Scuola Normale Superiore
Classe di Scienze Matematiche e Naturali
Corso di perfezionamento in Matematica

PhD thesis

Fractional minimal surfaces and Allen-Cahn equations

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2nd December 2018

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Abstract

In recent years fractional operators have received considerable attention both in pure and applied mathematics. They appear in biological observations, finance, crystal dislocation, digital image reconstruction and minimal surfaces.

In this thesis we study *nonlocal minimal surfaces* which are boundaries of sets minimizing certain integral norms and can be interpreted as a non-infinitesimal version of classical minimal surfaces. In particular, we consider critical points, with or withouth constraints, of suitable functionals, or approximations through diffuse models as the Allen-Cahn's.

In the first part of the thesis we prove an existence and multiplicity result for critical points of the fractional analogue of the Allen-Cahn equation in bounded domains. We bound the functional using a standard nonlocal tool: we split the domain in two regions and we analyze the three significative interactions. Then, the proof becomes an application of a classical Krasnoselskii's genus result.

Then, we consider a fractional mesoscopic model of phase transition i.e. the fractional Allen-Cahn equation with the addition of a mesoscopic term changing the 'pure phases' ± 1 in periodic functions. We investigate geometric properties of the interface of the associated minimal solutions. Then we construct minimal interfaces lying to a strip of prescribed direction and universal width. We provide a geometric and variational technique adapted to deal with nonlocal interactions.

In the last part of the thesis, we study functionals involving the fractional perimeter. In particular, first we study the localization of sets with constant nonlocal mean curvature and small prescribed volume in an open bounded domain, proving that these sets are 'sufficiently close' to critical points of a suitable potential. The proof is an application of the Lyupanov-Schmidt reduction to the fractional perimeter.

Finally, we consider the fractional perimeter in a half-space. We prove the existence of a minimal set with fixed volume and some of its properties as intersection with the hyperplane $\{x_N = 0\}$, symmetry, to be a graph in the x_N-direction and smoothness.

Contents

1 Introduction of the summary results

In recent years fractional operators have received a lot of attention both in pure and applied mathematics. The motivations are multiple: they appear in biological observations (for example when a predator decides that a nonlocal dispersive strategy to hunt its preys is more efficient) [64], in minimal surfaces [22], in crystal dislocation [12] and in finance [41]. In particular, from a probabilistic point of view, the fractional Laplacian is an infinitesimal generator of Lévy processes, see [10].

Fractional operators generalize classical ones, because if their order is given by the parameter $s \in (0,1)$, when $s \to 0^+$ we obtain the identity, while if $s \to 1^-$ we recover(after proper scaling) the classical local operator. For these reasons in the first part of this thesis we are interested in studying an elliptic nonlinear equation with fractional diffusion of the form

$$(-\Delta)^s u = W'(u) \quad \text{in } \Omega \subseteq \mathbb{R}^N \tag{1.0.1}$$

with $s \in (0,1)$, $(-\Delta)^s$ the fractional Laplacian (defined in (2.1.2)) and $W(u) := \frac{(1-|u|^2)^2}{4}$ the well known double-well potential. The interest versus this equation, known as the *fractional Allen-Cahn equation*, is due to the fact that it models the process of phase separation in iron alloys, along with order-disorder transitions, and the fractional exponent $s \in (0,1)$ allows us to consider long-range particle interactions (producing, depending on the value of s, local or nonlocal effect, see [82, 84]).

In the last years many aspects of the fractional Allen-Cahn equation has been studied. As it concerns existence, uniqueness and qualitative properties of (1.0.1) we refer to [20], where Cabré and Sire studied a more general equation of the form

$$(-\Delta)^s u + G'(u) = 0 \quad \text{in } \mathbb{R}^N,$$

where G denotes the potential associated to a nonlinearity f.

Then, some authors investigated multiplicity results of nontrivial solution for

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s u + u = h(u) & \text{in } \lambda\Omega, \\ u = 0 & \text{on } \partial(\lambda\Omega), \end{cases} \tag{1.0.2}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $\lambda \in \mathbb{R}^+$, $N > 2s$ with $s \in (0,1)$, and $h(u)$ has a subcritical growth (see [48]), or for

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s u + V(z)u = f(u) & \text{in } \mathbb{R}^N, \\ u \in H^s(\mathbb{R}^N) & \\ u(z) > 0 & z \in \mathbb{R}^N, \end{cases} \tag{1.0.3}$$

where $N > 2s$, the potential $V : \mathbb{R}^N \to \mathbb{R}$ and the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ satisfy suitable assumptions (see [49]).

Moreover, in [73], Passaseo studied the functional

$$f_\varepsilon(u) := \varepsilon \int_\Omega |Du|^2 \,\mathrm{d}x + \frac{1}{\varepsilon} \int_\Omega G(u)\,\mathrm{d}x, \tag{1.0.4}$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded domain, $u \in H^1(\Omega)$, $G \in C^2(\mathbb{R};\mathbb{R}^+)$ with exactly two zeros, and $\varepsilon \in \mathbb{R}^+$, showing that the number of critical points for f_ε goes to infinity as $\varepsilon \to 0$.

Afterwards, in [57] and [63], Guaraco and Mantoulidis used a min-max approach to study (1.0.4) as $\varepsilon \to 0$.

Motivated by these results, we addressed an existence and multiplicity results for the energy

$$F_\varepsilon(u) := \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{\varepsilon^{2s}}\int_\Omega W(u)\,\mathrm{d}x, & \text{if } s \in (0,1/2),\\ \dfrac{1}{2}\dfrac{1}{|\log\varepsilon|}\displaystyle\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+1}}\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{|\varepsilon\log\varepsilon|}\int_\Omega W(u)\,\mathrm{d}x, & \text{if } s = 1/2,\\ \dfrac{\varepsilon^{2s-1}}{2}\displaystyle\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{\varepsilon}\int_\Omega W(u)\,\mathrm{d}x, & \text{if } s \in (1/2,1), \end{cases} \tag{1.0.5}$$

that is the fractional counterpart in Ω of (1.0.4), with $\Omega \subseteq \mathbb{R}^N$ that is a bounded domain, W is the double-well potential (see (3.0.2) for more details), $u \in H^s(\Omega)$ and $\varepsilon \in \mathbb{R}^+$.

In particular, in the same spirit as [73], in Chapter 3, we consider the functional F_ε and we prove the following

Theorem 1.1. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain. Then there exist two sequences of positive numbers $\{\varepsilon_k\}_{k\in\mathbb{N}}$, $\{c_k\}_{k\in\mathbb{N}}$ such that for every $\varepsilon \in (0,\varepsilon_k)$, the functional F_ε has at least k pairs*

$$(-u_{1,\varepsilon}, u_{1,\varepsilon}), \dots, (-u_{k,\varepsilon}, u_{k,\varepsilon})$$

of critical points, all of them different from the constant pair $(-1,1)$ satisfying

$$-1 \le u_{i,\varepsilon}(x) \le 1 \quad \forall\, x \in \Omega,\ \forall\, \varepsilon \in (0,\varepsilon_k),\ i = 1,\dots k;$$
$$F_\varepsilon(u_{i,\varepsilon}) \le c_k \quad \forall\, \varepsilon \in (0,\varepsilon_k),\ i = 1,\dots,k.$$

Moreover, for all $\varepsilon \in (0,\varepsilon_k)$ and all $i = 1,\dots,k$ we have

$$F_\varepsilon(u_{i,\varepsilon}) \ge \min\left\{ F_\varepsilon(u) : u \in H^s(\Omega), -1 \le u(x) \le 1 \quad \text{for } x \in \Omega, \int_\Omega u\,\mathrm{d}x = 0 \right\}. \tag{1.0.6}$$

Another interesting problem related to the fractional Allen-Cahn equation concerns *plane-like minimizers*, i.e. minimizers that stay at a finite distance from a plane along every direction. About that, in [31], Cozzi and Valdinoci studied the functional

$$\mathtt{E}(u) := \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u(x) - u(y)|^2 K(x,y) \, \mathrm{d}x \, \mathrm{d}y + \int_{\mathbb{R}^N} W(x, u(x)) \, \mathrm{d}x \,, \tag{1.0.7}$$

where K is a kernel comparable to that one of the fractional laplacian and W is the double-well potential. In particular they constructed minimizers of $\mathtt{E}$ with interfaces in a slab of prescribed direction and bounded size (independently of the direction).

This type of problem was first studied by Caffarelli and De La Llave in [24] where the authors considered an elliptic integrand $\mathfrak{I}$ (but also functionals involving volume terms) in $\mathbb{R}^N$ or in suitable manifolds, periodic under integer translations, and they proved that for any plane in $\mathbb{R}^N$ there exists at least one minimizer of $\mathfrak{I}$ with a bounded distance from this plane.

The analogous result of [31] for $s = 1$ was proved in [86], where the first addendum of $\mathtt{E}$ is replaced by

$$\int \langle A(x) \nabla u(x), \nabla u(x) \rangle \, \mathrm{d}x \,,$$

with A bounded and uniformly elliptic matrix. Some other generalizations were analyzed in [74, 60, 13].

Then, in [69], Novaga and Valdinoci considered the Allen-Cahn energy with the addition of a 'mesoscopic term' H which is 'neutral' in the average and at each point it prefers one of the two phases, i.e.

$$E_{\Omega}(u) := \int_{\Omega} \Big(|\nabla u(x)|^2 + W(x, u(x)) + H(x) u(x) \Big) \, \mathrm{d}x \,, \tag{1.0.8}$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded domain, $N \geq 2$, $u \in H^1(\Omega)$, W is the standard double-well potential and $H \in L^\infty(\mathbb{R}^N)$.

They investigated geometric properties of the interfaces of the associated minimal solutions and they gave density estimates for the level sets. This allowed them to construct, in the periodic setting, minimal interfaces near a prescribed strip.

In the same spirit of [31] and [69], we studied in Chapter 4 the fractional Allen-Cahn energy with the addition of a 'mesoscopic term' H, i.e.

$$\mathcal{E}(u) := \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u(x) - u(y)|^2 K(x,y) \, \mathrm{d}x \, \mathrm{d}y + \int_{\mathbb{R}^N} W(x, u(x)) \, \mathrm{d}x + \int_{\mathbb{R}^N} H(x) u(x) \, \mathrm{d}x \,,$$

where K is a kernel comparable to that one of the fractional laplacian, W is the double-well potential and $H \in L^\infty(\mathbb{R}^N)$ (see Chapter 4 for more details).

For this functional we construct minimal interfaced near a strip of universal size:

Theorem 1.2. *Let $s \in (0,1)$, $\delta_0 \in (0, 1/10)$ and $N \geq 2$. Given $\theta \in (0, 1 - \delta_0)$, there exists $M_0 > 0$ depending only on θ and on universal quantities, such that for any $\omega \in \mathbb{R}^N \setminus \{0\}$, there is a class A-minimizer u_ω of $\mathcal{E}$ for which we have*

$$\{|u_\omega| < \theta\} \subset \Big\{ x \in \mathbb{R}^N : \frac{\omega}{|\omega|} \cdot x \in [0, M_0] \Big\}.$$

Moreover,

- *if* $\omega \in \mathbb{Q}^N \setminus \{0\}$, u_ω *is periodic with respect to* $\sim_\omega$*;*
- *if* $\omega \in \mathbb{R}^N \setminus \mathbb{Q}^N$, u_ω *is the uniform limit on compact subsets of* $\mathbb{R}^N$ *of a sequence of periodic class A-minimizers.*

We refer to Definition 4.4 and (4.0.2) for the notions of class A-minimizer and function $\sim_\omega$ periodic respectively.

In addition to the study of the properties which characterize the solutions of the fractional Allen-Cahn equation (1.0.1), it is also interesting to observe that, if $\Omega \subset \mathbb{R}^N$ is a bounded domain, the complete version of F_ε (defined in (1.0.5)) is given by $\mathcal{I}_{s,\Omega,\varepsilon} : H^s(\Omega) \to \mathbb{R} \cup \{+\infty\}$,

$$\mathcal{I}_{s,\Omega,\varepsilon}(u) := \begin{cases} \mathcal{K}(u,\Omega) + \varepsilon^{-2s} \int_\Omega W(u)\,\mathrm{d}x & \text{if } s \in (0,1/2), \\ |\varepsilon \log \varepsilon|^{-1} \big(\varepsilon^{2s} \mathcal{K}(u,\Omega) + \int_\Omega W(u)\,\mathrm{d}x \big) & \text{if } s = 1/2, \\ \varepsilon^{2s-1} \mathcal{K}(u,\Omega) + \frac{1}{\varepsilon} \int_\Omega W(u)\,\mathrm{d}x & \text{if } s \in (1/2,1), \end{cases} \tag{1.0.9}$$

where $\varepsilon > 0$, W is the double-well potential and

$$\mathcal{K}(u,\Omega) := \frac{1}{2} \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y + \int_\Omega \int_{\Omega^C} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y\,.$$

(1.0.9) is the fractional counterpart of the functional studied by Modica and Mortola in [22], where the authors proved the Γ-convergence of the energy to De Giorgi's perimeter (defined in [52]).

In the same way Savin and Valdinoci in [82] considered the functional $\mathcal{I}_{s,\Omega,\varepsilon}$ showing that if $s \in [1/2,1)$, then $\mathcal{I}_{s,\Omega,\varepsilon}$ Γ-converges to the classical perimeter, while if $s \in (0,1/2)$ and $u\big|_\Omega = \chi_E - \chi_{\mathbb{R}^N \setminus E}$ for some set $E \subset \Omega$, then $\mathcal{I}_{s,\Omega,\varepsilon}$ Γ-converges to the *fractional perimeter* (localized with respect to Ω)

$$P_s(E,\Omega) := \int_{E\cap\Omega} \int_{E^C} \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^{N+2s}} + \int_{E\setminus\Omega} \int_{E^C\cap\Omega} \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^{N+2s}}. \tag{1.0.10}$$

Moreover, Ambrosio, De Philippis and Martinazzi analyzed in [6] the link between the fractional perimeter and the classical De Giorgi's perimeter, showing the equi-coercivity, the Γ-convergence of the fractional perimeter, when s approaches 1/2, to the classical perimeter (up to a scaling factor), and they deduced a local convergence result for minimizers.

Therefore the fractional perimeter, defined for a measurable set $E \subset \mathbb{R}^N$, as

$$P_s(E) := \int_E \int_{E^C} \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^{N+2s}}, \tag{1.0.11}$$

where $N > 2$, $s \in (0,1/2)$ and E^C that is the complement of E, is a (nonlocal) variation of the classical notion of perimeter which takes into account also a long-range interactions between sets, and hence it is of great interest from a mathematical point of view. Additionaly, the fractional perimeter has a relevant role in many applications.

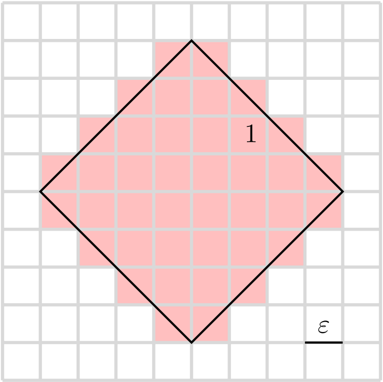


Figure 1.1: Discrepancy between classical perimeter and fractional perimeter in a bitmap.

For example, if we consider a bitmap, that is a digitalized image in which every pixel can only be black or white, we can easily see that the fractional perimeter is more accurate than the classical one to analyze digitalized images (see [39, 27]).

To observe this fact, we take a grid of square pixels of small side $\varepsilon > 0$ and a black square E of side 1 rotated by 45 degrees with respect to the orientation of the pixels. Then we digitalize the square and we see a numerical error due to the pixels intersecting the square which become black, see Figure 1.1. Computing the (classical) perimeter of the original square and that one of the digitalized image we notice an error of a factor $\sqrt{2}$ since the perimeter of the first is 4 and that one of the second is $4\sqrt{2}$ (independently on ε).

If we use the fractional perimeter (for example with $s = 0.48$ so that it is very close to the classical perimeter thanks to [6]), we get a much better approximation. Indeed, in this case, the discrepancy $D_s(\varepsilon)$ between the fractional perimeter of the original square and that one of the digitalized image is bounded by above by the sum of "boundary pixels", whose number is $4/\varepsilon$. Moreover, the intersection of one pixel with its complement is given, for $N = 2$, by the scaling factor ε^{2-2s} (obtained by (1.0.11)). Therefore, for $C > 0$, we obtain that $D_s(\varepsilon) \leq C\varepsilon^{1-2s} \to 0$ as $\varepsilon \to 0$.

For all these reasons, in the last part of this thesis we focus on the study of some properties holding for minimizers of the fractional perimeter, whose boundaries are called *nonlocal constant mean curvature surfaces.* They appear in the study of fractals [87], cellular automata [58, 25] and phase transitions [16, 82].

First we study fractional isoperimetric problems. Their standard version consists in the study of least-area sets contained in a fixed region (a ball, the Euclidean space, ...). More precisely, if we consider a N-dimensional manifold M^N, with or without boundary, the goal would be to find, among all the compact hypersurfaces $\Sigma \subset M$ which contain a region Ω of volume $V(\Omega) = m \in (0, V(M))$, those of minimal area $A(\Sigma)$. Such a region Ω is called an *isoperimetric region* and its boundary Σ is said an *isoperimetric hypersurface.*

For this problem, a first general existence and regularity result can be obtained combining the works of Almgren with those of Gonzalez, Massari, Tamanini and Grüter (see [2, 52, 53]). We also refer the reader to [76], where one can find an interesting survey about the various topologies of the minimizers.

Beyond the existence and the regularity problem, it is also interesting to study the geometry and the topology of the solutions, and to give a qualitative description of the isoperimetric regions. As it concerns these issues, we recall that in 2000 Morgan and Johnson showed in [67] that a region of small prescribed volume in a smooth and compact Riemannian manifold has asymptotically (as the volume tends to zero) at least as much perimeter as a round ball.

Afterwards, regarding critical points of the perimeter relative to a given set, in [43], Fall proved the existence of surfaces similar to half spheres surrounding a small volume near nondegenerate critical points of the mean curvature of the smooth boundary of an open set in $\mathbb{R}^3$. Moreover he showed that the boundary mean curvature determines the main terms studying the problem with a Lyapunov-Schmidt reduction.

Then, in [42] he proved that isoperimetric regions with small volume in a bounded smooth domain Ω are near global maxima of the mean curvature of Ω.

Results of the same kind were shown in [40] and [88]. The authors considered closed manifolds, proving that isoperimetric regions with small volume located near the maxima of scalar curvature. In [88] Ye also showed a viceversa: for every critical points p of the scalar curvature there exists a neighborhood of p foliated by constant mean curvature hypersurfaces. Moreover, in [85], Taylor studied the boundary regularity for the capillarity problem.

In the last years the increase of the interest for the fractional operator has led many mathematicians to study isoperimetric problems even in a fractional setting.

In [46], Figalli, Fusco, Maggi, Millot and Morini generalized to the fractional setting a well known quantitative isoperimetric inequality which holds for the classical perimeter. Indeed, in the Euclidean framework, we know that among all sets of prescribed measure, the balls have the least perimeter, i.e. for any $E \subset \mathbb{R}^N$ borel set of finite Lebesgue measure, it results

$$N|B_1|^{\frac{1}{N}}|E|^{\frac{N-1}{N}} \leq P(E), \tag{1.0.12}$$

with B_1 denoting the unit ball of $\mathbb{R}^N$ with center at the origin and $P(E)$ is the De Giorgi's perimeter of E. The equality in (1.0.12) holds if and only if E is a ball.

Fusco, Millot and Morini proved in [50] an analogous result for fractional perimeter P_s (defined in (1.0.11)), then Figalli, Fusco, Maggi, Millot and Morini improved it, showing the following result:

Theorem 1.3. *[46, Theorem 1.1] For every $N \geq 2$ and $s_0 \in (0, 1/2)$ there exists $C(N, s_0) > 0$ such that*

$$P_s(E) \geq \frac{P_s(B_1)}{|B_1|^{\frac{N-2s}{N}}}|E|^{\frac{N-2s}{N}}\left\{1 + \frac{A(E)^2}{C(N,s)}\right\} \tag{1.0.13}$$

whenever $s \in [s_0, 1/2]$ and $0 < |E| < \infty$.

As in [50],

$$A(E) := \inf\left\{ \frac{|E\triangle(B_{r_E}(x))|}{|E|} : x \in \mathbb{R}^N \right\}$$

is the Fraenkel asymmetry of E and measures the normalized L^1-distance of E from the set of balls of volume $|E|$, while $r_E := (|E|/|B_1|)^{1/N}$ so that $|E| = |B_{r_E}|$, where B_{r_E} is the ball of radius r_E and center at the origin.

In the same spirit of extension of classical results to the fractional setting, we mention a paper of Maggi and Valdinoci. In [61] they modify the classical Gauss free energy functional used in capillarity theory by considering surface tension energies of nonlocal type.

In this way, the authors analyzed a family of problems including an interesting nonlocal isoperimetric problem. In particular, taking $\Omega \subset \mathbb{R}^N$ and $\sigma \in (-1, 1)$, Maggi and Valdinoci studied the nonlocal capillarity energy of $E \subset \Omega$ defined as

$$\mathcal{E}(E) := \int_E \int_{E^C\cap\Omega} K(x,y)\,\mathrm{d}x\,\mathrm{d}y + \sigma \int_E \int_{\Omega^C} K(x,y)\,\mathrm{d}x\,\mathrm{d}y\,,$$

with $K : \mathbb{R}^N \setminus \{0\} \to [0, +\infty)$ that is an interaction kernel, i.e. an even function such that

$$\frac{\chi_{B_\varepsilon}(z)}{\lambda|z|^{N+2s}} \leq K(z) \leq \frac{\lambda}{|z|^{N+2s}} \quad \forall\, z \in \mathbb{R}^N \setminus \{0\}, \tag{1.0.14}$$

where $N \geq 2$, $s \in (0, 1/2)$, $\lambda \geq 1$, $\varepsilon \in [0, \infty)$ and $B_\varepsilon(x)$ that is the ball of center x and radius ε. They gave existence and regularity results, density estimates and new equilibrium conditions with respect to those of the classical Gauss free energy.

Motivated by the existence of these results, in Chapter 5, we want to study the localization of sets with constant nonlocal mean curvature and small prescribed volume in an open bounded domain proving this

Theorem 1.4. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set with smooth boundary and $s \in (0, 1/2)$.*

For x in a given compact set Θ of Ω, set

$$V_\Omega(x) := \int_{\mathbb{R}^N\setminus\Omega} \frac{1}{|x-y|^{N+2s}}\,\mathrm{d}y\,.$$

Then for every strict local extremal or non-degenerate critical point x_0 of V_Ω in Ω, there exists $\overline{\varepsilon} > 0$ such that for every $0 < \varepsilon < \overline{\varepsilon}$ there exist spherical-shaped surfaces with constant H_s^Ω curvature and enclosing volume identically equal to ε, approaching x_0 as $\varepsilon \to 0$.

We refer to Section 2.2.1 for the definition of H_s^Ω, which is the fractional counterpart of the well known mean curvature.

Moreover, knowing only the topology of the domain, we can also deduce a multiplicity result:

Corollary 1.5. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set with smooth boundary. Then there exists $\overline{\varepsilon} > 0$ such that for every $0 < \varepsilon < \overline{\varepsilon}$ there exist at least cat(Ω) spherical-shaped surfaces with constant H_s^Ω curvature and enclosing volume identically equal to ε.*

We write cat(Ω) to denote the Lusternik-Schnirelman category of the set Ω (see [59] and Section 2.5 for more details).

Then, in the second part of Chapter 5 we want to study the existence and some properties of sets with fixed volume $m \in (0, +\infty)$ which minimize the fractional perimeter in a half-space. We notice that, recently, in [65] Mihaila showed the axial symmetry of *smooth* critical points of the fractional perimeter in an half-space, using a variant of the moving plane method.

Our main result will be the following:

Theorem 1.6. *Let $s \in (0, 1/2)$. There exists a minimizer E for*

$$\left\{ \bar{P}_s(E, \mathbb{R}^N_+) := \int_E \int_{\mathbb{R}^N_+ \setminus E} \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^{N+2s}}, \quad E \text{ measurable set with } |E| = m \right\}$$

$$m \in (0, +\infty), \quad (1.0.15)$$

where $\mathbb{R}^N_+ := \{x \in \mathbb{R}^N : x_N > 0\}$ denotes the half-space. Moreover ∂E is a radially decreasing symmetric graph of class C^∞ in the interior, intersecting orthogonally the hyperplane $\{x_N = 0\}$.

1.1 Overview of the thesis

This thesis is organized in five chapters.

In Chapter 2 we introduce some notation, the setting and some preliminary results.

In Chapter 3 (whose results are published in [70]) we consider the fractional Allen-Cahn energy (1.0.5) in a bounded domain and we prove Theorem 1.1. To do this we get a bound by above on F_ε through a nonlocal estimate obtained splitting the domain in two suitable regions and evaluating F_ε in the three possible interactions. Then we show the validity of Palais-Smale condition and we apply a classical Krasnoselskii's genus tool to prove the existence and multiplicity results for minimizers of (1.0.5).

In Chapter 4 (whose results are published in [71]) we study a fractional mesoscopic model of phase transition in a periodic medium. We prove an important result about the regularity of minimizers of the associated functional, an energy estimate and some geometric properties. Then we give a proof of Theorem 1.2 (first under the additional assumption that K has a fast decay at infinity then for general kernels) both for rational and irrational vectors.

In Chapter 5 (whose results are published in [62]) first we study the localization of sets with constant nonlocal mean curvature and small prescribed volume in a bounded open set with smooth boundary. We prove Theorem 1.4 as an application of Lyapunov-Schmidt reduction and Corollary 1.5 through a result about the Lusternik-Schnirelman category. Then, in the second part of this chapter, we consider the fractional perimeter in a half-space, proving the existence of a minimal set with fixed volume and some of its properties as symmetry, to be a graph in the x_N-direction, smoothness and intersection with the hyperplane $\{x_N = 0\}$ in Theorem 1.6.

2 Notation and preliminary results

In this chapter we want to introduce the framework that will be used throughout this thesis.

2.1 Functional spaces

Let $\Omega \subseteq \mathbb{R}^N$ be an open set and $s \in (0,1)$. For any $p \in [1,+\infty)$ we define

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) : \frac{|u(x)-u(y)|}{|x-y|^{N/p+s}} \in L^p(\Omega \times \Omega) \right\} \tag{2.1.1}$$

as an intermediate Banach space between $L^p(\Omega)$ and $W^{1,p}(\Omega)$ endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} := \left(\int_\Omega |u|^p \, \mathrm{d}x + \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \, \mathrm{d}x \, \mathrm{d}y \right)^{1/p}.$$

The term

$$[u]_{W^{s,p}(\Omega)} := \left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \, \mathrm{d}x \, \mathrm{d}y \right)^{1/p}$$

is called the Gagliardo (semi)norm. If $p = 2$ we define

$$W^{s,2}(\Omega) := H^s(\Omega)$$

which is a Hilbert space.

This is an important space because it is related to the fractional Laplacian operator $(-\Delta)^s$:

Definition 2.1. We consider the Schwartz space of rapidly decaying functions defined as

$$\mathcal{S}(\mathbb{R}^N) := \left\{ f \in C^\infty(\mathbb{R}^N) : \forall\, \alpha, \beta \in \mathbb{N}_0^N, \ \sup_{x \in \mathbb{R}^N} |x^\alpha D^\beta f(x)| < \infty \right\}.$$

Taken $s \in (0,1)$, for any $u \in \mathcal{S}(\mathbb{R}^N)$, we define the *fractional laplacian of* u as

$$(-\Delta)^s u(x) := C(N,s) P.V. \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}} \, \mathrm{d}y \,, \tag{2.1.2}$$

where *P.V.* denotes the principal value, i.e.

$$(-\Delta)^s u(x) := C(N,s) \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{N+2s}} \, \mathrm{d}y \,,$$

where $B_\varepsilon(x)$ denotes a ball of radius ε and center $x \in \mathbb{R}^N$ and $C(N,s)$ is a dimensional constant depending on N and s given by

$$C(N,s) := \left(\int_{\mathbb{R}^N} \frac{1-\cos(\xi_1)}{|\xi|^{N+2s}} \, \mathrm{d}\xi \right)^{-1}. \tag{2.1.3}$$

As in the classical case, if $0 < s \le s' < 1$, the space $W^{s',p}$ is continuously embedded into $W^{s,p}$:

Proposition 2.2. *[35, Proposition 2.1]. Let $p \in [1,+\infty)$ and $0 < s \le s' < 1$. Let Ω be an open set in $\mathbb{R}^N$ and $u : \Omega \to \mathbb{R}$ be a measurable function. Then*

$$\|u\|_{W^{s,p}(\Omega)} \le C \|u\|_{W^{s',p}(\Omega)}$$

for suitable positive constant $C = C(N,s,p) \ge 1$. In particular

$$W^{s',p}(\Omega) \hookrightarrow W^{s,p}(\Omega).$$

Moreover, the space $W^{1,p}$ is continuously embedded in $W^{s,p}$:

Proposition 2.3. *[35, Proposition 2.2] Let $p \in [1,+\infty)$ and $s \in (0,1)$. Let Ω be an open set in $\mathbb{R}^N$ of class $C^{0,1}$ and $u : \Omega \to \mathbb{R}$ be a measurable function. Then*

$$\|u\|_{W^{s,p}(\Omega)} \le C \|u\|_{W^{1,p}(\Omega)}$$

for suitable positive constant $C = C(N,s,p) \ge 1$. In particular

$$W^{1,p}(\Omega) \hookrightarrow W^{s,p}(\Omega).$$

Definition 2.4. [35, Section 5] Let $s \in (0,1)$ and $p \in [1,+\infty)$. We say that an open set $\Omega \subseteq \mathbb{R}^N$ is an *extension domain* for $W^{s,p}$ if there exists $C = C(N,p,s,\Omega) > 0$ such that for every $u \in W^{s,p}(\Omega)$ there exists $\tilde{u} \in W^{s,p}(\mathbb{R}^N)$ with $\tilde{u}(x) = u(x)$ for all $x \in \Omega$ and $\|\tilde{u}\|_{W^{s,p}(\mathbb{R}^N)} \le C\|u\|_{W^{s,p}(\Omega)}$.

We point out that an arbitrary open set is not an extension domain for $W^{s,p}$, but any open set of class $C^{0,1}$ with bounded boundary it is.

If we have an extension domain, we have the following continuous embeddings (see also [35]):

Theorem 2.5. *[34, Theorem 4.53]. Let $s \in (0,1)$ and let $p \in (1,+\infty)$. Let $\Omega \subset \mathbb{R}^N$ be a $C^{0,1}$ set. We have:*

- *if $sp < N$, then $W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ for every $q \le Np/(N-sp)$;*
- *if $sp = N$, then $W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ for every $q < +\infty$;*
- *if $sp > N$, then $W^{s,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ and, more precisely,*

  $$W^{s,p}(\Omega) \hookrightarrow C_b^{0,s-N/p}(\Omega),$$

  where for $\lambda \in (0,1]$ we denote with $C_b^{0,\lambda}(\Omega)$ the space of bounded Hölder continuous functions of order λ on Ω.

As it concerns the compact embeddings we have this

Theorem 2.6. *[34, Theorem 4.54]. Let* $s \in [0,1)$*,* $p > 1$ *and* $N \geq 1$*. Let* $\Omega \subset \mathbb{R}^N$ *be a* $C^{0,1}$ *set.*

- *If* $sp < N$*, then the embedding of* $W^{s,p}(\Omega)$ *into* $L^k(\Omega)$ *is compact for every* $k < Np/(N - sp)$*;*
- *if* $sp = N$*, then the embedding of* $W^{s,p}(\Omega)$ *into* $L^k(\Omega)$ *is compact* $k < +\infty$*;*
- *if* $sp > N$*, then the embedding of* $W^{s,p}(\Omega)$ *in* $C_b^{0,\lambda}(\Omega)$ *is compact for every* $\lambda < s - N/p$*.*

When $s > 1$ and it is not integer we write $s = m + \sigma$, where m is an integer and $\sigma \in (0,1)$. In this case

$$W^{s,p}(\Omega) := \{u \in W^{m,p}(\Omega) : D^\alpha u \in W^{\sigma,p}(\Omega) \text{ for any } \alpha : |\alpha| = m\}.$$

This is a Banach space with respect to the norm

$$\|u\|_{W^{s,p}(\Omega)} := \left(\|u\|^p_{W^{m,p}(\Omega)} + \sum_{|\alpha|=m} \|D^\alpha u\|^p_{W^{\sigma,p}(\Omega)} \right)^{1/p}. \tag{2.1.4}$$

Obviously, if $s = m$ integer, the space $W^{s,p}(\Omega)$ coincides with the Sobolev space $W^{m,p}(\Omega)$.

