_{-2} \\
f_{10} & f_9 & f_8 & f_7 & f_6 & f_5 & f_4 & f_3 & f_2 & f_1 & f_0 & f_{-1} \\
f_{11} & f_{10} & f_9 & f_8 & f_7 & f_6 & f_5 & f_4 & f_3 & f_2 & f_1 & f_0
\end{array}\right]$$

$$P_{4,3} = \left[\begin{array}{ccc|ccc|ccc|ccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]$$

$$P'_{4,3} = \left[\begin{array}{ccc|ccc|ccc|ccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]$$

$$P_{3,2}T_{12}(f)P'_{3,2} = \left[\begin{array}{ccc|ccc|ccc|ccc}
f_0 & 0 & 0 & f_{-3} & 0 & 0 & f_{-6} & 0 & 0 & f_{-9} & 0 & 0 \\
f_1 & 0 & 0 & f_{-2} & 0 & 0 & f_{-5} & 0 & 0 & f_{-8} & 0 & 0 \\
f_2 & 0 & 0 & f_{-1} & 0 & 0 & f_{-4} & 0 & 0 & f_{-7} & 0 & 0 \\
\hline
f_3 & 0 & 0 & f_0 & 0 & 0 & f_{-3} & 0 & 0 & f_{-6} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]$$

The last step before going into the analysis of the symbols for the sequence is to bring $T_{n,g}(f)$ into a principal minor of $P_{n,g}T_{ng}(f)P'_{n,g}$ that can be done through a permutation of the columns. In fact we can consider a permutation $\tau \in S_{gn}$ such that $\tau(1+g(t-1)) = t$ for every $t \le n$, and write the permutation matrix

$$R_{n,g} = [\delta_{\tau(i),j}]_{i,j=1}^{ng} = \begin{cases} 1 & \tau(i) = j, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 9.1.3.

$$[P_{n,g}T_{ng}(f)P'_{n,g}R_{n,g}]_{i,j} = \begin{cases} [T_{n,g}(f)]_{i,j} & i,j \le n, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Throughout this proof, we repetitively use Lemma 9.1.1. First, suppose $i > n$.

$$[P_{n,g}T_{ng}(f)P'_{n,g}R_{n,g}]_{i,j} = \sum_{k=1}^{ng}[P_{n,g}]_{i,k}[T_{ng}(f)P'_{n,g}R_{n,g}]_{k,j} = 0.$$

If we instead suppose $j > n$, then

$$\begin{aligned}
[P_{n,g}T_{ng}(f)P'_{n,g}R_{n,g}]_{i,j} &= \sum_{k=1}^{ng}[P_{n,g}T_{ng}(f)P'_{n,g}]_{i,k}[R_{n,g}]_{k,j} \\
&= \sum_{t=1}^{n}[P_{n,g}T_{ng}(f)P'_{n,g}]_{i,1+g(t-1)}[R_{n,g}]_{1+g(t-1),j} = 0.
\end{aligned}$$

The only remaining case is $i,j \le n$ for which

$$\begin{aligned}
[P_{n,g}T_{ng}(f)P'_{n,g}R_{n,g}]_{i,j} &= \sum_{k=1}^{ng}[P_{n,g}T_{ng}(f)P'_{n,g}]_{i,k}[R_{n,g}]_{k,j} \\
&= \sum_{t=1}^{n}[P_{n,g}T_{ng}(f)P'_{n,g}]_{i,1+g(t-1)}[R_{n,g}]_{1+g(t-1),j} \\
&= [P_{n,g}T_{ng}(f)P'_{n,g}]_{i,1+g(j-1)}[R_{n,g}]_{1+g(j-1),j} \\
&= [T_{n,g}(f)]_{i,j}.
\end{aligned}$$

□

Example 9.1.4

▶

$$R_{4,3} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P_{4,3}T_{12}(f)P'_{4,3}R_{4,3} = \left[\begin{array}{cccc|cccccccc} f_0 & f_{-3} & f_{-6} & f_{-9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ f_1 & f_{-2} & f_{-5} & f_{-8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ f_2 & f_{-1} & f_{-4} & f_{-7} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ f_3 & f_0 & f_{-3} & f_{-6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

We can now start to analyse the asymptotic behaviour of the singular values of the sequences. First of all, we recognise that $T_{ng}(f)$ is actually a block Toeplitz sequence with symbol $1 \otimes F(\theta) : [-\pi, \pi] \to \mathbb{C}^{g\times g}$, defined by its Fourier coefficients

$$F_k = [f_{gk+\ell-m}]_{\ell,m=1}^{s}.$$

Note that it may not be possible to define F via its Fourier coefficients when f is only in L^1, so we need to assume at least $f \in L^2[-\pi, \pi]$.

Lemma 9.1.5. *$T_{ng}(f) = T_n(F)$ for every n, so*

$$\{T_{ng}(f)\} \sim_\sigma 1 \otimes F(\theta).$$

Proof. For every $1 \le \ell, m \le g$ and $1 \le i, j \le n$ we have

$$[T_n(F)]_{\ell+g(i-1),m+g(j-1)} = [F_{i-j}]_{\ell,m} = f_{g(i-j)+\ell-m} = [T_n(f)]_{\ell+g(i-1),m+g(j-1)}.$$

□

We also recognize that $P'_{n,g} = D_n(E_{1,1})$, where $E_{1,1}$ is the constant binary matrix of size g that have a single non-zero entry in position $(1, 1)$.

Lemma 9.1.6. *$P'_{n,g} = D_n(E_{1,1})$ for every n, so*

$$\{P'_{n,g}\} \sim_\sigma E_{1,1} \otimes 1.$$

Proof. From the definition, $P'_{n,g}$ is a diagonal matrix, like $D_n(E_{1,1})$ so we have just to compare the diagonal entries. For every $1 \le \ell \le g$ and $1 \le i \le n$ we have

$$[D_n(E_{1,1})]_{\ell+g(i-1),\ell+g(i-1)} = \delta_{\ell,1} = [P'_{n,g}]_{\ell+g(i-1),\ell+g(i-1)}.$$

□

The sequence of $P_{n,g}$ requires some extra work, since it is not always a block sampling sequence, but it differs from one up to a zero-distributed sequence.

Lemma 9.1.7. *There exists a zero-distributed sequence $\{Z_n\}_n$ such that*

$$P_{n,g} = D_n(\chi_{[0,1/g]} I_g) + Z_n.$$

Proof. Notice that both $P_{n,g}$ and $D_n(\chi_{[0,1/g]} I_g)$ are binary and diagonal sequences, so $Z_n = P_{n,g} - D_n(\chi_{[0,1/g]} I_g)$ is also a diagonal sequence with entries $0, 1, -1$. We want to bound the rank of Z_n, so we need to look up for the non-zero entries on the diagonal, or also said, the number of different entries between $P_{n,g}$ and $D_n(\chi_{[0,1/g]} I_g)$.

If $i \le g\lfloor n/g \rfloor$, then we have

$$i \le n \implies [P_{n,g}]_{i,i} = 1, \qquad i/ng \le 1/g \implies [D_n(\chi_{[0,1/g]} I_g)]_{i,i} = 1 \qquad \implies \qquad [Z_n]_{i,i} = 0.$$

On the other hand, if $i > g\lceil n/g \rceil$, then

$$i > n \implies [P_{n,g}]_{i,i} = 0, \qquad i/ng > 1/g \implies [D_n(\chi_{[0,1/g]} I_g)]_{i,i} = 0 \qquad \implies \qquad [Z_n]_{i,i} = 0.$$

In other words, the only non-zero elements of Z_n can be in number at most $g\lceil n/g \rceil - g\lfloor n/g \rfloor \le g$ and as a consequence,

$$\operatorname{rk} Z_n \le g = o(gn).$$

Hence, we conclude that $\{Z_n\}_n$ is zero-distributed. □

It is now easy to sum up all the results into a symbol for $\{P_{n,g} T_{ng}(f) P'_{n,g} R_{n,g}\}_n$.

Lemma 9.1.8.

$$\{P_{n,g} T_{ng}(f) P'_{n,g} R_{n,g}\}_n \sim_\sigma \chi_{[0,1/g]}(x) F(\theta) E_{1,1}.$$

Proof. Using Lemmas 9.1.7, 9.1.6 and 9.1.5 we can write

$$P_{n,g} T_{ng}(f) P'_{n,g} = (D_n(\chi_{[0,1/g]} I_s) + Z_n) T_n(F) D_n(E_{1,1})$$

and using the axioms **GLT 1,3,4** for block GLT sequences, we conclude that

$$\{P_{n,g} T_{ng}(f) P'_{n,g}\}_n \sim_\sigma \chi_{[0,1/g]}(x) F(\theta) E_{1,1}.$$

Eventually, notice that $R_{n,g}$ is an unitary matrix, so it does not change the singular values of the matrices when multiplied, and the thesis follows. □

From Lemma 9.1.3, we know that $T_{n,g}(f)$ is a principal minor of $A_n := P_{n,g} T_{ng}(f) P'_{n,g} R_{n,g}$ and every other entry is zero, so the singular values of A_n are the same of $T_{n,g}(f)$ except for additional $ng - n$ zeros. Moreover, notice that the symbol

$$\chi_{[0,1/g]}(x) F(\theta) E_{1,1} = \chi_{[0,1/g]}(x) \left[\begin{array}{c|c} F_{1,1}(\theta) & \\ \vdots & 0 \\ F_{g,1}(\theta) & \end{array} \right]$$

has at most one non-zero singular value for every (x, θ), given by $\chi_{[0,1/g]}(x) \|v(\theta)\|_2$, where v is the first column of F. Using all these informations, we can finally derive a symbol for $T_{n,g}(f)$.

Theorem 9.1.9.

$$\{T_{n,g}(f)\}_n \sim_\sigma \chi_{[0,1/g]}(x) \|v(\theta)\|_2.$$

Proof. Let us check the ergodic formula relative to matrix-valued symbols for for the sequence A_n, defined as $P_{n,g}T_{ng}(f)P'_{n,g}R_{n,g}$. Recall that the singular values of A_n are the same of $T_{n,g}(f)$ except for additional $ng-n$ zeros. Given any function $G \in C_c(\mathbb{C})$,

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{ng}\sum_{i=1}^{ng} G(\sigma_i(A_n)) &= \lim_{n\to\infty} \frac{1}{ng}\sum_{i=1}^{n} G(\sigma_i(T_{n,g}(f))) + \frac{ng-n}{ng}G(0) \\
&= \frac{g-1}{g}G(0) + \frac{1}{g}\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} G(\sigma_i(T_{n,g}(f))), \\
\lim_{n\to\infty} \frac{1}{ng}\sum_{i=1}^{ng} G(\sigma_i(A_n)) &= \frac{1}{\mu(D)}\int_D \sum_{i=1}^{g} \frac{G(\sigma_i(\chi_{[0,1/g]}(x)G(\theta)E_{1,1}))}{g} \mathrm{d}(x,\theta) \\
&= \frac{1}{\mu()D)}\int_D \frac{g-1}{g}G(0)\mathrm{d}(x,\theta) + \frac{1}{\mu(D)}\int_D \frac{G(\chi_{[0,1/g]}(x)\|v(\theta)\|_2)}{g}\mathrm{d}(x,\theta) \\
&= \frac{g-1}{g}G(0) + \frac{1}{g}\frac{1}{\mu(D)}\int_D G(\chi_{[0,1/g]}(x)\|v(\theta)\|_2)\mathrm{d}(x,\theta).
\end{aligned}$$

As a consequence, for every $G \in C_c(\mathbb{C})$

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} G(\sigma_i(T_{n,g}(f))) &= \frac{1}{\mu(D)}\int_D G(\chi_{[0,1/g]}(x)\|v(\theta)\|_2)\mathrm{d}(x,\theta) \\
&\implies \{T_{n,g}(f)\}_n \sim_\sigma \chi_{[0,1/g]}(x)\|v(\theta)\|_2,
\end{aligned}$$

and by definition

$$\{T_{n,g}(f)\}_n \sim_\sigma \chi_{[0,1/g]}(x)\|v(\theta)\|_2.$$

□

9.1.2 Precedent Result

In [64], a different argument was used to obtain a symbol for $\{T_{n,g}(f)\}_n$. Here we will show, with a formal computation, that the result we just derived coincides with the one in [64].

With our notation, the precedent result was

$$\{T_{n,g}(f)\}_n \sim_\sigma \chi_{[0,1/g]}(x)\sqrt{\frac{1}{g}\sum_{j=0}^{g-1}\left|f\left(\frac{\theta+2\pi j}{g}\right)\right|^2}$$

so we just need to prove the following.

Lemma 9.1.10.

$$\|v(\theta)\|_2^2 = \frac{1}{g}\sum_{j=0}^{g-1}\left|f\left(\frac{\theta+2\pi j}{g}\right)\right|^2$$

Proof.

$$\begin{aligned}
\frac{1}{g}\sum_{j=0}^{g-1}\left|f\left(\frac{\theta+2\pi j}{g}\right)\right|^2 &= \frac{1}{g}\sum_{j=0}^{g-1}\left|\sum_k e^{k\mathrm{i}\frac{\theta+2\pi j}{g}} f_k\right|^2 \\
&= \frac{1}{g}\sum_{j=0}^{g-1}\sum_{k,h} e^{(k-h)\mathrm{i}\frac{\theta+2\pi j}{g}} f_k\overline{f}_h \\
&= \frac{1}{g}\sum_{k,h} e^{(k-h)\mathrm{i}\frac{\theta}{g}} f_k\overline{f}_h \sum_{j=0}^{g-1} e^{(k-h)\mathrm{i}\frac{2\pi j}{g}} \\
&= \sum_{k,t} e^{t\mathrm{i}\theta} f_k\overline{f}_{k-tg},
\end{aligned}$$

$$\begin{aligned}
\|v\|_2^2 &= \sum_{j=1}^{g} |F_{j,1}|^2 \\
&= \sum_{j=1}^{g} \left| \sum_{r} f_{gr+j-1} e^{r\mathrm{i}\theta} \right|^2 \\
&= \sum_{j=1}^{g} \sum_{r,h} f_{gr+j-1} \overline{f}_{gh+j-1} e^{(r-h)\mathrm{i}\theta} \\
&= \sum_{j=1}^{g} \sum_{r,t} f_{gr+j-1} \overline{f}_{gr-gt+j-1} e^{t\mathrm{i}\theta} \\
&= \sum_{k,t} f_k \overline{f}_{k-gt} e^{ti\theta}.
\end{aligned}$$

□

Note that under our hypothesis, that is $f \in L^2$, all the computations are actually justified.

9.2 Multidimensional Case

Let us define a $\boldsymbol{g}$-Toeplitz matrix as follows:

$$A_{\boldsymbol{n}} = [a_{\boldsymbol{r}-\boldsymbol{1}-\boldsymbol{g}(\boldsymbol{s}-\boldsymbol{1})}]_{\boldsymbol{r},\boldsymbol{s}=\boldsymbol{1}}^{\boldsymbol{n}}$$

where $\boldsymbol{g} = (g_1, \ldots, g_d)$ is a d-index. Suppose that $a_{\boldsymbol{k}}$ is the $\boldsymbol{k}$-th Fourier coefficient of a d-multivariate function $f : [-\pi, \pi]^d \to \mathbb{C}$, and $f \in L^2$. We denote the matrix as $T_{\boldsymbol{n},\boldsymbol{g}}(f) = A_{\boldsymbol{n}}$.

Analogously to the unidimensional case, here we derive the asymptotic singular value distribution for the sequence of $\boldsymbol{g}$-Toeplitz matrices $\{T_{\boldsymbol{n},\boldsymbol{g}}(f)\}_n$. Here we provide a fast lookout on the steps we go through to obtain the symbol:

- notice that every matrix $T_{\boldsymbol{n},\boldsymbol{g}}$ is a non-principal minor of the Toeplitz matrix $T_{\boldsymbol{ng}}(f)$, for which we know a symbol up to a permutation,
- produce two sequences of block diagonal matrices $P_{\boldsymbol{n},\boldsymbol{g}}$ and $P'_{\boldsymbol{n},\boldsymbol{g}}$ both enjoying a symbol up to a permutation, such that $P_{\boldsymbol{n},\boldsymbol{g}} T_{\boldsymbol{ng}}(f) P'_{\boldsymbol{n},\boldsymbol{g}}$ zeroes out the entries not belonging to $T_{\boldsymbol{n},\boldsymbol{g}}$,
- deduce two sequences of permutation matrices $R_{\boldsymbol{n},\boldsymbol{g}}$ and $R'_{\boldsymbol{n},\boldsymbol{g}}$ such that $T_{\boldsymbol{n},\boldsymbol{g}}$ is a principal minor of $A_{\boldsymbol{n}} = R'_{\boldsymbol{n},\boldsymbol{g}} P_{\boldsymbol{n},\boldsymbol{g}} T_{\boldsymbol{ng}}(f) P'_{\boldsymbol{n},\boldsymbol{g}} R_{\boldsymbol{n},\boldsymbol{g}}$,
- derive the symbol of $\{A_{\boldsymbol{n}}\}_n$ and restrict it to obtain the symbol of $\{T_{\boldsymbol{n},\boldsymbol{g}}(f)\}_n$.

9.2.1 Symbol Derivation

We start by noticing that $T_{\boldsymbol{n},\boldsymbol{g}}(f)$ is a submatrix of the Toeplitz matrix $T_{\boldsymbol{ng}}(f)$, since

$$[T_{\boldsymbol{n},\boldsymbol{g}}(f)]_{\boldsymbol{i},\boldsymbol{j}} = f_{\boldsymbol{i}-\boldsymbol{1}-\boldsymbol{g}(\boldsymbol{j}-\boldsymbol{1})} = [T_{\boldsymbol{ng}}(f)]_{\boldsymbol{i},\boldsymbol{1}+\boldsymbol{g}(\boldsymbol{j}-\boldsymbol{1})}$$

so we can define $P_{\boldsymbol{n},\boldsymbol{g}}$ and $P'_{\boldsymbol{n},\boldsymbol{g}}$ diagonal binary matrix of dimension $N(\boldsymbol{ng})$ that zeros out all elements of $T_{\boldsymbol{ng}}(f)$ not belonging to $T_{\boldsymbol{n},\boldsymbol{g}}(f)$. In particular

$$[P_{\boldsymbol{n},\boldsymbol{g}}]_{\boldsymbol{i},\boldsymbol{j}} = \begin{cases} 1 & \boldsymbol{i} = \boldsymbol{j} \le \boldsymbol{n}, \\ 0 & \text{otherwise}, \end{cases} \qquad [P'_{\boldsymbol{n},\boldsymbol{g}}]_{\boldsymbol{i},\boldsymbol{j}} = \begin{cases} 1 & \boldsymbol{i} = \boldsymbol{j} \equiv \boldsymbol{1} \pmod{\boldsymbol{g}}, \\ 0 & \text{otherwise}. \end{cases}$$

Lemma 9.2.1.

$$[P_{\boldsymbol{n},\boldsymbol{g}} T_{\boldsymbol{ng}}(f) P'_{\boldsymbol{n},\boldsymbol{g}}]_{\boldsymbol{i},\boldsymbol{j}} = \begin{cases} [T_{\boldsymbol{n},\boldsymbol{g}}(f)]_{\boldsymbol{i},\boldsymbol{t}} & \boldsymbol{i} \le \boldsymbol{n}, \quad \boldsymbol{j} = \boldsymbol{1} + \boldsymbol{g}(\boldsymbol{t} - \boldsymbol{1}) \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. If $\boldsymbol{i} > \boldsymbol{n}$, then

$$[P_{\boldsymbol{n},\boldsymbol{g}} T_{\boldsymbol{ng}}(f) P'_{\boldsymbol{n},\boldsymbol{g}}]_{\boldsymbol{i},\boldsymbol{j}} = \sum_{\boldsymbol{k}=\boldsymbol{1}}^{\boldsymbol{ng}} [P_{\boldsymbol{n},\boldsymbol{g}}]_{\boldsymbol{i},\boldsymbol{k}} [T_{\boldsymbol{ng}}(f) P'_{\boldsymbol{n},\boldsymbol{g}}]_{\boldsymbol{k},\boldsymbol{j}} = 0$$

and similarly, if $\boldsymbol{j} \not\equiv \boldsymbol{1} \pmod{\boldsymbol{g}}$, then

$$[P_{\boldsymbol{n},\boldsymbol{g}} T_{\boldsymbol{ng}}(f) P'_{\boldsymbol{n},\boldsymbol{g}}]_{\boldsymbol{i},\boldsymbol{j}} = \sum_{\boldsymbol{k}=1}^{\boldsymbol{ng}} [P_{\boldsymbol{n},\boldsymbol{g}} T_{\boldsymbol{ng}}(f)]_{\boldsymbol{i},\boldsymbol{k}} [P'_{\boldsymbol{n},\boldsymbol{g}}]_{\boldsymbol{k},\boldsymbol{j}} = 0.$$

The only remaining case is $\boldsymbol{i} \le \boldsymbol{n}$ and $\boldsymbol{j} \equiv \boldsymbol{1} \pmod{\boldsymbol{g}}$, that can be rewritten as $\boldsymbol{j} = \boldsymbol{1} + \boldsymbol{g}(\boldsymbol{t} - \boldsymbol{1})$ for some $\boldsymbol{t} = \boldsymbol{1}, \dots, \boldsymbol{n}$.

$$\begin{aligned}
[P_{\boldsymbol{n},\boldsymbol{g}} T_{\boldsymbol{ng}}(f) P'_{\boldsymbol{n},\boldsymbol{g}}]_{\boldsymbol{i},\boldsymbol{j}} &= \sum_{\boldsymbol{k}=\boldsymbol{1}}^{\boldsymbol{ng}} [P_{\boldsymbol{n},\boldsymbol{g}}]_{\boldsymbol{i},\boldsymbol{k}} [T_{\boldsymbol{ng}}(f) P'_{\boldsymbol{n},\boldsymbol{g}}]_{\boldsymbol{k},\boldsymbol{j}} \\
&= [T_{\boldsymbol{ng}}(f) P'_{\boldsymbol{n},\boldsymbol{g}}]_{\boldsymbol{i},\boldsymbol{j}} \\
&= \sum_{\boldsymbol{k}=\boldsymbol{1}}^{\boldsymbol{ng}} [T_{\boldsymbol{ng}}(f)]_{\boldsymbol{i},\boldsymbol{k}} [P'_{\boldsymbol{n},\boldsymbol{g}}]_{\boldsymbol{k},\boldsymbol{j}} \\
&= [T_{\boldsymbol{ng}}(f)]_{\boldsymbol{i},\boldsymbol{j}} = [T_{\boldsymbol{n},\boldsymbol{g}}(f)]_{\boldsymbol{i},\boldsymbol{t}}.
\end{aligned}$$

□

The last step before going into the analysis of the symbols for the sequence is to bring $T_{\boldsymbol{n},\boldsymbol{g}}(f)$ into a principal minor of $P_{\boldsymbol{n},\boldsymbol{g}} T_{\boldsymbol{ng}}(f) P'_{\boldsymbol{n},\boldsymbol{g}}$ that can be done through a permutation of columns and rows. We won't write them explicitly, or prove they work, but it is evident that they exist due to Lemma 9.2.1. We call them $R_{\boldsymbol{n},\boldsymbol{g}}$ and $R'_{\boldsymbol{n},\boldsymbol{g}}$, and we denote $A_{\boldsymbol{n}} = R'_{\boldsymbol{n},\boldsymbol{g}} P_{\boldsymbol{n},\boldsymbol{g}} T_{\boldsymbol{ng}}(f) P'_{\boldsymbol{n},\boldsymbol{g}} R_{\boldsymbol{n},\boldsymbol{g}}$.

We can now start to analyse the asymptotic behaviour of the singular values of the sequences. First of all, we need to introduce another permutation matrix from [14], the matrix $\Gamma_{\boldsymbol{n},\boldsymbol{g}}$. Notice that a d-index $\boldsymbol{1} \le \boldsymbol{i} \le \boldsymbol{ng}$ must have $i_k \le n_k g_k$ for every $k \le d$. It means that we can express every i_k as a 2-index $i_k = (a_k, b_k)$ where $\boldsymbol{1} \le (a_k, b_k) \le (n_k, g_k)$ and $i_k = b_k + g_k(a_k - 1)$. The permutation associated to $\Gamma_{\boldsymbol{n},\boldsymbol{g}}$ brings the multi-index $\boldsymbol{i}$ into a multi-index $\boldsymbol{j} = (a_1, \dots, a_d, b_1, \dots, b_d)$ where $\boldsymbol{1} \le (a_1, \dots, a_d) \le \boldsymbol{n}$ and $\boldsymbol{1} \le (b_1, \dots, b_d) \le \boldsymbol{g}$. It means that the matrix $\Gamma_{\boldsymbol{n},\boldsymbol{g}} T_{\boldsymbol{ng}}(f) \Gamma^T_{\boldsymbol{n},\boldsymbol{g}}$ can now be indexed by a $2d$-index, thus giving it a $N(\boldsymbol{g})$ block structure. In the next result, we in fact prove that this new sequence is now a block Toeplitz sequence.

Lemma 9.2.2. *The matrix $B_{\boldsymbol{n}} = \Gamma_{\boldsymbol{n},\boldsymbol{g}} T_{\boldsymbol{ng}}(f) \Gamma^T_{\boldsymbol{n},\boldsymbol{g}}$ has a Toeplitz block structure, with blocks of size $N(\boldsymbol{g})$. It is in particular the Toeplitz block sequence associated to the function $F : [-\pi, \pi]^d \to \mathbb{C}^{N(\boldsymbol{g}) \times N(\boldsymbol{g})}$ defined as*

$$F_{\boldsymbol{k}} = [f_{\boldsymbol{\ell}-\boldsymbol{m}+\boldsymbol{gk}}]_{\boldsymbol{\ell},\boldsymbol{m}=\boldsymbol{1}}^{\boldsymbol{s}},$$

so

$$\{B_{\boldsymbol{n}}\}_n = \{T_{\boldsymbol{n}}(F)\}_n \sim_\sigma 1 \otimes F(\boldsymbol{\theta}).$$

Proof. Since $B_{\boldsymbol{n}}$ has size $N(\boldsymbol{ng})$, we can use the $2d$-indices $\boldsymbol{j} = (a_1, \dots, a_d, b_1, \dots, b_d)$ with $\boldsymbol{1} \le (a_1, \dots, a_d) \le \boldsymbol{n}$ and $\boldsymbol{1} \le (b_1, \dots, b_d) \le \boldsymbol{g}$. For simplicity, we may write equivalently $\boldsymbol{j} = (\boldsymbol{a}, \boldsymbol{b})$. To prove that $B_{\boldsymbol{n}}$ is a $\boldsymbol{g}$-block Toeplitz matrix, we need to show that that

$$[B_{\boldsymbol{n}}]_{(\boldsymbol{a},\boldsymbol{b}),(\boldsymbol{a}',\boldsymbol{b}')} = [B_{\boldsymbol{n}}]_{(\boldsymbol{c},\boldsymbol{b}),(\boldsymbol{c}',\boldsymbol{b}')}$$

whenever $\boldsymbol{a} - \boldsymbol{a}' = \boldsymbol{c} - \boldsymbol{c}'$.

$$\begin{aligned}
[B_{\boldsymbol{n}}]_{(\boldsymbol{a},\boldsymbol{b}),(\boldsymbol{a}',\boldsymbol{b}')} &= [\Gamma_{\boldsymbol{n},\boldsymbol{g}} T_{\boldsymbol{ng}}(f) \Gamma^T_{\boldsymbol{n},\boldsymbol{g}}]_{(\boldsymbol{a},\boldsymbol{b}),(\boldsymbol{a}',\boldsymbol{b}')} \\
&= [T_{\boldsymbol{ng}}(f)]_{\boldsymbol{b}+\boldsymbol{g}(\boldsymbol{a}-\boldsymbol{1}),\boldsymbol{b}'+\boldsymbol{g}(\boldsymbol{a}'-\boldsymbol{1})} \\
&= f_{\boldsymbol{b}-\boldsymbol{b}'+\boldsymbol{g}(\boldsymbol{a}-\boldsymbol{a}')} \\
&= f_{\boldsymbol{b}-\boldsymbol{b}'+\boldsymbol{g}(\boldsymbol{c}-\boldsymbol{c}')} \\
&= [T_{\boldsymbol{ng}}(f)]_{\boldsymbol{b}+\boldsymbol{g}(\boldsymbol{c}-\boldsymbol{1}),\boldsymbol{b}'+\boldsymbol{g}(\boldsymbol{c}'-\boldsymbol{1})} \\
&= [\Gamma_{\boldsymbol{n},\boldsymbol{g}} T_{\boldsymbol{ng}}(f) \Gamma^T_{\boldsymbol{n},\boldsymbol{g}}]_{(\boldsymbol{c},\boldsymbol{b}),(\boldsymbol{c}',\boldsymbol{b}')} \\
&= [B_{\boldsymbol{n}}]_{(\boldsymbol{c},\boldsymbol{b}),(\boldsymbol{c}',\boldsymbol{b}')}.
\end{aligned}$$

We can thus associate the function $F : [-\pi, \pi]^d \to \mathbb{C}^{N(\boldsymbol{g})\times N(\boldsymbol{g})}$ defined by its Fourier coefficients as

$$F_{\boldsymbol{k}} = [B_{\boldsymbol{n}}]_{(\boldsymbol{k}+\boldsymbol{1},\cdot),(\boldsymbol{1},\cdot)} = [f_{\boldsymbol{\ell}-\boldsymbol{m}+\boldsymbol{g}\boldsymbol{k}}]_{\boldsymbol{\ell},\boldsymbol{m}=\boldsymbol{1}}^{\boldsymbol{s}}.$$

□

For $P'_{\boldsymbol{n},\boldsymbol{g}}$ we also need to apply $\Gamma_{\boldsymbol{n},\boldsymbol{g}}$ to obtain a block diagonal sampling sequence.

Lemma 9.2.3. $\Gamma_{\boldsymbol{n},\boldsymbol{g}} P'_{\boldsymbol{n},\boldsymbol{g}} \Gamma^T_{\boldsymbol{n},\boldsymbol{g}} = D_{\boldsymbol{n}}(E_{\boldsymbol{1},\boldsymbol{1}})$, *where* $E_{\boldsymbol{1},\boldsymbol{1}}$ *is the binary matrix of size* $N(\boldsymbol{g})$ *that has only one non-zero entry in position* $(\boldsymbol{1},\boldsymbol{1})$. *As a consequence,*

$$\{\Gamma_{\boldsymbol{n},\boldsymbol{g}} P'_{\boldsymbol{n},\boldsymbol{g}} \Gamma^T_{\boldsymbol{n},\boldsymbol{g}}\} \sim_\sigma E_{\boldsymbol{1},\boldsymbol{1}} \otimes 1.$$

Proof. From the definition, $P'_{\boldsymbol{n},\boldsymbol{g}}$ is a diagonal matrix, like $D_{\boldsymbol{n}}(E_{\boldsymbol{1},\boldsymbol{1}})$, and the same holds for $\Gamma_{\boldsymbol{n},\boldsymbol{g}} P'_{\boldsymbol{n},\boldsymbol{g}} \Gamma^T_{\boldsymbol{n},\boldsymbol{g}}$, so we have just to compare the diagonal entries. For every $\boldsymbol{1} \le \boldsymbol{b} \le \boldsymbol{g}$ and $\boldsymbol{1} \le \boldsymbol{a} \le \boldsymbol{n}$ we have

$$\begin{aligned}
[D_{\boldsymbol{n}}(E_{\boldsymbol{1},\boldsymbol{1}})]_{(\boldsymbol{a},\boldsymbol{b}),(\boldsymbol{a},\boldsymbol{b})} &= \delta_{\boldsymbol{b},\boldsymbol{1}} \\
&= [P'_{\boldsymbol{n},\boldsymbol{g}}]_{\boldsymbol{b}+\boldsymbol{g}(\boldsymbol{a}-\boldsymbol{1}),\boldsymbol{b}+\boldsymbol{g}(\boldsymbol{a}-\boldsymbol{1})} \\
&= [\Gamma_{\boldsymbol{n},\boldsymbol{g}} P'_{\boldsymbol{n},\boldsymbol{g}} \Gamma^T_{\boldsymbol{n},\boldsymbol{g}}]_{(\boldsymbol{a},\boldsymbol{b}),(\boldsymbol{a},\boldsymbol{b})}.
\end{aligned}$$

□

The sequence of $P_{\boldsymbol{n},\boldsymbol{g}}$ requires some extra work, since it is not always a block sampling sequence, but it differs from one up to a zero-distributed sequence and up to the permutation $\Gamma_{\boldsymbol{n},\boldsymbol{g}}$.

