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Some Variations on Ricci Flow

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Introduction

Since its appearance almost thirty–five years ago, the Ricci flow has given rise to several beautiful results, among which the wonderful proofs of both Poincaré and Thurston geometrization conjectures, started by Hamilton in the '80–'90 [49, 52, 54, 55, 57] and substantially completed by Perelman in 2002–2003 [69, 70, 71]. The central idea of evolving geometric quantities by means of parabolic PDEs in order to make them "better" can be traced back to Eells and Sampson [36], who evolved maps between Riemannian manifolds by the so called *harmonic map* flow, in order to get extremal points of the energy functional

$$\mathcal{E}(\varphi) = \frac{1}{2}\int_N \mathrm{tr}_h(\varphi^* g)\, d\mu_h\,, \tag{0.1}$$

where $\varphi : N \to M$ is a smooth map between two Riemannian manifolds (N, h) and (M, g) and $d\mu_h$ is the canonical volume form of the metric h. The critical points include, as special cases, geodesics, harmonic maps and minimal submanifolds.

Nowadays, many geometric flows have been studied, most of which deform the metric g of a Riemannian manifold M by means of a differential operator $L(g)$ depending on the curvature tensors of the metric. There is also a wide set, constantly growing, of ideas and techniques, developed for the Ricci flow, that can often be adapted to other geometric flows. We refer for this subject to the series of books by Chow, Knopf et alt. [25, 26, 27, 29]. In the first chapter of the present work we recall some of these methods which will be used subsequently.

One of the major benefits in using parabolic PDEs in Riemannian geometry is the possibility of deforming the metric in a controlled way: this allows to obtain *a priori* estimates of the solutions and ultimately study their behaviour closer and closer to their maximal time of existence. When the flow, as it happens for the Ricci flow on three–manifolds, determines and restricts strongly the geometric nature of the possible limit points, one can classify them and eventually get informations on the topological and differential structure of the underlying manifold. Regarding other flows, when they are studied for the first time, it is costumary to keep in mind the line of analysis of the Ricci flow as a general guide, while hoping for new features, which would make the exploration worthwhile.

Some of the geometric flows studied in the literature, as the harmonic map flow, arise naturally as gradient flows of a geometric functional, indeed, they were actually introduced in order to obtain informations on the critical points of such functionals. In these cases, not only PDEs techniques are useful, but also variational methods may help in studying the behaviour of solutions. Two well known gradient–like flows are the Yamabe flow (see [10, 75, 81], for instance), introduced by Hamilton to obtain a constant scalar curvature metric in any given conformal class of a Riemannian manifold (M, g), and the mean curvature flow (see [35, 63, 80] for instance) of a compact hypersurface in the Euclidean space. To give an idea on how the variational approach to geometric flows could be useful, we may recall the first paper by Perelman [69], where the author proved that the Ricci flow can be regarded as a gradient flow and, by means of a related monotonicity formula, he was able to show a control of the injectivity radius during the blow–up procedure, which was one of the major obstacles in Hamilton's approach to Poincaré conjecture.

The link between geometric analysis and topology, which proved strong and fruitful, is not the only one. As claimed by Perelman [69, Introduction and Section 1], the idea of a monotone functional for the Ricci flow was inspired by its connection with the *renormalization group* (RG) flow for two–dimensional non linear σ–models in Quantum Field Theory (see [4] for a survey on the connections between the RG–flow and other geometric flows).

This flow arises from the quantization of the classical (harmonic map) action defined by the functional $\mathcal{E}$ in formula (0.1), when (N, h) is a surface and (M, g) a Riemannian manifold of dimension $n \geq 3$. For physical reasons, the functional is multiplied by $\frac{1}{2\pi a}$, where the quantity $a > 0$ is the so–called *string coupling constant*. Since our knowlegde of the physical aspects is very limited, we refer to the lecture notes by Gawedzki [43] for an introduction to the renormalization group accessible to mathematicians, while the survey [16] offers a mathematical understanding of how the RG–flow origins from the above action and its interplay with the Ricci flow.

Here we mention only that in the "perturbative regime", that is when $|\mathrm{Rm}|a \ll 1$, namely the metric g of the target manifold M has "small" curvature compared to the coupling constant, the functions defining the flow can be expanded in powers of a and their first order truncation corresponds to (a multiple of) the Ricci flow [39, 40, 61]. It is actually believed that the Ricci flow provides a good approximation for the full RG–flow. However, since the exact quantization of non linear σ–models has not yet been shown to exist, it is hard to quantify the error between the first order and the full renormalization group flow, in particular when the Ricci flow develops singularities. From the physical perspective, there appears relevant to consider also flows generated by more general actions to avoid such singularities, as in [16, 68].