For these spaces, embedding theorems similar to the previous ones hold, see [34, Theorem 4.57] and [34, Theorem 4.58].

2.2 Nonlocal Minimal Surfaces

In this section we introduce the nonlocal minimal surfaces (or s-minimal surfaces) that are boundaries of the minimizers of the fractional perimeter.

They appear in phenomena when the particles get farther and farther apart, faster than the interaction potential decaying. So two particles which belong to different phases and stay away from the interface give a nontrivial contribute to the total interaction energy.

2.2.1 The fractional Perimeter

The notion of fractional perimeter was introduced by Caffarelli, Roquejoffre and Savin in [22], where they were motivated by the structure of interphases that arise in classical phase field models when very long space correlations are present.

Definition 2.7. For $0 < s < 1/2$ the *fractional perimeter* (or s-perimeter) of a measurable set $E \subseteq \mathbb{R}^N$ is defined as

$$P_s(E) := \int_E \int_{E^C} \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^{N+2s}}, \tag{2.2.1}$$

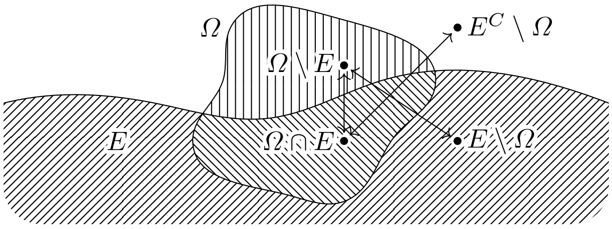


Figure 2.1: The interactions considered in the localized fractional perimeter.

where hereafter E^C will denote the complement of a set E. So, we say that a set $E \subset \mathbb{R}^N$ has finite s-perimeter if $P_s(E) < \infty$.

We point out that the fractional perimeter corresponds to the usual semi-norm of the characteristic function χ_E in the fractional Sobolev space $H^s(\mathbb{R}^N)$, that is

$$P_s(E) = \frac{1}{2}[\chi_E]^2_{H^s(\mathbb{R}^N)} = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\chi_E(x) - \chi_E(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y\,.$$

Moreover, by [6, Theorem 2], it is known that the fractional perimeter Γ-converges to De Giorgi's perimeter as $s \to 1/2$. Precisely, it holds

$$\Gamma - \lim_{s\uparrow 1/2}(1-2s)P_s(E) = \omega_{N-1}P(E), \tag{2.2.2}$$

where, here and in the following, ω_{N-1} denotes the $(N-1)$-dimensional measure of the unit sphere of $\mathbb{R}^{N-1}$.

The fractional perimeter can be localized to a bounded open set $\Omega \subseteq \mathbb{R}^N$ by taking away the contribution of points of E and E^C outside Ω, i.e.

$$P_s(E,\Omega) := \int_{E\cap\Omega}\int_{E^C} \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^{N+2s}} + \int_{E\cap\Omega^C}\int_{E^C\cap\Omega} \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^{N+2s}}, \tag{2.2.3}$$

where Ω^C denotes the complement of Ω.

Roughly speaking, the localized fractional perimeter represents the interaction of any point inside E with any point outside E where we "remove" possible infinite contributions to the energy which come from infinity (see Figure 2.1), since they do not contribute to the minimization.

Definition 2.8. We say that a set $E \subseteq \mathbb{R}^N$ is a *s-minimizer* for the fractional perimeter in Ω if

$$P_s(E,\Omega) \le P_s(F,\Omega) \tag{2.2.4}$$

for any measurable set F that coincides with E outside Ω, i.e. $F \setminus \Omega = E \setminus \Omega$.

The *boundaries* of s-minimizing sets are referred to as *nonlocal minimal surfaces.*

Remark 2.9. [22] The set $E \cap \Omega^C$ plays the role of 'boundary data' for $E \cap \Omega$. If $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain, $\inf P_s(\cdot,\Omega)$ is bounded by $P_s(E \setminus \Omega, \Omega) < \infty$.

The existence of these minimizer for the fractional perimeter is easily proved through the direct method of the Calculus of Variations. Indeed, the fractional perimeter is lower semicontinuous:

Proposition 2.10. *[22, Proposition 3.1]. If* $\chi_{E_n} \to \chi_E$ *in* L^1_{loc}, *then*

$$\liminf_{n\to+\infty} P_s(E_n, \Omega) \geq P_s(E, \Omega).$$

Hence, the following existence result holds:

Theorem 2.11. *[22, Theorem 3.2]. Let* $\Omega \subset \mathbb{R}^N$ *be a bounded Lipschitz domain,* $E_0 \subset \Omega^C$ *is a given set. There esists a set* E, *with* $E \cap \Omega^C = E_0$ *such that*

$$\inf_{F\cap\Omega^C=E_0} P_s(F, \Omega) = P_s(E, \Omega).$$

In [22] it is proved that s-minimizers satisfy a suitable integral equation (that is the Euler-Lagrange equation corresponding to the functional (2.2.3)). If E is a s-minimizer for P_s in Ω and ∂E is smooth enough, this equation results

$$\int_{\mathbb{R}^N} \frac{\chi_E(y) - \chi_{E^C}(y)}{|x-y|^{N+2s}} \,\mathrm{d}y = 0 \tag{2.2.5}$$

for any $x \in \Omega \cap \partial E$.

Hence, if $E \subseteq \mathbb{R}^N$ is an open set, in analogy with the classical minimal surfaces which have zero mean curvature, one defines the *nonlocal (or fractional) mean curvature*, briefly denoted with NMC, of ∂E at a point $x \in \partial E$ as

$$H_{s,\partial E}(x) := \int_{\mathbb{R}^N} \frac{\chi_E(y) - \chi_{E^C}(y)}{|x-y|^{N+2s}} \,\mathrm{d}y\,, \tag{2.2.6}$$

so that equation (2.2.5) can be written as $H_{s,\partial E}(x) = 0$.

We point out that the integral in (2.2.6) is understood in the principal value sense, hence defining

$$H^\delta_{s,\partial E}(x) := \int_{\mathbb{R}^N\setminus B_\delta(x)} \frac{\chi_E(y) - \chi_{E^C}(y)}{|x-y|^{N+2s}} \,\mathrm{d}y \tag{2.2.7}$$

we have

$$H_{s,\partial E} = \lim_{\delta\to 0} H^\delta_{s,\partial E}.$$

Note that, if $\partial E \in C^2$, the nonlocal mean curvature $H_{s,\partial E}$ is well-defined in a neighbourhood of x in the principal value sense and, in this case, it agrees with usual mean curvature in the limit as $s \to 1/2$ by the relation

$$\lim_{s\to 1/2} (1-2s) H_{s,\partial E} = \omega_{N-1} H_{\partial E},$$

where $H_{\partial E}$ denotes the classical mean curvature of ∂E, see [1, Theorem 12].

If E is smooth and compactly contained in Ω, let w be a smooth function defined on on ∂E, with small L^∞ norm. We call E_w the set whose boundary ∂E_w is parametrized by

$$\partial E_w = \{x + w(x)\nu_E(x) : \ x \in \partial E\} \tag{2.2.8}$$

where ν_E is a normal vector field to ∂E exterior to E.

The first variation of the fractional perimeter (2.2.3) along these normal perturbations is given by

$$\mathrm{d}_t P_s(E_{tw}, \Omega)_{|t=0} := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} P_s(E_{tw}, \Omega) = \int_{\partial E} (H_{s,\partial E}) w \, \mathrm{d}\sigma \,, \tag{2.2.9}$$

see [32], and this quantity vanishes for all such w if and only if

$$H_{s,\partial E}(x) = 0 \quad \text{for all } x \in \partial E \cap \Omega.$$

We point out that, besides (2.2.6), there are other ways to write the nonlocal mean curvature. For example, if $x \in \partial E$, setting $\tilde{\chi_E} := \chi_E - \chi_{E^C}$, we have

$$\begin{aligned} H_{s,\partial E}(x) &= \frac{1}{2} \int_{\mathbb{R}^N} \frac{\tilde{\chi_E}(x+y) - \tilde{\chi_E}(x-y)}{|y|^{N+2s}} \, \mathrm{d}y \\ &= \frac{1}{2} \int_{\mathbb{R}^N} \frac{\tilde{\chi_E}(x+y) - \tilde{\chi_E}(x-y) - 2\tilde{\chi_E}(x)}{|y|^{N+2s}} \, \mathrm{d}y \\ &= \frac{(-\Delta)^s \tilde{\chi_E}(x)}{C(N,s)}, \end{aligned} \tag{2.2.10}$$

where the first two integrals are understood in the principal value sense, $C(N,s)$ is defined in (2.1.3) and $(-\Delta)^s$ is the fractional Laplacian defined in (2.1.2). This representation is useful because it allows us to write the Euler-Lagrange equation as

$$(-\Delta)^s \tilde{\chi_E} = 0 \quad \text{along } \partial E.$$

Finally, as conclusion of this section, we recall the partial known results about the regularity theory of nonlocal minimal surfaces, (see [8] and [83]):

Theorem 2.12. *[16, Theorem 5.3] In the plane, s-minimizers are smooth, i.e.*

- *if E is a s-minimizer in $\Omega \subset \mathbb{R}^2$, then $\partial E \cap \Omega$ is a C^∞-curve.*
- *Let E be a s-minimizer in $\Omega \subset \mathbb{R}^N$, and let $\Sigma_E \subset \partial E \cap \Omega$ be its singular set. Then, denoting with $\mathcal{H}^d$ the d-dimensional Hausdorff measure, $\mathcal{H}^d(\Sigma_E) = 0$ for any $d > N-3$.*

Moreover, when s is close to 1/2, we have that

Theorem 2.13. *[16, Theorem 5.4] There exists $\varepsilon \in (0, 1/2)$ sucht that if $s \geq \frac{1}{2} - \varepsilon$, then*

- *if $N \leq 7$ any s-minimizer is of classe C^∞.*

- *If $N = 8$ any minimal surface is of class C^∞ except, at most, countably many isolated points.*
- *any s-minimal surface is of class C^∞ outside a closed set Σ of Hausdorff dimension $N - 8$.*

2.3 Classical and fractional Allen-Cahn equations

S. Allen and J. W. Cahn in the 1970s introduced the well-known *Allen-Cahn equation*

$$-\Delta u = u - u^3 \quad \text{in } \Omega \subseteq \mathbb{R}^N, \tag{2.3.1}$$

which describes a phase coexistence model, where u is the phase of the medium at $x \in \Omega$ and Ω represents the container.

It is easy to see that equation (2.3.1) has a variational structure, so its solutions can be found as critical points of the energy functional

$$\mathcal{I}_\Omega(u) := \frac{1}{2}\int_\Omega |\nabla u(x)|^2 \,\mathrm{d}x + \int_\Omega W(u(x))\,\mathrm{d}x\,, \tag{2.3.2}$$

where $W(r) := \frac{(1-r^2)^2}{4}$ is the well known double-well potential.

The first term of $\mathcal{I}_\Omega$ is an interfacial energy which prevents phase changes from point to point and 'wild' phase oscillations; the second term penalizes considerable deviations from the 'pure phases' ±1.

In the last thirty years of the 20^{th} century a lot of results about the Allen-Cahn equation are obtained: a Γ-convergence result (see [66]), energy and density estimates (see [23]), and locally uniform convergence of level sets (see [23]) are shown.

Recently with the growth of interest for fractional operators, a lot of mathematicians addressed their attention to the *fractional* counterpart of the *Allen-Cahn equation*, i.e.

$$(-\Delta)^s u = u - u^3 \quad \text{in } \Omega \subseteq \mathbb{R}^N, \tag{2.3.3}$$

where $s \in (0,1)$ and $(-\Delta)^s$ that is the fractional Laplacian introduced in (2.1.2). This model, different from the classical one, deals with longe-range interactions which can influence the coexistence of the two 'phases' introducing new phenomena.

However, as its classical counterpart, equation (2.3.3) has variational structure. In this case, up to scaling constants omitted for simplicity, the energy associated to the fractional Allen-Cahn equation is

$$\mathcal{I}_{s,\Omega}(u) := \frac{C_{N,s}}{4}\iint_{\mathcal{C}_\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y + \int_\Omega W(u(x))\,\mathrm{d}x\,, \tag{2.3.4}$$

where $C_{N,s}$ is defined in (2.1.3) and

$$\mathcal{C}_\Omega := (\Omega \times \Omega) \,\cup\, \big(\Omega \times (\mathbb{R}^N \setminus \Omega)\big) \,\cup\, \big((\mathbb{R}^N \setminus \Omega) \times \Omega\big). \tag{2.3.5}$$

It is interesting to observe that the interface term of (2.3.2) considers the points in Ω which can be regarded as the complement in $\mathbb{R}^N$ of the 'inactive' set $\mathbb{R}^N \setminus \Omega$, while in (2.3.3), $\mathcal{C}_\Omega$ collects the couples $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$ such that at least one of the points belongs to Ω (and hence $\mathcal{C}_\Omega$ takes into account the 'inactive' couples of points in $(\mathbb{R}^N \setminus \Omega) \times (\mathbb{R}^N \setminus \Omega)$).

Following [38], we recall some interesting results, previously analyzed for equation (2.3.1), obtained for the fractional Allen-Cahn equation (2.3.3).

For $\varepsilon > 0$ and $s \in (0,1)$, we define the functional $\mathcal{I}_{s,\Omega,\varepsilon} : H^s(\Omega) \to \mathbb{R}$ as

$$\mathcal{I}_{s,\Omega,\varepsilon}(u) := \begin{cases} \displaystyle\iint_{\mathcal{C}_\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{\varepsilon^{2s}}\int_\Omega W(u(x))\,\mathrm{d}x & \text{if } s \in \left(0, \tfrac{1}{2}\right), \\ \displaystyle\frac{1}{|\log\varepsilon|}\iint_{\mathcal{C}_\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{\varepsilon|\log\varepsilon|}\int_\Omega W(u(x))\,\mathrm{d}x & \text{if } s = \tfrac{1}{2}, \\ \displaystyle\varepsilon^{2s-1}\iint_{\mathcal{C}_\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{\varepsilon}\int_\Omega W(u(x))\,\mathrm{d}x & \text{if } s \in \left(\tfrac{1}{2}, 1\right), \end{cases} \tag{2.3.6}$$

that is the rescaled functional obtained from the use of the blow-down sequence

$$u_\varepsilon(x) := u\Big(\frac{x}{\varepsilon}\Big) \qquad \text{for } \varepsilon \to 0$$

in (2.3.4).

The Γ-convergence result for the fractional Allen-Cahn energy is the following:

Theorem 2.14. *[82, Theorem 1.5] If $\Omega \subseteq \mathbb{R}^N$ is a smooth domain and $u_\varepsilon : \Omega \to [-1, 1]$ is a sequence of minimizers for $\mathcal{I}_{s,\Omega,\varepsilon}$ such that*

$$\sup_{\varepsilon\in(0,1)} \mathcal{I}_{s,\Omega,\varepsilon}(u_\varepsilon) < +\infty,$$

then, up to a subsequence,

$$\lim_{\varepsilon\to 0} u_\varepsilon = u_0 := \chi_E - \chi_{E^C} \qquad \text{in } L^1(\Omega), \tag{2.3.7}$$

for some set $E \subseteq \mathbb{R}^N$.

If $s \in (0, 1/2)$ and u_ε converges weakly to u_0 in $\mathbb{R}^N \setminus \Omega$, then the set E minimizes the fractional perimeter P_s in Ω with respect to its datum in $\mathbb{R}^N \setminus \Omega$.

If $s \in [1/2, 1)$, the set E minimizes the perimeter in $\overline{\Omega}$ with respect to its boundary datum.

It is important to highlight that this theorem represents the nonlocal analogue of the classical Γ-convergence result proved in [66] with a fundamental difference: the same limit (2.3.7) holds but, depending on the parameter s, the limit set E has different features. Moreover, as remarked in [38], the Γ-convergence results stated in Theorem 2.14 are easier in the case $s \in (0, 1/2)$ since characteristic functions are admissible competitors with finite energy. Contrarily, if $s \in [1/2, 1)$, the proof is more difficult because it needs to reconstruct a local energy from all the nonlocal contributions.

As it concerns the fractional counterpart of the energy and density estimates we have this

Theorem 2.15. *[84, Theorem 1.3 and Theorem 1.4] Let $R \geq 1$ and B_R be the ball of radius R centered at the origin. If $u : B_{R+1} \to [-1,1]$ is a minimizer of $\mathcal{I}_{s,B_{R+1}}$ then*

$$\mathcal{I}_{s,B_{R+1}}(u) \leq \begin{cases} CR^{N-2s} & \text{if } s \in \left(0, \frac{1}{2}\right), \\ CR^{N-1}\log R & \text{if } s = \frac{1}{2}, \\ CR^{N-1} & \text{if } s \in \left(\frac{1}{2}, 1\right), \end{cases} \tag{2.3.8}$$

for some $C > 0$.

In addition, if $u(0) = 0$, the Lebesgue measure of $\{u > 1/2\}$ and $\{u < -1/2\}$ in B_R are both greater than cR^N for some $c > 0$.

It is interesting to note that, as in the classical case, the energy bound is influenced by the parameter s in the same way: for s large the estimate does not depends on s, while for s small the energy contributions coming from infinity add energy in a large ball.

Moreover we observe that the constants in Theorem 2.15 can depend on N and s and they are weaker than the constant of the classical case. However the estimates in Theorem 2.15 allow us to have this

Corollary 2.16. *[84, Corollary 1.7] If $\Omega \subseteq \mathbb{R}^N$ is a smooth domain, $E \subseteq \mathbb{R}^N$ and $u_\varepsilon : \Omega \to [-1,1]$ is a minimizer of $\mathcal{I}_{s,\Omega,\varepsilon}$ such that (2.3.7) holds true, then the set $\{|u_\varepsilon| \leq 1/2\}$ converges locally uniformly in Ω to ∂E as $\varepsilon \to 0$.*

2.3.1 De Giorgi's conjecture

Although we will not deal this topic in this thesis, we briefly discuss an important problem related to the Allen-Cahn equation: the well known *De Giorgi's conjecture.*

In 1979 De Giorgi conjectured the following

Conjecture 2.17. *[33] Let $u : \mathbb{R}^N \to [-1,1]$ be a solution of the Allen-Cahn equation (2.3.1) in the whole of $\mathbb{R}^N$ such that*

$$\frac{\partial u}{\partial x_N}(x) > 0 \quad \text{for all } x \in \mathbb{R}^N. \tag{2.3.9}$$

Is it true that u is $1D$ that is, denoting with S^{N-1} the $(N-1)$-dimensional sphere of $\mathbb{R}^N$, $u(x) = u_0(\omega \cdot x)$ for some $u_0 : \mathbb{R} \to \mathbb{R}$ and $\omega \in S^{N-1}$, at least for $N \leq 8$?

This conjecture was proved for $N = 2, 3$ (see [5, 51]) while the cases $N = 4, 8$ are still open. For $N = 4, \ldots, 8$ the conjecture was shown in [79] with the limit assumption

$$\lim_{x_N \to \pm\infty} u(x', x_N) = \pm 1. \tag{2.3.10}$$

A variant of the conjecture (known as Gibbons conjecture) with (2.3.9) replaced by a uniform limit assumption at infinity was showed independently in [44, 7, 9]. Moreover a variational variant with (2.3.9) replaced by a minimality assumption was proved in [79] when $N \leq 7$. The case $N = 8$ is still open, while a counterexample was given for $N \geq 9$ in [75].

In the fractional setting, Conjecture 2.17 is proved only in some cases:

Theorem 2.18. *Let* $u : \mathbb{R}^N \to [-1, 1]$ *be a solution of the fractional Allen-Cahn equation* (2.3.3) *in the whole of* $\mathbb{R}^N$ *such that*

$$\frac{\partial u}{\partial x_N}(x) > 0 \quad \textit{for all } x \in \mathbb{R}^N.$$

Suppose that either

$$N \leq 3 \qquad \textit{and} \qquad s \in (0, 1),$$

or

$$N = 4 \qquad \textit{and} \qquad s = \frac{1}{2}.$$

Then u *is* $1D$.

This theorem was proved in [21] when $N = 2$ and $s = \frac{1}{2}$, in [79, 20] when $N = 2$ and $s \in (0, 1)$, in [17] when $N = 3$ and $s = \frac{1}{2}$, in [18] when $N = 3$ and $s \in (\frac{1}{2}, 1)$, in [36] when $N = 3$ and $s \in (0, \frac{1}{2})$ and in [47] when $N = 4$ and $s = \frac{1}{2}$. For $N \geq 9$ and $s \in (\frac{1}{2}, 1)$ a counterexample to the validity of Theorem 2.18 was exibithed in [28]. In the other cases the problem is open (in higher dimensions Theorem 2.18 is proved with the additional limit assumption (2.3.10) by a collage of [80, 81, 37]).

By a superposition of the results in the same papers [80, 81, 37], one can prove the existence of $\varepsilon_0 \in (0, 1/2]$ such that the fractional variational version of De Giorgi's Conjecture was proved when $N \leq 7$ and $s \in (\frac{1}{2} - \varepsilon_0, 1)$ with the assumption (2.3.9) replaced by a minimality assumption.

Finally we mention an interesting result which holds for the fractional Allen-Cahn equation, but it is not true for the classical Allen-Cahn equation (see [75, Theorem 1]), revealing a purely nonlocal phenomenon:

Theorem 2.19. *[37] Let* $s \in (0, \frac{1}{2})$ *and* u *be a solution of* (2.3.3) *in* $\mathbb{R}^N$. *Then* u *is* $1D$.

As highlighted in [38], this theorem tell us that if we have a phase coexistence in this framework and we plug more energy into the system, then two situations can occur:

a) the two interfaces oscillate significantly at infinity (and hence the flatness assumption of Theorem 2.19 does not hold);

b) the graph of u can oscillate but, since from Theorem 2.19 u has to be $1D$, the phase separation occurrs along parallel hyperplanes with possible multiplicity.

Thanks to [28, Theorem 1.3] we know that Theorem 2.19 is false when $s \in (1/2, 1)$, while the case $s = 1/2$ is still open.

2.4 Introduction to the Lyapunov-Schmidt reduction

In this section we introduce the setting which we will use to apply the Lyapunov-Schmidt reduction, i.e. a tool that allows us to study a class of problems with a small (or large) parameter and variational structure.

If we denote with $B_1(\xi)$ a ball with center $\xi \in \mathbb{R}^N$ and unit radius and we take $w \in C^1(\partial B_1(\xi))$, we will write $\mathbb{B}(\xi, w)$ to indicate the set such that

$$\partial \mathbb{B}(\xi, w) := \{y \in \mathbb{R}^N : y = x + w(x)\nu_{B_1(\xi)}(x), x \in \partial B_1(\xi)\}, \tag{2.4.1}$$

where $\nu_{B_1(\xi)}$ is the outer unit normal to $\partial B_1(\xi)$. Then, if $\Omega \subseteq \mathbb{R}^N$ is an open and bounded set, we consider the fractional perimeter of a measurable set $E \subset \mathbb{R}^N$ in Ω as the interaction between E and its complement inside Ω only, i.e.

$$\bar{P}_s(E, \Omega) := \int_E \int_{\Omega \setminus E} \frac{\mathrm{d}x \, \mathrm{d}y}{|x-y|^{N+2s}}, \tag{2.4.2}$$

where $s \in (0, 1/2)$. In analogy with (2.2.6), we define the nonlocal mean curvature (in Ω) of ∂E at $x \in \partial E$ corresponding to (2.4.2) as

$$H^{\Omega}_{s,\partial E}(x) := \int_{\Omega} \frac{\chi_E(y) - \chi_{E^c \cap \Omega}(y)}{|x-y|^{N+2s}} \, \mathrm{d}y \, , \tag{2.4.3}$$

(see [61, Theorem 1.3 and Proposition 3.2 with $\sigma = 0$ and $g = 0$]) where, as usual, χ_E denotes the characteristic function of E, E^C is the complement of E, and the integral has to be understood in the principal value sense.

We also set

$$S_\xi := \partial B_1(\xi) \qquad \text{and} \qquad P^{\Omega}_{s,\xi}(w) := \bar{P}_s(\mathbb{B}(\xi, w), \Omega) \tag{2.4.4}$$

then, for $\beta \in (2s, 1)$ and $\varphi \in C^{1,\beta}(\partial \mathbb{B}(\xi, w))$, we set

$$\left(P^{\Omega}_{s,\xi}\right)'(w)[\varphi] := \int_{\partial \mathbb{B}(\xi,w)} H^{\Omega}_{s,\partial \mathbb{B}(\xi,w)} \varphi \, \mathrm{d}\sigma_w \tag{2.4.5}$$

where $d\sigma_w$ stands for the area element of $\partial \mathbb{B}(\xi, w(\xi))$.

Consider next the *spherical fractional Laplacian*

$$L_s \varphi(\theta) := P.V. \int_S \frac{\varphi(\theta) - \varphi(\sigma)}{|\theta - \sigma|^{N+2s}} \, \mathrm{d}\sigma \, ,$$

where $S := S^{N-1} = \partial B_1$ and $P.V.$ denotes the principal value.

It turns out that (see [19])

$$L_s : C^{1,\beta}(S) \to C^{\beta - s}(S). \tag{2.4.6}$$

The operator L_s has an increasing sequence of eigenvalues $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$ whose explicit expression is given by

$$\lambda_k := \frac{\pi^{(N-1)/2}\Gamma((1-2s)/2}{(1+2s)2^{2s}\Gamma((N+2s)/2)} \left(\frac{\Gamma\left(\frac{2k+N+2s}{2}\right)}{\Gamma\left(\frac{2k+N-2s-2}{2}\right)} - \frac{\Gamma\left(\frac{N+2s}{2}\right)}{\Gamma\left(\frac{N-2s-2}{2}\right)} \right), \tag{2.4.7}$$

see [78, Lemma 6.26], where Γ is the Euler Gamma function. The eigenfunctions are the usual spherical harmonics, i.e. one has

$$L_s\psi = \lambda_k\psi \qquad \text{for every } k \in \mathbb{N} \text{ and } \psi \in \mathcal{E}_k,$$

where $\mathcal{E}_k$ is the space of spherical harmonics of degree k and dimension $n_k := N_k - N_{k-2}$, with

$$N_k := \frac{(n+k-1)!}{(n-1)!k!} \quad \text{for } k \geq 0 \quad \text{and} \quad N_k = 0 \quad \text{for } k < 0.$$

We recall that $n_0 = 1$ and that $\mathcal{E}_0$ consists of constant functions, whereas $n_1 = N$ and $\mathcal{E}_1$ is spanned by the restrictions of the coordinate functions in $\mathbb{R}^N$ to the unit sphere S.

For sets that are suitable graphs over the unit sphere S of $\mathbb{R}^N$, we have the following result concerning nonlocal mean curvature relative to the whole space, see [19, Theorem 2.1, Lemma 5.1 and Theorem 5.2].

Proposition 2.20. *Given $\beta \in (2s, 1)$ we consider the family of functions*

$$\Upsilon := \left\{ \varphi \in C^{1,\beta}(S) \ : \ \|\varphi\|_{L^\infty(S)} < \frac{1}{2} \right\}.$$

Then the map $\varphi \mapsto H_{s,\partial\mathbb{B}(0,\varphi)}$ is a C^∞ function from Υ into $C^{\beta-2s}(S)$. Moreover, its linearization at $\varphi \equiv 0$ is given by

$$\varphi \longmapsto 2d_{N,s}(L_s - \lambda_1)\varphi, \tag{2.4.8}$$

where λ_1 is defined in (2.4.7) *and $d_{N,s} := \frac{1-2s}{(N-1)|B_1^{N-1}|}$ with B_1^{N-1} that is the unit ball in $\mathbb{R}^{N-1}$.*

Accordingly we have than every function in the kernel of the above linearized nonlocal mean curvature is a linear combination of first-order spherical harmonics, i.e. if $w \in \text{Ker}\,(L_s - \lambda_1)$, we have

$$w = \sum_{i=1}^{N} \lambda_i Y_i, \tag{2.4.9}$$

where $\{Y_i\}_{i=1,\cdots,N} \in \mathcal{E}_1$ and $\lambda_i \in \mathbb{R}$. Therefore, defining

$$W := \left\{ w \in C^{1,\beta}(S_\xi) : \int_{S_\xi} w\, Y_i = 0 \text{ for } i = 1, \ldots, N, \right\}, \tag{2.4.10}$$

it follows by Fredholm's theory that $L_s - \lambda_1$ is invertible on W.

As a consequence of the above proposition, using a perturbation argument, we deduce also the following result, for which we need to introduce some notation. Let Ω be a bounded set in $\mathbb{R}^N$. For $\varepsilon > 0$ we denote $\Omega_\varepsilon := \frac{1}{\varepsilon}\Omega$. Fix a compact set Θ in Ω, and let $\xi \in \frac{1}{\varepsilon}\Theta$. Then we consider the operator $L^{\Omega_\varepsilon}_{s,\xi}$ corresponding to the linearization of the nonlocal mean curvature at $B_1(\xi)$ relative to Ω_ε (defined as in (2.4.3)), namely the nonlocal operator such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} H^{\Omega_\varepsilon}_{s,\partial\mathbb{B}(\xi,t\varphi)} = (L^{\Omega_\varepsilon}_{s,\xi}\varphi).$$

We have the following result:

Proposition 2.21. *Let Ω, Θ, ξ and $L^{\Omega_\varepsilon}_{s,\xi}$ be as above, and let $\beta \in (2s,1)$. Consider the family of functions*

$$\Upsilon := \left\{\varphi \in C^{1,\beta}(S_\xi) \ : \ \|\varphi\|_{L^\infty(S_\xi)} < \frac{1}{2}\right\}.$$

Then the map $\varphi \mapsto H^{\Omega_\varepsilon}_{s,\partial\mathbb{B}(\xi,\varphi)}$ is a C^∞ function from Υ into $C^{\beta-2s}(S_\xi)$. Moreover, if W is as in (2.4.10)*, $L^{\Omega_\varepsilon}_{s,\xi}$ is invertible with uniformly bounded inverse on W.*

2.5 Genus and category of a set

In this last section we follow [4] to discuss briefly a theory introduced by Lusternik-Schnirelmann to deduce multiplicity results for critical points of a functional defined on a manifold M in connection with the topological properties of M. The main ingredient of this theory is a topological tool, called the Lusternik-Schnirelmann (or L-S) category.

Let M be a topological space.

Definition 2.22. [4, Definition 9.2] *The category of a set* $A \subseteq M$ with respect to M, denoted by $\mathrm{cat}_M(A)$, is the least integer k such that $A \subseteq A_1 \cup \cdots \cup A_k$ with A_i closed and contractible in M for every $i = 1, \cdots, k$.

We set $\mathrm{cat}(\varnothing) = 0$ and $\mathrm{cat}_M(A) = +\infty$ if there are no integers with the above property. We will use the notation $\mathrm{cat}(M)$ for $\mathrm{cat}_M(M)$ and $\bar{A}$ to indicate the topological closure of the set A.

Remark 2.23. From the previous definition, it is easy to see that $\mathrm{cat}_M(A) = \mathrm{cat}_M(\bar{A})$. Moreover, if $A \subset B \subset M$, we have that $\mathrm{cat}_M(A) \leq \mathrm{cat}_M(B)$, see [4, Lemma 9.6].