Lemma 9.2.4. *There exists a zero-distributed sequence* $\{Z_{\boldsymbol{n}}\}_n$ *such that*

$$\Gamma_{\boldsymbol{n},\boldsymbol{g}} P_{\boldsymbol{n},\boldsymbol{g}} \Gamma^T_{\boldsymbol{n},\boldsymbol{g}} = D_{\boldsymbol{n}}(\chi_{[0,\boldsymbol{1}/\boldsymbol{g}]} I_{\boldsymbol{g}}) + Z_{\boldsymbol{n}}.$$

Proof. Notice that both $\Gamma_{\boldsymbol{n},\boldsymbol{g}} P_{\boldsymbol{n},\boldsymbol{g}} \Gamma^T_{\boldsymbol{n},\boldsymbol{g}}$ and $D_{\boldsymbol{n}}(\chi_{[0,\boldsymbol{1}/\boldsymbol{g}]} I_{\boldsymbol{g}})$ are binary and diagonal sequences, so the rank of $Z_{\boldsymbol{n}}$ will be bounded by the number of different entries between $\Gamma_{\boldsymbol{n},\boldsymbol{g}} P_{\boldsymbol{n},\boldsymbol{g}} \Gamma^T_{\boldsymbol{n},\boldsymbol{g}}$ and $D_{\boldsymbol{n}}(\chi_{[0,\boldsymbol{1}/\boldsymbol{g}]} I_{\boldsymbol{g}})$.

If $\boldsymbol{a} \le \lfloor \boldsymbol{n}/\boldsymbol{g} \rfloor$, then we have

$$\boldsymbol{b} + \boldsymbol{g}(\boldsymbol{a}-\boldsymbol{1}) \le \boldsymbol{n} \implies [\Gamma_{\boldsymbol{n},\boldsymbol{g}} P_{\boldsymbol{n},\boldsymbol{g}} \Gamma^T_{\boldsymbol{n},\boldsymbol{g}}]_{(\boldsymbol{a},\boldsymbol{b}),(\boldsymbol{a},\boldsymbol{b})} = [P_{\boldsymbol{n},\boldsymbol{g}}]_{\boldsymbol{b}+\boldsymbol{g}(\boldsymbol{a}-\boldsymbol{1}),\boldsymbol{b}+\boldsymbol{g}(\boldsymbol{a}-\boldsymbol{1})} = 1,$$

$$\boldsymbol{a}/\boldsymbol{n} \le \boldsymbol{1}/\boldsymbol{g} \implies [D_{\boldsymbol{n}}(\chi_{[0,\boldsymbol{1}/\boldsymbol{g}]} I_{\boldsymbol{g}})]_{(\boldsymbol{a},\boldsymbol{b}),(\boldsymbol{a},\boldsymbol{b})} = 1 \qquad \implies \qquad [Z_{\boldsymbol{n}}]_{(\boldsymbol{a},\boldsymbol{b}),(\boldsymbol{a},\boldsymbol{b})} = 0.$$

On the other hand, if $\boldsymbol{a} \ge \lceil \boldsymbol{n}/\boldsymbol{g} \rceil + \boldsymbol{1}$, then

$$\boldsymbol{b} + \boldsymbol{g}(\boldsymbol{a}-\boldsymbol{1}) > \boldsymbol{n} \implies [\Gamma_{\boldsymbol{n},\boldsymbol{g}} P_{\boldsymbol{n},\boldsymbol{g}} \Gamma^T_{\boldsymbol{n},\boldsymbol{g}}]_{(\boldsymbol{a},\boldsymbol{b}),(\boldsymbol{a},\boldsymbol{b})} = [P_{\boldsymbol{n},\boldsymbol{g}}]_{\boldsymbol{b}+\boldsymbol{g}(\boldsymbol{a}-\boldsymbol{1}),\boldsymbol{b}+\boldsymbol{g}(\boldsymbol{a}-\boldsymbol{1})} = 0,$$

$$\boldsymbol{a}/\boldsymbol{n} > \boldsymbol{1}/\boldsymbol{g} \implies [D_{\boldsymbol{n}}(\chi_{[0,\boldsymbol{1}/\boldsymbol{g}]} I_{\boldsymbol{g}})]_{(\boldsymbol{a},\boldsymbol{b}),(\boldsymbol{a},\boldsymbol{b})} = 0 \qquad \implies \qquad [Z_{\boldsymbol{n}}]_{(\boldsymbol{a},\boldsymbol{b}),(\boldsymbol{a},\boldsymbol{b})} = 0.$$

In other words, the only non-zero elements of $Z_{\boldsymbol{n}}$ can have cardinality at most $N(\boldsymbol{g})N(\lceil \boldsymbol{n}/\boldsymbol{g} \rceil + \boldsymbol{1} - \lfloor \boldsymbol{n}/\boldsymbol{g} \rfloor) \le 2^d N(\boldsymbol{g})$ and as a consequence,

$$\operatorname{rk} Z_{\boldsymbol{n}} \le 2^d N(\boldsymbol{g}) = o(N(\boldsymbol{g}\boldsymbol{n})).$$

Hence we conclude that $\{Z_{\boldsymbol{n}}\}_n$ is zero-distributed. □

It is now easy to sum up all the results into a symbol for $R'_{\boldsymbol{n},\boldsymbol{g}} P_{\boldsymbol{n},\boldsymbol{g}} T_{\boldsymbol{n}\boldsymbol{g}}(f) P'_{\boldsymbol{n},\boldsymbol{g}} R_{\boldsymbol{n},\boldsymbol{g}}$.

Lemma 9.2.5.

$$\{R'_{\boldsymbol{n},\boldsymbol{g}} P_{\boldsymbol{n},\boldsymbol{g}} T_{\boldsymbol{n}\boldsymbol{g}}(f) P'_{\boldsymbol{n},\boldsymbol{g}} R_{\boldsymbol{n},\boldsymbol{g}}\}_n \sim_\sigma \chi_{[0,\boldsymbol{1}/\boldsymbol{g}]}(\boldsymbol{x}) F(\boldsymbol{\theta}) E_{\boldsymbol{1},\boldsymbol{1}}.$$

Proof. Using Lemmas 9.2.4, 9.2.3 and 9.2.2 we can write

$$\begin{aligned}
\Gamma_{\boldsymbol{n},\boldsymbol{g}} P_{\boldsymbol{n},\boldsymbol{g}} T_{\boldsymbol{n}\boldsymbol{g}}(f) P'_{\boldsymbol{n},\boldsymbol{g}} \Gamma^T_{\boldsymbol{n},\boldsymbol{g}} &= \Gamma_{\boldsymbol{n},\boldsymbol{g}} P_{\boldsymbol{n},\boldsymbol{g}} \Gamma^T_{\boldsymbol{n},\boldsymbol{g}} \Gamma_{\boldsymbol{n},\boldsymbol{g}} T_{\boldsymbol{n}\boldsymbol{g}}(f) \Gamma^T_{\boldsymbol{n},\boldsymbol{g}} \Gamma_{\boldsymbol{n},\boldsymbol{g}} P'_{\boldsymbol{n},\boldsymbol{g}} \Gamma^T_{\boldsymbol{n},\boldsymbol{g}} \\
&= (D_{\boldsymbol{n}}(\chi_{[0,\boldsymbol{1}/\boldsymbol{g}]} I_{\boldsymbol{g}}) + Z_{\boldsymbol{n}}) T_{\boldsymbol{n}}(F) D_{\boldsymbol{n}}(E_{\boldsymbol{1},\boldsymbol{1}})
\end{aligned}$$

and using the axioms **GLT 1,3,4** for block GLT generalized in the multivariate/multilevel case ([14]), we conclude that

$$\{\Gamma_{\boldsymbol{n},\boldsymbol{g}} P_{\boldsymbol{n},\boldsymbol{g}} T_{\boldsymbol{ng}}(f) P'_{\boldsymbol{n},\boldsymbol{g}} \Gamma^T_{\boldsymbol{n},\boldsymbol{g}}\}_n \sim_\sigma \chi_{[0,\boldsymbol{1}/\boldsymbol{g}]}(\boldsymbol{x}) F(\boldsymbol{\theta}) E_{\boldsymbol{1},\boldsymbol{1}}.$$

Eventually, notice that $R_{\boldsymbol{n},\boldsymbol{g}}\Gamma^T_{\boldsymbol{n},\boldsymbol{g}}$ and its transpose are unitary matrix, so they do not change the singular values of the matrices when multiplied, and the thesis follows. □

We know that $T_{\boldsymbol{n},\boldsymbol{g}}(f)$ is a principal minor of $A_{\boldsymbol{n}} = R'_{\boldsymbol{n},\boldsymbol{g}} P_{\boldsymbol{n},\boldsymbol{g}} T_{\boldsymbol{ng}}(f) P'_{\boldsymbol{n},\boldsymbol{g}} R_{\boldsymbol{n},\boldsymbol{g}}$ and every other entry is zero, so the singular values of $A_{\boldsymbol{n}}$ are the same of $T_{\boldsymbol{n},\boldsymbol{g}}(f)$ except for additional $N(\boldsymbol{ng}) - N(\boldsymbol{n})$ zeros. Moreover, notice that the symbol

$$\chi_{[0,\boldsymbol{1}/\boldsymbol{g}]}(\boldsymbol{x}) F(\boldsymbol{\theta}) E_{\boldsymbol{1},\boldsymbol{1}} = \chi_{[0,1/g]}(\boldsymbol{x}) \left[\begin{array}{c|c} F_{1,1}(\boldsymbol{\theta}) & \\ \vdots & 0 \\ F_{N(\boldsymbol{g}),1}(\boldsymbol{\theta}) & \end{array} \right]$$

has at most one non-zero singular value for every $(\boldsymbol{x},\boldsymbol{\theta})$, given by $\chi_{[0,\boldsymbol{1}/\boldsymbol{g}]}(\boldsymbol{x})\|v(\boldsymbol{\theta})\|_2$, where v is the first column of F. Using all these informations, we can finally derive a symbol for $T_{\boldsymbol{n},\boldsymbol{g}}(f)$.

Theorem 9.2.6.

$$\{T_{\boldsymbol{n},\boldsymbol{g}}(f)\}_n \sim_\sigma \chi_{[0,\boldsymbol{1}/\boldsymbol{g}]}(\boldsymbol{x})\|v(\boldsymbol{\theta})\|_2.$$

Proof. Let us check the ergodic relation relative to matrix-valued symbols for the sequence $A_{\boldsymbol{n}}$.

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{N(\boldsymbol{ng})} \sum_{\boldsymbol{i}=\boldsymbol{1}}^{N(\boldsymbol{ng})} G(\sigma_{\boldsymbol{i}}(A_{\boldsymbol{n}})) &= \lim_{n\to\infty} \frac{1}{N(\boldsymbol{ng})} \sum_{\boldsymbol{i}=\boldsymbol{1}}^{N(\boldsymbol{n})} G(\sigma_{\boldsymbol{i}}(T_{\boldsymbol{n},\boldsymbol{g}}(f))) + \frac{N(\boldsymbol{ng}) - N(\boldsymbol{n})}{N(\boldsymbol{ng})} G(0) \\
&= \frac{N(\boldsymbol{g}) - 1}{N(\boldsymbol{g})} G(0) + \frac{1}{N(\boldsymbol{g})} \lim_{n\to\infty} \frac{1}{N(\boldsymbol{n})} \sum_{\boldsymbol{i}=\boldsymbol{1}}^{\boldsymbol{n}} G(\sigma_{\boldsymbol{i}}(T_{\boldsymbol{n},\boldsymbol{g}}(f))) \\
&= \frac{1}{\mu(D)} \int_D \sum_{\boldsymbol{i}=\boldsymbol{1}}^{N(\boldsymbol{g})} \frac{G(\sigma_{\boldsymbol{i}}(\chi_{[0,\boldsymbol{1}/\boldsymbol{g}]}(\boldsymbol{x}) G(\boldsymbol{\theta}) E_{\boldsymbol{1},\boldsymbol{1}})}{N(\boldsymbol{g})} \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{\theta} \\
&= \frac{1}{\mu(D)} \int_D \frac{N(\boldsymbol{g}) - 1}{N(\boldsymbol{g})} G(0) \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{\theta} + \frac{1}{\mu(D)} \int_D \frac{G(\chi_{[0,\boldsymbol{1}/\boldsymbol{g}]}(\boldsymbol{x})\|v(\boldsymbol{\theta})\|_2)}{N(\boldsymbol{g})} \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{\theta} \\
&= \frac{N(\boldsymbol{g}) - 1}{N(\boldsymbol{g})} G(0) + \frac{1}{N(\boldsymbol{g})} \frac{1}{\mu(D)} \int_D G(\chi_{[0,\boldsymbol{1}/\boldsymbol{g}]}(\boldsymbol{x})\|v(\boldsymbol{\theta})\|_2) \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{\theta} \\
\implies \lim_{n\to\infty} \frac{1}{N(\boldsymbol{n})} \sum_{\boldsymbol{i}=\boldsymbol{1}}^{\boldsymbol{n}} G(\sigma_{\boldsymbol{i}}(T_{\boldsymbol{n},\boldsymbol{g}}(f))) &= \frac{1}{\mu(D)} \int_D G(\chi_{[0,\boldsymbol{1}/\boldsymbol{g}]}(\boldsymbol{x})\|v(\boldsymbol{\theta})\|_2) \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{\theta} \\
&\implies \{T_{\boldsymbol{n},\boldsymbol{g}}(f)\}_n \sim_\sigma \chi_{[0,\boldsymbol{1}/\boldsymbol{g}]}(\boldsymbol{x})\|v(\boldsymbol{\theta})\|_2.
\end{aligned}$$

□

Chapter 10

Future Works

In this chapter, we collect some of the open questions that can be found in the present document, and hints to possible future directions of research for theory and applications.

On the metric and analytic aspects, we cite Conjecture 3.2.17, that would be a fundamental perturbation result regarding the matrix-sequences. In particular, it is stronger than any result proved until now about perturbations of Hermitian sequences, and would be essential in expanding the possibilities of application for the entire GLT theory. In fact, since we are mainly interested in spectral distributions of matrix-sequences, our only option is to reduce ourselves to a Hermitian (or normal) sequence belonging to the GLT class, through perturbations, similarities or any other operation that is proved not to change the spectral symbol. In [8], other conjectures are stated in order to find a suitable distances that let us classify the perturbation of Hermitian sequences not affecting the symbol, for example by splitting real and imaginary part of the perturbation, and focusing on the second one.

In this context, Conjecture 5.3.20 is important, because it entices that any possible structure we may face can be transformed into a subset of the GLT sequences. In other words, it assures us that whenever we are looking for a symbol associated to a matrix-sequence, it is always possible to do so by applying an unitary transformation that brings it to a GLT sequence.

Notice moreover that we hinted how the building steps of the locally Toeplitz/circulant sequences uses only the semiseparable structure of these matrices, so new spaces may arise by considering different semiseparable starting families.

The GLT space is actually far to be completely understood. A simple open question is for example the following.

Conjecture 10.0.1. *Consider for every index $j \in \mathbb{Z}$ the function $a_j(x) \in L^1[0,1]$. In this case,*

$$\left\{ \sum_{j\in\mathbb{Z}} D_n(a_j) T_n(e^{\mathrm{i}j\theta}) \right\} \sim_{GLT} \sum_{j\in\mathbb{Z}} a_j(x) e^{\mathrm{i}j\theta}.$$

Moreover, if $f_j(x) = \overline{f_{-j}(x)}$ for every $j \in \mathbb{Z}$, then

$$\left\{ \sum_{j\in\mathbb{Z}} D_n(a_j) T_n(e^{\mathrm{i}j\theta}) \right\} \sim_{\lambda} \sum_{j\in\mathbb{Z}} a_j(x) e^{\mathrm{i}j\theta}.$$

A noticeable fact about the GLT is that, even though in general matrices do not commute, the GLT sequence $\{A_n B_n\}_n$ has always the same symbol of $\{B_n A_n\}_n$, or also said,

$$[\{A_n\}_n, \{B_n\}_n] := \{A_n B_n - B_n A_n\} \sim_{GLT} 0.$$

An useful criterion to check whether a s.u. sequence $\{A_n\}_n$ is not a GLT sequence, is to find a diagonal sampling sequence $\{D_n(a)\}_n$ or a Toeplitz one $\{T_n(f)\}_n$ such that the commutator is not zero-distributed. It has been proved that if $[\{A_n\}_n, \{B_n\}_n] \sim_{GLT} 0$ for every sequence $\{B_n\}_n$ of the form $\{D_n(a)\}_n$ with $a \in C^\infty[0,1]$

and every $\{T_n(f)\}_n$ with f a trigonometric polynomial, then $[\{A_n\}_n, \{B_n\}_n] \sim_{GLT} 0$ for every GLT sequence $\{B_n\}_n$. This is not a sufficient condition to conclude that $\{A_n\}_n$ is a GLT sequence, and in general there not exists yet a complete criterion for GLT sequences. In particular, an open question is the following.

Conjecture 10.0.2. *Given a sequence $\{A_n\}_n$ that admits a singular value symbol κ, assume the following conditions.*

- *$\{A_nB_n\}_n$ admits singular value symbol for every GLT sequence $\{B_n\}_n$.*
- *$[\{A_n\}_n, \{B_n\}_n] \sim_{GLT} 0$ for every GLT sequence $\{B_n\}_n$.*

In this case, $\{A_n\}_n$ is a GLT sequence.

If we now suppose that $\mathcal{A}$ is a s.a. that is also closed by conjugation, then $[\{A_n\}_n, \{A_n^H\}_n] \sim_{\mathfrak{A}} 0$ for every sequence $\{A_n\}_n$ admitting a symbol in $\mathcal{A}$. In a sense, it is hinting that $\{A_n\}_n$ is, up to zero-distributed sequences, a normal sequence, as also hypothesized by Conjecture 5.2.4. The link between normal sequences and GLT sequences has been stressed in this thesis, and seems to be a key point for the understanding of the structure of all the possible s.a., even though it is not clear how to exploit the relations.

In the thesis we explored how the symbols of sequences can be represented, and how every different form leads to new properties about the sequences. Every symbol has been shown to be representable through a finite measure of mass less than or equal to one, or, equivalently, by a measurable function with values in $\mathbb{C} \cup \{\infty\}$ and domain $[0, 1]$. If we identify $\mathbb{C} \cup \{\infty\}$ with the Riemann sphere, then the functions become maps between compact spaces, and new results may be derived from this setting, especially if we consider only continuous or highly regular symbols.

To study such spaces, we may thus need new metrics and new notions of convergence. A lead in this sense may come from the study of the derivatives of symbols developed in Appendix D, where we proposed generalization of the concept of a.c.s. convergence and a.c.s. equivalence to preserve the regularity of the symbols.

On this same topic, we notice that we always talked on asymptotic distribution of the eigenvalues, but seldom about how well or fast in general the sequences converge to the symbol. It has been already addressed in the particular cases of Toeplitz sequences associated to specific sets of symbols. We miss a general analysis on the matter, and we can notice from the arguments in Appendix D that it is linked to the distance from sequences with eigenvalues that are a perfect, or at least really good, sampling of the symbol, like diagonal, circulant, tridiagonal matrices, or in general coming from a τ-algebra.

Regarding the applications, it would be interesting to study families of graphs with good expansion properties and with an explicit construction, useful in code theory, random walks, etc. Examples are the Margulis graphs [55], Cayley graphs [62], and many others.

On other accounts, the immersed methods (or also called fictitious domain) seem also fit to be analysed through the use of reduced and block GLT sequences. For this reason, a complete investigation and formalization of the space of reduced block GLT sequence is needed and still missing, even if it seems to descend naturally from the theory developed in this thesis and the book [14].

As a last note, another nice expansion of the theory would be to consider discretizations of stochastic PDE, probably leading to matrices whose every entry is a random variable. This would lead to an other type of symbol that has not been considered in this document, that would probably be represented by a stochastic process.

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Appendix A

Equivalence rates on disks

Given a positive finite measure μ on $\mathbb{C}$, the notation $\{A_n\}_n \sim_\lambda \mu$ or $\{A_n\}_n \sim_\lambda \phi_\mu$ means

$$\lim_{n\to\infty} \frac{1}{s_n} \sum_{j=1}^{s_n} G(\lambda_j(A_n)) = \int_{\mathbb{C}} G \mathrm{d}\mu, \qquad \forall\, G \in C_c(\mathbb{C}). \tag{2.2}$$

What happens if we use characteristic functions instead of $G \in C_c(\mathbb{C})$?

Lemma A.0.1. *Let μ be a positive finite measure on $\mathbb{C}$. If $\{A_n\}_n \sim_\lambda \mu$, then the set*

$$E_{z_0} := \{\, r \in \mathbb{R}^+ \mid \chi_{B(z_0,r)} \text{ does not satisfy (2.2)} \,\}$$

contains at most numerable points for every $z_0 \in \mathbb{C}$.

Proof. Let $F : \mathbb{R}^+ \to \mathbb{R}^+$ defined as

$$F(r) := \mu(B(z_0, r)) = \int_{\mathbb{C}} \chi_{B(z_0,r)} \mathrm{d}\mu.$$

F is an increasing function, so it admits at most countable points of discontinuity. Let $r_0 \neq 0$ be a point of continuity for F and for every $\varepsilon > 0$ pick two functions $G_\varepsilon, G_{-\varepsilon} \in C_c(\mathbb{C})$ such that

$$\chi_{B(z_0,r_0)} \le G_\varepsilon \le \chi_{B(z_0,r_0+\varepsilon)}, \qquad \chi_{B(z_0,r_0-\varepsilon)} \le G_{-\varepsilon} \le \chi_{B(z_0,r_0)}.$$

Since $G_\varepsilon \in C_c(\mathbb{C})$, it satisfies (2.2), so

$$\frac{1}{s_n} \sum_{j=1}^{s_n} \chi_{B(z_0,r_0)}(\lambda_j(A_n)) \le \frac{1}{s_n} \sum_{j=1}^{s_n} G_\varepsilon(\lambda_j(A_n)),$$
$$\limsup_{n\to\infty} \frac{1}{s_n} \sum_{j=1}^{s_n} \chi_{B(z_0,r_0)}(\lambda_j(A_n)) \le \int_{\mathbb{C}} G_\varepsilon \mathrm{d}\mu \le \int_{\mathbb{C}} \chi_{B(z_0,r_0+\varepsilon)} \mathrm{d}\mu = F(r_0+\varepsilon),$$

and it holds for every $\varepsilon > 0$. F is continuous on r_0, so

$$\limsup_{n\to\infty} \frac{1}{s_n} \sum_{j=1}^{s_n} \chi_{B(z_0,r_0)}(\lambda_j(A_n)) \le \lim_{\varepsilon\to 0} F(r_0+\varepsilon) = F(r_0).$$

We can conduct the same analysis with $G_{-\varepsilon}$.

$$\frac{1}{s_n} \sum_{j=1}^{s_n} \chi_{B(z_0,r_0)}(\lambda_j(A_n)) \ge \frac{1}{s_n} \sum_{j=1}^{s_n} G_{-\varepsilon}(\lambda_j(A_n)),$$
$$\limsup_{n\to\infty} \frac{1}{s_n} \sum_{j=1}^{s_n} \chi_{B(z_0,r_0)}(\lambda_j(A_n)) \ge \int_{\mathbb{C}} G_{-\varepsilon} \mathrm{d}\mu \ge \int_{\mathbb{C}} \chi_{B(z_0,r_0-\varepsilon)} \mathrm{d}\mu = F(r_0-\varepsilon),$$

and it holds for every $\varepsilon > 0$. F is continuous on r_0, so

$$\limsup_{n\to\infty} \frac{1}{s_n} \sum_{j=1}^{s_n} \chi_{B(z_0,r_0)}(\lambda_j(A_n)) \geq \lim_{\varepsilon\to 0} F(r_0 - \varepsilon) = F(r_0).$$

We can then conclude that (2.2) holds with $\chi_{B(z_0,r)}$ since

$$\limsup_{n\to\infty} \frac{1}{s_n} \sum_{j=1}^{s_n} \chi_{B(z_0,r_0)}(\lambda_j(A_n)) = F(r_0) = \int_{\mathbb{C}} \chi_{B(z_0,r_0)} \mathrm{d}\mu.$$

If $\chi_{B(z_0,r)}$ does not satisfy (2.2), then r must be 0 or a point of discontinuity for F, so there are at most numerable elements in E_{z_0} for every $z_0 \in \mathbb{C}$. □

Actually, even the inverse proposition holds, but the proof is more convoluted and makes use of *fine coverings* and of the Besicovitch Covering Theorem.

Definition A.0.2. *Let $\mathscr{F}$ be a family of closed balls that covers a metric space E. We say that $\mathscr{F}$ is a* ***fine cover*** *if and only if each $x \in E$ is the centre of a closed ball in $\mathscr{F}$ with arbitrarily small radius.*

Theorem A.0.3 (Besicovitch Covering Theorem, [19])**.** *Let μ be a locally finite measure on $\mathbb{R}^n$. Let $E \subseteq \mathbb{R}^n$ be a Borel set and $\varepsilon > 0$. Take $\mathscr{F}$ a fine cover of E. Then there exists a disjoint subfamily $\mathscr{F}'$ which covers μ-almost all of E, and satisfies the inequality*

$$\sum_{B\in\mathscr{F}'} \mu(B) \leq \mu(E) + \varepsilon.$$

Lemma A.0.4. *Let μ be a positive finite measure on $\mathbb{C}$ and suppose that the set*

$$R_{z_0} := \{\, r \in \mathbb{R}^+ \mid \chi_{B(z_0,r)} \textit{ does not satisfy (2.2)} \,\}$$

has Lebesgue measure zero for every $z_0 \in \mathbb{C}$. In this case, $\{A_n\}_n \sim_\lambda \mu$.

Proof. Let $G \in C_c(\mathbb{C})$ and let $E = \operatorname{supp}(G)$. G is uniformly continuous, so given a number $\varepsilon > 0$ we can find a δ such that $|z - z_0| \leq \delta \implies |G(z) - G(z_0)| \leq \varepsilon$.

The set E is closed, so it is a Borel set in $\mathbb{C} = \mathbb{R}^2$. A fine cover $\mathscr{F}$ of E is given by all the closed balls $B(z, r)$ where $z \in E$, $r \notin E_z$ and $0 < r < \delta$. Using Theorem A.0.3, we obtain a subfamily $\mathscr{F}'$ of disjoint balls that covers E almost everywhere according to μ, and such that

$$\sum_{B\in\mathscr{F}'} \mu(B) \leq \mu(E) + \varepsilon.$$

Notice that $\mathscr{F}'$ is at most countable, otherwise the sum $\sum_{B\in\mathscr{F}'} \mu(B)$ would be infinite. Enumerate the elements of $\mathscr{F}'$ as

$$\mathscr{F}' = \{\, B_1, B_2, B_3, \dots \}$$

and call z_n and r_n the centre and radius of B_n for every n. Let F_n be the function

$$F_n(z) := \begin{cases} G(z_i), & z \in B_i, \quad i \leq n, \\ 0, & \textit{otherwise}. \end{cases}$$

Notice that

$$\begin{aligned} \int_{\mathbb{C}} |G - F_m| \mathrm{d}\mu &= \sum_{i=1}^{m} \int_{B_i} |G - F_m| \mathrm{d}\mu + \sum_{i=m}^{\infty} \int_{B_i} |G| \mathrm{d}\mu \\ &\leq \sum_{i=1}^{m} \int_{B_i} |G(z) - G(z_i)| \mathrm{d}\mu + \sum_{i=m}^{\infty} \mu(B_i) \|G\|_\infty \\ &\leq \sum_{i=1}^{m} \mu(B_i)\varepsilon + \sum_{i=m}^{\infty} \mu(B_i) \|G\|_\infty \end{aligned}$$

but the sum of $\mu(B_i)$ is bounded by $\mu(E)+\varepsilon$ so when we let m go to infinity, the term $\sum_{i=m}^{\infty}\mu(B_i)$ goes to zero. Consequentially,

$$\begin{aligned}\limsup_{m\to\infty}\int_{\mathbb{C}}|G-F_m|\mathrm{d}\mu &\le \limsup_{m\to\infty}\left(\sum_{i=1}^{m}\mu(B_i)\varepsilon+\sum_{i=m}^{\infty}\mu(B_i)\|G\|_\infty\right)\\ &\le (\mu(E)+\varepsilon)\varepsilon.\end{aligned}$$

Call now

$$V_m = E\cup\bigcup_{i=1}^{m}B_i.$$

Every V_m is closed and bounded, since it is a finite union of closed bounded sets, so we can find an open ball U_m such that $V_m\subseteq U_m$. Moreover we can suppose that $\mu(U_m)=\mu(\overline{U}_m)$ and that the characteristic function of $\overline{U}_m$ satisfies (2.2) from the hypothesis. The set $U_m\setminus V_m$ is open, and

$$\{\, B(x,r) \mid B(x,r)\subseteq U_m\setminus V_m,\ \chi_{B(x,r)} \text{ satisfies (2.2}\,\}$$

is a fine cover. Using Theorem A.0.3 again, we find a finite subfamily of disjoint closed balls $\{\,\widehat{B}_1,\dots,\widehat{B}_k\,\}$ contained in $U_m\setminus V_m$ such that

$$\mu((U_m\setminus V_m)\setminus\cup_{i=1}^{k}\widehat{B}_i)=\mu((U_m\setminus\cup_{i=1}^{k}\widehat{B}_i)\setminus V_m)<\widehat{\varepsilon}.$$

Notice that $\widehat{B}_i$ are disjoint from themselves and also from V_m. We can then compute an upper bound for the number of eigenvalues inside E and outside $\cup_{i=1}^{m}B_i$.

$$\begin{aligned}\frac{1}{s_n}\sum_{j=1}^{s_n}\chi_{E\setminus\cup_{i=1}^{m}B_i}(\lambda_j(A_n)) &\le \frac{1}{s_n}\sum_{j=1}^{s_n}\chi_{U_m}(\lambda_j(A_n))-\sum_{i=1}^{k}\frac{1}{s_n}\sum_{j=1}^{s_n}\chi_{\widehat{B}_i}(\lambda_j(A_n))-\sum_{i=1}^{m}\frac{1}{s_n}\sum_{j=1}^{s_n}\chi_{B_i}(\lambda_j(A_n))\\ &\le \frac{1}{s_n}\sum_{j=1}^{s_n}\chi_{\overline{U}_m}(\lambda_j(A_n))-\sum_{i=1}^{k}\frac{1}{s_n}\sum_{j=1}^{s_n}\chi_{\widehat{B}_i}(\lambda_j(A_n))-\sum_{i=1}^{m}\frac{1}{s_n}\sum_{j=1}^{s_n}\chi_{B_i}(\lambda_j(A_n)).\end{aligned}$$