In the first part of this work, namely Chapter 2, we concentrate on the flow arising from the second order truncation, that we denote by $\mathrm{RG}^{2,a}$–flow or two–loop RG–flow, whose equation (see [59]) is

$$\partial_t g_{ik} = -2\mathrm{R}_{ik} - a\mathrm{R}_{ijlm}\mathrm{R}_{kstu}g^{js}g^{lt}g^{mu}\,, \tag{0.2}$$

where the two tensors R_{ij} and R_{ijkl} are respectively the Ricci and the Riemann tensor of the metric g on a compact manifold M. An immediate motivation for this choice is the fact that, under the Ricci flow, the norm of the Riemann tensor must blow up at a finite singular time, therefore higher order terms in the expansion of the RG–flow should become relevant. Moreover, besides the physical motivations (see [61]), the problem is of mathematical interest of its own: in a way, this flow is a natural (non linear) generalization of the Ricci flow (recovered as a limit when the parameter a goes to zero), which has physical meaning and therefore might present nice properties, such as stability (see [48]). We finally underline that, albeit fully non linear, this flow involves only a second order operator, while the higher order truncations contain also the derivatives of the Riemann tensor, making the analysis harder.

The central idea is to find out whether and when the $\mathrm{RG}^{2,a}$–flow behaves qualitywise differently from the Ricci flow. At a first glance, we observe that the quadratic term makes the flow no more invariant by parabolic rescaling but this feature is not necessarily bad, as it could lead to new ways to form and resolve singularities (see [45, 67]). In [46], Gimre, Guenther and Isenberg focus on particular classes of solutions, beginning with the flow on n–dimensional compact manifolds with constant sectional curvature, and they observe that in negative curvature, the asymptotic behaviour of the flow depends on the value of the coupling constant a and of the sectional curvature and that, unlike the Ricci flow, a manifold with very negative sectional curvature, namely $\mathrm{K} < -\frac{2}{a}$, will collapse to a point in finite time. In the same paper, the authors also focus on three–dimensional locally homogeneous spaces, where the strong assumptions on the geometry of the initial metric allow to reduce the tensorial PDE to a system of ODEs; in some of these cases the Ricci flow and the $\mathrm{RG}^{2,a}$–flow show radically different solutions.

In [67], Oliynyk studies the behaviour of the two–loop RG–flow on a compact Riemannian surface, proving that a short time solution exists if the Gauss curvature of the initial metric satisfies a suitable condition. Given these results, in the first part of our study, we are going to focus on the general short time existence of solutions to the $\mathrm{RG}^{2,a}$–flow in dimension at least three, showing the following theorem.

THEOREM . [31] *Let (M, g_0) be a compact, smooth, three–dimensional Riemannian manifold and $a \in \mathbb{R}$. Assume that the sectional curvature K_0 of the initial metric g_0 satisfies*

$$1 + 2a\mathrm{K}_0(X, Y) > 0 \tag{0.3}$$

for every point $p \in M$ and vectors $X, Y \in T_pM$. Then, there exists a unique smooth solution $g(t)$ of equation (0.2) with $g(0) = g_0$, for $t \in [0, T)$.

Unlike the Ricci flow, the RG–flow is not weakly parabolic for *any* initial metric g_0 (and coupling costant a), while, if condition (0.3) is satisfied, the degeneracies are only due to the diffeomorphism invariance of the curvature tensors (see [49] and Lemma 1.3.1) and can be faced with DeTurck's trick (Section 1.3), leading to existence and uniqueness of a short time solution. Moreover, we remark that our result was extended using the same method to the n–dimensional case by Gimre, Guenther and Isenberg in [47].

A detailed survey on past results and open problems can be found in [45]: in Section 2.2 we address one of these problems, by computing the evolution equations, in the three–dimensional setting, of (minus) the Einstein tensor, whose eigenvalues are the sectional curvatures, in order to see if the parabolicity condition is preserved for some classes of initial data. Unfortunately (see Remark 2.2.2), up to now, such investigation has not given useful informations in this direction.

