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Integral Chow ring and cohomological invariants of stacks of hyperelliptic curves

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The study of stacks is strongly recommended to people who would have been flagellants in earlier times.

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Introduction

Overview

The main objects of investigation of this thesis are *moduli stacks of hyperelliptic curves*. The moduli stack of hyperelliptic curves of genus g is denoted $\mathcal{H}_g$, and its objects consist of the datum of a family of smooth curves of genus g and an involution, such that the quotient of the family of curves with respect to the involution is a family of smooth rational curves.

We study two types of invariants of $\mathcal{H}_g$: the *integral Chow ring* and the *graded-commutative ring of cohomological invariants*. For what concerns the first invariant, our main result is the following computation.

THEOREM A. *Let $g \geq 3$ be an odd number and let k_0 be a field of characteristic 0 or $> 2g+2$. Then we have:*

$$CH(\mathcal{H}_g) = \mathbb{Z}[\tau, c_2, c_3]/(4(2g+1)\tau, 8\tau^2 - 2g(g+1)c_2, 2c_3)$$

where the degree of τ is 1, the degree of c_2 is 2 and the degree of c_3 is 3.

We also provide a geometric interpretation of the generators in terms of Chern classes of vector bundles over $\mathcal{H}_g$.

Our second main result is the following computation of the additive structure of $\mathrm{Inv}^\bullet(\mathcal{H}_g)$ with coefficients in $\mathbb{F}_2$.

THEOREM B. *Let k_0 be an algebraically closed field of characteristic $\neq 2$, and let $g \geq 3$ be an odd number.*

Then the graded-commutative ring of cohomological invariants $\mathrm{Inv}^\bullet(\mathcal{H}_g)$ with coefficients in $\mathbb{F}_2$, regarded as a graded $\mathbb{F}_2$-vector space, has a basis given by the elements

$$1, x_1, w_2, x_2, ..., x_{g+1}, x_{g+2}$$

where the degree of each x_i is i and w_2 is the second Stiefel-Whitney class coming from $\mathrm{Inv}^\bullet(\mathcal{B}\mathrm{PGL}_2)$.

Finally, we focus on the geometric meaning and multiplicative structure of cohomological invariants. The following theorem is obtained in collaboration with Roberto Pirisi. In what follows, we indicate as $\{n\}$ the equivalence class of a non-zero element n of a field K inside $K^*/(K^*)^2$.

THEOREM C (joint with R. Pirisi). *Fix a base field k_0 of characteristic $\neq 2$ and an even number $g \geq 2$.*

Then there exist $2g+2$ cohomological invariants β_i for $i = 1, \dots, 2g+2$ and an exceptional cohomological invariant η_{g+2} such that:

(1) The invariants $1, \beta_1, \dots, \beta_{g+1}, \eta_{g+2}$ are $\mathbb{F}_2$-linearly independent, $\beta_i = 0$ for $i > g+2$ and $\beta_{g+2} = \{2\} \cdot \beta_{g+1}$.

(2) *The multiplicative structure is given by the formulas:*

$$\beta_r \cdot \beta_s = \{-1\}^{m(r,s)}\beta_{r+s-m}$$
$$\beta_i \cdot \eta_{g+2} = 0 \text{ for } i \neq g+1$$
$$\beta_{g+1} \cdot \eta_{g+2} = \{-1\}^{g+1}\eta_{g+2}$$
$$\eta_{g+2} \cdot \eta_{g+2} = \{-1\}^{g+2}\eta_{g+2}$$

where $m(r,s)$ is computed as follows: write r in dyadic form as $\sum 2^i$ for $i \in R \subset \{0,1,2,\dots\}$, and write s as $\sum 2^i$ for $i \in S \subset \{0,1,2,\dots\}$. Then $m(r,s) = \sum 2^i$ for $i \in R \cap S$.

(3) *If k_0 is algebraically closed, then the invariants $1, \beta_1, \dots, \beta_{g+1}, \eta_{g+2}$ form a basis of $\mathrm{Inv}^\bullet(\mathcal{H}_g)$ as an $\mathbb{F}_2$-vector space.*

In the next section we give a more detailed description of the contents of the thesis, together with some background informations.

Motivations and discussion of the main results

Moduli theory and algebraic stacks. Moduli theory is a fascinating area of algebraic geometry: its roots can be traced back to the pioneering observations of Riemann on the dimension of parameter spaces (nowadays called moduli spaces) of Riemann surfaces of fixed genus. The development of moduli theory has prompted a lot of further research in algebraic geometry over the years.

The idea behind moduli theory is the following: given a class of geometrical objects that share some common properties (e.g. Riemann surfaces of fixed genus, closed subschemes of a fixed algebraic variety, coherent quotient sheaves of a fixed coherent sheaf), we can look for a *moduli scheme* that "parametrizes" those objects.

A first attempt to formally define what "parametrizes" means would be to ask for a bijection between geometric points of the moduli scheme and isomorphism classes of the objects. This request ends to be too naive, as it is easy to see that the disjoint union of points over the set of isomorphism classes of objects of the given kind satifies this property.

We want more: a moduli scheme should be a scheme whose geometry captures the behaviour of our objects *in families*. The language of category theory can help us to formulate in a precise way what we are looking for:

Definition. *Let $\mathcal{F} : (\mathrm{Sch})^{\mathrm{op}} \to (\mathrm{Set})$ be a functor from the opposite category of schemes to the category of sets. We say that a scheme* F represents the functor *$\mathcal{F}$ if there is an equivalence $\mathrm{Hom}(-,\mathrm{F}) \to \mathcal{F}(-)$.*

In particular, if the functor $\mathcal{F}$ comes from a moduli problem and it is represented by a scheme F*, we say that* F *is a* fine moduli scheme*.*

Examples of fine moduli schemes are:

(1) *Grassmannians*: consider the moduli functor

$$S \longmapsto \left\{ \begin{matrix} \varphi : \mathcal{O}_S^{\oplus n} \to \mathcal{V} \text{ such that} \\ \varphi \text{ is surjective and} \\ \mathcal{V} \text{ is locally free of rank } m \end{matrix} \right\}$$

The associated fine moduli scheme is the Grassmannian $\mathrm{Gr}(m,n)$.

(2) *Hilbert scheme*: let X be a relatively projective scheme over a noetherian base B. We can consider the moduli functor from the category $(\mathrm{Sch/B})^{\mathrm{op}}$

to (Set) defined as follows:

$$S \longmapsto \begin{Bmatrix} \text{Closed subschemes } Z \subset X \times S \\ \text{flat and proper over } S \end{Bmatrix}$$

This functor is represented by the Hilbert scheme $\mathrm{Hilb}_{X/B}$ (see [Gro95b]).

(3) *Quot scheme*: let X be a relatively projective scheme over a noetherian base B and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. We can consider the following functor:

$$S \longmapsto \begin{Bmatrix} \text{Surjective } \mathcal{O}_{X\times S}\text{-linear morphisms } \varphi : \mathcal{F}_S \to \mathcal{E} \\ \text{with } \mathcal{E} \text{ coherent sheaf on } X \times S \\ \text{properly supported and flat over } S \end{Bmatrix}$$

The associated fine moduli scheme is the Quot scheme $\mathrm{Quot}_{\mathcal{E},X/B}$ (see [Gro95b]).

Consider the moduli functor of Riemann surfaces of fixed genus, or over a general base the moduli functor of algebraic smooth curves of fixed genus, defined as

$$\mathcal{M}_g : S \longmapsto \frac{\begin{Bmatrix} C \to S \text{ proper and smooth morphisms} \\ \text{whose geometric fibres are curves of genus } g \end{Bmatrix}}{\text{isomorphism}}$$

where by curves we mean proper and geometrically connected schemes of dimension 1: then this functor cannot be represented by a fine moduli scheme.

Algebraic stacks were introduced to somehow solve this issue: they had been first envisioned by Grothendieck ([Gro95a]) and independently by Mumford in [Mum65], and then formally defined in the subsequent works of Deligne and Mumford ([DM69]) and M. Artin ([Art74]).

Let us explain the basic ideas behind algebraic stacks by focusing on the particular case of moduli of smooth curves of fixed genus: observe that the moduli functor we exhibited above can be upgraded to a pseudofunctor from the category $(\mathrm{Sch})^{\mathrm{op}}$ to the category of groupoids (Grpd) by removing the identification between isomorphic families.

More precisely, we define $\mathcal{M}_g$ as the pseudofunctor that sends a scheme S to the groupoid whose objects are given by families of curves over S and whose morphisms are given by isomorphisms of the families over the base. The pullback morphisms are induced by taking the pullback of families of curves.

This is not a strict functor, because the composition of pullback morphisms is only isomorphic to the pullback of the composition, and it is not stricty equal to it.

As shown in [DM69], the pseudofunctor $\mathcal{M}_g$ enjoys some very nice properties, namely:

(1) The natural diagonal transformation $\delta : \mathcal{M}_g \to \mathcal{M}_g \times \mathcal{M}_g$ is *representable*: this boils down to the claim that, given any two families of curves $C \to B$ and $C' \to B$, the isomorphism functor $Isom_B(C, C')$ that sends a scheme S to the set of isomorphisms between the families $C_S \to S$ and $C'_S \to S$ is represented by a separated and quasi-compact scheme.

(2) There exists a natural transformation $\Phi : h_U \to \mathcal{M}_g$ from a functor of points of a scheme U such that Φ is *representable, étale and surjective*: this means that given any other scheme T and a natural transformation $h_T \to \mathcal{M}_g$, the fibred product $h_U \times_{\mathcal{M}_g} h_T$ is equivalent to the functor of points of a scheme S and in addition the corresponding morphism of

schemes $S \to T$ is étale and surjective.

A pseudofunctor satisfying these properties is a *Deligne-Mumford stack*, a particular class of algebraic stacks. Therefore, $\mathcal{M}_g$ is a Deligne-Mumford stack that basically by definition represents the moduli pseudofunctor of smooth curves of fixed genus.

Though abstract in nature, we can still do geometry on Deligne-Mumford stacks. Many geometric invariants originally defined only for schemes had been generalized to this setting, e.g. cohomology groups, Chow groups, K-theory, derived categories, and so on. The extension of these notions to the realm of Deligne-Mumford or algebraic stacks had been the result of many efforts in the last fifty years ([Mum65], [Gil84], [Vis89], [Kre99] and others).

This thesis deals with a specific Deligne-Mumford stack, *the stack* $\mathcal{H}_g$ *of hyperelliptic curves of genus* g, and it aims at investigating two specific invariants of this stack: the *integral Chow ring* $CH(\mathcal{H}_g)$ and the graded-commutative ring of *cohomological invariants* $\mathrm{Inv}^\bullet(\mathcal{H}_g)$.

Integral Chow ring of $\mathcal{H}_g$. The notion of Chow groups plays a central role in intersection theory. Given a scheme of finite type over a field, its Chow groups can be thought as an homology theory built up from algebraic cycles (i.e. algebraic closed subvarieties) instead of topological cycles, and with homological equivalence relation substituted by rational equivalence relation.

The idea of Chow groups was proposed by Severi in the Thirties, but a formal definition first appeared in the Chow's paper [Cho56]. In the second half of the last century Fulton, MacPherson and others gave a firm foundation to this theory (see [Ful98]), exploiting the language of schemes developed by Grothendieck and his collaborators.

Chow groups of a smooth variety over a field inherit an additional structure of graded ring, with the multiplication given by the intersection product.

The moduli stack of smooth curves $\mathcal{M}_g$ (resp. the proper moduli stack of stable curves $\overline{\mathcal{M}}_g$) admits a so called *coarse moduli scheme* M_g (resp. $\overline{M}_g$), i.e. there exists a morphism $\mathcal{M}_g \to M_g$ (resp. $\overline{\mathcal{M}}_g \to \overline{M}_g$) that induces a bijection on geometric points and it is initial among all the morphisms from $\mathcal{M}_g$ (resp. $\overline{\mathcal{M}}_g$) to a scheme.

The coarse moduli schemes M_g and $\overline{M}_g$ are not smooth, but in the landmark paper [Mum83] Mumford showed that there is a well defined intersection product on their Chow groups with rational coefficients, i.e. the Chow groups tensorized with $\mathbb{Q}$. Actually, this is an aspect of a more general phenomenon: in [Vis89] Vistoli defined the notion of rational Chow ring of a smooth Deligne-Mumford stack, and he also showed that this coincides with the rational Chow ring of its coarse moduli scheme.

In the paper [Mum83] Mumford also computed an explicit presentation of the rational Chow rings $CH(M_2)_{\mathbb{Q}}$ and $CH(\overline{M}_2)_{\mathbb{Q}}$ in terms of generators and relations, saying that *it seems very worthwile to work out* $CH(\overline{M}_g)$ *for other small values of* g*, in order to get some feeling for the properties of these rings and their relation to the geometry of* $\overline{M}_g$.

Since then, other rational Chow rings of moduli of curves had been computed: in [Fab90a], [Fab90b] Faber gave a presentation of $CH(M_3)_{\mathbb{Q}}$, $CH(\overline{M}_3)_{\mathbb{Q}}$ and $CH(M_4)_{\mathbb{Q}}$. In [Iza95] Izadi computed the rational Chow ring of M_5, and in [PV15] Penev and Vakil obtained an explicit presentation of $CH(M_6)_{\mathbb{Q}}$.

Generalizing some ideas of Totaro contained in [Tot99], Edidin and Graham developed in [EG98] an intersection theory with integral coefficients for quotient stacks, which are a specific class of algebraic stacks. This theory applies in particular to the cases of moduli of curves.

Edidin and Graham also showed that for Deligne-Mumford stacks which are quotient stacks, the tensor product of their integral Chow ring with $\mathbb{Q}$ and the rational Chow ring as defined by Vistoli are isomorphic.

In the last twenty years, some integral Chow rings of moduli stack of curves had been computed: in [EG98] Edidin and Graham determined $CH(\overline{\mathcal{M}}_{1,1})$, the integral Chow ring of the moduli stack of stable elliptic curves, and Vistoli in the appendix [Vis98] computed $CH(\mathcal{M}_2)$, the integral Chow ring of the moduli stack of curves of genus 2. Edidin and Fulghesu obtained in [EF08] an explicit presentation of the integral Chow ring of the moduli stack of at most 1-nodal rational curves, and in [EF09] they computed the integral Chow ring of the stack of hyperelliptic curves of even genus.

The first main result of this thesis is an explicit presentation of the integral Chow ring of $\mathcal{H}_g$, the moduli stack of hyperelliptic curves of genus g, when $g \geq 3$ is an odd number.

THEOREM A (Theorem 2.5.2). *Let k_0 be a field of characteristic 0 or $> 2g+1$, and let $\mathcal{H}_g$ be the stack over k_0 of hyperelliptic curves of odd genus g, with $g \geq 3$. Then we have:*

$$CH(\mathcal{H}_g) = \mathbb{Z}[\tau, c_2, c_3]/(4(2g+1)\tau, 8\tau^2 - 2g(g+1)c_2, 2c_3)$$

where the degree of τ is 1, the degree of c_2 is 2 and the degree of c_3 is 3.

We also provide a geometrical interpretation of the generators of this ring.

The content of the theorem above had already been presented in the paper [FV11], but recently Pirisi pointed out a mistake in the proof of [FV11, lemma 5.6] which is crucial in order to complete the computation. Actually, we prove that the proof of [FV11, lemma 5.6] cannot be fixed, because its consequences, in particular [FV11, lemma 5.3], are wrong.

Cohomological invariants of $\mathcal{H}_g$. The second part of the thesis focuses on another invariant of algebraic stacks, the graded-commutative ring of cohomological invariants.

The idea behind the notion of cohomological invariants comes from the theory of characteristic classes in topology: first introduced by Stiefel and Whitney as a tool to study vector bundles on manifolds, characteristic classes were later generalized to principal bundles.

Fix a topological group G and a cohomology theory H: then characteristic classes are a functorial way to associate to every principal G-bundle over a topological space X a cohomology class in $H(X)$. In other terms, characteristic classes are natural transformations from the functor

$$\mathcal{B}G : (\mathrm{Top}) \longrightarrow (\mathrm{Set}), \quad X \longmapsto \{G\text{-bundles over } X\} / \simeq$$

to the cohomology functor $X \mapsto H(X)$.

Cohomological invariants first appeared as a reformulation of this idea in an algebraic setting. More precisely, fix a positive number p, a field k_0 such that p is invertible in k_0 and an algebraic group G. Then we replace the category (Top) with (Field$/k_0$), the category of field extensions of k_0, and the cohomology theory H with

$$\mathrm{H}^\bullet : (\mathrm{Field}/k_0) \longrightarrow (\mathrm{GrCommRing}), \quad K \longmapsto \oplus_i H^i_{\acute{e}t}(\mathrm{Spec}(K), \mu_p^{\otimes i})$$

where μ_p denotes the group of p^{th}-roots of unity. We also substitute $\mathcal{B}G$ with its algebro-geometrical counterpart, namely the classifying stack $\mathcal{B}G$ or, better, its functor of points:

$$P_{\mathcal{B}G} : (\mathrm{Field}/k_0) \longrightarrow (\mathrm{Set}), \quad K \longmapsto \{G\text{-torsors over } \mathrm{Spec}(K)\} / \simeq$$

Cohomological invariants are then defined by copying the definition of characteristic classes in topology:

DEFINITION ([GMS03, Def. 1.1]). *A cohomological invariant of $\mathcal{B}G$ with coefficients in $\mathbb{F}_p$ is a natural transformation of functors*

$$P_{\mathcal{B}G} \longrightarrow \mathrm{H}^{\bullet} := \oplus_i H^i_{\text{ét}}(-, \mu_p^{\otimes i})$$

The graded-commutative ring of cohomological invariants is denoted $\mathrm{Inv}^{\bullet}(\mathcal{B}G)$.

The first appearance of cohomological invariants, though not in this formulation, can be traced back to the seminal paper of Witt [Wit37] and since then they have been extensively studied (see [GMS03]).

In the recent work [Pir18a], Pirisi extended the notion of cohomological invariants from classifying stacks to smooth algebraic stacks over k_0:

DEFINITION ([Pir18a, Def. 1.1]). *Let $\mathcal{X}$ be a smooth algebraic stack over k_0. Then a cohomological invariant of $\mathcal{X}$ is a natural transformation*

$$P_{\mathcal{X}} \longrightarrow \mathrm{H}^{\bullet}$$

from the functor of points of $\mathcal{X}$ to $\mathrm{H}^{\bullet}$ which satisfies a certain continuity condition (see [Pir18a, Def. 1.1]*).*

The graded-commutative ring of cohomological invariants of a smooth algebraic stack $\mathcal{X}$ is denoted $\mathrm{Inv}^{\bullet}(\mathcal{X})$.

The second main result of this thesis is the following:

THEOREM B (Theorem 3.1.1). *Let k_0 be an algebraically closed field of characteristic $\neq 2$, and let $\mathcal{H}_g$ denotes the moduli stack over k_0 of smooth hyperelliptic curves of odd genus $g \geq 3$.*

Then the graded-commutative ring of cohomological invariants $\mathrm{Inv}^{\bullet}(\mathcal{H}_g)$ *with coefficients in $\mathbb{F}_2$, regarded as a graded $\mathbb{F}_2$-vector space, has a basis given by the elements*

$$1, x_1, w_2, x_2, ..., x_{g+1}, x_{g+2}$$

where the degree of each x_i is i and w_2 is the second Stiefel-Whitney class coming from $\mathrm{Inv}^{\bullet}(\mathcal{B}\mathrm{PGL}_2)$.

The cohomological invariants $\mathrm{Inv}^{\bullet}(\mathcal{H}_g)$ with coefficients in $\mathbb{F}_p$, when g is even or equal to three, had already been computed by Pirisi in [Pir17] and [Pir18b] when the base field is algebraically closed and its characteristic does not divide p.

When g is odd and $p \neq 2$, the generators of $\mathrm{Inv}^{\bullet}(\mathcal{H}_g)$ can also be deduced from the main result of [Pir18b]. The importance of the case $p = 2$ is due to the fact that under this assumption, the cohomological invariants of $\mathcal{H}_g$ present a richer structure.

The third main result of this thesis concerns the multiplicative structure of $\mathrm{Inv}^{\bullet}(\mathcal{H}_g)$ with coefficients in $\mathbb{F}_2$ when $g \geq 2$ is even.

THEOREM C (Theorem 4.2.14, joint with R. Pirisi). *Fix a base field k_0 of characteristic $\neq 2$ and an even number $g \geq 2$.*

Then there exist $2g + 2$ cohomological invariants β_i for $i = 1, \dots, 2g + 2$ and an exceptional cohomological invariant η_{g+2} such that:

(1) The invariants $1, \beta_1, \dots, \beta_{g+1}, \eta_{g+2}$ are $\mathbb{F}_2$-linearly independent, $\beta_i = 0$ for $i > g + 2$ and $\beta_{g+2} = \{2\} \cdot \beta_{g+1}$.

(2) *The multiplicative structure is given by the formulas:*

$$\beta_r \cdot \beta_s = \{-1\}^{m(r,s)}\beta_{r+s-m}$$
$$\beta_i \cdot \eta_{g+2} = 0 \text{ for } i \neq g+1$$
$$\beta_{g+1} \cdot \eta_{g+2} = \{-1\}^{g+1}\eta_{g+2}$$
$$\eta_{g+2} \cdot \eta_{g+2} = \{-1\}^{g+2}\eta_{g+2}$$

where $m(r,s)$ *is computed as follows: write* r *in dyadic form as* $\sum 2^i$ *for* $i \in R \subset \{0,1,2,\dots\}$*, and write* s *as* $\sum 2^i$ *for* $i \in S \subset \{0,1,2,\dots\}$*. Then* $m(r,s) = \sum 2^i$ *for* $i \in R \cap S$*.*

(3) *If* k_0 *is algebraically closed, then the invariants* $1, \beta_1, \dots, \beta_{g+1}, \eta_{g+2}$ *form a basis of* $\mathrm{Inv}^\bullet(\mathcal{H}_g)$ *as an* $\mathbb{F}_2$*-vector space.*

Methods

The starting point for our computations is the presentation of the moduli stack of hyperelliptic curves of odd genus as a quotient stack contained in [**AV04**]: let $\mathbb{A}(1,2g+2)$ be the affine space of binary forms of degree $2g+2$, and let $\mathrm{PGL}_2 \times \mathbb{G}_m$ act on this scheme as follows:

$$(A,\lambda)\cdot f(x,y) := \lambda^{-2}\det(A)^g f(A^{-1}(x,y))$$

Observe that the determinant of an element of PGL_2 is not well defined, but the action above is well defined though. Let $\Delta'_{1,2g+2}$ be the divisor in $\mathbb{A}(1,2g+2)$ of singular forms, which is invariant with respect to the action above. Then [**AV04**, cor. 4.7] tells us that:

$$\mathcal{H}_g \simeq [(\mathbb{A}(1,2g+2)\setminus \Delta'_{1,2g+2})/\mathrm{PGL}_2\times\mathbb{G}_m]$$

This implies that:

$$CH(\mathcal{H}_g) \simeq CH_{\mathrm{PGL}_2\times\mathbb{G}_m}(\mathbb{A}(1,2g+2)\setminus\Delta'_{1,2g+2})$$

where on the right we have the $\mathrm{PGL}_2\times\mathbb{G}_m$-equivariant Chow ring of the scheme $\mathbb{A}(1,2g+2)\setminus\Delta'_{1,2g+2}$ as defined by Edidin and Graham in [**EG98**].

Moreover, by [**Pir18a**, th. 4.9] we also deduce:

$$\mathrm{Inv}^\bullet(\mathcal{H}_g) \simeq A^0_{\mathrm{PGL}_2\times\mathbb{G}_m}(\mathbb{A}(1,2g+2)\setminus\Delta'_{1,2g+2})$$

where on the right we have the zeroth $\mathrm{PGL}_2\times\mathbb{G}_m$-equivariant Chow group with coefficients in étale cohomology, as defined by Rost in [**Ros96**].

The group PGL_2 is a non-special group, i.e. there exist PGL_2-torsors over certain base schemes that are not Zariski-locally trivial but only étale-locally trivial. Equivariant computations involving non-special groups may be hard to handle. For instance, not every projective space endowed with an action of PGL_2 can be seen as the projectivization of a PGL_2-representation: this makes the computation of the PGL_2-equivariant Chow ring of $\mathbb{P}^n$ a non trivial challenge.

On the other hand, equivariant computations involving a special group, i.e. a group G such that every G-torsor can be trivialized Zariski-locally, are more approachable. An important example of special group is the general linear group GL_n. A key result of the present work is the following theorem, where a presentation of $\mathcal{H}_g$ as a quotient stack with respect to the action of a special group is explicitly obtained:

Theorem. *Let* $g \geq 3$ *be an odd number, and let* $\mathbb{A}(2,2)_3$ *be the scheme of quadratic forms in three variables of rank* 3*, endowed with the* $\mathrm{GL}_3\times\mathbb{G}_m$*-action given by the formula:*

$$(A,\lambda)\cdot q(x,y,z) := \det(A)q(A^{-1}(x,y,z))$$

Then there exists a $\mathrm{GL}_3 \times \mathbb{G}_m$*-equivariant vector bundle* $V_{g+1,3}$ *over* $\mathbb{A}(2,2)_3$ *and an invariant divisor* D'_3 *inside* $V_{g+1,3}$ *such that*

$$\mathcal{H}_g = [(V_{g+1,3} \setminus D'_3)/\mathrm{GL}_3 \times \mathbb{G}_m]$$

To obtain the presentation above, we use a fairly standard construction applied to the case of PGL_2-schemes, to which we refer to as the construction of a GL_3-counterpart of a PGL_2-scheme (and of the $\mathrm{GL}_3 \times \mathbb{G}_m$-counterpart of a $\mathrm{PGL}_2 \times \mathbb{G}_m$-scheme): given a PGL_2-scheme X, the GL_3-counterpart is a GL_3-scheme Y such that $[X/\mathrm{PGL}_2] \simeq [Y/\mathrm{GL}_3]$ (and similarly for $\mathrm{PGL}_2 \times \mathbb{G}_m$-scheme).

The new presentation is then exploited in the following two ways: on one side, it shows that the integral Chow ring is isomorphic to the $\mathrm{GL}_3 \times \mathbb{G}_m$-equivariant Chow ring of $V_{g+1,3} \setminus D'_3$, and the fact that $\mathrm{GL}_3 \times \mathbb{G}_m$ is special enables us to reduce to the action of the subtorus of diagonal matrices: in this setting, we have much more invariant subvarieties that we can use to make computations.

On the other side, the new presentation shows that the moduli stack $\mathcal{H}_g$ is an open substack of the quotient stack $[V_{g+1}/\mathrm{GL}_3 \times \mathbb{G}_m]$, where V_{g+1} is a vector bundle over the scheme $\mathbb{A}(2,2)_{[3,1]}$ parametrizing non-zero quadratic forms. This new feature turns to be particularly helpful when it comes to compute the zeroth equivariant Chow group with coefficients of $V_{g+1,3} \setminus D'_3$.

To investigate the multiplicative structure of the cohomological invariants of $\mathcal{H}_g$ for $g \geq 2$ an even number, we exploit the existence of a well defined morphism

$$W : \mathcal{H}_g \longrightarrow \mathcal{B}S_{2g+2}$$

that sends a family of hyperelliptic curves $C \to S$ with involution ι to $W_{C/S} \to S$, where $W_{C/S}$ is the Weierstrass divisor, i.e. the ramification locus of the quotient morphism $C \to C/\langle\iota\rangle$.

This fact will allow us to deduce the multiplicative structure of $\mathcal{H}_g$ from the one of $\mathcal{B}S_{2g+2}$.

Description of contents

Chapter 1: preliminary results. In chapter 1 we collect both introductory materials and new technical results.

In section 1.1 we give a quick introduction to equivariant Chow groups, mainly following [**EG98**]: in particular, we sketch their definition and main properties and we exhibit some explicit computations.

In section 1.2 we discuss equivariant Chow group with coefficients in étale cohomology, as defined in [**Ros96**]: we outline their construction and some useful results.

In section 1.3, we construct the GL_3-counterpart of a PGL_2-scheme and we apply this to those cases that are relevant to our purposes.

Chapter 2: the Chow ring of the stack of hyperelliptic curves of odd genus. In chapter 2 we compute the integral Chow ring of $\mathcal{H}_g$, the stack of hyperelliptic curves of genus g, where $g \geq 3$ is an odd number.

In section 2.1 we give a new presentation of $\mathcal{H}_g$, for $g \geq 3$ an odd number. We also introduce a class of vector bundles, denoted V_n, which will play a central role in the rest of the dissertation.

In section 2.2, we study some intersection theoretical properties of the projective bundles $\mathbb{P}(V_n)$.

In section 2.3 we begin the computation of the Chow ring of $\mathcal{H}_g$, obtaining the generators and some relations.

Section 2.4 is the technical core of the chapter: here is where the new presentation of $\mathcal{H}_g$ will prove to be particularly useful in order to find other relations in $CH(\mathcal{H}_g)$.

The computation of $CH(\mathcal{H}_g)$ is completed in section 2.5, where we also provide a geometrical interpretation of the generating cycles of $CH(\mathcal{H}_g)$.

Chapter 3: cohomological invariants of the stack of hyperelliptic curves of odd genus. In chapter 3 we compute the $\mathbb{F}_2$-module structure of the graded-commutative ring of cohomological invariants with coefficients in $\mathbb{F}_2$ of $\mathcal{H}_g$, for $g > 3$ an odd number.

In section 3.1 we prove the main theorem of the paper, assuming for the moment the key lemma 3.1.3, whose proof is postponed to section 3.4. The strategy of proof is similar to the one contained in [Pir18b]. The remainder of the chapter is devoted to develop the theory necessary to prove the key lemma 3.1.3.

In section 3.2 we study the geometry of the fundamental divisor D of the projective bundle $\mathbb{P}(V_n)$.

The observations made in this section are then applied in section 3.3 in order to do some intersection theoretical computations useful to prove the key lemma 3.1.3: the proof is finally completed in section 3.4.

Chapter 4: Cohomological invariants of the stack of hyperelliptic curves of even genus, multiplicative structure. In chapter 4 we study the multiplicative structure of $\mathrm{Inv}^\bullet(\mathcal{H}_g)$ with coefficients in $\mathbb{F}_2$ when $g \geq 2$ is an even number.

In section 4.1 we recall the classical theory of cohomological invariants for étale algebras and quadratic forms.

In section 4.2 we first investigate the multiplicative structure of $\mathrm{Inv}^\bullet(\mathcal{B}_g)$, where $\mathcal{B}_g$ is the stack of vector bundles $E \to S$ of rank 2 with a Cartier divisor $D \subset \mathbb{P}(E)$ which is étale of degree $2g+2$ over S. We derive from this results part of the multiplicative structure of $\mathrm{Inv}^\bullet(\mathcal{H}_g)$.

Afterward, we give a construction of what we call the exceptional cohomological invariant, i.e. a cohomological invariant of $\mathcal{H}_g$ that does not come from $\mathcal{B}_g$, and we complete then the computation of the multiplicative structure of $\mathrm{Inv}^\bullet(\mathcal{H}_g)$.

At the end of thesis the reader can find a brief chapter where we outline some possible future directions of research.

Assumptions and notation

Every scheme is assumed to be of finite type over $\mathrm{Spec}(k_0)$. The assumptions on the base field k_0 will be specified in each section, as they depend on the specific results we will be discussing.

A variety is a separated and integral scheme.

Every algebraic group is assumed to be linear.

The word *curve* will mean a proper, geometrically connected scheme of dimension 1.

If X is a variety, the notation $\mathrm{H}^\bullet(X)$ will stand for the graded-commutative ring $\oplus_i H^i_{\acute{e}t}(\xi_X, \mu_p^{\otimes i})$, where ξ_X is the generic point of X and p is a positive number that will be specified in every section.

Sometimes, we will write $\mathrm{H}^\bullet(R)$, where R is a finitely generated k_0-domain, to indicate $\mathrm{H}^\bullet(\mathrm{Spec}(R))$. Observe that if k_0 is algebraically closed $\mathrm{H}^\bullet(k_0) \simeq \mathbb{F}_p$.

The Chow groups with coefficients in $\mathrm{H}^\bullet$ will be denoted $A^i(-, \mathrm{H}^\bullet)$ or $A_i(-, \mathrm{H}^\bullet)$, whether we adopt the grading by codimension or dimension. At a certain point we will use the shorthand $A^i(-)$ to denote Chow groups with coefficients in $\mathrm{H}^\bullet$ of codimension i, and we will drop the apex to indicate the direct sum of Chow groups with coefficients of every codimension. A Chow group with coefficients is said to be trivial when it is isomorphic to $\mathrm{H}^\bullet(k_0)$. Similarly, the graded-commutative ring of cohomological invariants of a stack is said to be trivial when it is isomorphic to $\mathrm{H}^\bullet(k_0)$.

To denote the G-equivariant Chow groups with coefficients in $\mathrm{H}^\bullet$ we will write $A^i_G(-, \mathrm{H}^\bullet)$ or $A^i_G(-)$, and we will write A^i_G to indicate $A^i_G(\mathrm{Spec}(k_0))$. Similar notations will be used for Chow groups $CH^i(-)$. We will write $CH^i(-)_{\mathbb{F}_p}$ for the tensor product $CH^i(-) \otimes \mathbb{F}_p$.

Given an element x in a field K, its class in $K^*/(K^*)^2$ will be denoted $\{x\}$. In particular, for $K = k_0$, the element $\{x\}$ regarded as a constant cohomological invariant has degree 1.

We will denote $\mathbb{A}(n, d)$ the affine space of forms in $n + 1$ variables of degree d, and $\mathbb{P}(n, d)$ its projectivization.

Throughout the thesis, a relevant role will be played by $\mathbb{A}(2, 2)$, the space of quadratic ternary forms. The subschemes of $\mathbb{A}(2, 2)$ parametrising forms of rank r will be denoted $\mathbb{A}(2, 2)_r$, and the subschemes parametrising forms of rank in the interval $[b, a]$ will be denoted $\mathbb{A}(2, 2)_{[a,b]}$.

Given a relative scheme $X \to \mathbb{A}(2, 2)$, its pullback to $\mathbb{A}(2, 2)_r$ or $\mathbb{A}(2, 2)_{[a,b]}$ will be denoted X_r or $X_{[a,b]}$. At any rate, these definitions will be frequently repeated along the thesis.

Similarly, we will denote $\mathbb{P}(2, 2)_r$ (resp. $\mathbb{P}(2, 2)_{[a,b]}$) the subscheme of $\mathbb{P}(2, 2)$ parametrising conics of rank r (resp. of rank r such that $a \geq r \geq b$). If X is a scheme over $\mathbb{P}(2, 2)$, its pullback to $\mathbb{P}(2, 2)_r$ (resp. $\mathbb{P}(2, 2)_{[a,b]}$) will be denoted X_r (resp. $X_{[a,b]}$).

CHAPTER 1

Preliminary results

In the first two sections we give an overview of Chow groups and Chow groups with coefficients. In the last section we introduce the notion of GL_3-counterpart of a PGL_2-scheme and we outline some applications.

1.1. Equivariant Chow groups

We give here a brief introduction to equivariant Chow groups. These were first introduced in the papers [Tot99] and [EG98]. A nice account of this theory can be found in [FV18, sections 2-4].

Let X be a scheme of finite type over the ground field k_0, and let G be an algebraic group acting on it. By [EG98, lemma 9], for any $i > 0$ we can always find a G-representation V with an open subscheme $U \subset V$ such that G acts freely on U and the complement $V \setminus U$ has codimension $> i$.

Suppose that X is equidimensional and quasi-projective, and that the action of G is linearizable: by [EG98, prop. 23] there exists a G-torsor $X \times U \to (X \times U)/G$ in the category of schemes. We can define the G-equivariant Chow group of X of codimension i as:

$$CH^i_G(X) := CH^i((X \times U)/G)$$

This definition does not depend on the chosen representation ([EG98, prop. 1]).

Equivariant Chow groups enjoy most of the properties of standard Chow groups:

Proposition 1.1.1. *Let X and Y be equidimensional, quasi-projective schemes of finite type over k_0, endowed with a linearizable G-action. Then we have:*

(1) Proper pushforward*: every G-equivariant, proper morphism $f : X \to Y$ induces a homomorphism of groups*

$$f_* : CH^G_i(X) \longrightarrow CH^G_i(Y)$$

such that $f_*[V] = [k(V) : k(f(V))][f(V)]$ for every subvariety V ($[k(V) : k(f(V))] = 0$ if $\dim(V) > \dim(f(V))$).

(2) Flat pullback*: every G-equivariant, flat morphism $f : X \to Y$ of relative constant dimension induces a homomorphism of groups*

$$f^* : CH^i_G(Y) \longrightarrow CH^i_G(X)$$

such that $f^*[V] = [f^{-1}V]$, where the term on the right is the fundamental class of an equidimensional closed subscheme.

(3) Localization exact sequence*: given a closed, G-invariant subscheme $Z \overset{i}{\hookrightarrow} X$ whose open complement is $U \overset{j}{\hookrightarrow} X$, there exists an exact sequence*

$$CH^G_i(Z) \xrightarrow{i_*} CH^G_i(X) \xrightarrow{j^*} CH^G_i(U) \to 0$$

(4) Compatibility*: given a cartesian square of G-schemes*

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ Y' & \xrightarrow{f'} & X' \end{array}$$

where the horizontal morphisms are G-equivariant and proper, and the vertical morphisms are G-equivariant and flat of relative constant dimension d, we get a commutative diagram:

$$\begin{array}{ccc} CH_k^G(Y') & \xrightarrow{f'_*} & CH_k^G(X') \\ \downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle g'^*} \\ CH_{k+d}^G(Y) & \xrightarrow{f_*} & CH_{k+d}^G(X) \end{array}$$

(5) Homotopy invariance*: if $\pi : E \to X$ is a G-equivariant, finite rank vector bundle, then we have an isomorphism*

$$\pi^* : CH_G^i(X) \simeq CH_G^i(E)$$

(6) Projective bundle formula*: If $\mathbb{P}(E) \to X$ is the projectivization of a G-equivariant , finite rank vector bundle, then for $i < \mathrm{rk}(E)$ we have:*

$$CH_G^i(\mathbb{P}(E)) \simeq \oplus_{j=0}^{i} CH_G^j(X)$$

(7) Ring structure*: if X is smooth, then $CH_G(X)$ inherits a graded ring structure.*

(8) Projection formula*: if $f : X \to Y$ is a G-equivariant, proper and flat morphism between smooth schemes, then we have:*

$$f_*(\xi \cdot f^*\eta) = f_*\xi \cdot \eta$$

(9) Gysin homomorphism*: every G-equivariant, local complete intersection morphism $f : X \to Y$ induces a homomorphism of groups:*

$$f^! : CH_G^i(Y) \longrightarrow CH_G^i(X)$$

such that, if f is a closed embedding and V is a subvariety which intersect transversally Y, then $f^![V] = [V \cap X]$.

Given an equivariant vector bundle $E \to X$ of rank r, we can define the *equivariant Chern classes* of E as homomorphisms:

$$c_i^G(E) : CH_G^k(X) \longrightarrow CH_G^{k+i}(X)$$

where i ranges from 0 to r. If X is equidimensional and $[X]$ denotes its cycle class, the cycles $c_1^G(E)([X])$ will also be called equivariant Chern classes of E.

If X is smooth, we can restate the projective bundle formula 1.1.1.(6) by saying that, given an equivariant vector bundle $E \to X$ of rank r, we have:

$$CH_G(\mathbb{P}(E)) = CH_G(X)[h]/(h^r + c_1^G(E)h^{r-1} + c_2^G(E)h^{r-2} + \cdots + c_r^G(E)$$

where $h = c_1^G(\mathcal{O}(1))$.

If $f : X \to Y$ is a G-equivariant flat morphism between equidimensional schemes and $E \to Y$ is an equivariant vector bundle, we have:

$$f^* c_i^G(E) = c_i^G(f^*E)$$

This enables us to formulate a projection formula also for not necessarily flat morphism.

PROPOSITION 1.1.2. *Let $f : X \to Y$ be a proper morphism between G-schemes of finite type, and let $E \to X$ be an equivariant vector bundle. Then for any ξ in $CH_i^G(X)$ and for any k, we have:*

$$f_*(c_k^G(f^*E)(\xi)) = c_k^G(E)(f_*\xi)$$

The next proposition is of fundamental importance for computational purposes.

PROPOSITION 1.1.3. [**EG98**, prop. 6] *Let G be a connected reductive special group with maximal subtorus T split and Weyl group W. Then for any G-scheme X we have:*

$$CH_G(X) = CH_T(X)^W$$

Next we list some explicit presentations of equivariant Chow rings.

PROPOSITION 1.1.4. [**EG98**, pg. 14]

(1) Let $T = \mathbb{G}_m^{\oplus n}$ be a split torus. Then

$$CH_T(\mathrm{Spec}(k_0)) = \mathbb{Z}[\lambda_1, \dots, \lambda_n]$$

where λ_i is the equivariant first Chern class of the T-representation induced by the projection $T \to \mathbb{G}_m$ on the i^{th} factor.

(2) We have:

$$CH_{\mathrm{GL}_n}(\mathrm{Spec}(k_0)) = \mathbb{Z}[c_1, \dots, c_n]$$

where c_i is the i^{th} equivariant Chern class of the standard representation of GL_n.

REMARK 1.1.5. There is a natural inclusion $CH_{\mathrm{GL}_n}(X) \hookrightarrow CH_{\mathbb{G}_m^{\oplus n}}(X)$ which sends each c_i into the elementary symmetric polynomial of degree i in the variables $\lambda_1, \dots, \lambda_n$. Given a GL_n-scheme X, let U be an open subscheme of a representation such that the complement has codimension $> i$ and the geometric quotient $(X \times U)/\mathrm{GL}_n$ exists in the category of schemes: then there is a flat morphism $(X \times U)/\mathbb{G}_m^{\oplus n} \to (X \times U)/\mathrm{GL}_n$, and the inclusion of equivariant Chow rings above is induced by the pullback along this flat map.

PROPOSITION 1.1.6. [**FV18**, lemma 2.1] *Let G be a special algebraic group and let $T \subset G$ be a maximal subtorus. Let X be a smooth G-scheme and $I \subset CH_G(X)$ an ideal. Then:*

$$I \cdot CH_T(X) \cap CH_G(X) = I$$

The following lemma is useful to perform computations:

LEMMA 1.1.7. [**EF09**, lemma 2.4] *Let E be a T-representation and let $H \subset \mathbb{P}(E)$ be a T-invariant hypersurface defined by the homogeneous equation $f = 0$, with f in $\mathrm{Sym}^d E^\vee$. Let $\chi : T \to \mathbb{G}_m$ be the character such that for every element t of T we have $t \cdot f = \chi^{-1}(t)f$. Then:*

$$[H] = c_1^T(\mathcal{O}(d)) + c_1(\chi)$$

inside $CH_T^1(\mathbb{P}(E))$, where $c_1(\chi)$ is the first Chern class of the 1-dimensional T-representation associated to the character χ.

Take a quasi-projective scheme X endowed with a linearized G-action and let $L \to X$ be an equivariant line bundle. Define L^* as the complement in L of the zero section. Observe that L^* inherits the G-action and that it has a natural structure of $\mathbb{G}_m$-torsor, where $\mathbb{G}_m$ acts by scalar multiplication.

This second fact can also be seen by noting that $L^* \simeq \underline{\mathrm{Isom}}_X(L, \mathbb{A}^1_X)$ and that $\mathbb{G}_m$ acts on the second scheme as follows: given a morphism $S \to X$, a trivialization $\varphi : L_S \simeq \mathbb{A}^1_S$ and an element $\lambda : \mathbb{A}^1_S \simeq \mathbb{A}^1_S$ of $\mathbb{G}_m(S)$, we can define $\lambda \cdot \varphi := \lambda \circ \varphi$.

Proposition 1.1.8. [**Vis98**, pg. 638] *Let $L \to X$ be an equivariant line bundle over a G-scheme, and let L^* be the associated $\mathbb{G}_m$-torsor. Then we have:*

$$CH^i_G(L^*) = CH^i_G(X)/(c_1^G(L)(CH^{i-1}_G(X))$$

Sketch of proof. Applying the localization exact sequence (prop. 1.1.1.(3)) for the zero section $X \hookrightarrow L$ we get:

$$CH_G(X) \longrightarrow CH_G(L) \longrightarrow CH_G(L^*) \longrightarrow 0$$

Homotopy invariance (proposition 1.1.1.(5)) tells us that $CH_G(L) \simeq CH_G(X)$. To complete the proof, we only have to observe that the induced morphism

$$CH_G(X) \longrightarrow CH_G(L) \simeq CH_G(X)$$

coincides by construction with the first Chern class of L. □

1.2. Equivariant Chow groups with coefficients

In this section, we fix a positive number p and an algebraically closed base field k_0 such that p is invertible in k_0. Every scheme is assumed to be of finite type over k_0 and equidimensional.

1.2.1. Main definitions and properties. We will collect together some basic definitions and useful properties of equivariant Chow groups with coefficients in $\mathrm{H}^\bullet$. Our interest in these groups is due to the following result:

Theorem 1.2.1. [**Pir18a**, th. 4.9] *If X is a smooth, quasi-projective scheme endowed with a linearized action of an algebraic group G, then we have*

$$A^0_G(X, \mathrm{H}^\bullet) \simeq \mathrm{Inv}^\bullet([X/G])$$

where $A^0_G(-, \mathrm{H}^\bullet)$ is the zeroth equivariant Chow group with coefficients in $\mathrm{H}^\bullet$, endowed with its natural structure of graded-commutative ring, and $\mathrm{Inv}^\bullet(-)$ is the graded-commutative ring of cohomological invariants.