Remember that $B_i,\widehat{B}_i$ and $\overline{U}_m$ are chosen so that their characteristic function satisfy (2.2), so

$$\begin{aligned}&\limsup_{n\to\infty}\frac{1}{s_n}\sum_{j=1}^{s_n}\chi_{E\setminus\cup_{i=1}^{m}B_i}(\lambda_j(A_n))\le\\ &\qquad\le\lim_{n\to\infty}\left(\frac{1}{s_n}\sum_{j=1}^{s_n}\chi_{\overline{U}_m}(\lambda_j(A_n))-\sum_{i=1}^{k}\frac{1}{s_n}\sum_{j=1}^{s_n}\chi_{\widehat{B}_i}(\lambda_j(A_n))-\sum_{i=1}^{m}\frac{1}{s_n}\sum_{j=1}^{s_n}\chi_{B_i}(\lambda_j(A_n))\right)\\ &\qquad=\mu(\overline{U}_m)-\sum_{i=1}^{k}\mu(\widehat{B}_i)-\sum_{i=1}^{m}\mu(B_i)\\ &\qquad=\mu(U_m\setminus V_m)-\sum_{i=1}^{k}\mu(\widehat{B}_i)+\mu(V_m)-\sum_{i=1}^{m}\mu(B_i)\\ &\qquad<\widehat{\varepsilon}+\mu(E\setminus\cup_{i=1}^{m}B_i)\end{aligned}$$

and it works for every $\widehat{\varepsilon}>0$. We can conclude that

$$\limsup_{n\to\infty}\frac{1}{s_n}\sum_{j=1}^{s_n}\chi_{E\setminus\cup_{i=1}^{m}B_i}(\lambda_j(A_n))\le\mu(E\setminus\cup_{i=1}^{m}B_i).$$

Now we can finally prove the thesis.

$$\begin{aligned}\left|\frac{1}{s_n}\sum_{j=1}^{s_n}G(\lambda_j(A_n))-\int_{\mathbb{C}}G\mathrm{d}\mu\right| \le&\frac{1}{s_n}\left|\sum_{j=1}^{s_n}G(\lambda_j(A_n))-\sum_{j=1}^{s_n}F_m(\lambda_j(A_n))\right|+\left|\frac{1}{s_n}\sum_{j=1}^{s_n}F_m(\lambda_j(A_n))-\int_{\mathbb{C}}F_m\mathrm{d}\mu\right|\\ &+\left|\int_{\mathbb{C}}F_m\mathrm{d}\mu-\int_{\mathbb{C}}G\mathrm{d}\mu\right|.\end{aligned}$$

We already know

$$\limsup_{m\to\infty} \int_{\mathbb{C}} |G - F_m| \mathrm{d}\mu \le (\mu(E) + \varepsilon)\varepsilon$$

and

$$\lim_{m\to\infty} \mu(E \setminus \cup_{i=1}^m B_i) = 0,$$

so we can choose m such that

$$\int_{\mathbb{C}} |G - F_m| \mathrm{d}\mu \le 2(\mu(E) + \varepsilon)\varepsilon, \qquad \mu(E \setminus \cup_{i=1}^m B_i) < \varepsilon.$$

Moreover, every F_m is a linear combination of characteristic functions $\chi_{B(z,r)}$ that satisfy (2.2) and as a consequence,

$$\lim_{n\to\infty} \left| \frac{1}{s_n} \sum_{j=1}^{s_n} F_m(\lambda_j(A_n)) - \int_{\mathbb{C}} F_m \mathrm{d}\mu \right| = 0.$$

For the last term of the sum, we find that

$$\begin{aligned}
\frac{1}{s_n} \left| \sum_{j=1}^{s_n} G(\lambda_j(A_n)) - \sum_{j=1}^{s_n} F_m(\lambda_j(A_n)) \right| &\le \\
&\le \frac{1}{s_n} \sum_{j=1}^{s_n} |G(\lambda_j(A_n)) - F_m(\lambda_j(A_n))| \\
&\le \frac{1}{s_n} \left(\sum_{\lambda_j(A_n) \in E \setminus \cup_{i=1}^m B_i} |G(\lambda_j(A_n))| + \sum_{i=1}^m \sum_{\lambda_j(A_n) \in B_i} |G(\lambda_j(A_n)) - F_m(\lambda_j(A_n))| \right) \\
&\le \|G\|_\infty \frac{\#\{\, j \mid \lambda_j(A_n) \in E \setminus \cup_{i=1}^m B_i \,\}}{s_n} + \frac{1}{s_n} \sum_{i=1}^m \sum_{\lambda_j(A_n) \in B_i} |G(\lambda_j(A_n)) - G(z_i)| \\
&\le \|G\|_\infty \frac{\sum_{j=1}^{s_n} \chi_{E \setminus \cup_{i=1}^m B_i}(\lambda_j(A_n))}{s_n} + \varepsilon
\end{aligned}$$

and if we take the limits,

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{s_n} \left| \sum_{j=1}^{s_n} G(\lambda_j(A_n)) - \sum_{j=1}^{s_n} F_m(\lambda_j(A_n)) \right| &\le \|G\|_\infty \limsup_{n\to\infty} \frac{\sum_{j=1}^{s_n} \chi_{E \setminus \cup_{i=1}^m B_i}(\lambda_j(A_n))}{s_n} + \varepsilon \\
&\le \|G\|_\infty \mu(E \setminus \cup_{i=1}^m B_i) + \varepsilon \\
&\le \varepsilon(1 + \|G\|_\infty).
\end{aligned}$$

We can conclude that

$$\begin{aligned}
\left| \frac{1}{s_n} \sum_{j=1}^{s_n} G(\lambda_j(A_n)) - \int_{\mathbb{C}} G \mathrm{d}\mu \right| \le& \|G\|_\infty \frac{\sum_{j=1}^{s_n} \chi_{E \setminus \cup_{i=1}^m B_i}(\lambda_j(A_n))}{s_n} + \varepsilon + \left| \frac{1}{s_n} \sum_{j=1}^{s_n} F_m(\lambda_j(A_n)) - \int_{\mathbb{C}} F_m \mathrm{d}\mu \right| \\
&+ 2(\mu(E) + \varepsilon)\varepsilon
\end{aligned}$$

and

$$\limsup_{n\to\infty} \left| \frac{1}{s_n} \sum_{j=1}^{s_n} G(\lambda_j(A_n)) - \int_{\mathbb{C}} G \mathrm{d}\mu \right| \le \varepsilon(1 + \|G\|_\infty) + 2(\mu(E) + \varepsilon)\varepsilon$$

for every $\varepsilon > 0$. As a consequence

$$\lim_{n\to\infty} \left| \frac{1}{s_n} \sum_{j=1}^{s_n} G(\lambda_j(A_n)) - \int_{\mathbb{C}} G \mathrm{d}\mu \right| = 0$$

for every $G \in C_c(\mathbb{C})$ and $\{A_n\}_n \sim_\lambda \mu$.

$\square$

Notice that the set R_{z_0} can also be much larger then the one in the hypothesis, since we used only the following properties:

- 0 is a limit point for $R(z_0)^C$,
- For every $M > 0$, the set $R(z_0)^C \cap [M, +\infty]$ is uncountable.

Probably, with a bit more effort, we could prove that the result holds under the condition that the closed balls for which (2.2) holds, are a fine cover of $\mathbb{C}$. In any case, Lemma A.0.1 and Lemma A.0.4 let us state the following result.

Theorem A.0.5. *Given a matrix-sequence $\{A_n\}_n$ and a finite measure μ, the following are equivalent.*

- $\{A_n\}_n \sim_\lambda \mu$,
- *the set*
$$E_{z_0} := \{\, r \in \mathbb{R}^+ \mid \chi_{B(z_0,r)} \text{ does not satisfy (2.2)} \}$$
has Lebesgue measure zero for every $z_0 \in \mathbb{C}$.

Appendix B

S.U. and S.P.S.

Definition B.0.1. *A space X endowed with a probability measure μ is called standard probability space (s.p.s.) if there exist*

- *a compact interval $I \subseteq \mathbb{R}$ with Lebesgue measure $\ell := \ell_1$,*
- *a countable set $E = \{x_i\}_{i \in \mathbb{N}}$ with atomic measure ν,*

such that X and $Y := I \sqcup E$ are isomorphic modulo zero, meaning that there exists a bijective measurable and measure preserving function

$$\varphi : X \setminus A_1 \to \; Y \setminus A_2$$

such that φ^{-1} is still measurable and measure preserving, where $A_1 \subseteq X$, $\mu(A_1) = 0$ and $A_2 \subseteq Y$, $\ell \oplus \nu(A_2) = 0$.

Theorem B.0.2. *The following results hold.*

- *Every complete probability measure on the Borel set of a Polish space turns it into a standard probability space.*
- *Any measurable subset of a standard probability space is a standard probability space. It is assumed that the set is not a null set, and is endowed with the conditional measure.*

In particular,

- *the space $\mathbb{R}^d$ endowed with any probability measure is a s.p.s. for any d,*
- *any Lebesgue measurable set $D \subseteq \mathbb{R}^d$ with $0 < \ell_d(D) < \infty$ turns a s.p.s. when endowed with the normalized measure $\ell_d/\ell_d(D)$.*

Theorem B.0.3 (Riesz)**.** *Let $\phi : C_c(X) \to \mathbb{R}$ be a positive linear and continuous function, where X is an Hausdorff and locally compact space. There exists an uniquely determined Radon measure μ such that*

$$\phi(G) = \int_X G \mathrm{d}\mu \qquad \forall G \in C_c(X).$$

Lemma B.0.4. *Given any Lebesgue measurable set $D \subseteq \mathbb{R}^d$ with $0 < \ell_d(D) < \infty$ and any probability measure μ on $\mathbb{C}$, there exists a function $\kappa \in \mathscr{M}_D$ such that $\phi_\mu \equiv \phi_\kappa$.*

Proof. In [7] we showed that there exists a function $h \in \mathscr{M}_{[0,1]}$ such that $\phi_\mu \equiv \phi_h$. From Theorem B.0.2, the space D endowed with the conditional Lebesgue measure is a s.p.s. so there exist $A_1 \subseteq [0,1]$ and $A_2 \subseteq D$ zero measure sets and an isomorphism modulo zero

$$\varphi : D \setminus A_2 \to [0,1] \setminus A_1.$$

Let $\kappa : D \to \mathbb{C}$ be a measurable function defined as

$$\kappa(x) = \begin{cases} 0, & x \in A_2, \\ h(\varphi(x)), & x \notin A_2. \end{cases}$$

Given any Borel set $U \in \mathscr{B}$ in $\mathbb{C}$, we have

$$\ell_d(\kappa^{-1}(U)) = \begin{cases} \ell_d((h \circ \varphi)^{-1}(U)), & 0 \notin U, \\ \ell_d((h \circ \varphi)^{-1}(U) \sqcup A_2) = \ell_d((h \circ \varphi)^{-1}(U)), & 0 \in U, \end{cases}$$

since $\ell_d(A_2) = 0$, so $\ell_d(\kappa^{-1}(U)) = \ell_d((h \circ \varphi)^{-1}(U))$ in any case. Note that

$$(h \circ \varphi)^{-1}(U) = \varphi^{-1}\left(h^{-1}(U) \setminus A_1\right)$$

and since φ^{-1} preserves the measures, we conclude that

$$\frac{\ell_d(\kappa^{-1}(U))}{\ell_d(D)} = \frac{\ell_d((h \circ \varphi)^{-1}(U))}{\ell_d(D)} = \frac{\ell_d(\varphi^{-1}\left(h^{-1}(U) \setminus A_1\right))}{\ell_d(D)} = \ell(h^{-1}(U) \setminus A_1) = \ell(h^{-1}(U)).$$

This is enough to apply Lemma 2.2.2 and discover that $\phi_\kappa = \phi_h = \phi_\mu$. $\square$

Definition B.0.5. *Let $\{A_n\}_n$ be a matrix-sequence. We say that $\{A_n\}_n$ is* ***sparsely unbounded*** *(s.u.) if for every $M > 0$ there exists n_M such that, for $n \geq n_M$,*

$$\frac{\#\{i \in \{1, \dots, s_n\} : \sigma_i(A_n) > M\}}{s_n} \leq r(M),$$

where $\lim_{M \to \infty} r(M) = 0$.
Moreover, we say that $\{A_n\}_n$ is ***spectrally sparsely unbounded*** *(s.s.u.) if for every $M > 0$ there exists n_M such that, for $n \geq n_M$,*

$$\frac{\#\{i \in \{1, \dots, s_n\} : |\lambda_i(A_n)| > M\}}{s_n} \leq r(M),$$

where $\lim_{M \to \infty} r(M) = 0$.

Lemma B.0.6. *Let $D \subseteq \mathbb{R}^d$ be any measurable set with $0 < \ell_d(D) < \infty$. Given a sequence $\{A_n\}_n \sim_\lambda \phi$ for some functional $\phi : C_C(\mathbb{C}) \to \mathbb{R}$, then*

$$\{A_n\}_n \text{ s.s.u.} \iff \exists \kappa \in \mathscr{M}_D : \phi = \phi_\kappa.$$

Given a sequence $\{A_n\}_n \sim_\sigma \phi$ for some functional $\phi : C_C(\mathbb{R}) \to \mathbb{R}$, then

$$\{A_n\}_n \text{ s.u.} \iff \exists \kappa \in \mathscr{M}_D : \phi = \phi_\kappa.$$

Proof. In case $\{A_n\}_n \sim_\sigma \phi_\kappa$, we have $\{A_n\}_n \sim_\sigma \kappa$ and we know from [52, Proposition 5.4] that $\{A_n\}_n$ is s.u. The same proof shows that if $\{A_n\}_n \sim_\lambda \kappa$, then $\{A_n\}_n$ is s.s.u.

In [7] it has been shown that if $\{A_n\}_n \sim_\lambda \phi$ then there exists an unique Radon measure μ on $\mathbb{C}$ such that $\{A_n\}_n \sim_\lambda \mu$, and it always is a finite measure with mass $|\mu| \leq 1$. If we consider a function $G \in C_c(\mathbb{C})$ such that $1 \geq G(z) \geq \chi_{|z| \leq M}(z)$ for every $z \in \mathbb{C}$, then, from the definition of $\{A_n\}_n \sim_\lambda \mu$,

$$\begin{aligned} |\mu| \geq \int_{\mathbb{C}} G \mathrm{d}\mu = \lim_{n \to \infty} \frac{1}{s_n} \sum_{j=1}^{s_n} G(\lambda_j(A_n)) \\ &\geq \limsup_{n \to \infty} \frac{\#\{i \in \{1, \dots, s_n\} : |\lambda_i(A_n)| \leq M\}}{s_n} \\ &= 1 - \liminf_{n \to \infty} \frac{\#\{i \in \{1, \dots, s_n\} : |\lambda_i(A_n)| > M\}}{s_n} \\ &\geq 1 - r(M). \end{aligned}$$

As a consequence, $1 \geq |\mu| \geq 1 - \lim_{M \to \infty} r(M) = 1$, so μ is a probability measure. From Lemma B.0.4 we conclude that there exists $\kappa \in \mathscr{M}_D$ such that $\phi = \phi_\mu = \phi_\kappa$. $\square$

Appendix C

Closure Results

Here we will prove that the set of sequences that enjoy a singular value symbol is closed with respect to the a.c.s. convergence. To prove the result, we need some auxiliary results first.

Lemma C.0.1. *If $d_{a.c.s.}(\{A_n\}_n, \{B_n\}_n) \le s$, then there exist $\{R_n\}_n, \{N_n\}_n$ such that*

$$A_n = B_n + R_n + N_n, \qquad \limsup_{n\to\infty} \mathrm{rk}(R_n)/n + \|N_n\| \le s.$$

In particular, we have

$$\limsup_{n\to\infty} \mathrm{rk}(R_n)/n \le s, \qquad \limsup_{n\to\infty} \|N_n\| \le s.$$

Proof. By definition of a.c.s. distance, we have

$$d_{a.c.s.}(\{A_n\}_n, \{B_n\}_n) = \limsup_{n\to\infty} \min_{i=1:n+1} \left\{ \sigma_i(A_n - B_n) + \frac{i-1}{n} \right\} \le s$$

and as a consequence, for every $\delta > 0$ there exists n_δ such that for every $n \le n_\delta$,

$$\min_{i=1:n+1} \left\{ \sigma_i(A_n - B_n) + \frac{i-1}{n} \right\} \le s + \delta \implies A_n - B_n = R_{n,\delta} + N_{n,\delta}, \qquad \mathrm{rk}(R_{n,\delta}) + \|N_{n,\delta}\| \le s + \delta,$$

where $R_{n,\delta}, N_{n,\delta}$ are defined as in [6, Lemma 2.2]. We can thus consider $\delta_m = 2^{-m}$ and suppose without loss of generality that $n_{2^{-m}}$ are increasing in m. If

$$R_n := R_{n,2^{-m}}, \qquad N_n := N_{n,2^{-m}} \qquad \forall n_{2^{-m+1}} \le n < n_{2^{-m}}$$

then

$$\limsup_{n\to\infty} \mathrm{rk}(R_n)/n + \|N_n\| \le s.$$

□

Lemma C.0.2. *If $d_{a.c.s.}(\{A_n\}_n, \{B_n\}_n) \le s$ and $\{A_n\}_n \sim_\sigma k$, $\{B_n\}_n \sim_\sigma h$ with h, k nonnegative decreasing functions on $[0,1]$, then*

$$h(x - s) + s \ge k(x) \ge h(x + s) - s$$

for almost every x in $(s, 1-s)$.

Proof. In Lemma C.0.1, we proved the existence of $\{R_n\}_n, \{N_n\}_n$ such that

$$A_n = B_n + R_n + N_n, \qquad \limsup_{n\to\infty} \mathrm{rk}(R_n)/n \le s, \qquad \limsup_{n\to\infty} \|N_n\| \le s.$$

Let $\delta > 0$ and n_δ an index such that for every $n \ge n_\delta$

$$\mathrm{rk}(R_n) \le n(s + \delta), \qquad \|N_n\| \le s + \delta.$$

Using the interlacing theorem for singular values Theorem 3.1.1, we know that

$$\sigma_{i-\lfloor n(s+\delta)\rfloor}(B_n) \geq \sigma_i(B_n+R_n) \geq \sigma_{i+\lfloor n(s+\delta)\rfloor}(B_n) \qquad \forall n-\lfloor n(s+\delta)\rfloor \geq i \geq \lfloor n(s+\delta)\rfloor+1$$

and by the Weyl's perturbation theorem for singular values Theorem 3.1.3 we have

$$\sigma_i(B_n+R_n)-s-\delta \leq \sigma_i(B_n+R_n+N_n) \leq \sigma_i(B_n+R_n)+s+\delta \qquad \forall i.$$

We can thus infer that

$$\sigma_{i+\lfloor n(s+\delta)\rfloor}(B_n)-s-\delta \leq \sigma_i(A_n) \leq \sigma_{i-\lfloor n(s+\delta)\rfloor}(B_n)+s+\delta \qquad \forall n-\lfloor n(s+\delta)\rfloor \geq i \geq \lfloor n(s+\delta)\rfloor+1.$$

Let f_{A_n} be a piecewise linear function that interpolates the singular values of A_n on $[0,1]$ and similarly for f_{B_n}. Notice that if D_n contains the singular values of A_n in decreasing order, then $f_{A_n}=f_{D_n}$ and $\{D_n\}_n \sim_\lambda k$. Using [6, Lemma 4.2], we obtain that $f_{A_n}\to k$ and $f_{B_n}\to h$. As a consequence,

$$f_{A_n}(i/n)=\sigma_i(A_n) \leq \sigma_{i-\lfloor n(s+\delta)\rfloor}(B_n)+s+\delta = f_{B_n}\left(\frac{i-\lfloor n(s+\delta)\rfloor}{n}\right)+s+\delta$$

and

$$f_{A_n}(i/n)=\sigma_i(A_n) \geq \sigma_{i+\lfloor n(s+\delta)\rfloor}(B_n)-s-\delta = f_{B_n}\left(\frac{i+\lfloor n(s+\delta)\rfloor}{n}\right)-s-\delta$$

for every i such that $n-\lfloor n(s+\delta)\rfloor \geq i \geq \lfloor n(s+\delta)\rfloor+1$. Since f_{A_n} and f_{B_n} are piecewise linear, we can conclude that

$$f_{B_n}\left(x+\frac{\lfloor n(s+\delta)\rfloor}{n}\right)-s-\delta \leq f_{A_n}(x) \leq f_{B_n}\left(x-\frac{\lfloor n(s+\delta)\rfloor}{n}\right)+s+\delta$$

and

$$f_{B_n}\left(x+(s+\delta)\right)-s-\delta \leq f_{A_n}(x) \leq f_{B_n}\left(x-(s+\delta)\right)+s+\delta$$

for every $x\in(s+\delta,1-(s+\delta))$. Since $f_{A_n}\to k$ and $f_{B_n}\to h$ it follows that

$$h\left(x+(s+\delta)\right)-s-\delta \leq k(x) \leq h\left(x-(s+\delta)\right)+s+\delta$$

for every $x\in(s+\delta,1-(s+\delta))$ and every $\delta>0$. When we now consider $y\in(s,1-s)$ we know that $y\in(s+\delta,1-(s+\delta))$ when δ is small enough, meaning that

$$\lim_{\delta\to0^+} h\left(y+(s+\delta)\right)-s \leq k(y) \leq \lim_{\delta\to0^+} h\left(y-(s+\delta)\right)+s.$$

Every decreasing function is also continuous a.e., so up to negligible points (y such that h is not continuous at $y+s$ or $y-s$) we obtain that

$$h\left(y+s\right)-s \leq k(y) \leq h\left(y-s\right)+s.$$

□

Given f,g decreasing function on $[0,1]$, call

$$\widetilde{d}(f,g):=\inf_{s>0}\{s|g(x-s)+s\geq f(x)\geq g(x+s)-s, \quad \widetilde{\forall}x\in(s,1-s)\}, \qquad \overline{d}(f,g):=\max\{\widetilde{d}(f,g),\widetilde{d}(g,f)\}$$

where "$\widetilde{\forall}$" means "for almost every".

Lemma C.0.3. *Given a sequence of nonnegative decreasing functions f_n, suppose that they are a Cauchy sequence with respect to $\overline{d}$. In this case, there exists a limit function f such that $f_n\to f$ in measure.*

Proof. From the definition of Cauchy sequence, for every $s>0$ we can find a function f_{N_s} such that $\overline{d}(f_n,f_m)<s$ for every $n,m\geq N_s$. Notice that we can always take N_s such that $s'\leq s \implies N_{s'}\geq N_s$. Let us thus fix $s<1/4$, and call $M=f_{N_s}(s)+s$. From the definition of $\overline{d}$, we know that

$$M\geq f_{N_s}(x-s)+s\geq f_m(x) \qquad \widetilde{\forall}x\in[2s,1-2s], \quad \forall m\geq N.$$

In particular, all $f_m(x)$ for $x>2s$ and $m\geq N_s$ are bounded by the same constant M.

Take now f_n, f_m with $n, m \geq N_s$ such that $\overline{d}(f_n, f_m) < \varepsilon < s$. Let us divide the interval $[2s, 1-2s]$ into $k = \lfloor 1/\sqrt{\varepsilon} \rfloor$ subintervals of length $\geq \sqrt{\varepsilon}$, delimited by the points $x_0 = 2s, x_1, \ldots, x_{k-1}, x_k = 1-2s$ where $x_{i+1} - x_i = (1-4s)/k$. Note that

$$\#\{i | f_n(x_i) - f_n(x_{i+1}) > \sqrt[4]{\varepsilon}, 0 < i < k\} \leq M/\sqrt[4]{\varepsilon}$$

and that if $f_n(x_i) - f_n(x_{i+1}) \leq \sqrt[4]{\varepsilon}$, then for almost every $y \in [x_i + \varepsilon, x_{i+1} - \varepsilon]$

$$\begin{aligned} f_n(y-\varepsilon) + \varepsilon \geq f_m(y) \geq f_n(y+\varepsilon) - \varepsilon \implies |f_m(y) - f_n(y)| &\leq \max\{f_n(y-\varepsilon) - f_n(y), f_n(y) - f_n(y+\varepsilon)\} + \varepsilon \\ &\leq f_n(y-\varepsilon) - f_n(y+\varepsilon) + \varepsilon \\ &\leq f_n(x_i) - f_n(x_{i+1}) + \varepsilon \\ &\leq \sqrt[4]{\varepsilon} + \varepsilon. \end{aligned}$$

We can thus deduce an upper bound for the measure of the points where $|f_m(y) - f_n(y)| \geq \sqrt[4]{\varepsilon} + \varepsilon$.

$$\mu \left\{ y \in [2s, 1-2s] \mid |f_m(y) - f_n(y)| > \sqrt[4]{\varepsilon} + \varepsilon \right\} \leq \left(\frac{M}{\sqrt[4]{\varepsilon}} + 1 \right) \frac{1-4s}{k} + 2\varepsilon \left(k - \frac{M}{\sqrt[4]{\varepsilon}} \right).$$

Notice that

$$\varepsilon < s < \frac{1}{4} \implies \frac{1}{2\sqrt{\varepsilon}} > 1 \implies \frac{1}{\sqrt{\varepsilon}} \geq k = \left\lfloor \frac{1}{\sqrt{\varepsilon}} \right\rfloor > \frac{1}{\sqrt{\varepsilon}} - 1 > \frac{1}{2\sqrt{\varepsilon}}$$

so

$$\left(\frac{M}{\sqrt[4]{\varepsilon}} + 1 \right) \frac{1-4s}{k} + 2\varepsilon \left(k - \frac{M}{\sqrt[4]{\varepsilon}} \right) \leq M(1-4s)2\sqrt[4]{\varepsilon} + 2(1-4s)\sqrt{\varepsilon} + 2\sqrt{\varepsilon} - 2M\sqrt[4]{\varepsilon^3} \leq 2(M+2)\sqrt[4]{\varepsilon}.$$

Using the above relation, we have that

$$d_{mea}(f_n, f_m) \leq 4s + 2(M+2)\sqrt[4]{\varepsilon} + \sqrt[4]{\varepsilon} + \varepsilon \leq 4s + 2(M+3)\sqrt[4]{\varepsilon}.$$

This is enough to prove that f_n is a Cauchy sequence for d_{mea}. In fact, given any $\delta > 0$, consider $s < \delta/8$ and take $\varepsilon < s$ such that $2(M_s + 3)\sqrt[4]{\varepsilon} \leq \delta/2$. It turns out that for every $n, m \geq N_\varepsilon$ we have

$$d_{mea}(f_n, f_m) \leq 4s + 2(M_s + 3)\sqrt[4]{\varepsilon} \leq \delta.$$

As a consequence, f_n converge in measure to f.

□

Theorem C.0.4. *Given $\{B_{n,m}\}_{n,m} \sim_\sigma \kappa_m$ matrix-sequences with nonnegative and decreasing symbols κ_m on $[0,1]$ and such that $\{B_{n,m}\}_{n,m} \xrightarrow{a.c.s.} \{A_n\}_n$, the sequence κ_m converge in measure to a function κ.*

Proof. We know that $\{B_{n,m}\}_{n,m}$ is a Cauchy sequence for $d_{a.c.s.}$. By Lemma C.0.2, we infer that k_m are a Cauchy sequence for $\overline{d}$, and Lemma C.0.3 let us conclude that κ_m converge with respect to d_{mea}. □

We are now ready to prove the main result.

Theorem C.0.5. *Let $\{A_n\}_n, \{B_{n,m}\}_{n,m} \in \mathscr{E}$ and let $\kappa_m : D_m \subseteq \mathbb{R}^{t_m} \to \mathbb{C}$ be measurable functions defined on sets D_m with $0 < \ell_{t_m}(D_m) < \infty$. Suppose that*

1. *$\{B_{n,m}\}_{n,m} \sim_\sigma \kappa_m$ for every m,*
2. *$\{B_{n,m}\}_{n,m} \xrightarrow{a.c.s.} \{A_n\}_n$.*

Then there exists a measurable function $k : [0,1] \to \mathbb{C}$ that is a singular value symbol for $\{A_n\}_n$.

Proof. Suppose that ξ_m is the decreasing rearrangement of κ_m for every m. By [6, Lemma 4.1], we know that $\{B_{n,m}\}_{n,m} \sim_\sigma \xi_m$. Using Theorem C.0.4, we know that ξ_m converge to a function ξ in measure, and using the closure result [52, Corollary 5.1] we conclude that $\{A_n\}_n \sim_\sigma h$. □

Appendix D

Derivatives

For every matrix $A \in \mathbb{C}^{n\times n}$ define the function

$$p^k(A) := \min_{i=1,\dots,n+1} \left\{ \frac{i-1}{n} + n^k \sigma_i(A) \right\}.$$

Lemma D.0.1.

$$d_{a.c.s.^k}(\{A_n\}_n, \{B_n\}_n) := \limsup_{n\to\infty} p^k(A_n - B_n).$$

induces a complete pseudometric.

Proof. Using Theorem 3.1.13, we need only to prove that $p^k(A+B) \le p^k(A) + p^k(B)$. Suppose that r, s are the indexes of the minima for A, B respectively. If $r+s-1 > n$, then

$$p^k(A) + p^k(B) = \frac{r-1}{n} + n^k\sigma_r(A) + \frac{s-1}{n} + n^k\sigma_s(B) \ge 1 \ge p^k(A+B)$$

otherwise

$$p^k(A) + p^k(B) \ge n^k(\sigma_r(A) + \sigma_s(B)) + \frac{s+r-2}{n} \ge n^k\sigma_{r+s-1}(A+B) + \frac{s+r-2}{n} \ge p^k(A+B).$$

□

Notice that

$$d_{a.c.s.^k}(\{A_n\}_n, \{B_n\}_n) = d_{a.c.s.}(\{n^k A_n\}_n, \{n^k B_n\}_n).$$

Let J_n be the nilpotent Jordan block with rank $n-1$. Let us define

$$T_x : \mathscr{E} \to \mathscr{E} \qquad T_\theta : \mathscr{E} \to \mathscr{E}$$

$$T_x(\{A_n\}_n) = \{nJ_nA_nJ_n^T - nA_n\}_n \qquad T_\theta(\{A_n\}_n) = \{S_n \circ A_n\}, \quad (S_n)_{i,j} = \mathrm{i}(j-i)$$

Notice that if $v \in \mathbb{R}^n$ is a vector with $v_i = i$, then $S_n = \mathrm{i}ev^T - \mathrm{i}ve^T$, so it is a rank 2 matrix. For clarity, we will use T_x and T_θ also as maps of matrices:

$$T_x(A) = nJ_nAJ_n^T - nA \qquad T_\theta(A) = S_n \circ A.$$

Lemma D.0.2. *Suppose $a \in C^1[0,1]$ and $f \in C^1_{per}[-\pi,\pi]$. The following are true:*

$$T_x(\{D_n(a)\}_n) \sim_{GLT} a'(x) \otimes 1, \qquad T_\theta(\{D_n(a)\}_n) \sim_{GLT} 0,$$
$$T_\theta(\{T_n(f)\}_n) \sim_{GLT} 1 \otimes f'(\theta), \qquad T_x(\{T_n(f)\}_n) \sim_{GLT} 0.$$

Proof. First of all, notice that if D is diagonal

$$D = diag(d_1, d_2, \dots, s_n) \implies T_x(D) = n \cdot diag(d_2 - d_1, d_3 - d_2, \dots, s_n - s_{n-1}, -s_n).$$

Given now $a \in C^1[0,1]$, we may notice that

$$n(a_{i+1} - a_i) = \frac{a\left(\frac{i+1}{n}\right) - a\left(\frac{i}{n}\right)}{\frac{1}{n}} = a'(s), \qquad s \in \left[\frac{i}{n}, \frac{i+1}{n}\right]$$

but $a'(x)$ is uniformly continuous so for every n there exists $\delta_n > 0$ such that

$$|x - y| \leq \frac{1}{n} \implies |a'(x) - a'(y)| \leq \delta_n, \qquad \delta_n = o_n(1).$$

Consequently,

$$\|T_x(D_n(a)) - D_n(a') - R_n\| \leq \delta_n = o(1)$$

where R_n is a rank 1 matrix that takes care of the last entry. We conclude that

$$T_x(\{D_n(a)\}_n) \equiv_{acs} \{D_n(a')\}_n \sim_{GLT} a'(x) \otimes 1.$$

We can also notice that $T_\theta(\{D_n(a)\}_n) = \{0_n\}_n \sim_{GLT} 0$.