Then, assuming that

$$M = F^{-1}(0), \text{ where } F \in C^{1,1}(E,\mathbb{R}) \text{ with } E \supset M \quad \text{and} \quad F'(u) \neq 0 \ \forall\, u \in M, \tag{2.5.1}$$

we set

$$\mathrm{cat}_k(M) = \sup\{\mathrm{cat}_M(A) : A \subset M \text{ and } A \text{ is compact}\}.$$

Note that if M is compact, $\mathrm{cat}_k(M) = \mathrm{cat}(M)$.

We also recall the definition of the Palais-Smale condition (or (PS)-condition).

Definition 2.24. Let H be a Hilbert space and $J \in C^1(H)$. Every subsequence $\{u_n\}_{n\in\mathbb{N}}$ such that

$$\{J(u_n)\} \text{ is bounded} \quad \text{and} \quad J'(u_n) \to 0 \text{ in } H^{-1}(\Omega) \tag{2.5.2}$$

is relatively compact. If a sequence satisfies (2.5.2), it is called Palais-Smale sequence (or (PS)-sequence).

With this setting at hand, we can state an important result about the Lusternik-Schnirelman category:

Theorem 2.25. *[4, Theorem 9.10] Let* (2.5.1) *holds, let $J \in C^{1,1}(E,\mathbb{R})$ be bounded from below on M and let J satisfy (PS)-condition. Then J has at least $cat_k(M)$ critical points on M.*

Remark 2.26. If M has boundary, under the same assumptions of Theorem 2.25, one can still find at least $\mathrm{cat}_k(M)$ critical points for J provided ∇J is non zero on ∂M and points in the outward direction.

Actually, this interesting theory does not give any new result when M is the unit sphere S in a infinite dimensional Hilbert space because $\mathrm{cat}(S) = 1$. So it is useful to introduce another topological tool which will substitute the category in the sense of even simmetry:

Definition 2.27. [4, Definition 10.1] Let H be a Hilbert space and E be a closed subset of $H \setminus \{0\}$, symmetric with respect to 0 (i.e. $E = -E$).

We call *genus* of E in H, indicated with $\mathrm{gen}_H(E)$, the least integer m such that there exists $\phi \in C(H;\mathbb{R}^m)$ such that ϕ is odd and $\phi(x) \neq 0$ for all $x \in E$.

We set $\mathrm{gen}_H(E) = +\infty$ if there are no integer with the above property and $\mathrm{gen}_H(\emptyset) = 0$.

We recall that, if S^N is a N-dimensional sphere of H with centre in zero, it results $\mathrm{gen}_H(S^N) = N + 1$.

A remarkable result about the genus is the following:

Theorem 2.28. *[4, Proposition 10.8] Let H be a Hilbert space and $f : H \to \mathbb{R}$ be an even C^2-functional satisfying the (PS)-condition.*

Set $f^c := \{u \in H : f(u) \leq c\}$ for all $c \in \mathbb{R}$. Then, for all c_1, $c_2 \in \mathbb{R}$, such that $c_1 \leq c_2 < f(0)$, we have

$$\mathrm{gen}_H(f^{c_2}) \leq \mathrm{gen}_H(f^{c_1}) + \#\{(-u_i, u_i) : c_1 \leq f(u_i) \leq c_2,\, f'(u_i) = 0\},$$

where, if A is a set, we indicate with $\#A$ the cardinality of A.

3 Multiplicity of critical points for the fractional Allen-Cahn energy

In this chapter we present an existence and multiplicity result for critical points of the functional

$$F_\varepsilon(u) := \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{\varepsilon^{2s}}\int_\Omega W(u)\,\mathrm{d}x, & \text{if } s\in(0,1/2),\\ \dfrac{1}{2}\dfrac{1}{|\log\varepsilon|}\displaystyle\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+1}}\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{|\varepsilon\log\varepsilon|}\int_\Omega W(u)\,\mathrm{d}x, & \text{if } s=1/2,\\ \dfrac{\varepsilon^{2s-1}}{2}\displaystyle\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{\varepsilon}\int_\Omega W(u)\,\mathrm{d}x, & \text{if } s\in(1/2,1), \end{cases} \tag{3.0.1}$$

where $\Omega\subset\mathbb{R}^N$ is a smooth and bounded domain, $u\in H^s(\Omega)$, and $\varepsilon\in\mathbb{R}^+$.

The map $W:\mathbb{R}\to\mathbb{R}^+$ is the standard double-well potential, i.e. an even function such that

$$\begin{gathered} W\in C^2(\mathbb{R};\mathbb{R}^+),\quad W(\pm1)=0,\quad W>0 \text{ in } (-1,1),\\ W'(\pm1)=0,\quad W''(\pm1)>0. \end{gathered} \tag{3.0.2}$$

Hence, F_ε is the contribution in Ω of the energy associated to the fractional Allen-Cahn equation. It is the fractional counterpart of the functional studied by Modica and Mortola in [66], where they proved the Γ-convergence of their energy to De Giorgi's perimeter. An analogous result of Γ-convergence for a functional as (3.0.1) is discussed by Valdinoci-Savin in [82].

Passaseo studied in [73] the classical analogue of our functional, i.e.

$$f_\varepsilon(u) = \varepsilon\int_\Omega |Du|^2\,\mathrm{d}x + \frac{1}{\varepsilon}\int_\Omega G(u)\,\mathrm{d}x \tag{3.0.3}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded domain, $u\in H^1(\Omega)$, ε is a positive parameter and $G\in C^2(\mathbb{R};\mathbb{R}^+)$ is a nonnegative function with two zeros, α and β. He proved that the number of critical points for f_ε goes to ∞ as $\varepsilon\to0$.

Our goal is to extend Passaseo's result to the fractional counterpart given by F_ε. In particular we want to show the following

Theorem 3.1. *Let $\Omega\subset\mathbb{R}^N$ be a smooth bounded domain and W be a function satisfying* (3.0.2). *Then there exist two sequences of positive numbers $\{\varepsilon_k\}_{k\in\mathbb{N}}$, $\{c_k\}_{k\in\mathbb{N}}$ such that, for every $\varepsilon\in(0,\varepsilon_k)$, the functional F_ε has at least k pairs*

$$(-u_{1,\varepsilon},u_{1,\varepsilon}),\ldots,(-u_{k,\varepsilon},u_{k,\varepsilon})$$

of critical points, all of them different from the constant pair $(-1, 1)$ *satisfying*

$$\begin{gathered} -1 \leq u_{i,\varepsilon}(x) \leq 1 \quad \forall\, x \in \Omega,\ \forall\, \varepsilon \in (0, \varepsilon_k),\ i = 1, \dots k; \\ F_\varepsilon(u_{i,\varepsilon}) \leq c_k \quad \forall\, \varepsilon \in (0, \varepsilon_k),\ i = 1, \dots, k. \end{gathered}$$

Moreover, for every $\varepsilon \in (0, \varepsilon_k)$ *and* $i = 1, \dots, k$ *we have*

$$F_\varepsilon(u_{i,\varepsilon}) \geq \min\left\{ F_\varepsilon(u) : u \in H^s(\Omega), -1 \leq u(x) \leq 1 \text{ for } x \in \Omega, \int_\Omega u \, \mathrm{d}x = 0 \right\}. \quad (3.0.4)$$

First of all we observe that critical points of Theorem 3.1 do not include constant functions:

Remark 3.2. Notice that for every $\varepsilon > 0$, the function $u \equiv 0$ is obviously a critical point for the functional F_ε, but it is not included among the ones given by Theorem 3.1. Indeed if $s \in (1/2, 1)$, but for the other cases it is similar, we have

$$F_\varepsilon(0) = \frac{1}{\varepsilon} W(0) |\Omega| \to +\infty \quad \text{as } \varepsilon \to 0.$$

Moreover since $\inf\{W(t) : W'(t) = 0, -1 < t < 1\} > 0$, one can deduce that the critical points given by Theorem 3.1 are not constant functions. Indeed, if $u_\varepsilon = c_\varepsilon$ is a constant critical point for F_ε (distinct from ±1), it must be $W'(c_\varepsilon) = 0$ and $-1 < c_\varepsilon < 1$. Therefore

$$W(c_\varepsilon) \geq \inf\{W(t) : W'(t) = 0, -1 < t < 1\} > 0 \quad (3.0.5)$$

and thus, considering the functional related to $s \in (1/2, 1)$, but the other cases are similar, we would get

$$F_\varepsilon(c_\varepsilon) = \frac{1}{\varepsilon} W(c_\varepsilon) |\Omega| \to +\infty \quad \text{as } \varepsilon \to 0, \quad (3.0.6)$$

in contradiction with $F_\varepsilon(c_\varepsilon) \leq c_k$ for all $\varepsilon \in (0, \varepsilon_k)$.

Remark 3.3. Supposing, without loss of generality, that Ω is a connected domain, for all $\varepsilon > 0$ it results

$$\min\left\{ F_\varepsilon(u) : u \in H^s(\Omega), -1 \leq u(x) \leq 1 \quad \forall x \in \Omega, \int_\Omega u \, \mathrm{d}x = 0 \right\} > 0. \quad (3.0.7)$$

Indeed, let $\bar{u}$ be a minimizing function and let us assume $F_\varepsilon(\bar{u}) = 0$. Recalling the definition of F_ε it has to be

$$\int_\Omega \int_\Omega \frac{|\bar{u}(x) - \bar{u}(y)|^2}{|x - y|^{N+2s}} \, \mathrm{d}x \, \mathrm{d}y \equiv 0 \quad \text{and} \quad W(\bar{u}) \equiv 0. \quad (3.0.8)$$

From the first equality and the fact that $\int_\Omega \bar{u} \, \mathrm{d}x = 0$ it follows that $\bar{u} \equiv 0$, but this contradicts the second equality in (3.0.8) since $W(0) > 0$.

3.1 Estimate from above of F_ε

To prove Theorem 3.1 we need to introduce some notation and a preliminary result which allow us to obtain a bound from above of the functional F_ε.

Definition 3.4. Fixed $k > 0$ integer, for every $\lambda = (\lambda^{(0)}, \dots, \lambda^{(k)}) \in \mathbb{R}^{k+1}$ we define the function $\varphi_\lambda : \mathbb{R} \to \mathbb{R}$ as

$$\varphi_\lambda(t) := \sum_{m=0}^{k} \lambda^{(m)} \cos(mt).$$

For every $\lambda \in \mathbb{R}^{k+1}$ with $|\lambda|_{\mathbb{R}^{k+1}} = 1$ and $\varepsilon > 0$, let $L_\varepsilon(\varphi_\lambda) : \mathbb{R} \to \mathbb{R}$ be the function given by

$$L_\varepsilon(\varphi_\lambda)(t) := \frac{1}{2\varepsilon} \int_{t-\varepsilon}^{t+\varepsilon} \frac{\varphi_\lambda(\tau)}{|\varphi_\lambda(\tau)|} \, d\tau.$$

Note that $L_\varepsilon(\varphi_\lambda)$ is well defined because for all $\lambda \in \mathbb{R}^{k+1}$ with $|\lambda|_{\mathbb{R}^{k+1}} = 1$ the function φ_λ has only isolated zeros.

Now, for $x = (x_1, \cdots, x_N) \in \Omega \subset \mathbb{R}^N$, we denote by P_1 the projection onto the first component, and we set

$$S_\varepsilon^k := \{L_\varepsilon(\varphi_\lambda) \circ P_1 : \lambda \in \mathbb{R}^{k+1}, |\lambda|_{\mathbb{R}^{k+1}} = 1\}.$$

Lemma 3.5. *[73, Lemma 2.4] Let us fix $a, b \in \mathbb{R}$ with $a < b$ and consider*

$$\chi(\varphi_\lambda) := \#\{t \in [a,b] : \varphi_\lambda(t) = 0\}$$

for $\lambda \in \mathbb{R}^{k+1}$ with $|\lambda|_{\mathbb{R}^{k+1}} = 1$. Then, for every $k \in \mathbb{N}$, we have

$$\sup\{\chi(\varphi_\lambda) : \lambda \in \mathbb{R}^{k+1}, |\lambda|_{\mathbb{R}^{k+1}} = 1\} < +\infty.$$

Lemma 3.6. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and W be a function satisfying* (3.0.2). *Then, for every $k \in \mathbb{N}$ there exists a constant $c_k > 0$ such that*

$$F_\varepsilon(f) \le c_k \quad \forall\, f \in S_\varepsilon^k. \tag{3.1.1}$$

Proof. Let $u_{\lambda,\varepsilon} := L_\varepsilon(\varphi_\lambda) \circ P_1 \in S_\varepsilon^k$ and call

$$\begin{gathered} a := \inf P_1(\Omega), \quad b := \sup P_1(\Omega), \\ Z_\lambda := \{t \in [a,b] : \varphi_\lambda(t) = 0\}, \\ Z_{\lambda,\varepsilon} := \{t \in \mathbb{R} : \operatorname{dist}(t, Z_\lambda) < \varepsilon\}. \end{gathered}$$

Note that, for $x \in \Omega$,

(i) if $P_1(x) \notin Z_{\lambda,\varepsilon}$, then $|u_{\lambda,\varepsilon}(x)| = 1$ and $Du_{\lambda,\varepsilon}(x) = 0$;

(ii) if $P_1(x) \in Z_{\lambda,\varepsilon}$, then $|u_{\lambda,\varepsilon}(x)| \le 1$ and $|Du_{\lambda,\varepsilon}(x)| \le \frac{1}{\varepsilon}$.

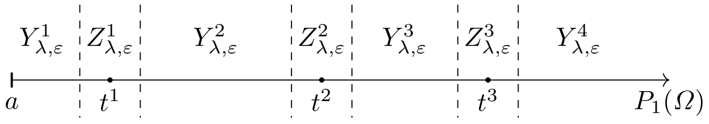


Figure 3.1: The partition of $P_1(\varOmega)$.

We want to evaluate $F_\varepsilon(u_{\lambda,\varepsilon})$, analyzing the contributions given by two terms of the functional.

Since $\varOmega$ is bounded, we can suppose that it is included in a cube Q of side large enough. Then, denoting with $Y_{\lambda,\varepsilon} := Z^C_{\lambda,\varepsilon}$ the complement of $Z_{\lambda,\varepsilon}$, for x, $y \in \varOmega$, we have three cases:

(a) $P_1(x) \in Y_{\lambda,\varepsilon}$ and $P_1(y) \in Y_{\lambda,\varepsilon}$;

(b) $P_1(x) \in Z_{\lambda,\varepsilon}$ and $P_1(y) \in Y_{\lambda,\varepsilon}$;

(c) $P_1(x) \in Z_{\lambda,\varepsilon}$ and $P_1(y) \in Z_{\lambda,\varepsilon}$.

From Lemma 3.5, we can set $k := \max\{\chi(\varphi_\lambda) : \lambda \in \mathbb{R}^{k+1}, |\lambda|_{\mathbb{R}^{k+1}} = 1\}$, so that

$$Z_{\lambda,\varepsilon} = \bigcup_{i=1}^{k} Z^i_{\lambda,\varepsilon} \quad \text{and} \quad Y_{\lambda,\varepsilon} \subseteq \bigcup_{i=1}^{k+1} Y^i_{\lambda,\varepsilon},$$

where, for all $i = 1, \cdots, k$, we denote $Z^i_\lambda := \{t^i \in [a,b] : \varphi_\lambda(t^i) = 0\}$, $Z^i_{\lambda,\varepsilon} := \{t \in \mathbb{R} : \operatorname{dist}\,(t, Z^i_\lambda) < \varepsilon\}$ and $Y^i_{\lambda,\varepsilon}$ are as in Figure 3.1.

Now, calling $\check{Z}_{\lambda,\varepsilon} := P_1^{-1}(Z_{\lambda,\varepsilon}) \cap \varOmega$ and $\check{Y}_{\lambda,\varepsilon} := P_1^{-1}(Y_{\lambda,\varepsilon}) \cap \varOmega$, we observe that

$$\int_{\check{Y}_{\lambda,\varepsilon}} W(u_{\lambda,\varepsilon})\,\mathrm{d}x = 0 \tag{3.1.2}$$

and, defining $\rho := \sup\{|x| : x \in \varOmega\}$, $M := \max\{W(t) : |t| \le 1\}$, and $c_N := \omega_{N-1}\rho^{N-1}$, we get

$$\int_{\check{Z}_{\lambda,\varepsilon}} W(u_{\lambda,\varepsilon})\,\mathrm{d}x \le M|\check{Z}_{\lambda,\varepsilon}| \le 2\varepsilon k M c_N < +\infty, \tag{3.1.3}$$

since $\chi(\varphi_\lambda) = \#(Z_\lambda)$, $Z_{\lambda,\varepsilon} = \cup_{t\in Z_\lambda}]t-\varepsilon, t+\varepsilon[$ and from Lemma 3.5, $\chi(\varphi_\lambda) \le k < +\infty$.

At this point it remains to estimate

$$\int_\varOmega \int_\varOmega \frac{|u_{\lambda,\varepsilon}(x) - u_{\lambda,\varepsilon}(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y,$$

so we analyze it in the three cases distinguished above:

Case (a). We have

$$\int_{\check{Y}_{\lambda,\varepsilon}} \int_{\check{Y}_{\lambda,\varepsilon}} \frac{|u_{\lambda,\varepsilon}(x) - u_{\lambda,\varepsilon}(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \le \sum_{\substack{i,j=1\\ i\neq j}}^{k+1} \int_{\check{Y}^i_{\lambda,\varepsilon}} \int_{\check{Y}^j_{\lambda,\varepsilon}} \frac{|u_{\lambda,\varepsilon}(x) - u_{\lambda,\varepsilon}(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y. \tag{3.1.4}$$

Since the bigger contribution in the interaction comes from two successive strips $\check{Y}^i_{\lambda,\varepsilon}$ and $\check{Y}^{i+1}_{\lambda,\varepsilon}$ for $i = 1, \cdots, k$ which are at least 2ε away, we denote with $Q_- := Q \cap P_1^{-1}(\{t < 0\})$, with $Q_+ := Q \cap P_1^{-1}(\{t > 2\varepsilon\})$ and we can write

$$\begin{aligned}
&\sum_{\substack{i,j=1\\ i\neq j}}^{k+1} \int_{\check{Y}^i_{\lambda,\varepsilon}} \int_{\check{Y}^j_{\lambda,\varepsilon}} \frac{|u_{\lambda,\varepsilon}(x) - u_{\lambda,\varepsilon}(y)|^2}{|x-y|^{N+2s}} \,\mathrm{d}x\,\mathrm{d}y \\
&\leq (k+1)^2 \int_{Q_-} \int_{Q_+} \frac{4}{|x-y|^{N+2s}} \,\mathrm{d}x\,\mathrm{d}y\,.
\end{aligned} \tag{3.1.5}$$

Then we split Q_- in n strips of width $\varepsilon > 0$, with n of order $1/\varepsilon$ and using polar coordinates, we obtain

$$\begin{aligned}
(k+1)^2 \int_{Q_-} \int_{Q_+} \frac{4}{|x-y|^{N+2s}} \,\mathrm{d}x\,\mathrm{d}y &\leq 4n(k+1)^2 c_N^2 \int_{-2\varepsilon}^{-\varepsilon} \int_{-2x_1}^{+\infty} r^{-2s-1} \,\mathrm{d}r\,\mathrm{d}x_1 \\
&= \frac{2}{s} n(k+1)^2 c_N^2 \int_{-2\varepsilon}^{-\varepsilon} (-2x_1)^{-2s} \,\mathrm{d}x_1\,.
\end{aligned} \tag{3.1.6}$$

Now, depending on the value of $s \in (0,1)$, we distinguish two cases:

(j) if $s \neq 1/2$, we have

$$\frac{2}{s} n(k+1)^2 c_N^2 \int_{-2\varepsilon}^{-\varepsilon} (-2x_1)^{-2s} \,\mathrm{d}x_1 = \frac{2^{1-2s} n(k+1)^2 c_N^2}{s(1-2s)} \cdot \varepsilon^{1-2s} (2^{1-2s} - 1). \tag{3.1.7}$$

(jj) If $s = 1/2$,

$$\begin{aligned}
\frac{2}{s} n(k+1)^2 c_N^2 \int_{-2\varepsilon}^{-\varepsilon} (-2x_1)^{-2s} \,\mathrm{d}x_1 &= 4n(k+1)^2 c_N^2 \int_{-2\varepsilon}^{-\varepsilon} (-2x_1)^{-1} \,\mathrm{d}x_1 \\
&= 2n(k+1)^2 c_N^2 \log 2.
\end{aligned} \tag{3.1.8}$$

Case (b). We note that $\check{Y}^i_{\lambda,\varepsilon} \subseteq Q \setminus \check{Z}^i_{\lambda,\varepsilon}$, thus

$$\begin{aligned}
&\int_{\check{Z}_{\lambda,\varepsilon}}\int_{\check{Y}_{\lambda,\varepsilon}} \frac{|u_{\lambda,\varepsilon}(x)-u_{\lambda,\varepsilon}(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y\\
&\le \sum_{i=1}^{k}\int_{\check{Z}^i_{\lambda,\varepsilon}}\int_{Q\setminus\check{Z}^i_{\lambda,\varepsilon}} \frac{|u_{\lambda,\varepsilon}(x)-u_{\lambda,\varepsilon}(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y\\
&\le 2c_N\varepsilon\sum_{i=1}^{k}\sup_{x\in\check{Z}^i_{\lambda,\varepsilon}}\int_{Q\setminus\check{Z}^i_{\lambda,\varepsilon}} \frac{\min\{1/\varepsilon^2|x-y|^2,4\}}{|x-y|^{N+2s}}\,\mathrm{d}y\\
&\le 2k\varepsilon c_N^2\left(\int_0^{2\varepsilon}\frac{1}{\varepsilon^2}r^{1-2s}\,\mathrm{d}r+\int_{2\varepsilon}^{+\infty}4r^{-1-2s}\,\mathrm{d}r\right)\\
&= k\left(\frac{2}{\varepsilon}\cdot\left.\frac{r^{2-2s}}{2-2s}\right]_0^{2\varepsilon}+8\varepsilon\left.\frac{r^{-2s}}{-2s}\right]_{2\varepsilon}^{+\infty}\right)c_N^2\\
&= k\varepsilon^{1-2s}\left(\frac{2^{2-2s}}{1-s}+\frac{2^{2-2s}}{s}\right)c_N^2.
\end{aligned} \tag{3.1.9}$$

Case (c). It results

$$\begin{aligned}
&\int_{\check{Z}_{\lambda,\varepsilon}}\int_{\check{Z}_{\lambda,\varepsilon}} \frac{|u_{\lambda,\varepsilon}(x)-u_{\lambda,\varepsilon}(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y\\
&=\sum_{i=1}^{k}\int_{\check{Z}^i_{\lambda,\varepsilon}}\int_{\check{Z}^i_{\lambda,\varepsilon}} \frac{|u_{\lambda,\varepsilon}(x)-u_{\lambda,\varepsilon}(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y\\
&\quad+\sum_{\substack{i,j=1\\ i\neq j}}^{k}\int_{\check{Z}^j_{\lambda,\varepsilon}}\int_{\check{Z}^i_{\lambda,\varepsilon}} \frac{|u_{\lambda,\varepsilon}(x)-u_{\lambda,\varepsilon}(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned} \tag{3.1.10}$$

Concerning the first term of the right-hand side, we have

$$\begin{aligned}
&\sum_{i=1}^{k}\int_{\check{Z}^i_{\lambda,\varepsilon}}\int_{\check{Z}^i_{\lambda,\varepsilon}} \frac{|u_{\lambda,\varepsilon}(x)-u_{\lambda,\varepsilon}(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y\\
&\le \frac{1}{\varepsilon^2}\sum_{i=1}^{k}|\check{Z}^i_{\lambda,\varepsilon}|c_N\int_0^{2\varepsilon}r^{1-2s}\,\mathrm{d}r \le kc_N^2\frac{2^{2-2s}}{1-s}\varepsilon^{1-2s}.
\end{aligned} \tag{3.1.11}$$

The other term is estimated as in Case (b).

Hence, by (3.1.5), (3.1.7), (3.1.8), (3.1.9) and (3.1.11), we obtain the following estimates for the functionals F_ε: if $s \in (0,1/2)$, we have

$$\begin{aligned} F_\varepsilon(u_{\lambda,\varepsilon}) &= \frac{1}{2}\int_\Omega\int_\Omega \frac{|u_{\lambda,\varepsilon}(x)-u_{\lambda,\varepsilon}(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{\varepsilon^{2s}}\int_\Omega W(u_{\lambda,\varepsilon})\,\mathrm{d}x \\ &\le k\biggl(c_N^2\biggl(\frac{2^{2-2s}}{1-s}+\frac{2^{2-2s}}{s}+\frac{2^{1-2s}}{1-s}\biggr)+2c_NM\biggr)\varepsilon^{1-2s} \\ &+ \frac{2^{-2s}n(k+1)^2}{s(1-2s)}(2^{1-2s}-1)c_N^2\varepsilon^{1-2s} \\ &\le k\biggl(c_N^2\biggl(\frac{2^{2-2s}}{1-s}+\frac{2^{2-2s}}{s}+\frac{2^{1-2s}}{1-s}\biggr)+2c_NM\biggr) \\ &+ \frac{2^{-2s}n(k+1)^2}{s(1-2s)}(2^{1-2s}-1)c_N^2. \end{aligned} \tag{3.1.12}$$

If $s = 1/2$ we get

$$\begin{aligned} F_\varepsilon(u_\lambda,\varepsilon) &= \frac{1}{2|\log\varepsilon|}\int_\Omega\int_\Omega \frac{|u_{\lambda,\varepsilon}(x)-u_{\lambda,\varepsilon}(y)|^2}{|x-y|^{N+1}}\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{|\varepsilon\log\varepsilon|}\int_\Omega W(u_{\lambda,\varepsilon})\,\mathrm{d}x \\ &\le \frac{1}{|\log\varepsilon|}\bigl(k(10c_N^2+2Mc_N)+n(k+1)^2c_N^2\log 2\bigr) \\ &\le k(10c_N^2+2Mc_N)+n(k+1)^2c_N^2\log 2. \end{aligned} \tag{3.1.13}$$

Finally, if $s \in (1/2,1)$ it results

$$\begin{aligned} F_\varepsilon(u_\lambda,\varepsilon) &= \frac{\varepsilon^{2s-1}}{2}\int_\Omega\int_\Omega \frac{|u_{\lambda,\varepsilon}(x)-u_{\lambda,\varepsilon}(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{\varepsilon}\int_\Omega W(u_{\lambda,\varepsilon})\,\mathrm{d}x \\ &\le k\biggl(c_N^2\biggl(\frac{2^{2-2s}}{1-s}+\frac{2^{2-2s}}{s}+\frac{2^{1-2s}}{1-s}\biggr)+2Mc_N\biggr)+\frac{2^{-2s}n(k+1)^2}{s(1-2s)}(2^{1-2s}-1)c_N^2, \end{aligned} \tag{3.1.14}$$

and the proof is complete. ☐

We now state a technical lemma, that will be useful to prove our main result.

Lemma 3.7. *For every $\varepsilon > 0$ and $k \in \mathbb{N}$ the set S_ε^k verifies the following properties:*

(a) S_ε^k is a compact subset of $H^s(\Omega)$;

(b) $S_\varepsilon^k = -S_\varepsilon^k$;

(c) for all $k \in \mathbb{N}$ there exists $\bar\varepsilon_k > 0$ such that $0 \notin S_\varepsilon^k$ $\forall\, \varepsilon \in (0,\bar\varepsilon_k)$;

(d) for all $k \in \mathbb{N}$ and $\forall\, \varepsilon > 0$ such that $0 \notin S_\varepsilon^k$, gen $(S_\varepsilon^k) \ge k+1$.

Proof. The points (b), (c) and (d) are proved in [73]. For (a) we use [73, Lemma 2.8] and the fact that $H^1(\Omega)$ is continuously embedded in $H^s(\Omega)$ for all $s \in (0,1)$, see Proposition 2.3. ☐

3.2 Proof of Theorem 3.1

In this section we will show the proof of Theorem 3.1. To do this we will use a classical result about the genus, i.e. Theorem 2.28.

To apply this result, however, we need to prove the following

Lemma 3.8. *The functional* (3.0.1) *satisfies the* (PS)*-condition.*

Proof. We will show the lemma for $s \in (1/2, 1)$ being the other cases analogous.

Since W is quadratic, there exist α, $\beta > 0$ such that

$$W(u) \geq \alpha u + \beta \quad \forall\, u \in \mathbb{R}. \tag{3.2.1}$$

From Lemma 3.6 we know that $\{F_\varepsilon(u_n)\}_{n\in\mathbb{N}}$ is bounded, hence (3.2.1) implies that $\|u_n\|_{H^s(\Omega)}$ is bounded, so that $u_n \rightharpoonup u$ in $H^s(\Omega)$ and $u_n \to u$ in L^q from Theorem 2.6, $\forall\, q < \frac{2N}{N-2s}$. Therefore $u_n \to u$ a.e. $x \in \Omega$.

We claim that u is a critical point of F_ε. Indeed for all $v \in H^s(\Omega)$,

$$\begin{aligned} F'_\varepsilon(u)(v) &= \varepsilon^{2s-1} \int_\Omega \int_\Omega \frac{u(x)-u(y)}{|x-y|^{N+2s}}(v(x)-v(y))\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + \frac{1}{\varepsilon}\int_\Omega W'(u)v\,\mathrm{d}x \\ &= \lim_{n\to\infty}\Bigg(\varepsilon^{2s-1}\int_\Omega\int_\Omega \frac{u_n(x)-u_n(y)}{|x-y|^{N+2s}}(v(x)-v(y))\,\mathrm{d}x\,\mathrm{d}y \\ &\qquad\qquad + \frac{1}{\varepsilon}\int_\Omega W'(u_n)v\,\mathrm{d}x\Bigg) = 0, \end{aligned} \tag{3.2.2}$$

since $u_n \rightharpoonup u$ in $H^s(\Omega)$ and, by hypothesis, $F'_\varepsilon(u_n) \to 0$. This implies that $F'_\varepsilon(u_n)(u_n - u) + F'_\varepsilon(u)(u_n - u) \to 0$, but

$$\begin{aligned} &F'_\varepsilon(u_n)(u_n-u) + F'_\varepsilon(u)(u_n-u) \\ &= \varepsilon^{2s-1}\int_\Omega\int_\Omega \frac{u_n(x)-u_n(y)}{|x-y|^{N+2s}}(u_n(x)-u(x)-u_n(y)+u(y))\,\mathrm{d}x\,\mathrm{d}y \\ &- \varepsilon^{2s-1}\int_\Omega\int_\Omega \frac{u(x)-u(y)}{|x-y|^{N+2s}}(u_n(x)-u(x)-u_n(y)+u(y))\,\mathrm{d}x\,\mathrm{d}y \\ &+ \frac{1}{\varepsilon}\int_\Omega [W'(u_n)-W'(u)](u_n-u)\,\mathrm{d}x, \end{aligned} \tag{3.2.3}$$

and the second term on the right hand side tends to 0. In particular we obtain

$$\int_\Omega\int_\Omega \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \to \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y.$$

Hence $\|u_n\|^2_{H^s(\Omega)} \to \|u\|^2_{H^s(\Omega)}$ and since $u_n \rightharpoonup u$ in $H^s(\Omega)$, the proof of the lemma is complete. ☐

We are now able to prove our main result.