First, let us sketch the construction of the standard Chow groups with coefficients in $\mathrm{H}^\bullet$. The paper of Rost [**Ros96**] is devoted to the foundation of the more general theory of Chow groups with coefficients in a cycle module M. Here we are only interested in the case $M = \mathrm{H}^\bullet$, though most of what we say is true for any cycle module. The proof that $\mathrm{H}^\bullet$ is a cycle module in the sense of [**Ros96**, def. 2.1] can be found in [**Ros96**, remarks 1.11 and 2.5].

Another nice introduction to Chow groups with coefficients in $\mathrm{H}^\bullet$ is [**Gui07**, sec. 2]: in particular, the equivariant case is discussed in [**Gui07**, sec. 2.3].

Remark 1.2.2. The hypothesis that the characteristic of the base field does not divide p is necessary for $\mathrm{H}^\bullet$ to be a cycle module.

Without this hypothesis, many things can go wrong: for instance, the proof that $\mathrm{H}^\bullet$ satisfies the axiom $D3$ (see [**Ros96**, remark 1.11]) does not work, as we do not have a norm residue homomorphism.

This is a consequence of the fact that the sequence of sheaves in the étale topology:

$$0 \longrightarrow \mu_p \longrightarrow \mathbb{G}_m \xrightarrow{\cdot p} \mathbb{G}_m \longrightarrow 0$$

is not exact if we remove the assumption on the characteristic of the base field (see [**Sta19**, Tag 03PM]).

A possible way to circumvent this issue could be to consider cohomology in the fppf or syntomic topology, but we have not checked all the details.

Let X be a scheme, d a positive integer and define:

$$C_i(X, \mathrm{H}^d) = \oplus_{x\in X^{(i)}} H^d_{\acute{e}t}(k(x), \mu_p^{\otimes d}))$$

where the sum is taken over all the points x of X having dimension equal to i. We define $C_i(X, \mathrm{H}^\bullet) := \oplus_{d\geq 0} C_i(X, \mathrm{H}^d)$. In this way $C_i(X, \mathrm{H}^\bullet)$ has a natural bigrading given by the cohomological degree d and the dimension i.

For every i ranging from 0 to the dimension of X and for every $d \geq 1$, there exists a differential:

$$\delta_i^d : C_i(X, \mathrm{H}^d) \longrightarrow C_{i+1}(X, \mathrm{H}^{d-1})$$

whose precise definition can be found in [**Ros96**, (3.2)]. We set $\delta_i^0 := 0$. By taking the direct sum over $d \geq 0$ of the δ_d^i, we can define:

$$\delta_i : C_i(X, \mathrm{H}^\bullet) \longrightarrow C_{i+1}(X, \mathrm{H}^\bullet)$$

The Chow groups with coefficients are then defined as:

$$A_i(X, \mathrm{H}^\bullet) := \ker(\delta_i)/\mathrm{im}(\delta_{i-1})$$

As we are assuming every scheme to be equidimensional, it makes sense to introduce also the codimensionally-graded Chow groups with coefficients $A^i(X, \mathrm{H}^\bullet)$: these are equal by definition to $A_{n-i}(X, \mathrm{H}^\bullet)$, where n is the dimension of X.

Chow groups with coefficients have two natural gradings, one given by codimension and the other given by the cohomological degree: an element α has codimension i and degree d if it is an element of $A^i(X, \mathrm{H}^d) := \ker(\delta^d_{n-i})/\mathrm{im}(\delta^d_{n-i-1})$.

The whole theory of Chow groups with coefficients has an equivariant counterpart, first introduced in [**Gui07**, sec. 2.3]. Let G be an algebraic group acting on a scheme X. Suppose moreover that X is quasi-projective with linearized G-action. Using the same ideas of [**EG98**], one can define the equivariant groups $A^i_G(X, \mathrm{H}^\bullet)$ as follows: take a representation V of G such that G acts freely on an open subscheme $U \subset V$ whose complement has codimension greater than $i + 1$. By [**EG98**, lemma 9] such a representation always exists. Then we define:

$$A^i_G(X, \mathrm{H}^\bullet) := A^i((X \times U)/G, \mathrm{H}^\bullet)$$

The content of [**EG98**, prop. 23] assures us that the quotient $(X \times U)/G$ exists in the category of schemes, and by the double filtration argument used in the proof of [**EG98**, prop. 1] we see that the definition above does not depend on the choice of U.

We list now some properties of equivariant Chow groups with coefficients that will be frequently used:

PROPOSITION 1.2.3. *Let X and Y be equidimensional, quasi-projective schemes of finite type over k_0, endowed with a linearized G-action. Then we have:*

(1) $CH^G_i(X) \otimes \mathbb{F}_p = A^G_i(X, \mathrm{H}^0)$.

(2) Proper pushforward*: every G-equivariant, proper morphism $f : X \to Y$ induces a homomorphism of groups*

$$f_* : A^G_i(X, \mathrm{H}^\bullet) \longrightarrow A^G_i(Y, \mathrm{H}^\bullet)$$

which preserves the cohomological degree.

(3) Flat pullback*: every G-equivariant, flat morphism $f : X \to Y$ of relative constant dimension induces a homomorphism of groups*

$$f^* : A^i_G(Y, \mathrm{H}^\bullet) \longrightarrow A^i_G(X, \mathrm{H}^\bullet)$$

which preserves the cohomological degree.

(4) Localization exact sequence*: given a closed, G-invariant subscheme $Z \overset{i}{\hookrightarrow} X$ whose open complement is $U \overset{j}{\hookrightarrow} X$, there exists a long exact sequence*

$$\cdots \to A_i^G(X, \mathrm{H}^\bullet) \xrightarrow{j^*} A_i^G(U, \mathrm{H}^\bullet) \xrightarrow{\partial} A_{i-1}^G(Z, \mathrm{H}^\bullet) \xrightarrow{i_*} A_{i-1}^G(X, \mathrm{H}^\bullet) \to \cdots$$

The boundary homomorphism ∂ has cohomological degree -1, whereas the other homomorphisms have cohomological degree zero.

(5) Compatibility*: given a cartesian square of G-schemes*

$$\begin{array}{ccc} Y & \xrightarrow{i} & X \\ \downarrow & & \downarrow \\ Y' & \xrightarrow{i'} & X' \end{array}$$

where all the morphisms are G-equivariant closed embeddings, we get a commutative square

$$\begin{array}{ccc} A_k^G(Y' \setminus Y, \mathrm{H}^\bullet) & \xrightarrow{i''_*} & A_k^G(X' \setminus X, \mathrm{H}^\bullet) \\ \downarrow \partial & & \downarrow \partial \\ A_{k-1}^G(Y, \mathrm{H}^\bullet) & \xrightarrow{i_*} & A_{k-1}^G(X, \mathrm{H}^\bullet) \end{array}$$

where i'' is the restriction of i' to $Y' \setminus Y$.

(6) Homotopy invariance*: if $\pi : E \to X$ is a G-equivariant, finite rank vector bundle, then we have an isomorphism*

$$\pi^* : A_G^i(X, \mathrm{H}^\bullet) \simeq A_G^i(E, \mathrm{H}^\bullet)$$

which preserves the cohomological degrees.

(7) Projective bundle formula*: If $\mathbb{P}(E) \to X$ is the projectivization of a G-equivariant , finite rank vector bundle, then for $i < \mathrm{rk}(E)$ we have:*

$$A_G^i(\mathbb{P}(E), \mathrm{H}^\bullet) \simeq \oplus_{j=0}^i A_G^j(X, \mathrm{H}^\bullet)$$

The isomorphism above preserves the cohomological degrees.

(8) Ring structure*: if X is smooth, then $A_G(X, \mathrm{H}^\bullet)$ inherits the structure of a graded-commutative $\mathbb{F}_p$-algebra, where the graded-commutativity should be understood in the following sense: if α has codimension i and degree d, and β has codimension j and degree e, then $\alpha \cdot \beta = (-1)^{de} \beta \cdot \alpha$, and the product has codimension $i + j$ and degree $d + e$.*

Sketch of proof. All the properties above, which are formulated for equivariant Chow groups with coefficients, follow from the analogous properties for the non-equivariant ones, hence it is enough to prove (1)-(7) in this second setting.

To prove (1), observe that $\ker(\delta_i^0) = C_i(X, \mathrm{H}^0)$. We have

$$H^0_{\text{ét}}(k(x), \mathbb{F}_p) = \mathbb{F}_p$$

hence $C_i(X, \mathrm{H}^0) = Z_i(X) \otimes \mathbb{F}_p$, the group of i-dimensional cycles mod p.

Recall that $H^1_{\text{ét}}(k(y), \mu_p) = k(y)^*/(k(y)^*)^p$, so that the group $C_{i-1}(X, \mathrm{H}^1)$ can be seen as a quotient of the group of non-zero rational functions on $i-1$-dimensional subvarieties of X.

Unwinding the definition of the differential δ_{i-1}^1 (see [**Ros96**, (3.2)]), we deduce that $\delta_{i-1}^1(\overline{\varphi}) = \mathrm{div}(\overline{\varphi})$, where $\overline{\varphi}$ is the equivalence class in $k(y)^*/(k(y)^*)^p$ of a rational function φ defined over a subvariety $Y \subset X$ whose generic point is y , and $\mathrm{div}(\overline{\varphi})$ is the associated divisor, which is a well defined element of $Z_i(X) \otimes \mathbb{F}_p$. This implies that:

$$(Z_i(X)/\sim_{\mathrm{rat}}) \otimes \mathbb{F}_p \simeq \ker(\delta_i^0)/\mathrm{im}(\delta_{i-1}^1)$$

where $\sim_{\mathrm{rat}}$ is the rational equivalence relation, and it proves (1).

The proofs of (2)-(5) are the content of [**Ros96**, sec. 4], (6) is [**Ros96**, prop. 8.6], (7) is [**Pir17**, prop. 2.4] and (8) is [**Ros96**, th. 14.6]. □

There is also a well defined theory of equivariant Chern classes for equivariant Chow groups with coefficients in $\mathrm{H}^\bullet$, which resembles the theory of Chern classes for the usual Chow groups. The foundation of this theory can be found in [**Pir17**, sec. 2.1],

Let X be a quasi-projective scheme endowed with a linearized G-action and let $E \to X$ be an equivariant vector bundle of rank r. Then for every i the vector bundle E induces a degree-preserving homomorphism of equivariant Chow groups with coefficients:

$$c_i^G(E) : A_*^G(X, \mathrm{H}^\bullet) \longrightarrow A_{*-i}^G(X, \mathrm{H}^\bullet)$$

which is the i^{th} *equivariant Chern class* of E.

Using the language of Chern classes, the projective bundle formula 1.2.3.(7) can be restated (see [**Pir17**, prop. 2.4]) by saying that, if $\pi : \mathbb{P}(E) \to X$ is the projectivization of an equivariant vector bundle, then we have:

$$A_G^i(\mathbb{P}(E), \mathrm{H}^\bullet) \simeq \oplus_{j=0}^i c_1^G(\mathcal{O}_{\mathbb{P}(E)}(1))^{i-j}(\pi^* A_G^j(X, \mathrm{H}^\bullet))$$

If X is smooth, there exist cycles γ_i in $A_G^i(X, \mathrm{H}^0)$ such that the equivariant Chern classes are equal to the multiplication by γ_i (see [**Pir17**, cor. 2.6]). Therefore, in this case it makes perfect sense to think of equivariant Chern classes as elements in $A_G(X, \mathrm{H}^\bullet)$.

In particular, for $\pi : \mathbb{P}(E) \to X$ the projectivization of an equivariant vector bundle of rank r over a smooth scheme, we can consider the element $h = c_1^G(\mathcal{O}(1))$ in $A_G^1(\mathbb{P}(E), \mathrm{H}^0)$ and proposition 1.2.3.(7) can be reformulated as follows:

$$A_G(\mathbb{P}(E), \mathrm{H}^\bullet) \simeq A_G(X, \mathrm{H}^\bullet)[h]/(h^r + c_1^G(E)h^{r-1} + c_2^G(E)h^{r-2} + \cdots + c_r^G(E))$$

This isomorphism preserves the cohomological degree.

1.2.2. Examples. We collect here some computations of equivariant Chow groups with coefficients which will be useful for our purposes.

PROPOSITION 1.2.4. [**Pir17**, prop. 2.11][**Pir18b**, prop. 3.1, prop. 3.2] *We have:*

(1) Let E be the standard representation of GL_3, regarded as an equivariant vector bundle of rank 3 over $\mathrm{Spec}(k_0)$. Then

$$A_{\mathrm{GL}_3}(\mathrm{Spec}(k_0), \mathrm{H}^\bullet) = \mathbb{F}_p[c_1, c_2, c_3]$$

where $c_i := c_i^{\mathrm{GL}_3}(E)$. In particular, $A_{\mathrm{GL}_3}(\mathrm{Spec}(k_0), \mathrm{H}^\bullet)$ is concentrated in degree 0.

(2) Fix $p = 2$ and let E be the standard representation of GL_3, on which PGL_2 acts via the adjoint representation. Set $c_i := c_i^{\mathrm{PGL}_2}(E)$. Then

$$A_{\mathrm{PGL}_2}(\mathrm{Spec}(k_0), \mathrm{H}^\bullet) = \mathbb{F}_2[c_2, c_3] \oplus \mathbb{F}_2[c_2, c_3] \cdot w_2 \oplus \mathbb{F}_2[c_2, c_3] \cdot \tau_1$$

as an $\mathbb{F}_2$-vector space. The generator w_2 has codimension 0 and degree 2, the generator τ_1 has codimension 1 and degree 1 and $c_1 = 0$.

(3) Fix $p = 2$. Then $A^0_{\mathrm{PGL}_2}(\mathbb{P}^1, \mathrm{H}^\bullet) \simeq \mathbb{F}_2$.

REMARK 1.2.5. Proposition 1.2.3.(7) and 1.2.4.(3) are not in contradiction, as $\mathbb{P}^1$ is not the projectivization of any PGL_2-representation of rank 2.

Here it is an analogue of proposition 1.1.8.

PROPOSITION 1.2.6. *Let X be an equidimensonal, quasi-projective scheme over k_0, endowed with a linearized G-action, and let $L \to X$ be an equivariant line bundle, with associated $\mathbb{G}_m$-torsor L^*. Then we have:*

$$A_G(L^*, \mathrm{H}^\bullet) \simeq \big(A_G(X, \mathrm{H}^\bullet)/\mathrm{im}(c_1^G(L))\big) \oplus \ker(c_1^G(L))[1]$$

as $\mathbb{F}_p$-vector spaces.

PROOF. Here $A^n_G(-)$ stands for $A^n_G(-, \mathrm{H}^\bullet)$. Using proposition 1.2.3.(4) applied to the zero section $i : X \hookrightarrow L$, we get:

$$\cdots \to A_G^{n-1}(X) \xrightarrow{i_*} A_G^n(L) \to A_G^n(L^*) \to A_G^n(X) \xrightarrow{i_*} A_G^{n+1}(X) \to \cdots$$

From proposition 1.2.3.(6), we know that there exists an isomorphism $(\pi^*)^{-1}$: $A_G(L) \simeq A_G(X)$. By definition of the Chern classes (see in particular [Pir17, def. 2.2]) we have $c_1^G(L) = (\pi^*)^{-1} \circ i_*$, so that the exact sequence above can be rewritten as:

$$\cdots \to A_G^{n-1}(X) \xrightarrow{c_1^G(L)} A_G^n(X) \to A_G^n(L^*) \to A_G^n(X) \xrightarrow{c_1^G(L)} A_G^{n+1}(X) \to \cdots$$

This implies the formula for $A_G(L^*)$. □

1.3. GL_3-counterpart of PGL_2-schemes

In this section we introduce the notion of GL_3-counterpart of a PGL_2-scheme (definition 1.3.5). This is a particular case of a fairly simple but interesting phenomenon: given an injective morphism of algebraic groups $H \hookrightarrow G$, there is a functor F from the category of H-schemes to the category of G-schemes such that the quotient stacks $[X/H]$ and $[F(X)/G]$ are isomorphic. What we do is to apply this fact to the injective morphism $\mathrm{PGL}_2 \hookrightarrow \mathrm{GL}_3$ and to give some explicit constructions.

We also state a simple result (proposition 1.3.8) which enables us to replace morphisms between PGL_2-equivariant Chow groups of PGL_2-schemes with morphisms between GL_3-equivariant Chow groups of their GL_3-counterparts.

We will denote $\mathbb{A}(n, d)$ the affine space of homogeneous forms in $n+1$ variables of degree d, and $\mathbb{P}(n, d)$ will stand for the projectivization of $\mathbb{P}(n, d)$.

We will denote $\mathbb{A}(2, 2)_r$ the locally closed subscheme of $\mathbb{A}(2, 2)$ of forms of rank r, and $\mathbb{A}(2, 2)_{[a,b]}$ will stand for the subscheme of forms of rank r with $a \geq r \geq b$. In particular, the scheme $\mathbb{A}(2, 2)_3$ parametrizes smooth forms.

Given a scheme X over $\mathbb{A}(2, 2)_{[3,1]}$, we will denote the pullback of X to $\mathbb{A}(2, 2)_r$ (resp. $\mathbb{A}(2, 2)_{[a,b]}$) as X_r (resp. $X_{[a,b]}$).

1.3.1. Basic definitions and some properties. Let $f : X \to X'$ be a PGL_2-equivariant morphism between two schemes of finite type over a base field k_0. We can form the quotient stacks $[X/\mathrm{PGL}_2]$ and $[X'/\mathrm{PGL}_2]$.

The morphism f induces a representable morphism between the two quotient stacks. Observe that both $[X/\mathrm{PGL}_2]$ and $[X'/\mathrm{PGL}_2]$ are stacks over $\mathcal{B}\mathrm{PGL}_2$, and that both structure morphisms are representable.

We have:

PROPOSITION 1.3.1. *Let $\mathbb{A}(2, 2)_3$ denote the scheme parametrising smooth quadratic ternary forms, endowed with the GL_3-action defined by the formula:*

$$A \cdot q(x, y, z) := \det(A) q(A^{-1}(x, y, z))$$

Then $[\mathbb{A}(2, 2)_3/\mathrm{GL}_3] \simeq \mathcal{B}\mathrm{PGL}_2$.

We start with two technical lemmas:

LEMMA 1.3.2. *Let L be an invertible sheaf on a scheme $\pi : X \to S$ such that π_*L is a globally generated locally free sheaf of rank $n+1$. Then giving an isomorphism $\pi_*L \simeq \mathcal{O}_S^{\oplus n+1}$ induces a morphism $f : X \to \mathbb{P}^n_S$ and an isomorphism $f^*\mathcal{O}(1) \simeq L$, and vice versa.*

PROOF. One implication is obvious. For the other one, suppose to have a morphism $f : X \to \mathbb{P}^n_S$ and an isomorphism $f^*\mathcal{O}(1) \simeq L$. For functoriality of the pushforward we obtain an isomorphism $\pi_*L \simeq \pi_*f^*\mathcal{O}(1)$. We want to produce an isomorphism of the sheaf on the right with $\mathcal{O}_S^{\oplus n+1}$.

Let $\mathrm{pr}_2 : \mathbb{P}^n \times S \to S$ be the canonical projection. Then we have a surjective morphism of locally free sheaves $\mathrm{pr}_2^*\mathrm{pr}_{2*}\mathcal{O}(1) \to \mathcal{O}(1)$ that composed with f^* and π_* induces a surjective morphism of locally free sheaves $\alpha : \pi_*\pi^*\mathrm{pr}_{2*}\mathcal{O}(1) \to \pi_*f^*\mathcal{O}(1)$. Observe that we have a canonical chain of isomorphisms:

$$\mathcal{O}_S^{\oplus n+1} \simeq \pi_*\pi^*\mathcal{O}_S^{\oplus n+1} \simeq \pi_*\pi^*\mathrm{pr}_{2*}\mathcal{O}(1)$$

Thus, composing these isomorphisms with α we get a surjective morphism of locally free sheaves $\mathcal{O}_S^{\oplus n+1} \to \pi_*f^*\mathcal{O}(1)$. The hypotheses imply that this morphism is actually an isomorphism, and we are done. □

LEMMA 1.3.3. *Let $\pi : C \to S$ be a smooth and proper morphism whose fibres are curves of genus zero. Then the sheaf $\pi_*\omega_{C/S}^{-1}$ is a locally free sheaf on S of rank 3 which satisfies the base change property. The morphism $\pi^*\pi_*\omega_{C/S}^{-1} \to \omega_{C/S}^{-1}$ is surjective and induces a closed immersion $C \hookrightarrow \mathbb{P}(\pi_*\omega_{C/S}^{-1})$.*

PROOF. Follows from the base change theorem in cohomology ([Har77, Th. III.12.11]) applied to $\pi_*\omega_{C/S}^{-1}$. □

Consider the prestack in groupoids over the category (Sch/k_0) of schemes

$$\mathcal{E}(S) = \left\{ (\pi : C \to S, \alpha : \pi_*\omega_{C/S}^{-1} \simeq \mathcal{O}_S^{\oplus 3}) \right\}$$

where $C \to S$ is a family of smooth rational curves, and the morphisms

$$(C \to S, \alpha : \pi_*\omega_{C/S}^{-1} \simeq \mathcal{O}_S^{\oplus 3}) \to (C' \to S', \alpha' : \pi'_*\omega_{C'/S'}^{-1} \simeq \mathcal{O}_{S'}^{\oplus 3})$$

are given by triples $(\varphi : S' \to S, \psi : C' \simeq \varphi^*C, \phi : \pi'_*\omega_{C'/S'}^{-1} \simeq \varphi^*\pi_*\omega_{C/S}^{-1})$, where ϕ must commute with α and α'. It can be easily checked that this prestack is equivalent to a sheaf.

Observe that there is a free and transitive action of GL_3 on $\mathcal{E}$, which turns $\mathcal{E}$ into a GL_3-torsor sheaf over $\mathcal{B}\mathrm{PGL}_2$. Consider also the auxiliary prestack

$$\mathcal{E}'(S) = \left\{ ((\mathbf{D}), \beta : i^*\mathcal{O}(1) \simeq \omega_{C/S}^{-1}) \right\}$$

where $(\mathbf{D})$ is a commutative diagram of the form

$$\begin{array}{ccc} C & \xrightarrow{\ i\ } & \mathbb{P}^2_S \\ \downarrow & \swarrow & \\ S & & \end{array}$$

with $C \to S$ a family of smooth rational curves, and i a closed immersion. Recall that $\mathbb{A}(2,2)_3$ is the scheme parametrising quadratic ternary forms of rank 3.

LEMMA 1.3.4. *There are isomorphisms $\mathcal{E} \simeq \mathcal{E}' \simeq \mathbb{A}(2,2)_3$.*

PROOF. The first isomorphism follows from lemma 1.3.2: given a pair $(\pi : C \to S, \alpha : \pi_*\omega_{C/S}^{-1} \simeq \mathcal{O}_S^{\oplus 3})$, the trivialization α determines a closed embedding

$i : C \hookrightarrow \mathbb{P}^2_S$ and an isomorphism $\beta : i^*\mathcal{O}(1) \simeq \omega_{C/S}^{-1}$, hence an object of $\mathcal{E}'(S)$, and viceversa.

Suppose to have a commutative triangle

$$\begin{array}{ccc} C & \overset{i}{\hookrightarrow} & \mathbb{P}^2_S \\ \downarrow & \swarrow & \\ S & & \end{array}$$

and an isomorphism $\varphi : i^*\mathcal{O}(1) \simeq \omega_{C/S}^{-1}$. This one can be seen as a non-zero section of $H^0(C, i^*\mathcal{O}(1) \otimes \omega_{C/S})$. We have the following chain of isomorphisms:

$$\begin{aligned} H^0(C, i^*\mathcal{O}(1) \otimes \omega_{C/S}) &\simeq H^0(\mathbb{P}^2_S, i_*(i^*\mathcal{O}(1) \otimes \omega_{C/S})) \\ &\simeq H^0(\mathbb{P}^2_S, i_*(i^*(\mathcal{O}(1) \otimes \omega_{\mathbb{P}^2_S} \otimes \mathcal{I}^{-1})) \\ &\simeq H^0(\mathbb{P}^2_S, \mathcal{O}(-2) \otimes \mathcal{I}^{-1} \otimes i_*\mathcal{O}_C) \end{aligned}$$

where $\mathcal{I}$ denotes the ideal sheaf of $i(C) \subset \mathbb{P}^2_S$ and in the last line we used the projection formula and the canonical isomorphism $\omega_{\mathbb{P}^2} \simeq \mathcal{O}(-3)$. If $L := \mathcal{O}(-2) \otimes \mathcal{I}^{-1}$, then by twisting the exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{\mathbb{P}^2_S} \to i_*\mathcal{O}_C \to 0$$

by L and by taking the associated long exact sequence in cohomology, we easily deduce the isomorphism

$$H^0(\mathbb{P}^2_S, L \otimes i_*\mathcal{O}_C) \simeq H^0(\mathbb{P}^2_S, L)$$

Now observe that a non-zero global section of L induces an isomorphism $\mathcal{I} \simeq \mathcal{O}(-2)$, and vice versa.

Thus, by dualizing the injective morphism of sheaves $\mathcal{I} \hookrightarrow \mathcal{O}_{\mathbb{P}^2_S}$ and by applying the isomorphism above, we obtain a morphism $\mathcal{O}_{\mathbb{P}^2_S} \to \mathcal{O}(2)$, which is equivalent to choosing a global section q of $\mathcal{O}(2)$, that will be smooth because of the hypotheses on C.

It is easy to check that the induced morphism

$$\mathcal{E}' \longrightarrow \mathbb{A}(2,2)_3, \quad ((\boldsymbol{D}), \varphi) \longmapsto q$$

is an isomorphism, whose inverse is given by sending q to the object $((\boldsymbol{D}), \varphi)$, where $(\boldsymbol{D})$ is the commutative triangle

$$\begin{array}{ccc} Q & \overset{i}{\hookrightarrow} & \mathbb{P}^2_S \\ \downarrow & \swarrow & \\ S & & \end{array}$$

and the isomorphism $\varphi : i^*\mathcal{O}(1) \simeq \omega_{Q/S}^{-1}$ is induced by

$$\mathcal{I}_Q \simeq \omega_{\mathbb{P}^2_S}(1) \simeq \mathcal{O}(-2)$$

□

Proof of prop. 1.3.1. From lemma 1.3.4 we know that $\mathbb{A}(2,2)_3 \simeq \mathcal{E}$, and $\mathcal{E}$ is a GL_3-torsor over $\mathcal{B}\mathrm{PGL}_2$ by construction. We only have to check that the action of GL_3 on $\mathbb{A}(2,2)_3$ is the correct one, but this immediately follows from the isomorphism $\mathcal{I} \simeq \omega_{\mathbb{P}^2_S}(1)$ seen in the proof of lemma 1.3.4. □

In other terms, there is a morphism $\mathbb{A}(2,2)_3 \to \mathcal{B}\mathrm{PGL}_2$ which is a GL_3-torsor. We can pull back along this torsor the map $[X/\mathrm{PGL}_2] \to [X'/\mathrm{PGL}_2]$: what we get is a GL_3-equivariant morphism $g : Y \to Y'$ between GL_3-schemes.

DEFINITION 1.3.5. Let X and X' be two schemes of finite type over $\mathrm{Spec}(k_0)$ endowed with a PGL_2-action, and let $f : X \to X'$ be a PGL_2-equivariant morphism. Then the GL_3-equivariant morphism $g : Y \to Y'$ obtained with the construction above is the GL_3-*counterpart of* f.

The GL_3-scheme Y (resp. Y') is the GL_3-*counterpart of* X (resp. X').

The construction of GL_3-counterparts is functorial. Observe also that if f is proper (resp. flat) then g will also be proper (resp. flat).

There is another way to think of GL_3-counterparts. The inclusion $i : \mathrm{PGL}_2 \hookrightarrow \mathrm{GL}_3$ induces a representable morphism of classifying stacks $\mathcal{B}\mathrm{PGL}_2 \to \mathcal{B}\mathrm{GL}_3$: this morphism sends a PGL_2-torsor $P \to S$ to the associated GL_3-torsor $P \times^{\mathrm{PGL}_2} \mathrm{GL}_3 \to S$, where $P \times^{\mathrm{PGL}_2} \mathrm{GL}_3 := (P \times \mathrm{GL}_3)/\mathrm{PGL}_2$ and the (right) action on the product is defined by the formula:

$$(x, g) \cdot h := (h^{-1}x, gi(h))$$

In general, given an algebraic group G, the functor $X \mapsto [X/G]$ induces an equivalence between the category of G-equivariant schemes over k_0 and the category of algebraic stacks over $\mathcal{B}G$ with representable structure morphism: therefore, given a PGL_2-scheme X, we get a representable morphism $[X/\mathrm{PGL}_2] \to \mathcal{B}\mathrm{PGL}_2 \to \mathcal{B}\mathrm{GL}_3$, hence a GL_3-equivariant scheme over k_0.

Let X be a scheme of finite type over k_0, endowed with an action of $\mathrm{PGL}_2 \times \mathbb{G}_m$. We can form the cartesian diagram

$$\begin{array}{ccc} X & \longrightarrow & [X/\mathrm{PGL}_2 \times \mathbb{G}_m] \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k_0) & \longrightarrow & \mathcal{B}(\mathrm{PGL}_2 \times \mathbb{G}_m) \end{array}$$

Regarding $\mathbb{A}(2,2)_3$ as a $\mathrm{GL}_3 \times \mathbb{G}_m$-scheme, where the action of $\mathbb{G}_m$ is the trivial one, we have that $\mathbb{A}(2,2)_3 \to \mathcal{B}(\mathrm{PGL}_2 \times \mathbb{G}_m)$ is a $\mathrm{GL}_3 \times \mathbb{G}_m$-torsor. We can take the pullback of $[X/\mathrm{PGL}_2 \times \mathbb{G}_m]$ along this last morphism, obtaining in this way a scheme Y.

DEFINITION 1.3.6. Let X be a scheme of finite type over $\mathrm{Spec}(k_0)$ endowed with a $\mathrm{PGL}_2 \times \mathbb{G}_m$-action. Then the scheme Y obtained with the construction above is the $\mathrm{GL}_3 \times \mathbb{G}_m$-*counterpart of* X. In particular, we have that $[Y/\mathrm{GL}_3 \times \mathbb{G}_m] \simeq [X/\mathrm{PGL}_2 \times \mathbb{G}_m]$.

DEFINITION 1.3.7. Let X and X' be two schemes of finite type over $\mathrm{Spec}(k_0)$ endowed with a $\mathrm{PGL}_2 \times \mathbb{G}_m$-action, and let $f : X \to X'$ be a $\mathrm{PGL}_2 \times \mathbb{G}_m$-equivariant morphism. The pullback $g : Y \to Y'$ of the induced morphism $[X/\mathrm{PGL}_2 \times \mathbb{G}_m] \to [X'/\mathrm{PGL}_2 \times \mathbb{G}_m]$ along the map $\mathbb{A}(2,2)_3 \to \mathcal{B}(\mathrm{PGL}_2 \times \mathbb{G}_m)$ is called the $\mathrm{GL}_3 \times \mathbb{G}_m$-*counterpart of* f.

The following proposition is immediate to prove:

PROPOSITION 1.3.8. *Let* $f : X \to X'$ *be a* PGL_2*-equivariant proper morphism between two* PGL_2*-schemes, and let* $g : Y \to Y'$ *be its* GL_3*-counterpart. Then we have:*

(1) a commutative diagram of equivariant Chow groups of the form

$$\begin{array}{ccc} CH_i^{\mathrm{PGL}_2}(X) & \xrightarrow{f_*} & CH_i^{\mathrm{PGL}_2}(X') \\ \downarrow & & \downarrow \\ CH_i^{\mathrm{GL}_3}(Y) & \xrightarrow{g_*} & CH_i^{\mathrm{GL}_3}(Y') \end{array}$$

where the vertical arrows are isomorphisms.

(2) *a commutative diagram of equivariant Chow groups with coefficients of the form*

$$\begin{array}{ccc} A_i^{\mathrm{PGL}_2}(X, \mathrm{H}^\bullet) & \xrightarrow{f_*} & A_i^{\mathrm{PGL}_2}(X', \mathrm{H}^\bullet) \\ \downarrow & & \downarrow \\ A_i^{\mathrm{GL}_3}(Y, \mathrm{H}^\bullet) & \xrightarrow{g_*} & A_i^{\mathrm{GL}_3}(Y', \mathrm{H}^\bullet) \end{array}$$

where the vertical arrows are isomorphisms.

(3) *The same results hold for* $\mathrm{PGL}_2 \times \mathbb{G}_m$*-equivariant morphisms and their* $\mathrm{GL}_3 \times \mathbb{G}_m$*-counterpart.*

1.3.2. Applications. We apply now the machinery above to a particular case. Let $\mathbb{P}(1, 2n)$ be the projective space of binary forms of degree $2n$. This scheme has a natural action of PGL_2 given by $A \cdot f(x, y) = f(A^{-1}(x, y))$. We want to find its GL_3-counterpart.

Consider the scheme $\mathbb{A}(2, 2)_{[3,1]} = \mathbb{A}(2, 2) \setminus \{0\}$ of non-zero quadratic ternary forms. This is a $\mathbb{G}_m$-torsor over the projective space $\mathbb{P}(2, 2)$ of plane conics.

DEFINITION 1.3.9.

(1) We denote $Q \subset \mathbb{P}(2, 2) \times \mathbb{P}^2$ the universal conic over $\mathbb{P}^2$.
(2) We denote $\widehat{Q} \subset \mathbb{A}(2, 2)_{[3,1]} \times \mathbb{P}^2$ the pullback of the universal conic $Q \to \mathbb{P}(2, 2)$ along the $\mathbb{G}_m$-torsor $\mathbb{A}(2, 2)_{[3,1]} \to \mathbb{P}(2, 2)$.

Let pr_1 and pr_2 be respectively the projection on the first and on the second factor of $\mathbb{P}(2, 2) \times \mathbb{P}^2$. We have a short exact sequence of locally free sheaves:

$$0 \longrightarrow \mathcal{I}_Q \otimes \mathrm{pr}_2^*\mathcal{O}(n) \longrightarrow \mathrm{pr}_2^*\mathcal{O}(n) \longrightarrow \mathrm{pr}_2^*\mathcal{O}(n)|_Q \longrightarrow 0$$

where $\mathcal{I}_Q \simeq \mathrm{pr}_1^*\mathcal{O}(-1) \otimes \mathrm{pr}_2^*\mathcal{O}(-2)$ is the ideal sheaf of Q.

Pushing everything forward along pr_1, and applying the projection formula for sheaves (see [**Har77**, ex. III.8.3]) we get:

$$0 \to \mathrm{pr}_{1*}\mathrm{pr}_2^*\mathcal{O}(n-2) \otimes \mathcal{O}(-1) \to \mathrm{pr}_{1*}\mathrm{pr}_2^*\mathcal{O}(n) \to \mathrm{pr}_{1*}(\mathrm{pr}_2^*\mathcal{O}(n)|_Q) \to 0$$

where exactness on the right is due to the vanishing of $R^1\mathrm{pr}_{1*}\mathrm{pr}_2^*\mathcal{O}(n-2)$, which follows from the base change theorem in cohomology (see [**Har77**, th. III.12.11]) combined with the well known vanishing of $H^1(\mathbb{P}^2, \mathcal{O}(n-2))$.

Observe that the first two sheaves appearing in the exact sequence above are locally free. From this we deduce that also the one on the right is locally free.

DEFINITION 1.3.10.

(1) We define $\overline{V}_n$ as the vector bundle associated to the locally free sheaf
$$\overline{\mathcal{V}}_n := \mathrm{pr}_{1*}(\mathrm{pr}_2^*\mathcal{O}(n)|_Q)$$
Its projectivization is denoted $\mathbb{P}(\overline{V}_n)$.
(2) We define $V_n \to \mathbb{A}(2, 2)_{[3,1]}$ as the vector bundle obtained by pulling back $\overline{V}_n$ along the $\mathbb{G}_m$-torsor $\mathbb{A}(2, 2)_{[3,1]} \to \mathbb{P}(2, 2)$. Its projectivization is denoted $\mathbb{P}(V_n)$.

Equivalently, V_n is the vector bundle associated to the locally free sheaf

$$\mathcal{V}_n := \mathrm{pr}_{1*}(\mathrm{pr}_2^*\mathcal{O}(n)|_{\widehat{Q}})$$

Another way to think of the vector bundle V_n is as follows: consider the following injective morphism of (trivial) vector bundles over $\mathbb{A}(2, 2)_{[3,1]}$:

$$\mathbb{A}(2, 2)_{[3,1]} \times \mathbb{A}(2, n-2) \longrightarrow \mathbb{A}(2, 2)_{[3,1]} \times \mathbb{A}(2, n), \quad (q, f) \longmapsto (q, qf)$$

Then the vector bundle V_n is equal to the cokernel of the morphism above.

REMARK 1.3.11. The points of $\mathbb{P}(\overline{V}_n)$ can be thought as pairs $(q,[f])$, where q is the projective equivalence class of a non-zero quadratic ternary form (equivalently, a conic) and $[f]$ is the equivalence class of a non-zero ternary form of degree n, where $f \sim f'$ if and only if g divides $f - f'$ or f' is a non-zero scalar multiple of f. The same description holds for the points of $\mathbb{P}(V_n)$, with the exception that in this case we really look at the ternary quadratic form q and not at the associated conic.

PROPOSITION 1.3.12. *The* GL$_3$*-counterpart of* $\mathbb{P}(1,2n)$ *is* $\mathbb{P}(V_n)_3$*, endowed with the* GL$_3$*-action:*

$$A\cdot(q,[f]) := (\det(A)q(A^{-1}(x,y,z)), [f(A^{-1}(x,y,z))])$$

where q is a smooth ternary forms of degree 2 and f is a representative of the equivalence class $[f]$ of a ternary form of degree n.

PROOF. The quotient $[\mathbb{P}(1,2n)/\mathrm{PGL}_2]$ is the stack whose objects are pairs $(C \to S, D)$, where $C \to S$ is a family of smooth rational curves and $D \subset C$ is a divisor such that the induced morphism $D \to S$ is flat and finite of degree $2n$.

Therefore the GL$_3$-counterpart of $\mathbb{P}(1,2n)$ is the fibred product:

$$[\mathbb{P}(1,2n)/\mathrm{PGL}_2] \times_{\mathcal{B}\mathrm{PGL}_2} \mathbb{A}(2,2)_3$$

This can be described as the stack whose objects are triples $((\mathbf{D}),\beta,D)$, where $((\mathbf{D}),i)$ is an object of $\mathbb{A}(2,2)_3 \simeq \mathcal{E}'$ (see lemma 1.3.4) and $D \subset C$ is as before.

First we construct a morphism from this stack to $\mathbb{P}(V_n)_3$. Clearly, there is a natural morphism $p : [\mathbb{P}(1,2n)/\mathrm{PGL}_2] \times_{\mathcal{B}\mathrm{PGL}_2} \mathbb{A}(2,2)_3 \to \mathbb{A}(2,2)_3$.

Given an object $((\mathbf{D}),\beta,D)$, consider the injective morphism of sheaves $\mathcal{I}_D \hookrightarrow \mathcal{O}_C$. After twisting by $\mathcal{O}(n)$ and pushing forward along $\pi : C \to S$, we get the injective morphism:

$$\pi_*\mathcal{I}_D(n) \longrightarrow \pi_*\mathcal{O}_C(n)$$

By cohomology and base change (see [Har77, Th. III.12.11]) we see that the first sheaf is invertible, whilst the second one, using the projection formula for vector bundles, can easily be proved to be isomorphic to $p^*V_{n,3}$. The well known characterization of the morphisms to projective bundles yields a morphism of sets

$$[\mathbb{P}(1,2n)/\mathrm{PGL}_2] \times_{\mathcal{B}\mathrm{PGL}_2} \mathbb{A}(2,2)_3(S) \to \mathbb{P}(V_n)_3(S)$$

As everything is functorial, we get a morphism from the GL$_3$-counterpart of $\mathbb{P}(1,2n)$ to $\mathbb{P}(V_n)_3$.

To construct an inverse of this morphism, consider the pullback $\pi : \mathcal{Q} \to \mathbb{P}(\overline{V}_n)$ of the universal conic $Q \to \mathbb{P}(2,2)$. We have a cartesian diagram:

$$\begin{array}{ccc} \mathcal{Q} & \xrightarrow{\ \rho\ } & Q \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle p} \\ \mathbb{P}(\overline{V}_n) & \xrightarrow{\ r\ } & \mathbb{P}(2,2) \end{array}$$

The Euler exact sequence for $\mathbb{P}(\overline{V}_n)$, pulled back to $\mathcal{Q}$, together with the definition of the locally free sheaf $\overline{\mathcal{V}}_n$ associated to the vector bundle $\overline{V}_n$ (see definition 1.3.10), yields an injective morphism:

$$\pi^*\mathcal{O}_{\mathbb{P}(\overline{V}_n)}(-1) \longrightarrow \pi^*r^*p_*(\mathrm{pr}_2^*\mathcal{O}_{\mathbb{P}^2}(n)|_Q) = \rho^*p^*p_*(\mathrm{pr}_2^*\mathcal{O}_{\mathbb{P}^2}(n)|_Q)$$

Observe that there is a surjective morphism

$$\rho^*p^*p_*(\mathrm{pr}_2^*\mathcal{O}_{\mathbb{P}^2}(n)|_Q) \longrightarrow \rho^*\mathrm{pr}_2^*\mathcal{O}_{\mathbb{P}^2}(n)|_Q$$

Hence, after twisting by $\rho^*\mathrm{pr}_2^*\mathcal{O}_{\mathbb{P}^2}(-n)|_Q$ we can construct a morphism:

$$\pi^*\mathcal{O}_{\mathbb{P}(\overline{V}_n)}(-1)\otimes\rho^*\mathrm{pr}_2^*\mathcal{O}_{\mathbb{P}^2}(-n)|_Q \longrightarrow \mathcal{O}_{\mathcal{Q}}$$

It is immediate to verify that this defines a Cartier divisor $\overline{\mathcal{D}}\subset\mathcal{Q}$: the restricted morphism $\overline{\mathcal{D}}_3\to\mathbb{P}(\overline{V}_n)_3$ is finite of degree $2n$.

Therefore, if we denote $\widehat{\mathcal{Q}}_3$ the pullback of $\mathcal{Q}$ to $\mathbb{P}(V_n)_3$ and $\mathcal{D}_3$ the pullback of $\overline{\mathcal{D}}$, the pair $(\widehat{\mathcal{Q}}_3,\mathcal{D}_3)$ is an object of $[\mathbb{P}(1,2n)/\mathrm{PGL}_2]\times_{\mathcal{B}\mathrm{PGL}_2}\mathbb{A}(2,2)_3(\mathbb{P}(V_n)_3)$, hence induces a morphism $\mathbb{P}(V_n)_3\to[\mathbb{P}(1,2n)/\mathrm{PGL}_2]\times_{\mathcal{B}\mathrm{PGL}_2}\mathbb{A}(2,2)_3$. It is easy to check that this second morphism is the inverse of the first morphism that we defined, which concludes the proof of the proposition. □

Remark 1.3.13. Another way to think of the points of $\mathbb{P}(V_n)_3$ is as pairs (q,E), where E is an effective divisor of degree $2n$ of the smooth plane conic C defined by the equation $q=0$.

Let F and G be the plane curves respectively defined by the equations $f=0$ and $g=0$, and suppose that they do not contain C as an irreducible component.

By the classical Noether's theorem $AF+BG$, the intersection of F with C is equal to the intersection of G with C if and only if the difference $f-g$ is divisible by q, or in other terms if and only if $f-g$ is in the image of $\mathbb{A}(2,2)_3\times\mathbb{A}(2,n-2)\to\mathbb{A}(2,2)_3\times\mathbb{A}(2,n)$.

From this we deduce that the points of $\mathbb{P}(V_n)_3$ are in bijection with the pairs (q,E), where E is an effective divisor of degree $2n$.

This can be reformulated by saying that $\mathbb{P}(V_n)_3$ is the relative Hilbert scheme of points $\mathrm{Hilb}^{2n}_{\widehat{Q}_3/\mathbb{A}(2,2)_3}$, where $\widehat{Q}_3$ is the pullback of the universal conic Q over $\mathbb{P}(2,2)$: the isomorphism is given by the universal object $(\widehat{\mathcal{Q}_3},\mathcal{D}_3)$, where $\mathcal{D}_3$ is the universal divisor constructed in the proof of proposition 1.3.12.

Similarly, the scheme $\mathbb{P}(\overline{V}_n)_3$ is the relative Hilbert scheme $\mathrm{Hilb}^{2n}_{Q_3/\mathbb{P}(2,2)_3}$.

Observe also that the scheme $\mathbb{P}(1,2n)$ can be thought as the Hilbert scheme $\mathrm{Hilb}^{2n}_{\mathbb{P}^1_k}$ of $2n$ points on $\mathbb{P}^1$. Its quotient $[\mathbb{P}(1,2n)/\mathrm{PGL}_2]$ can be identified with the Hilbert stack $\mathrm{Hilb}^{2n}_{P/\mathcal{B}\mathrm{PGL}_2}$ of $2n$ points relative to the universal torsor P over the classifying stack $\mathcal{B}\mathrm{PGL}_2$.