Given $f \in C^1_{per}[-\pi, \pi]$, with Fourier series $f(\theta) = \sum_k f_k e^{\mathrm{i}k\theta}$, we know that the Fourier series of its derivative is $f'(\theta) = \sum_k \mathrm{i}k f_k e^{\mathrm{i}k\theta}$, so

$$T_\theta(\{T_n(f)\}_n) = \{T_n(f')\}_n \sim_{GLT} 1 \otimes f'(\theta).$$

Eventually, we can explicitly write the action of T_x on any matrix

$$T_x(A)_{i,j} = n \cdot \begin{cases} -a_{i,j} & i = n \text{ or } j = n \\ a_{i+1,j+1} - a_{i,j} & \text{otherwise} \end{cases}$$

and on a Toeplitz matrix,

$$T_x(T_n(f))_{i,j} = n \cdot \begin{cases} -f_{i-j} & i = n \text{ or } j = n \\ 0 & \text{otherwise} \end{cases}$$

that has rank ≤ 2, so

$$T_x(\{T_n(f)\}_n) \sim_{GLT} 0.$$

□

T_x and T_θ seem to behave like partial derivatives. In particular, they respect also Leibniz rule.

Lemma D.0.3. *For every couple of sequences $\{A_n\}_n, \{B_n\}_n$,*

$$T_\theta(\{A_nB_n\}_n) = T_\theta(\{A_n\}_n)\{B_n\}_n + \{A_n\}_n T_\theta(\{B_n\}_n).$$

If $T_x(\{A_n\}_n)$ and $T_x(\{B_n\}_n)$ are sparsely unbounded, then

$$T_x(\{A_nB_n\}_n) \equiv_{acs} T_x(\{A_n\}_n)\{B_n\}_n + \{A_n\}_n T_x(\{B_n\}_n).$$

Proof. Notice that $s_{i,j} = s_{i,k} + s_{k,j}$, so

$$\begin{aligned}[S \circ (AB)]_{i,j} = s_{i,j} \sum_k a_{i,k} b_{k,j} &= \sum_k (s_{i,k} + s_{k,j}) a_{i,k} b_{k,j} \\ &= \sum_k s_{i,k} a_{i,k} b_{k,j} + \sum_k a_{i,k} s_{k,j} b_{k,j} = [(S \circ A)B + A(S \circ B)]_{i,j}\end{aligned}$$

and it directly proves that

$$T_\theta(\{A_nB_n\}_n) = T_\theta(\{A_n\}_n)\{B_n\}_n + \{A_n\}_n T_\theta(\{B_n\}_n).$$

Suppose now that $T_x(\{A_n\}_n)$ and $T_x(\{B_n\}_n)$ are sparsely unbounded.

$$
\begin{aligned}
T_x(\{A_nB_n\}_n) &= \{nJ_nA_nB_nJ_n^T - nA_nB_n\}_n,\\
T_x(\{A_n\}_n)\{B_n\}_n + \{A_n\}_nT_x(\{B_n\}_n) &= \{nJ_nA_nJ_n^TB_n - nA_nB_n + nA_nJ_nB_nJ_n^T - nA_nB_n\}_n,\\
\left\{\frac{1}{n}I_n\right\}_n T_x(\{A_n\}_n)T_x(\{B_n\}_n) &= \{nJ_nA_nJ_n^TJ_nB_nJ_n^T - nJ_nA_nJ_n^TB_n - nA_nJ_nB_nJ_n^T + nA_nB_n\}_n.
\end{aligned}
$$

$$
\begin{aligned}
\{nJ_nA_nJ_n^TJ_nB_nJ_n^T - nA_nB_n\}_n &= T_x(\{A_n\}_n)\{B_n\}_n + \{A_n\}_nT_x(\{B_n\}_n) + \left\{\frac{1}{n}I_n\right\}_n T_x(\{A_n\}_n)T_x(\{B_n\}_n)\\
&= T_x(\{A_nB_n\}_n) - \{nJ_nA_ne_1e_1^TB_nJ_n^T\}_n.
\end{aligned}
$$

Notice that $nJ_nA_ne_1e_1^TB_nJ_n^T$ has rank ≤ 1 and $\left\{\frac{1}{n}I_n\right\}_n T_x(\{A_n\}_n)T_x(\{B_n\}_n)$ is zero-distributed since both $T_x(\{A_n\}_n)$ and $T_x(\{B_n\}_n)$ are sparsely unbounded. This let us conclude that

$$T_x(\{A_nB_n\}_n) \equiv_{acs} T_x(\{A_n\}_n)\{B_n\}_n + \{A_n\}_nT_x(\{B_n\}_n).$$

□

Corollary D.0.4. *Let $\mathscr{A}$ be the $\mathbb{C}$-algebra generated by $\{D_n(a)\}_n$ and $\{T_n(f)\}_n$ with $a \in C^1[0,1]$ and $f \in C^1_{per}[-\pi,\pi]$. Every $\{A_n\}_n \in \mathscr{A}$ is a GLT sequence, and $\{A_n\}_n \sim_{GLT} \kappa$ implies that*

$$T_x(\{A_n\}_n) \sim_{GLT} {}^{\partial\kappa}\!/_{\partial x}, \qquad T_\theta(\{A_n\}_n) \sim_{GLT} {}^{\partial\kappa}\!/_{\partial\theta}.$$

Proof. The maps T_x and T_θ are linear, and using Lemma D.0.2 and D.0.3, we obtain the thesis only if $T_x(\{A_n\}_n)$ is sparsely unbounded on $\mathscr{A}$. Given any $\{A_n\}_n \in \mathscr{A}$, it can be represented as a polynomial in the generating sequences, but Lemma D.0.2 proves that T_x on generators produces sequences with GLT symbols, so $T_x(\{A_n\}_n)$ also owns a GLT symbol, and in particular it is sparsely unbounded. □

Sadly, there exists zero-distributed sequences that do not behave well with T_x and T_θ. For example,

$$A_n = \begin{pmatrix} 1/n & & & \\ & -1/n & & \\ & & \ddots & \\ & & & (-1)^{n+1}1/n \end{pmatrix}, \qquad T_x(A_n) = \begin{pmatrix} 2 & & & \\ & -2 & & \\ & & \ddots & \\ & & & (-1)^{n+1} \end{pmatrix}$$

where $\{A_n\}_n \sim_{GLT} 0$ and $\{T_x(A_n)\}_n$ is not even a GLT sequence. An other example is

$$A_{2n} = \frac{1}{\log(n)}\begin{pmatrix} 0_n & I_n \\ 0_n & 0_n \end{pmatrix}, \quad A_{2n+1} = \frac{1}{\log(n)}\begin{pmatrix} 0_n & I_{n,n+1} \\ 0_{n+1,n} & 0_{n+1} \end{pmatrix} \implies \{A_n\}_n \sim_{GLT} 0$$

but the matrices $T_\theta(A_n)$ have half of the singular values that increase as $n/\log(n)$ so the sequence $\{T_\theta(A_n)\}_n$ is not even sparsely unbounded.

In order to regain a concept of zero-distributed sequence, we need to trade the a.c.s. distance with another distance, $d_{a.c.s.^1}$, defined as follows.

Lemma D.0.5. *Two sequences $\{A_n\}_n$ and $\{B_n\}_n$ are equivalent for $d_{a.c.s.^1}$, that is*

$$d_{a.c.s.^1}(\{A_n\}_n, \{B_n\}_n) = 0$$

if and only if

$$A_n - B_n = R_n + N_n, \qquad \mathrm{rk}(R_n) = o(n), \quad \|N_n\|n = o(1).$$

In this case, we say that $\{A_n - B_n\}$ is zero-1 distributed and $\{A_n\}_n \equiv_1 \{B_n\}_n$.

Proof. Remember that

$$d_{a.c.s.^1}(\{A_n\}_n, \{B_n\}_n) = \limsup_{n\to\infty} \min_{i=1,\dots,n+1} \left\{\frac{i-1}{n} + n\sigma_i(A_n - B_n)\right\}.$$

Notice that if

$$A_n - B_n = R_n + N_n, \qquad \mathrm{rk}(R_n) = o(n), \quad \|N_n\|n = o(1).$$

then

$$\begin{aligned}
d_{a.c.s.^1}(\{A_n\}_n, \{B_n\}_n) &= d_{a.c.s.^1}(\{R_n + N_n\}_n, \{0_n\}_n) \\
&\le d_{a.c.s.^1}(\{R_n + N_n\}_n, \{N_n\}_n) + d_{a.c.s.^1}(\{N_n\}_n, \{0_n\}_n) \\
&= d_{a.c.s.^1}(\{R_n\}_n, \{0_n\}_n) + d_{a.c.s.^1}(\{N_n\}_n, \{0_n\}_n) \\
&\le \limsup_{n\to\infty} \frac{\mathrm{rk}(R_n)}{n} + \limsup_{n\to\infty} n\|N_n\| = 0
\end{aligned}$$

On the other hand, if we split the SVD of $A_n - B_n$ at the index i that realizes the minimum in

$$\min_{i=1,\dots,n+1} \left\{ \frac{i-1}{n} + n\sigma_i(A_n - B_n) \right\}$$

we obtain two sequences R_n, N_n with

$$0 = d_{a.c.s.^1}(\{A_n\}_n, \{B_n\}_n) = \limsup_{n\to\infty} \frac{\mathrm{rk}(R_n)}{n} + n\|N_n\| \implies \mathrm{rk}(R_n) = o(n), \quad \|N_n\|n = o(1).$$

□

The set of zero-1 distributed sequences is denoted as $\mathscr{Z}^1$ and is contained in $\mathscr{Z}$ since $d_{a.c.s.}(\cdot,\cdot) \le d_{a.c.s.^1}(\cdot,\cdot)$. They take the place of zero-distributed sequences when dealing with C^1 symbols.

Lemma D.0.6. *Given any matrix $A \in \mathbb{C}^{n\times m}$, we have*

$$\mathrm{rk}\, T_x(A) \le 2\,\mathrm{rk}\, A, \quad \|T_x(A)\| \le 2n\|A\|, \quad \mathrm{rk}\, T_\theta(A) \le 2\,\mathrm{rk}\, A, \quad \|T_\theta(A)\| \le 2n\|A\|.$$

Proof. The relations on T_x are simple to prove. The relations on T_θ require some properties of Hadamard product.

$$[vw^T \circ xy^T]_{i,j} = v_i w_j x_i y_j = [(v\circ x)(w \circ y)^T]_{i,j} \implies vw^T \circ xy^T = (v\circ x)(w\circ y)^T$$

so a simple computation leads to

$$\mathrm{rk}(T_\theta(A)) = \mathrm{rk}(S \circ A) =\le 2\,\mathrm{rk}(A).$$

Another property of Hadamard product is

$$\begin{aligned}
\|vw^T \circ A\| &= \max_{\|x\|=\|y\|=1} |x^T(vw^T \circ A)y| = \max_{\|x\|=\|y\|=1} \left| \sum_{i,j} x_i v_i a_{i,j} w_j y_j \right| \\
&= \max_{\|x\|=\|y\|=1} \left| (x\circ v)^T A (w \circ y) \right| \le \max_{\|x\|=\|y\|=1} \|x \circ v\| \|A\| \|w \circ y\| \\
&\le \max_{\|x\|=\|y\|=1} \|x\| \|v\|_\infty \|A\| \|w\|_\infty \|y\| = \|v\|_\infty \|A\| \|w\|_\infty
\end{aligned}$$

We can thus conclude

$$\|T_\theta(A)\| \le 2n\|A\|.$$

□

Corollary D.0.7. *If $\{Z_n\}_n$ is a zero-1 distributed sequence, then*

$$T_x(\{Z_n\}_n) \sim_{GLT} 0, \qquad T_\theta(\{Z_n\}_n) \sim_{GLT} 0.$$

Proof. T_x and T_θ are linear maps, and Lemma D.0.5 let us decompose $\{Z_n\}_n = \{R_n\}_n + \{N_n\}_n$. Notice that

$$\begin{aligned}
\mathrm{rk}(T_x(R_n)) \le 2\,\mathrm{rk}(R_n) &\implies \{T_x(R_n)\}_n \sim_{GLT} 0, \\
\|T_x(N_n)\| \le 2n\|N_n\| &\implies \{T_x(N_n)\}_n \sim_{GLT} 0
\end{aligned}$$

so $\{T_x(Z_n)\}_n \sim_{GLT} 0$. Moreover,

$$\begin{aligned} \operatorname{rk}(T_\theta(R_n)) \le 2\operatorname{rk}(R_n) &\implies \{T_\theta(R_n)\}_n \sim_{GLT} 0, \\ \|T_\theta(N_n)\| \le 2n\|N_n\| &\implies \{T_\theta(N_n)\}_n \sim_{GLT} 0 \end{aligned}$$

so $\{T_\theta(Z_n)\}_n \sim_{GLT} 0$. □

The last result lets us expand the algebra $\mathscr{A}$ of Corollary D.0.4 adding the set $\mathscr{Z}^1$. $\mathscr{B} = \mathscr{A} + \mathscr{Z}^1$ is still a $\mathbb{C}$-algebra, and T_x and T_θ still act as partial derivatives.

Lemma D.0.8.

$$d_{a.c.s.^1}(\{B_{n,m}\}_{n,m}, \{0_n\}_n) \to 0 \implies d_{a.c.s.}(T_x(\{B_{n,m}\}_{n,m}), \{0_n\}_n) \to 0, \quad d_{a.c.s.}(T_\theta(\{B_{n,m}\}_{n,m}), \{0_n\}_n) \to 0.$$

Proof. Repeating the argument of Lemma D.0.5, we can split every matrix $B_{n,m}$ into the sum of $R_{n,m}$ and $N_{n,m}$, where

$$\operatorname{rk}(R_{n,m}) = i - 1 = -1 + \arg\min_{i=1,\dots,n+1} \left\{ \frac{i-1}{n} + n\sigma_i(B_{n,m}) \right\}, \qquad \|N_{n,m}\| = \sigma_i(B_{n,m}).$$

We know that $d_{a.c.s.^1}(\{B_{n,m}\}_{n,m}, \{0_n\}_n) \to 0$, so

$$\lim_{m\to\infty} \limsup_{n\to\infty} \frac{\operatorname{rk}(R_{n,m})}{n} + n\|N_{n,m}\| = 0 \implies \limsup_{n\to\infty} \frac{\operatorname{rk}(R_{n,m})}{n} + n\|N_{n,m}\| = \widetilde{c}(m) \to 0.$$

If $c(m) = \widetilde{c}(m) + 1/m$, then $c(m) \to 0$ and we know that for every m there exists an index n_m for which

$$\frac{\operatorname{rk}(R_{n,m})}{n} \le c(m), \qquad n\|N_{n,m}\| \le c(m) \qquad \forall n \ge n_m.$$

If we apply T_x or T_θ, Lemma D.0.6 shows that we have the same bounds on the images, so we use the letter T to indicate any of the two maps.

$$T(B_{n,m}) = T(R_{n,m}) + T(N_{n,m}), \qquad \frac{\operatorname{rk}(T(R_{n,m}))}{n} \le 2c(m), \quad \|T(N_{n,m})\| \le c(m) \qquad \forall n \ge n_m$$

and this is enough to conclude that

$$d_{a.c.s.}(T(\{B_{n,m}\}_{n,m}), \{0_n\}_n) \to 0.$$

□

Corollary D.0.9.

$$\{B_{n,m}\}_{n,m} \xrightarrow{a.c.s.^1} \{A_n\}_n \implies \{T_x(B_{n,m})\}_{n,m} \xrightarrow{a.c.s.} \{T_x(A_n)\}_n, \quad \{T_\theta(B_{n,m})\}_{n,m} \xrightarrow{a.c.s.} \{T_\theta(A_n)\}_n.$$

Corollary D.0.10. *Suppose that $\{B_{n,m}\}_{n,m} \xrightarrow{a.c.s.^1} \{A_n\}_n$ and that $\{B_{n,m}\}_{n,m} \sim_{GLT} \kappa_m$ with $\{B_{n,m}\}_n \in \mathscr{B}$ for every m. Then it is true that*

- $T(\{B_{n,m}\}_{n,m}) \xrightarrow{a.c.s.} T(\{A_n\}_n)$,
- $\kappa_m \to \kappa$,
- $\{A_n\}_n \sim_{GLT} \kappa$,
- $k'_m \to \xi$,
- $T(\{A_n\}_n) \sim_{GLT} \xi$,

where T is T_x or T_θ and κ'_m is the respective partial derivative.

In the last corollary, we showed that if we close $\mathscr{B}$ with respect to the a.c.s.[1] convergence, we obtain GLT sequences such that their images through T are still GLT sequences. An open question is whether the symbols of the limit sequence $\{A_n\}_n \sim_{GLT} \kappa$ and its image $T(\{A_n\}_n) \sim_{GLT} \xi$ are linked or not.

Appendix E

Banded Sequences

From now on, we say that a sequence $\{A_n\}_n$ is almost-Hermitian if there exists an Hermitian sequence $\{\widetilde{A}_n\}_n$ such that $\|A_n - \widetilde{A}_n\|_2 = o(\sqrt{n})$.

Lemma E.0.1. *Suppose $\{A_n\}_n \sim_{GLT} k$ is an almost-Hermitian sequence. In this case, k is real valued and*

$$\{A_n\}_n \sim_\lambda k.$$

Proof. The hypothesis $\|A_n - \widetilde{A}_n\|_2 = o(\sqrt{n})$ implies that $\{A_n - \widetilde{A}_n\}_n$ is zero-distributed, so $\{\widetilde{A}_n\}_n \sim_{GLT} k$, but it is an Hermitian sequence, so k is real and $\{\widetilde{A}_n\}_n \sim_\lambda k$. Using **GLT 2**, we conclude that $\{A_n\}_n \sim_\lambda \kappa$. $\square$

Lemma E.0.2. *The set of almost-Hermitian sequences is a real vectorial space.*

Proof. If $\{A_n\}_n$ is an almost-Hermitian sequence and $c \in \mathbb{R}$, then

$$\|cA_n - c\widetilde{A}_n\|_2 = |c|\|A_n - \widetilde{A}_n\|_2 = o(\sqrt{n}).$$

If $\{B_n\}_n$ is also almost-Hermitian, then

$$\|A_n + B_n - \widetilde{A}_n - \widetilde{B}_n\| \le \|A_n - \widetilde{A}_n\|_2 + \|B_n - \widetilde{B}_n\|_2 = o(\sqrt{n}).$$

$\square$

Lemma E.0.3. *Let $a : [0,1] \to \mathbb{C}$ be a bounded function and call α-gap of a the function*

$$\omega_\alpha(x) := \sup_{z \in B_\alpha(x) \cap [0,1]} |a(z)| - \inf_{z \in B_\alpha(x) \cap [0,1]} |a(z)|.$$

If a is continuous a.e., then also ω_α is continuous a.e.

Proof. Let E be the set of discontinuity points for a, and define

$$Z := \{\, x \in [0,1] \mid x - \alpha \in E \text{ or } x + \alpha \in E \,\}.$$

Note that the measure of Z is at least two times the measure of E, so it is zero. Let now $x \in Z^C$ and $a(x) = b$. We know that both $x - \alpha$ and $x + \alpha$ (when they are inside $[0,1]$) are continuity points for a, so given any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|a(x-\alpha) - a(x-\alpha+y)| \le \varepsilon, \quad |a(x+\alpha) - a(x+\alpha+y)| \le \varepsilon \qquad \forall |y| < \delta.$$

As a consequence, for every $0 < y < \delta$ the following holds

$$
\begin{aligned}
&\sup_{z\in B_\alpha(x+y)\cap[0,1]} |a(z)| = \max\left\{ \sup_{z\in(x+y-\alpha,x+\alpha)\cap[0,1]} |a(z)|, \sup_{z\in[x+\alpha,x+\alpha+y)\cap[0,1]} |a(z)| \right\}\\
\implies &\sup_{z\in B_\alpha(x+y)\cap[0,1]} |a(z)| \le \max\left\{ \sup_{z\in B_\alpha(x)\cap[0,1]} |a(z)|, |a(x+\alpha)|+\varepsilon \right\} \le \sup_{z\in B_\alpha(x)\cap[0,1]} |a(z)| + \varepsilon,\\
&\sup_{z\in B_\alpha(x+y)\cap[0,1]} |a(z)| \ge \sup_{z\in B_\alpha(x)\cap[0,1]} |a(z)| - \varepsilon,\\
&\inf_{z\in B_\alpha(x+y)\cap[0,1]} |a(z)| = \min\left\{ \inf_{z\in(x+y-\alpha,x+\alpha)\cap[0,1]} |a(z)|, \inf_{z\in[x+\alpha,x+\alpha+y)\cap[0,1]} |a(z)| \right\}\\
\implies &\inf_{z\in B_\alpha(x+y)\cap[0,1]} |a(z)| \ge \min\left\{ \inf_{z\in B_\alpha(x)\cap[0,1]} |a(z)|, |a(x+\alpha)|-\varepsilon \right\} \ge \inf_{z\in B_\alpha(x)\cap[0,1]} |a(z)| - \varepsilon,\\
&\inf_{z\in B_\alpha(x+y)\cap[0,1]} |a(z)| \le \inf_{z\in B_\alpha(x)\cap[0,1]} |a(z)| + \varepsilon.
\end{aligned}
$$

An analogous argument let us conclude the same for $-\delta < y < 0$, so $|\omega_\alpha(x) - \omega_\alpha(x+y)|$ is bounded by

$$
\left| \sup_{z\in B_\alpha(x)\cap[0,1]} |a(z)| - \sup_{z\in B_\alpha(x+y)\cap[0,1]} |a(z)| \right| + \left| \inf_{z\in B_\alpha(x+y)\cap[0,1]} |a(z)| - \inf_{z\in B_\alpha(x)\cap[0,1]} |a(z)| \right| \le 2\varepsilon.
$$

This is enough to prove that ω_α is continuous at x, and consequently ω_α is continuous on Z^C, that is, it is continuous a.e. $\square$

Lemma E.0.4. *Given a sequence of Hermitian sequences $\{B_{n,m}\}_n$ with GLT symbol k_m, suppose that there exists a sequence $\{B_n\}_n$ with*

$$\lim_{m\to\infty} \limsup_{n\to\infty} \frac{1}{n} \|B_{n,m} - B_n\|_2^2 = 0.$$

In this case, there exists a limit function $k_m \to k$ and $\{B_n\}_n \sim_\lambda k$.

Proof. First of all, notice that the condition

$$\lim_{m\to\infty} \limsup_{n\to\infty} \frac{1}{n} \|B_{n,m} - B_n\|_2^2 = 0$$

directly implies that $\{B_{n,m}\}_n \xrightarrow{a.c.s.} \{B_n\}_n$ and consequentially k_m converge to a function k that is the GLT symbol for $\{B_n\}_n$. Let us now estimate the norm of the imaginary part of $\{B_n\}_n$.

$$
\begin{aligned}
\|\Im(B_n)\|_2 &= \frac{1}{2}\|B_n - B_n^H\|_2\\
&\le \frac{1}{2}\left(\|B_n - B_{n,m}\|_2 + \|B_{n,m} - B_{n,m}^H\|_2 + \|B_{n,m}^H - B_n^H\|_2 \right)\\
&= \|B_n - B_{n,m}\|_2\\
\implies \lim_{m\to\infty} \limsup_{n\to\infty} \frac{1}{n}\|\Im(B_n)\|_2^2 &= \lim_{m\to\infty} \limsup_{n\to\infty} \frac{1}{n}\|B_{n,m} - B_n\|_2^2 = 0\\
\implies \limsup_{n\to\infty} \frac{1}{n}\|\Im(B_n)\|_2^2 &= 0\\
\implies \|\Im(B_n)\|_2 &= o(\sqrt{n}).
\end{aligned}
$$

Since $B_n = \Re(B_n) + \mathrm{i}\Im(B_n)$, we can use Lemma E.0.1 and conclude that $\{B_n\}_n \sim_\lambda k$. $\square$

Actually, the existence of a spectral symbol k for $\{B_n\}_n$ holds even if $\{B_{n,m}\}_n \sim_\lambda k_m$ without the hypothesis that they are GLT sequences. In such case, though, we can say that $k_m \to k$ only if all the functions k_m, k are decreasing.

Theorem E.0.5. *Given any Riemann Integrable function $a : [0,1] \to \mathbb{C}$ and any natural number m, denote*

$$A_n(a,m) := D_n(a)T_n(e^{\mathrm{i}m\theta}) + D_n(a)^H T_n(e^{\mathrm{i}m\theta})^H.$$

In this case, $\{A_n(a,m)\}_n$ is almost-Hermitian and

$$\{A_n(a,m)\}_n \sim_\lambda 2\Re(a(x)e^{\mathrm{i}m\theta}).$$

Proof. First of all, from the theory of GLT sequences, we know that

$$\{A_n(a,m)\}_n \sim_{GLT} 2\Re(a(x)e^{\mathrm{i}m\theta})$$

and if $m = 0$, then $A_n(a,m)$ is Hermitian for every n, so the thesis follows. Suppose now that $m > 0$ and define the Hermitian matrix $\widetilde{A}_n(a,m)$ as

$$[\widetilde{A}_n(a,m)]_{i,j} = \begin{cases} a(j/n), & i-j = m, \\ \overline{a}(i/n), & i-j = -m, \\ 0, & \text{otherwise.} \end{cases}$$

Let $Z_n = A_n(a,m) - \widetilde{A}_n(a_m)$, and notice that

$$[Z_n]_{i,j} = \begin{cases} a(i/n) - a(j/n), & i-j = m, \\ 0, & \text{otherwise.} \end{cases}$$

Notice that if ω_α is the α-gap relative to a, then $\omega_\alpha \xrightarrow{\alpha\to 0} 0$ a.e. since

$$a \text{ continuous on } x \implies \omega_\alpha(x) \xrightarrow{\alpha\to 0} 0.$$

We can thus fix $\varepsilon > 0$ and find α_ε (that we call α for simplicity) such that, if we call

$$E := \{\, \omega_\alpha(x) \geq \varepsilon \,\},$$

then $\mu(E) < \varepsilon/2$. Notice that a is bounded and continuous a.e., so by Lemma E.0.3, ω_α is also continuous a.e. and thus E^C is an open set up to a negligible set. Every open set can be approximated from the inside by a finite union of open intervals, so we can take $G \subseteq E^C$ a finite union of open intervals with measure $\mu(G) > \mu(E) - \varepsilon/2 > 1 - \varepsilon$. The set G is Peano-Jordan measurable (also said, it has negligible boundary) so it is well described by dense sets, meaning that

$$\lim_{n\to\infty} \frac{1}{n} \#\{\, i \mid 1 \leq i \leq n, i/n \in G \,\} = \mu(G) > 1 - \varepsilon.$$

Let N be an index such that $m/N < \alpha$ and

$$\frac{1}{n} \#\{\, i \mid 1 \leq i \leq n, i/n \in G \,\} = \mu(G) > 1 - 2\varepsilon \forall n > N.$$

If $\|a\|_\infty = M$, we have

$$\begin{aligned} \|Z_n\|_2^2 &= \sum_{j=1}^{n-m} |a((j+m)/n) - a(j/n)|^2 \\ &= \sum_{j\leq n-m}^{j/n\notin G} |a((j+m)/n) - a(j/n)|^2 + \sum_{j\leq n-m}^{j/n\in G} |a((j+m)/n) - a(j/n)|^2 \\ &\leq 2\varepsilon n M^2 + n\varepsilon^2 \end{aligned}$$

for every $n \geq N$. As a consequence

$$\limsup_{n\to\infty} \frac{1}{n} \|Z_n\|_2^2 \leq 2\varepsilon M^2 + \varepsilon^2$$

for every $\varepsilon > 0$, so

$$\limsup_{n\to\infty} \frac{1}{n} \|Z_n\|_2^2 = 0.$$

$\{A_n(a,m)\}_n$ is thus almost-Hermitian and Lemma E.0.1 let us conclude that $\{A_n(a,m)\}_n \sim_\lambda 2\Re(a(x)e^{\mathrm{i}m\theta})$. $\square$

Lemma E.0.6. *Given $m+1$ Riemann Integrable function $a_0, a_1, \dots, a_m$, the sequence*

$$\sum_{k=0}^{m} \{D_n(a_k)T_n(e^{\mathrm{i}k\theta}) + D_n(a_k)^H T_n(e^{\mathrm{i}k\theta})^H\}_n$$

is almost-Hermitian and

$$\sum_{k=0}^{m} \{D_n(a_k)T_n(e^{\mathrm{i}k\theta}) + D_n(a_k)^H T_n(e^{\mathrm{i}k\theta})^H\}_n \sim_\lambda \sum_{k=0}^{m} 2\Re(a_k(x)e^{\mathrm{i}k\theta})$$

Proof. From Theorem E.0.5, we know that the sequences $\{A_n(a,k)\}_n$ are almost-Hermitian for every k and Lemma E.0.2 let us conclude also that $\sum_{k=0}^{m}\{A_n(a,k)\}_n$ is almost-Hermitian. Since it is also a GLT sequence with symbol $\sum_{k=0}^{m} 2\Re(a_k(x)e^{\mathrm{i}k\theta})$, the thesis follows from Lemma E.0.1. $\square$

Appendix F

Reduced GLT

We require that Ω is contained in $[0,1]^d$, and we work in the restricted euclidean topology and Lebesgue measure ℓ_d of $[0,1]^d$, unless specified differently.

F.1 Characteristic Sequences

Lemma F.1.1. *If Ω is a Peano-Jordan measurable set, then*

$$\lim_{n\to\infty} \frac{\mathrm{rk}(D_{\boldsymbol{n}}(\chi_\Omega))}{N(\boldsymbol{n})} = \ell_d(\Omega).$$

Proof. We know that

$$\{D_{\boldsymbol{n}}(\chi_\Omega)\}_n \sim_\sigma \chi_\Omega$$

so in particular, if we consider a continuous function $G : \mathbb{R} \to \mathbb{C}$ with compact support and such that $G(1) = 1$, $G(0) = 0$, then

$$\lim_{n\to\infty} \frac{\mathrm{rk}(D_{\boldsymbol{n}}(\chi_\Omega))}{N(\boldsymbol{n})} = \lim_{n\to\infty} \frac{1}{N(\boldsymbol{n})} \sum_{i=1}^{N(\boldsymbol{n})} G(\sigma_i(D_{\boldsymbol{n}}(\chi_\Omega))) = \int_{[0,1]^d} G(\chi_\Omega(x))\mathrm{d}x = \ell_d(\Omega).$$

□

Call

$$K_c = \{\, p \in [0,1]^d \mid d(p, \partial\Omega) \le c \,\}$$

the set of points whose distance from $\partial\Omega$ is at most $c \ge 0$. In the next result, we prove that K_c contains few points $z_{\boldsymbol{i}}$ when c tends to zero, so that in the applications we can ignore the conditions that arise from grid points that are close enough to the boundary.