Proof of Theorem 3.1. As usual we prove the theorem only for $s \in (1/2, 1)$. Consider $\overline{W} \in C^2(\mathbb{R}; \mathbb{R}^+)$ another even function, satisfying the following properties:

$$\overline{W} = W \quad \forall\, t \in [-1, 1] \quad \text{and} \quad t\overline{W}'(t) > 0 \text{ for } |t| > 1.$$

This asymptotic behaviour guarantees that

$$\overline{F}_\varepsilon(u) := \frac{\varepsilon^{2s-1}}{2} \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \,\mathrm{d}x \,\mathrm{d}y + \frac{1}{\varepsilon} \int_\Omega \overline{W}(u) \,\mathrm{d}x$$

is a C^2-functional which satisfies the (PS)-condition. We claim that for every critical point $\overline{u} \in H^s(\Omega)$ of the functional $\overline{F}_\varepsilon$, it holds $|\overline{u}(x)| \leq 1$ for all $x \in \Omega$, which implies that $\overline{u}$ is a critical point for the functional F_ε too. Indeed, if $\overline{u}$ is a critical point for $\overline{F}_\varepsilon$, for all $v \in H^s(\Omega)$, we have that

$$\varepsilon^{2s-1} \int_\Omega \int_\Omega \frac{\overline{u}(x) - \overline{u}(y)}{|x - y|^{N+2s}} (v(x) - v(y)) \,\mathrm{d}x \,\mathrm{d}y + \frac{1}{\varepsilon} \int_\Omega \overline{W}'(\overline{u}) v \,\mathrm{d}x = 0.$$

In particular, if we set $\hat{u} := \max\{\min\{\overline{u}, 1\}, -1\}$, choosing $v = \overline{u} - \hat{u}$, it results

$$\varepsilon^{2s-1} \int_\Omega \int_\Omega \frac{\overline{u}(x) - \overline{u}(y)}{|x - y|^{N+2s}} (\overline{u}(x) - \hat{u}(x) - \overline{u}(y) + \hat{u}(y)) \,\mathrm{d}x \,\mathrm{d}y + \frac{1}{\varepsilon} \int_\Omega \overline{W}'(\overline{u})(\overline{u} - \hat{u}) \,\mathrm{d}x = 0, \tag{3.2.4}$$

with

$$\begin{aligned} \int_\Omega \int_\Omega \frac{\overline{u}(x) - \overline{u}(y)}{|x - y|^{N+2s}} &(\overline{u}(x) - \hat{u}(x) - \overline{u}(y) + \hat{u}(y)) \,\mathrm{d}x \,\mathrm{d}y \\ &= \int_\Omega \int_\Omega \frac{|\overline{u}(x) - \overline{u}(y)|^2}{|x - y|^{n+2s}} \,\mathrm{d}x \,\mathrm{d}y \geq 0, \end{aligned} \tag{3.2.5}$$

and

$$\int_\Omega \overline{W}'(\overline{u})(\overline{u} - \hat{u}) \,\mathrm{d}x > \int_\Omega \overline{W}'(\overline{u} - \hat{u})(\overline{u} - \hat{u}) \,\mathrm{d}x > 0 \qquad \text{if } \overline{u} - \hat{u} \not\equiv 0 \text{ in } \Omega,$$

since $t\overline{W}'(t) > 0$ for $|t| > 1$. It follows that $\overline{u} = \hat{u}$, that is $|\overline{u}(x)| \leq 1$ for almost every $x \in \Omega$ as desired.

At this point we take $\varepsilon_k > 0$ such that $\varepsilon_k < \frac{1}{c_k} W(0)|\Omega|$, where c_k is the constant introduced in Lemma 3.6. Then, for every $\varepsilon \in (0, \varepsilon_k)$ we can apply Theorem 2.28 to the functional $\overline{F}_\varepsilon$ with $\overline{c}_1 < 0$ and $c_2 = c_k$, observing that $\overline{F}_\varepsilon(0) = \frac{1}{\varepsilon} W(0)|\Omega| > c_k$ for all $\varepsilon \in (0, \varepsilon_k)$. In this way, since $\mathrm{gen}(\overline{F}_\varepsilon^{\overline{c}_1}) = \mathrm{gen}(\emptyset) = 0$, and $\mathrm{gen}(\overline{F}_\varepsilon^{\overline{c}_k}) \geq \mathrm{gen}(S_\varepsilon^k) \geq k + 1$ from Lemma 3.7 and the fact that $S_\varepsilon^k \subseteq \overline{F}_\varepsilon^{\overline{c}_k} \subseteq H^s(\Omega) \setminus \{0\}$, we obtain that for every $\varepsilon \in (0, \varepsilon_k)$, the functional $\overline{F}_\varepsilon$ has at least $(k + 1)$ pairs $(-u_{0,\varepsilon}, u_{0,\varepsilon}), \ldots, (-u_{k,\varepsilon}, u_{k,\varepsilon})$ of critical points with $\overline{F}_\varepsilon(u_{i,\varepsilon}) \leq c_k$ for all $i = 0, 1, \ldots, k$.

Note that these $(k+1)$ pairs of critical points include also this one given by the minimizers ± 1. Thus we can assume that $(-u_{0,\varepsilon}, u_{0,\varepsilon}) = (-1, +1)$.

On the contrary, if Ω is a connected domain, the other solutions are not minimizers for the functional $\overline{F}_\varepsilon$. Indeed it results

$$\overline{F}_\varepsilon(u_{i,\varepsilon}) > 0 \quad \forall\, \varepsilon \in (0, \varepsilon_k) \text{ and } i = 0, 1, \ldots, k$$

because if $F_\varepsilon(u_{i,\varepsilon}) = \overline{F}_\varepsilon(u_{i,\varepsilon}) = 0$, we should have

$$\int_\Omega \int_\Omega \frac{|u_{i,\varepsilon}(x) - u_{i,\varepsilon}(y)|^2}{|x-y|^{N+2s}}\, \mathrm{d}x\, \mathrm{d}y = 0 \quad \text{and} \quad \overline{W}(u_{i,\varepsilon}) = 0 \ \text{ in } \Omega$$

and hence $u_{i,\varepsilon}$ should be a constant function with value $+1$ or -1.

Moreover let us remark that for all $\varepsilon \in (0, \varepsilon_k)$ and $i = 1, \ldots k$ we have

$$F_\varepsilon(u_{i,\varepsilon}) \geq \min\left\{ \overline{F}_\varepsilon(u) : u \in H^s(\Omega), \int_\Omega u\, \mathrm{d}x = 0 \right\}. \tag{3.2.6}$$

To see this fact, as discussed above, we assume that

$$\min\left\{ \overline{F}_\varepsilon(u) : u \in H^s(\Omega), \int_\Omega u\, \mathrm{d}x = 0 \right\} > 0,$$

otherwise (3.2.6) would be obvious. Then, for every $\overline{c}_1 > 0$ such that

$$\overline{c}_1 < \min\left\{ \overline{F}_\varepsilon(u) : u \in H^s(\Omega), \int_\Omega u\, \mathrm{d}x = 0 \right\},$$

we would have $\operatorname{gen}\left(\overline{F}_\varepsilon^{\overline{c}_1}\right) = 1$ because below c_1 the mean is non zero and we can use it as odd function into $\mathbb{R}^1$ in the genus definition, i.e. Definition 2.27. Therefore, if (3.2.6) were false, the solutions would belong to a set of genus one, in contradiction with their construction in Theorem 2.28. Now, it remains to prove (3.0.4). Let us replace the function $\overline{W}$ appearing in the definition of functional $\overline{F}_\varepsilon$ by a sequence of functions $\{\overline{W}_j\}_{j\in\mathbb{N}}$ and denote by $\{\overline{F}^j_\varepsilon\}_{j\in\mathbb{N}}$ the corresponding sequence of new functionals. Moreover suppose that, for all $j \in \mathbb{N}$, the functions $\overline{W}_j$ satisfy the same properties of $\overline{W}$ and that

$$\lim_{j\to\infty} \overline{W}_j(t) = +\infty \quad \text{for } |t| > 1. \tag{3.2.7}$$

Then property (3.2.6) holds for the higher critical values of the functional $\overline{F}^j_\varepsilon$ for all $j \in \mathbb{N}$. Thus (3.0.4) follows for j large enough, since

$$\begin{aligned} F_\varepsilon(u_{i,\varepsilon}) &\geq \lim_{j\to\infty} \min\left\{ \overline{F}^j_\varepsilon(u) : u \in H^s(\Omega), \int_\Omega u\, \mathrm{d}x = 0 \right\} \\ &= \min\left\{ F_\varepsilon(u) : u \in H^s(\Omega), |u(x)| \leq 1\ \forall x \in \Omega, \int_\Omega u\, \mathrm{d}x = 0 \right\} \end{aligned}$$

thanks to (3.2.7). □

4 Minimizers for a fractional Allen-Cahn equation in a periodic medium

In this chapter, which corresponds to [71], we study the solutions of a fractional mesoscopic model of phase transitions in a periodic medium, i.e. for $N \geq 2$ we consider the energy functional

$$\mathcal{E}(u) := \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x)-u(y)|^2 K(x,y)\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^N} W(x,u(x))\,\mathrm{d}x + \int_{\mathbb{R}^N} H(x)u(x)\,\mathrm{d}x\,. \tag{4.0.1}$$

The function $K : \mathbb{R}^N \times \mathbb{R}^N \to [0,+\infty]$ is measurable, symmetric and comparable to the kernel of the fractional laplacian, i.e.

$$K(x,y) = K(y,x) \quad \text{for a.e. } x,y \in \mathbb{R}^N \tag{K1}$$

and, denoting with $\chi_{(0,1)}$ the characteristic function of the interval $(0,1)$,

$$\frac{\lambda\chi_{(0,1)}(|x-y|)}{|x-y|^{N+2s}} \leq K(x,y) \leq \frac{\Lambda}{|x-y|^{N+2s}} \quad \text{for a.e. } x,y \in \mathbb{R}^N, \tag{K2}$$

for some $\Lambda \geq \lambda > 0$ and $s \in (0,1)$.

The function $H \in L^\infty(\mathbb{R}^N)$ is a small perturbation of the fractional Allen-Cahn functional. So we assume that

$$\sup_{\mathbb{R}^N} |H| \leq \eta, \tag{H1}$$

for η sufficiently small, depending on N and on the structural constants of the problem. We also assume that H has zero-average and it is $\mathbb{Z}^N$-periodic, i.e.

$$\int_{[0,1]^N} H(x)\,\mathrm{d}x = 0 \tag{H2}$$

and

$$H(x+k) = H(x) \qquad \forall\, k \in \mathbb{Z}^N. \tag{H3}$$

The map $W : \mathbb{R}^N \times \mathbb{R} \to [0,+\infty)$ is the standard double well-potential, i.e. it is a bounded measurable function such that

$$W(x,\pm 1) = 0 \quad \text{for a.e. } x \in \mathbb{R}^N, \tag{W1}$$

and for any $\theta \in [0,1)$

$$\inf_{\substack{x \in \mathbb{R}^N \\ |r| \le \theta}} W(x,r) \ge \gamma(\theta) \tag{W2}$$

where $\gamma : [0,1) \to \mathbb{R}^+$ is a non-increasing function. We assume that W is differentiable in the second component, with partial derivative locally bounded in $r \in \mathbb{R}$ and uniformly in $x \in \mathbb{R}^N$, that is

$$W(x,r)|W_u(x,r)| \le W^* \quad \text{for a.e. } x \in \mathbb{R}^N \text{ and any } r \in [-1,1] \tag{W3}$$

for some $W^* > 0$. Moreover, since we want to model a periodic environment, we require both K and W to be periodic under integer translations:

$$K(x+k, y+k) = K(x,y) \quad \text{for a.e. } x, y \in \mathbb{R}^N \text{ and any } k \in \mathbb{Z}^N \tag{K3}$$

and

$$W(x+k, r) = W(x,r) \quad \text{for a.e. } x \in \mathbb{R}^N \text{ and any } k \in \mathbb{Z}^N, \tag{W4}$$

for any fixed $r \in \mathbb{R}$. Finally we require that

$$W_u(x, -1-r) \le -c \quad \text{and} \quad W_u(x, 1+r) \ge c \tag{W5}$$

for any $r \ge \delta_0$ with $\delta_0 \in (0, 1/10)$, and suitable $c > 0$, and

$$W(x, -1+r) = W(x, 1+r) \tag{W6}$$

for any $r \in [-\delta_0, \delta_0]$.

The functional (4.0.1) is composed by three terms (the first two give us the fractional Allen-Cahn equation):

- a 'kinetic interaction term' $|u(x) - u(y)|^2 K(x,y)$, which penalizes the phase changes of the system;
- a double-well potential term W, which penalizes considerable deviations from the 'pure phase' ±1;
- a 'mesoscopic term' Hu, which is 'neutral' in the average and at each point it prefers one of the two phases.

Hence we have a model of phase coexistence where $u : \mathbb{R}^N \to \mathbb{R}$ is a state parameter.

The fractional exponent $s \in (0,1)$ represents the fact that this model considers long-range particle interactions (and it can produce, depending on the value of s, local or non-local effect, see [82, 84]).

We are interested in plane-like minimizers, so our main goal is to construct minimal interfaces lying to a strip of universal size. To do this we need to introduce some terminology:

Definition 4.1. Fixed $\omega \in \mathbb{Q}^N \setminus \{0\}$, we define in $\mathbb{R}^N$ the relation

$$x \sim_\omega y \iff y - x = k \in \mathbb{Z}^N \text{ with } \omega \cdot k = 0.$$

It is easy to see that $\sim_\omega$ is an equivalence relation and we denote with

$$\tilde{\mathbb{R}}^N_\omega := \mathbb{R}^N/_{\sim_\omega}$$

the associated quotient space.

A function $u : \mathbb{R}^N \to \mathbb{R}$ is said to be periodic with respect to $\sim_\omega$ if

$$u(x) = u(y) \quad \text{for any } x, y \in \mathbb{R}^N \text{ such that } x \sim_\omega y. \tag{4.0.2}$$

When the context is clear, we will write $\sim$ and $\tilde{\mathbb{R}}^N$ to refer to $\sim_\omega$ and $\tilde{\mathbb{R}}^N_\omega$.

Then, we consider a set $\Omega \subseteq \mathbb{R}^N$ and we define *the total energy* $\mathcal{E}$ of $u : \mathbb{R}^N \to \mathbb{R}$ in Ω as

$$\mathcal{E}(u,\Omega) := \frac{1}{2}\iint_{\mathcal{C}_\Omega} |u(x)-u(y)|^2 K(x,y)\,\mathrm{d}x\,\mathrm{d}y + \int_\Omega W(x,u(x))\,\mathrm{d}x + \int_\Omega H(x)u(x)\,\mathrm{d}x\,, \tag{4.0.3}$$

where

$$\begin{aligned} \mathcal{C}_\Omega &:= (\mathbb{R}^N \times \mathbb{R}^N) \setminus ((\mathbb{R}^N \setminus \Omega) \times (\mathbb{R}^N \setminus \Omega)) \\ &= (\Omega \times \Omega) \cup (\Omega \times (\mathbb{R}^N \setminus \Omega)) \cup ((\mathbb{R}^N \setminus \Omega) \times \Omega). \end{aligned} \tag{4.0.4}$$

Observe that if $\Omega = \mathbb{R}^N$ the energy (4.0.3) coincides with (4.0.1).

Moreover, setting for all $U, V \subseteq \mathbb{R}^N$

$$\mathscr{K}(u;U;V) := \frac{1}{2}\int_U \int_V |u(x)-u(y)|^2 K(x,y)\,\mathrm{d}x\,\mathrm{d}y\,,$$

thanks to (K1), we can see $\mathcal{E}(u,\Omega)$ as the sum of the kinetic part

$$\mathcal{K}(u;\Omega;\Omega) + 2\mathcal{K}(u;\Omega;\mathbb{R}^N \setminus \Omega)$$

and the potential part

$$\mathscr{P}(u;\Omega) := \int_\Omega \Big(W(x,u(x)) + H(x)u(x)\Big)\,\mathrm{d}x\,.$$

Assuming from now on that every set and every function is measurable, we give the following

Definition 4.2. Let $\Omega \subseteq \mathbb{R}^N$ be a bounded set. A function u is a local minimizer of $\mathcal{E}$ in Ω if $\mathcal{E}(u,\Omega) < +\infty$ and

$$\mathcal{E}(u,\Omega) \le \mathcal{E}(v,\Omega)$$

for any $v \equiv u$ in $\mathbb{R}^N \setminus \Omega$.

Remark 4.3. [31, Remark 1.2] A minimizer u of Ω is also a minimizer on every subset of Ω.

Since our aim is to construct functions with minimizing properties in $\mathbb{R}^N$, we have to make precise how we extend Definition 4.2 to the full space.

Definition 4.4. A function u is called a class A-minimizer of the functional $\mathcal{E}$ if it is a minimizer of $\mathcal{E}$ in Ω for any bounded set $\Omega \subseteq \mathbb{R}^N$.

With this setting at hand, we can state our main result:

Theorem 4.5. *Let $s \in (0,1)$ and $N \geq 2$. Suppose that the kernel K and the potential W satisfy* (K1)-(K3) *and* (W1)-(W6) *respectively.*

Given $\theta \in (0, 1-\delta_0)$, there exists a positive constant M_0 depending only on θ and on universal quantities, such that, for any $\omega \in \mathbb{R}^N \setminus \{0\}$, there is a class A minimizer u_ω of the functional $\mathcal{E}$ for which we have

$$\{|u_\omega| < \theta\} \subset \Big\{x \in \mathbb{R}^N : \frac{\omega}{|\omega|} \cdot x \in [0, M_0]\Big\}.$$

Moreover,

- *if $\omega \in \mathbb{Q}^N \setminus \{0\}$, u_ω is periodic with respect to $\sim_\omega$;*
- *if $\omega \in \mathbb{R}^N \setminus \mathbb{Q}^N$, u_ω is the uniform limit on compact subsets of $\mathbb{R}^N$ of a sequence of periodic class A minimizers.*

Roughly speaking, this theorem tells us that given any vector $\omega \in \mathbb{R}^N \setminus \{0\}$ we look for minimizers having most of the transition between the pure states in a strip orthogonal to ω and of universal width.

We prove this result using geometric and variational tools introduced in [24] and [86] and then adapted in [31] to deal with nonlocal interactions. Fixed $\omega \in \mathbb{Q}^N \setminus \{0\}$ we will consider the strip

$$S_\omega^M := \{x \in \mathbb{R}^N : \omega \cdot x \in [0, M]\},$$

where $M > 0$, and the quotient space $\tilde{\mathbb{R}}^N$ which allows us to gain compactness. This will be necessary to obtain a minimizer u_ω^M w.r.t. periodic perturbations with support in S_ω^M. Thanks to geometrical arguments, if $M/|\omega|$ is larger than some universal parameter M_0, u_ω^M becomes a class A-minimizer for $\mathcal{E}$. Since M_0 does not depend on the fixed direction ω, we can pass to the limit on rational directions and deduce the result for an irrational vector $\omega \in \mathbb{R}^N \setminus \mathbb{Q}^N$.

We stress that the energy and density estimates is the standard technique to show that u_ω^M is a class A-minimizer. These estimates have been obtained in [18, 84] (in different settings), but their framework is different from ours. Thus we use the Hölderianity of local minimizers of $\mathcal{E}$ and an energy estimate.

Finally we point out that the addition of the term Hu to (1.0.7) changes the ‘pure phases’ from ±1 into periodic functions, introducing a considerable difference with respect to [31]. Indeed, this fact produces a volume term in the energy that requires a renormalization as in [69].

4.1 Regularity of the minimizers and energy estimate

In this section we want to prove that local minimizers of $\mathcal{E}$ are Hölder continuous functions with a growing energy inside large balls.

Let $\Omega \subseteq \mathbb{R}^N$ be an open and bounded set, $s \in (0,1)$ and K a measurable kernel satisfying (K1) and (K2). If $u : \mathbb{R}^N \to \mathbb{R}$ is a measurable function, we say that $u \in X(\Omega)$ if

$$u_{|\Omega} \in L^2(\Omega) \quad \text{and} \quad (x,y) \mapsto (u(x)-u(y))\sqrt{K(x,y)} \in L^2(\mathcal{C}_\Omega).$$

Then we denote with $X_0(\Omega)$ the subspace of $X(\Omega)$ given by functions vanishing a.e. outside Ω. It is easy to see that by (K2) it results $H^s(\mathbb{R}^N) \subset X(\Omega) \subseteq H^s(\Omega)$ and if $\Omega' \subseteq \Omega$ we have $X_0(\Omega') \subseteq X_0(\Omega) \subset H^s(\mathbb{R}^N)$.

Now we call

$$\mathcal{D}_K(u,\varphi) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (u(x)-u(y))(\varphi(x)-\varphi(y))K(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

observing that it is well-defined for example when $u \in X(\Omega)$ and $\varphi \in X_0(\Omega)$. Let $f \in L^2(\Omega)$. We call $u \in X(\Omega)$ a supersolution of

$$\mathcal{D}_k(u,\cdot) = f \quad \text{in } \Omega \tag{4.1.1}$$

if

$$\mathcal{D}_k(u,\varphi) \geq \langle f,\varphi\rangle_{L^2(\mathbb{R}^N)} \quad \text{for any non-negative } \varphi \in X_0(\Omega). \tag{4.1.2}$$

Similarly, we say that $u \in X(\Omega)$ is a subsolution of (4.1.1) if

$$\mathcal{D}_k(u,\varphi) \leq \langle f,\varphi\rangle_{L^2(\mathbb{R}^N)} \quad \text{for any non-negative } \varphi \in X_0(\Omega) \tag{4.1.3}$$

and we tell that $u \in X(\Omega)$ is a solution of (4.1.1) if

$$\mathcal{D}_k(u,\varphi) = \langle f,\varphi\rangle_{L^2(\mathbb{R}^N)} \quad \text{for any } \varphi \in X_0(\Omega). \tag{4.1.4}$$

Obviously u is a solution of (4.1.1) if it is a subsolution and a supersolution.

Thanks to these definitions we can show the regularity of the minimizers of $\mathcal{E}$.

Theorem 4.6. *Take $s_0 \in (0,1/2)$ and let $s \in [s_0, 1-s_0]$. If u is a bounded local minimizer of $\mathcal{E}$ in a bounded open set $\Omega \subseteq \mathbb{R}^N$, then $u \in C^{0,\alpha}_{loc}(\Omega)$ for some $\alpha \in (0,1)$. The exponent α only depends on N, s_0, λ and Λ, while the $C^{0,\alpha}$ norm of u on any $\Omega' \subset\subset \Omega$ may also depend on $\|u\|_{L^\infty(\mathbb{R}^N)}$, $\|W_r(\cdot,u)\|_{L^\infty(\Omega)}$, η and $\operatorname{dist}(\Omega',\partial\Omega)$.*

Proof. If we compute the first variation of (4.0.3) we have that u is a solution of the Euler-Lagrange equation (4.1.1) in Ω with $-f = W_r(\cdot,u) + H(\cdot)$. Since $\mathcal{E}(u,\Omega) < +\infty$ we have that $u \in X(\Omega)$. Moreover u, $H \in L^\infty(\mathbb{R}^N)$ and W_r locally bounded imply that f is bounded in Ω. So we can apply Theorem 2.1 of [31] to obtain $C^{0;\alpha}_{\text{loc}}$ regularity of u in Ω. (We point out that Theorem 4.6 can also be proved using [30, Theorem 2.4] which allows us to deduce the regularity of minimizers without using the Euler-Lagrange equation). ☐

Now we define

$$\Psi_s(R) := \begin{cases} R^{1-2s} & \text{if } s \in (0,1/2) \\ \log R & \text{if } s = 1/2 \\ 1 & \text{if } s \in (1/2,1) \end{cases} \tag{4.1.5}$$

and thanks to a well-known result in [84], based on the preliminary estimates in [72], we want to show the energy estimate for minimizers:

Theorem 4.7. *Let* $N \in \mathbb{N}$, $s \in (0,1)$, $x_0 \in \mathbb{R}^N$ *and* $R \geq 3$. *Suppose that* K *and* W *satisfy* (K1), (K2) *and* (W1), (W3), *respectively. If* $u : \mathbb{R}^N \to [-1-\delta_0, 1+\delta_0]$ *is a local minimizer of* $\mathcal{E}$ *in* $B_{R+2}(x_0)$, *then*

$$\mathcal{E}(u, B_R(x_0)) \leq C R^{N-1} \Psi_s(R), \tag{4.1.6}$$

for some constant $C > 0$ *which depends on* N, s, Λ *and* W^*.

Proof. Since u is a local minimizer of $\mathcal{E}$ in $B_{R+2}(x_0)$, we know that

$$\mathcal{E}(u, B_{R+2}(x_0)) \leq \mathcal{E}(v, B_{R+2}(x_0)) \tag{4.1.7}$$

for any $v \equiv u$ in $\mathbb{R}^N \setminus B_{R+2}(x_0)$.

Moreover u satisfies

$$(-\Delta)^s u + W_u(x,u) + H(x) = 0 \quad \text{in } B_{R+2}(x_0), \tag{4.1.8}$$

and hence, given every domain $V \subset U \subset B_R(x_0)$, thanks to interior regularity estimates we have that

$$\|u\|_{H^s(V)} \leq c\sqrt{|U|},$$

where $c > 0$ is a constant, see [26, 11, 29].

Now, being $|u| \leq 1 + \delta_0$, we can proceed as in [69, Proof of Theorem 1] to obtain our thesis. ☐

We conclude this section giving an auxiliary result that will be very useful in the next Section 4.2.

Lemma 4.8. *Let* $s \in (0,1)$, U, $V \subseteq \mathbb{R}^N$ *be measurable sets and* u, $v \in H^s_{loc}(\mathbb{R}^N)$. *Then*

$$\mathscr{K}(\min\{u,v\}; U; V) + \mathscr{K}(\max\{u,v\}; U; V) \leq \mathscr{K}(u; U; V) + \mathscr{K}(v; U; V), \tag{4.1.9}$$

and

$$\mathscr{P}(\min\{u,v\}; U) + \mathscr{P}(\max\{u,v\}; U) = \mathscr{P}(u; U) + \mathscr{P}(v; U). \tag{4.1.10}$$

Proof. The second identity is straightforward, while the first is proved in [31, Lemma 3.2]. ☐

4.2 Proof of Theorem 4.5 for rapidly decaying kernels

In this section we want to prove Theorem 4.5 assuming the following hypothesis on K:

$$K(x,y) \leq \frac{\Gamma}{|x-y|^{N+\beta}} \quad \text{for a.e. } x, y \in \mathbb{R}^N \text{ such that } |x-y| \geq \bar{R} \text{ with } \beta > 1, \tag{K4}$$

for some constant $\varGamma$, $\bar{R} > 0$. This assumption is only technical and we will remove it in the next section. However a fast decay of the kernel K at infinity due to the fact that $\beta > 1$ ensures us that there exists a competitor with finite energy in the large. Then, since geometric estimates will not depend on the quantities in (K4), we can use a limit procedure.

We start showing that the functional $\mathcal{E}$ has a minimizer among all periodic functions. Let $s \in (0,1)$, $Q := [0,1]^N$ and define Q-periodic functions in $H^s_{\mathrm{loc}}(\mathbb{R}^N)$ as

$$H^s_{\mathrm{per}}(Q) = \{u \in H^s_{\mathrm{loc}}(\mathbb{R}^N) \text{ such that } u(x+e_j) = u(x) \text{ for all } x \in \mathbb{R}^N\} \tag{4.2.1}$$

where $\{e_1, \cdots e_N\}$ is the standard Euclidean base of $\mathbb{R}^N$.

With this notation in hand, proceeding as in [69, Lemma 7], we have the following

Theorem 4.9. *Assume K and W as in Theorem 4.5. Then the functional $\mathcal{E}$ attains its minimum in $H^s_{per}(Q)$. Moreover if u is a minimizer, it is continuous and*

$$\Big||u(x)| - 1\Big| \le \delta_0 \tag{4.2.2}$$

for any $x \in Q$, as long as η is small enough.

Proof. Let consider $\{u_n\}_{n\in\mathbb{N}}$ be a minimizing sequence. By (H2) we may suppose that

$$\mathcal{E}(u_k, Q) \le \mathcal{E}(1, Q) = 0. \tag{4.2.3}$$

Then, from (W5) we have

$$\begin{aligned} \min\{W(x,1+s) - W(x,1+\delta_0), W(x,-1-s) - W(x,-1-\delta_0)\} &\ge c(s-\delta_0) \\ \ge |H(x)(\delta_0 - s)| \end{aligned} \tag{4.2.4}$$

for any $s \ge \delta_0$ and

$$W(x,r) + H(x)r \ge 0$$

as long as $|r| \ge C_0$ with C_0 sufficiently large if η is small enough. Accordingly, by (4.2.3),

$$\iint_{\mathcal{C}_Q} |u_k(x) - u_k(y)|^2 K(x,y)\,\mathrm{d}x\,\mathrm{d}y \le \int_{Q\cap\{|u_k|\le C_0\}} |H(x)u_k(x)|\,\mathrm{d}x \le C_0|Q|\eta. \tag{4.2.5}$$

Hence we define

$$u_k^*(x) = \begin{cases} u_k(x) & \text{if} \quad |u_k(x)| < 1+\delta_0 \\ 1+\delta_0 & \text{if} \quad u_k(x) \ge 1+\delta_0 \\ -1-\delta_0 & \text{if} \quad u_k(x) \le -1-\delta_0 \end{cases} \tag{4.2.6}$$

and thanks to (4.2.4) we get that $\mathcal{E}(u_k^*, Q) \le \mathcal{E}(u_k, Q)$.

So, up to replacing u_k with u_k^* we may assume that

$$|u_k| \le 1 + \delta_0. \tag{4.2.7}$$

By (4.2.5), (4.2.7) and the compact embedding of $H^s(Q)$ in $L^2(Q)$ we obtain that $u_k \to u$ in $L^2(Q)$, $u_k \rightharpoonup u$ in $H^s(Q)$ and, up to subsequences, $u_k \to u$ a.e. Therefore $u \in H^s_{\text{per}}(Q)$ and

$$\liminf_{k\to\infty} \iint_{\mathcal{C}_Q} |u_k(x) - u_k(y)|^2 K(x,y)\,\mathrm{d}x\,\mathrm{d}y \geq \iint_{\mathcal{C}_Q} |u(x) - u(y)|^2 K(x,y)\,\mathrm{d}x\,\mathrm{d}y\,.$$

Then Fatou's Lemma and the Dominated Convergence Theorem give us

$$\inf_{H^s_{\text{per}}(Q)} \mathcal{E}(\cdot, Q) = \liminf_{k\to\infty} \mathcal{E}(u_k, Q) \geq \mathcal{E}(u, Q)$$

i.e. u is the desired minimizer.

From Theorem 4.6 we have that u is continuous, so it remains to prove (4.2.2). To do this, we take $u \in H^s_{\text{per}}(Q)$ minimizer for $\mathcal{E}(\cdot, Q)$ and define

$$u^*(x) := \begin{cases} u(x) & \text{if } |u(x)| < 1 + \delta_0 \\ 1 + \delta_0 & \text{if } u(x) \geq 1 + \delta_0 \\ -1 - \delta_0 & \text{if } u(x) \leq -1 - \delta_0. \end{cases} \tag{4.2.8}$$

By (4.2.4) and since u is a minimizer, we have

$$0 \leq \mathcal{E}(u^*, Q) - \mathcal{E}(u, Q) \leq -\frac{c}{2}\Big[\int_{\{u>1+\delta_0\}} (u - 1 - \delta_0) + \int_{\{u<-1-\delta_0\}} (-u - 1 - \delta_0)\Big] \leq 0,$$

that is $|u| \leq 1 + \delta_0$. Then, if by contradiction

$$-1 + \delta_0 \leq u(x_0) \leq 1 - \delta_0 \quad \text{for some } x_0 \in Q,$$

the uniform continuity of u gives

$$-1 + \frac{\delta_0}{2} \leq u(x) \leq 1 - \frac{\delta_0}{2}$$

for any $x \in B_\rho(x_0)$ for a suitable, universal $\rho > 0$. As a consequence $W(x, u(x)) \geq \text{const}$ for $x \in B_\rho(x_0)$, from which

$$\mathcal{E}(u, Q) \geq \text{const} \cdot |B_\rho(x_0)| - \eta|Q| > 0 = \mathcal{E}(1, Q) \geq \mathcal{E}(u, Q),$$

which is a contradiction and proves (4.2.2). □

This theorem and (W6) imply that the functional $\mathcal{E}(\cdot, Q)$ admits two minimizers $u_\pm \in H^s_{\text{per}}(Q)$ such that $u_+ = u_- + 2$ and

$$\|u_\pm \mp 1\|_{L^\infty(Q)} =: \delta_\eta < \delta_0. \tag{4.2.9}$$

Note that if $W(x, \cdot)$ is strictly convex in $[1 - \delta_0, 1 + \delta_0]$ and $[-1 - \delta_0, -1 + \delta_0]$ these minimizers are the only global minimizers of $\mathcal{E}(\cdot, Q)$ in $H^s_{\text{per}}(Q)$ and from now on we assume that

$$\mathcal{E}(u_+, Q) = \mathcal{E}(u_-, Q). \tag{4.2.10}$$

Remark 4.10. Note that (W6) (required for example by [69]) implies (4.2.10).