So proposition 1.3.12 gives us the following presentation of this stack as a quotient stack:

$$\mathrm{Hilb}^{2n}_{P/\mathcal{B}\mathrm{PGL}_2}\simeq[\mathrm{Hilb}^{2n}_{\widehat{Q}_3/\mathbb{A}(2,2)_3}/\mathrm{GL}_3]$$

An interesting feature of this new presentation is that provides us with a natural way to partially extend the Hilbert stack $\mathrm{Hilb}^{2n}_{P/\mathcal{B}\mathrm{PGL}_2}$ over the stack of genus 0 and at most 1 nodal curves $\mathcal{M}_0^{\leq 1}$ by taking the quotient stack $[\mathbb{P}(V_n)_{[3,2]}/\mathrm{GL}_3]$, where $\mathbb{P}(V_n)_{[3,2]}$ denotes the restriction of $\mathbb{P}(V_n)$ over the open subscheme $\mathbb{A}(2,2)_{[3,2]}$ of quadratic forms of rank ≥ 2.

In the proof of proposition 1.3.12 we constructed a subscheme $\overline{\mathcal{D}}$ of $\mathbb{P}(\overline{V}_n)\times\mathbb{P}^2$, whose fibre over a point $(q,[f])$ of $\mathbb{P}(\overline{V}_n)$ is the subscheme of $\mathbb{P}^2$ defined by the homogeneous ideal $I=(q,f)$.

Definition 1.3.14.

(1) Let $\overline{\mathcal{D}}^{\mathrm{ram}}$ be the locus defined by the 0^{th} Fitting ideal of the sheaf of relative differentials $\Omega^1_{\overline{\mathcal{D}}/\mathbb{P}(\overline{V}_n)}$. Then we define $\overline{D}$ as the image of the projection of $\overline{\mathcal{D}}^{\mathrm{ram}}$ onto $\mathbb{P}(\overline{V}_n)$.
(2) We define *the fundamental divisor* D as the pullback of $\overline{D}\to\mathbb{P}(2,2)$ along the morphism $\mathbb{A}(2,2)_{[3,1]}\to\mathbb{P}(2,2)$.

REMARK 1.3.15. By [**Sta19**, Tag 0C3I], the subscheme $\overline{\mathcal{D}}^{\mathrm{ram}}$ is the ramification locus of $\overline{\mathcal{D}} \to \mathbb{P}(V_n)$.

Moreover, due to the fact that taking the scheme theoretic image commutes with flat base change ([**Sta19**, Tag 081I],) we deduce that D is the projection onto $\mathbb{A}(2,2)_{[3,1]}$ of the ramification locus of $\mathcal{D} \to \mathbb{A}(2,2)_{[3,1]}$.

We can think of the points of $\overline{D}$ as those pairs $(q,[f])$, where q is a ternary quadratic form defined up to scalar multiplication and $[f]$ is the equivalence class of a ternary form of degree n (see remark 1.3.11), such that closed subscheme of $\mathbb{P}^2$ determined by the homogeneous ideal $I=(q,f)$ is either singular or it contains a line.

Let D_3 be the restriction of D to $\mathbb{A}(2,2)_3$: then D_3 is a Cartier divisor whose points can be regarded as pairs $(q,[f])$ such that the subscheme in $\mathbb{P}^2$ defined by the ideal $I=(q,f)$ is singular.

Let $\Delta_{1,2n}$ be the divisor of singular forms in $\mathbb{P}(1,2n)$. Then proposition 1.3.12, together with proposition 1.3.8, immediately implies the following result:

COROLLARY 1.3.16. *The* GL$_3$*-counterpart of* $\Delta_{1,2n}$ *is* D_3*. We also have commutative diagrams*

$$\begin{array}{ccc} CH_i^{\mathrm{PGL}_2}(\Delta_{1,2n}) & \xrightarrow{i_*} & CH_i^{\mathrm{PGL}_2}(\mathbb{P}(1,2n)) \\ \downarrow & & \downarrow \\ CH_i^{\mathrm{GL}_3}(D_3) & \xrightarrow{i_*} & CH_i^{\mathrm{GL}_3}(\mathbb{P}(V_n)_3) \end{array}$$

$$\begin{array}{ccc} A_i^{\mathrm{PGL}_2}(\Delta_{1,2n}) & \xrightarrow{i_*} & A_i^{\mathrm{PGL}_2}(\mathbb{P}(1,2n)) \\ \downarrow & & \downarrow \\ A_i^{\mathrm{GL}_3}(D_3) & \xrightarrow{i_*} & A_i^{\mathrm{GL}_3}(\mathbb{P}(V_n)_3) \end{array}$$

where the vertical arrows are all isomorphisms.

Let $V_{n,3}$ be the restriction of the vector bundle $V_n \to \mathbb{A}(2,2)_{[3,1]}$ (see definition 1.3.10) over $\mathbb{A}(2,2)_3$.

PROPOSITION 1.3.17. *The* GL$_3$*-counterpart of* $\mathbb{A}(1,2n)$ *is* $V_{n,3}$*. Moreover, if we endow* $\mathbb{A}(1,2n)$ *with the* $\mathbb{G}_m$*-action given by scalar multiplication, then the* $\mathrm{GL}_3 \times \mathbb{G}_m$*-counterpart of* $\mathbb{A}(1,2n)$ *is* $V_{n,3}$*, where* $\mathbb{G}_m$ *acts as* $\lambda \cdot (q,f) := (q, \lambda f)$*.*

PROOF. Observe that $\mathbb{A}(1,2n) \setminus \{0\}$ is the total space of the $\mathbb{G}_m$-torsor associated to the tautological line bundle $\mathcal{O}_{\mathbb{P}(1,2n)}(-1)$ of $\mathbb{P}(1,2n)$. It is almost immediate to check, using proposition 1.3.12, that the GL$_3$-counterpart of the total space of this $\mathbb{G}_m$-torsor is equal to the total space of the $\mathbb{G}_m$-torsor associated to $\mathcal{O}_{\mathbb{P}(V_n)_3}(-1)$. From this the proposition easily follows. □

Let $\Delta'_{1,2n} \subset \mathbb{A}(1,2n)$ be the PGL$_2$-invariant, closed subscheme parametrising singular binary forms of degree $2n$. In other terms, the points of $\Delta'_{1,2n}$ correspond to global sections σ of $\mathcal{O}_{\mathbb{P}^1_k}(2n)$ with multiple roots. We want to find its GL$_3$-counterpart.

DEFINITION 1.3.18. We define the closed subscheme $D' \subset V_n$ as the closure of the pullback of the fundamental divisor $D \subset \mathbb{P}(V_n)$ (see definition 1.3.14) along the $\mathbb{G}_m$-torsor $V_n \setminus \sigma_0 \to \mathbb{P}(V_n)$, where σ_0 denotes the zero section of V_n.

The following result follows from corollary 1.3.16:

PROPOSITION 1.3.19. *Let* D'_3 *be the restriction of* D' *to* $\mathbb{P}(V_n)_3$*. Then we have:*

(1) D'_3 *is the* GL_3*-counterpart and the* $\mathrm{GL}_3 \times \mathbb{G}_m$*-counterpart of* $\Delta'_{1,2n}$.
(2) $V_{n,3} \backslash D'_3$ *is the* GL_3*-counterpart and the* $\mathrm{GL}_3 \times \mathbb{G}_m$*-counterpart of* $\mathbb{A}(1,2n) \backslash \Delta'_{1,2n}$.

CHAPTER 2

The Chow ring of the stack of hyperelliptic curves of odd genus

In this chapter we compute the integral Chow ring of the stack of hyperelliptic curves of odd genus g, when $g \geq 3$ is an odd number and the characteristic of the base field is equal to zero or $> 2g+2$. The main result is theorem 2.5.2.

2.1. A new presentation of $\mathcal{H}_g$ as a quotient stack

Fix a base field k_0 of characteristic $\neq 2$ and let $g \geq 3$ be an odd integer.

Recall that by a *family of smooth rational curves* over S we mean a proper and smooth scheme over a k_0-base scheme S such that every fiber is a curve of genus 0.

Then a *family of hyperelliptic curves* of genus g over S is defined as a pair $(C \to S, \iota)$ where $C \to S$ is a proper and smooth scheme over a base k_0-scheme S such that every fiber is a curve of genus g, and $\iota \in \mathrm{Aut}(C)$ is an involution such that $C/\langle \iota \rangle \to S$ is a family of smooth rational curves.

Let $\mathcal{H}_g$ be the moduli stack of smooth hyperelliptic curves of genus g, whose objects are families of hyperelliptic curves of genus g as defined above, and the morphisms are the isomorphisms over S (the condition of commuting with the involutions is automatically satisfied). The goal of this section is to give a presentation of this stack as a quotient stack $[U'/\mathrm{GL}_3 \times \mathbb{G}_m]$, where U' is an certain scheme that will be defined later. This is done in theorem 2.1.1.

2.1.1. Properties of hyperelliptic curves. We briefly recall some basic facts about hyperelliptic curves (for an extensive treament see [**KL79**]). Let $C \to S$ be a family of hyperelliptic curves of genus g. By definition there exists a global involution ι which induces the hyperelliptic involution on every geometric fiber.

There exists also a canonical, finite, surjective S-morphism $f : C \to C'$ of degree 2 that on each geometric fiber corresponds to taking the quotient w.r.t. the hyperelliptic involution. The scheme $C' \to S$ is a family of smooth rational curves. The morphism f can also be described as the canonical morphism $f : C \to \mathbb{P}(\pi_*\omega_{C/S})$ whose image is C'.

Families of hyperelliptic curves have a canonical subscheme $W_{C/S}$, called the *Weierstrass subscheme*, that is the ramification divisor of f endowed with the scheme structure given by the zeroth Fitting ideal of $\Omega^1_{C/C'}$. It is finite and étale over S of degree $2g+2$, and its associated line bundle, when seen as an effective Cartier divisor, is the dualizing sheaf ω_f relative to the finite morphism f. Clearly, f induces an isomorphism between $W_{C/S}$ and the branch divisor D on C'.

2.1.2. The main result. Let $\mathbb{A}(1, 2g+2) \setminus \Delta'_{1,2g+2}$ be the scheme parametrising smooth binary forms of degree $2g+2$. There is an action of $\mathrm{PGL}_2 \times \mathbb{G}_m$ over this scheme defined as follows:

$$(A, \lambda) \cdot (f(x, y)) = \lambda^{-2} \det(A)^{g+1} f(A^{-1}(x, y))$$

Observe that the action above is well defined, though the determinant of an element of PGL_2 is not. In [**AV04**, cor. 4.7] the authors proved that $\mathcal{H}_g$ is isomorphic to the quotient stack

$$[\mathbb{A}(1,2g+2) \setminus \Delta'_{1,2g+2}/\mathrm{PGL}_2 \times \mathbb{G}_m]$$

Therefore, a new presentation of $\mathcal{H}_g$ as a quotient stack with respect to the action of $\mathrm{GL}_3 \times \mathbb{G}_m$ can be obtained by finding a $\mathrm{GL}_3 \times \mathbb{G}_m$-counterpart of the $\mathrm{PGL}_2 \times \mathbb{G}_m$-scheme $\mathbb{A}(1,2g+2) \setminus \Delta'_{1,2g+2}$.

THEOREM 2.1.1. *Let U' be the complement in $(V_{g+1,3} \setminus \sigma_0)$ of D'_3, where $V_{g+1,3}$ is the vector bundle over $\mathbb{A}(2,2)_3$ introduced in definition 1.3.10, σ_0 is the zero section and D'_3 is as in definition 1.3.18. Then we have an isomorphism:*

$$\mathcal{H}_g \simeq [U'/\mathrm{GL}_3 \times \mathbb{G}_m]$$

where the action on U' is given by the formula

$$(A,\lambda)\cdot(q,f) = (\det(A)q(A^{-1}(x,y,z)), \lambda^{-2} f(A^{-1}(x,y,z)))$$

PROOF. Proposition 1.3.19.(2) tells us that a $\mathrm{GL}_3 \times \mathbb{G}_m$-counterpart of $\mathbb{A}(1,2g+2) \setminus \Delta'_{1,2g+2}$, with $\mathbb{G}_m$ acting by simple multiplication, is $V_{g+1,3} \setminus D'_3$, where the action of $\mathrm{GL}_3 \times \mathbb{G}_m$ is:

$$(A,\lambda)\cdot(q,f) := (\det(A)q(A^{-1}(x,y,z), \lambda f(A^{-1}(x,y,z))$$

It is easy to see that the $\mathrm{GL}_3 \times \mathbb{G}_m$-counterpart of $\mathbb{A}(1,2g+2) \setminus \Delta'_{1,2g+2}$ with $\mathbb{G}_m$ acting by λ^{-2} is $V_{g+1,3} \setminus D'_3$, with $\mathbb{G}_m$ acting by multiplication for λ^{-2}. □

The theorem above can be rephrased by saying that $V_{g+1,3} \setminus D'_3$ is a $\mathrm{GL}_3 \times \mathbb{G}_m$-torsor over $\mathcal{H}_g$. It is well known that to every $\mathrm{GL}_3 \times \mathbb{G}_m$-torsor over a base X one can associate a rank 4 vector bundle of the form $\mathcal{E} \oplus \mathcal{L}$, where $\mathcal{E}$ is a rank 3 vector bundle and $\mathcal{L}$ is a line bundle, such that the total space of the original torsor will be equal to $\mathrm{Isom}(\mathcal{E}, \mathcal{O}^{\oplus 3}) \times_X \mathrm{Isom}(\mathcal{L}, \mathcal{O})$. We want to find the vector bundle over $\mathcal{H}_g$ associated to $V_{g+1,3} \setminus D'_3$.

Observe that $V_{g+1,3} \setminus D'_3$, seen as a stack in sets, has as objects the triples (S, q, f) where:

- S is a scheme.
- q is a global section of $\mathcal{O}_{\mathbb{P}^2_S}(2)$ whose zero locus $Q \subset \mathbb{P}^2_S$ is smooth over S.
- f is a global section of $\mathcal{O}_Q(g+1)$ over Q.

and $\mathrm{GL}_3 \times \mathbb{G}_m$ acts as described in theorem 2.1.1. This stack is equivalent to the stack $\mathcal{P}$ whose objects are

$$((\boldsymbol{D}), \varphi, L, \sigma, \alpha)$$

where:

(1) $(\boldsymbol{D})$ is a commutative diagram of the form

$$\begin{array}{ccc} C' & \xrightarrow{\ i\ } & \mathbb{P}^2_S \\ {\scriptstyle\pi}\downarrow & \swarrow & \\ S & & \end{array}$$

with $C' \to S$ a family of smooth rational curves and i a closed immersion.

(2) $\varphi : i^*\mathcal{O}(1) \simeq T_{C'/S}$.

(3) L is a line bundle over C' of degree $-(g+1)/2$.

(4) σ is a global section of $L^{-\otimes 2}$.

(5) $\alpha : \pi_*(L^{-1} \otimes T_{C'/S}^{-\otimes(g+1)/2}) \simeq \mathcal{O}_S$.

The elements (1) and (2) above induce by lemma 1.3.2 an isomorphism

$$\beta : \pi_* T_{C'/S} \simeq \mathcal{O}_S^{\oplus 3}$$

and vice versa. Therefore, it is easy to prove that the stack $\mathcal{P}$ is equivalent to the stack $\mathcal{P}'$ whose objects are

$$(\pi : C' \to S, L, \sigma, \alpha, \beta)$$

where:

- $\pi : C' \to S$ is a family of smooth rational curves.
- L is a line bundle of degree $-(g+1)/2$ over C'.
- σ is a global section of $L^{-\otimes 2}$.
- $\alpha : \pi_*(L^{-1} \otimes T_{C'/S}^{-\otimes(g+1)/2}) \simeq \mathcal{O}_S$.
- $\beta : \pi_* T_{C'/S} \simeq \mathcal{O}_S^{\oplus 3}$.

Let $\mathcal{H}_g^{\sim}$ be the stack whose objects are triples $(C' \to S, L, \sigma)$, where $C' \to S$ is a family of smooth rational curves, L is a line bundle on C' of degree $-g-1$ and σ is a global section of $L^{-\otimes 2}$ whose support is étale on S. In [**AV04**, prop. 3.4], the authors proved that $\mathcal{H}_g \simeq \mathcal{H}_g^{\sim}$.

There is a morphism $\mathcal{P}' \to \mathcal{H}_g^{\sim}$ defined as:

$$(\pi : C' \to S, L, \sigma, \alpha, \beta) \longmapsto (\pi : C' \to S, L, \sigma)$$

that realizes $\mathcal{P}'$ as a $\mathrm{GL}_3 \times \mathbb{G}_m$-torsor over $\mathcal{H}_g^{\sim}$, because GL_3 acts by multiplication on β and $\mathbb{G}_m$ acts by multiplication on α.

This description of $\mathcal{P}' \to \mathcal{H}_g^{\sim}$ allows us to determine the associated rank 4 vector bundle: it coincides with $\mathcal{E} \oplus \mathcal{L}$, where $\mathcal{E}$ is the rank 3 vector bundle over $\mathcal{H}_g^{\sim}$ functorially defined as:

$$\mathcal{E} : (\pi : C' \to S, L, \sigma) \longmapsto \pi_* T_{C'/S}$$

and $\mathcal{L}$ is the line bundle over $\mathcal{H}_g^{\sim}$ functorially defined as

$$\mathcal{L} : (\pi : C' \to S, L, \sigma) \longmapsto \pi_*(L^{-1} \otimes T_{C'/S}^{-\otimes(g+1)/2})$$

We may ask for a description of the vector bundles $\mathcal{E}$ and $\mathcal{L}$ as vector bundles over $\mathcal{H}_g$. This can be easily deduced from the description we gave before: indeed, if $C \to S$ is a family of hyperelliptic curves of genus g which is a double cover of $C' \to S$ via the morphism $\eta : C \to C'$, and if $W_{C/S}$ is the associated Weierstrass divisor, then:

(1) $\eta^* T_{C'/S} \simeq \omega_{C/S}^{-1} \otimes \mathcal{O}(W_{C/S})$.
(2) $\eta^* L \simeq \mathcal{O}\left(-\frac{g+1}{2} W_{C/S}\right)$.

From the formulas above it can be easily deduced that the vector bundle $\mathcal{E}$, seen as a vector bundle over $\mathcal{H}_g$, is functorially defined as

$$\mathcal{E}((\pi : C \to S, \iota)) = \pi_* \omega_{C/S}^{-1}\left(W_{C/S}\right)$$

whereas $\mathcal{L}$, seen as a line bundle over $\mathcal{H}_g$, is functorially defined as

$$\mathcal{L}((\pi : C \to S, \iota)) = \pi_* \omega_{C/S}^{\otimes \frac{g+1}{2}} \left(\frac{1-g}{2} W_{C/S} \right)$$

These considerations will be used at the end of the chapter in order to provide a geometrical description of the generators of the Chow ring of $\mathcal{H}_g$.

2.2. Intersection theory of $\mathbb{P}(V_n)_3$

The aim of this section is to study the vector bundles $\mathbb{P}(V_n)_3$ on $\mathbb{A}(2,2)_3$ that were introduced in the previous section (see definition 1.3.10). In particular, in the first subsection we concentrate on the geometry of $\mathbb{P}(V_n)_3$, and we show that over certain particular open subschemes of $\mathbb{A}(2,2)_3$ the vector bundles $V_{n,3}$ become trivial (lemma 2.2.2). We also study some interesting morphisms between $\mathbb{P}(V_n)_3$ for different n.

In the second subsection we do some computations in the T-equivariant Chow ring of $\mathbb{P}(V_n)_3$, where $T \subset \mathrm{GL}_3$ is the subgroup of diagonal matrices, focusing on the cycle classes of some specific T-invariant subvarieties (lemmas 2.2.7, 2.2.8 and 2.2.9).

2.2.1. Properties of $\mathbb{P}(V_n)_3$. We will use the following notational shorthand: an underlined letter $\underline{i}$ will indicate a triple (i_0, i_1, i_2), and the expression $\underline{X}^{\underline{i}}$ will indicate the monomial $X_0^{i_0} X_1^{i_1} X_2^{i_2}$.

A form f of degree n in three variables with coefficients in a ring R can then be expressed as $f = \sum b_{\underline{i}} \underline{X}^{\underline{i}}$, where the $b_{\underline{i}}$ are elements of R and the sum is taken over the triples $\underline{i}$ such that $|i| := i_0 + i_1 + i_2 = n$.

The coefficients $b_{\underline{i}}$ give us coordinates in $\mathbb{A}(2, n)$, thus homogeneous coordinates in $\mathbb{P}(2, n)$. The symbols $a_{\underline{i}}$ will be used only for the coefficients of quadrics, or equivalently for the coordinates of $\mathbb{A}(2, 2)$.

Finally, we say that $\underline{i} \leq \underline{j}$ iff $i_\alpha \leq j_\alpha$ for $\alpha = 0, 1, 2$. This is equivalent to the condition $\underline{X}^{\underline{i}} | \underline{X}^{\underline{j}}$.

DEFINITION 2.2.1. We define $\mathbb{A}(2, 2)_3^{\underline{i}}$ as the open subscheme of $\mathbb{A}(2, 2)_3$ where the coordinate $a_{\underline{i}}$ is not zero. We also define $Y^{\underline{i}}$ as the complement of $\mathbb{A}(2, 2)_3^{\underline{i}}$.

The open subschemes $\mathbb{A}(2, 2)_3^{(0,2,0)}$, $\mathbb{A}(2, 2)_3^{(0,1,1)}$ and $\mathbb{A}(2, 2)_3^{(0,0,2)}$ are an open covering of $\mathbb{A}(2, 2)_3$: indeed, a point not in the union of these three open subschemes will necessarily parametrise a quadric divisible by X_0, thus not smooth. The open subschemes $\mathbb{A}(2, 2)_3^{\underline{i}}$ share another property, expressed in the following lemma:

LEMMA 2.2.2. *The projective bundles $\mathbb{P}(V_n)_3$ are trivial over the open sets $\mathbb{A}(2, 2)_3^{\underline{i}}$.*

The proof of the lemma above relies on a lemma of linear algebra concerning the vector spaces of forms in three variables of fixed degree.

LEMMA 2.2.3. *Let $\underline{i}$ be a triple such that $|i| = 2$ and let B_n be the set of monomials of degree n in three variables not divisible by $\underline{X}^{\underline{i}}$.*

Fix a quadric q in three variables with non-zero coefficient $a_{\underline{i}}$. Define B'_n to be the set of polynomials obtained by multiplying q with a monomial of degree $n-2$ in three variables. Then the two sets B_n and B'_n are disjoint and $B_n \cup B'_n$ is a base for the vector space of homogeneous polynomials of degree n in three variables with coefficients in a field k.

PROOF. The fact that B_n and B'_n are disjoint is obvious, because every monomial in B'_n is divisible by $\underline{X}^{\underline{i}}$.

The monomials form a base for the vector space of homogeneous polynomials of degree n in three variables. Let M be the matrix representing the unique linear transformation that sends the base of monomials to the set $B_n \cup B'_n$ in the following way: monomials not divisible by $\underline{X}^{\underline{i}}$ are sent to themselves, and monomials of the form $\underline{X}^{\underline{i}} f$ are sent to qf.

Observe that, after possibly reordering the monomials of the form $\underline{X}^{\underline{i}} f$, the matrix M will be upper triangular, with either 1 or $a_{\underline{i}}$ on the diagonal: this shows that the determinant of M is invertible, thus $B'_n \cup B_n$ is a base. □

PROOF OF LEMMA 2.2.2. From lemma 2.2.3 we see that over $\mathbb{A}(2, 2)_3^{\underline{i}}$ the coordinates $b_{\underline{k}}$, for $|k| = n$ and $\underline{i} \not\leq \underline{k}$, induce a trivialization of the vector bundle $V_n|_{\mathbb{A}(2,2)_3^{\underline{i}}}$. □

Consider the following morphisms:

$$\pi_{n,m} : \mathbb{P}(1, 2n) \times \mathbb{P}(1, 2m) \longrightarrow \mathbb{P}(1, 4n + 2m), \quad (f, g) \longmapsto f^2 g$$

whose GL_3-counterparts are

$$\pi'_{n,m}:\mathbb{P}(V_n)_3\times_{\mathbb{A}(2,2)_3}\mathbb{P}(V_m)_3\longrightarrow\mathbb{P}(V_{2n+m})_3,\quad (q,f,g)\longmapsto(q,f^2g)$$

Applying proposition 1.3.8 we deduce the following commutative diagrams of equivariant Chow rings:

$$\begin{array}{ccc} CH_{\mathrm{GL}_3}(\mathbb{P}(V_n)_3)\otimes_{CH_{\mathrm{GL}_3}(\mathbb{A}(2,2)_3)}CH_{\mathrm{GL}_3}(\mathbb{P}(V_m)_3) & \xrightarrow{\pi'_{n,m*}} & CH_{\mathrm{GL}_3}(\mathbb{P}(V_{2n+m})_3) \\ \downarrow\wr\!\wr & & \downarrow\wr\!\wr \\ CH_{\mathrm{PGL}_2}(\mathbb{P}(1,2n))\otimes_{CH_{PGLt}}CH_{\mathrm{PGL}_2}(\mathbb{P}(1,2m)) & \xrightarrow{\pi_{n,m*}} & CH_{\mathrm{PGL}_2}(\mathbb{P}(1,4n+2m)) \end{array}$$

Observe that we also have the following class of closed linear immersions of projective bundles:

$$j_{n,r,l}:\mathbb{P}(V_{n-r-l})_3\hookrightarrow\mathbb{P}(V_n)_3,\quad (q,f)\longmapsto(q,X_0^rX_1^lf)$$

These are not GL_3-equivariant but only T-equivariant, where T is the maximal subtorus of diagonal matrices.

Definition 2.2.4. The T-invariant, closed subscheme $W_{n,r,l}\subset\mathbb{P}(V_n)_3$ is defined as the schematic image of $j_{n,r,l}$.

2.2.2. Computations in the T-equivariant Chow ring of $\mathbb{P}(V_n)_3$. Let us introduce another little piece of notation: with $\underline{\lambda}$ we mean the triple $(\lambda_1,\lambda_2,\lambda_3)$. If $\underline{i}$ is another triple (most of the times, we will have $\underline{i}=(i_0,i_1,i_2)$), we indicate with $\underline{i}\cdot\underline{\lambda}$ their scalar product.

As already observed, there is a well defined action of GL_3 on $\mathbb{P}(V_n)_3$ and therefore an induced action of the split torus T of diagonal matrices. We can consider the T-equivariant Chow ring $CH_T(\mathbb{P}(V_n)_3)$. If $Z\subset\mathbb{P}(V_n)_3$ is a T-invariant subvariety, its T-equivariant cycle class will be denoted $[Z]_T$.

Recall from proposition 1.1.4 that we have:

$$CH_T(\mathrm{Spec}(k_0))\simeq\mathbb{Z}[\lambda_1,\lambda_2,\lambda_3]$$

where λ_i is the first Chern class of the representation associated to the i^{th}-projection $T\to\mathbb{G}_m$. Let σ_i be the elementary symmetric polynomial of degree i in λ_1, λ_2 and λ_3.

Proposition 2.2.5. *Suppose that the characteristic of k_0 is $\neq 2$. Then we have*

$$CH_T(\mathbb{A}(2,2)_3)\simeq\mathbb{Z}[\lambda_1,\lambda_2,\lambda_3]/(\sigma_1,2\sigma_3)$$

Proof. Observe that $\mathbb{A}(2,2)_3$ is a $\mathbb{G}_m$-torsor over $\mathbb{P}(2,2)_3$, whose associated line bundle is $\mathcal{O}(-1)\otimes\mathbb{D}$, where $\mathbb{D}$ is the pullback of the determinant representation of GL_3 regarded as a T-representation. Applying proposition 1.1.8, we deduce:

$$CH_T(\mathbb{A}(2,2)_3)=CH_T(\mathbb{P}(2,2)_3)/(\sigma_1-h)$$

where h is the restriction of the hyperplane section of $\mathbb{P}(2,2)$.

The localization exact sequence (proposition 1.1.1.(3)) for the open embedding $\mathbb{P}(2,2)_3\hookrightarrow\mathbb{P}(2,2)$ is:

$$CH_T(\mathbb{P}(2,2)_{[2,1]})\xrightarrow{i_*}CH_T(\mathbb{P}(2,2))\longrightarrow CH_T(\mathbb{P}(2,2)_3)\longrightarrow 0$$

By the projective bundle formula (proposition 1.1.1.(6)) we see that the ring in the middle is isomorphic to $CH_T(\mathrm{Spec}(k_0))[h]/(f)$, where $f=\prod_{a+b+c=2}(a\lambda_1+b\lambda_2+c\lambda_3)$ is the equivariant top Chern class of the T-representation $\mathbb{A}(2,2)=\mathrm{Sym}^2E^\vee$ (here E is the standard representation of GL_3 regarded as a T-representation).

Let W be the closed subscheme inside $\mathbb{P}(2,2)\times\mathbb{P}^2$ defined as:

$$W=\{(q,p)\text{ such that }q_{X_0}(p)=q_{X_1}(p)=q_{X_2}(p)=0\}$$

Observe that $W|_{\mathbb{P}(2,2)_2} \to \mathbb{P}(2,2)_2$ is an isomorphism, and also the morphism $\varphi : \mathbb{P}(2,1) \to \mathbb{P}(2,2)_1$ that sends a linear form to its square is an isomorphism. If we denote $\pi : W \to \mathbb{P}(2,2)$ the induced projection, these last two remarks imply that $\mathrm{im}(i_*) = \mathrm{im}(\pi_*) + \mathrm{im}(\varphi_*)$.

The generators of $\mathrm{im}(\varphi_*)$ had been computed in [EF08, pg. 10], and they are:

$$4(h^3 - 2\sigma_1 h^2 + (\sigma_1^2 + \sigma_2)h + (\sigma_3 - \sigma_1\sigma_2))$$
$$2h(h^3 - 2\sigma_1 h^2 + (\sigma_1^2 + \sigma_2)h + (\sigma_3 - \sigma_1\sigma_2))$$
$$h^2(h^3 - 2\sigma_1 h^2 + (\sigma_1^2 + \sigma_2)h + (\sigma_3 - \sigma_1\sigma_2))$$

To compute the generators of $\mathrm{im}(\pi_*)$, first observe that W is a projective subbundle of $\mathbb{P}(2,2) \times \mathbb{P}^2 \to \mathbb{P}^2$. This implies that the generators of $CH_T(W)$ as a $CH_T(\mathrm{Spec}(k_0))$-module are of the form $h^i \cdot t^j$, where t is the pullback of the hyperplane section of $\mathbb{P}^2$, and h is the restriction of the hyperplane section of $\mathbb{P}(2,2)$ to W.

The projection formula (proposition 1.1.1.(8)) tells us that $\pi_*(h^i \cdot t^j) = h^i \cdot \pi_* t^j$, so we only have to compute $\pi_*([W]_T \cdot t^j)$ for $j = 0, 1, 2$. We can write:

$$[W]_T = \xi_3 + \xi_2 t + \xi_1 t^2$$

where ξ_i come from $CH_T(\mathbb{P}(2,2))$. Applying the compatibility formula (proposition 1.1.1.(4)) we deduce that $\pi_*([W]_T \cdot t^j) = \xi_{1+j}$.

To compute $[W]_T$, we can apply lemma 1.1.7, because W is the complete intersection of the hypersurfaces:

$$H_1 := \{q_{X_0}(p) = 0\}\,,\ H_2 := \{q_{X_1}(p) = 0\}\,,\ H_3 := \{q_{X_2}(p) = 0\}$$

A simple computation gives us the following result:

$$\begin{aligned}
[W]_T &= (h + t - \lambda_1)(h + t - \lambda_2)(h + t - \lambda_3) \\
&= (h+t)^3 - \sigma_1(h+t)^2 + \sigma_2(h+t) - \sigma_3 \\
&= (h^3 - \sigma_1 h^2 + \sigma_2 h - \sigma_3) + t(3h^2 - 2\sigma_1 h + \sigma_2) + t^2(3h - \sigma_1) + t^3 \\
&= (h^3 - \sigma_1 h^2 + \sigma_2 h - 2\sigma_3) + t(3h^2 - 2\sigma_1 h) + t^2(3h - 2\sigma_1)
\end{aligned}$$

where in the last equality we used the relation $t^3 = -\sigma_1 t^2 - \sigma_2 t - \sigma_3$. We deduce:

$$\xi_3 = h^3 - \sigma_1 h^2 + \sigma_2 h - 2\sigma_3$$
$$\xi_2 = 3h^2 - 2\sigma_1 h$$
$$\xi_1 = 3h - 2\sigma_1$$

Putting everything together, we get the desired conclusion. □

The projective bundle formula (proposition 1.1.1.(6)) together with the previous proposition implies that

$$CH_T(\mathbb{P}(V_n)_3) \simeq \mathbb{Z}[\lambda_1, \lambda_2, \lambda_3, h_n]/(\sigma_1, 2\sigma_3, p_n(h_n))$$

where h_n is the hyperplane section of $\mathbb{P}(V_n)_3$ and $p_n(h_n)$ the T-equivariant top Chern class of $V_{n,3}$.

Let $\mathbb{A}(2,2)_3^{\underline{i}}$ and $Y^{\underline{i}}$ be as in definition 2.2.1: the T-equivariant Chow ring of $\mathbb{P}(V_n)_3|_{\mathbb{A}(2,2)_3^{\underline{i}}}$ can be easily computed.

LEMMA 2.2.6. *We have* $CH_T(\mathbb{P}(V_n)_3|_{\mathbb{A}(2,2)_3^{\underline{i}}}) \simeq CH_T(\mathbb{P}(V_n)_3)/(\underline{i} \cdot \underline{\lambda})$.

PROOF. From the localization exact sequence (proposition 1.1.1.(3)) we get:

$$CH_T(\mathbb{P}(V_n)_3|_{Y^{\underline{i}}}) \xrightarrow{i_*} CH_T(\mathbb{P}(V_n)_3) \xrightarrow{j^*} CH_T(\mathbb{P}(V_n)_3|_{\mathbb{A}(2,2)_3^{\underline{i}}}) \to 0$$

We want to prove that $\mathrm{im}(i_*) = (\underline{i} \cdot \underline{\lambda})$.

Let t be the hyperplane section of $\mathbb{P}(V_n)_3|_{Y^{\underline{i}}}$, so that the T-equivariant Chow ring of $\mathbb{P}(V_n)_3|_{Y^{\underline{i}}}$ is generated as $CH_T(\mathrm{Spec}(k_0))$-module by elements of the form $p^{'*}\alpha \cdot t^d$ for $d \leq 2n$, where p' is the projection map to $Y^{\underline{i}}$.

This implies that $\mathrm{im}(i_*)$ is generated by the elements $i_*(p^{'*}\alpha \cdot t^d)$. Observe that $i^*h_n = t$. From the cartesian square

$$\begin{array}{ccc} \mathbb{P}(V_n)_3|_{Y^{\underline{i}}} & \xrightarrow{\ i\ } & \mathbb{P}(V_n)_3 \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ Y^{\underline{i}} & \xrightarrow{\ i'\ } & \mathbb{A}(2,2)_3 \end{array}$$

and the projection and compatibility formula (proposition 1.1.1.(4),(8)) we deduce:

$$\begin{aligned} i_*(p^{'*}\alpha \cdot t^d) &= i_*(p^{'*}\alpha \cdot i^*h_n^d) \\ &= i_*p^{'*}\alpha \cdot h_n^d = p^*i'_*\alpha \cdot h_n^d \end{aligned}$$

This means that $\mathrm{im}(i_*) = p^*(\mathrm{im}(i'_*))$.

Observe that $Y^{\underline{i}}$ is an open subscheme of a representation of T, namely the vector subspace of forms of degree 2 with the coefficient $a_{\underline{i}}$ equal to zero. This implies that $CH_T(Y^{\underline{i}})$ is generated by 1 as a $CH_T(\mathrm{Spec}(k_0))$-module, from which we deduce that $\mathrm{im}(i'_*)$ is generated, as an ideal, by $[Y^{\underline{i}}]_T$.

Because of the fact that the closure of $Y^{\underline{i}}$ inside $\mathbb{A}(2,2)$ is defined as the zero locus of the coordinate $a_{\underline{i}}$, the class of this closure can be computed using lemma 1.1.7: we obtain $[\overline{Y^{\underline{i}}}]_T = c_1 - \underline{i} \cdot \underline{\lambda}$. From the explicit presentation of $CH_T(\mathbb{A}(2,2)_3)$ that we have obtained in proposition 2.2.5 we deduce that $[Y^{\underline{i}}]_T = -\underline{i} \cdot \underline{\lambda}$. □

We previously defined T-invariant closed subschemes $W_{n,r,l} \subset \mathbb{P}(V_n)_3$ (see definition 2.2.4). The cycle classes $[W_{n,r,l}]_T$ have degree $2r + 2l$. We can pull back them via the open immersion

$$j : \mathbb{P}(V_n)_3|_{\mathbb{A}(2,2)_3^{(0,0,2)}} \simeq \mathbb{P}^{2n} \times \mathbb{A}(2,2)_3^{(0,0,2)} \hookrightarrow \mathbb{P}(V_n)_3$$

so that they can be actually computed. Indeed, here we have homogeneous coordinates given by the coefficients $b_{\underline{k}}$, for $|k| = n$ and $(0,0,2) \nleq \underline{k}$ and we see that

$$j^{-1}W_{n,r,l} = \{b_{\underline{k}} = 0 \text{ for } k_0 < r \text{ or } k_1 < l\}$$

from which we deduce that they are complete intersection. Using lemma 1.1.7 we get:

$$j^*[W_{n,r,l}]_T = \prod (h_n - \underline{k} \cdot \underline{\lambda}) \text{ for } \underline{k} \text{ s.t. } |\underline{k}| = n,\ k_2 < 2,\ k_0 < r \text{ or } k_1 < l$$

Combining this with lemma 2.2.6 we deduce:

Lemma 2.2.7. *We have*

$$[W_{n,r,l}]_T = \prod (h_n - \underline{k} \cdot \underline{\lambda}) + 2\lambda_3\xi \textit{ for } \underline{k} \textit{ s.t. } |\underline{k}| = n,\ k_2 < 2,\ k_0 < r \textit{ or } k_1 < l$$

where ξ is an element of $CH_T(\mathbb{P}(V_n)_3)$.

In particular, all these classes are monic in h_n. Another useful property is the following: the set-theoretic intersection of $W_{n,r,0}$ and $W_{n,0,l}$ is exactly $W_{n,r,l}$. This is also true at the level of Chow rings: indeed $W_{n,r,l}$ is the only component of $W_{n,r,0} \cap W_{n,0,l}$, all the varieties involved are smooth and it is easy to check that the intersection is transversal. This implies the following result:

Lemma 2.2.8. *We have $[W_{n,r,0}]_T \cdot [W_{n,0,l}]_T = [W_{n,r,l}]_T$.*

Recall that we have defined the morphism

$$\pi'_{n,m} : \mathbb{P}(V_n)_3 \times_{\mathbb{A}(2,2)_3} \mathbb{P}(V_m)_3 \longrightarrow \mathbb{P}(V_{2n+m})_3$$

as $\pi'_{n,m}(q, f, g) = (q, f^2 g)$. If we restrict this morphism to $W_{n,1,n-1} \times_{\mathbb{A}(2,2)_3} \mathbb{P}(V_m)_3$ we obtain a birational surjective morphism

$$W_{n,1,n-1} \times_{\mathbb{A}(2,2)_3} \mathbb{P}(V_m)_3 \longrightarrow W_{2n+m,2,2n-2}$$

This implies the following result:

LEMMA 2.2.9. *We have* $\pi'_{n,m*}([W_{n,1,n-1} \times_{\mathbb{A}(2,2)_3} \mathbb{P}(V_m)_3]_T) = [W_{2n+m,2,n-2}]_T$.

2.3. The Chow ring of $\mathcal{H}_g$: generators and first relations

The goal of this section is to do the first steps in the computation of the Chow ring of $\mathcal{H}_g$, finding the generators and some relations. The intermediate result that we find is the content of corollary 2.3.12.

2.3.1. Some preliminary results. In this subsection we prove two technical results that we will be used frequently.

The following proposition is a generalization of the classical construction of the projection morphism from a projective subspace.

PROPOSITION 2.3.1. *Let X be a scheme of finite type over k_0 and suppose to have an exact sequence of locally free sheaves over X of the form:*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow \mathcal{G} \longrightarrow 0$$

We can form the projective bundle $\pi : \mathbb{P}(\mathcal{G}) \to X$.

Consider the flat morphism $f : \mathbb{P}(\mathcal{E}) \setminus \mathbb{P}(\mathcal{F}) \to X$ obtained by composing the open embedding of $\mathbb{P}(\mathcal{E}) \setminus \mathbb{P}(\mathcal{F})$ into $\mathbb{P}(\mathcal{E})$ with the projection of this projective bundle onto X.

Then there exists a canonical morphism $\beta : \mathbb{P}(\mathcal{E}) \setminus \mathbb{P}(\mathcal{F}) \to \mathbb{P}(\mathcal{G})$. Moreover, if there is a section $\sigma : \mathcal{E} \to \mathcal{F}$, the scheme $\mathbb{P}(\mathcal{E}) \setminus \mathbb{P}(\mathcal{F})$ is isomorphic to the vector bundle over $\mathbb{P}(\mathcal{G})$ associated to the locally free sheaf $\pi^*\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{G})}(1)$.

PROOF. To define a morphism $\beta : \mathbb{P}(\mathcal{E}) \setminus \mathbb{P}(\mathcal{F}) \to \mathbb{P}(\mathcal{G})$ over X we have to find an invertible sheaf $\mathcal{L}$ over $\mathbb{P}(\mathcal{E}) \setminus \mathbb{P}(\mathcal{F})$ and an embedding of locally free sheaves $\mathcal{L} \hookrightarrow f^*\mathcal{G}$.

Let $\rho : \mathbb{P}(\mathcal{E}) \to X$ be the projection morphism. We have a canonical inclusion $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1) \hookrightarrow \rho^*\mathcal{E}$. Restricting this morphism to the open subscheme $\mathbb{P}(\mathcal{E}) \setminus \mathbb{P}(\mathcal{F})$ and composing it with the morphism $f^*\mathcal{E} \to f^*\mathcal{G}$, we get:

$$\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1)|_{\mathbb{P}(\mathcal{E})\setminus\mathbb{P}(\mathcal{F})} \longrightarrow f^*\mathcal{G}$$

This last morphism is injective because the image of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1)|_{\mathbb{P}(\mathcal{E})\setminus\mathbb{P}(\mathcal{F})}$ inside $f^*\mathcal{E}$ does not intersect the image of $f^*\mathcal{F}$.

To prove the second claim of the proposition, is enough to show that on the site of $\mathbb{P}(\mathcal{G})$-schemes there is an isomorphism of sheaves:

$$\mathrm{Hom}(-, \mathbb{P}(\mathcal{E}) \setminus \mathbb{P}(\mathcal{F})) \simeq \pi^*\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{G})}(1)$$

Take a $\mathbb{P}(\mathcal{G})$-scheme S and a morphism $\beta : S \to \mathbb{P}(\mathcal{E}) \setminus \mathbb{P}(\mathcal{F})$ over $\mathbb{P}(\mathcal{G})$: due to the fact that $\mathbb{P}(\mathcal{G})$ is a scheme over X, the morphism β can be regarded as a morphism of X-schemes, and it induces an inclusion

$$\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1)|_S \hookrightarrow \mathcal{E}|_S$$

The fact that β is a morphism of $\mathbb{P}(\mathcal{G})$-schemes implies that:

(1) $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1)|_S \simeq \mathcal{O}_{\mathbb{P}(\mathcal{G})}(-1)|_S$.
(2) After identifying $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1)|_S$ with $\mathcal{O}_{\mathbb{P}(\mathcal{G})}(-1)|_S$, the composition of β with the morphism $\mathcal{E}|_S \to \mathcal{G}|_S$ is equal to the inclusion $\mathcal{O}_{\mathbb{P}(\mathcal{G})}(-1)|_S \hookrightarrow \mathcal{G}|_S$.

Therefore, the sheaf $\mathrm{Hom}(-, \mathbb{P}(\mathcal{E}) \setminus \mathbb{P}(\mathcal{F}))$ on the site of $\mathbb{P}(\mathcal{G})$-schemes is isomorphic to the sheaf

$$\mathbf{F} : S \longmapsto \left\{ \mathcal{O}_{\mathbb{P}(\mathcal{G})}(-1)|_S \hookrightarrow \mathcal{E}|_S \text{ that commutes with the morphisms to } \mathcal{G}|_S \right\}$$

Using the section $\sigma : \mathcal{E} \to \mathcal{F}$ we can construct an isomorphism $\mathbf{F} \simeq \pi^*\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{G})}(1)$ as follows: in one direction, we send an inclusion $i : \mathcal{O}_{\mathbb{P}(\mathcal{G})}(-1)|_S \hookrightarrow \mathcal{E}|_S$ to $\sigma \circ i$, and then we twits by $\mathcal{O}_{\mathbb{P}(\mathcal{G})}(-1)|_S$. In this way we get a morphism

$$\mathcal{O}_S \longrightarrow \mathcal{F}|_S \otimes \mathcal{O}_{\mathbb{P}(\mathcal{G})}(1)|_S$$

In the other direction, given a morphism $\mathcal{O}_S \to \mathcal{F}|_S \otimes \mathcal{O}_{\mathbb{P}(\mathcal{G})}(1)|_S$ we first twist it by $\mathcal{O}_{\mathbb{P}(\mathcal{G})}(-1)|_S$, obtaining in this way a morphism $\alpha : \mathcal{O}_{\mathbb{P}(\mathcal{G})}(-1)|_S \to \mathcal{F}|_S$.

After that, we take the unique lifting $i : \mathcal{O}_{\mathbb{P}(\mathcal{G})}(-1)|_S \to \mathcal{E}|_S$ of the canonical inclusion $\mathcal{O}_{\mathbb{P}(\mathcal{G})}(-1)|_S \hookrightarrow \mathcal{G}|_S$ having the property that $\sigma \circ i = \alpha$.