Lemma F.1.2. *Given a sequence h_n of real nonnegative numbers converging to zero, and a Peano-Jordan measurable set Ω, then*

$$\lim_{n\to\infty} \frac{\mathrm{rk}(D_{\boldsymbol{n}}(\chi_{K_{h_n}}))}{N(\boldsymbol{n})} = 0.$$

Proof. Remember that $\partial\Omega$ is always a closed set contained into $[0,1]^d$. Notice that K_c converge to $K_0 = \partial\Omega$ as c tends to zero, so we know that

$$\lim_{c\to 0} \ell_d(K_c) = \ell_d(\partial\Omega) = 0.$$

K_c is a closed subset of $[0,1]^d$ for every c since

$$p \notin K_c \implies p \notin [0,1]^d \vee d(p, \partial\Omega) > c$$

and in both case there's an open neighbourhood of p disjoint from K_c. Moreover, if $c > 0$ then

$$p \in \partial K_c \implies p \in \partial[0,1]^d \vee d(p, \partial\Omega) = c$$

and it is known that the set of points at fixed positive distance from a closed set is negligible [44], so we can conclude that $\ell_d(\partial K_c) = 0$. This is actually true also for K_0 since

$$\partial K_0 = \partial\partial\Omega \subseteq \partial\Omega \implies \ell_d(\partial K_0) \le \ell_d(\partial\Omega) = 0.$$

We can thus use Lemma F.1.1 to infer that for every $c \ge 0$

$$\lim_{n\to\infty} \frac{\operatorname{rk}(D_{\boldsymbol{n}}(\chi_{K_c}))}{N(\boldsymbol{n})} = \ell_d(K_c).$$

Notice that if $h_n < h_m$ then $K_{h_n} \subseteq K_{h_m}$ and consequently $\operatorname{rk}(D_{\boldsymbol{n}}(\chi_{K_{h_n}})) \le \operatorname{rk}(D_{\boldsymbol{n}}(\chi_{K_{h_m}}))$. When we fix an index $m > 0$, we know that definitively $h_n < h_m$ since h_n are converging to zero, so the following relation holds

$$\limsup_{n\to\infty} \frac{\operatorname{rk}(D_{\boldsymbol{n}}(\chi_{K_{h_n}}))}{N(\boldsymbol{n})} \le \limsup_{n\to\infty} \frac{\operatorname{rk}(D_{\boldsymbol{n}}(\chi_{K_{h_m}}))}{N(\boldsymbol{n})} = \ell_d(K_{h_m}) \qquad \forall m$$

$$\implies \limsup_{n\to\infty} \frac{\operatorname{rk}(D_{\boldsymbol{n}}(\chi_{K_{h_n}}))}{N(\boldsymbol{n})} \le \inf_{m\in\mathbb{N}} \ell_d(K_{h_m}) = 0.$$

$\square$

Define Ξ_n as the grid composed by points of the form

$$\frac{\boldsymbol{i}}{\boldsymbol{n}+\boldsymbol{1}} = \left(\frac{i_1}{n_1+1}, \frac{i_2}{n_2+1}, \dots, \frac{i_d}{n_d+1}\right), \qquad i_j = 0,1,2,\dots,n_j,n_j+1, \quad j = 1,2,\dots,d.$$

Consequentially we define a new diagonal matrix associated to Ω

$$I_{\boldsymbol{n}}(\chi_\Omega) := \operatorname{diag}\left(\chi_\Omega\left(\frac{\boldsymbol{i}}{\boldsymbol{n}+\boldsymbol{1}}\right)\right)_{\boldsymbol{i}=\boldsymbol{1},\dots,\boldsymbol{n}}$$

that has dimension $N(\boldsymbol{n}) \times N(\boldsymbol{n})$, the same as $D_{\boldsymbol{n}}(\chi_\Omega)$. More in general, for any continuous a.e. function $a : [0,1]^d \to \mathbb{C}$ we denote

$$I_{\boldsymbol{n}}(a) := \operatorname{diag}\left(a\left(\frac{\boldsymbol{i}}{\boldsymbol{n}+\boldsymbol{1}}\right)\right)_{\boldsymbol{i}=\boldsymbol{1},\dots,\boldsymbol{n}}$$

so that $I_{\boldsymbol{n}}(a)$ and $D_{\boldsymbol{n}}(a)$ have the same dimension, and can actually be proved that they enjoy the same GLT and spectral symbol.

Lemma F.1.3. *If $a : [0,1]^d \to \mathbb{C}$ is a continuous a.e. function, then*

$$\{I_{\boldsymbol{n}}(a)\}_n \sim_{GLT} a.$$

Proof. Notice that $a : [0,1]^d \to \mathbb{C}$ is a continuous a.e. if and only if when we split it into real and imaginary part $a = a_1 + \mathrm{i}a_2$, both the real functions a_1 and a_2 are continuous a.e.. In the same way, we can split a_1 and a_2 in their positive and negative parts, and they are still continuous a.e.. By linearity of the GLT sequences, we can thus suppose that $a : [0,1]^d \to \mathbb{R}^+$, since it is sufficient to prove the general thesis.

The proof is divided into 3 steps, where we prove that the statement holds first when a is continuous, then when a is Riemann-integrable and eventually when a is continuous a.e..

Step 1. Suppose a is continuous and call ω_a its continuity module. Notice that

$$\left\|\frac{\boldsymbol{i}}{\boldsymbol{n}} - \frac{\boldsymbol{i}}{\boldsymbol{n}+\boldsymbol{1}}\right\|_2^2 \le \sum_{k=1}^d \left(\frac{i_k}{n_k(n_k+1)}\right)^2 \le \sum_{k=1}^d \frac{1}{n_k^2} =: h_n^2 \xrightarrow{n\to\infty} 0,$$

so we can obtain a bound on the norm of $I_{\boldsymbol{n}}(a) - D_{\boldsymbol{n}}(a)$ as

$$\|I_{\boldsymbol{n}}(a) - D_{\boldsymbol{n}}(a)\| = \max_{\boldsymbol{i}=\boldsymbol{1},\dots,\boldsymbol{n}} \left|a\left(\frac{\boldsymbol{i}}{\boldsymbol{n}+\boldsymbol{1}}\right) - a\left(\frac{\boldsymbol{i}}{\boldsymbol{n}}\right)\right| \le \omega_a(h_n) \xrightarrow{n\to\infty} 0.$$

This is enough to prove that $\{I_{\boldsymbol{n}}(a) - D_{\boldsymbol{n}}(a)\}_n$ is zero-distributed and consequentially $\{I_{\boldsymbol{n}}(a)\}_n \sim_{GLT} a$.

Step 2. Suppose a is Riemann-integrable, and consider a sequence of continuous function a_m converging to a in L^1 norm. A continuous function is in particular Riemann-integrable, so $a_m - a$ is also Riemann-integrable and we can compute

$$N(\boldsymbol{n})^{-1}\|I_{\boldsymbol{n}}(a_m) - I_{\boldsymbol{n}}(a)\|_1 = \frac{1}{N(\boldsymbol{n})}\sum_{\boldsymbol{i}=\boldsymbol{1}}^{\boldsymbol{n}}\left|a_m\left(\frac{\boldsymbol{i}}{\boldsymbol{n}+\boldsymbol{1}}\right) - a\left(\frac{\boldsymbol{i}}{\boldsymbol{n}+\boldsymbol{1}}\right)\right| \xrightarrow{n\to\infty} \|a - a_m\|_1 \xrightarrow{m\to\infty} 0.$$

We can thus write the difference as $\|I_{\boldsymbol{n}}(a_m) - I_{\boldsymbol{n}}(a)\|_1 = N(\boldsymbol{n})\varepsilon(n,m)$ where $\lim_{m\to\infty}\lim_{n\to\infty}\varepsilon(n,m) = 0$ and using **ACS 6** from [53, section 6], we discover that $\{I_{\boldsymbol{n}}(a_m)\}_n \xrightarrow{a.c.s.} \{I_{\boldsymbol{n}}(a)\}_n$. We know from Step 1 that $\{I_{\boldsymbol{n}}(a_m)\}_n \sim_{GLT} a_m$ for every m, and $a_m \xrightarrow{m\to\infty} a$ in measure, so we conclude that $\{I_{\boldsymbol{n}}(a)\}_n \sim_{GLT} a$.

Step 3. Suppose a is continuous a.e and call $a_m(x) := \max\{a(x), m\}$ its truncated function for every $m \in \mathbb{N}$. Notice that a_m are still continuous a.e. and also bounded, thus Riemann-integrable. Moreover, since a is measurable we know that

$$\ell_d\{x|a(x) > m\} =: h_m \xrightarrow{m\to\infty} 0.$$

We know from Step 2 that $\{I_{\boldsymbol{n}}(a_m)\}_n \sim_{GLT} a_m$ for every m, so we can fix $1 > \varepsilon > 0$ and consider $G_m(x)$ continuous and compact supported functions such that $\chi_{[0,m-\varepsilon]} \le G_m \le \chi_{[-\varepsilon,m]}$ to obtain

$$\begin{aligned}
N(\boldsymbol{n})^{-1}\operatorname{rk}(I_{\boldsymbol{n}}(a_m) - I_{\boldsymbol{n}}(a)) &= N(\boldsymbol{n})^{-1}\#\left\{\boldsymbol{i}\;\middle|\; a\left(\frac{\boldsymbol{i}}{\boldsymbol{n}+\boldsymbol{1}}\right) > m,\, \boldsymbol{1}\le\boldsymbol{i}\le\boldsymbol{n}\right\}\\
&= 1 - N(\boldsymbol{n})^{-1}\#\left\{\boldsymbol{i}\;\middle|\; a\left(\frac{\boldsymbol{i}}{\boldsymbol{n}+\boldsymbol{1}}\right) \le m,\, \boldsymbol{1}\le\boldsymbol{i}\le\boldsymbol{n}\right\}\\
&\le 1 - N(\boldsymbol{n})^{-1}\sum_{\boldsymbol{i}=\boldsymbol{1}}^{\boldsymbol{n}} G_m(\sigma_{\boldsymbol{i}}(D_{\boldsymbol{n}}(A_m))).
\end{aligned}$$

Note that $G_m(m) = 0$, so $G_m(a_m) = G_m(a)$ and taking the limits of the preceding relations, one can see that

$$\begin{aligned}
\limsup_{n\to\infty} N(\boldsymbol{n})^{-1}\operatorname{rk}(I_{\boldsymbol{n}}(a_m) - I_{\boldsymbol{n}}(a)) &\le 1 - \frac{1}{(2\pi)^d}\int_{[0,1]^d\times[-\pi,\pi]^d} G_m(a_m(x))\mathrm{d}x\\
&= 1 - \frac{1}{(2\pi)^d}\int_{[0,1]^d\times[-\pi,\pi]^d} G_m(a(x))\mathrm{d}x\\
&\le 1 - \frac{1}{(2\pi)^d}\int_{[0,1]^d\times[-\pi,\pi]^d} \chi_{[0,m-\varepsilon]}(a(x))\mathrm{d}x\\
&\le 1 - \frac{(2\pi)^d - h_{m-1}}{(2\pi)^d} =: c(m) \xrightarrow{m\to\infty} 0.
\end{aligned}$$

Consequently, for every m we can find n_m such that for every $n > n_m$, $\operatorname{rk}(I_{\boldsymbol{n}}(a_m) - I_{\boldsymbol{n}}(a)) \le c(m)N(\boldsymbol{n})$ with $c(m) \xrightarrow{m\to\infty} 0$, and it leads to $\{I_{\boldsymbol{n}}(a_m)\}_n \xrightarrow{a.c.s.} \{I_{\boldsymbol{n}}(a)\}_n$. We know that $a_m \xrightarrow{m\to\infty} a$ in measure, so we conclude that $\{I_{\boldsymbol{n}}(a)\}_n \sim_{GLT} a$. ☐

This result shows that for every $a : [0,1]^d \to \mathbb{C}$ continuous a.e. function, the sequences $\{I_{\boldsymbol{n}}(a)\}_n$ and $\{D_{\boldsymbol{n}}(a)\}_n$ have the same GLT (and consequently, spectral) symbol. In particular, if Ω is Peano-Jordan measurable, then χ_Ω is continuous a.e., so $\{I_{\boldsymbol{n}}(\chi_\Omega)\}_n \sim_{GLT} \chi_\Omega$. Moreover, it is also possible show that the difference $I_{\boldsymbol{n}}(\chi_\Omega) - D_{\boldsymbol{n}}(\chi_\Omega)$ has rank negligible when compared to $N(\boldsymbol{n})$.

Lemma F.1.4. *If Ω is Peano-Jordan measurable, then*

$$\operatorname{rk}\left(I_{\boldsymbol{n}}(\chi_\Omega) - D_{\boldsymbol{n}}(\chi_\Omega)\right) = o(N(\boldsymbol{n})).$$

Proof. It is enough to show that

$$E_n := \left\{\boldsymbol{i}\;\middle|\; \chi_\Omega\left(\frac{\boldsymbol{i}}{\boldsymbol{n}}\right) \ne \chi_\Omega\left(\frac{\boldsymbol{i}}{\boldsymbol{n}+\boldsymbol{1}}\right), \boldsymbol{1}\le\boldsymbol{i}\le\boldsymbol{n}\right\}$$

has cardinality negligible when compared to $N(\boldsymbol{n})$, since

$$\# E_n = \operatorname{rk}\left(I_{\boldsymbol{n}}(\chi_\Omega) - D_{\boldsymbol{n}}(\chi_\Omega)\right).$$

Note that if $\boldsymbol{i} \in E_n$ then there's a point of the boundary $\partial\Omega$ on the segment connecting the points $\boldsymbol{i}/\boldsymbol{n}$ and $\boldsymbol{i}/(\boldsymbol{n}+\mathbf{1})$. The distance between the two points is always bounded and tends to zero when n goes to infinity

$$\left\| \frac{\boldsymbol{i}}{\boldsymbol{n}} - \frac{\boldsymbol{i}}{\boldsymbol{n}+\mathbf{1}} \right\|_2^2 \le \sum_{k=1}^{d} \left(\frac{i_k}{n_k(n_k+1)} \right)^2 \le \sum_{k=1}^{d} \frac{1}{n_k^2} =: h_n^2 \xrightarrow{n\to\infty} 0.$$

It means that for every $\boldsymbol{i} \in E_n$ we have $d(\boldsymbol{i}/\boldsymbol{n}, \partial\Omega) \le h_n$, so Lemma F.1.2 let us conclude that

$$\begin{aligned}
\limsup_{n\to\infty} \frac{\# E_n}{N(\boldsymbol{n})} &= \limsup_{n\to\infty} \frac{\# \left\{ \frac{\boldsymbol{i}}{\boldsymbol{n}} \,\middle|\, \chi_\Omega\left(\frac{\boldsymbol{i}}{\boldsymbol{n}}\right) \neq \chi_\Omega\left(\frac{\boldsymbol{i}}{\boldsymbol{n}+\mathbf{1}}\right), \mathbf{1} \le \boldsymbol{i} \le \boldsymbol{n} \right\}}{N(\boldsymbol{n})} \\
&\le \limsup_{n\to\infty} \frac{\# \left\{ \frac{\boldsymbol{i}}{\boldsymbol{n}} \mid d\left(\frac{\boldsymbol{i}}{\boldsymbol{n}}, \partial\Omega\right) \le h_n, \mathbf{1} \le \boldsymbol{i} \le \boldsymbol{n} \right\}}{N(\boldsymbol{n})} \\
&= \limsup_{n\to\infty} \frac{\operatorname{rk}(D_{\boldsymbol{n}}(\chi_{K_{h_n}}))}{N(\boldsymbol{n})} = 0
\end{aligned}$$

□

The quantity

$$d_n^\Omega := \operatorname{rk}(I_{\boldsymbol{n}}(\chi_\Omega))$$

is important since it counts the number of grid points inside Ω. As a corollary, we find again the same results of Lemma F.1.1 and Lemma F.1.2, referred to the sequence $\{I_{\boldsymbol{n}}(\chi_\Omega)\}_n$. We will not prove them, since the arguments are the same we used in the proofs of Lemma F.1.1 and Lemma F.1.2.

Corollary F.1.5. *If Ω is a Peano-Jordan measurable set, then*

$$\lim_{n\to\infty} \frac{d_n^\Omega}{N(\boldsymbol{n})} = \ell_d(\Omega).$$

Corollary F.1.6. *Given a sequence h_n of real nonnegative numbers converging to zero, and a Peano-Jordan measurable set Ω, then*

$$\lim_{n\to\infty} \frac{d_n^{K_{h_n}}}{N(\boldsymbol{n})} = 0.$$

In particular, if $\ell_d(\Omega) > 0$, then

$$\lim_{n\to\infty} \frac{d_n^{K_{h_n}}}{d_n^\Omega} = 0.$$

Note that if $h_n = 0$ for every n, we have $K_{h_n} = K_0 = \partial\Omega$ for every n, so $d_n^{\partial\Omega} = o(d_n^\Omega) = o(N(\boldsymbol{n}))$. Notice that Corollary F.1.5 shows $\lim_{n\to\infty} d_n^\Omega = +\infty$ whenever the measure of Ω is not zero, so from now on, we suppose that $\ell_d(\Omega) > 0$.

F.2 Restriction and Expansion Operators

If we fix a Peano-Jordan measurable set Ω, then consider the map

$$Z_\Omega : \{A_{\boldsymbol{n}}\}_n \mapsto \{I_{\boldsymbol{n}}(\chi_\Omega) A_{\boldsymbol{n}} I_{\boldsymbol{n}}(\chi_\Omega)\}_n.$$

From now on, we abuse the notation and write $Z_\Omega(A_{\boldsymbol{n}})$ for the matrix $I_{\boldsymbol{n}}(\chi_\Omega) A_{\boldsymbol{n}} I_{\boldsymbol{n}}(\chi_\Omega)$. If we call $\mathscr{G}_d$ the set of d-dimensional GLT sequences, notice that $Z_\Omega(\mathscr{G}_d) \subseteq \mathscr{G}_d$ since it multiplies a GLT sequence with other GLT sequences, as shown in Lemma F.1.4. Some properties of this operation are

- Z_Ω is linear,

- Z_Ω is idempotent,
- if $\{A_{\boldsymbol{n}}\}_n \sim_{GLT} k(x,\theta)$, then $Z_\Omega(\{A_{\boldsymbol{n}}\}_n) \sim_{GLT} k(x,\theta)\chi_\Omega(x)$,
- if $\{A_{\boldsymbol{n}}\}_n$ is a real sequence, then $Z_\Omega(\{A_{\boldsymbol{n}}\}_n)$ is still real,
- if $\{A_{\boldsymbol{n}}\}_n$ is a Hermitian sequence, then $Z_\Omega(\{A_{\boldsymbol{n}}\}_n)$ is still Hermitian.

If we associate each multi-index $\boldsymbol{i}$ in the matrix $A_{\boldsymbol{n}}$ to the point $\frac{\boldsymbol{i}}{\boldsymbol{n}+\boldsymbol{1}} \in \Xi_n$, then Z_Ω sets to zero every row and column corresponding to a point not in Ω. We can thus try to delete the zero rows and columns in the matrices, and obtain a matrix with size $d_n^\Omega \times d_n^\Omega$.

Given a set Ω with negligible boundary, we consider $I_{\boldsymbol{n}}(\chi_\Omega)$ and we enumerate the non-zero rows and the zero rows through two strictly increasing functions

$$\phi_{\boldsymbol{n}} : \{1,2,\dots,d_n^\Omega\} \to \{\boldsymbol{1},\dots,\boldsymbol{n}\} \qquad \psi_{\boldsymbol{n}} : \{d_n^\Omega+1, d_n^\Omega+2,\dots,N(\boldsymbol{n})\} \to \{\boldsymbol{1},\dots,\boldsymbol{n}\}$$

such that the $\phi_{\boldsymbol{n}}(j)$-th row of $I_{\boldsymbol{n}}(\chi_\Omega)$ is non-zero for every j, and the $\psi_{\boldsymbol{n}}(j)$-th row of $I_{\boldsymbol{n}}(\chi_\Omega)$ is zero for every j. In particular, the images of $\phi_{\boldsymbol{n}}$ and $\psi_{\boldsymbol{n}}$ correspond to the set of points $\boldsymbol{i}/(\boldsymbol{n}+\boldsymbol{1})$ in Ξ_n respectively belonging and not belonging to Ω. Notice that $\phi_{\boldsymbol{n}}$ and $\psi_{\boldsymbol{n}}$ are uniquely determined by their properties.

For every $\boldsymbol{n}$, we define a rectangular matrix $\Pi_{\boldsymbol{n},\Omega}$ of size $d_n^\Omega \times N(\boldsymbol{n})$ as

$$(\Pi_{\boldsymbol{n},\Omega})_{i,\boldsymbol{j}} := (I_{\boldsymbol{n}}(\chi_\Omega))_{\phi_{\boldsymbol{n}}(i),\boldsymbol{j}}$$

so that, for any matrix $A_{\boldsymbol{n}}$ of size $N(\boldsymbol{n}) \times N(\boldsymbol{n})$, we can delete the rows and columns corresponding to points not belonging to Ω through the map

$$R_\Omega : \{A_{\boldsymbol{n}}\}_n \mapsto \{\Pi_{\boldsymbol{n},\Omega} A_{\boldsymbol{n}} (\Pi_{\boldsymbol{n},\Omega})^T\}_n$$

and add zero rows and columns corresponding to points not belonging to Ω to any matrix $S_{\boldsymbol{n}}^\Omega$ of size $d_n^\Omega \times d_n^\Omega$ through the map

$$E_\Omega : \{S_{\boldsymbol{n}}^\Omega\}_n \mapsto \{(\Pi_{\boldsymbol{n},\Omega})^T S_{\boldsymbol{n}}^\Omega \Pi_{\boldsymbol{n},\Omega}\}_n.$$

We will use the notation $R_\Omega(A_{\boldsymbol{n}})$ for $\Pi_{\boldsymbol{n},\Omega} A_{\boldsymbol{n}} (\Pi_{\boldsymbol{n},\Omega})^T$ and the notation $E_\Omega(S_{\boldsymbol{n}}^\Omega)$ for $(\Pi_{\boldsymbol{n},\Omega})^T S_{\boldsymbol{n}}^\Omega \Pi_{\boldsymbol{n},\Omega}$. Moreover, unless differently specified, we use the exponent Ω to distinguish the sequences $\{S_{\boldsymbol{n}}^\Omega\}_n$ of size $d_n^\Omega \times d_n^\Omega$ from classical sequences $\{A_{\boldsymbol{n}}\}_n$ of size $N(\boldsymbol{n}) \times N(\boldsymbol{n})$.

Note that the operators $E_\Omega, R_\Omega, Z_\Omega$, the matrices $\Pi_{\boldsymbol{n},\Omega}, I_{\boldsymbol{n}}(\chi_\Omega)$ and the quantity d_n^Ω can be defined for any measurable set Ω, even if not Peano-Jordan measurable.

F.2.1 Effects on the Sequences

Let us check some basic properties of the matrices $\Pi_{\boldsymbol{n},\Omega}, I_{\boldsymbol{n}}(\chi_\Omega)$ and the operators $E_\Omega, R_\Omega, Z_\Omega$.

Lemma F.2.1. *For every index $\boldsymbol{n}$, we have*

- $(\Pi_{\boldsymbol{n},\Omega})^T \Pi_{\boldsymbol{n},\Omega} = I_{\boldsymbol{n}}(\chi_\Omega)$,
- $\Pi_{\boldsymbol{n},\Omega} (\Pi_{\boldsymbol{n},\Omega})^T = I_{\boldsymbol{n}}^\Omega$.

In particular, given any matrix $A_{\boldsymbol{n}}$ of size $N(\boldsymbol{n}) \times N(\boldsymbol{n})$, and any matrix $S_{\boldsymbol{n}}^\Omega$ of size $d_n^\Omega \times d_n^\Omega$, we have

- $R_\Omega(A_{\boldsymbol{n}}) = R_\Omega \circ Z_\Omega(A_{\boldsymbol{n}})$,
- $R_\Omega(E_\Omega(S_{\boldsymbol{n}}^\Omega)) = S_{\boldsymbol{n}}^\Omega$,
- $E_\Omega(R_\Omega(A_{\boldsymbol{n}})) = Z_\Omega(A_{\boldsymbol{n}})$,
- $Z_\Omega(E_\Omega(S_{\boldsymbol{n}}^\Omega)) = E_\Omega(S_{\boldsymbol{n}}^\Omega)$.

Moreover $(E_\Omega(S_{\boldsymbol{n}}^\Omega))^H = E_\Omega((S_{\boldsymbol{n}}^\Omega)^H)$ and $(R_\Omega(A_{\boldsymbol{n}}))^H = R_\Omega(A_{\boldsymbol{n}}^H)$, so

- *$S_{\boldsymbol{n}}^\Omega$ Hermitian $\implies$ $E_\Omega(S_{\boldsymbol{n}}^\Omega)$ Hermitian,*
- *$A_{\boldsymbol{n}}$ Hermitian $\implies$ $R_\Omega(A_{\boldsymbol{n}})$ Hermitian.*

Proof.

$$
\begin{aligned}
((\Pi_{\boldsymbol{n},\Omega})^T\Pi_{\boldsymbol{n},\Omega})_{\boldsymbol{i},\boldsymbol{j}} &= \sum_{k=1}^{d_n^\Omega} (\Pi_{\boldsymbol{n},\Omega})_{k,\boldsymbol{i}}(\Pi_{\boldsymbol{n},\Omega})_{k,\boldsymbol{j}} \\
&= \sum_{k=1}^{d_n^\Omega} (I_{\boldsymbol{n}}(\chi_\Omega))_{\phi_{\boldsymbol{n}}(k),\boldsymbol{i}}(I_{\boldsymbol{n}}(\chi_\Omega))_{\phi_{\boldsymbol{n}}(k),\boldsymbol{j}} \\
&= \begin{cases} 1 & \boldsymbol{i}=\boldsymbol{j} \in Range(\phi_{\boldsymbol{n}}) \\ 0 & \text{otherwise} \end{cases} = (I_{\boldsymbol{n}}(\chi_\Omega))_{\boldsymbol{i},\boldsymbol{j}},
\end{aligned}
$$

$$
\begin{aligned}
(\Pi_{\boldsymbol{n},\Omega}(\Pi_{\boldsymbol{n},\Omega})^T)_{i,j} &= \sum_{\boldsymbol{k}=\boldsymbol{1}}^{\boldsymbol{n}} (\Pi_{\boldsymbol{n},\Omega})_{i,\boldsymbol{k}}(\Pi_{\boldsymbol{n},\Omega})_{j,\boldsymbol{k}} \\
&= \sum_{\boldsymbol{k}=\boldsymbol{1}}^{\boldsymbol{n}} (I_{\boldsymbol{n}}(\chi_\Omega))_{\phi_{\boldsymbol{n}}(i),\boldsymbol{k}}(I_{\boldsymbol{n}}(\chi_\Omega))_{\phi_{\boldsymbol{n}}(j),\boldsymbol{k}} \\
&= \delta_{\phi_{\boldsymbol{n}}(i),\phi_{\boldsymbol{n}}(j)} = \delta_{i,j},
\end{aligned}
$$

$$
\begin{aligned}
R_\Omega(A_{\boldsymbol{n}}) &= \Pi_{\boldsymbol{n},\Omega}A_{\boldsymbol{n}}(\Pi_{\boldsymbol{n},\Omega})^T \\
&= I_{\boldsymbol{n}}^\Omega \Pi_{\boldsymbol{n},\Omega}A_{\boldsymbol{n}}(\Pi_{\boldsymbol{n},\Omega})^T I_{\boldsymbol{n}}^\Omega \\
&= \Pi_{\boldsymbol{n},\Omega}(\Pi_{\boldsymbol{n},\Omega})^T\Pi_{\boldsymbol{n},\Omega}A_{\boldsymbol{n}}(\Pi_{\boldsymbol{n},\Omega})^T\Pi_{\boldsymbol{n},\Omega}(\Pi_{\boldsymbol{n},\Omega})^T \\
&= \Pi_{\boldsymbol{n},\Omega}I_{\boldsymbol{n}}(\chi_\Omega)A_{\boldsymbol{n}}I_{\boldsymbol{n}}(\chi_\Omega)(\Pi_{\boldsymbol{n},\Omega})^T \\
&= R_\Omega \circ Z_\Omega(A_{\boldsymbol{n}}),
\end{aligned}
$$

$$
\begin{aligned}
R_\Omega(E_\Omega(S_{\boldsymbol{n}}^\Omega)) &= \Pi_{\boldsymbol{n},\Omega}(\Pi_{\boldsymbol{n},\Omega})^T S_{\boldsymbol{n}}^\Omega \Pi_{\boldsymbol{n},\Omega}(\Pi_{\boldsymbol{n},\Omega})^T \\
&= I_{\boldsymbol{n}}^\Omega S_{\boldsymbol{n}}^\Omega I_{\boldsymbol{n}}^\Omega \\
&= S_{\boldsymbol{n}}^\Omega,
\end{aligned}
$$

$$
\begin{aligned}
E_\Omega(R_\Omega(A_{\boldsymbol{n}})) &= (\Pi_{\boldsymbol{n},\Omega})^T\Pi_{\boldsymbol{n},\Omega}A_{\boldsymbol{n}}(\Pi_{\boldsymbol{n},\Omega})^T\Pi_{\boldsymbol{n},\Omega} \\
&= I_{\boldsymbol{n}}(\chi_\Omega)A_{\boldsymbol{n}}I_{\boldsymbol{n}}(\chi_\Omega) \\
&= Z_\Omega(A_{\boldsymbol{n}}),
\end{aligned}
$$

$$
\begin{gathered}
R_\Omega(E_\Omega(S_{\boldsymbol{n}}^\Omega)) = S_{\boldsymbol{n}}^\Omega \implies E_\Omega(S_{\boldsymbol{n}}^\Omega) = E_\Omega(R_\Omega(E_\Omega(S_{\boldsymbol{n}}^\Omega))) = Z_\Omega(E_\Omega(S_{\boldsymbol{n}}^\Omega)), \\
(E_\Omega(S_{\boldsymbol{n}}^\Omega))^H = ((\Pi_{\boldsymbol{n},\Omega})^T S_{\boldsymbol{n}}^\Omega \Pi_{\boldsymbol{n},\Omega})^H = (\Pi_{\boldsymbol{n},\Omega})^T (S_{\boldsymbol{n}}^\Omega)^H \Pi_{\boldsymbol{n},\Omega} = E_\Omega((S_{\boldsymbol{n}}^\Omega)^H), \\
(R_\Omega(A_{\boldsymbol{n}}))^H = (\Pi_{\boldsymbol{n},\Omega}A_{\boldsymbol{n}}(\Pi_{\boldsymbol{n},\Omega})^T)^H = \Pi_{\boldsymbol{n},\Omega}A_{\boldsymbol{n}}^H(\Pi_{\boldsymbol{n},\Omega})^T = R_\Omega(A_{\boldsymbol{n}}^H).
\end{gathered}
$$

□

The operator R_Ω has the job to extract a principal minor from the matrices, so it is easy to see that it makes the norm drop.

Lemma F.2.2. *For every* $1 \le p \le \infty$,

$$\|R_\Omega(A)\|_p \le \|A\|_p.$$

Proof. The matrices $\Pi_{\boldsymbol{n},\Omega}$ are unitary, so we can apply the Cauchy interlacing theorem and find that

$$\sigma_i(R_\Omega(A)) \le \sigma_i(A) \qquad \forall 1 \le i \le d_n^\Omega.$$

The thesis easily follows from the definition of p-Schatten norm. □

The map R_Ω applied to $Z_\Omega(A_{\boldsymbol{n}})$ has the effect to delete only rows and columns that are already zero, and we can easily tell the behaviour of their singular values and eigenvalues.

Lemma F.2.3. *There exists a permutation matrix P of size $N(\boldsymbol{n}) \times N(\boldsymbol{n})$ such that for every matrix $A_{\boldsymbol{n}}$ of size $N(\boldsymbol{n}) \times N(\boldsymbol{n})$,*

$$PZ_\Omega(A_{\boldsymbol{n}})P^T = \begin{pmatrix} R_\Omega(A_{\boldsymbol{n}}) & 0 \\ 0 & 0 \end{pmatrix}.$$

In particular, $Z_\Omega(A_{\boldsymbol{n}})$ has the same eigenvalues and singular values of the matrix $R_\Omega(A_{\boldsymbol{n}})$ except for $N(\boldsymbol{n}) - d_n^\Omega$ null eigenvalues and singular values.