The second flow studied in this thesis was first suggested by Bourguignon in [9, Question 3.24], inspired by some unpublished work of Lichnerowicz in the sixties and the paper of Aubin [2]. The evolution equation of the Ricci–Bourguignon (RB) flow is the following

$$\frac{\partial}{\partial t} g = -2\big(\mathrm{Ric}_g - \rho \mathrm{R}_g g\big)\,, \tag{0.4}$$

where ρ is a real constant, Ric and R are respectively the Ricci tensor and the scalar curvature of the metric g on a smooth compact manifold M of dimension n.

Going back to Hamilton's first idea, we consider the Einstein–Hilbert functional [5, Section 4.C], that is, the integral of the scalar curvature

$$\mathcal{EH}(g) = \int_M \mathrm{R}_g \, d\mu_g \, .$$

It is well known (see for instance [5, Proposition 4.17]) that the Euler–Lagrange equation of this functional is

$$\mathrm{E}_g = \mathrm{Ric}_g - \frac{1}{2}\mathrm{R}_g g = 0,$$

where E_g is called Einstein tensor of the metric g, and the gradient flow associated to this equation behaves badly from a PDEs point of view; therefore a solution for a general initial metric g_0 on the manifold M cannot be expected. In [49], Hamilton modified the above equation simply by dropping the scalar curvature term, obtaining the Ricci flow $\frac{\partial}{\partial t} g = -2\mathrm{Ric}_g$. The motivation for this choice becomes clear if we consider the expression of the Ricci tensor in harmonic coordinates around a point p of the manifold M (see again [5, Section 5.E]), that is,

$$\mathrm{Ric}_{ij} = -\frac{1}{2}\Delta(g_{ij}) + \mathrm{LOT}\,,$$

where $\Delta(g_{ij})$ is the coordinate Laplacian operator (not the metric Laplacian) of the function g_{ij} and LOT stands for lower order terms. Hence, the Ricci flow equation can be regarded as a heat type equation that will eventually "improve" the initial metric.

Another possibility is to substitute the constant $1/2$ before the conformal term $\mathrm{R}_g g$ with a generic real constant and this choice gives origin to the (ρ–dependent) family of RB–flows of equation (0.4) suggested by Bourguignon. Besides the Ricci flow, which corresponds to the value $\rho = 0$, there are some other notable flows when the right hand side of equation (0.4) corresponds, for instance, to the Einstein tensor ($\rho = 1/2$), to the traceless Ricci tensor ($\rho = 1/n$) or to the Schouten tensor ($\rho = 1/2(n-1)$). Moreover, this family of geometric flows can be seen as an interpolation between the Ricci flow and the Yamabe flow, obtained as a limit when $\rho \to -\infty$. We do not address this last aspect in the thesis, focusing instead on the basic features of the Ricci–Bourguignon flow, but, for future investigations, it might be interesting to merge some methods proper of the Yamabe flow into the *corpus* of the Ricci flow techniques and check whether and when the RB–flow is more similar to one or the other "extremal" flows.

In Chapter 3 we begin to study the RB–flow and analyse its general properties: we mention that all the results of this chapter can be found in the paper [18], written by the author with Catino, Djadli, Mantegazza and Mazzieri.

The first question we answer is the short time existence of solutions, stated in the following theorem.

THEOREM . [18, Theorem 2.1] *Let* $\rho < 1/2(n-1)$*. Then, the evolution equation* (3.1) *admits a unique solution for a short time for any initial smooth,* n*–dimensional, compact Riemannian manifold* (M, g_0)*.*

As we did for the renormalization group flow, we prove the theorem by relating the RB–flow, which is degenerate parabolic because of the diffeomorphism invariance (see [49] and Lemma 1.3.1), to its non degenerate version, namely the DeTurck–Ricci–Bourguignon flow and proving that this latter admits a short time solution if $\rho < 1/2(n-1)$. When $\rho > 1/2(n-1)$, the principal symbol of the operator involved has one negative eigenvalue, then the general theory does not guarantee the existence of a short time solution for every initial metric g_0. The "borderline" case of the Schouten flow, corresponding to $\rho = 1/2(n-1)$, presents instead only an extra null eigenvalue, besides the ones due to diffeomorphism invariance, and it is currently unknown whether this flow has or not a short time solution for any initial metric. Recently Delay [32] gave some evidence that DeTurck's trick might fail for the Schouten tensor.