This construction gives us a well defined isomorphism of sheaves $\mathbf{F} \simeq \pi^*\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{G})}(1)$. $\square$

The next proposition will be used to compute equivariant cycle classes of invariant subvarieties.

Proposition 2.3.2. *Suppose to have a flat,* GL_3*-equivariant morphism* $f : \overline{U} \to \mathbb{P}(2,2)$ *and form the cartesian diagram*

$$\begin{array}{ccc} U & \xrightarrow{\quad q \quad} & \overline{U} \\ \downarrow & & \downarrow f \\ \mathbb{A}(2,2)_{[3,1]} & \longrightarrow & \mathbb{P}(2,2) \end{array}$$

Denote $\mathbb{P}(V_n)_U$ *the pullback of* $\mathbb{P}(V_n)$ *(see definition 1.3.10) along* $U \to \mathbb{A}(2,2)_{[3,1]}$. *Then we have:*

(1) *There exists a closed subscheme* $Y \subset U \times \mathbb{P}(2,n)$ *and a flat morphism*

$$p : U \times \mathbb{P}(2,n) \setminus Y \longrightarrow \mathbb{P}(V_n)_U$$

.

(2) *There exists an isomorphism*

$$\Phi : CH^k_{\mathrm{GL}_3}(\mathbb{P}(V_n)_U) \longrightarrow CH^k_{\mathrm{GL}_3}(\overline{U} \times \mathbb{P}(2,n))/(\mathrm{pr}_1^* f^* s - c_1)$$

where $s = c_1^{\mathrm{GL}_3}(\mathcal{O}_{\mathbb{P}(2,2)}(1))$ *and* $k < 2n$.

(3) *Let* $Z \subset \mathbb{P}(V_n)_U$ *be a subvariety of codimension* $< 2n$ *and let* $\overline{Z} \subset \overline{U} \times \mathbb{P}(2,n)$ *be a subvariety such that*

$$(q \times \mathrm{id})^{-1}(\overline{Z}|_{\overline{U} \times \mathbb{P}(2,n) \setminus q(Y)}) = p^{-1}(Z)$$

Then $\Phi[Z] = [\overline{Z}]$ *modulo the relation* $\mathrm{pr}_1^* f^* s - c_1 = 0$.

Proof. Consider the diagram:

$$\begin{array}{ccccc} & & \mathbb{P}(2,2) \times \mathbb{P}^2 & & \\ & \swarrow^{\mathrm{pr}_2} & & \searrow^{\mathrm{pr}_1} & \\ \mathbb{P}(2,2) & & & & \mathbb{P}^2 \end{array}$$

Recall from definition 1.3.10 that we have an exact sequence of locally free sheaves on $\mathbb{P}(2,2)$:

$$0 \longrightarrow \mathrm{pr}_{1*}\mathrm{pr}_2^*\mathcal{O}(n-2) \otimes \mathcal{O}_{\mathbb{P}(2,2)}(-1) \longrightarrow \mathrm{pr}_{1*}\mathrm{pr}_2^*\mathcal{O}(n-2) \longrightarrow \overline{\mathcal{V}}_n \longrightarrow 0$$

and that V_n is the vector bundle associated to the locally free sheaf $\mathcal{V}_n$, the pullback of $\overline{\mathcal{V}}_n$ to $\mathbb{A}(2,2)_{[3,1]}$.

From this we deduce the following exact sequence of locally free sheaves on U:

$$0 \longrightarrow \mathrm{pr}_{1*}\mathrm{pr}_2^*\mathcal{O}(n-2)_U \longrightarrow \mathrm{pr}_{1*}\mathrm{pr}_2^*\mathcal{O}(n-2)_U \longrightarrow \mathcal{V}_{n_U} \longrightarrow 0$$

Therefore, we can apply proposition 2.3.1, which gives us a morphism:

$$p : U \times \mathbb{P}(2,n) \setminus U \times \mathbb{P}(2,n-2) \longrightarrow \mathbb{P}(V_n)|_U$$

This proves (1), with $Y := U \times \mathbb{P}(2,n-2)$.

It is easy to see that p^* induces an isomorphism at the level of Chow groups. Using the localization exact sequence (proposition 1.1.1.(3)) we get an isomorphism:

$$\Phi_1 : CH^k_{\mathrm{GL}_3}(\mathbb{P}(V_n)|_U) \xrightarrow{p^*} CH^k_{\mathrm{GL}_3}(U \times \mathbb{P}(2,n) \setminus Y) \xrightarrow{(j^*)^{-1}} CH^k_{\mathrm{GL}_3}(U \times \mathbb{P}(2,n))$$

where j is the open embedding $(U \times \mathbb{P}(2,n)) \setminus Y \hookrightarrow U \times \mathbb{P}(2,n)$.

Observe that $U \to \overline{U}$ is the $\mathbb{G}_m$-torsor associated to the equivariant invertible sheaf $f^*\mathcal{O}_{\mathbb{P}(2,2)}(-1) \otimes \mathbb{D}$, where $\mathbb{D}$ is the determinant representation of GL_3, whose equivariant first Chern class is c_1, the generator of $CH^1_{\mathrm{GL}_3}$ (see proposition 1.1.4.(2)). From this we deduce that $U \times \mathbb{P}(2,n) \to \overline{U} \times \mathbb{P}(2,n)$ is the $\mathbb{G}_m$-torsor associated to the line bundle $\mathrm{pr}_1^* f^*\mathcal{O}_{\mathbb{P}(2,2)}(-1) \otimes \mathbb{D}$. Applying proposition 1.1.8, we get:

$$\Phi_2 : CH^k_{\mathrm{GL}_3}(U \times \mathbb{P}(2,n)) \simeq CH^k_{\mathrm{GL}_3}(\overline{U} \times \mathbb{P}(2,n))/(f^*\mathrm{pr}_1^* s - c_1)$$

We define $\Phi := \Phi_2 \circ \Phi_1$. This proves (2).

To prove (3), is enough to observe that with those assumptions we have:

$$\Phi_1[Z] = (j^*)^{-1}p^*[Z] = (q \times \mathrm{id})^*[\overline{Z}] = \Phi_2^{-1}[\overline{Z}]$$

□

2.3.2. Computation of the generators. Theorem 2.1.1 tells us that the Chow ring $CH(\mathcal{H}_g)$ is isomorphic to the equivariant Chow ring $CH_{\mathrm{GL}_3 \times \mathbb{G}_m}(U')$, where U' is the open subscheme of $V_{g+1,3}$ consisting of all the pairs (q,f) such that the intersection of the plane curves $\{q=0\}$ and $\{f=0\}$ is smooth, i.e. the intersection consists of $2g+2$ distinct points.

Let D_3' be the complement of U' in $V_{g+1,3}$. It is easy to see that D_3' is a closed subscheme of codimension 1. In this section, we will always assume $n = g+1$. The localization exact sequence in this case is

$$CH_{\mathrm{GL}_3 \times \mathbb{G}_m}(D_3') \xrightarrow{i_*} CH_{\mathrm{GL}_3 \times \mathbb{G}_m}(V_{n,3}) \xrightarrow{j^*} CH_{\mathrm{GL}_3 \times \mathbb{G}_m}(U') \to 0$$

Observe that $V_{n,3}$ is a $\mathrm{GL}_3 \times \mathbb{G}_m$-equivariant vector bundle over $\mathbb{A}(2,2)_3$, hence from proposition 1.1.1.(6) we deduce

(1) $$CH_{\mathrm{GL}_3 \times \mathbb{G}_m}(V_{n,3}) \simeq CH_{\mathrm{GL}_3 \times \mathbb{G}_m}(\mathbb{A}(2,2)_3)$$

with $\mathbb{G}_m$ acting trivially. We know an explicit presentation of $CH_T(\mathbb{A}(2,2))_3$ thanks to proposition 2.2.5: applying proposition 1.1.3 we get

(2) $$CH_{\mathrm{GL}_3 \times \mathbb{G}_m}(\mathbb{A}(2,2)_3) \simeq \mathbb{Z}[\tau, c_2, c_3]/(2c_3)$$

where τ is the first Chern class of the standard $\mathbb{G}_m$-representation and c_2, c_3 are respectively the second and the third Chern class of the standard GL_3-representation.

We have found a set of generators for $CH(\mathcal{H}_g)$. To find the relations among them, we have to compute the generators of the ideal $\mathrm{im}(i_*)$ inside the equivariant Chow ring of $V_{n,3}$.

Consider the projective bundle $\mathbb{P}(V_n)_3$ and its open subscheme U, whose preimage in $V_{n,3} \setminus \sigma_0$ is exactly U' (recall that $\sigma_0 : \mathbb{A}(2,2)_3 \to V_n$ is the zero section). Observe that $V_{n,3} \setminus \sigma_0$ is equivariantly isomorphic to the $\mathbb{G}_m$-torsor over $\mathbb{P}(V_n)_3$

associated to the line bundle $\mathcal{O}(-1)\otimes E^{-\otimes 2}$, where E is the standard representation of $\mathbb{G}_m$ pulled back to $\mathbb{P}(V_n)_3$. The same thing holds for U' over U. Applying proposition 1.1.8 we obtain:

$$CH_{\mathrm{GL}_3\times\mathbb{G}_m}(U') \simeq CH_{\mathrm{GL}_3\times\mathbb{G}_m}(U)/(-h_n-2\tau)$$

where h_n is the restriction of the hyperplane section of $\mathbb{P}(2,2)$.

This means that, if $p(h_n)$ is a relation in $CH_{\mathrm{GL}_3\times\mathbb{G}_m}(U)$, then $p(-2\tau)$ is a relation in $CH_{\mathrm{GL}_3\times\mathbb{G}_m}(U')$, and all the relations in this last ring are obtained from relations in the Chow ring of U in this way.

The action of $\mathbb{G}_m$ on $\mathbb{P}(V_n)_3$ is trivial, so that we can restrict ourselves to consider only the GL_3-action. Is then enough, in order to determine the equivariant Chow ring of U', to compute the GL_3-equivariant Chow ring of U.

Again, we have the localization exact sequence (proposition 1.1.1.(4)):

$$CH_{\mathrm{GL}_3}(D_3) \xrightarrow{i_*} CH_{\mathrm{GL}_3}(\mathbb{P}(V_n)_3) \xrightarrow{j^*} CH_{\mathrm{GL}_3}(U) \to 0$$

The projective bundle formula (proposition 1.1.1.(6)) tells us that the ring in the middle is isomorphic to:

$$CH_{\mathrm{GL}_3}(\mathbb{A}(2,2)_3)[h_n]/(p_n(h_n)) \simeq \mathbb{Z}[c_2,c_3,h_n]/(2c_3,p_n(h_n)) \tag{3}$$

where $p_n(h_n)$ is monic of degree $2n+1$. We have to compute the generators of the ideal $\mathrm{im}(i_*)$.

Observe that the closed subscheme D_3 admits a stratification

$$D_{3,n} \subset D_{3,n-1} \subset \ldots \subset D_{3,1} = D_3$$

where $D_{3,m}$ is the locus of pairs (q,f) such that $Q\cap F = 2E+E'$, with $\deg(E)=m$ (here Q is the vanishing locus of q and F is the vanishing locus of f). All these sets are clearly GL_3-invariant. Observe moreover that $D_{3,2s}$ coincides with the image of the equivariant, proper morphism

$$\pi'_{2s}:\mathbb{P}(V_s)_3\times_{\mathbb{A}(2,2)_3}\mathbb{P}(V_{n-2s})_3 \longrightarrow \mathbb{P}(V_n)_3,\quad (q,f,g)\longmapsto (q,f^2g)$$

which coincides with the morphism $\pi'_{s,n-2s}$ that we have defined in subsection 2.2.1. This induces a scheme structure on $D_{3,2s}$.

Consider also the GL_3-invariant closed subscheme $\mathcal{Y}_1\subset\mathbb{P}(V_1)_3$ defined as:

$$\mathcal{Y}_1 = \{(q,l) \text{ such that the plane curve } \{q=0\} \text{ is tangent to the line } \{l=0\}\}$$

and let us define the closed subschemes $\mathcal{Y}_{2s+1}$ as the image of the morphisms

$$\phi:\mathcal{Y}_1\times_{\mathbb{A}(2,2)_3}\mathbb{P}(V_s)_3 \longrightarrow \mathbb{P}(V_{2s+1})_3,\quad (q,l,f)\longmapsto (q,lf^2)$$

We can think of $\mathcal{Y}_{2s+1}$ as the locus of quadrics plus a divisor of the form $2E$, with $\deg(E)=2s+1$.

Consider the equivariant morphism:

$$\psi_{n,m}:\mathbb{P}(1,2n)\times\mathbb{P}(1,2m) \longrightarrow \mathbb{P}(1,2n+2m),\quad (f,g)\longmapsto fg$$

It is immediate to verify that its GL_3-counterpart is given by:

$$\psi'_{n,m}:\mathbb{P}(V_n)_3\times_{\mathbb{A}(2,2)_3}\mathbb{P}(V_m)_3 \longrightarrow \mathbb{P}(V_{n+m})_3,\quad (q,f,g)\longmapsto (q,fg)$$

Restricting the morphism $\psi'_{2s+1,n-2s-1}$ to $\mathcal{Y}_{2s+1}\times_{\mathbb{A}(2,2)_3}$ we obtain a proper morphism

$$\pi'_{2s+1}:\mathcal{Y}_{2s+1}\times_{\mathbb{A}(2,2)_3}\mathbb{P}(V_{n-2s-1})_3 \longrightarrow \mathbb{P}(V_n)_3$$

whose image is $D_{3,2s+1}$. This induces the scheme structure on $D_{3,2s+1}$.

The stratification defined above resembles the stratification

$$\Delta_{n,2n}\subset\ldots\subset\Delta_{1,2n}\subset\mathbb{P}(1,2n)$$

that has been introduced in [**FV11**, pg. 5]. Indeed, we have that $D_{3,s}$ is the GL_3-counterpart of $\Delta_{s,2n}$. Furthermore, it is easy to see that the GL_3-counterparts of the morphisms

$$\pi_{2s} : \mathbb{P}(1, 2s) \times \mathbb{P}(1, 2n-4s) \longrightarrow \mathbb{P}(1, 2n), \quad (f, g) \longmapsto f^2 g$$

are exactly the morphisms $\pi'_{2s} : \mathbb{P}(V_s)_3 \times_{\mathbb{A}(2,2)_3} \mathbb{P}(V_{n-2s})_3 \to \mathbb{P}(V_n)_3$. Applying proposition 1.3.8 we obtain the commutative diagram:

$$\begin{array}{ccc} CH_{\mathrm{PGL}_2}(\mathbb{P}(1,2s)\times\mathbb{P}(1,2n-4s)) & \xrightarrow{\simeq} & CH_{\mathrm{GL}_3}(\mathbb{P}(V_s)_3 \times_{\mathbb{A}(2,2)_3} \mathbb{P}(V_{n-2s})_3) \\ \downarrow & & \downarrow \\ CH_{\mathrm{PGL}_2}(\mathbb{P}(1,2n)) & \xrightarrow{\simeq} & CH_{\mathrm{GL}_3}(\mathbb{P}(V_n)_3) \end{array}$$

We also have that $\mathcal{Y}_1$ is the GL_3-counterpart of $\mathbb{P}^1$, as we can think of $\mathcal{Y}_1$ as the tautological conic over $\mathbb{A}(2,2)_3$, and in general the closed subschemes $\mathcal{Y}_{2s+1} \subset \mathbb{P}(V_{2s+1})$ are the GL_3-counterparts of $\mathbb{P}(1, 2s+1)$ sitting inside $\mathbb{P}(1, 4s+2)$ via the square map, so that we have

$$\begin{array}{ccc} CH_{\mathrm{PGL}_2}(\mathbb{P}(1,2s+1)\times\mathbb{P}(1,2n-4s-2)) & \xrightarrow{\simeq} & CH_{\mathrm{GL}_3}((\mathcal{Y}_{2s+1}) \times_{\mathbb{A}(2,2)_3} \mathbb{P}(V_{n-2s-1})_3) \\ \downarrow & & \downarrow \\ CH_{\mathrm{PGL}_2}(\mathbb{P}(1,4s+2)\times\mathbb{P}(1,2n-4s-2)) & \xrightarrow{\simeq} & CH_{\mathrm{GL}_3}(\mathbb{P}(V_{2s+1})_3 \times_{\mathbb{A}(2,2)_3} \mathbb{P}(V_{n-2s-1})_3) \\ \downarrow & & \downarrow \\ CH_{\mathrm{PGL}_2}(\mathbb{P}(1,2n)) & \xrightarrow{\simeq} & CH_{\mathrm{GL}_3}(\mathbb{P}(V_n)_3) \end{array}$$

Combining [**FV11**, lemma 3.1] with the diagrams above we obtain:

LEMMA 2.3.3. *Suppose that the characteristic of the base field k_0 is 0 or $> 2g+2$. Then the ideal $\mathrm{im}(i_*)$ is the sum of the ideals $\mathrm{im}(\pi'_{s*})$, and these ideals are equal to $\mathrm{im}(\pi_{s*})$ via the isomorphisms of the diagrams above.*

From now on, we will assume that the characteristic of k_0 is $> 2g+2$.

2.3.3. Computation of the first relations. The following result allows us to compute the first relations in the Chow ring of $\mathcal{H}_g$:

PROPOSITION 2.3.4. *We have $\mathrm{im}(\pi'_{1*}) = (2h_n^2 - 2n(n-1)c_2, 4(n-2)h_n)$.*

The remainder of this subsection is devoted to prove the proposition above.

Recall that in the proof of proposition 1.3.12 we constructed a closed subscheme of $\mathbb{P}(\overline{V}_n)\times\mathbb{P}^2$ that we called $\overline{\mathcal{D}}$: let $\mathcal{D}$ be its pullback to $\mathbb{P}(V_n)\times\mathbb{P}^2$. We can think of the points of $\mathcal{D}$ as triples $(q, [f], p)$ such that the plane curves $\{q = 0\}$ and $\{f = 0\}$ intersect in p.

Let $\mathcal{Z}$ be the ramification locus of $\mathcal{D}_3 \to \mathbb{P}(V_n)_3$, i.e. the locus defined by the 0^{th}-Fitting ideal of $\Omega^1_{\mathcal{D}_3/\mathbb{P}(V_n)_3}$.

Observe that it is GL_3-invariant, where GL_3 acts on $\mathbb{P}^2$ in the standard way (we can think of $\mathbb{P}^2$ as the projectivization of the standard representation of GL_3).

The image of $\mathcal{Z}$ via the projection on $\mathbb{P}(V_n)_3$ is D_3, and moreover $\mathrm{pr}_1 : \mathcal{Z} \to D_3$ is injective over $D_{3,1} \setminus D_{3,2}$. Before going on, let us pause a moment to study the geometry of $\mathcal{Z}$. We start with a well known technical result.

LEMMA 2.3.5. *Let p be a point of $\mathbb{P}^2$ and let Q, F be the plane projective curves defined by the homogeneous polynomials q and f. Let $J(q, f)$ be the 2×3-jacobian matrix, and suppose that Q and F intersect in p. Then their intersection is transversal iff there exists one 2×2- minor of $J(q, f)$ whose determinant does not vanish in p.*

Let us denote the determinant of the minor of $J(q,f)$ obtained by removing the column with the partial derivatives w.r.t. X_0 (resp. X_1 and X_2) as $\det_0 J(q,f)$ (resp. $\det_1 J(q,f)$ and $\det_2 J(q,f)$). Then we have the following equational characterization of $\mathcal{Z}$, which directly follows from the lemma above:

LEMMA 2.3.6. *Consider $\mathcal{Z}$ restricted to $\mathbb{P}(V_n)|_{\mathbb{A}(2,2)_3^{\underline{l}}} \times \mathbb{P}^2$, where $\underline{l} = (1,0,0)$ (resp. $(0,1,0)$ and $(0,0,1)$). Then $\mathcal{Z}$ is defined by the following equations in p and in the coefficients $a_{\underline{j}}$, $b_{\underline{k}}$, for $\underline{l} \nleq \underline{k}$, of q and f:*

- $q(p) = 0$
- $f(p) = 0$
- $\det_i J(q,f)(p) = 0$, *for $i = 0$ (resp. $i = 1$ and $i = 2$)*

A first step in the proof of proposition 2.3.4 is the following, which enables us to work with the morphism $\rho : \mathcal{Z} \to \mathbb{P}(V_n)_3$ rather than $\pi_1' : \mathcal{Y}_1 \times_{\mathbb{A}(2,2)_3} \mathbb{P}(V_{n-1})_3 \to \mathbb{P}(V_n)_3$.

LEMMA 2.3.7. *We have $\mathcal{Y}_1 \times_{\mathbb{A}(2,2)_3} \mathbb{P}(V_{n-1})_3 \simeq \mathcal{Z}$ and the isomorphism commutes with the morphisms to $\mathbb{P}(V_n)_3$.*

PROOF. Consider the morphism $\mathcal{Y}_1 \times_{\mathbb{A}(2,2)_3} \mathbb{P}(V_{n-1})_3 \to \mathbb{P}(V_n)_3 \times \mathbb{P}^2$ which sends a triple $(q, l, [f])$ to $(q, [lf], p)$ where p is the point of tangency of Q and L.

By construction, the image of this morphism is $\mathcal{Z}$. We want to define an inverse $\mathcal{Z} \to \mathcal{Y}_1 \times_{\mathcal{S}} \mathbb{P}(V_{n-1})$: this is done by sending a triple $(q, [f], p)$ of $\mathcal{Z}$ to $(q, l, [fl^{-1}])$, where $L = \{l = 0\}$ is the only line tangent to Q in p. Details are omitted. $\square$

COROLLARY 2.3.8. *We have $\mathrm{im}(\pi_{1*}') = \mathrm{im}(\rho_*)$.*

In order to prove proposition 2.3.4 is then equivalent to compute $\mathrm{im}(\rho_*)$. Let us call t the hyperplane section of $\mathbb{P}^2$, so that the equivariant Chow ring $CH_{\mathrm{GL}_3}(\mathbb{P}(V_n)_3 \times \mathbb{P}^2)$ is generated as $CH_{\mathrm{GL}_3}(\mathbb{A}(2,2)_3)$-algebra by $\mathrm{pr}_1^* h_n$ and $\mathrm{pr}_2^* t$: with a little abuse of notation we will keep calling these cycles h_n and t.

The class $[\mathcal{Z}]$ can then be written as a polynomial in t of degree 3 with coefficients in $CH_{\mathrm{GL}_3}(\mathbb{P}(V_n)_3)$: indeed, the dimension of $\mathcal{Y}_1 \times_{\mathbb{A}(2,2)_3} \mathbb{P}(V_{n-1})_3$ is equal to $2n+4$, which by lemma 2.3.7 is equal to the dimension of $\mathcal{Z}$, so that we deduce that the codimension of $\mathcal{Z}$ in $\mathbb{P}(V_n)_3 \times \mathbb{P}^2$ is equal to 3. We then have $[\mathcal{Z}] = \beta_1(h_n)t^2 + \beta_2(h_n)t + \beta_3(h_n)$.

LEMMA 2.3.9. *We have $\mathrm{im}(\rho_*) = (\beta_1(h_n), \beta_2(h_n), \beta_3(h_n))$.*

The lemma above reduces the computation of the generators of $\mathrm{im}(\rho_*)$ to the computation of the class $[\mathcal{Z}]$ inside $CH_{\mathrm{GL}_3}(\mathbb{P}(V_n)_3 \times \mathbb{P}^2)$. Before proving lemma 2.3.9 we need some preliminary results.

LEMMA 2.3.10. *The closed subscheme $\mathcal{Z}$ is a projective subbundle of $\mathbb{P}(V_n)_3 \times \mathbb{P}^2$ over the universal smooth quadric $\widehat{Q}_3 \subset \mathbb{P}^2 \times \mathbb{A}(2,2)_3$.*

PROOF. First recall that $\widehat{Q}_3 = \{(q,p) \text{ such that } q(p) = 0\}$. We can work Zariski-locally on $\widehat{Q}_3$, so that $\mathcal{Z}$ is described by the equations of lemma 2.3.6, which are linear in the coefficients of f: this proves the lemma. $\square$

LEMMA 2.3.11. *The image of $i_* : CH_{\mathrm{GL}_3}(\mathcal{Z}) \to CH_{\mathrm{GL}_3}(\mathbb{P}(V_n)_3 \times \mathbb{P}^2)$ is generated as an ideal by $i_*1 = [\mathcal{Z}]$.*

PROOF. We claim that the equivariant Chow ring of $\mathcal{Z}$ is generated as a $i^*\mathrm{pr}_2^*(CH_{\mathrm{GL}_3}(\mathbb{P}^2))$-algebra by i^*h_n, so that every element is a sum of monomials of the form $i^*\mathrm{pr}_2^*\xi \cdot i^*h_n^r$. If this is the case, the projection formula (proposition 1.1.1.(8)) imply that $i_*(i^*\mathrm{pr}_2^*\xi \cdot i^*h_n^r) = \mathrm{pr}_2^*\xi \cdot h_n^r \cdot [\mathcal{Z}]$, and we are done.

From lemma 2.3.10 we know that the equivariant Chow ring of $\mathcal{Z}$ is generated by i^*h_n as $CH_{\mathrm{GL}_3}(\widehat{Q}_3)$-algebra, where $\widehat{Q}_3$ is the universal smooth quadric and $CH_{\mathrm{GL}_3}(\widehat{Q}_3)$ acts via the pullback morphism induced by the projection $\phi : \mathcal{Z} \to \widehat{Q}_3$.

Consider the trivial vector bundle $\mathbb{P}^2 \times \mathbb{A}(2,2)$ over $\mathbb{P}^2$, which contains the vector subbundle $\widehat{Q}$ defined by the equation $q(p) = 0$, which is linear in the coefficients of q. By homotopy invariance (proposition 1.1.1.(5)), the equivariant Chow ring of $\widehat{Q}$ is isomorphic to the one of $\mathbb{P}^2$ via the pullback along the projection map.

Moreover $\widehat{Q}_3$ is an open subscheme of $\widehat{Q}$, thus from the localization exact sequence (see proposition 1.1.1.(3)) we deduce that $CH_{\mathrm{GL}_3}(\widehat{Q}_3)$ is generated by 1 as a $j^*\mathrm{pr}_2^*CH_{\mathrm{GL}_3}(\mathbb{P}^2)$-module, where $j : \widehat{Q}_3 \hookrightarrow \mathbb{A}(2,2)_3 \times \mathbb{P}^2$ is the closed embedding.

To finish the proof, one has only to observe that $\phi^*j^*\mathrm{pr}_2^* = i^*\mathrm{pr}_2^*$. □

Proof of lemma 2.3.9. Due to the fact that $\rho = \mathrm{pr}_1 \circ i$, from lemma 2.3.11 we deduce that $\mathrm{im}(\rho_*)$ is generated, as an ideal, by elements of the form

$$\mathrm{pr}_{1*}([\mathcal{Z}] \cdot \mathrm{pr}_1^*\xi \cdot \mathrm{pr}_2^*t^k) = \xi \cdot \mathrm{pr}_{1*}([\mathcal{Z}] \cdot \mathrm{pr}_2^*t^k)$$

The compatibility formula (proposition 1.1.1.(4)) applied to the cartesian diagram

$$\begin{array}{ccc} \mathbb{P}(V_n)_3 \times \mathbb{P}^2 & \xrightarrow{\mathrm{pr}_2} & \mathbb{P}^2 \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow \\ \mathbb{P}(V_n)_3 & \longrightarrow & \mathrm{Spec}(k_0) \end{array}$$

implies that $\mathrm{pr}_{1*}\mathrm{pr}_2^*t^d$ it is zero unless $d = 2$, in which case is equal to 1.

Therefore, we have $\mathrm{pr}_{1*}([\mathcal{Z}]\cdot\mathrm{pr}_2^*t^k) = \beta_{1+k}$ where $[\mathcal{Z}] = \beta_3+\beta_2\cdot\mathrm{pr}_2^*t+\beta_1\cdot\mathrm{pr}_2^*t^2$. This finishes the proof. □

Thanks to lemma 2.3.9 to prove proposition 2.3.4 we only have to compute the GL_3-equivariant cycle class of $\mathcal{Z}$.

Let $\mathcal{Z}'$ be the GL_3-invariant, closed subscheme of $\mathbb{P}(2,2) \times \mathbb{P}(2,n) \times \mathbb{P}^2$ whose points are triples (q,f,p) such that

$$q(p) = f(p) = 0, \quad \det_i J(q,f)(p) = 0, \;\; i = 0,1,2$$

We can apply proposition 2.3.2 to the morphism $\mathbb{P}(2,2)_3 \times \mathbb{P}^2 \to \mathbb{P}(2,2)$: this tells us that there is an isomorphism

$$\Phi : CH^3_{\mathrm{GL}_3}(\mathbb{P}(V_n)_3 \times \mathbb{P}^2) \simeq CH^3_{\mathrm{GL}_3}(\mathbb{P}(2,2)_3 \times \mathbb{P}(2,n) \times \mathbb{P}^2)/(s - c_1)$$

where s is the hyperplane section of $\mathbb{P}(2,2)$. Moreover, it also tells us $\Phi[\mathcal{Z}] = [\mathcal{Z}']$ modulo the relations $s - c_1 = 0$, $c_1 = 0$ and $2c_3 = 0$. Therefore, all we have to do is to compute the cycle class of $\mathcal{Z}'$.

Observe that $\mathcal{Z}'$ it is not a complete intersection, but it becomes so if we restrict to the open subscheme of $\mathbb{P}^2$ consisting of points where one of the homogeneous coordinates does not vanish. Let us consider the auxiliary cycle class $[\mathcal{Z}'_2]$, where $\mathcal{Z}'_2$ is the closed subscheme defined by the equations

$$q(p) = f(p) = 0, \quad \det_2 J(q,f)(p) = 0$$

It is easy to check that this locus has two irreducible components, $\mathcal{Z}'$ and $\mathcal{W}_2$, where $\mathcal{W}_2$ is defined by the equations $q(p) = 0$, $f(p) = 0$ and $p_2 = 0$ (here p_2 stands for the third homogeneous coordinate of $\mathbb{P}^2$).

Thus, we would like to write $[\mathcal{Z}'] = [\mathcal{Z}'_2] - [\mathcal{W}_2]$. Unfortunately, the subschemes $\mathcal{Z}'_2$ and $\mathcal{W}_2$ are not GL_3-invariant.

We can use the following trick: we first pass to the action of the maximal subtorus $T \subset \mathrm{GL}_3$, and we observe that $\mathcal{Z}'_2$ and $\mathcal{W}_2$ are equivariant with respect to the T-action. Then we compute their T-equivariant classes in the T-equivariant Chow ring of $\mathbb{P}(2,n)\times\mathbb{P}(2,2)\times\mathbb{P}^2$: by proposition 1.1.3, their difference will be S_3-invariant, hence after expressing it in terms of elementary symmetric polynomials σ_i and substituting σ_i with c_i we will obtain the GL_3-equivariant cycle $[\mathcal{Z}']$.

Both $\mathcal{Z}'_2$ and $\mathcal{W}_2$ are complete intersections, hence their cycle classes can be computed using lemma 1.1.7:

$$\begin{aligned}[]
[\mathcal{Z}'_2]_T-[\mathcal{W}_2]_T &= (s+2t)(h_n+nt)(s+h_n+nt-\lambda_1-\lambda_2)\\
&\quad-(s+2t)(h_n+nt)(t+\lambda_3)=\\
&=(s+2t)(h_n+nt)(s+h_n+(n-1)t-\sigma_1)
\end{aligned}$$

Observe that the cyle classes of $\mathcal{Z}'_2$ and $\mathcal{W}$ were not symmetric with respect to λ_i but they become so when combined together, precisely how we expected.

In order to find the coefficients β_1,β_2 and β_3 we have to put this expression in its canonical form, by using the relation $t^3=-c_3-c_2t-c_1t^2$ coming from the equivariant Chow ring of $\mathbb{P}^2$. In the end we obtain:

$$\begin{aligned}[]
[\mathcal{Z}'] =&(s^2h_n+sh_n^2-sh_nc_1-2n(n-1)c_3)+\\
&((n-1)s^2+(2n+1)sh_n-nsc_1+2h_n^2-2h_nc_1-2n(n-1)c_2)t\\
&(((n-1)^2+3n-1)s+(4n-2)h_n-2nc_1-2n(n-1)c_1)t^2
\end{aligned}$$

Substituting $s=c_1=0$ and $2c_3=0$, we obtain $\beta_1=(4n-2)h_n$, $\beta_2=2h_n^2-2n(n-1)c_2$ and $\beta_3=0$. This proves proposition 2.3.4.

Corollary 2.3.12. *The Chow ring of $\mathcal{H}_g$ is a quotient of the ring*

$$\mathbb{Z}[\tau,c_2,c_3]/(4(2g+1)\tau,8\tau^2-2g(g+1)c_2,2c_3)$$

Proof. The only thing we need to prove is that $p_n(-2\tau)$ is contained in the ideal above. This works exactly as in [FV11, proposition 6.4]. □

The next section will be devoted to check if there are other relations in the Chow ring of $\mathcal{H}_g$, or if they all come from the pullback to $CH_{\mathrm{GL}_3\times\mathbb{G}_m}(V_n)$ of $\mathrm{im}(\pi'_{1*})$.

2.4. Other generators of $\mathrm{im}(i_*)$

As before, we assume char(k_0)=0 or $>2g+2$. The value of n is always assumed to be even.

In this section, we first compute $\mathrm{im}(i_*)$ with $\mathbb{Z}[\frac{1}{2}]$-coefficients, which means that we do the computations in the equivariant Chow ring tensored over $\mathbb{Z}$ with $\mathbb{Z}[\frac{1}{2}]$. The main result is the following proposition:

Proposition 2.4.1. *We have* $\mathrm{im}(\pi'_{r*})\otimes_{\mathbb{Z}}\mathbb{Z}[\frac{1}{2}]\subset\mathrm{im}(\pi'_{1*})\otimes_{\mathbb{Z}}\mathbb{Z}[\frac{1}{2}]$.

In other terms, using $\mathbb{Z}[\frac{1}{2}]$-coefficients, the ideal $\mathrm{im}(i_*)$ coincides with the ideal $\mathrm{im}(\pi'_{1*})$, whose generators we computed in the previous section (proposition 2.3.4).

Next, we pass to $\mathbb{Z}_{(2)}$-coefficients. What we deduce at the end is the following result:

Proposition 2.4.2. *We have* $\mathrm{im}(i_*)\otimes_{\mathbb{Z}}\mathbb{Z}_{(2)}=(2h_n^2-2n(n-1)c_2,4(n-2)h_n,\pi'_{2*}(h_1^2\times[\mathbb{P}(V_{n-2})]))$ *and the inclusion* $\mathrm{im}(\pi'_{1*})\otimes_{\mathbb{Z}}\mathbb{Z}_{(2)}\subset\mathrm{im}(i_*)\otimes_{\mathbb{Z}}\mathbb{Z}_{(2)}$ *is strict.*

The last two propositions together imply:

Corollary 2.4.3. *We have* $\mathrm{im}(i_*)=(2h_n^2-2n(n-1)c_2,4(n-2)h_n,\pi'_{2*}(h_1^2\times[\mathbb{P}(V_{n-2})]))$.

Using proposition 1.3.8 we see that the corollary above, interpreted in the PGL_2-equivariant setting, says that the image of

$$i_* : CH_{\mathrm{PGL}_2}(\Delta_{1,2n}) \longrightarrow CH_{\mathrm{PGL}_2}(\mathbb{P}(1,2n))$$

is equal to the image of

$$\pi_{1*} : CH_{\mathrm{PGL}_2}(\mathbb{P}(1,1) \times \mathbb{P}(1,2n-2)) \longrightarrow CH_{\mathrm{PGL}_2}(\mathbb{P}(1,2n))$$

plus the cycle $\pi_{2*}(H^2 \times 1)$, where H is the hyperplane section of $\mathbb{P}(1,2)$ and the morphism π_2 is

$$\pi_2 : \mathbb{P}(1,2) \times \mathbb{P}(1,2n-4) \longrightarrow \mathbb{P}(1,2n), \quad (f,g) \longmapsto f^2 g$$

and the inclusion $\mathrm{im}(\pi_{1*}) \subset \mathrm{im}(i_*)$ is strict. Instead, in [**FV11**, proposition 5.3] is stated that $\mathrm{im}(\pi_{r*}) \subset \mathrm{im}(\pi_{1*})$.

To prove proposition 2.4.2, we initially work with GL_3-equivariant Chow rings, we pass then to the T-equivariant ones, and we complete the computation of $\mathrm{im}(i_*^T) \otimes_{\mathbb{Z}} \mathbb{Z}_{(2)}$ in this different setting. Then, using what we have found exploiting the T-equivariant Chow rings, we go back to the GL_3-context and we finish the computation of the generators of $\mathrm{im}(i_*) \otimes_{\mathbb{Z}} \mathbb{Z}_{(2)}$.

We initially follow the path of [**FV11**] but at a certain point we diverge. Indeed, as said before, the computation is completed in the T-equivariant setting, mainly because we start working with cycle classes of subvarieties that are only T-invariant and not GL_3-invariant. In particular, these classes do not have an analogue in the PGL_2-equivariant setting that is adopted in [**FV11**]. This is where we really need the new presentation given by theorem 2.1.1.

2.4.1. Computations with $\mathbb{Z}[\frac{1}{2}]$-coefficients. Let us recall the content of [**FV11**, lemma 5.4]:

LEMMA 2.4.4. *Let X be a smooth scheme on which PGL_n acts, and consider the induced action of SL_n via the quotient map $\mathrm{SL}_n \to \mathrm{PGL}_n$. Then the kernel of the pullback map $CH_{\mathrm{PGL}_n}(X) \to CH_{\mathrm{SL}_n}(X)$ is of n-torsion.*

From now until the end of the current subsection, every Chow ring is assumed to be tensored over $\mathbb{Z}$ with $\mathbb{Z}[\frac{1}{2}]$. The strategy adopted here is substantially the same as the one used in [**FV11**]. Consider first the commutative square

$$\begin{array}{ccc} CH_{\mathrm{GL}_3}(\mathbb{P}(V_s)_3 \times_{\mathbb{A}(2,2)_3} \mathbb{P}(V_{n-2s})_3) & \longrightarrow & CH_{\mathrm{GL}_3}(\mathbb{P}(V_n)_3) \\ \downarrow \wr\! \| & & \downarrow \wr\! \| \\ CH_{\mathrm{PGL}_2}(\mathbb{P}(1,2s) \times \mathbb{P}(1,2n-4s)) & \longrightarrow & CH_{\mathrm{PGL}_2}(\mathbb{P}(1,2n)) \\ \downarrow & & \downarrow \\ CH_{\mathrm{SL}_2}(\mathbb{P}(1,2s) \times \mathbb{P}(1,2n-4s)) & \longrightarrow & CH_{\mathrm{SL}_2}(\mathbb{P}(1,2n)) \end{array}$$

where the three horizontal arrows are the pushforward along the maps π'_{2s}, π_{2s} and $\pi_{2s}^{\mathrm{SL}_2}$.

Recall that $\pi'_{2s}(q,[f],[g]) := (q,[f^2 g])$, and that $\pi_{2s}(f,g) = f^2 g$, and $\pi_{2s}^{\mathrm{SL}_2}$ is defined in the same way.

Then from lemma 2.4.4 we deduce that the two last vertical arrows, when using $\mathbb{Z}[\frac{1}{2}]$-coefficients, are injective, so that it is enough to prove proposition 2.4.1, $r = 2s$, for the SL_2-equivariant Chow rings. This can be done copying verbatim [**EF09**, section 4], with the additional relation $c_1 = 0$.

The case $r = 2s+1$ is handled similarly, using the commutative square

$$\begin{array}{ccc} CH_{\mathrm{GL}_3}(\mathcal{Y}_{2s+1} \times_{\mathbb{A}(2,2)_3} \mathbb{P}(V_{n-2s-1})_3) & \longrightarrow & CH_{\mathrm{GL}_3}(\mathbb{P}(V_n)_3) \\ \downarrow \wr\| & & \downarrow \wr\| \\ CH_{\mathrm{PGL}_2}(\mathbb{P}(1,2s+1) \times \mathbb{P}(1,2n-4s-2)) & \longrightarrow & CH_{\mathrm{PGL}_2}(\mathbb{P}(1,2n)) \\ \downarrow & & \downarrow \\ CH_{\mathrm{SL}_2}(\mathbb{P}(1,2s+1) \times \mathbb{P}(1,2n-4s-2)) & \longrightarrow & CH_{\mathrm{SL}_2}(\mathbb{P}(1,2n)) \end{array}$$

This concludes the proof of proposition 2.4.1 stated at the beginning of the section.

2.4.2. Computations with $\mathbb{Z}_{(2)}$-coefficients, first part. Throughout this subsection, we will assume that every Chow ring and every ideal appearing is tensored over $\mathbb{Z}$ with $\mathbb{Z}_{(2)}$, also when is not explicitly written. The main result of this section is the next proposition.

PROPOSITION 2.4.5. *The ideal $\mathrm{im}(i_*) \otimes_{\mathbb{Z}} \mathbb{Z}_{(2)}$ is equal to the sum of the ideal $\mathrm{im}(\pi'_{1*}) \otimes_{\mathbb{Z}} \mathbb{Z}_{(2)}$ and the ideal generated by the elements $\pi'_{2s*}(h_s^{2s} \cdot 1)$ for $s = 1, ..., n/2$*

Recall from lemma 2.3.3 that when the characteristic of k_0 is 0 or $> 2g+2$ the ideal $\mathrm{im}(i_*)$ is the sum of the ideals $\mathrm{im}(\pi'_{r*})$.

LEMMA 2.4.6. *We have $\mathrm{im}(\pi'_{2s+1*}) \otimes_{\mathbb{Z}} \mathbb{Z}_{(2)} \subset \mathrm{im}(\pi'_{1*}) \otimes_{\mathbb{Z}} \mathbb{Z}_{(2)}$.*

PROOF. Consider the commutative square

$$\begin{array}{ccc} \mathcal{Y}_1 \times_{\mathbb{A}(2,2)_3} \mathbb{P}(V_s)_3 \times_{\mathbb{A}(2,2)_3} \mathbb{P}(V_{n-2s-1})_3 & \longrightarrow & \mathcal{Y}_1 \times_{\mathbb{A}(2,2)_3} \mathbb{P}(V_{n-1})_3 \\ \downarrow \phi\times\mathrm{id} & & \downarrow \pi'_1 \\ \mathcal{Y}_{2s+1} \times_{\mathbb{A}(2,2)_3} \mathbb{P}(V_{n-2s-1})_3 & \xrightarrow{\pi'_{2s+1}} & \mathbb{P}(V_n)_3 \end{array}$$

where the top horizontal arrow ψ sends a tuple (q,l,f,g) to (q,l,f^2g). Observe that the vertical arrow on the left is finite of degree $2s+1$, thus the pushforward induces an isomorphism at the level of Chow rings (we are using $\mathbb{Z}_{(2)}$-coefficients). The commutativity of the square implies that

$$(\pi'_{2s+1})_* = (\pi'_1)_* \circ (\psi)_* \circ (\phi \times \mathrm{id})_*^{-1}$$

and this implies the lemma. □

Now we want to study the image of π'_{2s*}. Observe that this ideal is generated by the pushforward of the classes $h_{2s}^i \cdot h_{n-2s}^j$, for $i = 0, ..., 2s$ and $j = 0, ..., 2n-4s$, where we use the notational shorthand $h_{2s}^i \cdot h_{n-2s}^j$ to indicate what should be more correctly denoted as $\mathrm{pr}_1^* h_{2s}^i \cdot \mathrm{pr}_2^* h_{n-2s}^j$. An intermediate result is the following lemma:

LEMMA 2.4.7. *We have that $\pi'_{2s*}(h_s^i \cdot h_{n-2s}^j)$ is in $\mathrm{im}(\pi'_{1*})$ for $i = 0, ..., 2s-1$ and $j = 0, ..., 4n-2s$.*

In order to prove the lemma above we need a technical result, which can be found also in [FV11], with the exception that there the authors claim the result also for $i = 2s$. The proof works exactly in the same way.

LEMMA 2.4.8. *We have that $\pi'_{2s*}(h_s^i \cdot h_{n-2s}^j)$ is 2-divisible for $i = 0, ..., 2s-1$ and $j = 0, ..., 4n-2s$.*

Proof. We start with the case $s = n/2$. Observe that $\pi'_{n*}(h^i_{n/2})$, for $i = 0, ..., n-1$, is 2-divisible if and only if $\pi'_{n*}(h^i_{n/2}) \cdot h_n$ is 2-divisible. This follows from the uniqueness of the representation of cycles in $CH_T(\mathbb{P}(V_n)_3)$ as polynomials in h_n of degree less or equal to $2n$.