Proof. Let $B_{\boldsymbol{n}} = Z_\Omega(A_{\boldsymbol{n}})$ and $S_{\boldsymbol{n}}^\Omega = R_\Omega(A_{\boldsymbol{n}})$. If we define the permutation matrix P as

$$P_{\boldsymbol{i},\boldsymbol{j}} = \begin{cases} \delta_{\boldsymbol{j},\phi_{\boldsymbol{n}}(|\boldsymbol{i}|)} & |\boldsymbol{i}| \le d_n^\Omega, \\ \delta_{\boldsymbol{j},\psi_{\boldsymbol{n}}(|\boldsymbol{i}|)} & |\boldsymbol{i}| > d_n^\Omega, \end{cases}$$

then the matrix $PB_{\boldsymbol{n}}P^T$ can be written as

$$PB_{\boldsymbol{n}}P^T = \begin{pmatrix} S_{\boldsymbol{n}}^\Omega & 0 \\ 0 & 0 \end{pmatrix}.$$

In fact

$$\begin{aligned}
(PB_{\boldsymbol{n}}P^T)_{\boldsymbol{i},\boldsymbol{j}} &= (PI_{\boldsymbol{n}}(\chi_\Omega)A_{\boldsymbol{n}}I_{\boldsymbol{n}}(\chi_\Omega)P^T)_{\boldsymbol{i},\boldsymbol{j}} \\
&= \sum_{\boldsymbol{k}=\boldsymbol{1}}^{\boldsymbol{n}}\sum_{\boldsymbol{h}=\boldsymbol{1}}^{\boldsymbol{n}} P_{\boldsymbol{i},\boldsymbol{k}}(I_{\boldsymbol{n}}(\chi_\Omega))_{\boldsymbol{k},\boldsymbol{k}}(A_{\boldsymbol{n}})_{\boldsymbol{k},\boldsymbol{h}}(I_{\boldsymbol{n}}(\chi_\Omega))_{\boldsymbol{h},\boldsymbol{h}}P_{\boldsymbol{j},\boldsymbol{h}} \\
&= \begin{cases}
(I_{\boldsymbol{n}}(\chi_\Omega))_{\phi_{\boldsymbol{n}}(|\boldsymbol{i}|),\phi_{\boldsymbol{n}}(|\boldsymbol{i}|)}(A_{\boldsymbol{n}})_{\phi_{\boldsymbol{n}}(|\boldsymbol{i}|),\phi_{\boldsymbol{n}}(|\boldsymbol{j}|)}(I_{\boldsymbol{n}}(\chi_\Omega))_{\phi_{\boldsymbol{n}}(|\boldsymbol{j}|),\phi_{\boldsymbol{n}}(|\boldsymbol{j}|)} & |\boldsymbol{i}| \le d_n^\Omega, |\boldsymbol{j}| \le d_n^\Omega \\
(I_{\boldsymbol{n}}(\chi_\Omega))_{\psi_{\boldsymbol{n}}(|\boldsymbol{i}|),\psi_{\boldsymbol{n}}(|\boldsymbol{i}|)}(A_{\boldsymbol{n}})_{\psi_{\boldsymbol{n}}(|\boldsymbol{i}|),\phi_{\boldsymbol{n}}(|\boldsymbol{j}|)}(I_{\boldsymbol{n}}(\chi_\Omega))_{\phi_{\boldsymbol{n}}(|\boldsymbol{j}|),\phi_{\boldsymbol{n}}(|\boldsymbol{j}|)} & |\boldsymbol{i}| > d_n^\Omega, |\boldsymbol{j}| \le d_n^\Omega \\
(I_{\boldsymbol{n}}(\chi_\Omega))_{\phi_{\boldsymbol{n}}(|\boldsymbol{i}|),\phi_{\boldsymbol{n}}(|\boldsymbol{i}|)}(A_{\boldsymbol{n}})_{\phi_{\boldsymbol{n}}(|\boldsymbol{i}|),\psi_{\boldsymbol{n}}(|\boldsymbol{j}|)}(I_{\boldsymbol{n}}(\chi_\Omega))_{\psi_{\boldsymbol{n}}(|\boldsymbol{j}|),\psi_{\boldsymbol{n}}(|\boldsymbol{j}|)} & |\boldsymbol{i}| \le d_n^\Omega, |\boldsymbol{j}| > d_n^\Omega \\
(I_{\boldsymbol{n}}(\chi_\Omega))_{\psi_{\boldsymbol{n}}(|\boldsymbol{i}|),\psi_{\boldsymbol{n}}(|\boldsymbol{i}|)}(A_{\boldsymbol{n}})_{\psi_{\boldsymbol{n}}(|\boldsymbol{i}|),\psi_{\boldsymbol{n}}(|\boldsymbol{j}|)}(I_{\boldsymbol{n}}(\chi_\Omega))_{\psi_{\boldsymbol{n}}(|\boldsymbol{j}|),\psi_{\boldsymbol{n}}(|\boldsymbol{j}|)} & |\boldsymbol{i}| > d_n^\Omega, |\boldsymbol{j}| > d_n^\Omega
\end{cases} \\
&= \begin{cases}
(A_{\boldsymbol{n}})_{\phi_{\boldsymbol{n}}(|\boldsymbol{i}|),\phi_{\boldsymbol{n}}(|\boldsymbol{j}|)} & |\boldsymbol{i}| \le d_n^\Omega, |\boldsymbol{j}| \le d_n^\Omega \\
0 & |\boldsymbol{i}| > d_n^\Omega, |\boldsymbol{j}| \le d_n^\Omega \\
0 & |\boldsymbol{i}| \le d_n^\Omega, |\boldsymbol{j}| > d_n^\Omega \\
0 & |\boldsymbol{i}| > d_n^\Omega, |\boldsymbol{j}| > d_n^\Omega
\end{cases}
\end{aligned}$$

and

$$\begin{aligned}
(S_{\boldsymbol{n}}^\Omega)_{i,j} = R_\Omega(A_{\boldsymbol{n}}) &= (\Pi_{\boldsymbol{n},\Omega}A_{\boldsymbol{n}}(\Pi_{\boldsymbol{n},\Omega})^T)_{i,j} \\
&= \sum_{\boldsymbol{k}=\boldsymbol{1}}^{\boldsymbol{n}}\sum_{\boldsymbol{h}=\boldsymbol{1}}^{\boldsymbol{n}}(\Pi_{\boldsymbol{n},\Omega})_{i,\boldsymbol{k}}(A_{\boldsymbol{n}})_{\boldsymbol{k},\boldsymbol{h}}(\Pi_{\boldsymbol{n},\Omega})_{j,\boldsymbol{h}} \\
&= \sum_{\boldsymbol{k}=\boldsymbol{1}}^{\boldsymbol{n}}\sum_{\boldsymbol{h}=\boldsymbol{1}}^{\boldsymbol{n}}(I_{\boldsymbol{n}}(\chi_\Omega))_{\phi_{\boldsymbol{n}}(i),\boldsymbol{k}}(A_{\boldsymbol{n}})_{\boldsymbol{k},\boldsymbol{h}}(I_{\boldsymbol{n}}(\chi_\Omega))_{\phi_{\boldsymbol{n}}(j),\boldsymbol{h}} \\
&= (A_{\boldsymbol{n}})_{\phi_{\boldsymbol{n}}(i),\phi_{\boldsymbol{n}}(j)}.
\end{aligned}$$

The proof is thus concluded, since $S_{\boldsymbol{n}}^\Omega$ has the same eigenvalues and singular values of $B_{\boldsymbol{n}}$ except for $N(\boldsymbol{n}) - d_n^\Omega$ zeros. □

Corollary F.2.4. *There exists a permutation matrix P of size $N(\boldsymbol{n}) \times N(\boldsymbol{n})$ such that for every matrix $S_{\boldsymbol{n}}^\Omega$ of size $d_n^\Omega \times d_n^\Omega$,*

$$PE_\Omega(S_{\boldsymbol{n}}^\Omega)P^T = \begin{pmatrix} S_{\boldsymbol{n}}^\Omega & 0 \\ 0 & 0 \end{pmatrix}.$$

In particular, $E_\Omega(S_{\boldsymbol{n}}^\Omega)$ has the same eigenvalues and singular values of the matrix $S_{\boldsymbol{n}}^\Omega$ except for $N(\boldsymbol{n}) - d_n^\Omega$ null eigenvalues and singular values.

Proof. Let $A_{\boldsymbol{n}} = E_\Omega(S_{\boldsymbol{n}}^\Omega)$. Using Lemma F.2.1, we get

$$S_{\boldsymbol{n}}^\Omega = R_\Omega(E_\Omega(S_{\boldsymbol{n}}^\Omega)) = R_\Omega(A_{\boldsymbol{n}}), \qquad Z_\Omega(A_{\boldsymbol{n}}) = Z_\Omega(E_\Omega(S_{\boldsymbol{n}}^\Omega)) = E_\Omega(S_{\boldsymbol{n}}^\Omega) = A_{\boldsymbol{n}}.$$

As a consequence, we can apply Lemma F.2.3 on $A_{\boldsymbol{n}}$ to find a permutation matrix P such that

$$PA_{\boldsymbol{n}}P^T = \begin{pmatrix} S_{\boldsymbol{n}}^{\Omega} & 0 \\ 0 & 0 \end{pmatrix}$$

so $S_{\boldsymbol{n}}^{\Omega}$ has the same eigenvalues and singular values of $A_{\boldsymbol{n}}$ except for $N(\boldsymbol{n}) - d_n^{\Omega}$ zeros. □

Corollary F.2.5. *There exists a permutation matrix P of size $N(\boldsymbol{n}) \times N(\boldsymbol{n})$ such that for every matrix $A_{\boldsymbol{n}}$ of size $N(\boldsymbol{n}) \times N(\boldsymbol{n})$,*

$$PA_{\boldsymbol{n}}P^T = \begin{pmatrix} R_{\Omega}(A_{\boldsymbol{n}}) & * \\ * & * \end{pmatrix}.$$

Proof. using Lemma F.2.1,

$$Z_{\Omega}(I_{\boldsymbol{n}}(\chi_{\Omega})) = I_{\boldsymbol{n}}(\chi_{\Omega})I_{\boldsymbol{n}}(\chi_{\Omega})I_{\boldsymbol{n}}(\chi_{\Omega}) = I_{\boldsymbol{n}}(\chi_{\Omega}),$$

$$R_{\Omega}(I_{\boldsymbol{n}}(\chi_{\Omega})) = \Pi_{\boldsymbol{n},\Omega}I_{\boldsymbol{n}}(\chi_{\Omega})(\Pi_{\boldsymbol{n},\Omega})^T = \Pi_{\boldsymbol{n},\Omega}(\Pi_{\boldsymbol{n},\Omega})^T\Pi_{\boldsymbol{n},\Omega}(\Pi_{\boldsymbol{n},\Omega})^T = I_{\boldsymbol{n}}^{\Omega}I_{\boldsymbol{n}}^{\Omega} = I_{\boldsymbol{n}}^{\Omega},$$

so Lemma 6.3.3 shows that there exists P such that

$$PI_{\boldsymbol{n}}(\chi_{\Omega})P^T = PZ_{\Omega}(I_{\boldsymbol{n}}(\chi_{\Omega}))P^T = \begin{pmatrix} R_{\Omega}(I_{\boldsymbol{n}}(\chi_{\Omega})) & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I_{\boldsymbol{n}}^{\Omega} & 0 \\ 0 & 0 \end{pmatrix}.$$

As a consequence, we have that

$$\begin{aligned} \begin{pmatrix} R_{\Omega}(A_{\boldsymbol{n}}) & 0 \\ 0 & 0 \end{pmatrix} &= PZ_{\Omega}(A_{\boldsymbol{n}})P^T \\ &= PI_{\boldsymbol{n}}(\chi_{\Omega})A_{\boldsymbol{n}}I_{\boldsymbol{n}}(\chi_{\Omega})P^T \\ &= PI_{\boldsymbol{n}}(\chi_{\Omega})P^TPA_{\boldsymbol{n}}P^TPI_{\boldsymbol{n}}(\chi_{\Omega})P^T \\ &= \begin{pmatrix} I_{\boldsymbol{n}}^{\Omega} & 0 \\ 0 & 0 \end{pmatrix} PA_{\boldsymbol{n}}P^T \begin{pmatrix} I_{\boldsymbol{n}}^{\Omega} & 0 \\ 0 & 0 \end{pmatrix} \\ \implies \begin{pmatrix} R_{\Omega}(A_{\boldsymbol{n}}) & * \\ * & * \end{pmatrix} &= PA_{\boldsymbol{n}}P^T \end{aligned}$$

□

F.2.2 Effects on the Symbols

We have seen how R_{Ω}, E_{Ω} modify the sequences of matrices. Now we focus on how the symbols change though these operators. Let us start with the reduction operator R_{Ω}.

Lemma F.2.6. *Let $\{A_{\boldsymbol{n}}\}_n$ be a sequence with $A_{\boldsymbol{n}}$ of size $N(\boldsymbol{n}) \times N(\boldsymbol{n})$ that is a fixed point for the operator Z_{Ω}, and let $k : [0,1]^d \times [-\pi,\pi]^d \to \mathbb{C}$ be a measurable function with $k(x,\theta)|_{x \notin \Omega} = 0$. If $\{A_{\boldsymbol{n}}\}_n \sim_{\sigma} k$, then*

$$R_{\Omega}(\{A_{\boldsymbol{n}}\}_n) \sim_{\sigma} k(x,\theta)|_{x\in\Omega}.$$

If $\{A_{\boldsymbol{n}}\}_n \sim_{\lambda} k$, then

$$R_{\Omega}(\{A_{\boldsymbol{n}}\}_n) \sim_{\lambda} k(x,\theta)|_{x\in\Omega}.$$

Proof. Suppose that $\{A_{\boldsymbol{n}}\}_n \sim_{\sigma} k$. Consider any continuous function $G : \mathbb{R} \to \mathbb{C}$ with compact support, and call $S_{\boldsymbol{n}}^{\Omega} = R_{\Omega}(A_{\boldsymbol{n}})$. By hypothesis $A_{\boldsymbol{n}} = Z_{\Omega}(A_{\boldsymbol{n}})$, so we can use Lemma 6.3.3 and obtain

$$\frac{1}{d_n^{\Omega}}\sum_{i=1}^{d_n^{\Omega}} G(\sigma_i(S_{\boldsymbol{n}}^{\Omega})) = \frac{N(\boldsymbol{n})}{d_n^{\Omega}}\frac{1}{N(\boldsymbol{n})}\sum_{i=1}^{N(\boldsymbol{n})} G(\sigma_i(A_{\boldsymbol{n}})) - \frac{N(\boldsymbol{n}) - d_n^{\Omega}}{d_n^{\Omega}}G(0).$$

Notice that $\{A_{\boldsymbol{n}}\}_n \sim_\sigma k(x,\theta) = k(x,\theta)\chi_\Omega(x)$, so Corollary F.1.5 shows that

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{d_n^\Omega}\sum_{i=1}^{d_n^\Omega} G(\sigma_i(S_{\boldsymbol{n}}^\Omega)) &= \lim_{n\to\infty} \frac{N(\boldsymbol{n})}{d_n^\Omega}\frac{1}{N(\boldsymbol{n})}\sum_{i=1}^{N(\boldsymbol{n})} G(\sigma_i(A_{\boldsymbol{n}})) - \lim_{n\to\infty}\frac{N(\boldsymbol{n})-d_n^\Omega}{d_n^\Omega}G(0)\\ &= \frac{1}{\ell_d(\Omega)}\frac{1}{(2\pi)^d}\int_{[0,1]^d\times[-\pi,\pi]^d} G(|k(x,\theta)|)\mathrm{d}(x,\theta) - \frac{1-\ell_d(\Omega)}{\ell_d(\Omega)}G(0)\\ &= \frac{1}{\ell_d(\Omega)(2\pi)^d}\int_{\Omega\times[-\pi,\pi]^d} G(|k(x,\theta)|)\mathrm{d}(x,\theta) + \frac{\ell_d(\Omega^C)}{\ell_d(\Omega)}G(0) - \frac{1-\ell_d(\Omega)}{\ell_d(\Omega)}G(0)\\ &= \frac{1}{\ell_d(\Omega\times[-\pi,\pi]^d)}\int_{\Omega\times[-\pi,\pi]^d} G(|k(x,\theta)|)\mathrm{d}(x,\theta).\end{aligned}$$

The last formula holds for every continuous function G with compact support, so

$$R_\Omega(\{A_{\boldsymbol{n}}\}_n) = \{S_{\boldsymbol{n}}^\Omega\}_n \sim_\sigma k(x,\theta)|_{x\in\Omega}.$$

If we suppose $\{A_{\boldsymbol{n}}\}_n \sim_\lambda k$, the proof is analogous. Consider any continuous and compact supported function $G : \mathbb{C} \to \mathbb{C}$ and use Lemma 6.3.3 to show that

$$\frac{1}{d_n^\Omega}\sum_{i=1}^{d_n^\Omega} G(\lambda_i(S_{\boldsymbol{n}}^\Omega)) = \frac{N(\boldsymbol{n})}{d_n^\Omega}\frac{1}{N(\boldsymbol{n})}\sum_{i=1}^{N(\boldsymbol{n})} G(\lambda_i(A_{\boldsymbol{n}}))) - \frac{N(\boldsymbol{n})-d_n^\Omega}{d_n^\Omega}G(0),$$

and exploiting $\{B_{\boldsymbol{n}}\}_n \sim_\lambda k(x,\theta) = k(x,\theta)\chi_\Omega(x)$ and Corollary F.1.5, we conclude that

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{d_n^\Omega}\sum_{i=1}^{d_n^\Omega} G(\lambda_i(S_{\boldsymbol{n}}^\Omega)) &= \lim_{n\to\infty} \frac{N(\boldsymbol{n})}{d_n^\Omega}\frac{1}{N(\boldsymbol{n})}\sum_{i=1}^{N(\boldsymbol{n})} G(\lambda_i(A_{\boldsymbol{n}})) - \lim_{n\to\infty}\frac{N(\boldsymbol{n})-d_n^\Omega}{d_n^\Omega}G(0)\\ &= \frac{1}{\ell_d(\Omega)}\frac{1}{(2\pi)^d}\int_{[0,1]^d\times[-\pi,\pi]^d} G(k(x,\theta))\mathrm{d}(x,\theta) - \frac{1-\ell_d(\Omega)}{\ell_d(\Omega)}G(0)\\ &= \frac{1}{\ell_d(\Omega)(2\pi)^d}\int_{\Omega\times[-\pi,\pi]^d} G(k(x,\theta))\mathrm{d}(x,\theta) + \frac{\ell_d(\Omega^C)}{\ell_d(\Omega)}G(0) - \frac{1-\ell_d(\Omega)}{\ell_d(\Omega)}G(0)\\ &= \frac{1}{\ell_d(\Omega\times[-\pi,\pi]^d)}\int_{\Omega\times[-\pi,\pi]^d} G(k(x,\theta))\mathrm{d}(x,\theta).\end{aligned}$$

The last formula holds for every continuous function G with compact support, so

$$R_\Omega(\{A_{\boldsymbol{n}}\}_n) = \{S_{\boldsymbol{n}}^\Omega\}_n \sim_\lambda k(x,\theta)|_{x\in\Omega}.$$

□

On the contrary let us analyse the effects of the extension operator E_Ω.

Lemma F.2.7. *Let $\{S_{\boldsymbol{n}}^\Omega\}_n$ be a sequence with $S_{\boldsymbol{n}}^\Omega$ of size $d_n^\Omega \times d_n^\Omega$, let $\kappa : \Omega\times[-\pi,\pi]^d \to \mathbb{C}$ be a measurable function, and define*

$$\kappa'(x,\theta) = \begin{cases}\kappa(x,\theta), & x\in\Omega,\\ 0, & x\in[0,1]^d\setminus\Omega.\end{cases}$$

If $\{S_{\boldsymbol{n}}^\Omega\}_n \sim_\sigma k$, then

$$E_\Omega(\{S_{\boldsymbol{n}}^\Omega\}_n) \sim_\sigma \kappa'(x,\theta).$$

If $\{S_{\boldsymbol{n}}^\Omega\}_n \sim_\lambda \kappa$, then

$$E_\Omega(\{S_{\boldsymbol{n}}^\Omega\}_n) \sim_\lambda \kappa'(x,\theta).$$

Proof. Suppose that $\{S_{\boldsymbol{n}}^\Omega\}_n \sim_\sigma \kappa$, and denote $\{A_{\boldsymbol{n}}\}_n = E_\Omega(\{S_{\boldsymbol{n}}^\Omega\}_n)$. If we consider any continuous function $G : \mathbb{R} \to \mathbb{C}$ with compact support, then we can use Corollary F.2.4 on $\{S_{\boldsymbol{n}}^\Omega\}_n$ to obtain

$$\frac{1}{N(\boldsymbol{n})}\sum_{i=1}^{N(\boldsymbol{n})} G(\sigma_i(A_{\boldsymbol{n}})) = \frac{d_n^\Omega}{N(\boldsymbol{n})}\frac{1}{d_n^\Omega}\sum_{i=1}^{d_n^\Omega} G(\sigma_i(S_{\boldsymbol{n}}^\Omega)) + \frac{N(\boldsymbol{n})-d_n^\Omega}{N(\boldsymbol{n})}G(0).$$

As a consequence of Corollary F.1.5, we can show that

$$\lim_{n\to\infty} \frac{1}{N(\boldsymbol{n})} \sum_{i=1}^{N(\boldsymbol{n})} G(\sigma_i(A_{\boldsymbol{n}})) =$$

$$\begin{aligned}
&= \lim_{n\to\infty} \frac{d_n^\Omega}{N(\boldsymbol{n})} \frac{1}{d_n^\Omega} \sum_{i=1}^{d_n^\Omega} G(\sigma_i(S_{\boldsymbol{n}}^\Omega)) + \lim_{n\to\infty} \frac{N(\boldsymbol{n}) - d_n^\Omega}{N(\boldsymbol{n})} G(0) \\
&= \ell_d(\Omega) \frac{1}{\ell_{2d}(\Omega \times [-\pi,\pi]^d)} \int_{\Omega\times[-\pi,\pi]^d} G(|\kappa(x,\theta)|)\mathrm{d}(x,\theta) + (1-\ell_d(\Omega))G(0) \\
&= \frac{1}{(2\pi)^d} \int_{[0,1]^d\times[-\pi,\pi]^d} G(|\kappa'(x,\theta)|)\mathrm{d}(x,\theta) - \frac{\ell_{2d}(\Omega^C \times [-\pi,\pi]^d)}{(2\pi)^d} G(0) + (1-\ell_d(\Omega))G(0) \\
&= \frac{1}{\ell_{2d}([0,1]^d \times [-\pi,\pi]^d)} \int_{\Omega\times[-\pi,\pi]^d} G(|\kappa'(x,\theta)|)\mathrm{d}(x,\theta).
\end{aligned}$$

The last formula holds for every continuous function G with compact support, so

$$E_\Omega(\{S_{\boldsymbol{n}}^\Omega\}_n) = \{A_{\boldsymbol{n}}\}_n \sim_\sigma \kappa'(x,\theta).$$

If we suppose $\{S_{\boldsymbol{n}}^\Omega\}_n \sim_\lambda \kappa$, the proof is analogous. If we consider any continuous function $G : \mathbb{C} \to \mathbb{C}$ with compact support, then we can use Corollary F.2.4 on $\{S_{\boldsymbol{n}}^\Omega\}_n$ to obtain

$$\frac{1}{N(\boldsymbol{n})} \sum_{i=1}^{N(\boldsymbol{n})} G(\lambda_i(A_{\boldsymbol{n}})) = \frac{d_n^\Omega}{N(\boldsymbol{n})} \frac{1}{d_n^\Omega} \sum_{i=1}^{d_n^\Omega} G(\lambda_i(S_{\boldsymbol{n}}^\Omega)) + \frac{N(\boldsymbol{n}) - d_n^\Omega}{N(\boldsymbol{n})} G(0).$$

As a consequence of Corollary F.1.5, we can show that

$$\lim_{n\to\infty} \frac{1}{N(\boldsymbol{n})} \sum_{i=1}^{N(\boldsymbol{n})} G(\lambda_i(A_{\boldsymbol{n}})) =$$

$$\begin{aligned}
&= \lim_{n\to\infty} \frac{d_n^\Omega}{N(\boldsymbol{n})} \frac{1}{d_n^\Omega} \sum_{i=1}^{d_n^\Omega} G(\lambda_i(S_{\boldsymbol{n}}^\Omega)) + \lim_{n\to\infty} \frac{N(\boldsymbol{n}) - d_n^\Omega}{N(\boldsymbol{n})} G(0) \\
&= \ell_d(\Omega) \frac{1}{\ell_{2d}(\Omega \times [-\pi,\pi]^d)} \int_{\Omega\times[-\pi,\pi]^d} G(\kappa(x,\theta))\mathrm{d}(x,\theta) + (1-\ell_d(\Omega))G(0) \\
&= \frac{1}{(2\pi)^d} \int_{[0,1]^d\times[-\pi,\pi]^d} G(\kappa'(x,\theta))\mathrm{d}(x,\theta) - \frac{\ell_{2d}(\Omega^C \times [-\pi,\pi]^d)}{(2\pi)^d} G(0) + (1-\ell_d(\Omega))G(0) \\
&= \frac{1}{\ell_{2d}([0,1]^d \times [-\pi,\pi]^d)} \int_{\Omega\times[-\pi,\pi]^d} G(\kappa'(x,\theta))\mathrm{d}(x,\theta).
\end{aligned}$$

The last formula holds for every continuous function G with compact support, so

$$E_\Omega(\{S_{\boldsymbol{n}}^\Omega\}_n) = \{A_{\boldsymbol{n}}\}_n \sim_\lambda \kappa'(x,\theta).$$

□

F.2.3 Effects on the Convergence

The operators R_Ω and E_Ω link two different matrix-sequence spaces, so we can analyse how they affect the metrics and the convergences.

Lemma F.2.8. *Given a sequence $\{S_{\boldsymbol{n}}^\Omega\}_n$ with $S_{\boldsymbol{n}}^\Omega$ of size $d_n^\Omega \times d_n^\Omega$ and a sequence $\{A_{\boldsymbol{n}}\}_n$ with $A_{\boldsymbol{n}}$ of size $N(\boldsymbol{n}) \times N(\boldsymbol{n})$, we have*

$$\{S_{\boldsymbol{n}}^\Omega\}_n \sim_\sigma 0 \implies E_\Omega(\{S_{\boldsymbol{n}}^\Omega\}_n) \sim_\sigma 0, \qquad \{S_{\boldsymbol{n}}^\Omega\}_n \sim_\lambda 0 \implies E_\Omega(\{S_{\boldsymbol{n}}^\Omega\}_n) \sim_\lambda 0,$$

$$\{A_{\boldsymbol{n}}\}_n \sim_\sigma 0 \implies R_\Omega(\{A_{\boldsymbol{n}}\}_n) \sim_\sigma 0, \qquad \{A_{\boldsymbol{n}}\}_n \sim_\lambda 0 \implies R_\Omega(\{A_{\boldsymbol{n}}\}_n) \sim_\lambda 0.$$

Proof. Easy corollary of Lemma F.2.7 and Lemma F.2.6. □

Lemma F.2.9. *Given two sequences $\{A_{\boldsymbol{n}}\}_n$ and $\{B_{\boldsymbol{n}}\}_n$ with matrices of size $N(\boldsymbol{n}) \times N(\boldsymbol{n})$,*

$$d_{a.c.s.}\left(\{A_{\boldsymbol{n}}\}_n, \{B_{\boldsymbol{n}}\}_n\right) \geq \ell_d(\Omega) d_{a.c.s.}\left(R_\Omega(\{A_{\boldsymbol{n}}\}_n), R_\Omega(\{B_{\boldsymbol{n}}\}_n)\right).$$

In particular,

$$\{B_{\boldsymbol{n},m}\}_n \xrightarrow{a.c.s.} \{A_{\boldsymbol{n}}\}_n \implies R_\Omega(\{B_{\boldsymbol{n},m}\}_n) \xrightarrow{a.c.s.} R_\Omega(\{A_{\boldsymbol{n}}\}_n).$$

Proof. Let P be the permutation matrix in Corollary F.2.5, so that

$$P(A_{\boldsymbol{n}} - B_{\boldsymbol{n}})P^T = \begin{pmatrix} R_\Omega(A_{\boldsymbol{n}} - B_{\boldsymbol{n}}) & * \\ * & * \end{pmatrix}.$$

Using the Cauchy interlacing theorem for singular values (Theorem 3.1.1), we get that $\sigma_i(R_\Omega(A_{\boldsymbol{n}} - B_{\boldsymbol{n}})) \leq \sigma_i(P(A_{\boldsymbol{n}} - B_{\boldsymbol{n}})P^T) = \sigma_i(A_{\boldsymbol{n}} - B_{\boldsymbol{n}})$ for every $1 \leq i \leq d_n^\Omega$. We can thus use the definition of $d_{a.c.s.}$ and Corollary F.1.5 to obtain

$$\begin{aligned}
d_{a.c.s.}\left(\{A_{\boldsymbol{n}}\}_n, \{B_{\boldsymbol{n}}\}_n\right) &= \limsup_{n\to\infty} \min_{i=1,\dots,N(\boldsymbol{n})+1} \left\{ \frac{i-1}{N(\boldsymbol{n})} + \sigma_i(A_{\boldsymbol{n}} - B_{\boldsymbol{n}}) \right\} \\
&= \limsup_{n\to\infty} \min_{i=1,\dots,N(\boldsymbol{n})+1} \left\{ \frac{i-1}{d_n^\Omega} \frac{d_n^\Omega}{N(\boldsymbol{n})} + \sigma_i(A_{\boldsymbol{n}} - B_{\boldsymbol{n}}) \right\} \\
&\geq \limsup_{n\to\infty} \frac{d_n^\Omega}{N(\boldsymbol{n})} \min_{i=1,\dots,N(\boldsymbol{n})+1} \left\{ \frac{i-1}{d_n^\Omega} + \sigma_i(A_{\boldsymbol{n}} - B_{\boldsymbol{n}}) \right\} \\
&\geq \ell_d(\Omega) \limsup_{n\to\infty} \min \left\{ \min_{i=1,\dots,d_n^\Omega} \left\{ \frac{i-1}{d_n^\Omega} + \sigma_i(A_{\boldsymbol{n}} - B_{\boldsymbol{n}}) \right\}, 1 \right\} \\
&\geq \ell_d(\Omega) \limsup_{n\to\infty} \min \left\{ \min_{i=1,\dots,d_n^\Omega} \left\{ \frac{i-1}{d_n^\Omega} + \sigma_i(R_\Omega(A_{\boldsymbol{n}} - B_{\boldsymbol{n}})) \right\}, 1 \right\} \\
&= \ell_d(\Omega) \limsup_{n\to\infty} \min_{i=1,\dots,d_n^\Omega+1} \left\{ \frac{i-1}{d_n^\Omega} + \sigma_i(R_\Omega(A_{\boldsymbol{n}} - B_{\boldsymbol{n}})) \right\} \\
&= \ell_d(\Omega) d_{a.c.s.}\left(R_\Omega(\{A_{\boldsymbol{n}}\}_n), R_\Omega(\{B_{\boldsymbol{n}}\}_n)\right).
\end{aligned}$$

Consequentially,

$$\begin{aligned}
\{B_{\boldsymbol{n},m}\}_n \xrightarrow{a.c.s.} \{A_{\boldsymbol{n}}\}_n &\iff d_{a.c.s.}\left(\{B_{\boldsymbol{n},m}\}_n, \{A_{\boldsymbol{n}}\}_n\right) \to 0 \\
&\implies d_{a.c.s.}\left(R_\Omega(\{B_{\boldsymbol{n},m}\}_n), R_\Omega(\{A_{\boldsymbol{n}}\}_n)\right) \to 0 \\
&\iff R_\Omega(\{B_{\boldsymbol{n},m}\}_n) \xrightarrow{a.c.s.} R_\Omega(\{A_{\boldsymbol{n}}\}_n).
\end{aligned}$$