4.2.1 Minimization with respect to periodic perturbations

Given $\omega \in \mathbb{Q}^N \setminus \{0\}$ and $u : \mathbb{R}^N \to \mathbb{R}$ a measurable function, we say that $u \in L^2(\tilde{\mathbb{R}}^N)$ if $u \in L^2_{\mathrm{loc}}(\mathbb{R}^N)$ and u is periodic with respect to $\sim$.

Hence, taken A, B two real numbers such that $A < B$ and denoting with $\tilde{\mathbb{R}}^N$ any fundamental domain of the relation $\sim$, we define

$$\mathcal{A}^{A,B}_\omega := \{u \in L^2_{\mathrm{loc}}(\tilde{\mathbb{R}}^N) : u(x) \geq 1 - \delta_0 \text{ if } \omega \cdot x \leq A \text{ and } u(x) \leq -1 + \delta_0 \text{ if } \omega \cdot x \geq B\} \tag{4.2.11}$$

the set of admissible functions and we consider

$$\begin{aligned}\mathcal{F}_\omega(u) := \frac{1}{2} \int_{\tilde{\mathbb{R}}^N} \int_{\mathbb{R}^N} \Big(|u(x) - u(y)|^2 - |u_+(x) - u_+(y)|^2\Big) K(x,y)\, \mathrm{d}x\, \mathrm{d}y \\ + \int_{\tilde{\mathbb{R}}^N} \Big(W(x, u(x)) - W(x, u_+(x))\Big)\, \mathrm{d}x + \int_{\tilde{\mathbb{R}}^N} H(x)\Big(u(x) - u_+(x)\Big)\, \mathrm{d}x\,.\end{aligned} \tag{4.2.12}$$

We want to show that there exists an absolute minimizer of $\mathcal{F}_\omega$ in the class $\mathcal{A}^{A,B}_\omega$, i.e. there exists $u \in \mathcal{A}^{A,B}_\omega$ such that $\mathcal{F}_\omega(u) \leq \mathcal{F}_\omega(v)$ for any $v \in \mathcal{A}^{A,B}_\omega$.

First of all we prove that $\mathcal{F}_\omega$ is not identically infinity on $\mathcal{A}^{A,B}_\omega$:

Theorem 4.11. *Let $\bar{u} \in \mathcal{A}^{A,B}_\omega$ be defined as*

$$\bar{u}(x) := \begin{cases} u_+ & \text{if } \omega \cdot x \leq A \\ u_+ - \frac{(u_+ - u_-)}{B - A}((\omega \cdot x) - A) & \text{if } A < \omega \cdot x \leq B \\ u_- & \text{if } \omega \cdot x > B. \end{cases} \tag{4.2.13}$$

Then $\mathcal{F}_\omega(\bar{u}) < +\infty$.

Proof. Since the potential term of $\mathcal{F}_\omega$ vanishes at u_+ and u_- (thanks to (4.2.10)) for a.e. $x \in \mathbb{R}^N$, it is obviously finite if we evaluate it in $\bar{u}$. So we only have to estimate the kinetic term and thanks to (K2) and (K4) it is sufficient to prove that

$$\begin{aligned}\int_{\tilde{\mathbb{R}}^N} \Big(\int_{B_{\bar{R}}(x)} \frac{|\bar{u}(x) - \bar{u}(y)|^2 - |u_+(x) - u_+(y)|^2}{|x - y|^{N+2s}}\, \mathrm{d}y \\ + \int_{\mathbb{R}^N \setminus B_{\bar{R}}(x)} \frac{|\bar{u}(x) - \bar{u}(y)|^2 - |u_+(x) - u_+(y)|^2}{|x - y|^{N+\beta}}\, \mathrm{d}y \Big)\, \mathrm{d}x < +\infty,\end{aligned} \tag{4.2.14}$$

where as usual $B_{\bar{R}}(x)$ denotes the ball with radius $\bar{R}$ and center at x.

Less than an affine transformation we may assume $\omega = e_N$ and for simplicity we may also suppose that $A = 0$ and $B = 1$. In this framework $\tilde{\mathbb{R}}^N = [0,1]^{N-1} \times \mathbb{R}$. Accordingly (4.2.14) is equivalent to

$$I := \int_{[0,1]^{N-1} \times \mathbb{R}} \int_{B_{\bar{R}}(x)} \frac{|\bar{u}(x) - \bar{u}(y)|^2 - |u_+(x) - u_+(y)|^2}{|x - y|^{N+2s}}\, \mathrm{d}y\, \mathrm{d}x < +\infty \tag{4.2.15}$$

and

$$J := \int_{[0,1]^{N-1} \times \mathbb{R}} \int_{\mathbb{R}^N \setminus B_{\bar{R}}(x)} \frac{|\bar{u}(x) - \bar{u}(y)|^2 - |u_+(x) - u_+(y)|^2}{|x - y|^{N+\beta}}\, \mathrm{d}y\, \mathrm{d}x < +\infty. \tag{4.2.16}$$

Recalling the definition of $\bar{u}$ it follows that

$$I = \int_{[0,1]^{N-1}\times[-\bar{R},\bar{R}+1]} \int_{B_{\bar{R}}(x)} \frac{|\bar{u}(x)-\bar{u}(y)|^2 - |u_+(x)-u_+(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}y\,\mathrm{d}x \tag{4.2.17}$$

and being $\bar{u}$ Lipschitz, we get

$$\begin{aligned} I &\leq 4(1+\delta_0)^2 \int_{[0,1]^{N-1}\times[-\bar{R},\bar{R}+1]} \left(\int_{B_{\bar{R}}(x)} \frac{\mathrm{d}y}{|x-y|^{N+2s-2}} \right) \mathrm{d}x \\ &= \frac{2N\omega_N(1+\delta_0)^2}{1-s}(2\bar{R}+1)\bar{R}^{2-2s} \end{aligned} \tag{4.2.18}$$

where we remind that ω_N denotes the N-dimensional measure of the unit sphere of $\mathbb{R}^N$ and hence (4.2.15) follows.

Now to prove (4.2.16) we write $J = J_1 + J_2 + J_3$ with

$$\begin{aligned} J_1 &:= \int_{[0,1]^{N-1}\times[2,+\infty)} \left(\int_{\mathbb{R}^N\setminus B_{\bar{R}}(x)} \frac{|\bar{u}(x)-\bar{u}(y)|^2 - |u_+(x)-u_+(y)|^2}{|x-y|^{N+\beta}}\,\mathrm{d}y \right) \mathrm{d}x\,, \\ J_2 &:= \int_{[0,1]^{N-1}\times(-\infty,-1)} \left(\int_{\mathbb{R}^N\setminus B_{\bar{R}}(x)} \frac{|\bar{u}(x)-\bar{u}(y)|^2 - |u_+(x)-u_+(y)|^2}{|x-y|^{N+\beta}}\,\mathrm{d}y \right) \mathrm{d}x\,, \\ J_3 &:= \int_{[0,1]^{N-1}\times[-1,2]} \left(\int_{\mathbb{R}^N\setminus B_{\bar{R}}(x)} \frac{|\bar{u}(x)-\bar{u}(y)|^2 - |u_+(x)-u_+(y)|^2}{|x-y|^{N+\beta}}\,\mathrm{d}y \right) \mathrm{d}x\,. \end{aligned}$$

By the definition of $\bar{u}$ we have that

$$\begin{aligned} J_1 &\leq \int_{[0,1]^{N-1}\times[2,+\infty)} \left(\int_{\mathbb{R}^{N-1}\times(-\infty,1]} \frac{|u_-(x)-\bar{u}(y)|^2}{|x-y|^{N+\beta}}\,\mathrm{d}y \right) \mathrm{d}x \\ &\leq 4(1+\delta_0)^2 \int_{[0,1]^{N-1}\times[2,+\infty)} \left(\int_{\mathbb{R}^{N-1}\times(-\infty,1]} \frac{\mathrm{d}y}{|x-y|^{N+\beta}} \right) \mathrm{d}x\,. \end{aligned} \tag{4.2.19}$$

Writing $x = (x', x_N) \in \mathbb{R}^{N-1}\times\mathbb{R}$, $y = (y', y_N) \in \mathbb{R}^{N-1}\times\mathbb{R}$ and substituing $z' := (y'-x')/|x_N - y_N|$, we get

$$\begin{aligned} &\int_{\mathbb{R}^{N-1}\times(-\infty,1]} \frac{\mathrm{d}y}{|x-y|^{N+\beta}} \\ &\qquad = \int_{-\infty}^{1} |x_N - y_N|^{-N-\beta} \left[\int_{\mathbb{R}^{N-1}} \left(1 + \frac{|x'-y'|^2}{|x_N-y_N|^2} \right)^{-\frac{N+\beta}{2}} \mathrm{d}y' \right] \mathrm{d}y_N \\ &\qquad\quad = \int_{-\infty}^{1} |x_N - y_N|^{-1-\beta} \left[\int_{\mathbb{R}^{N-1}} (1+|z'|^2)^{-\frac{N+\beta}{2}}\,\mathrm{d}z' \right] \mathrm{d}y_N \\ &\qquad\qquad = \frac{\Theta}{\beta}(x_N-1)^{-\beta}, \end{aligned} \tag{4.2.20}$$

where

$$\varTheta := \int_{\mathbb{R}^{N-1}} (1+|z'|^2)^{-\frac{N+\beta}{2}}\,\mathrm{d}z' < +\infty.$$

Therefore

$$J_1 \le \frac{4(1+\delta_0)^2}{\beta}\varTheta\int_2^{+\infty}(x_N-1)^{-\beta}\,\mathrm{d}x_N = 4(1+\delta_0)^2\frac{\varTheta}{(\beta-1)\beta},$$

since $\beta > 1$. Analogously it is easy to see that J_2 is finite too. Thus we pass to estimate J_3. Since $\bar{u}$ is a bounded function we have

$$J_3 \le 4(1+\delta_0)^2\int_{[0,1]^{N-1}\times[-1,2]}\left(\int_{\mathbb{R}^N\setminus B_{\bar{R}}(x)}\frac{\mathrm{d}y}{|x-y|^{N+\beta}}\right)\mathrm{d}x = \frac{12N\omega_N}{\beta}(1+\delta_0)^2\bar{R}^{-\beta} \tag{4.2.21}$$

and (4.2.16) follows. ☐

Note that condition (K4) allows us to have the integrability of the first addendum of $\mathcal{F}_\omega$.

With this result in hand we can prove that

Theorem 4.12. *There exists an absolute minimizer of the functional $\mathcal{F}_\omega$ in the class $\mathcal{A}_\omega^{A,B}$.*

Proof. We use the standard direct method of the Calculus of variations.

By Theorem 4.11 and since $\mathcal{F}_\omega \ge 0$ we have that

$$m := \inf\{\mathcal{F}_\omega(u) : u \in \mathcal{A}_\omega^{A,B}\} \in [0,+\infty).$$

So, if $\{u_j\}_{j\in\mathbb{N}} \subseteq \mathcal{A}_\omega^{A,B}$ is a minimizing sequence, we may suppose that

$$|u_j| \le 1+\delta_0 \quad \text{a.e. in } \mathbb{R}^N. \tag{4.2.22}$$

Then we consider an integer $k > \max\{-A, B\}$ and the Lipschitz domains

$$\varOmega_k := \tilde{\mathbb{R}}^N \cap \{x \in \mathbb{R}^N : |\omega\cdot x| \le k\}.$$

Thanks to (4.2.22) and (K2) we obtain

$$\begin{aligned}
[u_j]^2_{H^s(\varOmega_k)} \le{}& \int_{\varOmega_k}\left(\int_{B_1(x)}\frac{|u_j(x)-u_j(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}y\right)\mathrm{d}x \\
&+ 4(1+\delta_0)^2\int_{\varOmega_k}\left(\int_{\mathbb{R}^N\setminus B_1(x)}\frac{\mathrm{d}y}{|x-y|^{N+2s}}\right)\mathrm{d}x \\
\le{}& \frac{2}{\lambda}\mathcal{F}_\omega(u_j,\varOmega_k) + \frac{1}{\lambda}\int_{\varOmega_k}\int_{\mathbb{R}^N}\frac{|u_+(x)-u_+(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y + \frac{2}{\lambda}\int_{\varOmega_k}W(x,u_+(x))\,\mathrm{d}x \\
&+ \frac{2}{\lambda}\int_{\varOmega_k}H(x)u_+(x)\,\mathrm{d}x - \frac{2}{\lambda}\int_{\varOmega_k}W(x,u_j(x))\,\mathrm{d}x - \frac{2}{\lambda}\int_{\varOmega_k}H(x)u_j(x)\,\mathrm{d}x \\
&+ 2\frac{(1+\delta_0)^2}{s}N\omega_N|\varOmega_k|,
\end{aligned} \tag{4.2.23}$$

where we denote with

$$\begin{aligned}\mathcal{F}_\omega(u,\Omega_k) &:= \frac{1}{2}\int_{\Omega_k}\int_{\mathbb{R}^N}\big(|u(x)-u(y)|^2-|u_+(x)-u_+(y)|^2\big)K(x,y)\,\mathrm{d}x\,\mathrm{d}y\\ &\quad+\int_{\Omega_k}(W(x,u(x))-W(x,u_+(x)))\,\mathrm{d}x+\int_{\Omega_k}H(x)(u(x)-u_+(x))\,\mathrm{d}x\,.\end{aligned} \tag{4.2.24}$$

Now we take $k\in\mathbb{N}$ such that $k\omega\in\mathbb{Z}^N$, so that Ω_k is a periodicity domain for u_+. From this and the fact that u_+ is minimizer for $\mathcal{E}$ on all the domains Ω_k, we get

$$0\le\mathcal{F}_\omega(u_j,\Omega_k)\le\mathcal{F}_\omega(u_j,\tilde{\mathbb{R}}^N),$$

so (4.2.23) becomes

$$\begin{aligned}[u_j]^2_{H^s(\Omega_k)} &\le \frac{2}{\lambda}\mathcal{F}_\omega(u_j)+\frac{2}{\lambda}\int_{\Omega_k}\int_{\mathbb{R}^N}\frac{|u_+(x)-u_+(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y+\frac{2}{\lambda}\int_{\Omega_k}W(x,u_+(x))\,\mathrm{d}x\\ &\quad+\frac{2}{\lambda}\int_{\Omega_k}H(x)u_+(x)\,\mathrm{d}x+2|\Omega_k|\bigg(\frac{(1+\delta_0)}{\lambda}\eta+\frac{(1+\delta_0)^2}{s}N\omega_N\bigg).\end{aligned} \tag{4.2.25}$$

Hence $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $H^s(\Omega_k)$ uniformly in j. Since $H^s(\Omega_k)\hookrightarrow\hookrightarrow L^2(\Omega_k)$ (see [35, Theorem 7.1]), less than extract a subsequence, $u_j\to u$ in $L^2(\Omega_k)$ and a.e. in Ω_k. Now we use a diagonal argument (on j and k) to find a subsequence $\{u^*_j\}_{j\in\mathbb{N}}$ of $\{u_j\}_{j\in\mathbb{N}}$ such that $u_j{}^*\to u$ a.e. in $\tilde{\mathbb{R}}^N$. We may identify the u^*_j's and u with their $\sim$-periodic extension to $\mathbb{R}^N$ so that the convergence will be in the full space $\mathbb{R}^N$.

As a consequence $u\in\mathcal{A}^{A,B}_\omega$ and using Fatou's Lemma we get $\mathcal{F}_\omega(u)=m$ that concludes the proof. ☐

4.2.2 The minimal minimizer

Define

$$\mathcal{M}^{A,B}_\omega:=\{u\in\mathcal{A}^{A,B}_\omega:\mathcal{F}_\omega(u)\le\mathcal{F}_\omega(v)\ \text{ for any } v\in\mathcal{A}^{A,B}_\omega\}$$

the set of the absolute minimizers of $\mathcal{F}_\omega$ in $\mathcal{A}^{A,B}_\omega$. Observe that from Theorem 4.12, $\mathcal{M}^{A,B}_\omega$ is not empty, hence we can introduce the following

Definition 4.13. We call $u^{A,B}_\omega$ a minimal minimizer when it is the infimum of $\mathcal{M}^{A,B}_\omega$ if we consider $\mathcal{M}^{A,B}_\omega$ subset of the partially ordered set $(\mathcal{A}^{A,B}_\omega,\le)$. In particular $u^{A,B}_\omega$ is the unique function of $\mathcal{A}^{A,B}_\omega$ such that

$$u^{A,B}_\omega\le u \text{ in } \mathbb{R}^N \text{ for every } u\in\mathcal{M}^{A,B}_\omega \tag{4.2.26}$$

and

$$\text{if } v\in\mathcal{A}^{A,B}_\omega \text{ is such that } v\le u \text{ in } \mathbb{R}^N \text{ for every } u\in\mathcal{M}^{A,B}_\omega,\ \text{then } v\le u^{A,B}_\omega \text{ in } \mathbb{R}^N. \tag{4.2.27}$$

The existence of $u_\omega^{A,B}$ is not obvious, so we will denote the rest of the section to show it.

First of all we need to prove that the minimum between two elements of $\mathcal{M}_\omega^{A,B}$ still belongs to $\mathcal{M}_\omega^{A,B}$. To obtain this, we follow [31], showing first a couple of auxiliary lemmas.

Lemma 4.14. *Let A, A', B, B' be real numbers such that $A < A'$ and $B < B'$ with $A < B$ and $A' < B'$. If $u \in \mathcal{M}_\omega^{A,B}$ and $v \in \mathcal{M}_\omega^{A',B'}$, then $\min\{u,v\} \in \mathcal{M}_\omega^{A,B}$.*

Proof. Observing that $\min\{u,v\} \in \mathcal{A}_\omega^{A,B}$ and $\max\{u,v\} \in \mathcal{A}_\omega^{A',B'}$ and using Lemma 4.8, we get

$$\mathcal{F}_\omega(\min\{u,v\}) + \mathcal{F}_\omega(\max\{u,v\}) \leq \mathcal{F}_\omega(u) + \mathcal{F}_\omega(v).$$

Now, since $v \in \mathcal{M}_\omega^{A',B'}$ we have

$$\mathcal{F}_\omega(\min\{u,v\}) + \mathcal{F}_\omega(\max\{u,v\}) \leq \mathcal{F}_\omega(u) + \mathcal{F}_\omega(\max\{u,v\})$$

and hence

$$\mathcal{F}_\omega(\min\{u,v\}) \leq \mathcal{F}_\omega(u),$$

that is $\min\{u,v\} \in \mathcal{M}_\omega^{A,B}$. □

As a consequence, if we choose $A = A'$ and $B = B'$ we obtain this

Corollary 4.15. *If u, $v \in \mathcal{M}_\omega^{A,B}$, then $\min\{u,v\} \in \mathcal{M}_\omega^{A,B}$.*

At this point we can show that $\mathcal{M}_\omega^{A,B}$ is also closed with respect to take the minimum among a countable family of its elements:

Lemma 4.16. *If $\{u_n\}_{n\in\mathbb{N}}$ is a sequence of elements in $\mathcal{M}_\omega^{A,B}$, then $\inf_{n\in\mathbb{N}} u_n \in \mathcal{M}_\omega^{A,B}$.*

Proof. Define $u_* := \inf_{j\in\mathbb{N}} u_j$ and inductively the sequence

$$v_j := \begin{cases} u_1 & \text{if } j = 1 \\ \min\{v_{j-1}, u_j\} & \text{if } j \geq 2. \end{cases} \tag{4.2.28}$$

Corollary 4.15 gives us that $\{v_j\}_{j\in\mathbb{N}} \subseteq \mathcal{M}_\omega^{A,B}$. On the other hand $v_j \to u_*$ a.e. in $\mathbb{R}^N$, so from an application of Fatou's Lemma we have that $u_* \in \mathcal{A}_\omega^{A,B}$ and

$$\mathcal{F}_\omega(u_*) \leq \lim_{j\to+\infty} \mathcal{F}_\omega(v_j) = \mathcal{F}_\omega(v_k)$$

for any $k \in \mathbb{N}$. Hence $u_* \in \mathcal{M}_\omega^{A,B}$. □

These results allow us to prove this

Proposition 4.17. *The minimal minimizer $u_\omega^{A,B}$ exists and belongs to $\mathcal{M}_\omega^{A,B}$.*

Proof. Since $\mathcal{M}_\omega^{A,B}$ is separable with respect to convergence a.e. (see [31, Proposition B.2]), for all $u \in \mathcal{M}_\omega^{A,B}$ we can find a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{M}_\omega^{A,B}$ from which we can extract a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ such that $u_{n_k} \to u$ a.e. in $\mathbb{R}^N$. We define

$$u_\omega^{A,B} := \inf_{n\in\mathbb{N}} u_n$$

and from Lemma 4.16 we get $u_\omega^{A,B} \in \mathcal{M}_\omega^{A,B}$.

We claim that $u_\omega^{A,B}$ is the minimal minimizer, that is we have to check (4.2.26) and (4.2.27).

Let $u \in \mathcal{M}_\omega^{A,B}$ and $\{u_{n_k}\}_{k\in\mathbb{N}}$ a subsequence of $\{u_n\}_{n\in\mathbb{N}}$ such that $u_{n_k} \to u$ a.e. in $\mathbb{R}^N$. By definition $u_\omega^{A,B} \le u_{n_k}$ in $\mathbb{R}^N$ for all $k \in \mathbb{N}$. Therefore, passing to the limit as $k \to +\infty$, we obtain (4.2.26).

In order to prove (4.2.27) we have to suppose the existence of $v \in \mathcal{A}_\omega^{A,B}$ such that $v \le u$ for all $u \in \mathcal{M}_\omega^{A,B}$. This implies $v \le u_n$ for all $n \in \mathbb{N}$. Hence $v \le u_\omega^{A,B}$ and (4.2.27) is proved. ☐

4.2.3 The doubling property

The doubling property, or no-symmetry breaking property, is an important feature of the minimal minimizer. In this subsection we want to show that $u_\omega^{A,B}$ is not only the minimal minimizer of $\mathcal{M}_\omega^{A,B}$, but also the minimal minimizer over the functions with periodicity multiple of $\sim$. To do this we introduce a few more notation.

We denote with $z_1, \cdots, z_{N-1} \in \mathbb{Z}^N$ some vectors spanning the $(N-1)$-dimensional lattice induced by $\sim$. If $k \in \mathbb{Z}^N$ is such that $\omega \cdot k = 0$ we can write

$$k = \sum_{i=1}^{N-1} \mu_i z_i,$$

with $\mu_1, \cdots, \mu_{N-1} \in \mathbb{Z}$. Then we take $m \in \mathbb{N}^{N-1}$ and we define the equivalence relation $\sim_m$ as

$$x \sim_m y \Leftrightarrow x - y = \sum_{i=1}^{N-1} \mu_i m_i z_i \quad \text{for } \mu_1, \cdots, \mu_{N-1} \in \mathbb{Z}.$$

We denote by $\tilde{\mathbb{R}}_m^N := \mathbb{R}^N/_{\sim_m}$ and with $L^2_{\text{loc}}(\tilde{\mathbb{R}}_m^N)$ the $\sim_m$ periodic functions of $L^2_{\text{loc}}(\tilde{\mathbb{R}}^N)$. Note that in $\tilde{\mathbb{R}}_m^N$ there are $m_1 \cdots m_{N-1}$ copies of $\tilde{\mathbb{R}}^N$ because the relation $\sim$ is stronger than $\sim_m$ and $L^2_{\text{loc}}(\tilde{\mathbb{R}}^N) \subseteq L^2_{\text{loc}}(\tilde{\mathbb{R}}_m^N)$.

We define the space

$$\mathcal{A}_{\omega,m}^{A,B} := \{u \in L^2_{\text{loc}}(\tilde{\mathbb{R}}_m^N) : u(x) \ge 1 - \delta_0 \text{ if } \omega \cdot x \le A \text{ and } u(x) \le -1 + \delta_0 \text{ if } \omega \cdot x \ge B\},$$

i.e. the admissible functions related to the new equivalence relation. Then we consider the functional

$$\begin{aligned}\mathcal{F}_{\omega,m}(u) := \frac{1}{2}\int_{\tilde{\mathbb{R}}_m^N}\int_{\mathbb{R}^N} \Big(|u(x)-u(y)|^2 - |u_+(x)-u_+(y)|^2\Big) K(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ + \int_{\tilde{\mathbb{R}}_m^N} \Big(W(x,u(x)) - W(x,u_+(x))\Big)\,\mathrm{d}x + \int_{\tilde{\mathbb{R}}_m^N} H(x)\Big(u(x)-u_+(x)\Big)\,\mathrm{d}x\end{aligned} \tag{4.2.29}$$

and the set of absolute minimizers

$$\mathcal{M}^{A,B}_{\omega,m} := \{u \in \mathcal{A}^{A,B}_{\omega,m} : \mathcal{F}_{\omega,m}(u) \leq \mathcal{F}_{\omega,m}(v) \text{ for any } v \in \mathcal{A}^{A,B}_{\omega,m}\}.$$

We call $u^{A,B}_{\omega,m}$ the minimal minimizer of $\mathcal{M}^{A,B}_{\omega,m}$ whose existence is assured by the same arguments of Subsection 4.2.2.

Finally we denote the translation of a function $u : \mathbb{R}^N \to \mathbb{R}$ in the direction $z \in \mathbb{R}^N$ by

$$\tau_z u(x) := u(x - z) \quad \text{for any } x \in \mathbb{R}^N. \tag{4.2.30}$$

At this point we can show that the minimal minimizer in a class of larger period coincides to that in a class of smaller period:

Proposition 4.18. *For any $m \in \mathbb{N}^{N-1}$, it results $u^{A,B}_{\omega,m} = u^{A,B}_{\omega}$.*

Proof. Proceeding as in [31, Proposition 4.3.1], we consider without loss of generality $m_1 = 2$ and $m_i = 1$ for every $i = 2, \cdots, N-1$. (The general case is analogous but the notation is much heavier).

First we show that $u^{A,B}_{\omega} \in \mathcal{M}^{A,B}_{\omega,m}$, since this implies that $u^{A,B}_{\omega,m} \leq u^{A,B}_{\omega}$. To do this we consider $\tau_{z_1} u^{A,B}_{\omega,m}$ (i.e. the translation of $u^{A,B}_{\omega}$ in the doubled direction of z_1) and we observe that it is an element of $\mathcal{M}^{A,B}_{\omega,m}$. Defining

$$\hat{u}^{A,B}_{\omega,m} := \min\{u^{A,B}_{\omega,m}, \tau_{z_1} u^{A,B}_{\omega,m}\}, \tag{4.2.31}$$

we may see that it is $\sim$-periodic, so $\hat{u}^{A,B}_{\omega,m} \in \mathcal{A}^{A,B}_{\omega}$. Then, using Lemma 4.8 and arguing as in the proof of Lemma 4.14, we have

$$\mathcal{F}_{\omega,m}(u^{A,B}_{\omega}) = 2\mathcal{F}_{\omega}(u^{A,B}_{\omega}) \leq 2\mathcal{F}_{\omega}(\hat{u}^{A,B}_{\omega,m}) = \mathcal{F}_{\omega,m}(\hat{u}^{A,B}_{\omega,m}) \leq \mathcal{F}_{\omega,m}(u^{A,B}_{\omega,m}).$$

As a consequence $u^{A,B}_{\omega} \in \mathcal{M}^{A,B}_{\omega,m}$ and so $u^{A,B}_{\omega,m} \leq u^{A,B}_{\omega}$, being $u^{A,B}_{\omega,m}$ the minimal minimizer of $\mathcal{M}^{A,B}_{\omega,m}$.

On the other hand, since $\hat{u}^{A,B}_{\omega,m} \in \mathcal{M}^{A,B}_{\omega,m}$ and $u^{A,B}_{\omega} \in \mathcal{A}^{A,B}_{\omega,m}$, we get

$$\mathcal{F}_{\omega}(\hat{u}^{A,B}_{\omega,m}) = \frac{1}{2}\mathcal{F}_{\omega,m}(\hat{u}^{A,B}_{\omega,m}) \leq \frac{1}{2}\mathcal{F}_{\omega,m}(u^{A,B}_{\omega}) = \mathcal{F}_{\omega}(u^{A,B}_{\omega}),$$

from which it follows that $\hat{u}^{A,B}_{\omega,m} \in \mathcal{M}^{A,B}_{\omega}$. Hence

$$u^{A,B}_{\omega} \leq \hat{u}^{A,B}_{\omega,m} \leq u^{A,B}_{\omega,m}$$

and the proof is complete. □

4.2.4 Minimization with respect to compact perturbations

In this subsection we want to construct a class A-minimizer for $\mathcal{E}$, so we have to prove that the elements of $\mathcal{M}^{A,B}_{\omega}$ are also minimizers of the energy $\mathcal{E}$ with respect to compact perturbations in the strip

$$S^{A,B}_{\omega} := \{x \in \mathbb{R}^N : \omega \cdot x \in [A, B]\}.$$

We call

$$\tilde{S}^{A,B}_{\omega,m} := S^{A,B}_{\omega}/{\sim_m}$$

the quotient of the strip with respect to the relation $\sim_m$ and we show a relation between $\mathcal{E}$ and $\mathcal{F}_{\omega,m}$.

Lemma 4.19. *Let $u \in \mathcal{A}^{A,B}_{\omega,m}$ be a bounded function such that $\mathcal{F}_{\omega,m}(u) < \infty$. For any $\Omega \subset\subset \tilde{S}^{A,B}_{\omega,m}$, consider v another bounded function such that $u = v$ in $\mathbb{R}^N \setminus \Omega$ and denote with $\varphi := v - u$. Calling $\tilde{v}$ and $\tilde{\varphi}$ the $\sim_m$-periodic extension to $\mathbb{R}^N$ of $v_{|_{\tilde{\mathbb{R}}^N_m}}$ and $\varphi_{|_{\tilde{\mathbb{R}}^N_m}}$ respectively, we have*

$$\mathcal{E}(v, \tilde{\mathbb{R}}^N_m) - \mathcal{E}(u, \tilde{\mathbb{R}}^N_m) = \mathcal{F}_{\omega,m}(\tilde{v}) - \mathcal{F}_{\omega,m}(u) + \int_{\tilde{\mathbb{R}}^N_m} \int_{\mathbb{R}^N \setminus \tilde{\mathbb{R}}^N_m} \tilde{\varphi}(x)\tilde{\varphi}(y) K(x,y) \, \mathrm{d}x \, \mathrm{d}y \,. \tag{4.2.32}$$

In particular if $u \in \mathcal{M}^{A,B}_{\omega,m}$,

$$\mathcal{E}(v, \tilde{\mathbb{R}}^N_m) - \mathcal{E}(u, \tilde{\mathbb{R}}^N_m) \geq \int_{\tilde{\mathbb{R}}^N_m} \int_{\mathbb{R}^N \setminus \tilde{\mathbb{R}}^N_m} \tilde{\varphi}(x)\tilde{\varphi}(y) K(x,y) \, \mathrm{d}x \, \mathrm{d}y \,. \tag{4.2.33}$$

Observe that, being φ compactly supported on $\tilde{S}^{A,B}_{\omega,m}$ and bounded, the right hand sides of (4.2.32) and (4.2.33) are finite (see [31, Lemma A.2 in Appendix A])

Proof. We prove the lemma in the case $m = (1, \cdots, 1)$ but the general case is analogous; moreover we only show (4.2.32) because then (4.2.33) follows noticing that $\tilde{v} \in \mathcal{A}^{A,B}_{\omega,m}$. Recalling the expression of $\mathcal{E}$ (see (4.0.1)), we start by computing $\mathcal{K}(v, \tilde{\mathbb{R}}^N, \mathbb{R}^N \setminus \tilde{\mathbb{R}}^N)$.