We also have that $\pi'^*_n h_n = 2h_{n/2}$, and from this we deduce that

$$\pi'_{n*}(h^i_{n/2}) \cdot h_n = \pi'_{n*}((h^i_{n/2}) \cdot \pi'^*_n h_n) = 2\pi'_{n*} h^{i+1}_{n/2}$$

Now consider the general case, and observe that we have a factorization of π'_{2s} as follows:

$$\mathbb{P}(V_s)_3 \times_{\mathbb{A}(2,2)_3} \mathbb{P}(V_{n-2s})_3 \xrightarrow{\pi''_{2s}\times \mathrm{id}} \mathbb{P}(V_{2s})_3 \times_{\mathbb{A}(2,2)_3} \mathbb{P}(V_{n-2s})_3 \xrightarrow{\pi_{2s,n-2s}} \mathbb{P}(V_n)_3$$

where $\pi''_{2s}((q, f)) = (q, f^2)$. At the level of Chow rings, the first morphism coincides with

$$\begin{array}{c} CH_{\mathrm{GL}_3}(\mathbb{P}(V_s)_3) \otimes_{CH_{\mathrm{GL}_3}(\mathbb{A}(2,2)_3)} CH_{\mathrm{GL}_3}(\mathbb{P}(V_{n-2s})_3) \\ \downarrow \pi''_{2s*}\otimes \mathrm{id}_* \\ CH_{\mathrm{GL}_3}(\mathbb{P}(V_{2s})_3) \otimes_{CH_{\mathrm{GL}_3}(\mathbb{A}(2,2)_3)} CH_{\mathrm{GL}_3}(\mathbb{P}(V_{n-2s})_3) \end{array}$$

From the previous case we deduce that

$$\pi'_{2s*}(h^i_s \cdot h^j_{n-2s}) = 2\pi_{2s,n-2s*}(\pi''_{2s*} h^{i+1}_s \cdot h^j_{n-2s})$$

which concludes the proof of the lemma. □

Remark 2.4.9. We cannot extend the previous lemma to the classes in which h^{2s}_s appears. Indeed, for $s = n/2$, write $\pi'_{n*} h^n_{n/2}$ as a polynomial $\alpha_0 h^{2n}_n + \alpha_1 h^{2n-1}_n + ... + \alpha_{2n}$. Then we have

$$\pi'_{n*} h^n_{n/2} \cdot h_n = (\alpha_0 h^{2n}_n + \alpha_1 h^{2n}_n + ...\alpha_{2n}) \cdot h_n = \alpha_0(h^{n+1}_n - p_n(h_n)) + ... + \alpha_{2n} h_n$$

and the first coefficient will always be 2-divisible, no matter if α_0 is even or not.

Proof of lemma 2.4.7. Consider again the commutative diagram

$$\begin{array}{ccc} CH_{\mathrm{GL}_3}(\mathbb{P}(V_s)_3 \times_{\mathbb{A}(2,2)_3} \mathbb{P}(V_{n-2s})_3) & \longrightarrow & CH_{\mathrm{GL}_3}(\mathbb{P}(V_n)_3) \\ \downarrow \wr\!\!\mid & & \downarrow \wr\!\!\mid \\ CH_{\mathrm{PGL}_2}(\mathbb{P}(1,2s) \times \mathbb{P}(1,2n-4s)) & \longrightarrow & CH_{\mathrm{PGL}_2}(\mathbb{P}(1,2n)) \\ \downarrow & & \downarrow \\ CH_{\mathrm{SL}_2}(\mathbb{P}(1,2s) \times \mathbb{P}(1,2n-4s)) & \longrightarrow & CH_{\mathrm{SL}_2}(\mathbb{P}(1,2n)) \end{array}$$

where the three horizontal arrows are respectively the pushforward along the morphisms π'_{2s}, π_{2s} and $\pi^{\mathrm{SL}_2}_{2s}$.

Recall that the kernel of the two last vertical maps is (c_3) and that, from [EF09], we already know that $\mathrm{im}(\pi^{\mathrm{SL}_2}_{2s*})$ is contained in $\mathrm{im}(\pi^{\mathrm{SL}_2}_{1*})$. This implies that there exists a cycle ξ such that $\pi'_{2s*}(h^i_s \cdot h^j_{n-2s}) + c_3 \cdot \xi$ is contained in $\mathrm{im}(\pi'_{1*})$.

Observe that this last ideal is contained in (2) and, by lemma 2.4.8, so is $\pi'_{2s*}(h^i_s \cdot h^j_{n-2s})$. From this we deduce that $c_3\xi$ is contained in (c_3) and (2), but $(c_3) \cap (2) = (0)$, thus $c_3 \cdot \xi = 0$ and consequently $\pi'_{2s*}(h^i_s \cdot h^j_{n-2s})$ is contained in $\mathrm{im}(\pi'_{1*})$. □

So far we have proved that the ideal $\mathrm{im}(i_*)$ is equal to the sum of the ideal $\mathrm{im}(\pi'_{1*})$ and the ideal generated by the elements $\pi'_{2s*}(h^{2s}_s \cdot h^j_{n-2s})$ for $s = 1, ..., n/2$ and $j = 0, ..., 2n-4s$. We are in position to prove the main result of this subsection.

Proof of proposition 2.4.5. The key observation is that $\pi'^*_{2s}h_n = 2h_s + h_{n-2s}$. This implies the following chain of equalities:

$$\begin{aligned}\pi'_{2s*}(h_s^{2s}\cdot h_{n-2s}^{j}) &= \pi'_{2s*}(h_s^{2s}\cdot h_{n-2s}^{j-1}\cdot(h_{n-2s}+2h_s-2h_s))\\ &= \pi'_{2s*}(h_s^{2s}\cdot h_{n-2s}^{j-1}\cdot \pi'^*_{2s}h_n) - 2\pi'_{2s*}(h_s^{2s+1}\cdot h_{n-2s}^{j-1})\\ &= \pi'_{2s*}(h_s^{2s}\cdot h_{n-2s}^{j-1})\cdot h_n - 2\pi'_{2s*}((\alpha_0 h_s^{2s}+\ldots+\alpha_{2s})\cdot h_{n-2s}^{j-1})\end{aligned}$$

By lemma 2.4.7 we see that the cycle $\pi'_{2s*}(h_s^{2s}\cdot h_{n-2s}^{j})$ is in the ideal $\mathrm{im}(\pi'_{1*}) + (\pi'_{2s*}(h_s^{2s}\cdot h_{n-2s}^{j-1}))$. Iterating this argument, we see that for every $j>0$ the cycle $\pi'_{2s*}(h_s^{2s}\cdot h_{n-2s}^{j})$ is contained in the ideal $\mathrm{im}(\pi'_{1*}) + (\pi'_{2s*}(h_s^{2s}\cdot 1))$. Applying this to every s we conclude the proof of the lemma. □

2.4.3. Interlude: computations in the T-equivariant setting. Let again $T\subset \mathrm{GL}_3$ be the maximal subtorus of diagonal matrices. In this subsection the fact that we work with the T-equivariant Chow ring will be essential. For the sake of clarity, the morphisms between T-equivariant Chow rings will be denoted with a T in the apex. Moreover, we will keep using $\mathbb{Z}_{(2)}$-coefficients, so that every ring and ideal is assumed to be tensored over $\mathbb{Z}$ with $\mathbb{Z}_{(2)}$, where not explicitly written.

What we have found in the last subsection implies that

$$\mathrm{im}(i_*^T) = \mathrm{im}(\pi'^T_{1*}) + (\pi'^T_{2s*}(h_s^{2s}\cdot 1), \pi'^T_{n*}(h^n_{n/2})),\quad s=1,\ldots,n/2-1$$

Recall that we defined in the third section the T-invariant subvarieties $W_{n,r,l}$ of $\mathbb{P}(V_n)_3$ whose points are the pairs $(q, X_0^r X_1^l f)$ (see definition 2.2.4). We will prove the following result:

Proposition 2.4.10. *Inside $CH_T(\mathbb{P}(V_n)_3)$, we have*

$$\mathrm{im}(i_*^T) = (2h_n^2 - 2n(n-1)c_2, 4(n-2)h_n, [W_{n,2,0}]_T)$$

All what we need in order to prove the proposition above is the following lemma:

Lemma 2.4.11. *We have* $\mathrm{im}(i_*^T) = \mathrm{im}(\pi'^T_{1*}) + ([W_{n,2,2s-2}]_T)_{s=1,\ldots,n/2}$.

Proof. From lemma 2.2.7 we know that the cycle class $[W_{s,1,s-1}]_T$ contained in the T-equivariant Chow ring of $\mathbb{P}(V_s)_3$ is a monic polynomial in h_s of degree $2s$. We already know from lemma 2.4.7 that the cycles $\pi'^T_{2s*}(h_s^i\cdot 1)$, for $i<2s$, are in $\mathrm{im}(\pi'^T_{1*})$.

Combining this with our initial observation, we get

$$\mathrm{im}(\pi'^T_{1*}) + \pi'_{2s*}(h_s^{2s}\cdot 1) \subset \mathrm{im}(\pi'^T_{1*}) + (\pi'^T_{2s*}([W_{s,1,s-1}]_T\times 1))$$

because we have

$$\pi'^T_{2s*}(h_s^{2s}\cdot 1) = \pi'^T_{2s*}([W_{s,1,s-1}]_T\times 1) - \sum \xi_i \pi'^T_{2s*}(h_s^i\cdot 1)$$

for $i<2s$. The other inclusion is obvious because the ideal on the right is by construction contained in $\mathrm{im}(i_*^T)$, that coincides with the ideal on the left.

To finish the proof of the lemma, is enough to observe that, by lemma 2.2.9, we have $\pi'^T_{2s*}([W_{s,1,s-1}]_T\times 1) = [W_{n,2,2s-2}]_T$. □

Proof of proposition 2.4.10. From lemma 2.2.8 we see that the ideal

$$([W_{n,2,2s-2}]_T),\ s=1,\ldots,n/2$$

is actually generated by $[W_{n,2,0}]_T$.

Together with lemma 2.4.11 this implies that $\mathrm{im}(i_*^T) = \mathrm{im}(\pi'^T_{1*}) + ([W_{n,2,0}]_T)$. The fact that the inclusion $\mathrm{im}(\pi'^T_{1*})\subset \mathrm{im}(i_*^T)$ is strict follows from the fact that $\mathrm{im}(\pi'^T_{1*})\subset (2)$ whereas $[W_{n,2,0}]_T$ is not 2-divisible, as lemma 2.2.7 shows. □

2.4.4. Computations with $\mathbb{Z}_{(2)}$-coefficients, part two. We want to deduce from proposition 2.4.10 what are the generators of $\mathrm{im}(i_*)\otimes_{\mathbb{Z}}\mathbb{Z}_{(2)}$. Again, all the ideals and the Chow rings will be assumed to be tensorized over $\mathbb{Z}$ with $\mathbb{Z}_{(2)}$, where not explicitly stated.

Observe that the ideal $\mathrm{im}(i_*^T)$ is equal to the sum of ideals $\mathrm{im}(\pi_{r*}^{'T})$, where r ranges from 1 to n. In particular, proposition 2.4.5 remains true also in the T-equivariant setting, so that we actually know that

$$\mathrm{im}(i_*^T)=\mathrm{im}(\pi_{1*}^{'T})+(\pi_{2s*}^{'T}(h_s^{2s}\cdot 1))_{s=1,\dots,n/2}$$

and from lemma 2.4.7, which also remains true in the T-equivariant setting, we know that the cycles of the form $\pi_{2s*}^{'T}(h_s^i\cdot h_{n-2s}^j)$ are in $\mathrm{im}(\pi_{1*}^{'T})$ for $i=0,\dots,2s-1$ and $j=0,\dots,4n-2s$.

We proved in proposition 2.4.10 that $\mathrm{im}(i_*^T)$ is equal to $\mathrm{im}(\pi_{1*}^{'T})$ plus the ideal generated by the cycle class $[W_{n,2,0}]_T=\pi_{2*}^{'T}([W_{1,1,0}]_T\times 1)$. The equality

$$[W_{1,1,0}]_T=(h_1-\lambda_2)(h_1-\lambda_3)=h_1^2-(\lambda_2+\lambda_3)h_1+\lambda_2\lambda_3$$

is immediate to check, using the fact that $\mathbb{P}(V_1)_3=\mathbb{P}(2,1)\times\mathbb{A}(2,2)_3$ and the usual formula for cycle classes of complete intersections (lemma 1.1.7)

Putting all together, we readily deduce that

$$\mathrm{im}(i_*^T)=\mathrm{im}(\pi_{1*}^{'T})+([W_{n,2,0}]_T)\subset\mathrm{im}(\pi_{1*}^{'T})+(\pi_{2*}^{'T}(h_1^2\cdot 1))\subset\mathrm{im}(i_*^T)$$

which implies the following result:

Corollary 2.4.12. *We have* $\mathrm{im}(i_*^T)=\mathrm{im}(\pi_{1*}^{'T})+(\pi_{2*}^{'T}(h_1^2\cdot 1))$

The corollary above gives explicit generators for the ideal $\mathrm{im}(i_*^T)$ that are also symmetric in the λ_i, hence using proposition 1.1.3 we deduce proposition 2.4.2, which was stated at the beginning of the section.

2.5. The Chow ring of $\mathcal{H}_g$: end of the computation

In this section we finish the computation of $CH(\mathcal{H}_g)$. Recall that in corollary 2.4.3 we proved that

$$\mathrm{im}(i_*)=(2h_n^2-2n(n-1)c_2,4(n-2)h_n,\pi'_{2*}(h_1^2\times[\mathbb{P}(V_{n-2})]))$$

In order to obtain the relations inside the Chow ring of $\mathcal{H}_g$, we need to pull back the generators of the ideal above along the $\mathbb{G}_m$-torsor $p:V_{n,3}\setminus\sigma_0\to\mathbb{P}(V_n)_3$, where σ_0 denotes the image of the zero section $\mathbb{A}(2,2)_3\to V_n$.

We have already computed some of these relations in the third section (see corollary 2.3.12). Let us call I the ideal appearing in that corollary, that is

$$I=(4(2g+1)\tau,8\tau^2-2g(g+1)c_2,2c_3)$$

By construction, it coincides with the pullback of the ideal $\mathrm{im}(\pi'_{1*})$. Unfortunately, the ideal $\mathrm{im}(i_*)$ is not equal to $\mathrm{im}(\pi'_{1*})$ inside $CH_{\mathrm{GL}_3}(\mathbb{P}(V_n)_3)$, so we can't conclude that I is the whole ideal of relations, though this claim may still be true, because when pulling back the only generator of $\mathrm{im}(i_*)$ not in $\mathrm{im}(\pi'_{1*})$ we may obtain a cycle contained in I.

Moreover, the fact that we do not know an explicit expression for the last generator, namely $\pi'_{2*}(h_1^2\times[\mathbb{P}(V_{n-2})_3])$, prevents us from finishing the computation in a direct way.

Recall that in the previous section we also deduced that

$$\mathrm{im}(i_*^T)=\mathrm{im}(\pi_{1*}^{'T})+([W_{n,2,0}]_T)$$

inside the T-equivariant Chow ring of $\mathbb{P}(V_n)_3$, and observe that the relations inside the Chow ring of $\mathcal{H}_g$ are exactly the pullback of the elements of $\mathrm{im}(i_*^T)$, symmetric in λ_i, along the $\mathbb{G}_m$-torsor $p:V_{n,3}\setminus\sigma_0\to\mathbb{P}(V_n)_3$.

If ξ is in $\mathrm{im}(i_*)$, seeing it as an element of the T-equivariant Chow ring through the embedding $CH_{\mathrm{GL}_3}(\mathbb{P}(V_n)_3) \hookrightarrow CH_T(\mathbb{P}(V_n)_3)$ (see remark 1.1.5), then we have $\xi = \alpha \cdot \pi'_{1*}\xi + \beta \cdot [W_{n,2,0}]_T$.

LEMMA 2.5.1. *Let $\xi = \alpha_0 h_n^{2n} + \alpha_1 h_n^{2n-1} + ... + \alpha_{2n}$ be a cycle in $\mathrm{im}(i_*)$, considered as an ideal in the GL_3-equivariant Chow ring of $\mathbb{P}(V_n)_3$, and suppose that α_{2n} is 2-divisible. Then $p^*\xi$ is in I.*

The proof of the lemma is postponed to the end of the section. Write ξ as a polynomial in h_n of degree less or equal to $2n$. If we prove that ξ, written in this form and evaluated in $h_n = 0$, is 2-divisible, then by lemma 2.5.1 we can conclude that $p^*\xi$ must be in I, thus I is the whole ideal of relations.

We already know that every element in the image of π'_{1*} is 2-divisible, so that we only need to check that $\beta \cdot [W_{n,2,0}]_T$, seen as a polynomial in h_n of degree less than or equal to $2n$ and evaluated in $h_n = 0$, is 2-divisible.

Clearly, it is enough to prove this claim when $\beta = h_n^d$, where $d = 0, ..., 2n$. For matters of clarity, let us work with $\mathbb{Z}/2\mathbb{Z}$-coefficients, so that what we need to prove is that $\beta \cdot [W_{n,2,0}]_T$, seen as a polynomial in h_n of degree less or equal to $2n$ and evaluated in $h_n = 0$, is equal to 0.

Write $[W_{n,2,0}]_T$ as $h_n^4 + \eta_1 \cdot h_n^3 + ... + \eta_4$. Observe that, with these coefficients, we have $h_n^{2n+1} = 0$. This implies that the product $h_n^d \cdot [W_{n,2,0}]_T$ is equal to

$$\chi_4 h_n^{d+4} + \chi_3 \eta_1 h_n^{d+3} + ... + \chi_0 \eta_4 h_n^d$$

where $\chi_j = 0$ for $j + d > 2n$, and equal to 1 otherwise. In particular, for $d > 0$ the evaluation of this polynomial in $h_n = 0$ is zero. In other terms, we have shown that $h_n^d \cdot [W_{n,2,0}]_T$, written as a polynomial in h_n of degree less or equal to $2n$ and evaluated in $h_n = 0$, is 2-divisible for $d > 0$.

Now we only need to prove that $[W_{n,2,0}]_T$ itself has this property. Recall from lemma 2.2.7 that

$$[W_{n,2,0}]_T = \prod (h_n - \underline{k} \cdot \underline{\lambda}) + 2\lambda_3\xi \text{ for } \underline{k} \text{ s.t. } |\underline{k}| = n,\ k_2 < 2,\ k_0 < 2$$

The 2-divisibility of $[W_{n,2,0}]_T$ when evaluated in $h_n = 0$ is then equivalent to studying the 2-divisibility of the product $\prod(-\underline{k} \cdot \underline{\lambda})$, where $|\underline{k}| = n$, $k_2 < 2$, $k_0 < 2$.

Observe that there are only four triples $\underline{k}$ that verify the conditions above, namely $(0, n, 0)$, $(0, n-1, 1)$, $(1, n-1, 0)$ and $(1, n-2, 1)$. This implies that the product above is a multiple of $n\lambda_2$, thus it is 2-divisible because $n = g + 1$ is even.

To conclude the proof of theorem 2.5.2 we only need to show the technical lemma 2.5.1.

PROOF OF LEMMA 2.5.1. We can assume that ξ is not in $\mathrm{im}(\pi'_{1*})$, otherwise the conclusion is obvious. Moreover, we can also assume that ξ is in $\mathrm{im}(\pi'_{2s*})$, because we have proved in the last section that $\mathrm{im}(\pi'_{2s+1*})$ is contained in $\mathrm{im}(\pi'_{1*})$.

Consider again the commutative diagram

$$\begin{array}{ccc} CH_{\mathrm{GL}_3}(\mathbb{P}(V_s)_3 \times_{\mathbb{A}(2,2)_3} \mathbb{P}(V_{n-2s})_3) & \longrightarrow & CH_{\mathrm{GL}_3}(\mathbb{P}(V_n)_3) \\ \downarrow \wr\!\| & & \downarrow \wr\!\| \\ CH_{\mathrm{PGL}_2}(\mathbb{P}(1,2s) \times \mathbb{P}(1,2n-4s)) & \longrightarrow & CH_{\mathrm{PGL}_2}(\mathbb{P}(1,2n)) \\ \downarrow & & \downarrow \\ CH_{\mathrm{SL}_2}(\mathbb{P}(1,2s) \times \mathbb{P}(1,2n-4s)) & \longrightarrow & CH_{\mathrm{SL}_2}(\mathbb{P}(1,2n)) \end{array}$$

Then it must be true that $\xi = \pi'_{1*}\zeta + c_3 \cdot \eta$: we know from [EF09] that the image of the last horizontal map is contained in the image of $\pi_{1*}^{\mathrm{SL}_2}$, and that the kernel of the last two vertical maps, which are surjective, is generated as an ideal by c_3.

Let us briefly comment on the surjectivity of these maps: first recall that $\mathbb{P}(1,2m)$ is the projectivization of a PGL_2-representation, hence $CH_{\mathrm{PGL}_2}(\mathbb{P}(1,2m))$ is generated by the hyperplane section and the Chern classes c_2 and c_3. Similarly, the ring $CH_{\mathrm{PGL}_2}(\mathbb{P}(1,2m)\times\mathbb{P}(1,2m'))$ is generated by the hyperplane sections of the two factors plus c_2 and c_3.

Moreover, it is well known that CH_{SL_2} is generated by c_2: therefore the SL_2-equivariant Chow ring of $\mathbb{P}(1,2m)\times\mathbb{P}(1,2m')$ is generated by c_2 and the hyperplane sections, and the surjectivity of the two last vertical maps in the diagram above follows, because all the projective spaces that we considered parametrise forms of even degree.

Observe that we can assume that η, seen as a polynomial in h_n, has only odd coefficients: indeed, if we write $\eta=\eta'+2\eta''$ then

$$c_3\cdot\eta=c_3\cdot\eta'+2c_3\cdot\eta''=c_3\cdot\eta'$$

We deduce then that η must be equal to $h_n\cdot\gamma$, because by hypothesis when we evaluate ξ in $h_n=0$ we must obtain something even, and η has only odd coefficients. In the end, we have that $\xi=\pi'_{1*}\zeta+h_n\cdot c_3\cdot\gamma$. We now pull back ξ to $V_{n,3}\setminus\sigma_0$, which we saw to be equivalent to substituting h_n with -2τ, so we get:

$$p^*\xi=p^*\pi'_{1*}\zeta-2\tau\cdot c_3\cdot p^*\gamma=p^*\pi'_{1*}\zeta$$

where in the last equality we used the relation $2c_3=0$. This concludes the proof of the lemma. □

Putting all together, we have finally proved the

THEOREM 2.5.2. *Let g be an odd integer ≥ 3. Suppose that the characteristic of the base field k_0 is 0 or $>2g+2$, and let $\mathcal{H}_g$ be the moduli stack over k_0 of hyperlliptic curves of genus g. Then we have:*

$$CH(\mathcal{H}_g)=\mathbb{Z}[\tau,c_2,c_3]/(4(2g+1)\tau,8\tau^2-2g(g+1)c_2,2c_3)$$

where the degree of τ is 1, the degree of c_2 is 2 and the degree of c_3 is 3.

We want to give a geometrical interpretation of the generators of $CH(\mathcal{H}_g)$. Recall that in order to do the computations of the last three sections we used the isomorphism $[U'/\mathrm{GL}_3\times\mathbb{G}_m]$ obtained in theorem 2.1.1, where U' is the open subscheme of V_{g+1} whose points are pairs (q,f) such that the plane curves $\{q=0\}$ and $\{f=0\}$ intersect transversely.

We also showed, at the end of section 2.1, that the rank 4 vector bundle over $\mathcal{H}_g$ associated to the $\mathrm{GL}_3\times\mathbb{G}_m$-torsor U' is the vector bundle $\mathcal{E}\oplus\mathcal{L}$, where $\mathcal{L}$ is the line bundle over $\mathcal{H}_g$ functorially defined as

$$\mathcal{L}((\pi:C\to S,\iota))=\pi_*\omega_{C/S}^{\otimes\frac{g+1}{2}}\left(\frac{1-g}{2}W\right)$$

and $\mathcal{E}$ is the rank 3 vector bundle over $\mathcal{H}_g$ functorially defined as

$$\mathcal{E}((\pi:C\to S,\iota))=\pi_*\omega_{C/S}^{\vee}(W)$$

Then by construction the generator τ coincides with $c_1(\mathcal{L})$ and c_2 and c_3 coincide respectively with $c_2(\mathcal{E})$ and $c_3(\mathcal{E})$, whereas $c_1(\mathcal{E})=0$. This analysis agrees with the one made in the last section of [FV11].

CHAPTER 3

Cohomological invariants of the stack of hyperelliptic curves of odd genus

In this chapter we deal with the computation of the graded $\mathbb{F}_2$-vector space structure of the cohomological invariants with coefficients in $\mathbb{F}_2$ of $\mathcal{H}_g$, the stack of hyperelliptic curves of odd genus, when $g > 3$ is an odd number. As already said in the introduction, this is the only case left open, because for g even, $g = 3$ or $p \neq 2$ this type of result can be found in [**Pir17**].

The main obstruction to generalize the computation of $\mathrm{Inv}^\bullet(\mathcal{H}_3)$ done by Pirisi in [**Pir18b**] consists in proving that a certain morphism of PGL_2-equivariant Chow groups with coefficients is zero.

More precisely, let $\mathbb{P}(1, 2n)$ denote the projective space of binary forms of degree $2n$, endowed with the GL_2-action

$$A \cdot f(x, y) := \det(A)^n f(A^{-1}(x, y))$$

This action descends to a well defined action of PGL_2 on the same scheme. Let $\Delta_{1,2n} \subset \mathbb{P}(1, 2n)$ be the closed, PGL_2-invariant subscheme parametrising singular forms. Then to extend the results of Pirisi on $\mathrm{Inv}^\bullet(\mathcal{H}_3)$ is enough to prove the following:

KEY LEMMA. *Let k_0 be an algebraically closed field of characteristic $\neq 2$, and let $i : \Delta_{1,2n} \hookrightarrow \mathbb{P}(1, 2n)$ be the inclusion of the subscheme of singular forms into the projective space of binary forms of degree $2n$ over k_0. Then the pushforward homomorphism:*

$$i_* : A^0_{\mathrm{PGL}_2}(\Delta_{1,2n}, \mathrm{H}^\bullet) \longrightarrow A^1_{\mathrm{PGL}_2}(\mathbb{P}(1, 2n), \mathrm{H}^\bullet)$$

between equivariant Chow groups with coefficients in $\mathrm{H}^\bullet := \oplus_i H^i_{\text{ét}}(-, \mu_2^{\otimes i})$ vanishes.

The notion of GL_3-counterpart of a PGL_2-scheme will play a central role in the proof of the key lemma. The main result is theorem 3.1.1.

3.1. Cohomological invariants of $\mathcal{H}_g$

In this section, we fix $p = 2$, so that $\mathrm{H}^\bullet(-) = \oplus_{d\geq 0} H^d_{\text{èt}}(-, \mu_2^{\otimes d})$. We assume the base field k_0 to be algebraically closed and of characteristic $\neq 2$. We will also adopt the shorter notation $A_G(-)$ instead of $A_G(-, \mathrm{H}^\bullet)$ to denote Chow groups with coefficients.

In this section we will prove our main theorem, which is the following:

THEOREM 3.1.1. *Let k_0 be an algebraically closed field of characteristic $\neq 2, 3$, and let $\mathcal{H}_g$ denotes the moduli stack of smooth hyperelliptic curves of odd genus $g \geq 3$ over k_0.*

Then the graded-commutative ring of cohomological invariants $\mathrm{Inv}^\bullet(\mathcal{H}_g)$ with coefficients in $\mathbb{F}_2$, regarded as a graded $\mathbb{F}_2$-vector space, has a basis given by the elements

$$1, x_1, w_2, x_2, ..., x_{g+1}, x_{g+2}$$

where the degree of each x_i is i and w_2 is the second Stiefel-Whitney class coming from $\mathrm{Inv}^\bullet(\mathcal{B}\mathrm{PGL}_2)$*.*

The case $g = 3$ is [**Pir18b**, th. 0.1]. Actually, the only obstruction to generalize the result contained there to any odd genus is given by [**Pir18b**, corollary 3.9], where the assumption $g = 3$ is strictly necessary. Once one generalizes that corollary, the computation of the cohomological invariants is basically done.

Therefore, what we present here is essentially a rewriting of the proof contained in [**Pir18b**, sec. 3]: the only difference is in the key lemma 3.1.3, which is a generalization of [**Pir18b**, corollary 3.9]. The proof of this key result, which is rather non-trivial, is postponed to section 3.4, as we need to develop more theory in order to complete it.

3.1.1. Setup. Let $\mathbb{A}(1, n)$ be the affine space of binary forms of degree n and let X_n be the open subscheme parametrising forms with distinct roots. Recall [**AV04**, cor. 4.7], which tells us the following:

$$\mathcal{H}_g \simeq [X_{2g+2}/\mathrm{PGL}_2 \times \mathbb{G}_m]$$

The action of $\mathrm{PGL}_2 \times \mathbb{G}_m$ on X_{2g+2} descends from the action of $\mathrm{GL}_2 \times \mathbb{G}_m$ defined by the formula:

$$(A, \lambda) \cdot f(x, y) := \lambda^{-2} \det(A)^{g+1} f(A^{-1}(x, y))$$

This presentation and theorem 1.2.1 imply that:

$$\mathrm{Inv}^\bullet(\mathcal{H}_g) \simeq A^0_{\mathrm{PGL}_2 \times \mathbb{G}_m}(X_{2g+2}) \tag{4}$$

Therefore, to obtain the cohomological invariants of $\mathcal{H}_g$ is enough to compute the codimension 0 part of an equivariant Chow ring with coefficients.

Let $\mathbb{P}(1, 2n)$ be the projective space of binary forms of degree $2n$, and denote $\Delta_{1,2n}$ the divisor in $\mathbb{P}(1, 2n)$ parametrising singular forms. We are first going to compute $A^0_{\mathrm{PGL}_2}(\mathbb{P}(1, 2g + 2) \setminus \Delta_{1,2g+2})$, and then we will use the fact that

$$X_{2g+2} \longrightarrow \mathbb{P}(1, 2g + 2) \setminus \Delta_{1,2g+2}$$

is a $\mathrm{PGL}_2 \times \mathbb{G}_m$-equivariant $\mathbb{G}_m$-torsor to get a presentation of $A^0_{\mathrm{PGL}_2 \times \mathbb{G}_m}(X_{2g+2})$.

3.1.2. Proof of the main theorem. As already stated at the beginning of this section, we will always be assuming $p = 2$. The proposition below is the starting point to determine $\mathrm{Inv}^\bullet(\mathcal{H}_g)$.

PROPOSITION 3.1.2. [**Pir18b**, cor. 3.10] *The graded-ring $A^0_{\mathrm{PGL}_2}(\mathbb{P}(1, 2n) \setminus \Delta_{1,2n})$ is freely generated as $\mathbb{F}_2$-module by $n + 2$ elements $1, x_1, ..., x_n, w_2$ where the degree of x_i is i and w_2 is the second Stiefel-Whitney class coming from the cohomological invariants of $\mathcal{B}\mathrm{PGL}_2$.*

The proof of this proposition is by induction on n. To set up the induction argument, we need the following technical lemma, which is of fundamental importance:

KEY LEMMA 3.1.3. *For $n \geq 1$ the boundary morphism*

$$\partial : A^0_{\mathrm{PGL}_2}(\mathbb{P}(1, 2n)) \longrightarrow A^0_{\mathrm{PGL}_2}(\Delta_{1,2n})$$

is surjective.

The key lemma above is proved by Pirisi only for $n \leq 8$ (see [**Pir18b**, cor. 3.9]): he shows that, for every $n \geq 1$ there exists an element g_{2n} in $A_{\mathrm{PGL}_2}(\mathbb{P}(1, 2n))$ such that, for every α in $A^0_{\mathrm{PGL}_2}(\Delta_{1,2n})$, we have $g_{2n} \cdot i_*\alpha = 0$ (what here is called $\mathbb{P}(1, 2n)$ is denoted P^{2n} by Pirisi).

From this, he deduces that if $i_*\alpha \neq 0$ then $g_{2n} \cdot c_1^G(\mathcal{O}(1))$ is divided by another element f_{2n} of $A_{\mathrm{PGL}_2}(\mathbb{P}(1,2n))$, and this cannot be the case for $2n \leq 8$ but can be true for $2n > 8$: the last two assertions follow from the explicit construction of f_{2n} and g_{2n} contained in [**Pir18b**, lm. 3.7, pr. 3.8].

In this way, Pirisi proves that $i_* = 0$ for $2n \leq 8$, which implies the key lemma 3.1.3 for $n \leq 4$. This allows him to prove proposition 3.1.2 only for this values of n, and to compute $\mathrm{Inv}^\bullet(\mathcal{H}_g)$ only for $g = 3$.

In section 3.4 we will prove key lemma 3.1.3 for every $n \geq 1$, adopting a completely different strategy. For the remainder of the section, we assume key lemma 3.1.3.

We will also need the following results:

Lemma 3.1.4. *We have:*

(1) for $n \geq 2$, there is an isomorphism

$$A^0_{\mathrm{PGL}_2}(\Delta_{1,2n}) \simeq A^0_{\mathrm{PGL}_2}((\mathbb{P}(1,2n-2) \setminus \Delta_{1,2n-2}) \times \mathbb{P}^1)$$

(2) for $n \geq 1$, the pullback morphism

$$A^0_{\mathrm{PGL}_2}(\mathbb{P}(1,2n) \setminus \Delta_{1,2n}) \to A^0_{\mathrm{PGL}_2}((\mathbb{P}(1,2n) \setminus \Delta_{1,2n}) \times \mathbb{P}^1)$$

is surjective with kernel generated by w_2, the Stiefel-Whitney class coming from the cohomological invariants of PGL_2*.*

Proof. Before proceeding with the proof, let us warn the reader that what is called $\mathbb{P}(1,2n)$ here is denoted P^{2n} in [**Pir18b**].

From [**Pir18b**, pr. 2.2] we know that

$$A^0_{\mathrm{PGL}_2}(\Delta_{1,2n} \setminus \Delta_{2,2n}) \simeq A^0_{\mathrm{PGL}_2}((\mathbb{P}(1,2n-2) \setminus \Delta_{1,2n-2}) \times \mathbb{P}^1)$$

where $\Delta_{2,2n}$ is the closed subscheme of $\mathbb{P}(1,2n)$ parametrising forms which have two double roots. To prove (1), it is then enough to show that $A^0_{\mathrm{PGL}_2}(\Delta_{1,2n}) \simeq A^0_{\mathrm{PGL}_2}(\Delta_{1,2n} \setminus \Delta_{2,2n})$.

The key lemma 3.1.3 together with [**Pir18b**, cor. 3.5] imply that the condition $S_1(2n)$ (resp. $S_2(2n)$) of [**Pir18b**, pg. 14] hold true for every $n \geq 1$ (resp. for every $n \geq 2$).

Condition $S_1(2n)$ is exactly (2), and condition $S_2(2n)$ says that the pullback morphism $A^0_{\mathrm{PGL}_2}(\Delta_{1,2n}) \to A^0_{\mathrm{PGL}_2}(\Delta_{1,2n} \setminus \Delta_{2,2n})$ is an isomorphism, from which we deduce (1). □

Proof of prop. 3.1.2. The key lemma 3.1.3, applied to the localization exact sequence (see proposition 1.2.3.(4)) associated to the closed subscheme $\Delta_{1,2n}$, gives us the following short exact sequence of $\mathbb{F}_2$-vector spaces:

$$0 \to A^0_{\mathrm{PGL}_2}(\mathbb{P}(1,2n)) \to A^0_{\mathrm{PGL}_2}(\mathbb{P}(1,2n) \setminus \Delta_{1,2n}) \xrightarrow{\partial} A^0_{\mathrm{PGL}_2}(\Delta_{1,2n}) \to 0$$

From this we deduce:

$$A^0_{\mathrm{PGL}_2}(\mathbb{P}(1,2n) \setminus \Delta_{1,2n}) \simeq A^0_{\mathrm{PGL}_2}(\mathbb{P}(1,2n)) \oplus A^0_{\mathrm{PGL}_2}(\Delta_{1,2n})[1]$$

where the notation $A_{\mathrm{PGL}_2}(\Delta_{1,2n})[1]$ means that everything is degree-shifted by one.

The projective space $\mathbb{P}(1,2n)$ is the projectivization of a PGL_2-representation, hence using the projective bundle formula (proposition 1.2.3.(7)) we deduce

$$A^0_{\mathrm{PGL}_2}(\mathbb{P}(1,2n)) \simeq A^0_{\mathrm{PGL}_2}$$

and the $\mathbb{F}_2$-vector space on the right is known (see proposition 1.2.4.(2)).

To compute $A^0_{\mathrm{PGL}_2}(\Delta_{1,2n})$, we proceed by induction on n. To prove the base case $n = 1$, observe that $\Delta_{1,2} \simeq \mathbb{P}^1$, and $A^0_{\mathrm{PGL}_2}(\mathbb{P}^1)$ is trivial by proposition 1.2.4.(3).

Lemma 3.1.4.(1) tells us that:

$$A^0_{\mathrm{PGL}_2}(\Delta_{1,2n}) \simeq A^0_{\mathrm{PGL}_2}((\mathbb{P}(1,2n-2)\setminus\Delta_{1,2n-2})\times\mathbb{P}^1)$$

and lemma 3.1.4.(2) says that the $\mathbb{F}_2$-vector space on the right is isomorphic to $A^0_{\mathrm{PGL}_2}(\mathbb{P}(1,2n-2)\setminus\Delta_{1,2n-2})/\mathbb{F}_2\cdot w_2$. Using the inductive hypothesis, we get the desired result. □

As announced at the beginning of the section, we will use proposition 3.1.2 to compute $A^0_{\mathrm{PGL}_2\times\mathbb{G}_m}(X_{2n})$. We will use the following result:

LEMMA 3.1.5. [**Pir18b**, proposition 2.3] *Let Y be a scheme endowed with an action of* PGL_2*, and let $\mathbb{G}_m$ act trivially on it. Then*

$$A_{\mathrm{PGL}_2\times\mathbb{G}_m}(Y)\simeq A_{\mathrm{PGL}_2}(Y)[t]$$

where t has codimension 1 *and degree* 0*. In particular, we get an isomorphism on the codimension zero part.*

We observed that $X_{2n}\to\mathbb{P}(1,2n)\setminus\Delta_{1,2n}$ is a $\mathrm{PGL}_2\times\mathbb{G}_m$-equivariant $\mathbb{G}_m$-torsor. Let L be the line bundle associated to X_{2n}.

LEMMA 3.1.6. *The codimension zero part of* $\ker(c_1^{\mathrm{PGL}_2\times\mathbb{G}_m}(L))$ *is generated as an $\mathbb{F}_2$-vector space by a single element x_n of degree n.*

PROOF. The proof of [**Pir18b**, th. 3.12], once we know the key lemma 3.1.3 for every $n\geq 1$, works for all $n\geq 1$. □

We now have all the elements necessary to prove the main result of the paper.

PROOF OF THEOREM 3.1.1. We know from (4) that

$$\mathrm{Inv}^\bullet(\mathcal{H}_g)\simeq A^0_{\mathrm{PGL}_2\times\mathbb{G}_m}(X_{2g+2})$$

Applying proposition 1.2.6 to the $\mathbb{G}_m$-torsor $X_{2g+2}\to\mathbb{P}(1,2g+2)\setminus\Delta_{1,2g+2}$, we get:

$$A^0_{\mathrm{PGL}_2\times\mathbb{G}_m}(X_{2g+2})\simeq A^0_{\mathrm{PGL}_2\times\mathbb{G}_m}(\mathbb{P}(1,2g+2)\setminus\Delta_{1,2g+2})\oplus\ker(c_1^{\mathrm{PGL}_2\times\mathbb{G}_m}(L))[1]$$

where L is the line bundle on $\mathbb{P}(1,2g+2)\setminus\Delta_{1,2g+2}$ associated to the $\mathbb{G}_m$-torsor X_{2g+2}, and we consider only the codimension zero part of $\ker(c_1^{\mathrm{PGL}_2\times\mathbb{G}_m}(L))$.

The first summand is isomorphic to $A^0_{\mathrm{PGL}_2}(\mathbb{P}(1,2g+2)\setminus\Delta_{1,2g+2})$ by lemma 3.1.5, and this $\mathbb{F}_2$-vector space can be computed using proposition 3.1.2 with $n=g+1$.

The second summand is also known, thanks to lemma 3.1.6. Putting all together, we get the desired conclusion. □

3.2. The geometry of the fundamental divisor

In this section we will study the geometry of D_2, the restriction of the fundamental divisor D (see definition 1.3.14) to $\mathbb{P}(V_n)_2$, and we will show that D_2 has two irreducible components (proposition 3.2.5).

We will also consider the proper transform $D'_{[2,1]}$ of $D_{[2,1]}$ inside a certain variety $\widetilde{\mathbb{P}}(V_n)_{[2,1]}$ equipped with a birational morphism to $\mathbb{P}(V_n)_{[2,1]}$ (see definition 3.2.11), and we will show that the restriction of $D'_{[2,1]}$ to the exceptional locus of this morphism has two irreducible components (proposition 3.2.14).

3.2.1. The geometry of D_2. Recall definition 1.3.10, where we introduced the vector bundles $\overline{V}_n$ and V_n defined respectively over $\mathbb{P}(2,2)$ and over $\mathbb{A}(2,2)_{[3,1]}$, the scheme of non-zero quadratic forms in three variables. The points of $\mathbb{P}(\overline{V}_n)$ can be thought as pairs $(q,[f])$, where q is the projective equivalence class of a non-zero quadratic ternary form and $[f]$ is the equivalence class of a non-zero ternary form of degree n (see remark 1.3.11).

Let $\overline{D}_2$ denote the pullback of $\overline{D} \to \mathbb{P}(2,2)$ to the subscheme of rank 2 conics $\mathbb{P}(2,2)_2$: then it follows from the last lines of remark 1.3.15 that the points of $\overline{D}_2$ are pairs $(q,[f])$ where $q = l_1 l_2$ is a quadratic form of rank 2 (well defined up to multiplication by a non-zero scalar) and the subscheme defined by the homogeneous ideal $I = (l_1 l_2, f)$ of $\mathbb{P}^2$ is either singular or has an irreducible component of dimension 1 (this happens when one of the linear factors of q divides f).

Definition 3.2.1.

(1) We define the subset $\overline{D}_2^1 \subset \mathbb{P}(\overline{V}_n)_2$ as the set of points $(q,[f])$ such that $q = l_1 l_2$, where l_1 and l_2 are two distinct linear forms, and the closed subscheme of $\mathbb{P}^2$ defined by the ideal $I = (l_1, l_2, f)$ is non-empty.

(2) We define the subset D_2^1 as the preimage of $\overline{D}_2^1$ along the $\mathbb{G}_m$-torsor $\mathbb{P}(V_n)_2 \to \mathbb{P}(\overline{V}_n)_2$.

Thanks to the next proposition, we can make the subsets $\overline{D}_2^1$ and D_2^1 into closed subschemes.

Proposition 3.2.2. *There is a scheme structure on $\overline{D}_2^1$ (resp. on D_2^1) which turns it into an irreducible divisor inside $\mathbb{P}(\overline{V}_n)_2$ (resp. $\mathbb{P}(V_n)_2$). Moreover, we have that $\overline{D}_2^1$ (resp. D_2^1) is an irreducible component of $\overline{D}_2$ (resp. D_2).*

Proof. It is enough to show that $\overline{D}_2^1$ is an irreducible divisor inside $\mathbb{P}(\overline{V}_n)_2$.

Let $\mathcal{Y}^1$ be the subscheme of $\mathbb{P}(2,1) \times \mathbb{P}(2,1) \times \mathbb{P}(2,n) \times \mathbb{P}^2$ defined as follows:

$$\mathcal{Y}^1 = \{(l_1, l_2, f, p) \text{ such that } l_1(p) = l_2(p) = f(p) = 0\}$$

Observe that $\mathcal{Y}^1 \to \mathbb{P}^2$ is a projective bundle over $\mathbb{P}^2$, hence it is irreducible, and its fibres have codimension 3 in $\mathbb{P}(2,1) \times \mathbb{P}(2,1) \times \mathbb{P}(2,n)$.

Call Y^1 the image of the projection morphism $\mathrm{pr}_{123} : \mathcal{Y}^1 \to \mathbb{P}(2,1) \times \mathbb{P}(2,1) \times \mathbb{P}(2,n)$, where by image we mean the smallest closed subscheme of $\mathbb{P}(2,1) \times \mathbb{P}(2,1) \times \mathbb{P}(2,n)$ through which pr_{123} factors (see [**Sta19**, Tag 01R7]).

The morphism $\mathcal{Y}^1 \to Y^1$ is generically finite, hence Y^1 is closed, irreducible and it has codimension 1. Moreover, due to the properness of pr_{123}, the points of Y^1 correspond to triples (l_1, l_2, f) such that the closed subscheme of $\mathbb{P}^2$ defined by the ideal $I = (l_1, l_2, f)$ is not empty.

Consider the projective morphism:

$$\pi : \mathbb{P}(2,1) \times \mathbb{P}(2,1) \times \mathbb{P}(2,n) \longrightarrow \mathbb{P}(2,2) \times \mathbb{P}(2,n)$$

which sends a triple (l_1, l_2, f) to $(l_1 l_2, f)$. Observe that the image of π is $\mathbb{P}(2,2)_{[2,1]} \times \mathbb{P}(2,n)$, and that π is finite of degree 2 over its image. This implies that $\pi(Y^1)$, regarded as a subscheme inside $\mathbb{P}(2,2)_{[2,1]} \times \mathbb{P}(2,n)$, is closed, irreducible, it has codimension 1 and its points correspond to pairs $(l_1 l_2, f)$ such that the closed subscheme of $\mathbb{P}^2$ defined by the ideal $I = (l_1, l_2, f)$ is not empty.

Recall that $\overline{V}_n$ is a quotient of the vector bundle associated to the locally free sheaf $\mathrm{pr}_{1*}\mathrm{pr}_2^*\mathcal{O}(n)$ by the subsheaf $\mathrm{pr}_{1*}\mathrm{pr}_2^*\mathcal{O}(n-2) \otimes \mathcal{O}(-1)$ (see definition 1.3.10.(1)). Applying proposition 2.3.1, we obtain a morphism:

$$(\mathbb{P}(2,2) \times \mathbb{P}(2,n)) \setminus \mathrm{im}(i) \longrightarrow \mathbb{P}(\overline{V}_n)$$

where $i : \mathbb{P}(2,2)\times\mathbb{P}(2,n-2) \hookrightarrow \mathbb{P}(2,2)\times\mathbb{P}(2,n)$ is the inclusion of (trivial) projective bundles which sends a pair (q,f) to (q,qf). The quotient morphism sends a pair (q,f) to $(q,[f])$, where $[f]$ denotes the equivalence class of f in the quotient.