□

Lemma F.2.10. *Given two sequences $\{A^\Omega_{\boldsymbol{n}}\}_n$ and $\{B^\Omega_{\boldsymbol{n}}\}_n$ with matrices of size $d_n^\Omega \times d_n^\Omega$,*

$$d_{a.c.s.}\left(\{A^\Omega_{\boldsymbol{n}}\}_n, \{B^\Omega_{\boldsymbol{n}}\}_n\right) \geq d_{a.c.s.}\left(E_\Omega(\{A^\Omega_{\boldsymbol{n}}\}_n), E_\Omega(\{B^\Omega_{\boldsymbol{n}}\}_n)\right) \geq \ell_d(\Omega) d_{a.c.s.}\left(\{A^\Omega_{\boldsymbol{n}}\}_n, \{B^\Omega_{\boldsymbol{n}}\}_n\right).$$

In particular,

$$\{B^\Omega_{\boldsymbol{n},m}\}_n \xrightarrow{a.c.s.} \{A^\Omega_{\boldsymbol{n}}\}_n \iff E_\Omega(\{B^\Omega_{\boldsymbol{n},m}\}_n) \xrightarrow{a.c.s.} E_\Omega(\{A^\Omega_{\boldsymbol{n}}\}_n).$$

Proof. Thanks to Lemma F.2.1, we know that $R_\Omega(E_\Omega(\{A^\Omega_{\boldsymbol{n}}\}_n)) = \{A^\Omega_{\boldsymbol{n}}\}_n$, and the same happens to $\{B^\Omega_{\boldsymbol{n}}\}_n$, so we can apply Lemma F.2.10 and obtain

$$d_{a.c.s.}\left(E_\Omega(\{A^\Omega_{\boldsymbol{n}}\}_n), E_\Omega(\{B^\Omega_{\boldsymbol{n}}\}_n)\right) \geq \ell_d(\Omega) d_{a.c.s.}\left(\{A^\Omega_{\boldsymbol{n}}\}_n, \{B^\Omega_{\boldsymbol{n}}\}_n\right).$$

On the other hand, since $Z_\Omega(E_\Omega(\{A^\Omega_{\boldsymbol{n}}\}_n - \{B^\Omega_{\boldsymbol{n}}\}_n)) = E_\Omega(\{A^\Omega_{\boldsymbol{n}}\}_n - \{B^\Omega_{\boldsymbol{n}}\}_n)$, Corollary F.2.4 assures us that the singular values of $\{A^\Omega_{\boldsymbol{n}}\}_n - \{B^\Omega_{\boldsymbol{n}}\}_n$ are the same of the singular values of $E_\Omega(\{A^\Omega_{\boldsymbol{n}}\}_n - \{B^\Omega_{\boldsymbol{n}}\}_n)$ except $N(\boldsymbol{n}) - d_n^\Omega$

for zeros. It means that

$$\begin{aligned}
d_{a.c.s.}\left(E_\Omega(\{A^\Omega_{\boldsymbol{n}}\}_n), E_\Omega(\{B^\Omega_{\boldsymbol{n}}\}_n)\right) &= \limsup_{n\to\infty} \min_{i=1,\dots,N(\boldsymbol{n})+1} \left\{ \frac{i-1}{N(\boldsymbol{n})} + \sigma_i(E_\Omega(A^\Omega_{\boldsymbol{n}} - B^\Omega_{\boldsymbol{n}}))\right\} \\
&= \limsup_{n\to\infty} \min_{i=1,\dots,d^\Omega_n+1} \left\{ \frac{i-1}{N(\boldsymbol{n})} + \sigma_i(E_\Omega(A^\Omega_{\boldsymbol{n}} - B^\Omega_{\boldsymbol{n}}))\right\} \\
&= \limsup_{n\to\infty} \min_{i=1,\dots,d^\Omega_n+1} \left\{ \frac{i-1}{N(\boldsymbol{n})} + \sigma_i(A^\Omega_{\boldsymbol{n}} - B^\Omega_{\boldsymbol{n}})\right\} \\
&\le \limsup_{n\to\infty} \min_{i=1,\dots,d^\Omega_n+1} \left\{ \frac{i-1}{d^\Omega_n} + \sigma_i(A^\Omega_{\boldsymbol{n}} - B^\Omega_{\boldsymbol{n}})\right\} \\
&= d_{a.c.s.}\left(\{A^\Omega_{\boldsymbol{n}}\}_n, \{B^\Omega_{\boldsymbol{n}}\}_n\right).
\end{aligned}$$

Consequentially,

$$\begin{aligned}
\{B^\Omega_{\boldsymbol{n},m}\}_n \xrightarrow{a.c.s.} \{A^\Omega_{\boldsymbol{n}}\}_n &\iff d_{a.c.s.}\left(\{B^\Omega_{\boldsymbol{n},m}\}_n, \{A^\Omega_{\boldsymbol{n}}\}_n\right) \to 0 \\
&\iff d_{a.c.s.}\left(E_\Omega(\{B^\Omega_{\boldsymbol{n},m}\}_n), E_\Omega(\{A^\Omega_{\boldsymbol{n}}\}_n)\right) \to 0 \\
&\iff E_\Omega(\{B^\Omega_{\boldsymbol{n},m}\}_n) \xrightarrow{a.c.s.} E_\Omega(\{A^\Omega_{\boldsymbol{n}}\}_n).
\end{aligned}$$

□

F.2.4 Different Grids

Suppose now that we want to choose a slight different set of points for every n, and we ask whether the resulting sequence of matrices still enjoys a symbol. Remember that the symmetric difference $\triangle$ between two sets is the set of elements belonging to only one of the two sets. In symbols, $A\triangle B = (A \setminus B) \cup (B \setminus A)$.

Lemma F.2.11. *Let Γ_n be a measurable set in $[0,1]^d$ (not necessarily Peano-Jordan measurable) and let Ω be a Peano-Jordan measurable set with positive measure in $[0,1]^d$. Suppose that*

$$d_n^{\Omega\triangle\Gamma_n} = o(N(\boldsymbol{n})).$$

Given a sequence $\{A_{\boldsymbol{n}}\}_n$ with $A_{\boldsymbol{n}}$ of size $N(\boldsymbol{n})\times N(\boldsymbol{n})$, and a measurable function k, we have that

$$R_\Omega(\{A_{\boldsymbol{n}}\}_n) \sim_\sigma k \iff \{R_{\Gamma_n}(A_{\boldsymbol{n}})\}_n \sim_\sigma k.$$

Moreover, if $A_{\boldsymbol{n}}$ are Hermitian, then

$$R_\Omega(\{A_{\boldsymbol{n}}\}_n) \sim_\lambda k \iff \{R_{\Gamma_n}(A_{\boldsymbol{n}})\}_n \sim_\lambda k.$$

Proof. Consider the difference

$$R_{\Omega\cup\Gamma_n}(Z_{\Omega\cap\Gamma_n}(A_{\boldsymbol{n}})) - R_{\Omega\cup\Gamma_n}(E_\Omega(R_\Omega(A_{\boldsymbol{n}}))) = R_{\Omega\cup\Gamma_n}(Z_{\Omega\cap\Gamma_n}(A_{\boldsymbol{n}}) - Z_\Omega(A_{\boldsymbol{n}})).$$

The matrix has at most $d_n^{\Omega\setminus\Gamma_n} \le d_n^{\Omega\triangle\Gamma_n} = o(N(\boldsymbol{n}))$ non-zero rows and columns, and from Corollary F.1.5, we infer also that $d_n^{\Omega\setminus\Gamma_n} = o(d_n^\Omega)$. Consequently, $d_n^{\Omega\setminus\Gamma_n} = o(d_n^{\Omega\cup\Gamma_n})$, so the sequence is zero-distributed. Moreover, the matrix $B_{\boldsymbol{n}}^{\Omega\cup\Gamma_n} := R_{\Omega\cup\Gamma_n}(E_\Omega(R_\Omega(A_{\boldsymbol{n}})))$ is actually $R_\Omega(A_{\boldsymbol{n}})$ with additional $d_n^{\Gamma_n\setminus\Omega} \le d_n^{\Omega\triangle\Gamma_n} = o(N(\boldsymbol{n}))$ zero columns and rows, so we just added few zero singular values, for which holds again $d_n^{\Gamma_n\setminus\Omega} = o(d_n^{\Omega\cup\Gamma_n})$. In particular, if we consider any continuous function $G:\mathbb{C}\to\mathbb{C}$ with compact support, then

$$\frac{1}{d_n^{\Gamma_n\cup\Omega}} \sum_{i=1}^{d_n^{\Gamma_n\cup\Omega}} G(\sigma_i(B_{\boldsymbol{n}}^{\Omega\cup\Gamma_n})) = \frac{d_n^{\Gamma_n\setminus\Omega}}{d_n^{\Gamma_n\cup\Omega}} G(0) + \frac{d_n^{\Gamma_n\cup\Omega} - d_n^{\Gamma_n\setminus\Omega}}{d_n^{\Gamma_n\cup\Omega}} \frac{1}{d_n^\Omega} \sum_{i=1}^{d_n^\Omega} G(\sigma_i(R_\Omega(A_{\boldsymbol{n}})))$$

and asymptotically we have

$$\lim_{n\to\infty} \frac{1}{d_n^{\Gamma_n\cup\Omega}} \sum_{i=1}^{d_n^{\Gamma_n\cup\Omega}} G(\sigma_i(B_{\boldsymbol{n}}^{\Omega\cup\Gamma_n})) = \lim_{n\to\infty} \frac{1}{d_n^\Omega} \sum_{i=1}^{d_n^\Omega} G(\sigma_i(R_\Omega(A_{\boldsymbol{n}}))).$$

It leads to

$$R_\Omega(\{A_{\boldsymbol{n}}\}_n) \sim_\sigma k \iff \{R_{\Omega\cup\Gamma_n}(E_\Omega(R_\Omega(A_{\boldsymbol{n}})))\}_n \sim_\sigma k \iff \{R_{\Omega\cup\Gamma_n}(Z_{\Omega\cap\Gamma_n}(A_{\boldsymbol{n}}))\}_n \sim_\sigma k$$

and the same argument can be applied to $R_{\Gamma_n}(A_{\boldsymbol{n}})$, so we can conclude that

$$R_\Omega(\{A_{\boldsymbol{n}}\}_n) \sim_\sigma k \iff \{R_{\Gamma_n}(A_{\boldsymbol{n}})\}_n \sim_\sigma k.$$

If $A_{\boldsymbol{n}}$ are Hermitian, then all the matrices considered until now are also Hermitian, so the same results apply to the spectral symbols and

$$R_\Omega(\{A_{\boldsymbol{n}}\}_n) \sim_\lambda k \iff \{R_{\Gamma_n}(A_{\boldsymbol{n}})\}_n \sim_\lambda k.$$

□

F.3 Definition and Properties of Reduced GLT Sequences

In the following propositions, we denote the image of R_Ω when applied to GLT sequences as $\mathscr{G}_d^\Omega := R_\Omega(\mathscr{G}_d)$, and we call it the space of *Reduced GLT* with respect to Ω.

F.3.1 Reduced GLT Symbol

Lemma F.3.1. *Given a GLT sequence $\{A_{\boldsymbol{n}}\}_n \sim_{GLT} k(x,\theta)$ with $k : [0,1]^d \times [-\pi,\pi]^d \to \mathbb{C}$, then*

$$R_\Omega(\{A_{\boldsymbol{n}}\}_n) \sim_\sigma k(x,\theta)|_{x\in\Omega}.$$

If $A_{\boldsymbol{n}}$ are also Hermitian matrices, then

$$R_\Omega(\{A_{\boldsymbol{n}}\}_n) \sim_\lambda k(x,\theta)|_{x\in\Omega}.$$

Proof. Thanks to Lemma F.2.1, we have $R_\Omega(\{A_{\boldsymbol{n}}\}_n) = R_\Omega(Z_\Omega(\{A_{\boldsymbol{n}}\}_n))$ and if we call $\{B_{\boldsymbol{n}}\}_n = Z_\Omega(\{A_{\boldsymbol{n}}\}_n)$, then $Z_\Omega(\{B_{\boldsymbol{n}}\}_n) = \{B_{\boldsymbol{n}}\}_n$ since Z_Ω is an idempotent operator, and $\{B_{\boldsymbol{n}}\}_n \sim_{GLT} k(x,\theta)\chi_\Omega(x)$, so in particular $\{B_{\boldsymbol{n}}\}_n \sim_\sigma k(x,\theta)\chi_\Omega(x)$. We can thus use Lemma F.2.6 and obtain that

$$R_\Omega(\{A_{\boldsymbol{n}}\}_n) = R_\Omega(\{B_{\boldsymbol{n}}\}_n) \sim_\sigma k(x,\theta)\chi_\Omega(x)|_{x\in\Omega} = k(x,\theta)|_{x\in\Omega}.$$

If $A_{\boldsymbol{n}}$ are Hermitian matrices, then also $\{B_{\boldsymbol{n}}\}_n = Z_\Omega(\{A_{\boldsymbol{n}}\}_n)$ is a Hermitian sequence, since Z_Ω preserves the Hermitianity, so $\{B_{\boldsymbol{n}}\}_n \sim_{GLT} k(x,\theta)\chi_\Omega(x) \implies \{B_{\boldsymbol{n}}\}_n \sim_\lambda k(x,\theta)\chi_\Omega(x)$. As before, $Z_\Omega(\{B_{\boldsymbol{n}}\}_n) = \{B_{\boldsymbol{n}}\}_n$ and Lemma F.2.6 assure us that

$$R_\Omega(\{A_{\boldsymbol{n}}\}_n) = R_\Omega(\{B_{\boldsymbol{n}}\}_n) \sim_\lambda k(x,\theta)\chi_\Omega(x)|_{x\in\Omega} = k(x,\theta)|_{x\in\Omega}.$$

□

Notice that the map R_Ω is not injective, but one can prove that all the GLT sequences with the same image have symbols that coincide on $\Omega \times [-\pi,\pi]$.

Lemma F.3.2. *Given $\{A_{\boldsymbol{n}}\}_n \sim_{GLT} k$, $\{B_{\boldsymbol{n}}\}_n \sim_{GLT} h$ such that $R_\Omega(\{A_{\boldsymbol{n}}\}_n) = R_\Omega(\{B_{\boldsymbol{n}}\}_n) = \{S_{\boldsymbol{n}}^\Omega\}_n \in \mathscr{G}_d^\Omega$, the symbols k, h coincide on $\Omega \times [-\pi,\pi]^d$.*

Proof. Since R_Ω is linear, we can use Lemma F.3.1 and obtain

$$\{A_{\boldsymbol{n}}\}_n - \{B_{\boldsymbol{n}}\}_n \sim_{GLT} k - h \implies R_\Omega(\{A_{\boldsymbol{n}}\}_n - \{B_{\boldsymbol{n}}\}_n) = \{S_{\boldsymbol{n}}^\Omega\}_n - \{S_{\boldsymbol{n}}^\Omega\}_n = \{0_{\boldsymbol{n}}^\Omega\}_n \sim_\sigma (k-h)|_{x\in\Omega} = \kappa.$$

Notice that if the set where $0 < L < |\kappa| < M$ has non-zero measure, then we can consider a nonnegative continuous function $G : \mathbb{R} \to \mathbb{C}$ with compact support such that $G(0) = 0$ and $G(x) > \delta > 0$ for every $x \in (L, M)$ to get an absurd

$$0 = \lim_{n\to\infty} \frac{1}{d_n^\Omega} G(\sigma_i(0_{\boldsymbol{n}})) = \frac{1}{\ell_d(\Omega)(2\pi)^d} \int_{\Omega\times[-\pi,\pi]^d} G(|\kappa(x,\theta)|)\mathrm{d}(x,\theta) \geq \frac{1}{\ell_d(\Omega)(2\pi)^d} \delta \ell_d\{|\kappa| > 0\} > 0.$$

We conclude that $\kappa = 0$, and so k, h coincide on $\Omega \times [-\pi,\pi]^d$. □

As a corollary, every GLT sequence mapped into $\{S^\Omega_{\boldsymbol{n}}\}_n$ possesses a symbol with a fixed value on $\Omega\times[-\pi,\pi]^d$, so we can associate to each reduced GLT sequence $\{S^\Omega_{\boldsymbol{n}}\}_n$ an unique symbol, called *Reduced GLT Symbol*, obtained as the restriction of any GLT symbol of the sequences in the counter-image $R_\Omega^{-1}(\{S^\Omega_{\boldsymbol{n}}\}_n)\cap\mathscr{G}_d$. From now on, we will use the notation $\{S^\Omega_{\boldsymbol{n}}\}_n \sim^\Omega_{GLT} s$ to indicate that $s:\Omega\times[-\pi,\pi]^d\to\mathbb{C}$ is the restriction of a symbol $k:[0,1]^d\times[-\pi,\pi]^d\to\mathbb{C}$ such that $\{A_{\boldsymbol{n}}\}_n\sim_{GLT} k$ and $R_\Omega(\{A_{\boldsymbol{n}}\}_n)=\{S^\Omega_{\boldsymbol{n}}\}_n$.

Given any reduced GLT sequence $\{S^\Omega_{\boldsymbol{n}}\}_n$, it is easy to produce a GLT sequence $\{A_{\boldsymbol{n}}\}_n$ such that $R_\Omega(\{A_{\boldsymbol{n}}\}_n)=\{S^\Omega_{\boldsymbol{n}}\}_n$ using the operator E_Ω. We can thus reverse Lemma F.3.1.

Lemma F.3.3. *If $\{S^\Omega_{\boldsymbol{n}}\}_n \sim^\Omega_{GLT} \kappa$, then $E_\Omega(\{S^\Omega_{\boldsymbol{n}}\}_n)\sim_{GLT} k(x,\theta)=\begin{cases}\kappa(x,\theta) & x\in\Omega,\\ 0 & x\notin\Omega.\end{cases}$ and $R_\Omega(E_\Omega(\{S^\Omega_{\boldsymbol{n}}\}_n))=\{S^\Omega_{\boldsymbol{n}}\}_n$.*

Proof. Since $\{S^\Omega_{\boldsymbol{n}}\}_n\in\mathscr{G}^\Omega_d=R_\Omega(\mathscr{G}_d)$, there exists a GLT sequence $\{A_{\boldsymbol{n}}\}_n\sim_{GLT} h$ such that $R_\Omega(\{A_{\boldsymbol{n}}\}_n)=\{S^\Omega_{\boldsymbol{n}}\}_n$, but thanks to Lemma F.2.1 we know that also $R_\Omega(Z_\Omega(\{A_{\boldsymbol{n}}\}_n))=\{S^\Omega_{\boldsymbol{n}}\}_n$ and

$$Z_\Omega(\{A_{\boldsymbol{n}}\}_n)\sim_{GLT} h(x,\theta)\chi_\Omega(x)=k(x,\theta)=\begin{cases}\kappa(x,\theta) & x\in\Omega,\\ 0 & x\notin\Omega.\end{cases}$$

Using again Lemma F.2.1, we can conclude, since

$$Z_\Omega(\{A_{\boldsymbol{n}}\}_n)=E_\Omega(R_\Omega(\{A_{\boldsymbol{n}}\}_n))=E_\Omega(\{S^\Omega_{\boldsymbol{n}}\}_n).$$

□

F.3.2 Axioms of Reduced GLT

Using the connection between $\mathscr{G}_d$ and $\mathscr{G}^\Omega_d$, we can prove that many properties of the first space transfer to the second.

Theorem F.3.4. *Suppose $\{A^\Omega_{\boldsymbol{n}}\}_n$, $\{B^\Omega_{\boldsymbol{n}}\}_n$ are reduced GLT sequences and $\{X^\Omega_{\boldsymbol{n}}\}_n,\{Y^\Omega_{\boldsymbol{n}}\}_n$ are sequences with $X^\Omega_{\boldsymbol{n}},Y^\Omega_{\boldsymbol{n}}\in\mathbb{C}^{d^\Omega_n\times d^\Omega_n}$.*

GLT 1. *If $\{A^\Omega_{\boldsymbol{n}}\}_n\sim^\Omega_{GLT}\kappa$ then $\{A^\Omega_{\boldsymbol{n}}\}_n\sim_\sigma\kappa$. If $\{A^\Omega_{\boldsymbol{n}}\}_n\sim^\Omega_{GLT}\kappa$ and each $A^\Omega_{\boldsymbol{n}}$ is Hermitian then $\{A^\Omega_{\boldsymbol{n}}\}_n\sim_\lambda\kappa$.*

GLT 2. *If $\{A^\Omega_{\boldsymbol{n}}\}_n\sim^\Omega_{GLT}\kappa$ and $\{A^\Omega_{\boldsymbol{n}}\}_n=\{X^\Omega_{\boldsymbol{n}}\}_n+\{Y^\Omega_{\boldsymbol{n}}\}_n$, where*

- *every $X^\Omega_{\boldsymbol{n}}$ is Hermitian,*
- *$(d^\Omega_n)^{-1}\|Y^\Omega_{\boldsymbol{n}}\|_2^2\to 0$,*

then $\{A^\Omega_{\boldsymbol{n}}\}_n\sim_\lambda\kappa$.

GLT 3. *Here we list three important examples of reduced GLT sequences.*

- *Given a function f in $L^1([-\pi,\pi]^d)$, its associated Toeplitz sequence is $\{T^\Omega_{\boldsymbol{n}}(f)\}_n=R_\Omega(\{T_{\boldsymbol{n}}(f)\}_n)$, where the elements are multidimensional Fourier coefficients of f:*

$$T_{\boldsymbol{n}}(f)=[f_{\boldsymbol{i}-\boldsymbol{j}}]^{\boldsymbol{n}}_{\boldsymbol{i},\boldsymbol{j}=\boldsymbol{1}},\qquad f_{\boldsymbol{k}}=\frac{1}{(2\pi)^d}\int_{-\pi}^{\pi}f(\theta)e^{-\mathrm{i}\boldsymbol{k}\cdot\theta}\mathrm{d}\theta.$$

$\{T^\Omega_{\boldsymbol{n}}(f)\}_n$ is a reduced GLT sequence with symbol $\kappa(x,\theta)=f(\theta)$.

- *Given an almost everywhere continuous function, $\widetilde{a}:[0,1]^d\to\mathbb{C}$ and its restriction $a=\widetilde{a}|_\Omega$, its associated diagonal sampling sequence $\{D^\Omega_{\boldsymbol{n}}(a)\}_n$ is defined as*

$$D^\Omega_{\boldsymbol{n}}(a)=\mathrm{diag}\left(\left\{a\left(\frac{\phi(i)}{\boldsymbol{n}+\boldsymbol{1}}\right)\right\}_{i=1}^{d^\Omega_n}\right).$$

$\{D^\Omega_{\boldsymbol{n}}(a)\}_n$ is a reduced GLT sequence with symbol $\kappa(x,\theta)=a(x)$.

- *Any zero-distributed sequence* $\{Y^{\Omega}_{\boldsymbol{n}}\}_n \sim_\sigma 0$ *is a reduced GLT sequence with symbol* $\kappa(x,\theta) = 0$.

GLT 4. *If* $\{A^{\Omega}_{\boldsymbol{n}}\}_n \sim^{\Omega}_{GLT} \kappa$ *and* $\{B^{\Omega}_{\boldsymbol{n}}\}_n \sim^{\Omega}_{GLT} \xi$*, then*

- $\{(A^{\Omega}_{\boldsymbol{n}})^H\}_n \sim^{\Omega}_{GLT} \overline{\kappa}$*, where* $(A^{\Omega}_{\boldsymbol{n}})^H$ *is the conjugate transpose of* $A^{\Omega}_{\boldsymbol{n}}$*,*
- $\{\alpha A^{\Omega}_{\boldsymbol{n}} + \beta B^{\Omega}_{\boldsymbol{n}}\}_n \sim^{\Omega}_{GLT} \alpha\kappa + \beta\xi$ *for all* $\alpha, \beta \in \mathbb{C}$*,*
- $\{A^{\Omega}_{\boldsymbol{n}} B^{\Omega}_{\boldsymbol{n}}\}_n \sim^{\Omega}_{GLT} \kappa\xi$*.*

GLT 5. *If* $\{A^{\Omega}_{\boldsymbol{n}}\}_n \sim^{\Omega}_{GLT} \kappa$ *and* $\kappa \neq 0$ *a.e., then* $\{(A^{\Omega}_{\boldsymbol{n}})^{\dagger}\}_n \sim^{\Omega}_{GLT} \kappa^{-1}$*, where* $(A^{\Omega}_{\boldsymbol{n}})^{\dagger}$ *is the Moore–Penrose pseudoinverse of* $A^{\Omega}_{\boldsymbol{n}}$*.*

GLT 6. *If* $\{A^{\Omega}_{\boldsymbol{n}}\}_n \sim^{\Omega}_{GLT} \kappa$ *and each* $A^{\Omega}_{\boldsymbol{n}}$ *is Hermitian, then* $\{f(A^{\Omega}_{\boldsymbol{n}})\}_n \sim^{\Omega}_{GLT} f(\kappa)$ *for all continuous functions* $f : \mathbb{C} \to \mathbb{C}$*.*

GLT 7. $\{A^{\Omega}_{\boldsymbol{n}}\}_n \sim^{\Omega}_{GLT} \kappa$ *if and only if there exist GLT sequences* $\{B_{\boldsymbol{n},m}\}_n \sim^{\Omega}_{GLT} \kappa_m$ *such that* κ_m *converges to* κ *in measure and* $\{B_{\boldsymbol{n},m}\}_n \xrightarrow{\text{a.c.s.}} \{A^{\Omega}_{\boldsymbol{n}}\}_n$ *as* $m \to \infty$*.*

GLT 8. *Suppose* $\{A^{\Omega}_{\boldsymbol{n}}\}_n \sim^{\Omega}_{GLT} \kappa$ *and* $\{B^{\Omega}_{\boldsymbol{n},m}\}_n \sim^{\Omega}_{GLT} \kappa_m$*, where both* $A^{\Omega}_{\boldsymbol{n}}$ *and* $B^{\Omega}_{\boldsymbol{n},m}$ *have the same size* $d^{\Omega}_n \times d^{\Omega}_n$*. Then,* $\{B^{\Omega}_{\boldsymbol{n},m}\}_n \xrightarrow{\text{a.c.s.}} \{A^{\Omega}_{\boldsymbol{n}}\}_n$ *as* $m \to \infty$ *if and only if* κ_m *converges to* κ *in measure.*

GLT 9. *If* $\{A^{\Omega}_{\boldsymbol{n}}\}_n \sim^{\Omega}_{GLT} \kappa$ *then there exist functions* $a_{i,m}, f_{i,m}$*,* $i = 1, \ldots, N_m$*, such that*

- $a_{i,m} \in C^{\infty}(\Omega)$ *and* $f_{i,m}$ *is a trigonometric polynomial,*
- $\sum_{i=1}^{N_m} a_{i,m}(x) f_{i,m}(\theta)$ *converges to* $\kappa(x,\theta)$ *a.e.,*
- $\left\{\sum_{i=1}^{N_m} D^{\Omega}_{\boldsymbol{n}}(a_{i,m}) T^{\Omega}_{\boldsymbol{n}}(f_{i,m})\right\}_n \xrightarrow{\text{a.c.s.}} \{A^{\Omega}_{\boldsymbol{n}}\}_n$ *as* $m \to \infty$*.*

Proof. Given $\{A^{\Omega}_{\boldsymbol{n}}\}_n \sim^{\Omega}_{GLT} \kappa$, $\{B^{\Omega}_{\boldsymbol{n}}\}_n \sim^{\Omega}_{GLT} \xi$, call

$$\{A_{\boldsymbol{n}}\}_n = E_{\Omega}(\{A^{\Omega}_{\boldsymbol{n}}\}_n) \sim_{GLT} \kappa', \quad \{B_{\boldsymbol{n}}\}_n = E_{\Omega}(\{B^{\Omega}_{\boldsymbol{n}}\}_n) \sim_{GLT} \xi',$$

where κ' and ξ' are the extension of κ and ξ as specified in Lemma F.3.3. We know that $\kappa'|_{x\in\Omega} = \kappa$, $\xi'|_{x\in\Omega} = \xi$ and $R_{\Omega}(\{A_{\boldsymbol{n}}\}_n) = \{A^{\Omega}_{\boldsymbol{n}}\}_n$, $R_{\Omega}(\{B_{\boldsymbol{n}}\}_n) = \{B^{\Omega}_{\boldsymbol{n}}\}_n$.

GLT 1. Using Lemma F.3.1, we know that $\{A^{\Omega}_{\boldsymbol{n}}\}_n \sim_\sigma \kappa'|_{x\in\Omega} = \kappa$. If $\{A^{\Omega}_{\boldsymbol{n}}\}_n$ is Hermitian, then $\{A_{\boldsymbol{n}}\}_n$ is Hermitian by Lemma F.2.1, so Lemma F.3.1 let us conclude that $\{A^{\Omega}_{\boldsymbol{n}}\}_n \sim_\lambda \kappa'|_{x\in\Omega} = \kappa$.