Proceeding as in [31, Lemma 4.4.1], we get

$$\begin{aligned} \mathscr{K}(v, \tilde{\mathbb{R}}^N, \mathbb{R}^N \setminus \tilde{\mathbb{R}}^N) &= \mathscr{K}(\tilde{v}, \tilde{\mathbb{R}}^N, \mathbb{R}^N \setminus \tilde{\mathbb{R}}^N) + \mathscr{K}(u, \tilde{\mathbb{R}}^N, \mathbb{R}^N \setminus \tilde{\mathbb{R}}^N) \\ &\quad - \mathscr{K}(v, \mathbb{R}^N \setminus \tilde{\mathbb{R}}^N, \tilde{\mathbb{R}}^N) + \int_{\tilde{\mathbb{R}}^N} \int_{\mathbb{R}^N \setminus \tilde{\mathbb{R}}^N} \tilde{\varphi}(x)\tilde{\varphi}(y) \mathscr{K}(x,y) \, \mathrm{d}x \, \mathrm{d}y \,. \end{aligned} \tag{4.2.34}$$

Then we note that

$$\mathscr{K}(v, \tilde{\mathbb{R}}^N, \tilde{\mathbb{R}}^N) = \mathscr{K}(\tilde{v}, \tilde{\mathbb{R}}^N, \tilde{\mathbb{R}}^N) \quad \text{and} \quad \mathscr{P}(v, \tilde{\mathbb{R}}^N) = \mathscr{P}(\tilde{v}, \tilde{\mathbb{R}}^N)$$

and recalling the definitions of $\mathcal{E}$ and $\mathcal{F}_\omega$ we conclude the proof. □

Now we are ready to prove that the absolute minimizers of $\mathcal{F}_{\omega,m}$ in $\mathcal{A}^{A,B}_{\omega,m}$ are also minimizers for $\mathcal{E}$ with respect to compact perturbations in $\tilde{S}^{A,B}_{\omega,m}$:

Proposition 4.20. *If $u \in \mathcal{M}^{A,B}_{\omega,m}$, then it is a local minimizer of $\mathcal{E}$ in every open set $\Omega \subset\subset \tilde{S}^{A,B}_{\omega,m}$, i.e.*

$$\mathcal{E}(u, \Omega) \leq \mathcal{E}(v, \Omega) \tag{4.2.35}$$

for all $v \equiv u$ in $\mathbb{R}^N \setminus \Omega$.

Proof. Without loss of generality we may suppose that $\mathcal{E}(v,\Omega) < +\infty$ and $|v| \leq 1+\delta_0$ a.e. in $\mathbb{R}^N$. Let $\varphi := v-u$ and note that spt $\varphi \subset \Omega$. We claim that (4.2.35) holds with Ω replaced by $\tilde{\mathbb{R}}^N_m$, i.e.

$$\mathcal{E}(u,\tilde{\mathbb{R}}^N_m) \leq \mathcal{E}(v,\tilde{\mathbb{R}}^N_m). \tag{4.2.36}$$

Then Remark 4.3 will imply (4.2.35).

To show (4.2.36) we observe that if φ is either non-negative or non-positive, then (4.2.36) is a direct consequence of (4.2.33). Moreover, if φ is sign-changing, we consider $\min\{u,u+\varphi\}$ and $\max\{u,u+\varphi\}$. From Lemma 4.8 we get

$$\mathcal{E}(\min\{u,u+\varphi\},\tilde{\mathbb{R}}^N_m)+\mathcal{E}(\max\{u,u+\varphi\},\tilde{\mathbb{R}}^N_m) \leq \mathcal{E}(u,\tilde{\mathbb{R}}^N_m)+\mathcal{E}(u+\varphi,\tilde{\mathbb{R}}^N_m).$$

Moreover, noticing that

$$\min\{u,u+\varphi\} = u-\varphi_- \quad \text{and} \quad \max\{u,u+\varphi\} = u+\varphi_+$$

and using (4.2.33), we obtain

$$\begin{aligned} 2\mathcal{E}(u,\tilde{\mathbb{R}}^N_m) &\leq \mathcal{E}(u-\varphi_-,\tilde{\mathbb{R}}^N_m)+\mathcal{E}(u+\varphi_+,\tilde{\mathbb{R}}^N_m) = \mathcal{E}(\min\{u,u+\varphi\},\tilde{\mathbb{R}}^N_m) \\ &\quad + \mathcal{E}(\max\{u,u+\varphi\},\tilde{\mathbb{R}}^N_m) \leq \mathcal{E}(u,\tilde{\mathbb{R}}^N_m)+\mathcal{E}(u+\varphi,\tilde{\mathbb{R}}^N_m) \end{aligned} \tag{4.2.37}$$

that is our thesis. ☐

As a consequence of this proposition and Subsection 4.2.3 we have the following

Corollary 4.21. *The minimal minimizer* $u^{A,B}_\omega$ *is a local minimizer of* $\mathcal{E}$ *for every* $\Omega \subset\subset S^{A,B}_\omega$.

Proof. Given Ω, consider $m \in \mathbb{N}^{N-1}$ such that $\Omega \subset\subset \tilde{S}^{A,B}_{\omega,m}$. Thanks to Proposition 4.18, $u^{A,B}_\omega$ is the minimal minimizer with respect to $\mathcal{M}^{A,B}_{\omega,m}$ and Proposition 4.20 implies that $u^{A,B}_\omega$ is a local minimizer of $\mathcal{E}$ in Ω. ☐

4.2.5 The Birkhoff property

In this subsection we recall a geometric property of the level sets of the minimal minimizer called the Birkhoff property, or non-self intersection property, representing the fact that the level sets of the minimal minimizers are ordered under translations.

We start giving some useful notation. We define

$$\tau_z E := E+z = \{x+z : x \in E\}$$

the translation of a set $E \subseteq \mathbb{R}^N$ with respect to $z \in \mathbb{R}^N$ and observe that for a sublevel set (and analogously for a superlevel set)

$$\tau_z\{u<\theta\} = \{\tau_z u < \theta\}.$$

Definition 4.22. We say that $E \subseteq \mathbb{R}^N$ has the Birkhoff property with respect to a vector $\bar{\omega} \in \mathbb{R}^N$ if

- $\tau_k E \subseteq E$ for any $k \in \mathbb{Z}^N$ such that $\bar{\omega} \cdot k \leq 0$, and
- $\tau_k E \supseteq E$ for any $k \in \mathbb{Z}^N$ such that $\bar{\omega} \cdot k \geq 0$.

We call *Birkhoff set* a set satisfying the Birkhoff property and we recall an useful result about these sets:

Proposition 4.23. *[31, Proposition 4.5.2] Let $E \subseteq \mathbb{R}^N$ a Birkhoff set with respect to $\bar{\omega} \in \mathbb{R}^N \setminus \{0\}$ and containing a ball $B_{\sqrt{N}}$ of radius $\sqrt{N}$. Then E contains a half-space including the center of the ball, is delimited by a hyperplane orthogonal to the vector $\bar{\omega}$ and is such that $\bar{\omega}$ points outside of it.*

Now proceeding as in [31, Proposition 4.5.3], we want to show that level sets of the minimal minimizer are Birkhoff sets.

Proposition 4.24. *Given $\theta \in \mathbb{R}$, the superlevel set $\{u_\omega^{A,B} > \theta\}$ has the Birkhoff property with respect to ω, i.e.*

- *$\{\tau_k u_\omega^{A,B} > \theta\} \subseteq \{u_\omega^{A,B} > \theta\}$, for any $k \in \mathbb{Z}^N$ such that $\omega \cdot k \leq 0$, and*
- *$\{\tau_k u_\omega^{A,B} > \theta\} \supseteq \{u_\omega^{A,B} > \theta\}$, for any $k \in \mathbb{Z}^N$ such that $\omega \cdot k \geq 0$.*

In the same way the sublevel set $\{u_\omega^{A,B} < \theta\}$ has the Birkhoff property with respect to $-\omega$.

Proposition 4.24 still holds if strict levels are replaced by the broad ones.

Proof. Denote with $v := \min\{u_\omega^{A,B}, \tau_k u_\omega^{A,B}\}$ and note that $\tau_k u_\omega^{A,B}$ is the minimal minimizer with respect to $\tau_k \mathcal{M}\omega^{A,B} = \mathcal{M}\omega^{A+\omega\cdot k, B+\omega\cdot k}$. Now, if $\omega \cdot k \leq 0$, from Lemma 4.14 we have that $v \in M_\omega^{A+\omega\cdot k, B+\omega\cdot k}$, so that $\tau_k u_\omega^{A,B} \leq v \leq u_\omega^{A,B}$. Therefore

$$\{\tau_k u_\omega^{A,B} > \theta\} \subseteq \{u_\omega^{A,B} > \theta\}.$$

Similarly, if $\omega \cdot k \geq 0$ we get that $v \in \mathcal{M}_\omega^{A,B}$ and hence

$$\{u_\omega^{A,B} > \theta\} \subseteq \{\tau_k u_\omega^{A,B} > \theta\}.$$

For the conclusion concerning the sublevel set $\{u_\omega^{A,B} \leq \theta\}$ and the superlevel set $\{u_\omega^{A,B} \geq \theta\}$ we can reason as in [31, Proposition 4.5.3]. ☐

4.2.6 Unconstrained and class A-minimization

From now on we consider strips of the form

$$S_\omega^M := S_\omega^{0,M} = \{x \in \mathbb{R}^N : \omega \cdot x \in [0, M]\}.$$

We denote the space of admissible functions $\mathcal{A}_\omega^{0,M}$ with $\mathcal{A}_\omega^M$, the absolute minimizers with $\mathcal{M}_\omega^M$ and the minimal minimizer with u_ω^M. Since we want to avoid narrow strips, we assume $M > 10|\omega|$.

The goal of this subsection is to show that, for large universal values of $M/|\omega|$, the minimal minimizer u_ω^M becomes unconstrained, i.e. it no longer feels boundary data prescribed outside S_ω^M, gaining additional minimizing properties in the whole $\mathbb{R}^N$. First of all we adapt the results of Section 4.1 to the minimal minimizer u_ω^M and, in view of Corollary 4.21, we have that u_ω is a local minimizer for $\mathcal{E}$ inside the strip S_ω^M. Thus, from Theorem 4.6, we get the existence of universal quantities $\alpha \in (0,1)$ and $C_1 \geq 1$ such that

$$\|u_\omega^M\|_{C^{0,\alpha}(S)} \leq C_1 \tag{4.2.38}$$

for any open $S \subset\subset S_\omega^M$ with $\operatorname{dist}(S, \partial S_\omega^M) \geq 1$. Then from Proposition 4.7, fixed $x_0 \in S_\omega^M$ and $R \geq 3$ such that $B_{R+2}(x_0) \subset\subset S_\omega^M$ we get that

$$\mathcal{E}(u_\omega^M, B_R(x_0)) \leq C_2 R^{N-1} \Psi_R(R) \tag{4.2.39}$$

where $C_2 > 0$ is a universal constant and $\Psi_R(R)$ is defined in (4.1.5).

These two inequalities have a crucial role to show the main result of this section:

Theorem 4.25. *There exists a universal constant $M_0 > 0$ such that if $M \geq M_0|\omega|$, the distance between the superlevel set $\{u_\omega^M > -1+\delta_0\}$ and the upper constraint $\{\omega \cdot x = M\}$ delimiting S_ω^M is at least* 1.

Proof. First of all we point out that during this proof we will denote balls B and cubes Q without expliciting their center. Then we claim that

$$\begin{aligned}&\exists\ M_0 \geq 8N \text{ universal constant such that, for any } M \geq M_0|\omega|, \text{ we find a ball}\\ &B_{\sqrt{N}}(\bar z) \subset\subset S_\omega^M \text{ for some } \bar z \in S_\omega^M \text{ on which either } u_\omega^M \geq 1-\delta_0 \text{ or } u_\omega^M \leq -1+\delta_0.\end{aligned} \tag{4.2.40}$$

Given $M \geq 8N|\omega|$, assume that for every ball $\tilde B_{\sqrt{N}} \subset\subset S_\omega^M$ we can find a point $\tilde x \in \tilde B_{\sqrt{N}}$ such that $|u_\omega^M(\tilde x)| < 1-\delta_0$. If we prove that $M/|\omega| \leq M_0$, then claim (4.2.40) follows.

Proceeding as in [31, Proposition 4.6.1] we take $k \geq 2$ to be the only integer such that

$$k \leq \frac{M}{4N|\omega|} < k+1. \tag{4.2.41}$$

Then we let $x_0 \in S_\omega^M$ be a point on the hyperplane $\left\{\omega \cdot x = \frac{M}{2}\right\}$ and $B = B_{Nk}(x_0)$. Thanks to (4.2.41) we get that $B \subset\subset S_\omega^M$ with

$$\operatorname{dist}(B, \partial S_\omega^M) = \frac{M}{2|\omega|} - Nk \geq Nk \geq 4. \tag{4.2.42}$$

Therefore we can apply (4.2.38) to obtain

$$\|u_\omega^M\|_{C^{0,\alpha}(B)} \leq C_1. \tag{4.2.43}$$

Now we consider Q a cube with center in x_0 and side $2\sqrt{N}k$. Clearly $Q \subset B$ and we can partition it (up a negligible set) into a collection $\{Q_j\}_{j=1}^{k^N}$ of cubes with sides $2\sqrt{N}$ parallel to those of Q. Then we call $B_j \subset Q_j$ the ball of radius $\sqrt{N}$ with the same center of Q_j. By our initial assumption, for every $j = 1, \cdots, k^N$, there exists $\tilde{x}_j \in B_j$ such that $|u_\omega^M(\tilde{x}_j)| < 1 - \delta_0$. We claim that

$$|u_\omega^M| < 1 - \delta_1 \quad \text{in } B_{r_0}(\tilde{x}_j) \tag{4.2.44}$$

for some $\delta_1 < \delta_0$ and some universal radius $r_0 \in (0, 1)$. Indeed, defining $r_0 := \left(\frac{\delta_0 - \delta_1}{C_1}\right)^{1/\alpha}$, by (4.2.43), we have

$$|u_\omega^M(x)| \le |u_\omega^M(\tilde{x}_j)| + C_1|x - \tilde{x}_j|^\alpha < 1 - \delta_0 + C_1 r_0^\alpha =: 1 - \delta_1,$$

for any $x \in B_{r_0}(\tilde{x}_j)$ and the claim is proved.

On the other hand, since $\tilde{x}_j \in B_j \subset Q_j$, we get

$$|B_{r_0}(\tilde{x}_j) \cap Q_j| \ge \frac{1}{2^N}|B_{r_0}(\tilde{x}_j)| = \frac{\omega_N}{2^N} r_0^N. \tag{4.2.45}$$

Therefore, from (4.2.44), (4.2.45) and (W2) we obtain

$$\begin{aligned} \mathscr{P}(u_\omega^M, B) &\ge \mathscr{P}(u_\omega^M, Q) = \sum_{j=1}^{k^N} \mathscr{P}(u_\omega^M, Q_j) \ge \sum_{j=1}^{k^N} \mathscr{P}(u_\omega^M, B_{r_0}(\tilde{x}_j) \cap Q_j) \\ &= \sum_{j=1}^{k^N} \int_{B_{r_0}(\tilde{x}_j) \cap Q_j} \Big(W(x, u_\omega^M(x)) + H(x) u_\omega^M(x) \Big)\, \mathrm{d}x \\ &\ge \Big[\gamma(1 - \delta_1) - \eta(1 - \delta_1) \Big] \sum_{j=1}^{k^N} |B_{r_0}(\tilde{x}_j) \cap Q_j| \\ &\ge \Big[\gamma(1 - \delta_1) - \eta(1 - \delta_1) \Big] \frac{\omega_N}{2^N} r_0^N k^N := C_3 k^N, \end{aligned} \tag{4.2.46}$$

where $C_3 > 0$ is a universal constant. Moreover from (4.2.39) (that we can apply to B thanks to (4.2.42))

$$\mathscr{P}(u_\omega^M, B) \le \mathcal{E}(u_\omega^M, B) \le C_2 (Nk)^{N-1} \Psi_s(Nk) \le C_4 k^{N-1} \Psi_s(k),$$

for some universal $C_4 > 0$. Comparing the last two inequalities and recalling (4.1.5), we deduce that k cannot be greater than a universal constant. By (4.2.41), the same holds for $M/|\omega|$, so (4.2.40) follows.

Thus it remains to show that u_ω^M cannot be greater or equal to $1 - \delta_0$ on $B_{\sqrt{N}}(\bar{z})$, showing that $u_\omega \le -1 + \delta_0$ in $B_{\sqrt{N}}(\bar{z})$.

Assume by contradiction that

$$u_\omega^M \ge 1 - \delta_0 \quad \text{in } B_{\sqrt{N}}(\bar{z}). \tag{4.2.47}$$

Using Proposition 4.24 we have that the set $\{u_\omega^M \ge 1 - \delta_0\}$ has the Birkhoff property with respect to ω. Then, (4.2.47) and Proposition 4.23 imply that the superlevel set contains the half-space $\Pi_- := \{\omega \cdot (x - \bar{z}) < 0\}$. Since $B_{\sqrt{N}}(\bar{z}) \subset S^M_\omega$, we can affirm that $\partial \Pi_-$ is at least at distance 1 from the level constraint $\{\omega \cdot x = 0\}$.

As a consequence if we suppose w.l.o.g. that $\omega_1 > 0$, the translation $\tau_{-e_1} u^M_\omega \in \mathcal{A}^M_\omega$. In view of the periodicity assumptions (K3), (W4) and (H3), we get that $\mathcal{F}_\omega(\tau_{-e_1} u^M_\omega) = \mathcal{F}_\omega(u^M_\omega)$ and so $\tau_{-e_1} u^M_\omega \in \mathcal{M}^M_\omega$. Then, since u^M_ω is the minimal minimizer, it results

$$u^M_\omega(x + e_1) = \tau_{-e_1} u^M_\omega \geq u^M_\omega(x) \quad \text{for a.e. } x \in \mathbb{R}^N.$$

Now, iterating this inequality we obtain

$$u^M_\omega(x + te_1) \geq u^M_\omega(x) \geq 1 - \delta_0 \quad \text{for a.e. } x \in \Pi_- \text{ and } t \in \mathbb{N}$$

or equivalently $u^M_\omega \geq 1 - \delta_0$ a.e. in $\mathbb{R}^N$ that contradicts the fact that $u^M_\omega \leq -1 + \delta_0$ in $\{\omega \cdot x \geq M\}$ by construction. As a consequence $u^M_\omega \leq -1 + \delta_0$ on the ball $B_{\sqrt{N}}(\bar{z})$, hence applying again Proposition 4.24 and Proposition 4.23 to the sublevel set $\{u^M_\omega \leq -1 + \delta_0\}$, we prove the theorem. □

Corollary 4.26. *If $M \geq M_0|\omega|$, then $u^M_\omega = u^{M+a}_\omega$ for all $a \geq 0$.*

Proof. Given $M \geq M_0|\omega|$ and $a \in [0, 1]$, we may apply Theorem 4.25 to the minimal minimizer u^{M+a}_ω to obtain that $u^{M+a}_\omega \leq -1 + \delta_0$ a.e. in the half-space $\{\omega \cdot x \geq M\}$. But then $u^{M+a}_\omega \in \mathcal{A}^M_\omega$ and by minimality of u^M_ω, we get that $\mathcal{F}_\omega(u^M_\omega) \leq \mathcal{F}_\omega(u^{M+a}_\omega)$.

On the other hand, obviously $u^M_\omega \in \mathcal{A}^{M+a}_\omega$, so $\mathcal{F}_\omega(u^{M+a}_\omega) \leq \mathcal{F}_\omega(u^M_\omega)$. Therefore u^M_ω and u^{M+a}_ω belong to $\mathcal{M}^M_\omega \cap \mathcal{M}^{M+a}_\omega$ and hence they are the same function. Iterating this argument we can extend this result to any $a \geq 0$. □

Roughly speaking, this corollary tells us that if $M/|\omega|$ is greater than the universal constant M_0 found in Theorem 4.25, the upper constraint $\{\omega \cdot x = M\}$ becomes irrelevant for the minimal minimizer u^M_ω which achieves values below $-1 + \delta_0$ well before touching the constraint.

In the next proposition we show that we have an analogous behaviour with the lower constraint $\{\omega \cdot x = 0\}$ and hence we get that the minimal minimizer u^M_ω is unconstrained.

Proposition 4.27. *If $M \geq M_0|\omega|$, then $u^M_\omega \in \mathcal{M}^{-a,M+a}_\omega$ for any $a \geq 0$, i.e. u^M_ω is unconstrained.*

Proof. Fix $k \in \mathbb{Z}^N$ such that $\omega \cdot k \geq a$. Let $v \in \mathcal{A}^{-a,M+a}_\omega$ and consider its translation $\tau_k v \in \mathcal{A}^{M+a+\omega \cdot k}_\omega$. Corollary 4.26 tells us that $\mathcal{F}_\omega(u^M_\omega) \leq \mathcal{F}_\omega(\tau_k v)$ and the thesis follows since by (K3), (W4) and (H3), we have that $\mathcal{F}_\omega(v) = \mathcal{F}_\omega(\tau_k v)$. □

We conclude this subsection combining the previous proposition with the results of Subsection 4.2.1 obtaining that u^M_ω is a class A minimizer.

Theorem 4.28. *If $M \geq M_0|\omega|$, then u^M_ω is a class A minimizer of the functional $\mathcal{E}$.*

Proof. The proof is similar to that of [31, Theorem 4.6.4]; we include it for completeness. Let $\Omega \subseteq \mathbb{R}^N$ be a bounded subset. Take $a \geq 0$ and $m \in \mathbb{Z}^{N-1}$ such that $\Omega \subset\subset \tilde{S}^{-a,M+a}_{\omega,m}$. From Proposition 4.18, it follows that $u^{-a,M+a}_{\omega}$ is the minimal minimizer of the class $\mathcal{M}^{-a,M+a}_{\omega,m}$. On the other hand Proposition 4.27 implies that $\mathcal{F}_\omega(u^M_\omega) = \mathcal{F}_\omega(u^{-a,M+a}_\omega)$. Then

$$\mathcal{F}_{\omega,m}(u^M_\omega) = c_m \mathcal{F}_\omega(u^M_\omega) = c_m \mathcal{F}_\omega(u^{-a,M+a}_\omega) = \mathcal{F}_{\omega,m}(u^{-a,M+a}_\omega),$$

where $c_m = \Pi^{N-1}_{i=1} m_i$.

Therefore $u^M_\omega \in \mathcal{M}^{-a,M+a}_{\omega,m}$ and Proposition 4.20 yields that u^M_ω is a local minimizer of $\mathcal{E}$ in Ω. ☐

4.2.7 The case of irrational directions

In this subsection we want to prove Theorem 4.5 with the assumption (K4), also for irrational vectors ω. We will use an approximation argument as in [31, Subsection 4.7].

Taken $\omega \in \mathbb{R}^N \setminus \mathbb{Q}^N$, we consider a sequence $\{\omega_j\}_{j\in\mathbb{N}} \subset \mathbb{Q}^N \setminus \{0\}$ such that $\omega_j \to \omega$. Denoting with u_j the class A minimizer given by our construction which corresponds to ω_j, we know that $u_j \in H^s_{\mathrm{loc}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $|u_j| \leq 1 + \delta_0$ in $\mathbb{R}^N$ and

$$\left\{x \in \mathbb{R}^N : |u_j(x)| \leq 1 - \delta_0\right\} \subseteq \left\{x \in \mathbb{R}^N : \frac{\omega_j}{|\omega_j|} \cdot x \in [0, M_0]\right\} \tag{4.2.48}$$

for any $j \in \mathbb{N}$. Moreover Theorem 4.6 implies that the u_j's are uniformly bounded in $C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ for some unversal $\alpha \in (0,1)$. So, thanks to Ascoli-Arzelà Theorem we can find a subsequence of $\{u_j\}_{j\in\mathbb{N}}$ (not relabeled) converging to some continuous function u, uniformly on compact subsets of $\mathbb{R}^N$ and $|u| \leq 1 + \delta_0$ in $\mathbb{R}^N$. Since condition (4.2.48) passes to the limit, the same inclusion holds if we replace u_j and w_j with u and w. Hence, to prove Theorem 4.5 we only need to check that u is a class A minimizer of $\mathcal{E}$. With this aim in mind we fix $R \geq 1$ and we claim that u is a local minimizer of $\mathcal{E}$ in B_R, i.e. $\mathcal{E}(u, B_R) < +\infty$ and

$$\mathcal{E}(u, B_R) \leq \mathcal{E}(u + \varphi, B_R) \quad \text{for any } \varphi \text{ such that } \operatorname{spt} \varphi \subset B_R. \tag{4.2.49}$$

Thanks to Remark 4.3, this will implies that u is a class A minimizer.

To show (4.2.49) we apply Theorem 4.7 to u_j so

$$\mathcal{E}(u_j, B_{R+1}) \leq C_R$$

for some constant $C_R > 0$ independent of j. Moreover by an application of Fatou's Lemma

$$\mathcal{E}(u, B_{R+\tau}) \leq \liminf_{j\to+\infty} \mathcal{E}(u_j, B_{R+\tau}) \tag{4.2.50}$$

for any $\tau \in [0,1]$. In particular

$$\mathcal{E}(u, B_R) \leq \mathcal{E}(u, B_{R+1}) \leq C_R < +\infty \tag{4.2.51}$$

because $\mathcal{E}(u,\cdot)$ is monotone non-decreasing with respect to set inclusion. As it concerns the right hand side of (4.2.50) we let $\{\varepsilon_j\}_{j\in\mathbb{N}}$ such that

$$\varepsilon_j := \|u_j - u\|_{L^\infty(B_{R+1})}.$$

It is easy to see that $\varepsilon_j \to 0$ and we may suppose that $\varepsilon_j \le 1/2$ for any $j \in \mathbb{N}$. Then we consider $\eta_j \in C_c^\infty(\mathbb{R}^N)$ a cut-off function such that $0 \le \eta_j \le 1$ in $\mathbb{R}^N$, $\eta_j = 1$ in B_R, spt $(\eta_j) \subseteq B_{R+\varepsilon_j}$ and $|\nabla \eta_j| \le 2/\varepsilon_j$ in $\mathbb{R}^N$. Take φ as in (4.2.49) and assume w.l.o.g. that $\varphi \in L^\infty(\mathbb{R}^N)$. We also suppose that $\mathcal{E}(u+\varphi, B_R) < +\infty$, otherwise (4.2.49) is obviously satisfied. Consequently, using (4.2.51), (K2) and the boundedness of H, u and φ, we get that $\varphi \in H^s(B_{R+1})$. At this point we define $v := u + \varphi$ and

$$v_j := \eta_j u + (1-\eta_j)u_j + \varphi \quad \text{in } \mathbb{R}^N.$$

Observe that $v_j = v$ in B_R and $v_j = u_j$ in $\mathbb{R}^N \setminus B_{R+\varepsilon_j}$, hence v_j is an admissible competitor for u_j in $B_{R+\varepsilon_j}$. Then, being u_j minimizer,

$$\mathcal{E}(u_j, B_{R+\varepsilon_j}) \le \mathcal{E}(v_j, B_{R+\varepsilon_j}). \tag{4.2.52}$$

Moreover $v_j \to v$ uniformly on compact subsets of $\mathbb{R}^N$ and

$$\|v_j - v\|_{L^\infty(B_{R+1})} \le \|u_j - u\|_{L^\infty(B_{R+1})} = \varepsilon_j.$$

Now, we want to deal with the right-hand side of (4.2.52). We can proceed as in [31, Pag. 32-34] to decompose $\mathcal{C}_{B_{R+\varepsilon_j}}$ and estimate $\mathcal{E}(v_j, B_{R+\varepsilon_j})$ in each of these regions. Then we can use

$$\mathscr{P}(v_j, B_{R+\varepsilon_j}) \le \mathscr{P}(v, B_R) + W^*|B_{R+\varepsilon_j} \setminus B_R| + \eta\|u\|_{L^\infty(\mathbb{R}^N)}|B_{R+\varepsilon_j} \setminus B_R|$$

to say that there exists a function $r : (0,1) \to (0,+\infty)$ for which

$$\lim_{\delta \to 0^+} r(\delta) = 0 \tag{4.2.53}$$

such that

$$\limsup_{j\to+\infty} \mathcal{E}(u_j, B_R) \le \mathcal{E}(v, B_R) + r(\delta).$$

Combining this inequality with (4.2.50) we have

$$\mathcal{E}(u, B_R) \le \mathcal{E}(v, B_R) + r(\delta)$$

and since δ is arbitrary and (4.2.53) holds, we obtain (4.2.49), i.e. u is a class A minimizer of $\mathcal{E}$.

4.3 Proof of Theorem 4.5 for general kernels

In this section we want to prove Theorem 4.5 also for kernel not satisfying condition (K4). Indeed none of the estimates that we showed there involve any of the parameters appearing in (K4). So we can use a limit argument similar to this of Section 4.2.7.

Let K be a kernel satisfying (K1), (K2), (K3) and consider a monotone increasing sequence $\{R_j\}_{j\in\mathbb{N}} \subset [2, +\infty)$ diverging to $+\infty$. We define

$$K_j(x, y) := K(x, y)\chi_{[0,R_j]}(|x - y|) \quad \text{for any } x, y \in \mathbb{R}^N$$

and we observe that it fulfills (K1), (K2), (K3). Moreover K_j satisfies (K4) with $\bar{R} = R_j$. Call $\mathcal{E}_j$ the energy functional (4.0.3) corresponding to K_j and, fixed a direction $\omega \in \mathbb{R}^N \setminus \{0\}$, we denote with u_j the plane-like class A-minimizer for $\mathcal{E}_j$ with direction ω. Since K_j verifies (K4) these minimizers exist thanks to Section 4.2. We have

$$\Big\{x \in \mathbb{R}^N : |u_j(x)| \le 1 - \delta_0\Big\} \subseteq \Big\{x \in \mathbb{R}^N : \frac{\omega}{|\omega|} \cdot x \in [0, M_0]\Big\}, \tag{4.3.1}$$

for a universal value $M_0 > 0$. We also know that $|u_j| \le 1 + \delta_0$ in $\mathbb{R}^N$ and, thanks to Theorem 4.6, $\|u_j\|_{C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)} \le C$ for some $\alpha \in (0, 1]$. We underline that, since K_j satisfies (K2) with the same structural constants, we can choose M_0, α and C independent of j. As a consequence, Ascoli-Arzelà Theorem implies that, up to a subsequence, $\{u_j\}$ converges to a continuous function u, uniformly on compact subsets of $\mathbb{R}^N$. The limit function u satisfies (4.3.1) and, if ω is rational, each u_j is $\sim$-periodic, hence u is $\sim$-periodic. To show that u is a class A minimizer, we fix $R \ge 1$ and we take a perturbation φ with spt $\varphi \subset\subset B_R$. We know that

$$\mathcal{E}_j(u_j, B_R) \le \mathcal{E}_j(u_j + \varphi, B_R) \quad \text{for any } j \in \mathbb{N}.$$

On the other hand from an application of Fatou's Lemma we get

$$\mathcal{E}(u, B_R) \le \liminf_{j\to+\infty} \mathcal{E}_j(u_j, B_R)$$

and following the reasoning of the Subsection 4.2.7 we have that

$$\limsup_{j\to+\infty} \mathcal{E}_j(u_j, B_R) \le \mathcal{E}(u + \varphi, B_R).$$

These two inequalities tell us that u is a class A minimizer of $\mathcal{E}$ so Theorem 4.5 is completely proved.

5 On critical points of the relative fractional perimeter

In this last chapter of this thesis we shift our attention to the fractional perimeter. In particular, we focus first on the localization of sets with constant nonlocal mean curvature (briefly denoted with CNMC sets) and small prescribed volume relative to an open bounded domain. Then, in the second part of the chapter we study the existence and some properties of sets minimizing the fractional perimeter in an half-space.