This morphism restricts to the morphism:

$$p : (\mathbb{P}(2,2)_{[2,1]} \times \mathbb{P}(2,n)) \setminus \mathrm{im}(i)_{[2,1]} \longrightarrow \mathbb{P}(\overline{V}_n)_{[2,1]}$$

We can restrict $\pi(Y^1)$ to $(\mathbb{P}(2,2)_{[2,1]} \times \mathbb{P}(2,n)) \setminus \mathrm{im}(i)_{[2,1]}$ and take its image via p, which we denote Z^1. By construction, we have that Z^1 is closed and irreducible. Moreover, by [**Sta19**, Tag 01R8.(4)], it contains $\overline{D}_2^1$ as an open, dense subset.

Observe that p is a topological quotient: this implies that $\overline{D}_2^1$ is closed because its preimage, which is the restriction of $\pi(Y^1)$ to $(\mathbb{P}(2,2)_{[2,1]} \times \mathbb{P}(2,n)) \setminus \mathrm{im}(i)_{[2,1]}$, is closed. We deduce that $\overline{D}_2^1 = Z^1$, hence that $\overline{D}_2^1$ is irreducible because Z^1 is so.

Again by proposition 2.3.1, the morphism p makes $(\mathbb{P}(2,2)_{[2,1]} \times \mathbb{P}(2,n)) \setminus \mathrm{im}(i)_{[2,1]}$ into a vector bundle over $\mathbb{P}(\overline{V}_n)_{[2,1]}$, whose restriction over $\overline{D}_2^1$ is $\pi(Y^1)$, which has codimension 1 in $(\mathbb{P}(2,2)_{[2,1]} \times \mathbb{P}(2,n)) \setminus \mathrm{im}(i)_{[2,1]}$: from this we deduce that $\overline{D}_2^1$ has codimension 1.

Finally, we have to show that $\overline{D}_2^1 \subset \overline{D}_2$: this follows from the fact that if $(l_1l_2,[f])$ is a point of $\overline{D}_2^1$, then the subscheme of $\mathbb{P}^2$ defined by the ideal $I = (l_1l_2, f)$ either contains a line or it has a singular point p, which is the unique point such that $l_1(p) = l_2(p) = 0$. □

DEFINITION 3.2.3.

(1) We define the subset $\overline{D}_2^2 \subset \mathbb{P}(\overline{V}_n)_2$ as the set of points $(q,[f])$ such that $q = l_1l_2$, where l_1 and l_2 are two linerarly independent linear forms, and there is an i such that the two plane curves defined respectively by the equations $l_i = 0$ and $f = 0$ do not intersect transversally. This is equivalent to the condition that the subscheme of $\mathbb{P}^2$ defined by the homogeneous ideal $I = (l_i, f)$ is either singular or a line.

(2) We define the subscheme D_2^2 as the preimage of $\overline{D}_2^2$ along the $\mathbb{G}_m$-torsor $\mathbb{P}(V_n)_2 \to \mathbb{P}(\overline{V}_n)_2$.

PROPOSITION 3.2.4. *There is a scheme structure on $\overline{D}_2^2$ (resp. on D_2^2) which turns it into an irreducible divisor inside $\mathbb{P}(\overline{V}_n)_2$ (resp. $\mathbb{P}(V_n)_2$). Moreover, we have that $\overline{D}_2^2$ (resp. D_2^2) is an irreducible component of $\overline{D}_2$ (resp. D_2).*

PROOF. It is enough to prove the proposition for $\overline{D}_2^2$.

We will argue as in the proof of proposition 3.2.2. Let $\mathcal{Y}^2$ be the subscheme of $\mathbb{P}(2,1) \times \mathbb{P}(2,n) \times \mathbb{P}^2$ defined as follows:

$$\mathcal{Y}^2 := \left\{(l,f,p) \text{ such that } l(p) = f(p) = \det_i J(l,f)(p) = 0 \text{ for } i = 1,2,3\right\}$$

where $\det_i J(l,f)$ denotes the i^{th}-minor of the Jacobian matrix associated to the forms l and f.

Zariski-locally, to verify if a triple (l,f,p) is in $\mathcal{Y}^2$, it is enough to check the vanishing of only one of the three minors of $J(l,f)$: for instance, if U_z denotes the open subscheme of $\mathbb{P}^2$ where the coordinate $z \neq 0$, then $\mathcal{Y}^2 \cap \mathbb{P}(2,1) \times \mathbb{P}(2,n) \times U_z$ is the locus where:

$$l(p) = f(p) = l_x(p)f_y(p) - l_y(p)f_x(p) = 0$$

Let $H \subset \mathbb{P}(2,1)\times\mathbb{P}^2$ be the universal line: then $\mathcal{Y}^2$ can be regarded as a codimension 2 projective subbundle of $\mathbb{P}(2,n) \times H$ over H, hence $\mathcal{Y}^2$ is irreducible.

Denote Y^2 the image of the projection of $\mathcal{Y}^2$ on $\mathbb{P}(2,1) \times \mathbb{P}(2,n)$. Due to the fact that $\mathcal{Y}^2 \to Y^2$ is generically finite, we obtain that Y^2 is irreducible, it has codimension 1 and its points are pairs of forms (l, f) such that the associated plane curves do not intersect transversally.

Consider the projective morphism:

$$\pi : \mathbb{P}(2,1) \times \mathbb{P}(2,1) \times \mathbb{P}(2,n) \longrightarrow \mathbb{P}(2,2) \times \mathbb{P}(2,n)$$

which sends a triple (l_1, l_2, f) to $(l_1 l_2, f)$. The scheme theoretic image $\pi(\mathbb{P}(2,1) \times Y^2)$ will be closed, irreducible, of codimension 1 and its points will be pairs $(l_1 l_2, f)$ such that there exists an $i \in \{1,2\}$ such that the plane curves defined respectively by the equations $l_i = 0$ and $f = 0$ do not intersect transversally.

From here the proof works exactly as the proof of proposition 3.2.2: one has only to substitute $\pi(Y^1)$ with $\pi(\mathbb{P}(2,1) \times Y^2)$ and $\overline{D}_2^1$ with $\overline{D}_2^2$. □

We are ready to prove the first main result of this section:

Proposition 3.2.5. *The closed subscheme $\overline{D}_2$ (resp. D_2) of $\mathbb{P}(\overline{V}_n)_2$ (resp. $\mathbb{P}(V_n)_2$) has two irreducible components, both* GL_3*-invariants and of codimension* 1*, which are $\overline{D}_2^1$ and $\overline{D}_2^2$ (resp. D_2^1 and D_2^2).*

Proof. We prove the proposition only for $\overline{D}_2$, as the other case easily follows from this one.

We know from propositions 3.2.2 and 3.2.4 that $\overline{D}_2^1$ and $\overline{D}_2^2$ are both irreducible, of codimension 1 and are contained in $\overline{D}_2$. The GL_3-invariance of both divisors is easy to check.

We have to show that $\overline{D}_2^1$ and $\overline{D}_2^2$ are the only irreducible components of $\overline{D}_2$. The points of $\overline{D}_2$ are pairs $(l_1 l_2, [f])$ such that $l_1 \neq l_2$ and the closed subscheme Z of $\mathbb{P}^2$ associated to the ideal $I = (l_1 l_2, f)$ is either singular or it contains a line (see remark 1.3.15). If Z contains a line, than the point $(q, [f])$ is in $\overline{D}_2^1 \cap \overline{D}_2^2$.

Otherwise, the subscheme Z must be supported on a set of points, and it must be singular: we can either have that the support of every singular point lays on only one of the two lines defined respectively by the equations $l_1 = 0$ and $l_2 = 0$, or that there exists a singular point whose support lays on the intersection of the two lines.

In the first case, there must exist a line, say the one defined by the equation $l_1 = 0$, such that its intersection with the plane curve F defined by the equation $f = 0$ is somewhere non-transversal, thus the point $(l_1 l_2, [f])$ is in $\overline{D}_2^2$.

In the second case, the only possibility left is that the plane curve F contains the intersection point of the two lines: this implies that $(l_1 l_2, [f])$ is in $\overline{D}_2^1$. □

Finally, we have the following simple result on D_1:

Proposition 3.2.6. *We have $\overline{D}_1 = \mathbb{P}(\overline{V}_n)_1$ and $D_1 = \mathbb{P}(V_n)_1$.*

Proof. All the points in $\mathbb{P}(\overline{V}_n)_1$ are of the form (l^2, f), hence their associated subschemes of $\mathbb{P}^2$ are all singular, and by construction the points in $\overline{D}_1$ are those pairs (l^2, f) such that the associated subschemes are either singular or contain a line (see remark 1.3.15).

The same argument works also for D_1. □

3.2.2. The geometry of D_1'. Recall (definition 1.3.9) that $Q \to \mathbb{P}(2,2)$ is the universal conic and $\widehat{Q} \to \mathbb{A}(2,2)_{[3,1]}$ is the pullback of Q along the $\mathbb{G}_m$-torsor $\mathbb{A}(2,2)_{[3,1]} \to \mathbb{P}(2,2)$.

Definition 3.2.7. We define $Q^{\mathrm{sing}}_{[2,1]}$ (resp. $\widehat{Q}^{\mathrm{sing}}_{[2,1]}$) as the closed subscheme of $Q_{[2,1]}$ (resp. $\widehat{Q}_{[2,1]}$, see definition 1.3.9) whose associated sheaf of ideals is the

1^{th}-Fitting ideal of the sheaf of relative differentials of

$$\Omega^1_{Q_{[2,1]}/\mathbb{P}(2,2)_{[2,1]}} \text{ (resp. } \Omega^1_{\widehat{Q}_{[2,1]}/\mathbb{A}(2,2)_{[2,1]}})$$

Observe that both $Q^{\text{sing}}_{[2,1]}$ and $\widehat{Q}^{\text{sing}}_{[2,1]}$ inherit a GL_3-action.

REMARK 3.2.8.

(1) We have:

$$Q^{\text{sing}}_{[2,1]} = \{(q,p) \text{ such that } q_x(p) = q_y(p) = q_z(p) = 0\} \subset \mathbb{P}(2,2)\times\mathbb{P}^2$$

If the characteristic of the base field is $\neq 2$, we deduce that $Q^{\text{sing}}_{[2,1]} \to \mathbb{P}^2$ is a projective bundle: actually, it is a projective subbundle of $\mathbb{P}(2,2)\times\mathbb{P}^2$.

(2) By [**Sta19**, Tag 0C3I] we have that $\widehat{Q}^{\text{sing}}_{[2,1]}$ is the pullback of $Q^{\text{sing}}_{[2,1]}$ along the $\mathbb{G}_m$-torsor $\mathbb{A}(2,2)_{[2,1]} \to \mathbb{P}(2,2)_{[2,1]}$.

We also know from [**Sta19**, Tag 0C3K] that $Q^{\text{sing}}_{[2,1]}$ (resp. $\widehat{Q}^{\text{sing}}_{[2,1]}$) is the subscheme of $Q_{[2,1]}$ (resp. $\widehat{Q}^{\text{sing}}_{[2,1]}$) where the projection onto $\mathbb{P}(2,2)_{[2,1]}$ (resp $\mathbb{A}(2,2)_{[2,1]}$) is not smooth.

PROPOSITION 3.2.9. *Suppose that the characteristic of the base field is $\neq 2$. Then the morphism $Q^{\text{sing}}_{[2,1]} \to \mathbb{P}(2,2)_{[2,1]}$ is an isomorphism over $\mathbb{P}(2,2)_2$. Moreover, the restriction $Q^{\text{sing}}_1 \to \mathbb{P}(2,2)_1$ is a projective bundle. Similarly, we have that $\widehat{Q}^{\text{sing}}_{[2,1]} \to \mathbb{A}(2,2)_{[2,1]}$ is an isomorphism over $\mathbb{A}(2,2)_2$ and the restriction $\widehat{Q}^{\text{sing}}_1 \to \mathbb{A}(2,2)_1$ is a projective bundle.*

PROOF. The restriction $Q^{\text{sing}}_2 \to \mathbb{P}(2,2)_2$ has a section, which sends a point $(l_1 l_2)$ in $\mathbb{P}(2,2)_2$ to the pair $(l_1 l_2, p)$ where p is the unique point in $\mathbb{P}^2$ such that $l_1(p) = l_2(p) = 0$. This proves that $Q^{\text{sing}}_2 \to \mathbb{P}(2,2)_2$ is actually an isomorphism.

The restriction Q^{sing}_1 can be described by the following equations:

$$Q^{\text{sing}}_1 = \left\{(l^2,p) \text{ such that } 2l(p)l_x(p) = 2l(p)l_y(p) = 2l(p)l_z(p) = 0\right\} \subset \mathbb{P}(2,2)_1\times\mathbb{P}^2$$

At least one of the partial derivatives of l, evaluated in p, is invertible, hence the subscheme can be equivalently describes as:

$$Q^{\text{sing}}_1 = \left\{(l^2,p) \text{ such that } l(p) = 0\right\}$$

This proves that $Q^{\text{sing}}_1 \to \mathbb{P}(2,2)_1$ is a projective subbundle of $\mathbb{P}(2,2)_1\times\mathbb{P}^2$.

The assertions on $\widehat{Q}^{\text{sing}}_{[2,1]}$ and $\widehat{Q}^{\text{sing}}_1$ easily follow from what we have just proved, because we have the cartesian diagrams:

$$\begin{array}{ccc} \widehat{Q}^{\text{sing}}_{[2,1]} & \longrightarrow & Q^{\text{sing}}_{[2,1]} \\ \downarrow & & \downarrow \\ \mathbb{A}(2,2)_{[2,1]} & \longrightarrow & \mathbb{P}(2,2)_{[2,1]} \end{array} \qquad \begin{array}{ccc} \widehat{Q}^{\text{sing}}_1 & \longrightarrow & Q^{\text{sing}}_1 \\ \downarrow & & \downarrow \\ \mathbb{A}(2,2)_1 & \longrightarrow & \mathbb{P}(2,2)_1 \end{array}$$

This concludes the proof. □

We are ready to give the main definitions of this subsection.

DEFINITION 3.2.10.

(1) We define $\widetilde{\mathbb{P}}(\overline{V}_n)_{[2,1]}$ as the pullback of $\mathbb{P}(\overline{V}_n)_{[2,1]}$ along the morphism $Q^{\text{sing}}_{[2,1]} \to \mathbb{P}(2,2)_{[2,1]}$. Similarly, we define $\widetilde{\mathbb{P}}(V_n)_{[2,1]}$ as the pullback of $\mathbb{P}(V_n)_{[2,1]}$ along the morphism $\widehat{Q}^{\text{sing}}_{[2,1]} \to \mathbb{A}(2,2)_{[2,1]}$.

(2) We define $\widetilde{\mathbb{P}}(\overline{V}_n)_1$ as the restriction of $\widetilde{\mathbb{P}}(\overline{V}_n)_{[2,1]}$ over $\mathbb{P}(2,2)_1$. Similarly we define $\widetilde{\mathbb{P}}(V_n)_1$ as the restriction of $\widetilde{\mathbb{P}}(V_n)_{[2,1]}$ to $\mathbb{A}(2,2)_1$.

DEFINITION 3.2.11.

(1) We define $\overline{D}'_{[2,1]}$ (resp. $D'_{[2,1]}$) as the closure of the preimage of $\overline{D}_2$ (resp. D_2) inside $\widetilde{\mathbb{P}}(\overline{V}_n)_{[2,1]}$ (resp. $\widetilde{\mathbb{P}}(V_n)_{[2,1]}$).

(2) We define $\overline{D}'_1$ (resp. D'_1) as the pullback of $\overline{D}'_{[2,1]}$ (resp. $D'_{[2,1]}$) to $\widetilde{\mathbb{P}}(\overline{V}_n)_1$ (resp. $\widetilde{\mathbb{P}}(V_n)_1$).

All the objects defined above inherit a GL_3-action or are GL_3-equivariant subschemes.

LEMMA 3.2.12. *The restriction $\widetilde{\mathbb{P}}(\overline{V}_n)_2$ of $\widetilde{\mathbb{P}}(\overline{V}_n)_{[2,1]}$ over Q_2^{sing} is isomorphic to $\mathbb{P}(\overline{V}_n)_2$.*

PROOF. Follows directly from proposition 3.2.9. □

Let $\overline{D}_2^1$ and $\overline{D}_2^2$ (resp. D_2^1 and D_2^2) be as in the definitions 3.2.1 and 3.2.3. Briefly, $\overline{D}_2^1$ is the subscheme of $\mathbb{P}(\overline{V}_n)_2$ whose points are pairs $(q,[f])$ with $q = l_1l_2$ a quadratic ternary form of rank 2 and f a ternary form of degree n such that there exists a point p in $\mathbb{P}^2$ with $l_1(p) = l_2(p) = f(p) = 0$.

On the other hand, the points of $\overline{D}_2^2$ are the pairs $(q,[f])$ with $q = l_1l_2$ and such that the plane curves $l_i = 0$ and $f = 0$ do not intersect transversely, for some $i \in \{1,2\}$.

DEFINITION 3.2.13.

(1) We define $\overline{D'}^i_{[2,1]}$ as the closure inside $\widetilde{\mathbb{P}}(\overline{V}_n)_{[2,1]}$ of the preimage of $\overline{D}_2^i$ for $i = 1,2$. We also define $\overline{D'}^i_1$ as the restriction of $\overline{D'}^i_{[2,1]}$ to $\widetilde{\mathbb{P}}(\overline{V}_n)_1$.

(2) We define $D'^i_{[2,1]}$ as the closure inside $\widetilde{\mathbb{P}}(V_n)_{[2,1]}$ of the preimage of D_2^i for $i = 1,2$. We also define D'^i_1 as the restriction of $D'^i_{[2,1]}$ to $\widetilde{\mathbb{P}}(V_n)_1$.

PROPOSITION 3.2.14. *We have that $\overline{D'}_1$ (resp. D'_1) is a codimension 1, GL_3-invariant closed subscheme of $\widetilde{\mathbb{P}}(\overline{V}_n)_1$ (resp. $\widetilde{\mathbb{P}}(V_n)_1$) with two GL_3-invariant irreducible components of codimension 1, which are $\overline{D'}^1_1$ and $\overline{D'}^2_1$ (resp. D'^1_1 and D'^2_1).*

PROOF. It is enough to prove the proposition for $\overline{D'}_1$. By proposition 3.2.5 we know that $\overline{D}_2^1$ and $\overline{D}_2^2$ are the only two irreducible components of $\overline{D}_2$: due to the fact that $\widetilde{\mathbb{P}}(\overline{V}_n)_{[2,1]} \to \mathbb{P}(\overline{V}_n)_{[2,1]}$ is an isomorphism over $\mathbb{P}(\overline{V}_n)_2$, we deduce that $\overline{D'}^i_{[2,1]}$, where $i = 1,2$ (see definition 3.2.13.(1)), are the only two irreducible components of $\overline{D'}_{[2,1]}$.

This implies that $\overline{D'}_1$ is the union of the two closed subschemes $\overline{D'}^1_1$ and $\overline{D'}^2_1$. We have to show that these two closed subschemes are distinct and have codimension 1: both these properties will be easy to verify once we know what are the points of $\overline{D'}^1_1$ and $\overline{D'}^2_1$.

Recall from definition 3.2.1 that the points of $\overline{D}_2^1$ are pairs $(l_1l_2,[f])$ with $l_1 \neq l_2$ linear forms and such that there exists a point p in $\mathbb{P}^2$ with $l_1(p) = l_2(p) = f(p) = 0$. Therefore, the points of the preimage $\overline{D'}^1_2$ of $\overline{D}_2^1$ inside $\widetilde{\mathbb{P}}(\overline{V}_n)_2$ (see lemma 3.2.12) are equal to triples $(l_1l_2,[f],p)$ with $l_1(p) = l_2(p) = f(p) = 0$.

Let R be a DVR over k_0 with quotient field K, and take a morphism $\mathrm{Spec}(R) \to \overline{D'}^1_{[2,1]}$ such that the induced map from $\mathrm{Spec}(K)$ factors through $\overline{D'}^1_2$ and the closed point lands in $\widetilde{\mathbb{P}}(\overline{V}_n)_1$: then the image of this point will be of the form $(l^2,[f],p)$ with $l(p) = f(p) = 0$, and every such point can be obtained in this way. This gives us a description of the points of $\overline{D'}^1_1$

On the other hand, the points in $\overline{D'}^2_2$ will be triples $(l_1l_2,[f],p)$ with $l_1(p)=l_2(p)=0$ and such that the plane curve F defined by the equation $f=0$ in $\mathbb{P}^2$ intersects non-transversely one of the two lines (see definition 3.2.3). Given a morphism $\mathrm{Spec}(R)\to\overline{D'}^2_{[2,1]}$ whose generic point lands in $\overline{D'}^2_2$, and whose closed point is sent to a point in $\widetilde{\mathbb{P}}(\overline{V}_n)_1$, such a point will be of the form $(l^2,[f],p)$ with $l(p)=0$ and the plane curve F intersecting non-transversely the line $l=0$. This gives us a description of the points of $\overline{D'}^2_1$.

Putting all together, we obtain that $\overline{D'}^1_1\neq\overline{D'}^2_1$.

Finally, propositions 3.2.2 and 3.2.4 tell us that both $\overline{D}^1_2$ and $\overline{D}^2_2$ have codimension 1 in $\mathbb{P}(\overline{V}_n)_2$, thus $\overline{D'}^1_{[2,1]}$ and $\overline{D'}^2_{[2,1]}$ have codimension 1 in $\widetilde{\mathbb{P}}(\overline{V}_n)_{[2,1]}$ by lemma 3.2.12. When restricting a subvariety to another, the codimension can only decrease: therefore, both $\overline{D'}^1_1$ and $\overline{D'}^2_1$ must have codimension 1 in $\widetilde{\mathbb{P}}(\overline{V}_n)_1$, as they do not coincide with the whole $\widetilde{\mathbb{P}}(\overline{V}_n)_1$. □

REMARK 3.2.15. From the proof of proposition 3.2.14 we deduce an explicit description of the points of D'^1_1 and D'^2_1. More precisely, a triple $(l^2,[f],p)$ is in D'^1_1 if and only if $l(p)=f(p)=0$, that is if the plane curves defined by the equations $l=0$ and $f=0$ intersect in p (this condition does not depend on the representative of the equivalence class $[f]$).

On the other hand, the points of D'^2_1 are the triples such that $l(p)=0$ and the plane curves $l=0$ and $f=0$ do not intersect transversally.

3.3. Some equivariant intersection theory

The main goal of this section is to compute the cycle classes of some schemes that we introduced in section 3.2, namely D^i_2 (see definitions 3.2.1 and 3.2.3) and D'^i_1 (see definitions 3.2.13) for $i=1,2$. These results will be used to give a proof of the key lemma 3.1.3 in section 3.4.

3.3.1. Cycle classes of D^i_2. From now on, we will always assume the characteristic of the base field to be $\neq 2$.

We will write $CH_{\mathrm{GL}_3}(X)_{\mathbb{F}_2}$ for $CH_{\mathrm{GL}_3}(X)\otimes\mathbb{F}_2$ and we will denote $\mathbb{A}(2,d)$ the affine space of forms in three variables of degree d.

As before, the notation $\mathbb{A}(2,2)_r$ will stand for the scheme of quadratic forms in three variables of rank r and $\mathbb{A}(2,2)_{[a,b]}$ for the scheme of quadratic forms of rank r with $a\geq r\geq b$. A similar notation will be adopted for the subschemes of $\mathbb{P}(2,2)$.

Recall that GL_3 acts on $\mathbb{A}(2,2)_3$ via the formula:

$$A\cdot q(x,y,z):=\det(A)q(A^{-1}(x,y,z))$$

and that $\mathbb{A}(2,2)_3$ is a GL_3-equivariant $\mathbb{G}_m$-torsor over $\mathbb{P}(2,2)_3$.

Recall that in definition 3.2.1 we introduced the divisor $D^1_2\subset\mathbb{P}(V_n)_2$ whose points correspond to pairs $(q,[f])$ where $q=l_1l_2$ is a rank 2 quadric and the subscheme of $\mathbb{P}^2$ associated to the homogeneous ideal $I=(l_1,l_2,f)$ is non-empty. In proposition 3.2.2 we proved that D^1_2 is irreducible.

Let $D^1_{[3,2]}$ be the embedding of D^1_2 inside $\mathbb{P}(V_n)_{[3,2]}$.

LEMMA 3.3.1. *We have* $[D^1_{[3,2]}]=c_1h\neq 0$ *in* $CH^2_{\mathrm{GL}_3}(\mathbb{P}(V_n)_{[3,2]})_{\mathbb{F}_2}$.

PROOF. Consider the GL_3-invariant, closed subscheme of $\mathbb{P}(2,2)\times\mathbb{P}(2,n)\times\mathbb{P}^2$ defined as:

$$\overline{\mathcal{Z}}^1:=\{(q,f,u)\text{ such that }q_x(u)=q_y(u)=q_z(u)=f(u)=0\}$$

Let $\overline{Z}^1$ be its image via the projection on $\mathbb{P}(2,2)\times\mathbb{P}(2,n)$.

By proposition 2.3.2.(2) applied to the morphism $\mathbb{P}(2,2)_{[3,2]} \hookrightarrow \mathbb{P}(2,2)$, we get an isomorphism:

$$\Phi : CH^2_{\mathrm{GL}_3}(\mathbb{P}(V_n)_{[3,2]}) \simeq CH^2_{\mathrm{GL}_3}(\mathbb{P}(2,2)_{[3,2]} \times \mathbb{P}(2,n))/(s - c_1)$$

where $s = c_1^{\mathrm{GL}_3}(\mathcal{O}_{\mathbb{P}(2,2)_{[3,2]}}(1))$.

Moreover, by proposition 2.3.2.(3), we have that the cycle class of the restriction of $\overline{Z}^1$ to $\mathbb{P}(2,2)_{[3,2]} \times \mathbb{P}(2,n)$ is equal to $\Phi[D^1_{[3,2]}]$, thus to prove the lemma is enough to compute $[\overline{Z}^1]$ inside $CH^2_{\mathrm{GL}_3}(\mathbb{P}(2,2)_{[3,2]} \times \mathbb{P}(2,n))_{\mathbb{F}_2}$ and substitute s with c_1.

Observe that $\mathrm{pr}_{12*}[\overline{\mathcal{Z}}^1] = [\overline{Z}^1]$ because pr_{12} restricted to $\overline{\mathcal{Z}}_1$ is an isomorphism.

We compute now the cycle class of $\overline{\mathcal{Z}}^1$ inside the T-equivariant Chow ring of $\mathbb{P}(2,2) \times \mathbb{P}(2,n) \times \mathbb{P}^2$, which is isomorphic to

$$\mathbb{Z}[\lambda_1, \lambda_2, \lambda_3, s, h, t]/(t^3 + t^2\sigma_1(\lambda_i) + t\sigma_2(\lambda_i) + \sigma_3(\lambda_i), R_s, R_h)$$

where $s = c_1^T(\mathcal{O}_{\mathbb{P}(2,2)}(1))$, $h = c_1^T(\mathcal{O}_{\mathbb{P}(2,n)}(1))$, $t = c_1^T(\mathcal{O}_{\mathbb{P}^2}(1))$, the λ_i come from $CH_T(\mathrm{Spec}(k_0))$ (see proposition 1.1.4.(1)), σ_i is the elementary symmetric polynomial of degree i in three variables and R_s and R_h are monic polynomials respectively in s and h, with coefficients in $CH_{\mathrm{GL}_3}(\mathrm{Spec}(k_0))$. To obtain this isomorphism, one has to apply two times the projective bundle formula (proposition 1.1.1.(6)).

The scheme $\overline{\mathcal{Z}}^1$ is a complete intersection of four T-invariant hypersurfaces, which are

$$H_1 := \{q_x(p) = 0\},\ H_2 := \{q_y(p) = 0\}$$
$$H_3 := \{q_z(p) = 0\},\ H_4 := \{f(p) = 0\}$$

Thanks to lemma 1.1.7, we can compute the cycle class of each hypersurface and then we can use the fact that $[\overline{\mathcal{Z}}^1]_T = \prod [H_i]_T$. The final result is:

$$[\overline{\mathcal{Z}}^1]_T = (s + t - \lambda_1)(s + t - \lambda_2)(s + t - \lambda_3)(h + nt)$$

Using the relation $t^3 + t^2\sigma_1(\lambda_i) + t\sigma_2(\lambda_i) + \sigma_3(\lambda_i) = 0$ and after tensoring with $\mathbb{F}_2$, we get:

$$[\overline{\mathcal{Z}}^1]_T = t^2(s^2 + sh + s\sigma_1(\lambda_i)) + t\xi_1 + \xi_2 \qquad \text{for } n \text{ odd}$$
$$[\overline{\mathcal{Z}}^1]_T = t^2(hs) + t\xi_1 + \xi_2 \qquad \text{for } n \text{ even}$$

We have a cartesian diagram:

$$\begin{array}{ccc} \mathbb{P}(2,2) \times \mathbb{P}(2,n) \times \mathbb{P}^2 & \longrightarrow & \mathbb{P}^2 \\ \downarrow & & \downarrow \\ \mathbb{P}(2,2) \times \mathbb{P}(2,n) & \longrightarrow & \mathrm{Spec}(k_0) \end{array}$$

Hence, using the compatibility (proposition 1.1.1.(4)) and the projection formula (proposition 1.1.1.(8)), we deduce:

$$[\overline{Z}^1]_T = s^2 + sh + s\sigma_1(\lambda_i) \qquad \text{for } n \text{ odd}$$
$$[\overline{Z}^1]_T = hs \qquad \text{for } n \text{ even}$$

inside $CH^2_T(\mathbb{P}(2,2) \times \mathbb{P}(2,n))_{\mathbb{F}_2}$. We apply now proposition 1.1.3 to deduce that:

$$[\overline{Z}^1] = s^2 + sh + sc_1 \qquad \text{for } n \text{ odd}$$
$$[\overline{Z}^1] = hs \qquad \text{for } n \text{ even}$$

inside $CH^2_{\mathrm{GL}_3}(\mathbb{P}(2,2) \times \mathbb{P}(2,n))_{\mathbb{F}_2}$, because the Weyl group associated to the maximal subtorus $T \subset \mathrm{GL}_3$ is the symmetric group.

Finally, substituting s with c_1, we get $[\overline{Z}^1] = c_1 h \bmod 2$, hence $[D^1_{[3,2]}] = c_1 h$. This last element is not zero because by the projective bundle formula (proposition 1.1.1.(6)) we have that $CH^1_{\mathrm{GL}_3}(\mathbb{A}(2,2)_{[3,2]})_{\mathbb{F}_2} \cdot h = \mathbb{F}_2 \cdot c_1 h$ is a non-zero direct summand. ☐

Recall that we defined another divisor D^2_2 in $\mathbb{P}(V_n)_2$, whose points are the pairs $(q,[f])$ such that $q = l_1 l_2$ has rank 2 and the plane curves $l_i = 0$ and $f = 0$ do not intersect transversally in $\mathbb{P}^2$ for some i (see definition 3.2.3).

By proposition 3.2.4 we know that D^2_2 is an irreducible divisor in $\mathbb{P}(V_n)_2$. Let $D^2_{[3,2]}$ be its embedding in $\mathbb{P}(V_n)_{[3,2]}$.

LEMMA 3.3.2. *We have* $[D^2_2] = 0$ *in* $CH^1_{\mathrm{GL}_3}(\mathbb{P}(V_n)_2)_{\mathbb{F}_2}$ *and* $[D^2_{[3,2]}] = 0$ *in* $CH^2_{\mathrm{GL}_3}(\mathbb{P}(V_n)_{[3,2]})_{\mathbb{F}_2}$.

PROOF. Clearly, it is enough to show that $[D^2_2] = 0$ in $CH^1_{\mathrm{GL}_3}(\mathbb{P}(V_n)_2)_{\mathbb{F}_2}$.

Recall that in the proof of proposition 3.2.4 we proved that there exists a closed subscheme $Y^2 \subset \mathbb{P}(2,1) \times \mathbb{P}(2,n)$ whose points are pairs (l,f) such that the plane curves $l = 0$ and $f = 0$ do not intersect transversally. Consider the image of $\mathbb{P}(2,1) \times Y^2$ via the morphism

$$\pi : \mathbb{P}(2,1) \times \mathbb{P}(2,1) \times \mathbb{P}(2,n) \longrightarrow \mathbb{P}(2,2)_{[2,1]} \times \mathbb{P}(2,n)$$

which sends (l_1, l_2, f) to $(l_1 l_2, f)$, and let $\overline{Z}^2$ be the restriction of this image to $\mathbb{P}(2,2)_2 \times \mathbb{P}(2,n)$.

By proposition 2.3.2 applied to the morphism $f : \mathbb{P}(2,2)_2 \to \mathbb{P}(2,2)$, we get an isomorphism:

$$\Phi : CH^1_{\mathrm{GL}_3}(\mathbb{P}(V_n)_2) \simeq CH^1_{\mathrm{GL}_3}(\mathbb{P}(2,2)_2 \times \mathbb{P}(2,n))/(f^* s - c_1)$$

such that $\Phi[D^2_2] = [\overline{Z}^2]$ modulo $f^* s - c_1$. Hence, it is enough to show that $[\overline{Z}^2] = 0$ in $CH_{\mathrm{GL}_3}(\mathbb{P}(2,2)_2 \times \mathbb{P}(2,n))_{\mathbb{F}_2}$.

We claim that $[Y^2] = 0$ in $CH^1_{\mathrm{GL}_3}(\mathbb{P}(2,1) \times \mathbb{P}(2,n))_{\mathbb{F}_2}$. From this the proposition would easily follow, because the morphism $\mathbb{P}(2,1) \times Y^2 \to \overline{Z}^2$ is generically bijective.

To compute $[Y^2]$ we will first find an explicit expression for $[\mathcal{Y}^2]$, regarded as an element of $CH_{\mathrm{GL}_3}(\mathbb{P}(2,1) \times \mathbb{P}(2,n) \times \mathbb{P}^2)$, where:

$$\mathcal{Y}^2 := \left\{ (l,f,p) \text{ such that } l(p) = f(p) = \det_i J(l,f)(p) = 0 \right\}$$

Here $\det_i J(l,f)$ stands for the determinant of the minor of the Jacobian matrix $J(l,f)$ obtained by removing the i^{th} column.

This would enable us to compute $[Y^2]$ because $\mathrm{pr}_{12*}[\mathcal{Y}^2] = [Y^2]$.

Observe that the scheme $\mathcal{Y}^2$ is not a complete intersection but, if we restrict to the open subscheme of $\mathbb{P}^2$ where $p_2 \neq 0$, then we need exactly three equations to describe the restriction over this open subscheme of $\mathcal{Y}^2$, namely $l(p) = f(p) = \det_3 J(l,f)(p) = 0$.

Consider the T-invariant subscheme

$$\mathcal{Y}^2_1 := \left\{ (l,f,p) \text{ such that } l(p) = f(p) = \det_3 J(l,f)(p) = 0 \right\}$$

where T is the usual subtorus of GL_3 made of the diagonal matrices. Then we have that $\mathcal{Y}^2_1$ has two irreducible components, which are $\mathcal{Y}^2$ and the T-invariant subscheme

$$\mathcal{Y}^2_2 := \{ (l,f,p) \text{ such that } l(p) = f(p) = p_2 = 0 \}$$

From this we deduce that $[\mathcal{Y}^2]_T = [\mathcal{Y}^2_1]_T - [\mathcal{Y}^2_2]_T$ in $CH_T(\mathbb{P}(2,1) \times \mathbb{P}(2,n) \times \mathbb{P}^2)$.

Observe that $\mathcal{Y}_1^2$ is a complete intersection of three T-invariant hypersurfaces H_1, H_2 and H_3 and that $\mathcal{Y}_2^2$ is a complete intersections of three T-invariant hypersurfaces H_1, H_2 and H_4, where:

$$\begin{aligned} H_1 &= \{l(p) = 0\} \\ H_2 &= \{f(p) = 0\} \\ H_3 &= \left\{\det_3 J(l, f)(p) = 0\right\} \\ H_4 &= \{p_2 = 0\} \end{aligned}$$

Hence $[\mathcal{Y}_1^2]_T = [H_1]_T[H_2]_T[H_3]_T$ and $[\mathcal{Y}_2^2]_T = [H_1]_T[H_2]_T[H_4]_T$. We have that $CH_T(\mathbb{P}(2,1) \times \mathbb{P}(2,n) \times \mathbb{P}^2)$ is generated as a $CH_T(\mathrm{Spec}(k_0))$-algebra by the following elements:

$$\begin{aligned} v &= \mathrm{pr}_1^* c_1^T(\mathcal{O}_{\mathbb{P}(2,1)}(1)) \\ h &= \mathrm{pr}_2^* c_1^T(\mathcal{O}_{\mathbb{P}(2,n)}(1)) \\ t &= \mathrm{pr}_3^* c_1^T(\mathcal{O}_{\mathbb{P}^2}(1)) \end{aligned}$$

We can compute $[H_i]_T$ using the formula given in lemma 1.1.7. In the end we get:

$$\begin{aligned} [\mathcal{Y}_1^2]_T - [\mathcal{Y}_2^2]_T &= (s+t)(h+nt)(s+h+(n-1)t - \lambda_1 - \lambda_2) \\ &\quad - (s+t)(h+nt)(t+\lambda_3) = \\ &= (s+t)(h+nt)(s+h+(n-2)t - \sigma_1(\lambda_i)) \end{aligned}$$

Expanding the expression above, using the identity $t^3 + \sigma_1(\lambda_i)t^2 + \sigma_2(\lambda_i)t + \sigma_3(\lambda_i) = 0$ and after tensoring with $\mathbb{F}_2$, we obtain that

$$[\mathcal{Y}^2]_T = t\xi_1 + \xi_2$$

for some ξ_i in $CH_T(\mathbb{P}(2,1) \times \mathbb{P}(2,n))$. The compatibility property (see proposition 1.1.1.(4)) applied to the diagram

$$\begin{array}{ccc} \mathbb{P}(2,1) \times \mathbb{P}(2,n) \times \mathbb{P}^2 & \longrightarrow & \mathbb{P}^2 \\ \downarrow & & \downarrow \\ \mathbb{P}(2,1) \times \mathbb{P}(2,n) & \longrightarrow & \mathrm{Spec}(k_0) \end{array}$$

implies that $[Y^2]_T = \mathrm{pr}_{12*}[\mathcal{Y}^2]_T = 0$.

By injectivity of the morphism $CH_{\mathrm{GL}_3}(\mathbb{P}(2,1) \times \mathbb{P}(2,n)) \hookrightarrow CH_T(\mathbb{P}(2,1) \times \mathbb{P}(2,n))$ (see proposition 1.1.3), we conclude that $[Y^2] = 0$. □

3.3.2. Cycle classes of $D_1'^i$. Let $\widehat{Q} \to \mathbb{A}(2,2)_{[3,1]}$ be the pullback of the universal conic $Q \to \mathbb{P}(2,2)$, and let $\widehat{Q}^{\mathrm{sing}}$ be the closed subscheme of singular points, i.e. the scheme defined by 1^{th}-Fitting ideal of $\Omega^1_{\widehat{Q}/\mathbb{A}(2,2)_{[3,1]}}$.

Recall that $\widehat{Q}^{\mathrm{sing}}_{[2,1]} \to \mathbb{A}(2,2)_{[2,1]}$ is a birational morphism, which is an isomorphism over $\mathbb{A}(2,2)_2$ (see lemma 3.2.12). Moreover, the fibre $\widehat{Q}_1^{\mathrm{sing}}$ over $\mathbb{A}(2,2)_1$ is a projective subbundle of $\mathbb{A}(2,2)_1 \times \mathbb{P}^2$, defined by the equation:

$$\widehat{Q}_1^{\mathrm{sing}} = \left\{(l^2, p) \text{ such that } l(p) = 0\right\}$$

Observe that all these schemes inherits a GL_3-action, which comes from the diagonal action of GL_3 on the product $\mathbb{A}(2,2) \times \mathbb{P}^2$.

In what follows, the projective bundle $\widehat{Q}_1^{\mathrm{sing}}$ will be simply denoted E_1, because we think of it as an exceptional divisor.

LEMMA 3.3.3. *We have*

$$CH_{\mathrm{GL}_3}(E_1) \simeq \mathbb{Z}[c_1, c_2, c_3, t, v]/(c_1 - 2v, f_t, f_v)$$

where t is the restriction to E_1 of $c_1^{\mathrm{GL}_3}(\mathcal{O}_{\mathbb{P}^2}(1))$, f_t is a polynomial of degree 2 monic in t and f_v is a polynomial of degree 3 monic in v.

PROOF. We have that $E_1 \to \mathbb{A}(2,2)_1$ is a projective bundle, hence applying the projection bundle formula (proposition 1.1.1.(6)) we get:

$$CH_{\mathrm{GL}_3}(E_1) \simeq CH_{\mathrm{GL}_3}(\mathbb{A}(2,2)_1)[t]/(f_t)$$

where t is the restriction to $E_1 \subset \mathbb{A}(2,2)_1 \times \mathbb{P}^2$ of $c_1^{\mathrm{GL}_3}(\mathcal{O}_{\mathbb{P}^2}(1))$.

Observe that $\mathbb{A}(2,2)_1$ is the $\mathbb{G}_m$-torsor over $\mathbb{P}(2,2)_1$ associated to the equivariant line bundle $\mathcal{O}_{\mathbb{P}(2,2)}(-1)|_{\mathbb{P}(2,2)_1} \otimes \mathbb{D}$, where $\mathbb{D}$ is the determinant representation of GL_3. Applying proposition 1.1.8 we deduce:

$$CH_{\mathrm{GL}_3}(\mathbb{A}(2,2)_1) \simeq CH_{\mathrm{GL}_3}(\mathbb{P}(2,2)_1)/(c_1 - s)$$

where $s = c_1^{\mathrm{GL}_3}(\mathcal{O}_{\mathbb{P}(2,2)}(1)|_{\mathbb{P}(2,2)_1})$.

There is an equivariant isomorphism

$$\varphi : \mathbb{P}(2,1) \longrightarrow \mathbb{P}(2,2)_1, \quad l \longmapsto l^2$$

which induces an isomorphism at the level of Chow rings:

$$\varphi^* : CH_{\mathrm{GL}_3}(\mathbb{P}(2,2)_1) \simeq CH_{\mathrm{GL}_3}(\mathbb{P}(2,1)) \simeq \mathbb{Z}[c_1, c_2, c_3, v]/(f_v)$$

where $v = c_1^{\mathrm{GL}_3}(\mathcal{O}_{\mathbb{P}(2,1)}(1))$.

Observe that $\varphi^*(\mathcal{O}_{\mathbb{P}(2,2)}(-1)|_{\mathbb{P}(2,2)_1} \otimes \mathbb{D}) = \mathcal{O}_{\mathbb{P}(2,1)}(-2) \otimes \mathbb{D}$, hence:

$$CH_{\mathrm{GL}_3}(\mathbb{P}(2,2)_1)/(s - c_1) \simeq \mathbb{Z}[c_1, c_2, c_3, v]/(f_v, c_1 - 2v)$$

This concludes the proof of the lemma. □

Recall from definition 3.2.10 that $\widetilde{\mathbb{P}}(V_n)_{[2,1]}$ denotes the pullback of $\mathbb{P}(V_n)_{[2,1]} \to \mathbb{A}(2,2)_{[2,1]}$ along the morphism $\widehat{Q}^{\mathrm{sing}}_{[2,1]} \to \mathbb{A}(2,2)_{[2,1]}$, and that $\widetilde{\mathbb{P}}(V_n)_1$ is the restriction of $\widetilde{\mathbb{P}}(V_n)_{[2,1]}$ to $E_1 = \widehat{Q}^{\mathrm{sing}}_1$.

We also introduced in definition 3.2.11 the divisor $D'_{[2,1]}$, which is the closure of the preimage of D_2 in $\widetilde{\mathbb{P}}(V_n)_{[2,1]}$. We also defined D'_1 as the restriction of $D'_{[2,1]}$ to E_1.

By proposition 3.2.14, we know that the divisor D'_1 has two irreducible components, denoted D'^1_1 and D'^2_1: the points of D'^1_1 are the triples $(l^2, [f], p)$ such that $l(p) = f(p) = 0$. The points of D'^2_1 are the triples $(l^2, [f], p)$ such that $l(p) = 0$ and the plane curves $l = 0$ and $f = 0$ do not intersect transversally (see definition 3.2.13 and remark 3.2.15).

We want to compute the cycle classes of these two components.

LEMMA 3.3.4. *We have $[D'^1_1] = h + nt$ in $CH^1_{\mathrm{GL}_3}(\widetilde{\mathbb{P}}(V_n)_1)$, where h denotes the hyperplane section of the projective bundle $\widetilde{\mathbb{P}}(V_n)_1 \to E_1$ and t comes from the hyperplane section of $\mathbb{P}^2$.*

PROOF. Consider the cartesian diagram:

$$\begin{array}{ccc} E_1 & \longrightarrow & \overline{E}_1 \\ \big\downarrow & & \big\downarrow g \\ \mathbb{A}(2,2)_{[3,1]} & \longrightarrow & \mathbb{P}(2,2) \end{array}$$

where $\overline{E}_1 = Q_1^{\mathrm{sing}}$. We can apply proposition 2.3.2 to the diagram above: we obtain an isomorphism

$$\Phi : CH_{\mathrm{GL}_3}(\widetilde{\mathbb{P}}(V_n)_1) \simeq CH_{\mathrm{GL}_3}(\overline{E}_1 \times \mathbb{P}(2,n))/(g^*s - c_1)$$

Let $\overline{Z'}^1 \subset \overline{E}_1 \times \mathbb{P}(2,n)$ be the GL_3-invariant subvariety defined as:

$$\overline{Z'}^1 := \left\{ (l^2, f, p) \text{ such that } l(p) = f(p) = 0 \right\}$$

Then proposition 2.3.2 also implies that $\Phi[D_1'^1] = [\overline{Z'}^1]$ modulo the relation $g^*s - c_1 = 0$.