GLT 2. Let $\{X_{\boldsymbol{n}}\}_n = E_{\Omega}(\{X^{\Omega}_{\boldsymbol{n}}\}_n)$ and $\{Y_{\boldsymbol{n}}\}_n = E_{\Omega}(\{Y^{\Omega}_{\boldsymbol{n}}\}_n)$. The operator E_{Ω} is linear, so $\{A_{\boldsymbol{n}}\}_n = \{X_{\boldsymbol{n}}\}_n + \{Y_{\boldsymbol{n}}\}_n$, where $\{A_{\boldsymbol{n}}\}_n \sim_{GLT} \kappa'$. Using Corollary F.2.4, we know that the singular values of $Y_{\boldsymbol{n}}$ are the same of $Y^{\Omega}_{\boldsymbol{n}}$ except for zero singular values. As a consequence,

$$\lim_{n\to+\infty} (N(\boldsymbol{n}))^{-1} \|Y_{\boldsymbol{n}}\|_2^2 = \lim_{n\to+\infty} \frac{d^{\Omega}_n}{N(\boldsymbol{n})} (d^{\Omega}_n)^{-1} \|Y^{\Omega}_{\boldsymbol{n}}\|_2^2 = \ell_d(\Omega) \cdot 0 = 0.$$

We can thus assert that $\{A_{\boldsymbol{n}}\}_n \sim_\lambda \kappa'$, but $\kappa'|_{x\notin\Omega} = 0$ and $R_{\Omega}(\{A_{\boldsymbol{n}}\}_n) = \{A^{\Omega}_{\boldsymbol{n}}\}_n$, so we can apply Lemma F.2.6 and conclude that

$$\{A^{\Omega}_{\boldsymbol{n}}\}_n = R_{\Omega}(\{A_{\boldsymbol{n}}\}_n) \sim_\lambda \kappa'|_{x\in\Omega} = \kappa.$$

GLT 3. We know that $\{T_{\boldsymbol{n}}(f)\}_n \sim_{GLT} f$, so Lemma F.3.1 assures us that

$$\{T^{\Omega}_{\boldsymbol{n}}(f)\}_n = R_{\Omega}(\{T_{\boldsymbol{n}}(f)\}_n) \sim^{\Omega}_{GLT} f(\theta).$$

Analogously, Lemma F.1.3 shows that $\{I_{\boldsymbol{n}}(\widetilde{a})\}_n \sim_{GLT} \widetilde{a}$ and it is easy to check that $\{D^{\Omega}_{\boldsymbol{n}}(a)\}_n = R_{\Omega}(\{I_{\boldsymbol{n}}(\widetilde{a})\}_n)$, so

$$\{D^{\Omega}_{\boldsymbol{n}}(a)\}_n \sim^{\Omega}_{GLT} a.$$

Moreover, Lemma F.2.8, shows that

$$\{Y^{\Omega}_{\boldsymbol{n}}\}_n \sim_\sigma 0 \implies E_{\Omega}(\{Y^{\Omega}_{\boldsymbol{n}}\}_n) \sim_\sigma 0 \implies E_{\Omega}(\{Y^{\Omega}_{\boldsymbol{n}}\}_n) \sim_{GLT} 0 \implies \{Y^{\Omega}_{\boldsymbol{n}}\}_n = R_{\Omega}(E_{\Omega}(\{Y^{\Omega}_{\boldsymbol{n}}\}_n)) \sim^{\Omega}_{GLT} 0.$$

GLT 4. Using Lemma F.2.1 and Lemma F.3.1, we know that

$$\{(A_{\boldsymbol{n}}^{\Omega})^H\}_n = (R_\Omega(\{A_{\boldsymbol{n}}\}_n))^H = R_\Omega(\{A_{\boldsymbol{n}}^H\}_n) \sim_{GLT}^{\Omega} \overline{\kappa}'|_{x\in\Omega} = \overline{\kappa}.$$

Moreover, R_Ω is linear, so we can apply Lemma F.3.1 on $\alpha\{A_{\boldsymbol{n}}\}_n + \beta\{B_{\boldsymbol{n}}\}_n \sim_{GLT} \alpha\kappa' + \beta\xi'$ and obtain

$$\{\alpha A_{\boldsymbol{n}}^{\Omega} + \beta B_{\boldsymbol{n}}^{\Omega}\}_n = R_\Omega(\alpha\{A_{\boldsymbol{n}}\}_n + \beta\{B_{\boldsymbol{n}}\}_n) \sim_{GLT}^{\Omega} \alpha\kappa' + \beta\xi'|_{x\in\Omega} = \alpha\kappa + \beta\xi.$$

In order to prove the last point, remember that $Z_\Omega(A_{\boldsymbol{n}}) = A_{\boldsymbol{n}}$, so we can use Lemma F.2.1 and obtain the relation

$$\begin{aligned}
R_\Omega(A_{\boldsymbol{n}}B_{\boldsymbol{n}}) &= \Pi_{\boldsymbol{n},\Omega}A_{\boldsymbol{n}}B_{\boldsymbol{n}}(\Pi_{\boldsymbol{n},\Omega})^T \\
&= \Pi_{\boldsymbol{n},\Omega}D_{\boldsymbol{n}}(\chi_\Omega)A_{\boldsymbol{n}}D_{\boldsymbol{n}}(\chi_\Omega)B_{\boldsymbol{n}}(\Pi_{\boldsymbol{n},\Omega})^T \\
&= \Pi_{\boldsymbol{n},\Omega}D_{\boldsymbol{n}}(\chi_\Omega)A_{\boldsymbol{n}}(D_{\boldsymbol{n}}(\chi_\Omega))^2B_{\boldsymbol{n}}(\Pi_{\boldsymbol{n},\Omega})^T \\
&= \Pi_{\boldsymbol{n},\Omega}A_{\boldsymbol{n}}D_{\boldsymbol{n}}(\chi_\Omega)B_{\boldsymbol{n}}(\Pi_{\boldsymbol{n},\Omega})^T \\
&= \Pi_{\boldsymbol{n},\Omega}A_{\boldsymbol{n}}(\Pi_{\boldsymbol{n},\Omega})^T\Pi_{\boldsymbol{n},\Omega}B_{\boldsymbol{n}}(\Pi_{\boldsymbol{n},\Omega})^T \\
&= R_\Omega(A_{\boldsymbol{n}})R_\Omega(B_{\boldsymbol{n}}).
\end{aligned}$$

Using Lemma F.3.1, we conclude that

$$\{A_{\boldsymbol{n}}^{\Omega}\}_n\{B_{\boldsymbol{n}}^{\Omega}\}_n = R_\Omega(\{A_{\boldsymbol{n}}\}_n)R_\Omega(\{B_{\boldsymbol{n}}\}_n) = R_\Omega(\{A_{\boldsymbol{n}}\}_n\{B_{\boldsymbol{n}}\}_n) \sim_{GLT}^{\Omega} \kappa'\xi'|_{x\in\Omega} = \kappa\xi.$$

GLT 5. Notice that $\partial\Omega = \partial(\Omega^C)$, so $\{D_{\boldsymbol{n}}(\chi_{\Omega^C})\}_n \sim_{GLT} \chi_{\Omega^C}$. If we define $\{C_{\boldsymbol{n}}\}_n = \{A_{\boldsymbol{n}}\}_n + \{D_{\boldsymbol{n}}(\chi_{\Omega^C})\}_n$, then

$$\{C_{\boldsymbol{n}}\}_n \sim_{GLT} \kappa'(x,\theta) + \chi_{\Omega^C}(x) = \begin{cases}\kappa & x\in\Omega,\\ 1 & x\notin\Omega,\end{cases}$$

so $\kappa'(x,\theta) + \chi_{\Omega^C}(x) = 0$ if and only if $x\in\Omega$ and $\kappa(x,\theta) = 0$. In particular it is different from zero a.e., so

$$\{C_{\boldsymbol{n}}^\dagger\}_n \sim_{GLT} (\kappa'(x,\theta) + \chi_{\Omega^C}(x))^{-1} = \begin{cases}\kappa^{-1} & x\in\Omega,\\ 1 & x\notin\Omega.\end{cases}$$

We know that $Z_\Omega(\{A_{\boldsymbol{n}}\}_n) = \{A_{\boldsymbol{n}}\}_n$ and using Lemma F.2.1,

$$Z_\Omega(D_{\boldsymbol{n}}(\chi_\Omega)) = D_{\boldsymbol{n}}(\chi_\Omega)D_{\boldsymbol{n}}(\chi_\Omega)D_{\boldsymbol{n}}(\chi_\Omega) = D_{\boldsymbol{n}}(\chi_\Omega),$$

$$R_\Omega(D_{\boldsymbol{n}}(\chi_\Omega)) = \Pi_{\boldsymbol{n},\Omega}D_{\boldsymbol{n}}(\chi_\Omega)(\Pi_{\boldsymbol{n},\Omega})^T = \Pi_{\boldsymbol{n},\Omega}(\Pi_{\boldsymbol{n},\Omega})^T\Pi_{\boldsymbol{n},\Omega}(\Pi_{\boldsymbol{n},\Omega})^T = I_{\boldsymbol{n}}^{\Omega}I_{\boldsymbol{n}}^{\Omega} = I_{\boldsymbol{n}}^{\Omega}.$$

Let P be the permutation matrix in Lemma 6.3.3, so that

$$PA_{\boldsymbol{n}}P^T = \begin{pmatrix}A_{\boldsymbol{n}}^{\Omega} & 0\\ 0 & 0\end{pmatrix}, \qquad PD_{\boldsymbol{n}}(\chi_{\Omega^C})P^T = P(I_{\boldsymbol{n}} - D_{\boldsymbol{n}}(\chi_\Omega))P^T = I_{\boldsymbol{n}} - \begin{pmatrix}I_{\boldsymbol{n}}^{\Omega} & 0\\ 0 & 0\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & I_{\boldsymbol{n}}^{\Omega^C}\end{pmatrix},$$

$$PC_{\boldsymbol{n}}P^T = P(A_{\boldsymbol{n}} + D_{\boldsymbol{n}}(\chi_{\Omega^C}))P^T = \begin{pmatrix}A_{\boldsymbol{n}}^{\Omega} & 0\\ 0 & I_{\boldsymbol{n}}^{\Omega^C}\end{pmatrix} \implies PC_{\boldsymbol{n}}^\dagger P^T = \begin{pmatrix}(A_{\boldsymbol{n}}^{\Omega})^\dagger & 0\\ 0 & I_{\boldsymbol{n}}^{\Omega^C}\end{pmatrix}.$$

Consequentially,

$$\begin{aligned}
\begin{pmatrix}R_\Omega(C_{\boldsymbol{n}}^\dagger) & 0\\ 0 & 0\end{pmatrix} &= PZ_\Omega(C_{\boldsymbol{n}}^\dagger)P^T \\
&= PD_{\boldsymbol{n}}(\chi_\Omega)C_{\boldsymbol{n}}^\dagger D_{\boldsymbol{n}}(\chi_\Omega)P^T \\
&= PD_{\boldsymbol{n}}(\chi_\Omega)P^T\begin{pmatrix}(A_{\boldsymbol{n}}^{\Omega})^\dagger & 0\\ 0 & I_{\boldsymbol{n}}^{\Omega^C}\end{pmatrix}PD_{\boldsymbol{n}}(\chi_\Omega)P^T \\
&= \begin{pmatrix}I_{\boldsymbol{n}}^{\Omega} & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}(A_{\boldsymbol{n}}^{\Omega})^\dagger & 0\\ 0 & I_{\boldsymbol{n}}^{\Omega^C}\end{pmatrix}\begin{pmatrix}I_{\boldsymbol{n}}^{\Omega} & 0\\ 0 & 0\end{pmatrix} \\
&= \begin{pmatrix}(A_{\boldsymbol{n}}^{\Omega})^\dagger & 0\\ 0 & 0\end{pmatrix}
\end{aligned}$$

and Lemma F.3.1 let us conclude that

$$\{(A_{\boldsymbol{n}}^{\Omega})^\dagger\}_n = R_\Omega(\{C_{\boldsymbol{n}}^\dagger\}_n) \sim_{GLT}^{\Omega} (\kappa'(x,\theta) + \chi_{\Omega^C}(x))^{-1}|_{x\in\Omega} = \kappa^{-1}.$$

GLT 6. If $A_{\boldsymbol{n}}^\Omega$ is Hermitian, then $A_{\boldsymbol{n}} = E_\Omega(A_{\boldsymbol{n}}^\Omega)$ is also Hermitian and $\{A_{\boldsymbol{n}}\}_n \sim_{GLT} \kappa'$, so

$$\{f(A_{\boldsymbol{n}})\}_n \sim_{GLT} f(\kappa') = \begin{cases} f(\kappa(x,\theta)) & x \in \Omega, \\ f(0) & x \notin \Omega. \end{cases}$$

Notice that, using Lemma 6.3.3,

$$PA_{\boldsymbol{n}}P^T = \begin{pmatrix} A_{\boldsymbol{n}}^\Omega & 0 \\ 0 & 0 \end{pmatrix} \implies Pf(A_{\boldsymbol{n}})P^T = f(PA_{\boldsymbol{n}}P^T) = \begin{pmatrix} f(A_{\boldsymbol{n}}^\Omega) & 0 \\ 0 & f(0)I_{\boldsymbol{n}}^{\Omega^C} \end{pmatrix},$$

so one can prove that

$$\begin{aligned} \begin{pmatrix} R_\Omega(f(A_{\boldsymbol{n}})) & 0 \\ 0 & 0 \end{pmatrix} &= PZ_\Omega(f(A_{\boldsymbol{n}}))P^T \\ &= PD_{\boldsymbol{n}}(\chi_\Omega)f(A_{\boldsymbol{n}})D_{\boldsymbol{n}}(\chi_\Omega)P^T \\ &= PD_{\boldsymbol{n}}(\chi_\Omega)P^T \begin{pmatrix} f(A_{\boldsymbol{n}}^\Omega) & 0 \\ 0 & f(0)I_{\boldsymbol{n}}^{\Omega^C} \end{pmatrix} PD_{\boldsymbol{n}}(\chi_\Omega)P^T \\ &= \begin{pmatrix} I_{\boldsymbol{n}}^\Omega & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} f(A_{\boldsymbol{n}}^\Omega) & 0 \\ 0 & f(0)I_{\boldsymbol{n}}^{\Omega^C} \end{pmatrix} \begin{pmatrix} I_{\boldsymbol{n}}^\Omega & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} f(A_{\boldsymbol{n}}^\Omega) & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

and consequentially Lemma F.3.1 let us conclude

$$\{f(A_{\boldsymbol{n}}^\Omega)\}_n = R_\Omega(\{f(A_{\boldsymbol{n}})\}_n) \sim_{GLT}^\Omega f(\kappa')|_{x\in\Omega} = f(\kappa).$$

GLT 7. Notice that if $\{A_{\boldsymbol{n}}^\Omega\}_n \sim_{GLT}^\Omega \kappa$ and $A_{\boldsymbol{n}}^\Omega = B_{\boldsymbol{n},m}^\Omega$ for every m, then $\{B_{\boldsymbol{n},m}^\Omega\}_n \sim_{GLT}^\Omega \kappa_m = \kappa$, κ_m converges to κ and $\{B_{\boldsymbol{n},m}^\Omega\}_n \xrightarrow{\text{a.c.s.}} \{A_{\boldsymbol{n}}^\Omega\}_n$.

On the opposite, assume there exist reduced GLT sequences $\{B_{\boldsymbol{n},m}^\Omega\}_n \sim_{GLT}^\Omega \kappa_m$ such that κ_m converges to κ in measure and $\{B_{\boldsymbol{n},m}^\Omega\}_n \xrightarrow{\text{a.c.s.}} \{A_{\boldsymbol{n}}^\Omega\}_n$. Let $B_{\boldsymbol{n},m} = E_\Omega(B_{\boldsymbol{n},m}^\Omega)$ and let κ'_m be the extension of κ given by Lemma F.3.3, so that $\{B_{\boldsymbol{n},m}\}_n \sim_{GLT} \kappa'_m$. Using Lemma F.2.10, we know that $\{B_{\boldsymbol{n},m}\}_n \xrightarrow{\text{a.c.s.}} E_\Omega(\{A_{\boldsymbol{n}}^\Omega\}_n)$, and moreover

$$\kappa'_m = \begin{cases} \kappa_m(x,\theta) & x \in \Omega, \\ 0 & x \notin \Omega, \end{cases} \to \kappa' = \begin{cases} \kappa(x,\theta) & x \in \Omega, \\ 0 & x \notin \Omega, \end{cases}$$

so $E_\Omega(\{A_{\boldsymbol{n}}^\Omega\}_n) \sim_{GLT} \kappa'$ and Lemma F.3.1 let us conclude that

$$R_\Omega(E_\Omega(\{A_{\boldsymbol{n}}^\Omega\}_n)) = \{A_{\boldsymbol{n}}^\Omega\}_n \sim_{GLT}^\Omega \kappa'|_{x\in\Omega} = \kappa.$$

GLT 8. Let $B_{\boldsymbol{n},m} = E_\Omega(B_{\boldsymbol{n},m}^\Omega)$ and let κ'_m be the extension of κ given by Lemma F.3.3, so that $\{B_{\boldsymbol{n},m}\}_n \sim_{GLT} \kappa'_m$. Using Lemma F.2.10, we know that

$$\{B_{\boldsymbol{n},m}^\Omega\}_n \xrightarrow{\text{a.c.s.}} \{A_{\boldsymbol{n}}^\Omega\}_n \iff \{B_{\boldsymbol{n},m}\}_n \xrightarrow{\text{a.c.s.}} E_\Omega(\{A_{\boldsymbol{n}}^\Omega\}_n) \iff \kappa'_m \to \kappa'.$$

All the functions κ'_m and κ' are zero outside Ω, and Ω has positive measure, so

$$\kappa'_m - \kappa' \to 0 \iff \kappa'_m - \kappa'|_{x\in\Omega} \to 0 \iff \kappa_m - \kappa \to 0 \iff \kappa_m \to \kappa.$$

GLT 9. The functions in $C^\infty(\Omega)$ are restrictions of functions in $C^\infty([0,1]^d)$, so we can consider $E_\Omega(\{A_{\boldsymbol{n}}^\Omega\}_n) \sim_{GLT} \kappa'$ and find $a'_{i,m} \in C^\infty([0,1]^d)$ and trigonometric polynomials $f_{i,m}$ such that $\sum_{i=1}^{N_m} a'_{i,m}(x)f_{i,m}(\theta)$ converges to $\kappa'(x,\theta)$ a.e., and if $a'_{i,m}|_{x\in\Omega} = a_{i,m}$, then $\sum_{i=1}^{N_m} a'_{i,m}(x)f_{i,m}(\theta)|_{x\in\Omega} = \sum_{i=1}^{N_m} a_{i,m}(x)f_{i,m}(\theta)$ converges to $\kappa'|_{x\in\Omega} = \kappa$ almost everywhere. Thanks to **GLT 3** we know that $\{D_{\boldsymbol{n}}^\Omega(a_{i,m})\}_n \sim_{GLT}^\Omega a_{i,m}$ and $\{T_{\boldsymbol{n}}^\Omega(f_{i,m})\}_n \sim_{GLT}^\Omega f_{i,m}$, so we can apply **GLT 4** and obtain

$$\left\{\sum_{i=1}^{N_m} D_{\boldsymbol{n}}^\Omega(a_{i,m})T_{\boldsymbol{n}}^\Omega(f_{i,m})\right\}_n \sim_{GLT}^\Omega \sum_{i=1}^{N_m} a_{i,m}(x)f_{i,m}(\theta) \to \kappa,$$

so that **GLT 8** let us conclude that

$$\left\{\sum_{i=1}^{N_m} D_{\boldsymbol{n}}^\Omega(a_{i,m})T_{\boldsymbol{n}}^\Omega(f_{i,m})\right\}_n \xrightarrow{\text{a.c.s.}} \{A_{\boldsymbol{n}}^\Omega\}_n.$$

$\square$

Appendix G

Distances on Block Sequences and Matrix-Valued Functions

If we consider the space $\mathscr{M}_D^r$ of measurable functions $\Upsilon : D \to \mathbb{C}^{r\times r}$, the convergence in measure is induced by the distance

$$\widetilde{d}_{mea,r}(\Upsilon, \Xi) = \widetilde{p}_{mea,r}(\Upsilon - \Xi), \qquad \widetilde{p}_{mea,r}(\Phi) = \sum_{i=1}^{r}\sum_{j=1}^{r} p_{mea}(\Phi_{i,j})$$

We now want to consider a different distance:

$$d_{mea,r}(\Upsilon, \Xi) = p_{mea,r}(\Upsilon - \Xi), \qquad p_{mea,r}(\Phi) = \inf_{x\geq 0}\left\{ x + \frac{1}{r}\sum_{i=1}^{r} \frac{\ell(\sigma_i(\Phi) > x)}{\ell(D)} \right\}$$

Lemma G.0.1. *$d_{mea,r}$ is a distance of the space $\mathscr{M}_D^r$ and it induces the convergence in measure.*

Proof. First of all, $d_{mea,r}(\Upsilon, \Upsilon) = 0$ and

$$d_{mea,r}(\Upsilon, \Xi) = 0 \implies p_{mea,r}(\Upsilon - \Xi) = 0 \implies$$

$$\forall n > 0\ \exists x_n \geq 0 \,:\, x_n + \frac{1}{r}\sum_{i=1}^{r} \frac{\ell(\sigma_i(\Upsilon - \Xi) > x_n)}{\ell(D)} < \frac{1}{n}$$

$$\implies \frac{\ell(\sigma_i(\Upsilon - \Xi) = 0)}{\ell(D)} = 1 \quad \forall i \implies \Upsilon = \Xi \quad a.e.$$

The expression is clearly symmetrical, so we only need to prove the triangular inequality.

Given any matrices A, B in $\mathbb{C}^{r\times r}$, and $y, z \geq 0$, suppose that $\sigma_i(A) > y \iff i \leq k_A$ and $\sigma_i(B) > z \iff i \leq k_B$, where we say that k_A is zero if $\sigma_1(A) \leq y$, and similarly for k_B. We can decompose A, B into

$$A = N_A + R_A, \quad \|N_A\| \leq y, \quad \mathrm{rk}(R_A) = k_A, \qquad B = N_B + R_B, \quad \|N_B\| \leq z, \quad \mathrm{rk}(R_B) = k_B$$

$$\implies A + B = (N_A + N_B) + (R_A + R_B), \quad \|N_A + N_B\| \leq y + z, \quad \mathrm{rk}(R_A + R_B) \leq k_A + k_B$$

and using the Cauchy Interlacing Theorem for singular values (Theorem 3.1.1) we obtain that $\sigma_i(A + B) > y + z \implies i \leq k_A + k_B$.

Given now $y, z \geq 0$ and two functions $u, v : D \to \mathbb{C}^{r\times r}$, let us call

$$A_i = \{\, p \in D \mid \sigma_i(u(p)) > y \,\}, \qquad B_i = \{\, p \in D \mid \sigma_i(v(p)) > z \,\}$$

$$C_i = \{\, p \in D \mid \sigma_i((u + v)(p)) > y + z \,\}.$$

Moreover, we use the convention $A_0 = B_0 = D$. Notice that

$$B_r \subseteq B_{r-1} \subseteq \ldots \subseteq B_1 \subseteq B_0$$

and the same holds for A_i, C_i. We define also the sets $E_m = B_{m-1} - B_m$ for $m = 1, 2, \dots, r$ and set $E_{r+1} = B_r$. Notice that E_i are all disjoint sets. Using the result on the singular values proved above, we know that

$$\begin{aligned}
C_i &\subseteq \{\, p \in D \mid k_{u(p)} + k_{v(p)} \geq i \,\} \\
&= \bigcup_{k=0}^{i} \big[\{\, p \in D \mid k_{u(p)} \geq k \,\} \cap \{\, p \in D \mid k_{v(p)} \geq i - k \,\} \big] \\
&= \bigcup_{k=0}^{i} \big[A_k \cap B_{i-k} \big] \\
&= \bigcup_{k=0}^{i} \big[A_k \cap (E_{r+1} \sqcup E_r \sqcup \cdots \sqcup E_{i-k+1}) \big] \\
&= \bigcup_{k=0}^{i} \coprod_{m=i-k+1}^{r+1} \big[A_k \cap E_m \big] = \coprod_{m=1}^{r+1} \bigcup_{k=\min\{0, i-m+1\}}^{i} \big[A_k \cap E_m \big] \\
&= \coprod_{m=1}^{r+1} \big[A_{\min\{0, i-m+1\}} \cap E_m \big] \\
&= \big[A_0 \cap (E_{r+1} \sqcup E_r \sqcup \cdots \sqcup E_{i+1}) \big] \sqcup \coprod_{m=1}^{i} \big[A_{i-m+1} \cap E_m \big] \\
&= B_i \sqcup \coprod_{m=1}^{i} \big[A_{i-m+1} \cap E_m \big] = B_i \sqcup \coprod_{m=1}^{i} \big[A_m \cap E_{i-m+1} \big].
\end{aligned}$$

If we now compute $\sum_i \ell(C_i)$ we obtain

$$\begin{aligned}
\sum_{i=1}^{r} \ell(C_i) &= \sum_{i=1}^{r} \left(\ell(B_i) + \sum_{m=1}^{i} \ell(A_m \cap E_{i-m+1}) \right) \\
&= \sum_{i=1}^{r} \ell(B_i) + \sum_{m=1}^{r} \sum_{i=m}^{r} \ell(A_m \cap E_{i-m+1}) \\
&\leq \sum_{i=1}^{r} \ell(B_i) + \sum_{m=1}^{r} \ell(A_m).
\end{aligned}$$

We can now prove the triangular inequality for $d_{mea,r}$. Given $\Upsilon, \Xi, \Phi : D \to \mathbb{C}^{r \times r}$ three measurable functions, let $\Psi = \Upsilon - \Xi$ and $\Theta = \Xi - \Phi$. Notice that $\Psi + \Theta = \Upsilon - \Phi$. If we take any $y, z \geq 0$, then we obtain

$$\begin{aligned}
y + \frac{1}{r} \sum_{i=1}^{r} \frac{\ell(\sigma_i(\Psi) > y)}{\ell(D)} + z + \frac{1}{r} \sum_{i=1}^{r} \frac{\ell(\sigma_i(\Theta) > z)}{\ell(D)} &= y + z + \frac{1}{r} \sum_{i=1}^{r} \frac{\ell(A_i) + \ell(B_i)}{\ell(D)} \\
&\leq y + z + \frac{1}{r} \sum_{i=1}^{r} \frac{\ell(C_i)}{\ell(D)} \\
&= y + z + \frac{1}{r} \sum_{i=1}^{r} \frac{\ell(\sigma_i(\Psi + \Theta) > y + z)}{\ell(D)}.
\end{aligned}$$

Taking the infimum with respect to y and z, we have

$$\begin{aligned}
d_{mea,r}(\Upsilon, \Xi) + d_{mea,r}(\Xi, \Phi) &= p_{mea,r}(\Psi) + p_{mea,r}(\Theta) \\
&= \inf_{y \geq 0} \left\{ y + \frac{1}{r} \sum_{i=1}^{r} \frac{\ell(\sigma_i(\Psi) > y)}{\ell(D)} \right\} + \inf_{z \geq 0} \left\{ z + \frac{1}{r} \sum_{i=1}^{r} \frac{\ell(\sigma_i(\Theta) > z)}{\ell(D)} \right\} \\
&\leq \inf_{y, z \geq 0} \left\{ y + z + \frac{1}{r} \sum_{i=1}^{r} \frac{\ell(\sigma_i(\Psi + \Theta) > y + z)}{\ell(D)} \right\} \\
&= p_{mea,r}(\Psi + \Theta) = d_{mea,r}(\Upsilon, \Phi).
\end{aligned}$$

To prove that $d_{mea,r}$ induces the convergence in measure, we need to show that for every sequence of measurable functions $\Upsilon_n : D \to \mathbb{C}^{r\times r}$ we have

$$p_{mea,r}(\Upsilon_n) \to 0 \iff \widetilde{p}_{mea,r}(\Upsilon_n) \to 0.$$

Suppose first that $p_{mea,r}(\Upsilon_n) \to 0$. It means that there exist a sequence $\{x_n\}_n$ of nonnegative numbers such that

$$x_n + \frac{1}{r}\sum_{i=1}^{r} \frac{\ell(\sigma_i(\Upsilon_n) > x_n)}{\ell(D)} \to 0.$$

In particular we have $x_n \to 0$ and $\ell(\sigma_i(\Upsilon_n) > x_n) \to 0$ for every i. We can in particular assume that $\|\Upsilon_n(p)\| \le x_n$ except for a set $E_n \subseteq D$ with $\ell(E_n)/\ell(D) \le 2^{-n}$. Recall that

$$|\Upsilon_n(p)_{i,j}| \le \|\Upsilon_n(p)\|_F \le \sqrt{r}\|\Upsilon_n(p)\|$$

for every index i, j, so we can conclude that

$$p_{mea}((\Upsilon_n)_{i,j}) = \inf\left\{\frac{\ell(E^C)}{\ell(D)} + \operatorname*{ess\,sup}_{E} |(\Upsilon_n)_{i,j}|\right\} \le \frac{\ell(E_n)}{\ell(D)} + \operatorname*{ess\,sup}_{E_n^C} |(\Upsilon_n)_{i,j}| \le 2^{-n} + \sqrt{r}x_n \to 0$$

$$\implies \widetilde{p}_{mea,r}(\Upsilon_n) = \sum_{i,j} p_{mea}((\Upsilon_n)_{i,j}) \to 0$$

If we suppose now that $\widetilde{p}_{mea,r}(\Upsilon_n) \to 0$, then every $p_{mea}((\Upsilon_n)_{i,j})$ converges to zero and we can find a sequence $L_n^{i,j}$ such that

$$\frac{\ell(\{\, x \in D \mid |(\Upsilon_n)_{i,j}| > L_n^{i,j}\,\})}{\ell(D)} + L_n^{i,j} \to 0.$$

In particular, $L_n \to 0$ and $|(\Upsilon_n)_{i,j}| > L_n$ on a set $E_n^{i,j}$ whose measure goes to zero. If $E_n = \cup_{i,j} E_n^{i,j}$, and $L_n = \max_{i,j} L_n^{i,j}$, then we know that $|(\Upsilon_n)_{i,j}| \le L_n$ for every i, j except for a set contained in E_n, whose measure goes to zero. Recall that

$$\|\Upsilon_n(p)\| \le \|\Upsilon_n(p)\|_F = \sqrt{\sum_{i,j} |\Upsilon_n(p)_{i,j}|^2} \le \max_{i,j} |\Upsilon_n(p)_{i,j}|$$

so we can conclude that

$$p_{mea,r}(\Upsilon_n) = \inf_{x\ge 0}\left\{x + \frac{1}{r}\sum_{i=1}^{r}\frac{\ell(\sigma_i(\Upsilon_n) > x)}{\ell(D)}\right\} \le L_n + \frac{1}{r}\sum_{i=1}^{r}\frac{\ell(\sigma_i(\Upsilon_n) > L_n)}{\ell(D)} \le L_n + \frac{1}{r}\sum_{i=1}^{r}\frac{\ell(E_n)}{\ell(D)} \to 0$$

□

Consider a measurable function $\Upsilon : D \to \mathbb{C}^{r\times r}$ where D is a measurable subset of $\mathbb{R}^s$ with positive and finite measure. We say that $\{A_n\}_n \sim_\sigma \Upsilon$ when

$$\lim_{n\to\infty} \frac{1}{s_n}\sum_{i=1}^{s_n} G(\sigma_i(A_n)) = \frac{1}{r\ell_s(D)}\int_D \sum_{i=1}^{r} G(\sigma_i(\Upsilon(\boldsymbol{x})))\mathrm{d}\boldsymbol{x},$$

for every function $G : \mathbb{R} \to \mathbb{C}$ which is continuous with compact support, where $A_n \in \mathbb{C}^{s_n\times s_n}$. Notice that the ergodic formula is well defined since Theorem 3.1.3 assures us that for any matrix A,

$$|\sigma_j(A+E) - \sigma_j(A)| \le \|E\|, \qquad \forall j,$$

and as a consequence the function $\sigma_j : \mathbb{C}^{n\times n} \to \mathbb{R}$ is continuous. Since Υ is a measurable functions, we obtain that also

$$\sigma_{j,f} : D \to \mathbb{R}^+ : x \mapsto \sigma_j(\Upsilon(x))$$

is measurable, and the integral in the ergodic formula is well defined for every i.

Let us define a new function $\Xi : D' \to \mathbb{R}$, where $D' = D \times [0, r] \subseteq \mathbb{R}^{s+1}$ defined as

$$\Xi(x, t) = \sigma_{\lceil t\rceil}(\Upsilon(x)) \qquad \forall x \in D,\ t \in [0, r].$$

Given $G \in C_c(\mathbb{R})$, we have that

$$\frac{1}{\ell(D')}\int_{D'} G(\Xi(y))\mathrm{d}y = \frac{1}{r\ell(D)}\sum_{i=1}^{r}\int_{D\times(i-1,i]} G(\sigma_i(\Upsilon(x)))\mathrm{d}(x,t) = \frac{1}{r\ell(D)}\int_D \sum_{i=1}^{r} G(\sigma_i(\Upsilon(x)))\mathrm{d}x$$

so we deduce that $\{A_n\}_n \sim_\sigma \Upsilon \iff \{A_n\}_n \sim_\sigma \Xi$.

Suppose $\{A_n\}_n \sim_\sigma \Upsilon \in \mathscr{M}_D^r$ and notice that $A_n \in \mathbb{C}^{nr\times nr}$ and $\{0_n\}_n$ is the sequence of zero matrices and growing size nr. As we have proved in the precedent section, $\{A_n\}_n \sim_\sigma \Upsilon \iff \{A_n\}_n \sim_\sigma \Xi$, and from the scalar case we already know that

$$\rho_{a.c.s.}(\{A_n\}_n) = p_{mea}(\Xi).$$

Let us now write formally $p_{mea}(\Xi)$.

$$\begin{aligned}
p_{mea}(\Xi) &= \inf_{L\geq 0}\left\{\frac{\ell(|\Xi| > L)}{\ell(D')} + L\right\} \\
&= \inf_{L\geq 0}\left\{L + \frac{1}{r\ell(D)}\sum_{i=1}^{n}\ell(\{\,(x,t)\in D' \mid t\in(i-1,i],\ |\Xi(x,t)| > L\,\})\right\} \\
&= \inf_{L\geq 0}\left\{L + \frac{1}{r\ell(D)}\sum_{i=1}^{n}\ell(\{\,x\in D \mid |\sigma_i(\Upsilon(x))| > L\,\})\right\} \\
&= p_{mea,r}(\Upsilon),
\end{aligned}$$

so

$$d_{mea,r}(\Upsilon, 0) = p_{mea,r}(\Upsilon) = \rho_{a.c.s.}(\{A_n\}_n) = d_{a.c.s.}\left(\{A_n\}_n, \{0_n\}_n\right).$$