5.1 Localization of sets with CNMC and small volume

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set. We consider the fractional perimeter of a measurable set $E \subset \mathbb{R}^N$ in Ω as the interaction between E and its complement inside Ω, i.e.

$$\bar{P}_s(E, \Omega) := \int_E \int_{\Omega \setminus E} \frac{\mathrm{d}x\, \mathrm{d}y}{|x-y|^{N+2s}}, \tag{5.1.1}$$

where $s \in (0, 1/2)$.

Notice that, with respect to the general definition of fractional perimeter given in (2.2.3), in this definition we are neglecting the interaction between $E \cap \Omega$ and $E^C \setminus \Omega$.

Similar to what we saw in Section 2.2.1, the nonlocal mean curvature (in Ω) of ∂E at $x \in \partial E$ corresponding to (5.1.1) is given by

$$H^{\Omega}_{s,\partial E}(x) := P.V. \int_{\Omega} \frac{\chi_E(y) - \chi_{E^C \cap \Omega}(y)}{|x-y|^{N+2s}}\, \mathrm{d}y\,. \tag{5.1.2}$$

Observe that, when $\Omega = \mathbb{R}^N$, (5.1.2) coincides with (2.2.6), so we will simply write $H_{s,\partial E}$ to refer to $H^{\mathbb{R}^N}_{s,\partial E}$.

The first main result of this chapter is to prove that sets with constant nonlocal mean curvature and prescribed small volume in a bounded open set with smooth boundary are sufficiently close to critical points of a suitable nonlocal potential:

Theorem 5.1. *Let $s \in (0, 1/2)$ and $\Omega \subseteq \mathbb{R}^N$ be a bounded open set with smooth boundary.*

For x in a given compact set Θ of Ω, set

$$V_{\Omega}(x) := \int_{\mathbb{R}^N \setminus \Omega} \frac{1}{|x-y|^{N+2s}}\, \mathrm{d}y\,.$$

Then for every strict local extremal or non-degenerate critical point x_0 of V_{Ω} in Ω, there exists $\overline{\varepsilon} > 0$ such that for every $0 < \varepsilon < \overline{\varepsilon}$ there exist spherical-shaped surfaces S_{ε}

with constant $H^{\Omega}_{s,\partial S_\varepsilon}$ *curvature and enclosing volume identically equal to* ε*, approaching* x_0 *as* $\varepsilon \to 0$.

We prove this result using the non-degeneracy of sheres with respect to the linearized nonlocal mean curvature equation, which follows from a result in [19]. Moreover, the central tool of the proof is a Lyapunov-Schmidt reduction which allows us to study a finite-dimensional problem, treated by carefully expanding the relative fractional perimeter of balls with small volume.

Then, thanks to classical results in min-max theory, we deduce a multiplicity result:

Corollary 5.2. *Let* $s \in (0, 1/2)$ *and* $\Omega \subseteq \mathbb{R}^N$ *be a bounded open set with smooth boundary. Then there exists* $\bar{\varepsilon} > 0$ *such that for every* $0 < \varepsilon < \bar{\varepsilon}$ *there exist at least cat*(Ω) *spherical-shaped surfaces* S_ε *with constant* $H^{\Omega}_{s,\partial S_\varepsilon}$ *curvature and enclosing volume identically equal to* ε.

Here $\mathrm{cat}(\Omega)$ denotes the Lusternik-Schnirelman category of the set Ω (see [59] and Section 2.5 for more details).

5.1.1 The Lyapunov-Schmidt reduction

In this section we show a finite-dimensional reduction, i.e. the Lyapunov-Schmidt reduction, which will determine the location of critical points of the relative fractional perimeter depending on s and the geometry of the domain. To obtain it, one of the main tools is the following asymptotic expansion of the relative s-perimeter.

From now on we consider $s \in (0, 1/2)$ and, for every $\varepsilon > 0$, we set $\Omega_\varepsilon := \dfrac{1}{\varepsilon}\Omega$. We aim to prove that the nonlocal mean curvature H^{Ω}_s is *sufficiently close* to $H^{\mathbb{R}^N}_s$.

Lemma 5.3. *Let* $\Theta \subseteq \Omega$ *be a fixed compact set. For all* $\varepsilon > 0$ *we consider* $B_1(\bar{x})$ *a ball of center* $\bar{x} \in \Theta_\varepsilon := \frac{1}{\varepsilon}\Theta$ *and unit radius. Then, for the fractional perimeter defined in* (5.1.1)*, the following expansion holds*

$$\bar{P}_s(B_1(\bar{x}), \Omega_\varepsilon) = P_s(B_1(\bar{x})) - \omega_N \varepsilon^{2s} V_\Omega(\varepsilon\bar{x}) + O(\varepsilon^{1+2s}) \qquad \text{as } \varepsilon \to 0, \tag{5.1.3}$$

where ω_N *is the volume of the* N*-dimensional unit ball and*

$$V_\Omega(\varepsilon\bar{x}) := \int_{\mathbb{R}^N \setminus \Omega} \frac{1}{|\bar{x} - y|^{N+2s}} \,\mathrm{d}y\,. \tag{5.1.4}$$

Moreover one has that

$$\nabla_{\bar{x}} \bar{P}_s(B_1(\bar{x}), \Omega_\varepsilon) = -\omega_N \varepsilon^{2s+1} \nabla_{\bar{x}} V_\Omega(\varepsilon\bar{x}) + O(\varepsilon^{2+2s}). \tag{5.1.5}$$

Proof. Taking ε small enough, we can assume $B_1(\bar{x}) \subset \Omega_\varepsilon$. Recalling (2.2.1), we have

$$\bar{P}_s(B_1(\bar{x}), \Omega_\varepsilon) - P_s(B_1(\bar{x})) = -\int_{B_1(\bar{x})} \int_{\mathbb{R}^N \setminus \Omega_\varepsilon} \frac{1}{|x - y|^{N+2s}} \,\mathrm{d}x \,\mathrm{d}y\,. \tag{5.1.6}$$

If we replace x with $\bar{x}$ in the last integrand, we obtain

$$\frac{1}{|x-y|^{N+2s}} = \frac{1}{|\bar{x}-y|^{N+2s}} + O\left(\frac{1}{|\bar{x}-y|^{N+2s+1}}\right) \qquad x \in B_1(\bar{x}), \quad y \in \mathbb{R}^N \setminus \Omega_\varepsilon .$$

Therefore

$$\begin{aligned}&\int_{B_1(\bar{x})} \int_{\mathbb{R}^N \setminus \Omega_\varepsilon} \frac{1}{|x-y|^{N+2s}} \, \mathrm{d}x \, \mathrm{d}y \\ &\qquad = \omega_N \int_{\mathbb{R}^N \setminus \Omega_\varepsilon} \frac{1}{|\bar{x}-y|^{N+2s}} \, \mathrm{d}y + \int_{\mathbb{R}^N \setminus \Omega_\varepsilon} \frac{O(1)}{|\bar{x}-y|^{N+2s+1}} \, \mathrm{d}y \, .\end{aligned}$$

From the latter formulas and a change of variables, we find that

$$\bar{P}_s(B_1(\bar{x}), \Omega_\varepsilon) - P_s(B_1(\bar{x})) = -\varepsilon^{2s} \omega_N \int_{\Omega^C} \frac{1}{|\bar{x}-y|^{N+2s}} \, \mathrm{d}y + O(\varepsilon^{1+2s}),$$

which concludes the proof of (5.1.3). Formula (5.1.5) follows in a similar manner. □

Now we want to evaluate the deviation of the nonlocal mean curvature from a constant, when it is computed relatively to a large domain. To do that, we define

$$\begin{aligned}&\tilde{H}_{s,\xi} : S^{N-1} \to \mathbb{R} \\ &\tilde{H}_{s,\xi}(x) := H^{\Omega_\varepsilon}_{s,S_\xi}(x+\xi).\end{aligned} \tag{5.1.7}$$

Lemma 5.4. *Let $\beta \in (2s,1)$. For the (relative) fractional mean curvature defined in* (5.1.2)*, the following expansion holds:*

$$\tilde{H}_{s,\xi} = c_{N,s} + O(\varepsilon^{2s}) \qquad \text{in } C^{\beta-2s}(S^{N-1}), \tag{5.1.8}$$

where $c_{N,s} := H_{s,S_\xi}$ and we recall that $S_\xi = \partial B_1(\xi)$ with $B_1(\xi)$ denoting the ball of center at ξ and unit radius. Moreover, one has that for all $i = 1, \ldots, N$,

$$\frac{\partial}{\partial \xi_i} \tilde{H}_{s,\xi} = O(\varepsilon^{2s+1}) \qquad \text{in } C^{\beta-2s}(S^{N-1}). \tag{5.1.9}$$

Proof. Using the definition of (relative) fractional mean curvature (see (5.1.2)) and [77, Lemma 2], for $x \in \partial B_1$, we can write

$$\tilde{H}_{s,\xi}(x) = c_{N,s} + \int_{\mathbb{R}^N \setminus \Omega_\varepsilon} \frac{\mathrm{d}y}{|x+\xi-y|^{N+2s}} . \tag{5.1.10}$$

where $c_{N,s} := H_{s,\xi}(\cdot + \xi)$.

Therefore we get that, for $x \in \partial B_1$,

$$\tilde{H}_{s,\xi}(x) = c_{N,s} + O(\varepsilon^{2s}). \tag{5.1.11}$$

Then, using (5.1.10) and differentiating with respect to ξ_i, we find that, for all $i = 1, \dots, N$,

$$\begin{aligned} \frac{\partial}{\partial \xi_i} \tilde{H}_{s,\xi} &= \frac{\partial}{\partial \xi_i} \left(c_{N,s} + \int_{\mathbb{R}^N \setminus \Omega_\varepsilon} \frac{\mathrm{d}y}{|x+\xi-y|^{N+2s}} \right) \\ &= O\left(\int_{\mathbb{R}^N \setminus \Omega_\varepsilon} \frac{\mathrm{d}y}{|x+\xi-y|^{N+2s+1}} \right) = O(\varepsilon^{2s+1}). \end{aligned} \tag{5.1.12}$$

Thus, we proved (5.1.8) and (5.1.9) in a pointwise sense. It is easy however to see that they also hold in the C^1 sense on the unit sphere S_ξ, and therefore also in $C^{\beta-2s}(S^{N-1})$. $\square$

At this point we can perform the finite-dimensional reduction of the problem, which is possible by the smallness of volume in the statement of Theorem 5.1.

We refer to [3] for a general treatment of the subject and to Section 2.4 for the setting used in the following.

Proposition 5.5. *Suppose that Ω is a smooth bounded set of $\mathbb{R}^N$, Θ a set compactly contained in Ω, and let $\beta \in (2s, 1)$. For $\varepsilon > 0$ small, let $\xi \in \Theta_\varepsilon$. Then there exist $w_\varepsilon : S_\xi \to \mathbb{R}$ in W and $\lambda = (\lambda_1, \cdots, \lambda_N) \in \mathbb{R}^N$ such that*

$$Vol(\mathbb{B}(\xi, w_\varepsilon)) = \omega_N; \qquad \int_{S_\xi} w_\varepsilon Y_i \, \mathrm{d}\sigma = 0; \qquad H^{\Omega_\varepsilon}_{s,\partial\mathbb{B}(\xi,w_\varepsilon)} = c + \sum_{i=1}^N \lambda_i Y_i,$$

where $c \in \mathbb{R}$ is close to $c_{N,s}$ and where $\{Y_i\}_{i=1,\cdots,N} \in \mathcal{E}_1$ (extended as zero-homogeneous function in a neighborhood of the unit sphere). Moreover, there exists $C > 0$ (depending on Θ, Ω, N and s) such that $\|w_\varepsilon\|_{C^{1,\beta}(S_\xi)} \leq C\varepsilon^{2s}$ and such that $\|\partial_\xi w_\varepsilon\|_{C^{1,\beta}(S_\xi)} \leq C\varepsilon^{2s+1}$.

To make the above formula for $H^{\Omega_\varepsilon}_s$ more precise, we mean that

$$H^{\Omega_\varepsilon}_{s,\partial\mathbb{B}(\xi,w_\varepsilon)}(\xi + x(1 + w_\varepsilon(x))) = c + \sum_{i=1}^N \lambda_i Y_i(x) \qquad \text{for every } x \in S_\xi.$$

Proof. Let us denote by $\overline{W}$ the family of functions in $C^{\beta-2s}(S_\xi)$ that are L^2-orthogonal, with respect to the standard volume element of S_ξ, to constants and to the first-order spherical harmonics. Notice that $\overline{W} \subseteq W$, see (2.4.10). Let us consider the two-component function $F_{\overline{W}} : \Theta_\varepsilon \times C^{1,\beta}(S_\xi) \to C^{\beta-2s}(S_\xi) \times \mathbb{R}$ defined by

$$F_{\overline{W}}(\xi, w) := \Big(P_{\overline{W}}(H^{\Omega_\varepsilon}_{s,\partial\mathbb{B}(\xi,w)}), Vol(\mathbb{B}(\xi, w)) - \omega_N \Big); \qquad w \in W,$$

where $\omega_N := Vol(B_1(\xi))$ and $P_{\overline{W}} : C^{\beta-2s}(S_\xi) \mapsto \overline{W}$ the orthogonal L^2-projection onto the space $\overline{W}$, with respect to the standard volume element of S_ξ. With this notation, we want to find $w \in W$ such that $F_{\overline{W}}(\xi, w) = (0, 0)$.

By Lemma 5.4 we have that

$$F_{\overline{W}}(\xi,0)=(O(\varepsilon^{2s}),0), \tag{5.1.13}$$

where the latter quantity is intended to be bounded by $C\varepsilon^{2s}$ in the $C^{\beta-2s}(S_\xi)$ sense. In our notation, the constant C is allowed to vary from one formula to the other.

By Proposition 2.21 and by the fact that

$$\frac{\mathrm{d}}{\mathrm{d}w}\bigg|_{w=0} Vol(\mathbb{B}(\xi,w))[\varphi]=\int_{S_\xi}\varphi\,d\sigma,$$

we have that $L_\xi:=\nabla_w F_{\overline{W}}(\xi,0)\in Inv(W,\overline{W}\times\mathbb{R})$ with $\|L_\xi^{-1}\|_{L(\overline{W}\times\mathbb{R},W)}\leq C$. Hence $F_{\overline{W}}(\xi,w)=(0,0)$ if and only if $F_{\overline{W}}(\xi,0)+L_\xi[w]-L_\xi[w]+F_{\overline{W}}(\xi,w)-F_{\overline{W}}(\xi,0)=(0,0)$, which can be written as

$$w=T_\xi(w):=-L_\xi^{-1}[F_{\overline{W}}(\xi,0)-L_\xi[w]+F_{\overline{W}}(\xi,w)-F_{\overline{W}}(\xi,0)].$$

Therefore $F_{\overline{W}}(\xi,w)=(0,0)$ if and only if w is a fixed point for T_ξ. Let us show that T_ξ is a contraction in a ball $B_{\overline{C}\varepsilon^{2s}}(\xi)$ centered at ξ with radius $\overline{C}\varepsilon^{2s}$ for $\overline{C}$ sufficiently large. From the definition of T_ξ, the estimate (5.1.13) and the fact that

$$\|L_\xi^{-1}\|_{L(\overline{W}\times\mathbb{R},W)}\leq C,$$

we have

$$\|T_\xi(0)\|_{C^{1,\beta}(S_\xi)}=\|L_\xi^{-1}[F_{\overline{W}}(\xi,0)]\|_{C^{1,\beta}(S_\xi)}\leq C^2\varepsilon^{2s}. \tag{5.1.14}$$

Then, taking w_1 and $w_2\in B_{\bar{C}\varepsilon^{2s}}(\xi)\subseteq W$ it follows that

$$\|T_\xi(w_1)-T_\xi(w_2)\|_{C^{1,\beta}(S_\xi)}\leq C\|F_{\overline{W}}(\xi,w_1)-F_{\overline{W}}(\xi,w_2)-L_\xi[w_1-w_2]\|_{C^{1,\beta}(S_\xi)}. \tag{5.1.15}$$

We notice that the function $w\mapsto Vol(\mathbb{B}(\xi,w))$ is a smooth function from the metric ball of radius $\frac{1}{2}$ in $C^{1,\beta}(S_\xi)$ into $\mathbb{R}$. Thanks also to the smoothness statement in Proposition 2.21, the right hand side in the latter formula can be bounded by

$$\begin{aligned}F_{\overline{W}}(\xi,w_1)-F_{\overline{W}}(\xi,w_2)-L_\xi[w_1-w_2]=\int_0^1\Big(&\nabla_w F_{\overline{W}}(\xi,w_2+s(w_1-w_2))\\ &-\nabla_w F_{\overline{W}}(\xi,0)[w_1-w_2]\Big)\,\mathrm{d}s\leq C\|w_1-w_2\|^2_{C^{1,\beta}(S_\xi)}.\end{aligned} \tag{5.1.16}$$

Hence, in $B_{\bar{C}\varepsilon^{2s}}(\xi)\subseteq W$ the Lipschitz constant of T_ξ is $C\bar{C}\varepsilon^{2s}$. So choosing first any $\bar{C}\geq 2C$, and then $\varepsilon>0$ small enough, we find therefore that T_ξ is a contraction in $B_{\bar{C}\varepsilon^{2s}}(\xi)$. As a consequence, there exists $w_\varepsilon:S_\xi\to\mathbb{R}$ in W such that $\|w_\varepsilon\|_{C^{1,\beta}(S_\xi)}\leq\bar{C}\varepsilon^{2s}$ and such that $F_{\overline{W}}(\xi,w_\varepsilon)=(0,0)$.

We also recall that the fixed point w_ε can be proved to be continuous and differentiable with respect to the parameter ξ, (see e.g. [14], Section 2.6).

Recall that $w_\varepsilon=w_\varepsilon(\xi)$ solves

$$Vol(\mathbb{B}(\xi,w_\varepsilon))=\omega_N\quad\text{and}\quad P_{\overline{W}}(H^{\Omega_\varepsilon}_{s,\partial\mathbb{B}(\xi,w_\varepsilon)})=0\qquad\text{for all }\xi\in\mathbb{R}^N.$$

We want next to differentiate the above relations with respect to ξ. For this purpose, it is convenient to fix an index i, and to consider the one-parameter family of centers

$$\xi(t) = (\xi_1, \dots, \xi_i + t, \dots, \xi_N) = \xi + t\mathbf{e}_i. \tag{5.1.17}$$

Our aim is to understand the variation of $\partial\mathbb{B}(\xi(t), w_\varepsilon(\xi(t)))$ normal to $\partial\mathbb{B}(\xi, w_\varepsilon(\xi))$. The above variation is characterized by a translation in the i-th component and by a variation of w_ε, which is in the radial direction with respect to the center ξ. Therefore, letting ν_{w_ε} denote the unit outer normal vector to $\partial\mathbb{B}(\xi, w_\varepsilon(\xi))$, the normal variation of $\partial\mathbb{B}(\xi(t), w_\varepsilon(\xi(t)))$ with respect to $\partial\mathbb{B}(\xi, w_\varepsilon(\xi))$ (computed at $t = 0$) is the scalar product between the pointwise shift $\mathbf{e}_i + \frac{\partial w_\varepsilon(\xi)}{\partial \xi_i}$ and the unit outer normal vector to $\partial\mathbb{B}(\xi, w_\varepsilon(\xi))$ that is ν_{w_ε}, i.e.

$$\nu_{w_\varepsilon} \cdot \mathbf{e}_i + \frac{\partial w_\varepsilon(\xi)}{\partial \xi_i}(x - \xi) \cdot \nu_{w_\varepsilon}, \qquad x \in S_\xi. \tag{5.1.18}$$

Hence we have that

$$\frac{\partial}{\partial \xi_i} Vol(\mathbb{B}(\xi, w_\varepsilon)) = 0 \quad \text{and} \quad P_{\overline{W}}\left(\frac{\partial}{\partial \xi_i} H^{\Omega_\varepsilon}_{s,\partial\mathbb{B}(\xi,w_\varepsilon(\xi))}\right)\left[\nu_{w_\varepsilon} \cdot \mathbf{e}_i + \frac{\partial w_\varepsilon(\xi)}{\partial \xi_i}(x - \xi) \cdot \nu_{w_\varepsilon}\right] = 0.$$

Using (5.1.9) and Proposition 2.21 one finds from the second equation in the latter formula that $\|v_{i,\varepsilon}\|_{C^{1,\beta}(S_\xi)} \leq C\varepsilon^{2s+1}$, where $v_{i,\varepsilon} = P_{\overline{W}} \partial_{\xi_i} w_\varepsilon$. Since $\dfrac{\partial w_\varepsilon}{\partial \xi_i} \in W$, it remains to control then the component of $\partial_{\xi_i} w_\varepsilon$ in the orthogonal complement of $\bar{W}$, namely its average. Let us write

$$\partial_{\xi_i} w_\varepsilon = v_{i,\varepsilon} + c_{i,\varepsilon} \qquad \text{with } c_{i,\varepsilon} \in \mathbb{R}.$$

From a direct computation we have that

$$0 = \frac{\partial}{\partial \xi_i} Vol(\mathbb{B}(\xi, w_\varepsilon)) = \int_{S_\xi} (1 + w_\varepsilon)^{N-1}(v_{i,\varepsilon} + c_{i,\varepsilon}) d\sigma.$$

Since we know that $\|v_{i,\varepsilon}\|_{C^{1,\beta}(S_\xi)} \leq C\varepsilon^{2s+1}$, it follows from the latter formula that also $|c_{i,\varepsilon}| \leq C\varepsilon^{2s+1}$. Therefore one deduces

$$\|\partial_{\xi_i} w_\varepsilon\|_{C^{1,\beta}(S_\xi)} \leq C\varepsilon^{2s+1}, \tag{5.1.19}$$

which is the desired conclusion, possibly relabelling the constant C. □

5.1.2 Proof of Theorem 5.1

In this subsection we prove Theorem 5.1 using the Lyapunov-Schmidt reduction. In particular, as first step, we show how to find ξ's so that the Lagrange multipliers λ_i in the statement of Proposition 5.5 vanish, obtaining surfaces with constant (relative) nonlocal mean curvature.

Proposition 5.6. *Let* $w_\varepsilon : S_\xi \to \mathbb{R}$ *given by Proposition 5.5. Recalling* (2.4.4), *for* $\xi \in \Theta_\varepsilon$ *we define* $\Phi_\xi := P_s^{\Omega_\varepsilon}(\mathbb{B}(\xi, w_\varepsilon))$. *Then, for* $\varepsilon > 0$ *sufficiently small, if* $\nabla_\xi {\Phi_\xi}_{|_{\xi=\bar{\xi}}} = 0$ *for some* $\bar{\xi} \in \Theta_\varepsilon$, *one has*

$$H^{\Omega_\varepsilon}_{s,\partial\mathbb{B}(\bar{\xi},w_\varepsilon)} \equiv c,$$

where $c = c(\varepsilon, \bar{\xi})$.

Proof. Recall that $w_\varepsilon = w_\varepsilon(\xi)$ solves

$$Vol(\mathbb{B}(\xi, w_\varepsilon)) = \omega_N \quad \text{and} \quad P_{\overline{W}}(H^{\Omega_\varepsilon}_{s,\partial\mathbb{B}(\xi,w_\varepsilon)}) = 0 \qquad \text{for all } \xi \in \mathbb{R}^N.$$

Since $Vol(\mathbb{B}(\xi, w_\varepsilon)) = \omega_N$ for any choice of ξ, it follows that the integral over $\partial\mathbb{B}(\xi, w_\varepsilon(\xi))$ of the normal variation vanishes, i.e. recalling (5.1.18), we have for $\xi = \bar{\xi}$

$$\int_{\partial\mathbb{B}(\bar{\xi},w_\varepsilon(\bar{\xi}))} \left[\nu_{w_\varepsilon} \cdot \mathbf{e}_i + \frac{\partial w_\varepsilon(\bar{\xi})}{\partial \xi_i}(x - \bar{\xi}) \cdot \nu_{w_\varepsilon}\right] d\sigma_{w_\varepsilon} = 0, \tag{5.1.20}$$

where $d\sigma_{w_\varepsilon}$stands for the area element of $\partial\mathbb{B}(\bar{\xi}, w_\varepsilon(\bar{\xi}))$.

For the same reason, recalling (2.2.9) and (5.1.17), we have that

$$\begin{aligned} &\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} P_s^{\Omega_\varepsilon}(\mathbb{B}(\xi(t), w_\varepsilon(\xi(t)))) \\ &\qquad = \int_{\partial\mathbb{B}(\bar{\xi},w_\varepsilon(\bar{\xi}))} H^{\Omega_\varepsilon}_{s,\partial\mathbb{B}(\bar{\xi},w_\varepsilon(\bar{\xi}))} \left[\nu_{w_\varepsilon} \cdot \mathbf{e}_i + \frac{\partial w_\varepsilon(\bar{\xi})}{\partial \xi_i}(x - \bar{\xi}) \cdot \nu_{w_\varepsilon}\right] d\sigma_{w_\varepsilon}. \end{aligned}$$

By our choice of $\bar{\xi}$ we have that, for all $i = 1, \dots, N$

$$\frac{\partial}{\partial \xi}_{i|_{\xi=\bar{\xi}}} \Phi_\xi = 0.$$

Recalling also that by Proposition 5.5, $H^{\Omega_\varepsilon}_{s,\partial\mathbb{B}(\bar{\xi},w_\varepsilon)} = c + \sum_{i=1}^N \lambda_i Y_i$ (see Section 2.4 for the definition of the first-order sphereical harmonics Y_i) , from (5.1.20) we have that for all $i = 1, \dots, N$

$$0 = \int_{\partial\mathbb{B}(\bar{\xi},w_\varepsilon(\bar{\xi}))} \left(\sum_{j=1}^N \lambda_j Y_j\right) \left[\nu_{w_\varepsilon} \cdot \mathbf{e}_i + \frac{\partial w_\varepsilon(\bar{\xi})}{\partial \xi_i}(x - \bar{\xi}) \cdot \nu_{w_\varepsilon}\right] d\sigma_{w_\varepsilon}. \tag{5.1.21}$$

Notice that by the estimates on w_ε and $\partial_\xi w_\varepsilon$ in Proposition 5.5 one has

$$\int_{\partial\mathbb{B}(\bar{\xi},w_\varepsilon(\bar{\xi}))} Y_j \left[\nu_{w_\varepsilon} \cdot \mathbf{e}_i + \frac{\partial w_\varepsilon(\bar{\xi})}{\partial \xi_i}(x - \bar{\xi}) \cdot \nu_{w_\varepsilon}\right] d\sigma_{w_\varepsilon} = \delta_{ij} + o_\varepsilon(1); \qquad i, j = 1, \dots, N.$$

Therefore the system (5.1.21) implies the vanishing of all λ_j's, which gives the desired conclusion. ☐

The next step is to show that fractional perimeter of $B_1(\xi)$ is sufficiently close to fractional perimeter of the deformed ball $\mathbb{B}(\xi, w_\varepsilon)$, also when we differentiate with respect to ξ.

Proposition 5.7. *Let $w_\varepsilon : S_\xi \to \mathbb{R}$ given by Proposition 5.5. The following Taylor expansion holds:*

$$P_s^{\Omega_\varepsilon}(\mathbb{B}(\xi, w_\varepsilon)) = P_s^{\Omega_\varepsilon}(B_1(\xi)) + O(\varepsilon^{4s}). \tag{5.1.22}$$

Moreover one has

$$\frac{\partial}{\partial \xi_i} P_s^{\Omega_\varepsilon}(\mathbb{B}(\xi, w_\varepsilon)) = \frac{\partial}{\partial \xi_i} P_s^{\Omega_\varepsilon}(B_1(\xi)) + O(\varepsilon^{1+4s}). \tag{5.1.23}$$

Proof. Recalling the notation introduced in Section 2.4 and thanks to the first statement of Lemma 5.4, we get that

$$\begin{aligned} P_s^{\Omega_\varepsilon}(\mathbb{B}(\xi, w_\varepsilon)) &= P_s^{\Omega_\varepsilon}(B_1(\xi)) + (P_{s,\xi}^{\Omega_\varepsilon})'(0)[w_\varepsilon] + P_s^{\Omega_\varepsilon}(\mathbb{B}(\xi, w_\varepsilon)) - (P_{s,\xi}^{\Omega_\varepsilon})'(0)[w_\varepsilon] \\ &\quad - P_s^{\Omega_\varepsilon}(B_1(\xi)) \\ &= P_s^{\Omega_\varepsilon}(B_1(\xi)) + O(\varepsilon^{4s}) + \int_0^1 \Big((P_{s,\xi}^{\Omega_\varepsilon})'(t\, w_\varepsilon) - (P_{s,\xi}^{\Omega_\varepsilon})'(0) \Big)[w_\varepsilon]\, \mathrm{d}t\,. \end{aligned} \tag{5.1.24}$$

Using the fact that the nonlocal mean curvature is smooth, we deduce then that

$$\int_0^1 \Big((P_{s,\xi}^{\Omega_\varepsilon})'(t\, w_\varepsilon) - (P_{s,\xi}^{\Omega_\varepsilon})'(0) \Big)[w_\varepsilon]\, \mathrm{d}t = O(\varepsilon^{4s}),$$

so the last two formulas imply (5.1.22).

To prove (5.1.23), we use the estimate $\|\partial_\xi w_\varepsilon\|_{C^{1,\beta}(S_\xi)} \le C\varepsilon^{2s+1}$ from Proposition 5.5. Calling τ_i the quantity in (5.1.18), we write that

$$\frac{\partial}{\partial \xi_i} P_s^{\Omega_\varepsilon}(\mathbb{B}(\xi, w_\varepsilon)) = (P_{s,\xi}^{\Omega_\varepsilon})'(w_\varepsilon)[\tau_i].$$

Taylor-expanding the latter quantity, we can write that

$$\begin{aligned} \frac{\partial}{\partial \xi_i} P_s^{\Omega_\varepsilon}(\mathbb{B}(\xi, w_\varepsilon)) &= (P_{s,\xi}^{\Omega_\varepsilon})'(0)[\tau_i] + \frac{1}{2}(P_{s,\xi}^{\Omega_\varepsilon})''(0)[\tau_i] + o(\varepsilon^{1+4s}) \\ &= \frac{\partial}{\partial \xi_i} P_s^{\Omega_\varepsilon}(B_1(\xi)) + O(\varepsilon^{1+4s}). \end{aligned} \tag{5.1.25}$$

This concludes the proof. □

With this result at hand, we are ready to prove the main theorem of this section.

Proof of Theorem 5.1. Suppose x_0 is a strict local extremal of V_Ω, without loss of generality a minimum. Then there exists an open set $\Upsilon \subset\subset \Omega$ such that $V_\Omega(x_0) <$

$\inf_{\partial\Upsilon} V_\Omega - \delta$ for some $\delta > 0$. Let Φ_ξ be defined as in Proposition 5.6: by the estimates (5.1.3) and (5.1.22) it follows that for every $\bar{x} \in \frac{1}{\varepsilon}\Upsilon$

$$\Phi_{\bar{x}} = P_s^{\mathbb{R}^N}(B_1(\bar{x})) - \omega_N \varepsilon^{2s} V_\Omega(\varepsilon\bar{x}) + O(\varepsilon^{1+2s}). \tag{5.1.26}$$

Since $P_s^{\mathbb{R}^N}(B_1(\bar{x})) = P_s^{\mathbb{R}^N}(B_1(\frac{x_0}{\varepsilon}))$, we get

$$\begin{aligned} \Phi_{\frac{x_0}{\varepsilon}} - \Phi_{\bar{x}} &= \omega_N \varepsilon^{2s}(V_\Omega(\varepsilon\bar{x}) - V_\Omega(x_0)) + O(\varepsilon^{1+2s}) \\ &\geq \omega_N \varepsilon^{2s}(\inf_{\partial\Upsilon} V_\Omega(\varepsilon\bar{x}) - V_\Omega(x_0)) + O(\varepsilon^{1+2s}) \\ &> \delta\omega_N \varepsilon^{2s} + O(\varepsilon^{1+2s}) \geq \delta\omega_N\varepsilon^{2s} + C\varepsilon^{1+2s} > 0 \end{aligned} \tag{5.1.27}$$

for $\varepsilon < \frac{\delta\omega_N}{C}$ where $C > 0$ is a constant. Hence, for ε sufficiently small

$$\Phi_{\frac{x_0}{\varepsilon}} > \sup_{\frac{1}{\varepsilon}\Upsilon} \Phi_{\cdot}.$$

As a consequence $\Phi_\cdot$ attains a maximum in the dilated domain $\frac{1}{\varepsilon}\Upsilon$, and the conclusion follows from Proposition 5.6.