Consider the cartesian diagram

$$\begin{array}{ccc} \varphi^*\overline{E}_1 & \longrightarrow & \overline{E}_1 \\ \downarrow & & \downarrow \\ \mathbb{P}(2,1) & \xrightarrow{\varphi} & \mathbb{P}(2,2)_1 \end{array}$$

where φ sends $[l]$ to $[l^2]$. The horizontal arrows are then equivariant isomorphisms.

We have an induced isomorphism:

$$\begin{aligned} \varphi^* : CH_{\mathrm{GL}_3}(\overline{E}_1 \times \mathbb{P}(2,n))/(g^*s - c_1) &\simeq CH_{\mathrm{GL}_3}(\varphi^*\overline{E}_1 \times \mathbb{P}(2,n))/(\varphi^*g^*s - c_1) \\ &\simeq \mathbb{Z}[c_1, c_2, c_3, t, v, h]/(2v - c_1, f_t, f_v, f_h) \end{aligned}$$

where $h = c_1^{\mathrm{GL}_3}(\mathcal{O}_{\mathbb{P}(2,n)}(1))$ and the other generators are as in lemma 3.3.3. The isomorphism φ^* sends $[\overline{Z'}^1]$ to $[\varphi^{-1}\overline{Z'}^1]$. To compute the cycle class of $\varphi^{-1}\overline{Z'}^1$ we consider the subscheme of $\mathbb{P}(2,1) \times \mathbb{P}(2,n) \times \mathbb{P}^2$ defined as:

$$\mathcal{W}^1 = \{(l, f, p) \text{ such that } f(p) = 0\}$$

If $i : \varphi^*\overline{E}_1 \times \mathbb{P}(2,n) \hookrightarrow \mathbb{P}(2,1) \times \mathbb{P}(2,n) \times \mathbb{P}^2$ is the closed immersion, then we have $i^![\mathcal{W}^1] = [\overline{\mathcal{Z}'}^1]$ (see proposition 1.1.1.(9)). Thanks to lemma 1.1.7, we get $[\mathcal{W}^1] = h + nt$, hence:

$$[\overline{\mathcal{Z}'}^1] = i^![\mathcal{W}^1] = h + nt$$

Here there is a little and quite common abuse of notation, as we are denoting t both the pullback of $c_1^{\mathrm{GL}_3}(\mathcal{O}_{\mathbb{P}^2}(1))$ to $\mathbb{P}(2,1) \times \mathbb{P}^2$ and its restriction to $\varphi^*\overline{E}_1$. This concludes the proof. □

Observe that $D_1'^1$ can also be seen as a codimension 2 integral subscheme of $\widetilde{\mathbb{P}}(V_n)_{[2,1]}$.

LEMMA 3.3.5. *The class* $[D_1'^1]$ *is non-zero in* $CH^2_{\mathrm{GL}_3}(\widetilde{\mathbb{P}}(V_n)_{[2,1]})_{\mathbb{F}_2}$.

To prove the lemma above, we need a technical result.

LEMMA 3.3.6. *The cycle class* $[E_1]$ *is not zero in* $CH^1_{\mathrm{GL}_3}(\widehat{Q}^{\mathrm{sing}}_{[2,1]})_{\mathbb{F}_2}$.

PROOF. From remark 3.2.8.(2) we know that there is a cartesian diagram:

$$\begin{array}{ccc} \widehat{Q}^{\mathrm{sing}}_{[2,1]} & \xrightarrow{f'} & Q^{\mathrm{sing}}_{[2,1]} \\ \downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle q} \\ \mathbb{A}(2,2)_{[3,1]} & \xrightarrow{f} & \mathbb{P}(2,2) \end{array}$$

Therefore, the top horizontal morphism is a $\mathbb{G}_m$-torsor whose associated equivariant line bundle is $q^*\mathcal{O}_{\mathbb{P}(2,2)}(-1) \otimes \mathbb{D}$, where the latter is the determinant representation of GL_3.

This implies (proposition 1.1.8) that the pullback morphism

$$f'^{*} : CH_{\mathrm{GL}_3}(Q^{\mathrm{sing}}_{[2,1]}) \longrightarrow CH_{\mathrm{GL}_3}(\widehat{Q}^{\mathrm{sing}}_{[2,1]})$$

is surjective with kernel the ideal $(c_1 - q^* s)$. In particular, to compute $[E_1]$ we can equivalently check that $[\overline{E}_1]$ is not zero modulo the ideal $(c_1 - q^* s)$, where $\overline{E}_1 = Q^{\mathrm{sing}}_1$. This is because we have the cartesian diagram:

$$\begin{array}{ccc} E_1 & \longrightarrow & \widehat{Q}^{\mathrm{sing}}_{[2,1]} \\ \downarrow & & \downarrow \\ \overline{E}_1 & \longrightarrow & Q^{\mathrm{sing}}_{[2,1]} \end{array}$$

The equivariant Chow ring of $Q^{\mathrm{sing}}_{[2,1]}$ can be easily determined applying the projective bundle formula (proposition 1.1.1.(6)), because $Q^{\mathrm{sing}}_{[2,1]}$ is a projective bundle over $\mathbb{P}^2$ (see remark 3.2.8). We have:

$$CH_{\mathrm{GL}_3}(Q^{\mathrm{sing}}_{[2,1]}) = \mathbb{Z}[c_1, c_2, c_3, s, t]/(R_s, R_t)$$

where t is the pullback of the hyperplane section of $\mathbb{P}^2$ and s is the restriction of the hyperplane section of $\mathbb{P}(2, 2)$.

Proposition 1.1.3 tells us that for any GL_3-scheme X we have an inclusion of $CH_{\mathrm{GL}_3}(X)$ inside $CH_T(X)$, where T denotes the subtorus of diagonal matrices: applying this to our case, we deduce that it is enough to show that $[\overline{E}_1]_T \neq 0$ modulo the ideal $(\sigma_1(\lambda_i) - s)$, where the λ_i are the generators of $CH_T(\mathrm{Spec}(k_0))$ (see proposition 1.1.4).

Let U be the open, T-invariant subscheme $\mathbb{P}^2 \setminus \{[1:0:0], [0:1:0]\}$ of $\mathbb{P}^2$. Denote $Q^{\mathrm{sing}}_{[2,1]}|_U$ (resp. $\overline{E}_1|_U$) the restriction over U of $Q^{\mathrm{sing}}_{[2,1]} \to \mathbb{P}^2$ (resp. $\overline{E}_1$). The localization exact sequence (proposition 1.1.1.(3)) implies that the restriction morphism induces an isomorphism

$$\begin{aligned} CH^1_T(Q^{\mathrm{sing}}_{[2,1]}|_U)/(\sigma_1(\lambda_i) - s) &\simeq CH^1_T(Q^{\mathrm{sing}}_{[2,1]})/(\sigma_1(\lambda_i) - s) \\ &\simeq \mathbb{Z}\langle \lambda_1, \lambda_2, \lambda_3, s, t\rangle \ /\langle \sigma_1(\lambda_i) - s\rangle \end{aligned}$$

so that we have reduced ourselves to show that $[\overline{E}_1|_U]_T \neq 0$ modulo $\sigma_1(\lambda_i) - s$.

We can write $[\overline{E}_1|_U]_T = ns + mt + \sum_{i=1}^{3} k_i \lambda_i$, where n, m and k_i are coefficients in $\mathbb{F}_2$. Observe that it is enough to prove $m \neq 0$. If $p : \overline{E}_1|_U \to U$ is the usual projection, then we have:

$$p_*([\overline{E}_1|_U] \cdot s^2) = np_* s^3 + \sum_{i=1}^{3} k_i \lambda_i \cdot p_* s^2 + mt \cdot p_* s^2 = \xi + mt$$

where ξ is linear polynomial in the λ_i. In the identity above we used several facts: the projection formula (proposition 1.1.1.(8)) to obtain $mp_*(t \cdot s^2) = mt \cdot p_* s^2$, the fact that for every projective bundle $p : \mathbb{P}(E) \to X$ whose fibres have dimension d we have $p_* c_1(O_E(1))^d = [X]$ (see [**Ful98**, prop. 3.1.(a).ii]) and finally that the relation $R_s = 0$ is a monic polynomial of degree 3 in s with coefficients in $CH_T(\mathrm{Spec}(k_0))$. If we prove that $m \neq 0$, we are done, because $\{\lambda_1, \lambda_2, \lambda_3, t\}$ is a basis for $CH^1_T(\mathbb{P}^2)_{\mathbb{F}_2}$.

Let H_1 be the T-invariant hyperplane in the projective bundle $\mathbb{P}(2, 2) \times U \to U$ whose points are those pairs (q, p) such that $q(1, 0, 0) = 0$. Similarly, define H_2 as the hyperplane whose points are pairs (q, p) such that $q(0, 1, 0) = 0$. The equations defining these two hyperplanes are respectively $q_{11} = 0$ and $q_{22} = 0$, where $q = q_{11}x^2 + q_{22}y^2 + \ldots$.

Lemma 1.1.7, together with the description above of H_i, tells us that $[H_i]_T = c_1^T(\mathcal{O}_{\mathbb{P}(2,2)}(1)) + 2\lambda_i$. Observe that

$$i : Q^{\mathrm{sing}}_{[2,1]}|_U \hookrightarrow \mathbb{P}(2,2) \times U$$

is a regular embedding because both the domain and the target are smooth, hence there is a well defined Gysin homomorphism (proposition 1.1.1.(9)):

$$i^! : CH_T(\mathbb{P}(2,2) \times U) \longrightarrow CH_T(Q^{\mathrm{sing}}_{[2,1]}|_U)$$

In particular, $i^!([H_1]_T \cdot [H_2]_T) = s^2$ in $CH_T(Q^{\mathrm{sing}}_{[2,1]}|_U)_{\mathbb{F}^2}$. On the other hand, we have that $i^!([H_1]_T[H_2]_T) = [H_1 \cap H_2 \cap Q^{\mathrm{sing}}_{[2,1]}|_U]_T$, where:

$$H_1 \cap H_2 \cap Q^{\mathrm{sing}}_{[2,1]}|_U = \left\{ \begin{matrix} (q,p) \text{ such that } q_x(p) = q_y(p) = 0 \\ q_z(p) = q(1,0,0) = q(0,1,0) = 0 \\ \text{and } p \neq [1:0:0], [0:1:0] \end{matrix} \right\}$$

We can write $p_*[\overline{E}_1]_T \cdot s^2 = p_*[\overline{E}_1 \cap H_1 \cap H_2]_T$, and $\overline{E}_1 \cap H_1 \cap H_2$ is sent by p onto the restriction to U of the unique line $L = \{z = 0\}$ that passes through $(1,0,0)$ and $(0,1,0)$. Moreover, $\overline{E}_1 \cap H_1 \cap H_2$ is 1:1 on L, therefore

$$p_*[\overline{E}_1 \cap H_1 \cap H_2]_T = [L]_T = t + \lambda_3$$

This proves that $[\overline{E}_1|_U]_T = t + \xi$ in $CH_T(Q^{\mathrm{sing}}_{[2,1]}|_U)_{\mathbb{F}_2}$, where ξ is a linear combination of λ_i and s, and it concludes the proof. □

Now we are ready to give a proof of lemma 3.3.5.

Proof of lemma 3.3.5. From lemma 3.3.4 we know that $[D'^1_1] = h + nt$ in $CH^1_{\mathrm{GL}_3}(\widetilde{\mathbb{P}}(V_n)_1)$, where h is the hyperplane section of the projective bundle $\widetilde{\mathbb{P}}(V_n)_1 \to E_1$ and t is the pullback of the hyperplane section of $\mathbb{P}^2$ along the morphism $\widetilde{\mathbb{P}}(V_n)_1 \to E_1 \to \mathbb{P}^2$.

Let $j : E_1 \to \widehat{Q}^{\mathrm{sing}}_{[2,1]}$ be the closed immersion, so that we have a cartesian square:

$$\begin{array}{ccc} \widetilde{\mathbb{P}}(V_n)_1 & \xrightarrow{j'} & \widetilde{\mathbb{P}}(V_n)_{[2,1]} \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ E_1 & \xrightarrow{j} & \widehat{Q}^{\mathrm{sing}}_{[2,1]} \end{array}$$

We want to prove that $j'_*[D'^1_1]$ is non-zero in $CH^2_{\mathrm{GL}_3}(\widetilde{\mathbb{P}}(V_n)_{[2,1]})_{\mathbb{F}^2}$.

We have:

$$\begin{aligned} j'_*[D'^1_1] &= j'_*h + nj'_*t = j'_*(h \cdot p'^*1) + j'_*p'^*c_1^{\mathrm{GL}_3}(\mathcal{O}_{\mathbb{P}^2}(1)) \\ &= h \cdot p^*j_*1 + p^*j_*c_1^{\mathrm{GL}_3}(\mathcal{O}_{\mathbb{P}^2}(1)) \end{aligned}$$

where in the second identity we applied the projection formula for not necessarily flat morphisms (proposition 1.1.2) to the proper morphism j': with a common abuse of notation, we used the symbol h to denote both the hyperplane section of $\widetilde{\mathbb{P}}(V_n)_{[2,1]}$ and its pullback along j.

The third identity is a consequence of the compatibility property (proposition 1.1.1.(4)).

The projective bundle formula (prop. 1.1.1.(6)) applied to $CH^2_{\mathrm{GL}_3}(\widetilde{\mathbb{P}}(V_n)_{[2,1]})$ tells us that this group decomposes as a direct sum

$$\begin{aligned} CH^2_{\mathrm{GL}_3}(\widetilde{\mathbb{P}}(V_n)_{[2,1]}) \simeq\ & p^*CH^2_{\mathrm{GL}_3}(\widetilde{\mathbb{A}}(2,2)_{[2,1]}) \oplus p^*CH^1_{\mathrm{GL}_3}(\widetilde{\mathbb{A}}(2,2)_{[2,1]}) \cdot h \\ & \oplus p^*CH^0_{\mathrm{GL}_3}(\widetilde{\mathbb{A}}(2,2)_{[2,1]}) \cdot h^2 \end{aligned}$$

Hence it is enough to check that $[E_1] \neq 0$ in $CH^1_{\mathrm{GL}_3}(\widehat{Q}^{\mathrm{sing}}_{[2,1]})$, which follows from lemma 3.3.6. □

We focus now on $D_1'^2$ (definition 3.2.13). Recall that the points of $D_1'^2$ can be seen as triples $(l^2, [f], p)$ such that the plane curves $l = 0$ and $f = 0$ does not intersect transversely at p. This property is clearly independent of the choice of a representative of $[f]$.

Observe that, just as for $D_1'^1$, this scheme can be seen both as a codimension 1 subvariety of $\widetilde{\mathbb{P}}(V_n)_1$ and as a codimension 2 subvariety of $\widetilde{\mathbb{P}}(V_n)_{[2,1]}$.

LEMMA 3.3.7. *We have that* $[D_1'^2] = 0$ *in* $CH^1_{\mathrm{GL}_3}(\widetilde{\mathbb{P}}(V_n)_1)_{\mathbb{F}_2}$. *The same thing holds in* $CH^2_{\mathrm{GL}_3}(\widetilde{\mathbb{P}}(V_n)_{[2,1]})_{\mathbb{F}_2}$.

PROOF. Clearly, the second assertion follows from the first one. Applying proposition 2.3.2 exactly in the same way as in the proof of lemma 3.3.4, we see that it is enough to show $[\overline{Z'}^2] = 0$ in $CH^1_{\mathrm{GL}_3}(\overline{E}_1 \times \mathbb{P}(2,n))/(g^*s - c_1)$, where g is the obvious morphism to $\mathbb{P}(2,2)$ and $\overline{Z'}^2$ is the subvariety whose points are triples (l^2, f, p) such that the plane curves $l = 0$ and $f = 0$ intersect non transversally in p.

As in the proof of lemma 3.3.4, we have a cartesian diagram

$$\begin{array}{ccc} \varphi^*\overline{E}_1 & \longrightarrow & \overline{E}_1 \\ \downarrow & & \downarrow \\ \mathbb{P}(2,1) & \xrightarrow{\varphi} & \mathbb{P}(2,2)_1 \end{array}$$

where φ sends $[l]$ to $[l^2]$, and the horizontal arrows are equivariant isomorphisms.

Therefore, there is an equivariant isomorphism $\psi : \varphi^*\overline{E}_1 \times \mathbb{P}(2,n) \simeq \overline{E}_1 \times \mathbb{P}(2,n)$. After identifying the equivariant Chow rings of these two schemes, we have that $[\overline{Z'}^2] = [\psi^{-1}\overline{Z'}^2]$.

Recall that in the proof of lemma 3.3.2 we introduced the subscheme Y^2 of $\mathbb{P}(2,1) \times \mathbb{P}(2,n)$ whose points are pairs (l, f) such that the plane curves $l = 0$ and $f = 0$ do not intersect transversally, and we also proved that $[Y^2] = 0$ in $CH^1_{\mathrm{GL}_3}(\mathbb{P}(2,1) \times \mathbb{P}(2,n))_{\mathbb{F}_2}$.

Observe that the embedding $i : \varphi^*\overline{E}_1 \times \mathbb{P}(2,n) \hookrightarrow \mathbb{P}(2,1) \times \mathbb{P}(2,n) \times \mathbb{P}^2$ is a local complete intersection, because both schemes are regular. We can apply proposition 1.1.1.(9), which tells us that there is a well defined Gysin homomorphism:

$$i^! : CH_{\mathrm{GL}_3}(\mathbb{P}(2,n) \times \mathbb{P}(2,1) \times \mathbb{P}^2) \longrightarrow CH_{\mathrm{GL}_3}(\varphi^*\overline{E}_1 \times \mathbb{P}(2,n))$$

In particular, we have $i^![Y^2 \times \mathbb{P}^2] = [\psi^{-1}\overline{Z'}^2]$, thus the latter cycle class is zero in $CH_{\mathrm{GL}_3}(\varphi^*\overline{E}_1 \times \mathbb{P}(2,n))_{\mathbb{F}_2}$. □

3.4. The key lemma

In this section, we will always be working with Chow groups with coefficients in $\mathbb{F}_2$. The goal is to prove the key lemma 3.1.3, which is the only missing ingredient for completing the computation of the cohomological invariants of $\mathcal{H}_g$ (see section 3.1). Let us restate here what we want to prove:

KEY LEMMA. *Fix an algebraically closed base field* k_0 *of characteristic* $\neq 2$. *Let* $\mathbb{P}(1, 2n)$ *be the projective space of degree* $2n$ *binary forms, endowed with the* PGL_2*-action:*

$$A \cdot f(x, y) = f(A^{-1}(x, y))$$

Let $\Delta_{1,2n}$ be the divisor parametrising singular forms and denote i the closed embedding $\Delta_{1,2n} \hookrightarrow \mathbb{P}(1,2n)$. Then $i_* : A^0_{\mathrm{PGL}_2}(\Delta_{1,2n}) \to A^1_{\mathrm{PGL}_2}(\mathbb{P}(1,2n))$ is zero for every n.

From now on, we assume that the characteristic of the base field k_0 is $\neq 2$.

3.4.1. Proof of the key lemma. Corollary 1.3.16 implies that

$$A^0_{\mathrm{PGL}_2}(\Delta_{1,2n}) \longrightarrow A^1_{\mathrm{PGL}_2}(\mathbb{P}(1,2n))$$

is zero if and only if is zero the morphism

$$i_* : A^0_{\mathrm{GL}_3}(D_3) \longrightarrow A^1_{\mathrm{GL}_3}(\mathbb{P}(V_n)_3)$$

If $\widehat{Q}_3$ denotes the pullback to $\mathbb{A}(2,2)_3$ of the universal smooth conic $Q_3 \to \mathbb{P}(2,2)_3$, then $\mathbb{P}(V_n)_3$ is isomorphic to the relative Hilbert scheme of points $\mathrm{Hilb}^{2n}_{\widehat{Q}_3/\mathbb{A}(2,2)_3}$ (see remark 1.3.13), and D_3 is the divisor parametrizing subschemes non-étale over the base.

We know almost nothing about $A^0_{\mathrm{GL}_3}(D_3)$ but, on the other side, we know a lot about $A^1_{\mathrm{GL}_3}(\mathbb{P}(V_n)_3)$. Indeed, from the formula for equivariant Chow rings with coefficients of projective bundles (proposition 1.2.3.(7)), we have

$$A^1_{\mathrm{GL}_3}(\mathbb{P}(V_n)_3) \simeq A^1_{\mathrm{GL}_3}(\mathbb{A}(2,2)_3) \oplus A^0_{\mathrm{GL}_3}(\mathbb{A}(2,2)_3) \cdot h$$

where h is equal to $c_1^{\mathrm{GL}_3}(\mathcal{O}(1))$, which is an element of codimension 1 and degree 0. Proposition 1.3.8 applied to the PGL_2-scheme $\mathrm{Spec}(k_0)$, whose GL_3-counterpart is $\mathbb{A}(2,2)_3$, tells us that $A_{\mathrm{PGL}_2}(\mathrm{Spec}(k_0)) \simeq A_{\mathrm{GL}_3}(\mathbb{A}(2,2)_3)$. Combining this with proposition 1.2.4, we readily deduce that

$$A^1_{\mathrm{GL}_3}(\mathbb{P}(V_n)_3) \simeq (CH^1_{\mathrm{GL}_3}(\mathbb{P}(V_n)_3)_{\mathbb{F}_2}) \oplus \mathbb{F}_2 \cdot \tau_1 \oplus \mathbb{F}_2 \cdot w_2 h$$

where the first addend coincides with the elements of degree 0, the element τ_1 has codimension and degree both equal to 1, and finally w_2 has codimension 0 and degree 2, so that $w_2 h$ has codimension 1 and degree 2.

Thanks to the fact that i_* preserves the degree (proposition 1.2.3.(2)), every element in $A^0_{\mathrm{GL}_3}(D_3)$ of degree greater than 2 will be sent by i_* to 0. We need to find out if there is any element α of degree smaller or equal to 2 such that $i_*\alpha$ is not zero.

Lemma 3.4.1. *The morphism $i_* : CH^0_{\mathrm{GL}_3}(D_3)_{\mathbb{F}_2} \to CH^1_{\mathrm{GL}_3}(\mathbb{P}(V_n)_3)_{\mathbb{F}_2}$ is zero.*

Proof. We have to show that the cycle class $[D_3] = 0$ in $CH^1_{\mathrm{GL}_3}(\mathbb{P}(V_n)_3)_{\mathbb{F}_2}$. From 2.3.4 we have that $[D_3] = 4(n-2)h$ in $CH_{\mathrm{GL}_3}(\mathbb{P}(V_n)_3)$, where h is equal to $c_1^{\mathrm{GL}_3}(\mathcal{O}_{\mathbb{P}(V_n)_3}(1))$. This implies the lemma. □

We have a closed embedding of $\mathbb{P}(V_n)_2$ inside $\mathbb{P}(V_n)_{[3,2]}$, whose open complement is $\mathbb{P}(V_n)_3$. Moreover, if we pullback $D_{[3,2]}$ along this closed embdding, we obtain D_2. By compatibility (proposition 1.2.3.(5)) we get:

$$\begin{array}{ccc} A^0_{\mathrm{GL}_3}(D_{[3,2]}) & \xrightarrow{i_*^{[3,2]}} & A^1_{\mathrm{GL}_3}(\mathbb{P}(V_n)_{[3,2]}) \\ \big\downarrow{\scriptstyle j_D^*} & & \big\downarrow{\scriptstyle j^*} \\ A^0_{\mathrm{GL}_3}(D_3) & \xrightarrow{i_*} & A^1_{\mathrm{GL}_3}(\mathbb{P}(V_n)_3) \\ \big\downarrow{\scriptstyle \partial_D} & & \big\downarrow{\scriptstyle \partial} \\ A^0_{\mathrm{GL}_3}(D_2) & \xrightarrow{i_*^2} & A^1_{\mathrm{GL}_3}(\mathbb{P}(V_n)_2) \\ \big\downarrow{\scriptstyle f_{D*}} & & \big\downarrow{\scriptstyle f_*} \\ A^1_{\mathrm{GL}_3}(D_{[3,2]}) & \xrightarrow{i_*^{[3,2]}} & A^2_{\mathrm{GL}_3}(\mathbb{P}(V_n)_{[3,2]}) \end{array} \tag{5}$$

Observe that the vertical sequences are exact.

LEMMA 3.4.2. *The Chow group with coefficients $A^1_{\mathrm{GL}_3}(\mathbb{P}(V_n)_{[3,2]})$ is concentrated in degree 0. In particular, if α is an element of $A^0_{\mathrm{GL}_3}(D_3)$ of degree greater than 0, then $i_*\alpha = 0$ if and only if $i^2_*(\partial_D\alpha) = 0$.*

PROOF. Using the projective bundle formula (proposition 1.2.3.(7)) we have

$$A^1_{\mathrm{GL}_3}(\mathbb{P}(V_n)_{[3,2]}) = A^1_{\mathrm{GL}_3}(\mathbb{A}(2,2)_{[3,2]}) \oplus A^0_{\mathrm{GL}_3}(\mathbb{A}(2,2)_{[3,2]}) \cdot h$$

where $h = c_1^{\mathrm{GL}_3}(\mathcal{O}_{\mathbb{P}(V_n)_{[3,2]}}(1))$. The scheme $\mathbb{A}(2,2)_{[3,2]}$ is an open subscheme of $\mathbb{A}(2,2)$ whose complement has codimension 3. The localization exact sequence (proposition 1.2.3.(4)) implies that:

$$A^i_{\mathrm{GL}_3}(\mathbb{A}(2,2)_{[3,2]}) = A^i_{\mathrm{GL}_3}(\mathbb{A}(2,2)) \text{ for } i = 0, 1$$

The GL_3-representation $\mathbb{A}(2,2)$ can be regarded as a GL_3-equivariant vector bundle over $\mathrm{Spec}(k_0)$, hence by homotopy invariance (proposition 1.2.3.(6)) we obtain:

$$A^i_{\mathrm{GL}_3}(\mathbb{A}(2,2)) = A^i_{\mathrm{GL}_3}(\mathrm{Spec}(k_0))$$

By proposition 1.2.4, there are no non-zero elements of degree greater than 0 in this ring, therefore $A^1_{\mathrm{GL}_3}(\mathbb{P}(V_n)_{[3,2]})$ is concentrated in degree 0.

Let α be an element of $A^0_{\mathrm{GL}_3}(D_3)$ of degree greater than 0. Then $i_*\alpha = 0$ implies that $i^2_*(\partial_D) = 0$ because of the commutativity of the middle square of diagram 5. On the other hand, if $i^2_*(\partial_D\alpha) = 0$ then the exactness of the vertical sequences of diagram 5 implies that $i_*\alpha = j^*\beta$ for some β in $A^1_{\mathrm{GL}_3}(\mathbb{P}(V_n)_{[3,2]})$.

By proposition 1.2.3.(2) and 1.2.3.(3), we have $\deg(\alpha) = \deg(i_*\alpha) = \deg(\beta)$, thus $\beta = 0$ because $A^1_{\mathrm{GL}_3}(\mathbb{P}(V_n)_{[3,2]})$ is concentrated in degree 0. □

PROPOSITION 3.4.3. *If α in $A^0_{\mathrm{GL}_3}(D_3)$ has degree 1, then $i_*\alpha = 0$.*

PROOF. By hypothesis, $\partial_D\alpha$ is a degree zero element of $A^0_{\mathrm{GL}_3}(D_2)$: the degree zero part of this group can be identified with $CH^0_{\mathrm{GL}_3}(D_2)_{\mathbb{F}_2}$ (proposition 1.2.3.(1)). From proposition 3.2.5 we deduce that

$$CH^0_{\mathrm{GL}_3}(D_2)_{\mathbb{F}_2} \simeq CH^0_{\mathrm{GL}_3}(D_2^1)_{\mathbb{F}_2} \oplus CH^0_{\mathrm{GL}_3}(D_2^2)_{\mathbb{F}_2} \simeq \mathbb{F}_2 \oplus \mathbb{F}_2$$

where D_2^1 and D_2^2 are the two irreducible components of D_2 (see definition 3.2.1 and 3.2.3).

Write $\partial_D\alpha = (n, m)$. From lemma 3.4.2 we have that $i_*\alpha = 0$ if and only if

$$0 = i^2_*(n, m) = n[D_2^1] + m[D_2^2] \in CH^1_{\mathrm{GL}_3}(\mathbb{P}(V_n)_2)_{\mathbb{F}_2}$$

Because of the exactness of the left vertical sequence of the diagram 5, we have $f_{D*}(\partial_D \alpha) = 0$. This implies that

$$0 = i_*^{[3,2]} f_{D*}(\partial_D \alpha) = n[D_2^1] + m[D_2^2]$$

in $CH^2_{\mathrm{GL}_3}(\mathbb{P}(V_n)_{[3,2]})$. By lemma 3.3.1 we know that $[D_1^1] = c_1 h \neq 0$ and lemma 3.3.2 tells us that $[D_2^2] = 0$: we deduce $0 = nc_1 h$, thus $n = 0$ and $\partial_D \alpha = (0, m)$

In particular $i_*^2 \partial_D \alpha = m[D_2^2]$ in $CH^1_{\mathrm{GL}_3}(\mathbb{P}(V_n)_2)$, and by lemma 3.3.2 we know that this last term is zero: this concludes the proof. $\square$

Let $\widehat{Q}^{\mathrm{sing}}_{[2,1]} \to \mathbb{A}(2,2)_{[2,1]}$ be the singular locus of $\widehat{Q}_{[2,1]} \to \mathbb{A}(2,2)_{[2,1]}$, where $\widehat{Q}_{[2,1]}$ is the pullback to $\mathbb{A}(2,2)_{[2,1]}$ of the universal conic $Q_{[2,1]} \to \mathbb{P}(2,2)_{[2,1]}$ restricted over the closed subscheme of conics of rank < 3 (see definition 3.2.7). Denote E_1 the restriction of $\widehat{Q}^{\mathrm{sing}}_{[2,1]} \to \mathbb{A}(2,2)_{[2,1]}$ over $\mathbb{A}(2,2)_1$. Recall that $\widehat{Q}^{\mathrm{sing}}_2 \to \mathbb{A}(2,2)_2$ is an isomorphism, and that $E_1 \to \mathbb{A}(2,2)_1$ is a projective bundle (see proposition 3.2.9).

Let $\widetilde{\mathbb{P}}(V_n)_{[2,1]}$ be the pullback of $\mathbb{P}(V_n)_{[2,1]} \to \mathbb{A}(2,2)_{[2,1]}$ along the morphism $\widehat{Q}^{\mathrm{sing}}_{[2,1]} \to \mathbb{A}(2,2)_{[2,1]}$, and define $\widetilde{\mathbb{P}}(V_n)_1$ as the restriction of $\widetilde{\mathbb{P}}(V_n)_{[2,1]}$ over E_1 (see definition 3.2.10). The restriction $\widetilde{\mathbb{P}}(V_n)_2$ of $\widetilde{\mathbb{P}}(V_n)_{[2,1]}$ over $\widehat{Q}^{\mathrm{sing}}_2$ is isomorphic to $\widetilde{\mathbb{P}}(V_n)_2$ (see lemma 3.2.12).

Finally, let $D'_{[2,1]}$ be as in definition 3.2.11 and let D'_1 be the restriction of $D'_{[2,1]}$ over E_1. Then by compatibility (proposition 1.2.3.(5)), we get the following commutative diagram, whose vertical sequences are exact:

$$\begin{array}{ccc}
A^0_{\mathrm{GL}_3}(D'_{[2,1]}) & \xrightarrow{i'_*} & A^1_{\mathrm{GL}_3}(\widetilde{\mathbb{P}}(V_n)_{[2,1]}) \\
\downarrow{\scriptstyle j_D^*} & & \downarrow{\scriptstyle j^*} \\
A^0_{\mathrm{GL}_3}(D_2) & \xrightarrow{i^2_*} & A^1_{\mathrm{GL}_3}(\mathbb{P}(V_n)_2) \\
\downarrow{\scriptstyle \partial_D} & & \downarrow{\scriptstyle \partial} \\
A^0_{\mathrm{GL}_3}(D'_1) & \xrightarrow{i'^1_*} & A^1_{\mathrm{GL}_3}(\widetilde{\mathbb{P}}(V_n)_1) \\
\downarrow{\scriptstyle f_{D*}} & & \downarrow{\scriptstyle f_*} \\
A^1_{\mathrm{GL}_3}(D'_{[2,1]}) & \xrightarrow{i'_*} & A^2_{\mathrm{GL}_3}(\widetilde{\mathbb{P}}(V_n)_{[2,1]})
\end{array} \tag{6}$$

Observe that lemma 3.2.12 allowed us to identify in the diagram above $A^0_{\mathrm{GL}_3}(D'_2) \simeq A^0_{\mathrm{GL}_3}(D_2)$ and $A^0_{\mathrm{GL}_3}(\widetilde{\mathbb{P}}(V_n)_2) \simeq A^0_{\mathrm{GL}_3}(\mathbb{P}(V_n)_2)$.

Lemma 3.4.4. *We have:*

- $A^0_{\mathrm{GL}_3}(\widehat{Q}^{\mathrm{sing}}_{[2,1]}) \simeq A^0_{\mathrm{GL}_3}(\mathbb{P}^2)$
- $A^1_{\mathrm{GL}_3}(\widehat{Q}^{\mathrm{sing}}_{[2,1]}) \simeq A^1_{\mathrm{GL}_3}(\mathbb{P}^2)$

Proof. Let $\widehat{Q}^{\mathrm{sing}}$ be the closed subscheme of $\mathbb{A}(2,2) \times \mathbb{P}^2$ defined as follows:

$$\widehat{Q}^{\mathrm{sing}} := \{(q,p) \text{ such that } q_x(p) = q_y(p) = q_z(p) = 0\}$$

Observe that $\widehat{Q}^{\mathrm{sing}} \to \mathbb{P}^2$ is a GL_3-equivariant vector subbundle of the (trivial) vector bundle $\mathbb{A}(2,2) \times \mathbb{P}^2 \to \mathbb{P}^2$. Therefore, for the homotopy invariance of the Chow groups with coefficients (see proposition 1.2.3.(6)) we get $A^1_{\mathrm{GL}_3}(\widehat{Q}^{\mathrm{sing}}) \simeq A^1_{\mathrm{GL}_3}(\mathbb{P}^2)$.

The scheme $\widehat{Q}^{\mathrm{sing}}_{[2,1]}$ is the complement in $\widehat{Q}^{\mathrm{sing}}$ of the zero section, which has codimension 3. The localization exact sequence (see proposition 1.2.3.(4)) implies that, if Z is a closed subscheme of codimension $> i$ of a scheme X, then the Chow group with coefficients of codimension i of X and $X \setminus Z$ coincide.

Applying this to our case, we deduce that $A^i_{\mathrm{GL}_3}(\widehat{Q}^{\mathrm{sing}}) \simeq A^i_{\mathrm{GL}_3}(\widehat{Q}^{\mathrm{sing}}_{[2,1]})$ for $i = 0, 1$. $\square$

LEMMA 3.4.5. *The Chow group with coefficients $A^1_{\mathrm{GL}_3}(\widetilde{\mathbb{P}}(V_n)_{[2,1]})$ is concentrated in degree* 0. *In particular, given an element β of $A^0_{\mathrm{GL}_3}(D_2)$ whose degree is greater than* 0, *then $i^2_*\beta = 0$ if and only if $i'^1_*(\partial_D \beta) = 0$.*

PROOF. Applying the projective bundle formula (proposition 1.2.3.(7)) we get:

$$A^1_{\mathrm{GL}_3}(\widetilde{\mathbb{P}}(V_n)_{[2,1]}) \simeq A^1_{\mathrm{GL}_3}(\widehat{Q}^{\mathrm{sing}}_{[2,1]}) \oplus A^0_{\mathrm{GL}_3}(\widehat{Q}^{\mathrm{sing}}_{[2,1]}) \cdot h$$

We know from lemma 3.4.4 that $A^i_{\mathrm{GL}_3}(\widehat{Q}^{\mathrm{sing}}_{[2,1]})$ is concentrated in degree 0 for $i = 0, 1$: this proves the first part of the lemma.

Given an element β in $A^0_{\mathrm{GL}_3}(D_2)$ of degree > 0, suppose that $i'^1_*(\partial_D \beta) = 0$. The commutativity of diagram 6 implies that $\partial(i^2_*\beta) = 0$, and the exactness of the right vertical sequence tells us that $i^2_*\beta = j^*\gamma$. Observe that

$$0 < \deg(\beta) = \deg(i^2_*\beta) = \deg(j^*\gamma) = \deg(\gamma)$$

because pushforwards and pullback preserve the degree (see proposition 1.2.3.(2) and 1.2.3.(3)). The group $A^1_{\mathrm{GL}_3}(\widetilde{\mathbb{P}}(V_n)_{[2,1]})$ is concentrated in degree 0, hence $\gamma = 0$ and this concludes the proof. $\square$

PROPOSITION 3.4.6. *If β is an element of degree* 1 *in $A^0_{\mathrm{GL}_3}(D_2)$, then $i^2_*\beta = 0$.*

PROOF. By lemma 3.4.5, we can equivalently show that $i'^1_*(\partial_D \beta) = 0$. Observe that $\partial_D(\beta)$ has degree 0. The degree 0 part of $A^0_{\mathrm{GL}_3}(D'_1)$ is isomorphic to $CH^0_{\mathrm{GL}_3}(D'_1)_{\mathbb{F}_2}$ by proposition 1.2.3.(1). Recall that D'_1 is the union of two irreducible components (see proposition 3.2.14), hence we have:

$$CH^0_{\mathrm{GL}_3}(D'_1)_{\mathbb{F}_2} = CH^0_{\mathrm{GL}_3}(D'^1_1)_{\mathbb{F}_2} \oplus CH^0_{\mathrm{GL}_3}(D'^2_1)_{\mathbb{F}_2}$$

By exactness of the left vertical sequence of diagram 6, we have that $f_{D*}(\partial_D \beta) = 0$, thus $i'_* f_{D*}(\partial_D \beta) = 0$. If we write $\partial_D \beta$ as (n, m), then we are saying that $n[D'^1_1] + m[D'^2_1] = 0$ in $CH^1_{\mathrm{GL}_3}(\widetilde{\mathbb{P}}(V_n)_{[2,1]})_{\mathbb{F}_2}$. By corollary 3.3.5 and lemma 3.3.7 we know that $[D'^1_1] \neq 0$ and $[D'^2_1] = 0$, thus $n = 0$.

We have proved that $\partial_D \beta = (0, m)$. This implies that $i'^1_*(\partial_D \beta) = m[D'^2_1]$, which is equal to zero by lemma 3.3.7. $\square$

We are ready to prove the key lemma 3.1.3.

PROOF OF KEY LEMMA 3.1.3. We want to prove that:

$$i_* : A^0_{\mathrm{GL}_3}(D_3) \longrightarrow A^1_{\mathrm{GL}_3}(\mathbb{P}(V_n)_3)$$

is zero. As already observed, $A^1_{\mathrm{GL}_3}(\mathbb{P}(V_n)_3)$ is concentrated in degree 0, 1 and 2, hence every element α in $A^0_{\mathrm{GL}_3}(D_3)$ of degree > 2 will be sent to 0, because i_* preserves the degree (see proposition 1.2.3.(1)).

Suppose $\deg(\alpha) = 0$. The morphism i_* restricted to the degree 0 part is equal to

$$i_* : CH^0_{\mathrm{GL}_3}(D_3)_{\mathbb{F}_2} \to CH^1_{\mathrm{GL}_3}(\mathbb{P}(V_n)_3)_{\mathbb{F}_2}$$

by proposition 1.2.3.(1), and we know that this morphism is zero by lemma 3.4.1.

Suppose $\deg(\alpha) = 1$. Then proposition 3.4.3 tells us that $i_*\alpha = 0$.

The only case left is when $\deg(\alpha) = 2$. By lemma 3.4.2 we have that $i_*\alpha = 0$ if and only if $i_*^2(\partial_D \alpha) = 0$. Observe that $\deg(\partial_D \alpha) = 1$, hence we can apply proposition 3.4.6 to deduce that $i_*^2(\partial_D \alpha) = 0$. This finishes the proof. □

CHAPTER 4

Cohomological invariants of the stack of hyperelliptic curves of even genus: multiplicative structure

In this chapter we investigate the multiplicative structure of $\mathrm{Inv}^{\bullet}(\mathcal{H}_g)$, the graded-commutative ring of cohomological invariants with coefficients in $\mathbb{F}_2$ of the stack of hyperelliptic curves of even genus $g \geq 2$. Our main result is theorem 4.2.14. All the results of this chapter had been obtained in collaboration with Roberto Pirisi.

4.1. Cohomological invariants of étale algebras

The goal of this section is to recall some results on cohomological invariants of classifying stacks. We will be mainly concerned with $\mathcal{B}S_{2n}$, regarded as the stack that classifies étale algebras of degree $2n$, and $\mathcal{B}O_{2n}$, the classifying stack of quadratic forms of degree $2n$. All this material can be found in [GMS03]: more precise references will be given along the way.

Whenever we write $\mathrm{Inv}^{\bullet}(-)$, we will always be thinking of cohomological invariants with coefficients in $\mathbb{F}_2$. Therefore, the base field k_0 will always be assumed to be of characteristic $\neq 2$.

Recall that the classifying stack $\mathcal{B}S_n$ over k_0, where S_n is the symmetric group on n elements, is equivalent to the stack whose objects are degree n étale morphisms $X \to S$ of k_0-schemes: the equivalence in given by the functor that sends a degree n étale morphism $X \to S$ to the S_n-torsor $\mathrm{Isom}_S(X, \coprod_{i=1}^n S) \to S$, where S_n acts on the right by permutation.

Cohomological invariants of $\mathcal{B}S_n$ with coefficients in $\mathbb{F}_2$ have been extensively studied (in particular, see [GMS03, ch. VII]). We briefly summarize here their main features.

Proposition 4.1.1.

(1) *There is a well defined cohomological invariant α_1 that sends an étale K-algebra $E \simeq K[x]/(x^2 - a)$ to the element $\{a\}$ in $K^*/(K^*)^2$ (we are using the well known identification of this group with $H^1_{\acute{e}t}(\mathrm{Spec}(K), \mu_2)$).*

(2) *We have $\mathrm{Inv}^{\bullet}(\mathcal{B}S_2) \simeq \mathrm{H}^{\bullet}(k_0)[\alpha_1]/(\alpha_1^2 - \{-1\} \cdot \alpha_1)$*

Proof. As the characteristic of k_0 is $\neq 2$, we have the following chain of isomorphisms:

$$\mathcal{B}S_2 \simeq \mathcal{B}\mu_2 \simeq [(\mathbb{A}^1 \setminus \{0\})/\mathbb{G}_m]$$

where $\mathbb{G}_m$ acts by multiplication by λ^2. As $\mathbb{G}_m$ is special, this implies that $\mathbb{A}^1 \setminus \{0\}$ is a smooth-Nisnevich cover and the cohomological invariants of $\mathcal{B}S_2$ are those of $\mathbb{A}^1 \setminus \{0\}$ which are invariant with respect to the action of $\mathbb{G}_m$.

The invariants of $\mathbb{A}^1 \setminus \{0\}$ can be easily computed using the long exact sequence associated to the open embedding $\mathbb{A}^1 \setminus \{0\} \subset \mathbb{A}^1$ (see proposition 1.2.3.(4)). We have:

$$0 \to A^0(\mathbb{A}^1) \to A^0(\mathbb{A}^1 \setminus \{0\}) \to A^0(\{0\}) \to A^1(\mathbb{A}^1)$$

The last morphism vanishes because $A^1(\mathbb{A}^1) \simeq \mathrm{H}^\bullet(k_0) \otimes CH^1(\mathbb{A}^1) = 0$.

Moreover, we have a splitting $A^0(\{0\}) \to A^0(\mathbb{A}^1 \setminus \{0\})$ given by multiplication by α_1', where α_1' is the invariant that sends a morphism $\mathrm{Spec}(K) \to \mathbb{A}^1 \setminus \{0\}$, which is equivalent to the datum of β in K^*, to the equivalence class of β in $H^1_{\acute{e}t}(\mathrm{Spec}(K), \mu_2) \simeq K^*/(K^*)^2$. The fact that $\partial(\alpha_1') = 1$ follows from the fact that α_1' does not extend to a cohomological invariant of $\mathbb{A}^1$, because it is not trivial. Therefore we have:

$$A^0(\mathbb{A}^1 \setminus \{0\}) \simeq \mathrm{H}^\bullet(k_0) \cdot 1 \oplus \mathrm{H}^\bullet(k_0) \cdot \alpha_1'$$

Observe that α_1' descends to an invariant α_1 of $[\mathbb{A}^1 \setminus \{0\}/\mathbb{G}_m]$, where $\mathbb{G}_m$ acts by multiplication for λ^2: if we identify $\mathcal{B}S_2$ with the classifying stack of quadratic étale algebras, we see that α_1 is the invariant described in (1).