Suppose now that x_0 is a non-degenerate critical point of V_Ω. From (5.1.5) and (5.1.23) one can find an open set $\Upsilon \subset\subset \Omega$ such that

$$\deg\left(\nabla\Phi_\cdot, \frac{1}{\varepsilon}\Upsilon, 0\right) \neq 0.$$

This implies that Φ_ξ has a critical point in $\frac{1}{\varepsilon}\Upsilon$, and the conclusion again follows from Proposition 5.6.

Since in both cases the set Υ containing x_0 can be taken arbitrarily small, the localization statement in the theorem is also proved. $\square$

Remark 5.8. From [3, Theorem 2.24] one has a relation between the Morse index of a critical point as found in Proposition 5.6 and the Morse index of the corresponding critical point of Φ. In our case, since round spheres are global minimizers for the fractional perimeter relative to $\mathbb{R}^N$, these two indices coincide.

To prove Corollary 5.2, we need the following lemma.

Lemma 5.9. *For all $x \in \partial\Omega$ one has*

$$\lim_{y\to x} V_\Omega(y) = +\infty,$$

and

$$\lim_{\Omega\ni y\to x} \nabla V_\Omega(y) \cdot \nu_\Omega(x) = +\infty,$$

where ν_Ω denotes the outer unit normal to $\partial\Omega$.

Proof. Letting $d := \operatorname{dist}(x, \partial\Omega)$ for all $x \in \Omega$, thanks to the change of variables $x' = \frac{x}{d}$, we get that

$$V_\Omega(x) = \int_{\mathbb{R}^N \setminus \Omega} \frac{1}{|x-y|^{N+2s}} \, \mathrm{d}y = \int_{\mathbb{R}^N \setminus (\Omega/d)} \frac{1}{|dx' - y'|^{N+2s}} \, \mathrm{d}y' \tag{5.1.28}$$

from which, if $d \to 0$, setting $\mathbb{R}^N_+ = \{x \in \mathbb{R}^N : x_N > 0\}$, we have

$$\int_{\mathbb{R}^N \setminus (\Omega/d)} \frac{1}{|dx' - y'|^{N+2s}} \, \mathrm{d}y' \to \int_{(\mathbb{R}^N_+)^C} \frac{1}{|y'|^{N+2s}} \, \mathrm{d}y' < +\infty,$$

i.e. V_Ω behaves asymptotically as d^{-N-2s} when $d \to 0$. With a similar proof, one finds that the component of ∇V_Ω normal to $\partial\Omega$ behaves as $d^{-N-2s-1}$. □

Proof of Corollary 5.2. Given $\delta > 0$ small enough, let us define the set $\Omega^\delta \subseteq \Omega$ by

$$\Omega^\delta := \{x \in \Omega \ : \ \operatorname{dist}(x, \partial\Omega) > \delta\}.$$

From Lemma 5.9 we have

$$\nabla V_\Omega \cdot \nu_{\Omega^\delta} > 0 \quad \text{on } \partial\Omega^\delta.$$

As in the proof of Theorem 5.1, it turns out that

$$\nabla \Phi_\cdot \cdot \nu_{\frac{1}{\varepsilon}\Omega^\delta} > 0 \quad \text{on } \partial \frac{1}{\varepsilon}\Omega^\delta.$$

Clearly, since $\bar{\Omega}$ is compact, the (PS)-condition holds. So the conclusion follows from Theorem 2.25 and Remark 2.26. □

Remark 5.10. It is interesting to see how the geometry of the domain (and not just the topology, as in Corollary 5.2) plays a role in order to obtain either uniqueness of multiplicity of solutions.

In this last part of the section we will prove uniqueness for the unit ball B_1, i.e. we will show that V_{B_1} has a unique critical point at the origin which is a non-degenarate minimum.

Secondly, we will give an example of dumble-bell domain, topologically equivalent to a ball, such that the reduced functional Φ_ξ (defined as in Proposition 5.6) has at least three critical points, while Corollary 5.2 would give us only one solution.

Lemma 5.11. *If B_1 is the unit ball of $\mathbb{R}^N$, then $0 \in B_1$ is a non-degenerate global minimum of V_{B_1} and it is the unique critical point.*

Proof. First of all we note that V_{B_1} is a radial function, i.e. $V_{B_1}(x) = v_{B_1}(|x|)$. Hence, since V_{B_1} is smooth in the interior of the ball, it follows that $v'_{B_1}(0) = 0$.

It is easily seen that

$$(\Delta V_{B_1})(0) = 2(1+s)(N+2s) \int_{B_1^C} \frac{1}{|y|^{N+2s+2}} \, \mathrm{d}y > 0,$$

where B_1^C denotes the complement of B_1. Therefore, since $v''_{B_1}(0) = \frac{1}{n}\Delta V_{B_1}(0)$, it follows that for fixed $\delta > 0$ one has $v''_{B_1}(t) > 0$ for $t \in [0, \delta]$, which implies the non-degeneracy of the origin as a critical point of V_{B_1}.

It remains to show the monotonicity of v_{B_1} in the whole interval $(0, 1)$, but since Lemma 5.9 holds, it is sufficient to show that

$$\frac{\mathrm{d}}{\mathrm{d}t} V_{B_1}(t\vec{e}_1) \neq 0 \quad \text{for } t \in [\delta, 1-\delta]. \tag{5.1.29}$$

Recalling the definition (5.1.4), we get

$$\frac{\mathrm{d}}{\mathrm{d}t} V_{B_1}(t\vec{e}_1) = \tilde{c}_{N,s} \int_{B_1^C} \frac{y_1 - t}{|y - t\vec{e}_1|^{N+2s+2}} \,\mathrm{d}y\,, \tag{5.1.30}$$

where $\tilde{c}_{N,s}$ is a constant depending only on N and s and $y = (y_1, y') \in \mathbb{R} \times \mathbb{R}^{N-1}$.

By Fubini's Theorem

$$\int_{B_1^C} \frac{y_1 - t}{|y - t\vec{e}_1|^{N+2s+2}} \,\mathrm{d}y = \int_{\mathbb{R}^{N-1}} \mathrm{d}y' \int_{\{y_1 : (y_1, y') \in B_1^C\}} \frac{y_1 - t}{|y - t\vec{e}_1|^{N+2s+2}} \,\mathrm{d}y\,. \tag{5.1.31}$$

Since $(y_1, y') \in B_1^c \times \mathbb{R}^{N-1}$, we have two cases:

1) if $|y'| \geq 1 \quad \Rightarrow \quad y_1 \in \mathbb{R}$;

2) if $|y'| < 1 \quad \Rightarrow \quad y_1 \leq -\sqrt{1 - |y'|^2} \ \vee \ y_1 \geq \sqrt{1 - |y'|^2}$.

In the first case we obtain by oddness

$$\int_{\{y_1 : (y_1, y') \in B_1^C\}} \frac{y_1 - t}{|y - t\vec{e}_1|^{N+2s+2}} \,\mathrm{d}y = \int_{\{y_1 \in \mathbb{R}\}} \frac{y_1 - t}{((y_1 - t)^2 + |y'|^2)^{(N+2s+2)/2}} \,\mathrm{d}y = 0. \tag{5.1.32}$$

In the second case, using the changes of variables $y_1 - t = s$ and $z = t - y_1$, we get

$$\begin{aligned}
&\int_{\{y_1 : (y_1, y') \in B_1^C\}} \frac{y_1 - t}{|y - t\vec{e}_1|^{N+2s+2}} \,\mathrm{d}y \\
&= \int_{\{y_1 \leq -\sqrt{1-|y'|^2}\}} \frac{y_1 - t}{|y - t\vec{e}_1|^{N+2s+2}} \,\mathrm{d}y + \int_{\{y_1 \geq \sqrt{1-|y'|^2}\}} \frac{y_1 - t}{|y - t\vec{e}_1|^{N+2s+2}} \,\mathrm{d}y \\
&= \int_{\{z \geq t + \sqrt{1-|y'|^2}\}} \frac{z}{(z^2 + |y'|^2)^{(N+2s+2)/2}} \,\mathrm{d}z \\
&+ \int_{\{s \geq \sqrt{1-|y'|^2} - t\}} \frac{s}{(s^2 + |y'|^2)^{(N+2s+2)/2}} \,\mathrm{d}y > 0,
\end{aligned} \tag{5.1.33}$$

since $\{z : z \geq t + \sqrt{1 - |y'|^2}\} \subseteq \{z : z \geq \sqrt{1 - |y'|^2} - t\}$ and since the first integral is negative.

Putting together (5.1.30), (5.1.31), (5.1.32) and (5.1.33) we obtain (5.1.29) which concludes the proof. □

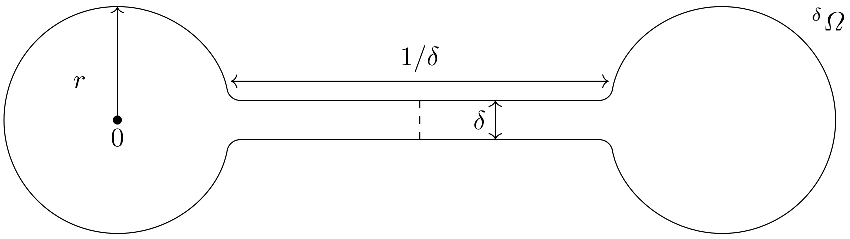


Figure 5.1: A dumb-bell domain $^{\delta}\Omega$.

Lemma 5.12. *Let Φ_ξ be defined as in Proposition 5.6. There exist a dumble-bell domain (as in Figure 5.1), with the same topology of the ball, such that Φ_ξ has at least three critical points.*

Sketch of the Proof. Given $\delta > 0$ small enough, we consider a sequence of domains $^{\delta}\Omega$ as in Figure 5.1. Fixed $r \in (0,1)$, it is easy to see that

$$V_{^{\delta}\Omega} \to V_{B_1} \quad \text{in } C^2(B_r(0)) \quad \text{as } \delta \to 0. \tag{5.1.34}$$

For δ small, by Lemma 5.11, we get that $V_{^{\delta}\Omega}$ has a unique non-degenerate minimum x_1 in $B_{r/2}(0)$ and there exists $\gamma > 0$ such that

$$\inf_{\partial B_r(0)} V_{^{\delta}\Omega} > \sup_{B_{r/2}(0)} V_{^{\delta}\Omega} + \gamma.$$

By symmetry, we have a non-degenerate minimum point x_2 in the other ball with the same properties. Recall also that from Lemma 5.9 that if $x \in \partial(^{\delta}\Omega)$, it holds

$$\lim_{^{\delta}\Omega \ni y \to x} V_{^{\delta}\Omega}(y) = +\infty.$$

Hence, from (5.1.26) (with a similar formula for the gradient in ξ) and the above observations, there exists a critical point of Φ other that x_1 and x_2, by Mountain Pass Theorem.

□

5.2 s-minimizers in an half-space

In this second part of the chapter we consider the fractional perimeter of a measurable set $E \subset \mathbb{R}^N$ in an half-space, proving the existence of a minimizer under fixed volume constraint. Then we characterize its intersection with the hyperplane $\{x_N = 0\}$ and we show some of its properties as smoothness and symmetry.

Let us consider a bounded open set with smooth boundary $\Omega \subseteq \mathbb{R}^N$, and $s \in (0, 1/2)$. We point out that, if

$$\bar{P}_s(E, \Omega) := \int_E \int_{\Omega \setminus E} \frac{\mathrm{d}x\, \mathrm{d}y}{|x-y|^{N+2s}}, \tag{5.2.1}$$

using the direct method of Calculus of Variations and the Sobolev embeddings (which hold for fractional spaces too, see [35]), it is easy to show that there exist minimizers for

$$\{\bar{P}_s(E,\Omega), |E|=m\} \quad m\in(0,+\infty), \tag{5.2.2}$$

see [22, Theorem 3.2].

Our goal is to show that minimizers exist also relatively to an half-space, and to characterize them to some extent. Thus, analogously to (5.2.1), we define the fractional perimeter in an half-space:

Definition 5.13. Let $s\in(0,1/2)$ and $E\subset\mathbb{R}^N$ be a measurable set. We denote with

$$\bar{P}_s(E,\mathbb{R}^N_+) := \int_E\int_{\mathbb{R}^N_+\setminus E}\frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^{N+2s}}, \tag{5.2.3}$$

where $\mathbb{R}^N_+=\{x\in\mathbb{R}^N : x_N>0\}$ is the half-space.

The main result of this section is the following:

Theorem 5.14. *There exists a minimizer E for the problem*

$$\inf\Big\{\bar{P}_s(A,\mathbb{R}^N_+),\ |A|=m\Big\}, \quad m\in(0,+\infty), \tag{5.2.4}$$

where $\mathbb{R}^N_+ := \{x\in\mathbb{R}^N : x_N>0\}$. Moreover ∂E is a radially-decreasing symmetric graph of class C^∞ in the interior, intersecting orthogonally the hyperplane $\{x_N=0\}$.

To prove this theorem first we will show the existence of a suitable rearranged minimizing sequence which is axially symmetric and graphical over the boundary hyperplane. After that, we will employ some results from [8], [22], [61] to prove a diameter bound and the smoothness of the minimizing limit.

5.2.1 Proof of Theorem 5.14

We begin by studying minimizers of

$$\{\bar{P}_s(E,\mathbb{R}^N_+):\ E\subseteq B_R^+,\ |E|=m\} \quad m\in(0,+\infty), \tag{5.2.5}$$

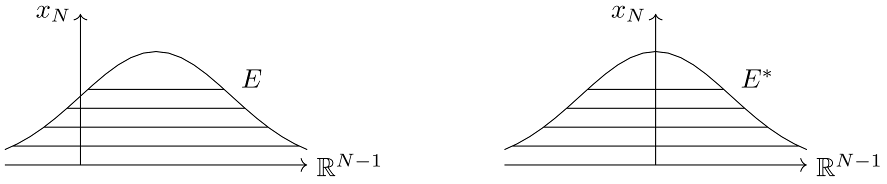
where $B_R^+ := B_R\cap\mathbb{R}^N_+$ denotes the open half-ball of radius $R>0$ centred at the origin with $|B_R^+|\geq m$. Without loss of generality we can assume that $m=1$ and, with the same reasoning used to show the existence of minimizers for (5.2.2), we can also prove the following result.

Proposition 5.15. *Problem* (5.2.5) *admits a minimizer $E\subseteq B_R^+$.*

Note that, since we look for minimizers in a half-ball, we can assume that minimizers of (5.2.5) are close sets.

We have next the following lemma.

Lemma 5.16. *If E is a minimizer for* (5.2.5)*, then dist* $(E,\{z_N=0\})=0$.



(a) Level set of u. **(b)** Level set of u^*.

Figure 5.2: The radially symmetric rearrangement of u.

Proof. By contradiction suppose that the minimizer $E \subseteq B_R^+$ does not intersect the plane $\{z_N = 0\}$. Then, if $e := (e_1, \cdots, e_N)$ is the canonical basis of $\mathbb{R}^N$ and $\lambda := \operatorname{dist}(E, \{z_N = 0\}) > 0$, we consider the shifted set $E - \lambda e_N$. Using the following change of variables (i.e. translating downwards the set E by $\lambda \vec{e}_N$)

$$\begin{aligned} E \ni x &\longmapsto x' = x - \lambda e_N \in E - \lambda e_N, \\ \mathbb{R}^N_+ \setminus E \ni y &\longmapsto y' = y - \lambda e_N \in \mathbb{R}^N_+ \setminus (E - \lambda e_N), \end{aligned}$$

we have

$$\begin{aligned} \bar{P}_s(E, \mathbb{R}^N_+) &= \int_E \int_{\mathbb{R}^N_+ \setminus E} \frac{\mathrm{d}x\,\mathrm{d}y}{|x - \lambda e_N - y + \lambda e_N|^{N+2s}} \\ &> \int_{E - \lambda e_N} \int_{\mathbb{R}^N_+ \setminus (E - \lambda e_N)} \frac{\mathrm{d}x\,\mathrm{d}y}{|x - y|^{N+2s}} = \bar{P}_s(E - \lambda e_N, \mathbb{R}^N_+). \end{aligned} \tag{5.2.6}$$

This is in contradiction to the minimality of E. ☐

Now we address to show the symmetry of minimizers of (5.2.5). To do this, we have to premise a couple of useful definitions.

Definition 5.17. Given a function $u : \mathbb{R}^N \to \mathbb{R}^+$, we define $u^* : \mathbb{R}^N \to \mathbb{R}^+$ the *radially symmetric rearrangement of u with respect to x_N* so that, given $x_N > 0$, $t > 0$, the superlevel set $\{u^*(\cdot, x_N) > t\}$ is a ball B in $\mathbb{R}^{N-1}$ centered at the origin and such that

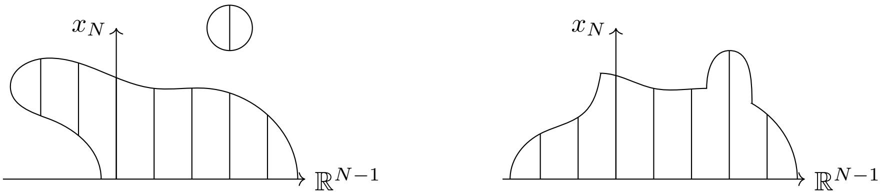
$$|\{u^*(\cdot, x_N) > t\}| = |\{u(\cdot, x_N) > t\}|,$$

as in Figure 5.2.

If $u = \chi_E$, we call E^* the ball such that $\chi_{E^*} = (\chi_E)^*$.

Definition 5.18. Given a function $u : \mathbb{R}^N \to \mathbb{R}^+$, we define $\hat{u} : \mathbb{R}^N \to \mathbb{R}^+$ to be *the decreasing rearrangement of u with respect to x_N*: given $x' > 0$, $t > 0$, $\{x_N : \hat{u}(x', x_N) > t\} \subseteq \mathbb{R}^+$ is a segment of the form $[0, \alpha)$ with $\alpha := |\{x_N : \hat{u}(x', x_N) > t\}|$, as in Figure 5.3.

If $u = \chi_E$, we call $\hat{E}$ the set such that $\chi_{\hat{E}} = \widehat{(\chi_E)}$. Notice that $\partial \hat{E}$ is a graph in the direction $\vec{e}_N$.



(a) Level set of u. **(b)** Level set of $\hat{u}$.

Figure 5.3: The decreasing rearrangement of u.

With these definitions at hand, we can show that minimizers of (5.2.5) are radially symmetric sets:

Lemma 5.19. *If E is a minimizer of* (5.2.5), *we have that*

$$\bar{P}_s(E^*, \mathbb{R}^N_+) \le \bar{P}_s(E, \mathbb{R}^N_+)$$

and the equality holds if and only if $E = E^*$.

Proof. Proceeding as in [68], we define

$$\mathcal{H}^s(\mathbb{R}^N_+) := \{u \in L^2(\mathbb{R}^N_+) : [u]_{\mathcal{H}^s(\mathbb{R}^N_+)} < +\infty\},$$

where

$$[u]^2_{\mathcal{H}^s(\mathbb{R}^N_+)} := \inf\bigg\{\int_{\mathbb{R}^N_+\times\mathbb{R}^+} (|\nabla_x v|^2 + |\partial_y v|^2) y^{1-2s}\,\mathrm{d}x\,\mathrm{d}y \\ : v \in H^1_{\mathrm{loc}}(\mathbb{R}^N_+ \times \mathbb{R}^+), v(\cdot, 0) = u(\cdot)\bigg\}. \tag{5.2.7}$$

The space $\mathcal{H}^s(\mathbb{R}^N_+)$ is endowed with the Hilbert norm

$$\|u\|^2_{\mathcal{H}^s(\mathbb{R}^N_+)} = \|u\|^2_{L^2(\mathbb{R}^N_+)} + [u]^2_{\mathcal{H}^s(\mathbb{R}^N_+)}.$$

According to (5.2.7) we get that

$$\bar{P}_s(E, \mathbb{R}^N_+) = \frac{1}{2}\inf\bigg\{\int_{\mathbb{R}^N_+\times\mathbb{R}^+} (|\nabla_x v|^2 + |\partial_y v|^2) y^{1-2s}\,\mathrm{d}x\,\mathrm{d}y \\ : v \in H^1_{\mathrm{loc}}(\mathbb{R}^N_+ \times \mathbb{R}^+), v(\cdot, 0) = \chi_E(\cdot)\bigg\} \tag{5.2.8}$$

and we define

$$H^1(\mathbb{R}^N_+ \times \mathbb{R}^+, y^{1-2s}\,\mathrm{d}y) := \big\{v \in H^1_{\mathrm{loc}}(\mathbb{R}^N_+ \times \mathbb{R}^+) \\ : \int_{\mathbb{R}^N_+\times\mathbb{R}^+} (|v|^2 + |\nabla_x v|^2 + |\partial_y v|^2) y^{1-2s}\,\mathrm{d}x\,\mathrm{d}y < \infty\bigg\}. \tag{5.2.9}$$

For all $v \in H^1(\mathbb{R}^N_+ \times \mathbb{R}^+, y^{1-2s}\,\mathrm{d}y)$, we set $v^*(\cdot, y) = [v(\cdot, y)]^*$. Then

a) since the symmetrization preserves characteristic functions, we have that

$$(\chi_E(\cdot))^* = \chi_{E^*}(\cdot); \tag{5.2.10}$$

b) from [15, Theorem 1] we get that

$$\int_{\mathbb{R}^N_+\times\mathbb{R}^+} (|\nabla_x v^*|^2 + |\partial_y v|^*)^2) y^{1-2s}\,\mathrm{d}x\,\mathrm{d}y \le \int_{\mathbb{R}^N_+\times\mathbb{R}^+} (|\nabla_x v|^2 + |\partial_y v|^2) y^{1-2s}\,\mathrm{d}x\,\mathrm{d}y\,. \tag{5.2.11}$$

Hence combining (5.2.8), (5.2.10) and (5.2.11) we deduce the desired conclusion. ☐

In a similar way we obtain also this

Lemma 5.20. *Let E be a minimizer of* (5.2.5)*. Then*

$$\bar{P}_s(\hat{E}, \mathbb{R}^N_+) \le \bar{P}_s(E, \mathbb{R}^N_+)$$

and the equality holds if and only if $E = \hat{E}$.

Proof. Proceeding as in Lemma 5.19 and denoting with $\hat{v}(\cdot, y) = [v(\widehat{\cdot, y})]$, we have that

$$(\chi_E\widehat{(\cdot)}) = \chi_{\hat{E}}(\cdot), \tag{5.2.12}$$

and from [15, Theorem 1] we get

$$\int_{\mathbb{R}^N_+\times\mathbb{R}^+} (|\nabla_x \hat{v}|^2 + (\hat{v_y})^2) y^{1-2s}\,\mathrm{d}x\,\mathrm{d}y \le \int_{\mathbb{R}^N_+\times\mathbb{R}^+} (|\nabla_x v|^2 + v_y^2) y^{1-2s}\,\mathrm{d}x\,\mathrm{d}y\,. \tag{5.2.13}$$

Recalling (5.2.8) and using (5.2.12) and (5.2.13) we conclude the proof. ☐

Remark 5.21. Note that from these two symmetrizations we obtain a connected minimizer for (5.2.5).

Now we prove an estimate on the diameter of minimizers of (5.2.5) which will allows us to deduce, as a corollary, that these minimizing sets are also minimizers for (5.2.4):

Theorem 5.22. *If E is a minimizer of* (5.2.5)*, then*

$$|diam\, E| \le \frac{2\sqrt{2}C_0}{r_0^{N-1}}, \tag{5.2.14}$$

where diam E denotes the diameter of the set E and both $C_0 > 0$ and $r_0 > 0$ come from [61, Theorem 1.7].

Proof. Thanks to Lemma 5.19 and Lemma 5.20 we can suppose that there exists $H > 0$ such that

$$[0, He_N] \subseteq\; E \tag{5.2.15}$$

and that, for all $t > 0$,

$$E_t := E \cap \{x_N = t\} = B_{R(t)}. \tag{5.2.16}$$

We consider the interval $[0, He_N]$ and we divide it in M subintervals of length at most $2r_0$, where $r_0 > 0$ comes from [61, Theorem 1.7] and, denoting with $\left[\,\cdot\,\right]$ the integer part of a real number, $M = \left[\frac{H}{2r_0}\right] + 1$. For every subinterval we take its center x^i where $i = 1, \cdots, M$. From [61, Theorem 1.7], for every x^i, there exist $C_0 > 0$ and a ball $B_{r_0}(x^i)$ with center at x^i and radius r_0 such that

$$|E \cap B_{r_0}(x^i)| \geq \frac{r_0^N}{C_0} > 0 \quad \text{for all } i = 1, \cdots, M.$$

Thus

$$1 = |E| \geq \frac{H}{2r_0} \cdot \frac{r_0^N}{C_0}.$$

and hence

$$H \leq \frac{2C_0}{r_0^{N-1}}. \tag{5.2.17}$$

We proceed similarly to estimate $R(t)$ for all $t > 0$, obtaining that

$$|R(t)| \leq \frac{2C_0}{r_0^{N-1}} \qquad \text{for all } t > 0. \tag{5.2.18}$$

Combinig (5.2.17) and (5.2.18), we deduce the thesis. □

Corollary 5.23. *Let E be a minimizer of* (5.2.5). *If $R > \frac{2\sqrt{2}C_0}{r_0^{N-1}}$ (where C_0, $r_0 > 0$ comes from [61, Theorem 1.7]) it is a free minimizer, i.e.*

$$E \subset B_R.$$

Finally we prove the regularity of sets minimizing (5.2.5):

Proposition 5.24. *Let E be a minimizer of* (5.2.5). *Then ∂E is C^∞.*

Proof. From Lemma 5.20 we know that ∂E is graph in the direction x_N. Then [8, Corollary 3] implies that ∂E is C^∞ outside a closed singular set of Hausdorff dimension $N - 8$. Moreover, since by Lemma 5.19 E is also radially decreasing and symmetric, the singular set has to be its highest point (in the x_N direction of E). Now we consider a blow up of E centered at the singular point and we obtain a singular and symmetrical cone C. By densities estimates (see [61, Theorem 1.7]) which hold for E, we get that $C \neq \varnothing$. Hence C is a lipschitz cone, so [45, Theorem 1] tells us that C is a halfspace. As a consequence ∂E is C^∞. □

Proof of Theorem 5.14. From Proposition 5.15 and Corollary 5.23 we have the existence of a minimizer for (5.2.4). Moreover, thanks to Lemma 5.19, Lemma 5.20, Proposition 5.24 and Lemma 5.16, we deduce the minimizer's properties. □

Remark 5.25. It would be interesting to know whether minimizers, or even critical points, of (5.2.4) are unique up to horizontal translations (see for instance [55, 56, 54] for similar uniqueness results).

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Ringraziamenti

In queste ultime pagine vorrei ringraziare chi in un modo o nell'altro ha contribuito alla realizzazione di questa tesi, che ha rappresentato per me un percorso di lavoro e grande crescita, durante il quale ho collaborato con matematici molto bravi, ma anche instaurato e consolidato dei rapporti profondi.

Un primo caloroso ringraziamento va ai miei due relatori A. Malchiodi e M. Novaga che in questi anni mi hanno sempre seguito, sopportato, aiutato e consigliato nei vari lavori, permettendomi anche di conoscere matematici di fama internazionale e fare esperienze uniche come andare in visita al M.I.T. È stato per me un grande onore lavorare con loro.

Grazie anche al relatore della mia Tesi Magistrale, D. Mugnai, che oltre a farmi avvicinare alla ricerca, mi ha proposto di scrivere una tesi che mi ha permesso di entrare in una scuola d'eccellenza come questa.

Ringrazio il Prof. M. Morini perché, anche se il lavoro di cui abbiamo discusso non è in questa tesi e probabilmente non verrà mai pubblicato, è stato gentile e disponibile a darmi chiarimenti su un suo paper e consigli su come affrontare alcune questioni del mio. Lo ringrazio, poi, perché con le sue osservazioni mi ha permesso di migliorare questa tesi.

Un ringraziamento speciale va poi al mio carissimo amico Luigi. Abbiamo cominciato questo percorso insieme e condiviso tanti bei momenti, ed è soprattutto grazie a lui e ai suoi consigli se non ho abbandonato la Scuola dopo la crisi dei primi mesi.

Grazie al mio amico Giorgio, di cui ho grande stima. Pur essendo stato un collega 'solo' per un anno e mezzo, è stato oggetto di un'infinità di problemi, dubbi e spiegazioni. Lo ringrazio per essere stato sempre disponibile, gentile e avermi insegnato tanto. È anche grazie a lui se sono riuscita a sistemare importanti questioni emerse in questo dottorato, e quindi se riesco a concludere questo percorso. Oltre ad aver conosciuto un ottimo matematico, ho acquisito un grande amico!

Ringrazio i miei fratelli Diletta e Nicolò che, ritenendomi la loro sorellona intelligente, mi sono sempre stati a fianco anche da lontano; mamma Sonia che fin da piccola mi ha dedicato tanto tempo, facendomi appassionare ai libri, allo studio e alla curiosità verso ciò che mi circondava; babbo Mauro che col suo 'essere speciale' mi fa sempre capire che è orgoglioso di me; le nonne e la zia che mi hanno sempre sostenuto.

Un GRAZIE va anche al mio fidanzato Marco, che è sempre presente ad aiutarmi con problemi di ogni genere, a insegnarmi sempre cose nuove oltre a supportarmi e consigliarmi nei momenti di sconforto. Conosciuto in Scuola come un fisico vicino d'ufficio, oggi è una presenza importante nella mia vita.

Grazie alle mie amiche di Sestino: Ilenia, Maria, Sofia e Alessia, che ormai da vari anni sono presenti durante i miei traguardi e nella vita.

Grazie ad Alberta, che è stata mia coinquilina in tutti questi quattro anni. Da una convivenza nata per caso, abbiamo creato una grande amicizia, che ci ha fatto condividere tante emozioni (belle e brutte) e tante esperienze. Grazie anche a Nicole, perché, anche se ci conosciamo da poco tempo, abbiamo talmente tante cose in comune (a partire dalle nostre stranezze e sfortune), che in poco tempo abbiamo instaurato una bellissima amicizia. Insomma, siete state (e spero continuerete ad essere anche se le nostre strade si divideranno), due persone importanti. È stato bello tornare a casa e raccontarsi le nostre giornate, fare uscite insieme, vedere tutto Özpetek, e soprattutto andare insieme al nostro amato Carrefour. Questi e tanti bei momenti hanno reso la mia vita di "dottoranda vecchiotta" molto più piacevole!

Grazie a Lucrezia che, da essere una semplice coinquilina è diventata un'amica. Anche se non siamo più 'sotto lo stesso tetto' da un po' di tempo, lei ti fa sempre sentire la sua presenza.

Grazie ad Alberto e Virginia, che da colleghi di università, sono oggi miei grandi amici. Nonostante siamo un po' distanti, riusciamo a rimanere sempre in contatto e, oggi addirittura si sono organizzati per non perdersi la mia discussione.

Infine, ma non per importanza, un grazie anche a me stessa, perché fino a poco tempo fa ero convinta che non sarei mai riuscita a discutere il mio dottorato. Invece ho affrontato le difficoltà incontrate e sono arrivata qui davanti a tutti oggi ad esporre la mia tesi, raggiungendo questo importante traguardo che, ottenuto poi da Normalista, è per me un'enorme soddisfazione!