We are left with proving that $\alpha_1^2 = \{-1\} \cdot \alpha_1$: observe that a quadratic étale algebra E can be regarded as a rank 2 K-vector space with a quadratic form defined as $y \mapsto \mathrm{Tr}_E(m_{y^2})$, where m_{y^2} denotes the multiplication by y^2, regarded as a K-linear application on the K-vector space E, and $\mathrm{Tr}_E(-)$ is the usual trace.

In particular, if we have $E \simeq K[x]/(x^2 - a)$, the associated quadratic form will be diagonal of the form $\langle 2, 2a \rangle$. This shows that, if $q : \mathcal{B}S_2 \to \mathcal{B}O_2$ is the morphism induced by the construction above, then $\alpha_1 = q^* w_1$, where w_1 is the first Stiefel-Whitney invariant of quadratic forms (see [GMS03, 17.1]), which sends a quadratic form to its determinant.

Then (2) follows from the relation $w_1^2 = \{-1\} \cdot w_1$ (see [GMS03, (19.3)]). □

The inclusion $S_2^{\times n} \hookrightarrow S_{2n}$ which sends σ_i to $(i, i+1)$ induces a morphism of classifying stacks $f : \mathcal{B}S_2^{\times n} \to \mathcal{B}S_{2n}$, hence we have a pullback morphism $f^* : \mathrm{Inv}^\bullet(\mathcal{B}S_{2n}) \to \mathrm{Inv}^\bullet(\mathcal{B}S_2^{\times n})$.

By proposition 4.1.1 we deduce that:

$$\mathrm{Inv}^\bullet(\mathcal{B}S_2^{\times n}) \simeq \mathrm{H}^\bullet(k_0) \cdot 1 \oplus \mathrm{H}^\bullet(k_0) \cdot \mathrm{pr}_1^* \alpha_1 \oplus \cdots \oplus \mathrm{H}^\bullet(k_0) \cdot \mathrm{pr}_n^* \alpha_1$$

as $\mathrm{H}^\bullet(k_0)$-module. Observe that there is an action of S_n on $\mathrm{Inv}^\bullet(\mathcal{B}S_2^{\times n})$ given by permuting the $\mathrm{pr}_i^* \alpha_1$.

Theorem 4.1.2. [GMS03, Th. 25.6] *The pullback morphism*

$$f^* : \mathrm{Inv}^\bullet(\mathcal{B}S_{2n}) \longrightarrow \mathrm{Inv}^\bullet(\mathcal{B}S_2^{\times n})$$

induces an isomorphism of $\mathrm{H}^\bullet(k_0)$*-algebras:*

$$\mathrm{Inv}^\bullet(\mathcal{B}S_{2n}) \simeq \mathrm{Inv}^\bullet(\mathcal{B}S_2^{\times n})^{S_n}$$

In particular $\mathrm{Inv}^\bullet(\mathcal{B}S_{2n})$ *is freely generated as* $\mathrm{H}^\bullet(k_0)$*-module by* $1, \alpha_1, \dots, \alpha_n$*, where* $\deg(\alpha_i) = i$*.*

To get some feeling for the cohomological invariants α_i appearing in theorem 4.1.2, take a multiquadratic étale algebra E of degree $2n$, i.e. E is isomorphic to a product of n quadratic étale algebras $E_j \simeq K[x]/(x^2 - a_j)$. Then $\alpha_i(E) = \sigma_i(\{a_1\}, \dots, \{a_n\})$, where σ_i is the i^{th}-elementary symmetric polynomial and the product is the cup product in $\mathrm{H}^\bullet(K)$.

There is another way to describe the cohomological invariants of $\mathcal{B}S_{2n}$, as showed in [GMS03, 25.7].

Definition 4.1.3. We define the morphism of classifying stack $q : \mathcal{B}S_{2n} \to \mathcal{B}O_{2n}$ as follows: given an étale morphism $f : T \to S$, we define the quadratic form q_T on the locally free $\mathcal{O}_S$-sheaf $f_*\mathcal{O}_T$ as the one that sends an element y to the trace of m_{y^2}, the $\mathcal{O}_S$-linear morphism given by multiplication for y^2.

In particular, if E is an étale K-algebra, we get a K-quadratic form $q_E := q(\mathrm{Spec}(E))$. Using the morphism q we can pull back cohomological invariants from $\mathcal{B}O_{2n}$ to $\mathcal{B}S_{2n}$.

Cohomological invariants of $\mathcal{B}O_n$, or equivalently cohomological invariants of non-degenerate quadratic forms of rank n, are discussed in [**GMS03**, 17]: we recall here just some basic facts.

Write $\langle a_1, \dots, a_n \rangle$ to indicate the diagonal quadratic form of rank n whose diagonal entries are $a_1, \dots, a_n$. Then we define the i^{th} Stiefel-Whitney invariant w_i as the cohomological invariant that sends $q \simeq \langle a_1, \dots, a_n \rangle$ to $\sigma_i(\{a_1\} \dots, \{a_n\})$, where as before σ_i is the elementary symmetric polynomial of degree i in the $\{a_j\}$, which we regard as elements of $K^*/(K^*)^2 \simeq H^1_{\text{ét}}(\mathrm{Spec}(K), \mu_2)$, and the product is the cup product of $\mathrm{H}^\bullet(K)$. It can be proved that w_i is well defined, i.e. does not depend on how we diagonalize q.

THEOREM 4.1.4. [**GMS03**, 17.3 and remark 17.4.(1)] *The graded-commutative* $\mathrm{H}^\bullet(k_0)$*-algebra* $\mathrm{Inv}^\bullet(\mathcal{B}O_n)$ *is generated as* $\mathrm{H}^\bullet(k_0)$*-module by the Stiefel-Whitney invariants* $1, w_1, \dots, w_n$

The multiplicative structure is given by the relations of the form

$$w_r \cdot w_s = \{-1\}^{m(r,s)} \cdot w_{r+s-m(r,s)}$$

where $m(r, s)$ *is obtained as follows: write* r *in dyadic form as* $\sum 2^i$ *for* $i \in R \subset \{0, 1, 2, \dots\}$*, and write* s *as* $\sum 2^i$ *for* $i \in S \subset \{0, 1, 2, \dots\}$*. Then* $m = \sum 2^i$ *for* $i \in R \cap S$*.*

The theorem above gives a rather complete picture of the graded-commutative ring $\mathrm{Inv}^\bullet(\mathcal{B}O_n)$. In particular, as observed in [**GMS03**, remark 17.4.(1)], the invariants w_{2^i} generate $\mathrm{Inv}^\bullet(\mathcal{B}O_n)$.

THEOREM 4.1.5. *Let* $q : \mathcal{B}S_{2n} \to \mathcal{B}O_{2n}$ *be the morphism introduced in definition 4.1.3.*

(1) *We have:*

$$\begin{aligned} &\alpha_i = q^* w_i + \{2\} \cdot q^* w_1 \cdot q^* w_{i-2} && \text{for } i \leq n \\ &q^* w_i = 0 && \text{for } i > n+1 \\ &q^* w_{n+1} = 0 && \text{if } n \text{ is even} \\ &q^* w_{n+1} = \{2\} \cdot q^* w_n && \text{if } n \text{ is odd} \end{aligned}$$

(2) *We have that* $\mathrm{Inv}^\bullet(\mathcal{B}S_{2n})$ *is freely generated as* $\mathrm{H}^\bullet(k_0)$*-module by*

$$1, q^* w_1, \dots, q^* w_n$$

In particular, the induced morphism of graded-commutative rings

$$q^* : \mathrm{Inv}^\bullet(\mathcal{B}O_{2n})/(w_{n+2}, \dots, w_{2n}) \longrightarrow \mathrm{Inv}^\bullet(\mathcal{B}S_{2n})$$

has kernel (w_{n+1}) *if* n *is even and has kernel equal to* $(w_{n+1} - \{2\} \cdot w_n)$ *if* n *is odd.*

PROOF. Point (1) is [**GMS03**, thm. 25.13]. For (2), observe that (1) implies that the morphism of $\mathrm{H}^\bullet(k_0)$-modules q^* is surjective with kernel equal to $(w_{n+1}, \dots, w_{2n})$ if n is even and equal to $(w_{n+1} - \{2\} \cdot w_n, w_{n+2}, \dots, w_{2n})$ if n is even. Therefore, if we mod out by the kernel, we get an isomorphism of $\mathrm{H}^\bullet(k_0)$-modules, hence of graded-commutative rings. □

The theorem above describes the multiplicative structure of $\mathrm{Inv}^\bullet(\mathcal{B}S_{2n})$, building on what we already know about $\mathrm{Inv}^\bullet(\mathcal{B}O_{2n})$ from theorem 4.1.4.

4.2. Multiplicative structure of $\mathrm{Inv}^\bullet(\mathcal{H}_g)$

We assume as usual that the base field k_0 has characteristic $\neq 2$. All the cohomological invariants are implicitly assumed to be with coefficients in $\mathbb{F}_2$.

In this section we study the multiplicative structure of $\mathrm{Inv}^\bullet(\mathcal{H}_g)$ for $g \geq 2$ even. Our main result is theorem 4.2.14.

4.2.1. Cohomological invariants of $\mathcal{B}_g$. Let g be any integer ≥ 2. Recall from [**AV04**, ex. 3.5] that the stack $\mathcal{H}_g$ is isomorphic to the stack $\mathcal{H}_g^\sim$ whose objects are triples $(\pi : C \to S, L, s)$, where $\pi : C \to S$ is a family of smooth rational curves, L is a line bundle of degree $-g-1$ and s is a global section of $L^{-\otimes 2}$ whose vanishing locus is étale over S.

For us it will be more convenient to work with $\mathcal{H}_g^\sim$ rather than $\mathcal{H}_g$, though everything we say can be restated in terms of objects of the latter stack.

Definition 4.2.1.

(1) We define $\mathcal{B}_g$ as the stack whose objects are triples $(\pi : C \to S, L, D)$ where $\pi : C \to S$ is a family of smooth rational curves, L is a line bundle on C of degree $-g-1$ and D is a divisor of C étale over S of degree $2g+2$.

(2) We define the morphism $F : \mathcal{H}_g^\sim \to \mathcal{B}_g$ as the one that sends a triple $(\pi : C \to S, L, s)$ to the triple $(\pi : C \to S, L, D_s)$ where D_s is the divisor defined as the vanishing locus of s.

(3) We define $\mathrm{pr} : \mathcal{B}_g \to \mathcal{B}S_{2g+2}$ as the morphism that sends a triple $(\pi : C \to S, L, D)$ to the étale morphism $D \to S$.

Observe that the morphism $F : \mathcal{H}_g^\sim \to \mathcal{B}_g$ makes $\mathcal{B}_g$ into a $\mathbb{G}_m$-torsor over $\mathcal{H}_g^\sim$, and in particular shows that $\mathcal{B}_g$ is a smooth Artin stack.

Proposition 4.2.2.

(1) The pullback morphism of graded-commutative $\mathrm{H}^\bullet(k_0)$*-algebras*

$$\mathrm{pr}^* : \mathrm{Inv}^\bullet(\mathcal{B}S_{2g+2}) \to \mathrm{Inv}^\bullet(\mathcal{B}_g)$$

is injective.

(2) If k_0 is algebraically closed, then pr^* is an isomorphism when g is even, and has cokernel equal to $\mathbb{F}_2 \cdot w_2$ when g is odd, where w_2 is the cohomological invariant pulled back via the morphism $\mathcal{B}_g \to \mathcal{B}\mathrm{PGL}_2$ that remembers only the family of smooth rational curves.

Proof. Suppose that $\mathrm{pr}^*\alpha_i \neq 0$ for $i = 1, \ldots, g+1$. We can rewrite $\mathrm{pr}^* = \oplus_{i=0}^{g+1} \mathrm{pr}_i^*$, where $\mathrm{pr}_i^* : \mathrm{Inv}^i(\mathcal{B}S_{2g+2}) \to \mathrm{Inv}^i(\mathcal{H}_g)$ is the homogeneous part of degree i. By theorem 4.1.2 we know that $\mathrm{Inv}^i(\mathcal{B}S_{2g+2}) \simeq \mathbb{F}_2 \cdot \alpha_i$, hence our assumption implies the injectivity of pr_i^* for each i, hence of pr^*.

Proving that $\mathrm{pr}^*\alpha_i \neq 0$ is equivalent to finding an étale K-algebra E of the form $K[x]/(f)$ such that $\alpha_i(E) \neq 0$: given such an algebra E, then $\mathrm{Spec}(E)$ will be by construction a divisor D in $\mathbb{P}^1_K$ of degree $2g+2$. The hyperelliptic curve $(C \to \mathrm{Spec}(K), \iota)$ obtained by taking the double cover of $\mathbb{P}^1_K$ ramified along D will be such that $\mathrm{pr}^*\alpha_i((C \to \mathrm{Spec}(K), \iota)) = \alpha_i(E) \neq 0$, which will imply that $\mathrm{pr}^*\alpha_i \neq 0$.

By [**GMS03**, 12.3] $\alpha_i(E^{\mathrm{ver}}) \neq 0$, where E^{ver} is the versal S_{2g+2}-torsor, which is the algebra

$$E^{\mathrm{ver}} \simeq k_0(c_1, \ldots, c_{2g+2})[t]/(t^n + c_1 t^{n-1} + \cdots + c_n)$$

over $K = k_0(c_1, \ldots, c_n)$ (see [GMS03, ex. 5.6.(2)]). This K-point thus lifts to a K-point of $\mathcal{H}_g$, and we are done.

For point (2), when k_0 is algebraically closed we know from [Pir17, cor. 4.7] and proposition 3.1.2 that for even g the $\mathbb{F}_2$-vector space $\mathrm{Inv}^\bullet(\mathcal{B}_g)$ is generated by $g+2$ elements $1, x_1, \ldots, x_{g+1}$, where $\deg(x_i) = i$, whereas if g is odd $\mathrm{Inv}^\bullet(\mathcal{B}_g)$ is generated by $g+2$ elements and the Stiefel-Whitney invariant w_2. This implies the surjectivity of pr^* when g is even, and that the cokernel is equal to $\mathbb{F}_2 \cdot w_2$ when g is odd. □

DEFINITION 4.2.3. We define the *Weierstrass morphism* $W : \mathcal{H}_g \to \mathcal{B}S_{2g+2}$ as the composition of $\mathrm{pr} \circ F$ with the isomorphism $\mathcal{H}_g \simeq \mathcal{H}_g^{\sim}$, where pr and F are the morphisms introduced in definition 4.2.1.

In other terms, the Weierstrass morphism sends a family of hyperelliptic curves $(C \to S, \iota)$ to the Weierstrass divisor $W_{C/S}$, regarded as an étale scheme over S.

COROLLARY 4.2.4. *The morphism of graded-commutative* $\mathrm{H}^\bullet(k_0)$*-algebras*

$$W^* : \mathrm{Inv}^\bullet(\mathcal{B}S_{2g+2}) \longrightarrow \mathrm{Inv}^\bullet(\mathcal{H}_g)$$

induced by the Weierstrass morphism (definition 4.2.3) is injective.

PROOF. Recall that $W = \mathrm{pr} \circ F$. From proposition 4.2.2.(1) we know that pr^* is injective, and F^* is injective because $F : \mathcal{H}_g \to \mathcal{B}_g$ is a $\mathbb{G}_m$-torsor, hence smooth-Nisnevich. □

DEFINITION 4.2.5. For $1 \leq i \leq 2g+2$ we define the cohomological invariant β_i of $\mathcal{H}_g$ as

$$\beta_i := W^* q^* w_i$$

where $W : \mathcal{H}_g \to \mathcal{B}S_{2g+2}$ is the Weierstrass morphism (definition 4.2.3) and $q : \mathcal{B}S_{2g+2} \to \mathcal{B}O_{2g+2}$ is the morphism introduced in definition 4.1.3.

PROPOSITION 4.2.6. *Let k_0 be a field of characteristic $\neq 2$ and fix a positive integer $g \geq 2$. Then the cohomological invariants of $\mathcal{H}_g$*

$$1, \beta_1, \ldots, \beta_{g+1}$$

are $\mathbb{F}_2$-linearly independent, $\beta_i = 0$ for $i > g+2$, $\beta_{g+2} = 0$ if g is odd and it is equal to $\{2\} \cdot \beta_{g+1}$ if g is even.

Their multiplicative structure is given by the formula:

$$\beta_r \cdot \beta_s = \{-1\}^m \beta_{r+s-m}$$

where m is computed as follows: write r in dyadic form as $\sum 2^i$ for $i \in R \subset \{0, 1, 2, \ldots\}$, and write s as $\sum 2^i$ for $i \in S \subset \{0, 1, 2, \ldots\}$. Then $m = \sum 2^i$ for $i \in R \cap S$.

PROOF. As $\deg(\beta_i) = i$, we can equivalently show that $\beta_i \neq 0$ for $i = 1, \ldots, g+1$. We know from corollary 4.2.4 that W^* is injective, hence we only have to check $q^* w_i \neq 0$ for $i = 1, \ldots, g+1$, which follows from theorem 4.1.5.(1).

The same theorem also implies the statements on β_i for $i > g+1$.

Finally, the multiplicative structure is deduced from the one of the Stiefel-Whitney invariants stated in theorem 4.1.4. □

4.2.2. The exceptional invariant. Suppose that $g \geq 2$ is even. Recall from the previous section that $\mathcal{B}_g$ is the smooth Artin stack whose objects are triples $(\pi : C \to S, L, D)$ where $\pi : C \to S$ is a family of smooth rational curves, L is a line bundle on C of degree $-g-1$ and D is a divisor of C étale over S of degree $2g+2$.

PROPOSITION 4.2.7.

(1) *Let* $(\pi : \mathcal{C}_g \to \mathcal{B}_g, \mathcal{L}_g, \mathcal{D}_g)$ *be the universal object over* $\mathcal{B}_g$*. Then* $\mathcal{C}_g$ *is a smooth Artin stack and*

$$\pi^* : \mathrm{Inv}^\bullet(\mathcal{B}_g) \longrightarrow \mathrm{Inv}^\bullet(\mathcal{C}_g)$$

is an isomorphism.

(2) *Let* $\mathcal{C}_g^\sim$ *be the pullback of* $\mathcal{C}_g$ *along* $F : \mathcal{H}_g^\sim \to \mathcal{B}_g$ *(see definition 4.2.1.(2)). Then:*

$$\mathrm{Inv}^\bullet(\mathcal{H}_g^\sim) \simeq \mathrm{Inv}^\bullet(\mathcal{C}_g^\sim)$$

In particular, the induced map of graded-commutative $\mathrm{H}^\bullet(k_0)$*-algebras* $\mathrm{Inv}^\bullet(\mathcal{B}_g) \to \mathrm{Inv}^\bullet(\mathcal{H}_g)$ *is injective.*

Proof. By definition $\mathcal{L}$ has degree $-g-1$, hence the line bundle on $\mathcal{C}_g$ defined as $\mathcal{M} := \omega_\pi^{\frac{g}{2}} \otimes \mathcal{L}_g^{-1}$ has degree 1. This implies that there is an isomorphism between $\mathbb{P}(\pi_*\mathcal{M}) \simeq \mathcal{C}_g$.

Observe that $\mathcal{B}_g$ is isomorphic to the stack $[(\mathbb{P}(1, 2g+2) \setminus \Delta_{1,2g+2})/\mathrm{GL}_2]$, hence $\mathcal{C}_g$ is the projectivization of an equivariant vector bundle E over $\mathbb{P}(1, 2g+2) \setminus \Delta_{1,2g+2}$. From this and the usual isomorphism between cohomological invariants of quotient stacks and equivariant 0-cycles with coefficients of the cover (see [**Pir18a**, thm. 4.16]) we deduce that $\mathrm{Inv}^\bullet(\mathcal{C}_g) \simeq A^0_{\mathrm{GL}_2}(\mathbb{P}(E))$. The isomorphism of cohomological invariants then follows from the projective bundle formula for Chow groups with coefficients (proposition 1.2.3.(7)).

This proves (1) and implies (2), because $\mathcal{C}_g^\sim \to \mathcal{H}_g^\sim$ will be the projectivization of a vector bundle as well. The last assertion is clear. □

Remark 4.2.8. The hypothesis of g being even is essential in the proof above, otherwise it is not true that $\mathcal{C}_g$ is the projectivization of a rank 2 vector bundle.

This is the main difference between the even and the odd case: if $(C \to S, \iota)$ is a family of hyperelliptic curves of genus g, the family of smooth rational curves $C/\iota \to S$ will always have trivial class in the Brauer group when g is even. This is not true in general for odd g.

The remainder of the section will be devoted to the construction of a cohomological invariant η_{g+2} of $\mathcal{C}_g^\sim$ that does not come from $\mathcal{B}_g$.

The objects of $\mathcal{C}_g$ are quadruples $(C \to S, L, D, p)$ where $(C \to S, L, D)$ is an object of $\mathcal{B}_g$ and $p : S \to C$ is a section of the family of smooth rational curves. Similarly, the objects of $\mathcal{C}_g^\sim$ are quadruples $(C \to S, L, s, p)$ where $(C \to S, L, s)$ is an object of $\mathcal{H}_g^\sim$ and $p : S \to C$ is a section of the family of smooth rational curves.

Let $\mathcal{U}_g$ be the open substack of $\mathcal{C}_g$ whose objects are those quadruples $(C \to S, L, D, p)$ such that D and p do not intersect. Analogously, let $\mathcal{U}_g^\sim$ be the open substack of $\mathcal{C}_g^\sim$ whose objects are those quadruples such that s never vanishes on the section p.

Lemma 4.2.9.

(1) *Let* B *be the Borel subgroup of* GL_2 *of upper triangular matrices. Then we have:*
- $\mathcal{C}_g \simeq [(\mathbb{P}(1, 2g+2) \setminus \Delta_{1,2g+2})/B]$
- $\mathcal{C}_g^\sim \simeq [(\mathbb{A}(1, 2g+2) \setminus \Delta'_{1,2g+2})/B]$

(2) *Let* U *be the* B*-invariant open subscheme of* $(\mathbb{A}(1, 2g+2) \setminus \Delta'_{1,2g+2})$ *whose points are those binary forms that does not vanish on* $[1 : 0]$*, and let* V *be the image of the projection of* U *in* $(\mathbb{P}(1, 2g+2) \setminus \Delta_{1,2g+2})$*. Then:*
- $\mathcal{U}_g \simeq [V/B]$
- $\mathcal{U}_g^\sim \simeq [U/B]$

Proof. Recall that if $(\pi : C \to S, L, D)$ is an object of $\mathcal{C}_g$, then the line bundle $E := \omega_\pi^{\otimes \frac{g}{2}} \otimes L^{-1}$ has degree one, hence $C \simeq \mathbb{P}(\pi_* E)$ canonically, and the set of sections $p : S \to C$ is then equivalent to the set of line bundles $M \subset E$.

Consider the B-torsor over $\mathcal{C}_g$ whose objects consist of the datum

$$(C \to S, L, D, p, \varphi)$$

where $(C \to S, L, D, p)$ is an object of $\mathcal{C}_g$ and φ is an isomorphism between flags:

$$\varphi : (M \subset E) \simeq (\mathcal{O}_S \subset \mathcal{O}_S^{\oplus 2})$$

The morphisms in this B-torsor consist of the datum of a morphism between objects of $\mathcal{C}_g$ that commutes with the trivializations of the respective flags.

There is an equivalence between this B-torsor and $\mathbb{P}(1, 2g+2) \setminus \Delta_{1,2g+2}$: given a datum $(C \to S, L, D, p, \varphi)$ as before, the trivialization φ of the flag induces an isomorphism of C with $\mathbb{P}^1_S$, of p with $\{\infty\} \times S$ and of L with $\mathcal{O}(-g-1)$.

This implies that there is a well defined morphism from the B-torsor to $\mathbb{P}(1, 2g+2) \setminus \Delta_{1,2g+2}$ that, after identifying C with $\mathbb{P}^1$, only remembers D, and that the composition of this with the morphism that sends the datum of a divisor $D \subset \mathbb{P}^1_S$ étale over S of degree $2g+2$ to the object $(\mathbb{P}^1_S, \mathcal{O}(-g-1), D, \{\infty\} \times S, \mathrm{id})$ is equivalent to the identity.

This proves that $\mathbb{P}(1, 2g+2) \setminus \Delta_{1,2g+2}$ is a B-torsor over $\mathcal{C}_g$, hence the first statement. The second one is proved exactly in the same way: the only difference is that instead of a divisor D, the datum of an object of $\mathcal{C}_g^\sim$ contains a global section s of $L^{-\otimes 2}$.

Finally, point (2) follows from the construction above. □

Remark 4.2.10. The lemma above in particular implies that $(\mathbb{A}(1, 2g+2) \setminus \Delta'_{1,2g+2}) \to \mathcal{C}_g^\sim$ is a smooth-Nisnevich cover, because the group B is special.

This fact will be relevant for the construction of the exceptional cohomological invariant η_{g+2} of $\mathcal{C}_g^\sim$ (i.e. a cohomological invariant which does not come from $\mathcal{B}_g$): we will first construct the exceptional invariant on the cover $\mathbb{A}(1, 2g+2) \setminus \Delta'_{1,2g+2}$, and subsequently we will show that is B-invariant, hence descends to an invariant of $\mathcal{C}_g^\sim$.

Lemma 4.2.11. *Let U and V be as in lemma 4.2.9. Then we have:*

$$\mathrm{Inv}^\bullet(U) \simeq \mathrm{Inv}^\bullet(V)[t]/(t^2 - \{-1\} \cdot t)$$

where t is a cohomological invariant of degree 1 that sends a binary form $f(x, y)$ with coefficients in a field K to $f(1, 0)$, regarded as an element of $K^*/(K^*)^2 \simeq H^1_{\text{ét}}(\mathrm{Spec}(K), \mu_2)$.

Proof. Let t be the degree 1 cohomological invariant of $\mathbb{G}_m$ which sends an element x of $\mathbb{G}_m(K) = K^*$ to its equivalence class in $K^*/(K^*)^2$. Then we claim that for any scheme X we have:

$$\mathrm{Inv}^\bullet(X \times \mathbb{G}_m) \simeq \mathrm{Inv}^\bullet(X)[t]/(t^2 - \{-1\} \cdot t)$$

Observe that $U \to V$ is a $\mathbb{G}_m$-torsor that can be trivialized via the isomorphism $U \simeq V \times \mathbb{G}_m$ that sends a section s of $\mathcal{O}_{\mathbb{P}^1_S}(2g+2)$ to the pair $(D \to S, s(\infty))$, where by $s(\infty)$ we mean the composition $S \to \{\infty\} \times S \to \mathbb{A}^1 \times S$ regarded as an element of $\mathcal{O}_S(S)$ (the trivialization $\mathcal{O}(2g+2)|_{\{\infty\}} \simeq \mathbb{A}^1$ is induced by the restriction of the section x^{2g+2}).

To prove the claim, we argue as follows: by proposition 1.2.6, we know that

$$\mathrm{Inv}^\bullet(X \times \mathbb{G}_m) \simeq (\mathrm{Inv}^\bullet(X)/c_1(L)) \oplus \ker(c_1(L))[1]$$

as $\mathbb{F}_2$-modules, where L is the line bundle associated to the trivial $\mathbb{G}_m$-torsor: observe that in our case we have $c_1(L) = 0$.

Actually, from the proof of proposition 1.2.6 we see that the left summand can be canonically identified with the submodule of invariants pulled back from X, and the right summand can be identified with the submodule of invariants such that the boundary morphism $\mathrm{Inv}^\bullet(X \times \mathbb{G}_m) \to \mathrm{Inv}^\bullet(X)$ induced by the open inclusion $X \times \mathbb{G}_m \subset X \times \mathbb{A}^1$ is not zero.

Let γ be a cohomological invariant pulled back from X: then $\partial(t \cdot \gamma) = \partial t \cdot \gamma$ because the pullback morphism $\mathrm{Inv}^\bullet(X) \to \mathrm{Inv}^\bullet(X \times \mathbb{G}_m)$ factorizes through $\mathrm{Inv}^\bullet(X \times \mathbb{A}^1)$. Moreover $\partial t = 1$, otherwise t would be an invariant of the whole $X \times \mathbb{A}^1$, which is not the case.

Putting all together, we have proved that the morphism of graded-commutative $\mathrm{H}^\bullet(k_0)$-algebras

$$\mathrm{Inv}^\bullet(X)[t] \longrightarrow \mathrm{Inv}^\bullet(X \times \mathbb{G}_m)$$

is surjective. We only have to prove that $t^2 = \{-1\} \cdot t$.

From [GMS03, ex. 17.5.(1)] we know that, given a field K over k_0, x and y in K^* such that $x + y \neq 0$, then we have $\{x\} \cdot \{y\} = \{x + y\} \cdot \{-xy\}$ in $\mathrm{H}^\bullet(K)$ (we are only considering cohomology groups with $\mathbb{F}_2$-coefficients).

From this we deduce

$$\{t\}^2 = \{2t\} \cdot \{-1\} = (\{2\} + \{t\}) \cdot \{-1\} = \{2\} \cdot \{-1\} + \{t\} \cdot \{-1\}$$

and that $0 = \{1\} \cdot \{1\} = \{2\} \cdot \{-1\}$, from which the relation above follows. □

PROPOSITION 4.2.12. *Let $f : U \to \mathcal{B}S_{2g+2}$ be the morphism that sends a global section s of $\mathcal{O}(2g + 2)$ to the étale morphism $D \to S$, where D is the vanishing locus of s. Let t be the degree 1 cohomological invariant of U introduced in lemma 4.2.11.*

Then the degree $g + 2$ cohomological invariant $t \cdot f^*\alpha_{g+1}$ of U is non-zero and descends to a cohomological invariant of $\mathcal{U}_g^\sim$, which moreover extends to a cohomological invariant of $\mathcal{C}_g^\sim$.

PROOF. The fact that $t \cdot f^*\alpha_{g+1}$ is non-zero follows from the computation of $\mathrm{Inv}^\bullet(U)$ of lemma 4.2.11.

By lemma 4.2.9.(1) and remark 4.2.10, we know that U is a smooth-Nisnevich cover of $\mathcal{U}_g^\sim$ and that a cohomological invariant of U descends if and only if it is B-invariant, where B is the group of upper triangular matrices.

Therefore, the degree 1 cohomological invariant t of U descends to an invariant of $\mathcal{U}_g^\sim$, because it is B-invariant: if $f(x, y)$ is a binary form of degree $2g + 2$ with coefficients in a field K and b is an element of B

$$b = \begin{pmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{pmatrix}$$

then $t(b \cdot f(x, y)) = [\det(b)^g b_{11}^{2g+2} f(1, 0)]$, which is equal to $[f(1, 0)]$ in $K^*/(K^*)^2$. This implies that $t \cdot f^*\alpha_{g+1}$ descends to $\mathcal{C}_g^\sim$.

Let Z be the complement of U in $\mathbb{A}(1, 2g + 2) \setminus \Delta'_{1,2g+2}$: then proposition 1.2.3.(4) gives us the following long exact sequence of equivariant Chow groups with coefficients:

$$0 \longrightarrow A_B^0(\mathbb{A}(1, 2g + 2) \setminus \Delta'_{1,2g+2}) \longrightarrow A_B^0(U) \longrightarrow A_B^0(Z)$$

To show that $t \cdot f^*\alpha_{g+1}$ comes from a global cohomological invariant, we have to show that $\partial(t \cdot f^*\alpha_{g+1}) = 0$.

We already know that $f^*\alpha_{g+1}$ extends, because is the restriction to U of the pullback of α_{g+1} along the morphism $\mathcal{C}_g^\sim \to \mathcal{B}_g$. We also know from the proof of lemma 4.2.11 that $\partial(t) = 1$. This implies that $\partial(t \cdot f^*\alpha_{g+1}) = \alpha_{g+1}$.

We have reduced ourselves to prove that α_{g+1} vanishes on the étale algebras that are in the image of $[Z/B] \to \mathcal{B}S_{2g+2}$.

Observe that these algebras will be of the form $E' \times K$: this is because by construction the section $p : \mathrm{Spec}(K) \to C$ of an object $(C \to \mathrm{Spec}(K), L, s, p)$ of $\mathcal{C}_g^\sim$ contained in the closed substack $[Z/B]$ will factorize through $D \subset C$, the vanishing locus of the section s.

Therefore, if $D = \mathrm{Spec}(E)$, the section p will give us a section of E, thus $E \simeq E' \times K$. Applying the formulas of [**GMS03**, 25.2], we deduce:

$$\alpha_{g+1}(E) = \alpha_{g+1}(E') = 0$$

because E' has degree $2g + 1$ and α_i vanishes on étale algebras of degree $< 2i$ (see [**GMS03**, th. 25.6]). This concludes the proof of the proposition. $\square$

DEFINITION 4.2.13. We define *the exceptional cohomological invariant* η_{g+2} of $\mathrm{Inv}^\bullet(\mathcal{H}_g)$ as the image via the isomorphism $\mathrm{Inv}^\bullet(\mathcal{C}_g^\sim) \simeq \mathrm{Inv}^\bullet(\mathcal{H}_g)$ of the degree $g + 2$ cohomological invariant of $\mathcal{C}_g^\sim$ constructed in proposition 4.2.12.

We are ready to prove the main result of the chapter.

THEOREM 4.2.14. *Fix a base field k_0 of characteristic $\neq 2$ and an even number $g \geq 2$.*

Let β_i for $i = 1, \dots, 2g + 2$ be the cohomological invariant of $\mathcal{H}_g$ introduced in definition 4.2.5 and let η_{g+2} be the exceptional cohomological invariant introduced in definition 4.2.13. Then:

(1) *The invariants $1, \beta_1, \dots, \beta_{g+1}, \eta_{g+2}$ are $\mathbb{F}_2$-linearly independent, $\beta_i = 0$ for $i > g + 2$ and $\beta_{g+2} = \{2\} \cdot \beta_{g+1}$.*

(2) *The multiplicative structure is given by the formulas:*

$$\beta_r \cdot \beta_s = \{-1\}^{m(r,s)} \beta_{r+s-m}$$
$$\beta_i \cdot \eta_{g+2} = 0 \text{ for } i \neq g + 1$$
$$\beta_{g+1} \cdot \eta_{g+2} = \{-1\}^{g+1} \eta_{g+2}$$
$$\eta_{g+2} \cdot \eta_{g+2} = \{-1\}^{g+2} \eta_{g+2}$$

where $m(r, s)$ is computed as follows: write r in dyadic form as $\sum 2^i$ for $i \in R \subset \{0, 1, 2, \dots\}$, and write s as $\sum 2^i$ for $i \in S \subset \{0, 1, 2, \dots\}$. Then $m(r, s) = \sum 2^i$ for $i \in R \cap S$.

(3) *If k_0 is algebraically closed, then the invariants $1, \beta_1, \dots, \beta_{g+1}, \eta_{g+2}$ generate $\mathrm{Inv}^\bullet(\mathcal{H}_g)$.*

PROOF. We already know (1) from proposition 4.2.6: the only thing to observe is that $\deg(\eta_{g+2}) = g + 2$, hence it is independent of the β_i for $i < g + 2$.

Proposition 4.2.6 also gives us the relations in (2) not involving η_{g+2}. To compute $\beta_i \cdot \eta_{g+2}$, we can pull back this invariant along the composition:

$$\mathcal{U}_g^\sim \hookrightarrow \mathcal{C}_g^\sim \longrightarrow \mathcal{H}_g^\sim \simeq \mathcal{H}_g$$

The induced morphism at the level of cohomological invariants is injective, because $\mathcal{U}_g \hookrightarrow \mathcal{C}_g^\sim$ is an open embedding, and the pullback morphism $\mathrm{Inv}^\bullet(\mathcal{H}_g^\sim) \to \mathrm{Inv}^\bullet(\mathcal{C}_g^\sim)$ is an isomorphism (see proposition 4.2.7.(2)). Therefore, it is enough to prove the formulas of (2) for the pullback of the invariants to $\mathcal{U}_g^\sim$.

The pullback of η_{g+2} to $\mathcal{U}_g^\sim$ is by construction (see definition 4.2.13) equal to $t \cdot f^* \alpha_{g+1}$, where t is the degree 1 invariant that sends an object

$$(C \to \mathrm{Spec}(K), L, s, p : \mathrm{Spec}(K) \to C)$$

to the evaluation in $K^*/(K^*)^2$ of s at the K-point p and $f : \mathcal{U}_g \to \mathcal{B}S_{2g+2}$ is the morphism that only remembers the K-étale algebra $\{s = 0\}$. This has been proved in proposition 4.2.12.

Moreover, the pullback of β_i is equal to $f^*q^*w_i$, where q is the morphism introduced in definition 4.1.3 and w_i is the Stiefel-Whitney invariant of degree i.

Recall from theorem 4.1.5 that α_{g+1} is equal to $q^*w_{g+1} + \{2\} \cdot q^*w_1 \cdot q^*w_{g-1}$. We know from 4.1.4 that $w_1 \cdot w_{g-1} = \{-1\} \cdot w_{g-1}$ and the formula [**GMS03**, ex. 17.5.(1)] tells us that $0 = \{1\}^2 = \{2\} \cdot \{-1\}$, hence $\alpha_{g+1} = q^*w_{g+1}$.

Theorem 4.1.4 tells us that $w_i \cdot w_{g+1} = \{-1\}^m w_{g+1+i-m}$: using the fact that $q^*w_j = 0$ for $j > g+2$, we deduce that $f^*q^*(w_i \cdot w_{g+1}) = 0$ unless $i - m = 0$ or 1 (by construction $m \leq i$).

Suppose that $m = i$: as we are supposing that $i \leq g+1$, this implies that $i = g+1$, in which case we have $w_{g+1}^2 = \{-1\}^{g+1} \cdot w_{g+1}$. From this we deduce:

$$\begin{aligned} f^*q^*(w_{g+1}) \cdot t \cdot f^*q^*(w_{g+1}) &= -t \cdot f^*q^*(w_{g+1}) \\ &= -t \cdot f^*q^*(\{-1\}^{g+1} \cdot w_{g+1} \\ &= \{-1\}^{g+1} t \cdot f^*q^*w_{g+1} \end{aligned}$$

which implies $\beta_{g+1} \cdot \eta_{g+1} = \{-1\}^{g+1}\eta_{g+1}$. Suppose $i - m = 1$, then:

$$\begin{aligned} w_i \cdot w_{g+1} &= \{-1\}^m \cdot w_{g+2} \\ &= \{-1\}^i \cdot \{2\} \cdot w_{g+1} \end{aligned}$$

Recall that $\{-1\} \cdot \{2\} = 0$, hence the last line above is zero unless $m = 0$ and $i = 1$: this can never happen, because both 1 and $g+1$ are odd, thus $m \geq 1$.

To conclude the proof of (2), observe that:

$$\begin{aligned} t \cdot f^*q^*w_{g+1} \cdot t \cdot f^*q^*w_{g+1} &= -t^2 \cdot f^*q^*w_{g+1}^2 \\ &= -\{-1\} \cdot t \cdot \{-1\}^{g+1} w_{g+1} \\ &= \{-1\}^{g+2} t \cdot w_{g+1} \end{aligned}$$

Finally, point (3) is a direct consequence of what we already proved and [**Pir17**, thm. 1.1]. ☐

REMARK 4.2.15. We may ask ourselves if the generators of $\mathrm{Inv}^\bullet(\mathcal{H}_g)$ as $\mathrm{H}^\bullet(k_0)$-algebra generate also when the base field is not algebraically closed.

Future directions

We outline here some future directions of research related to the themes discussed in the previous chapter.

4.2.3. Multiplicative structure of $\mathcal{H}_g$, g odd. The techniques used to investigate the multiplicative structure of $\mathrm{Inv}^\bullet(\mathcal{H}_g)$ for an even $g \geq 2$, and in particular the construction of the exceptional invariant η_{g+2} (see proposition 4.2.12), does not seem to apply in the odd genus case.

Let $g \geq 3$ be an odd number. The universal rational curve $\mathcal{C}_g^\sim \to \mathcal{H}_g^\sim$ is not a smooth-Nisnevich cover: this is a consequence of the fact that the quotient of a hyperelliptic curve of odd genus by the involution is a curve of genus zero that does not necessarily have a rational point.

What one might consider is the Hilbert stack $\mathcal{X}_g$ parametrising 0-dimensional subschemes Z of degree 2 relative to the morphism $\mathbb{P}(\pi_* T_{\mathcal{C}_g^\sim/\mathcal{H}_g^\sim}) \to \mathcal{H}_g^\sim$: this Hilbert stack is actually isomorphic to a projective bundle over $\mathcal{H}_g^\sim$, hence the pullback morphism induces an isomorphism at the level of cohomological invariants.

Inside $\mathcal{X}_g$ we may take the open substack $\mathcal{U}_g^\sim$ whose objects are those quadruples $(\pi : C \to S, L, s, Z)$ such that the family of smooth rational curves $\pi : C \to S$, once embedded in $\mathbb{P}(\pi_* T_{C/S})$, does not intersect the 0-dimensional substack Z, and the section s does not vanish on the intersection of C with the unique line in $\mathbb{P}(\pi_* T_{C/S})$ that contains Z.

There is a representation of $\mathcal{U}_g^\sim$ as a quotient stack $[U/T]$ where U is an open subscheme of $V_{g+1,3}$ such that the projection $U \to V \subset \mathbb{P}(V_{g+1})_3$ makes U into a trivial $\mathbb{G}_m$-torsor over V.

Similarly to what is done in proposition 4.2.12, we can use this fact to construct an exceptional cohomological invariant of degree $g+2$, but we are not able to prove that it extends to a global invariant.

Moreover, when g is odd there is another cohomological invariant in $\mathcal{H}_g$ that we must take into account to understand the multiplicative structure of $\mathrm{Inv}^\bullet(\mathcal{H}_g)$, namely the second Stiefel-Whitney invariant coming from the morphism $\mathcal{H}_g \to \mathcal{B}\mathrm{PGL}_2$. At the present moment, we do not have fully understood how to multiply this invariant with the other ones.

Therefore, the following seems to be an interesting line of future research:

QUESTION. *What is the multiplicative structure of $\mathcal{H}_g$ for $g \geq 3$ odd?*

4.2.4. Cohomological invariants of some compactifications of $\mathcal{H}_g$. Theorem 4.2.14 is telling us that when the base field k_0 is algebraically closed, the cohomological invariants of $\mathcal{H}_g$ for an even $g \geq 2$ are all constructed using the cohomological invariants of $\mathcal{B}S_{2g+2}$, exploiting the fundamental fact that the Weierstrass divisor of a family of hyperelliptic curves is an étale algebra over the base.

If we consider the Deligne-Mumford compactification $\overline{\mathcal{H}}_g$ by stable curves, there is no obvious way to extend the Weierstrass morphism 4.2.3 to the whole $\overline{\mathcal{H}}_g$, thus we expect that the cohomological invariants of $\mathcal{H}_g$ do not come from $\mathrm{Inv}^\bullet(\overline{\mathcal{H}}_g)$. The next proposition shows that this is actually the case.

Proposition 4.2.16. *Let k_0 be a field of characteristic $\neq 2$, and let $\overline{\mathcal{H}}_g$ be the Deligne-Mumford compactification of $\mathcal{H}_g$ via stable curves, where $g \geq 2$ is even. Then $\mathrm{Inv}^\bullet(\overline{\mathcal{H}_g})$ is trivial.*

Sketch of proof. We have an injective morphism of $\mathrm{H}^\bullet(k_0)$-modules:

$$\mathrm{Inv}^\bullet(\overline{\mathcal{H}}_g) \hookrightarrow \mathrm{Inv}^\bullet(\overline{\mathcal{H}}_g^o)$$

where $\overline{\mathcal{H}}_g^o$ is the open substack of families of stable hyperelliptic curves such that every geometric fibre is irreducible and has at most one singular point.

We want to prove that the right hand side is trivial.

For doing this, observe that we have an isomorphism:

$$\overline{\mathcal{H}}_g^o \simeq [\mathbb{A}(1, 2g+2) \setminus \Delta'_{2,2g+2} \cup Z)/\mathrm{GL}_2]$$

where $\Delta'_{2,2g+2}$ is the usual subscheme parametrising forms with at least two double roots, and Z is the subscheme parametrising forms with at least one triple point.

Both Z and $\Delta'_{2,2g+2}$ have codimension > 1 in $\mathbb{A}(1, 2g + 2)$, from which we deduce:

$$\mathrm{Inv}^\bullet(\overline{\mathcal{H}}_g^o) \simeq A^0_{\mathrm{GL}_2}(\mathbb{A}(1, 2g+2)) \simeq \mathrm{H}^\bullet(k_0)$$

This concludes the proof. □

Let $\widetilde{\mathcal{H}}_g$ be the compactification of $\mathcal{H}_g$ by admissible covers. The Weierstrass morphism can be extended to a morphism

$$\widetilde{W} : \widetilde{\mathcal{H}}_g \longrightarrow \mathcal{B}S_{2g+2}$$

Therefore, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Inv}^\bullet(\widetilde{\mathcal{H}}_g) & \longrightarrow & \mathrm{Inv}^\bullet(\mathcal{H}_g) \\ \uparrow {\scriptstyle \widetilde{W}^*} & \nearrow {\scriptstyle W^*} & \\ \mathrm{Inv}^\bullet(\mathcal{B}S_{2g+2}) & & \end{array}$$

We know from corollary 4.2.4 that W^* is injective, hence so it is $\widetilde{W}^*$.

Question. *Let $g \geq 2$ be an even number, and fix a base field k_0 of characteristic $\neq 2$. Does the exceptional invariant η_{g+2} of $\mathcal{H}_g$ defined in 4.2.13 extend to a cohomological invariant of the compactified stack $\widetilde{\mathcal{H}}_g$?*

An answer to this question would allow us to describe $\mathrm{Inv}^\bullet(\widetilde{\mathcal{H}}_g)$

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