The second natural step is looking at what happens to the geometry of the manifold during the flow, hence we compute the evolution equations for the Riemann tensor, the Ricci tensor and the scalar curvature and we find out that, for $\rho < 1/2(n-1)$, they all satisfy a reaction–diffusion equation during the RB–flow. This generalizes the analogous fact for the Ricci flow and enables us to find geometric conditions which are preserved, although, for $\rho \neq 0$, the differential operators involved are more non linear than the (time–dependent) metric Laplacian $\Delta_{g(t)}$. In Section 1.5 we provide the natural extension of Hamilton's maximum principle for tensors to this class of elliptic operators and in Section 3.3 we prove the following result, analogous to [49, Corollary 7.6 and Lemma 11.11] and [50, Theorem 8.3].

THEOREM . [18, Proposition 4.1, 4.10] *Let* $(M, g(t))_{t\in[0,T)}$ *be a maximal solution of the RB–flow* (0.4) *and* $\rho < \frac{1}{2(n-1)}$*.*
The minimum of the scalar curvature is non decreasing along the flow. In particular if $\mathrm{R}_{g(0)} \geq \alpha$ *for some* $\alpha \in \mathbb{R}$*, then* $\mathrm{R}_{g(t)} \geq \alpha$ *for every* $t \in [0, T)$*. Moreover if* $\alpha > 0$ *then* $T < \frac{n}{2(1-n\rho)\alpha}$*.*
Finally, if the curvature operator $\mathcal{R}_{g(0)}$ *of the initial metric is non negative, it remains so during the flow.*

Moreover, by means of the strong maximum principle for functions, we prove (Proposition 3.3.4) that any ancient solution to the RB–flow with $\rho < 1/2(n-1)$ either it is Ricci–flat or it has positive scalar curvature.

So far, these are the only conditions preserved in general dimension n and, as for the Ricci flow, the main obstacle is the complexity of the 0–order term in the evolution equations of the curvature tensors. In the last ten years, a work by Bohm and Wilking [7] has shown how to exploit the algebraic structure of the space of curvature operators to find new preserved conditions from the old ones. Moreover, in their proof of the $\frac{1}{4}$–pinched differentiable sphere theorem [13], Brendle and Schoen exhibited a new preserved condition in general dimension $n \geq 4$, namely the non negative *isotropic curvature*, which was already known to be preserved in dimension 4 by Hamilton [56] (see also the PhD thesis of Nguyen [65, 66]). A recent survey on this set of ideas may be found in [76]. Therefore, we hope that future investigations could lead to other preserved conditions peculiar to the Ricci–Bourguignon flow.

In the second part of Chapter 3, we focus on the long time behaviour of solutions. Another feature that the Ricci–Bourguignon flow shares with the Ricci flow is a smoothing property, that allows, given a uniform bound on the norm of the Riemann tensor on $M \times [0, T)$, to obtain *a priori* estimates on the covariant derivatives of the Riemann tensor for any time $t > 0$. The difference in the operator involved in the evolution equation plays here a significant role; indeed, we are unable to directly estimate the pointwise norm of the covariant derivatives $\nabla^k \mathrm{Rm}$, but we have to look for integral bounds. Nonetheless, we are able to prove the following "Bando–Bernstein–Shi" type of global estimates.

THEOREM. [18, Theorem 5.6] *Assume* $\rho < \frac{1}{2(n-1)}$. *If* $g(t)$ *is a compact solution of the RB–flow for* $t \in [0,T)$ *such that*

$$\sup_{(x,t)\in M\times[0,T)} |\mathrm{Rm}_{g(t)}(x)| \leq K\,,$$

then for all $k \in \mathbb{N}$ *there exists a constant* $C(n,\rho,k,K,T)$ *such that for all* $t \in (0,T]$

$$\|\nabla^k \mathrm{Rm}_{g(t)}\|_2^2 \leq \frac{C}{t^k} \sup_{t\in[0,T)} \|\mathrm{Rm}_{g(t)}\|_2^2\,.$$

Then, following Hamilton [49, Sections 12–14], we prove some general interpolation inequalities for tensors in Section 1.6 and show that a compact solution of the RB–flow existing up to a finite maximal time T must have unbounded Riemann tensor when approaching T.

THEOREM. [18, Theorem 5.7] *Assume* $\rho < \frac{1}{1-2(n-1)\rho}$. *If* $g(t)$ *is a compact solution of the RB–flow on a maximal time interval* $[0,T)$, $T < +\infty$, *then*

$$\limsup_{t\to T} \max_M |\mathrm{Rm}_{g(t)}| \;=\; +\infty\,.$$

We point out here that, since the square norm of the Riemann tensor does not satisfy a heat type equation as it happens for the Ricci flow (see [30, Lemma 6.1 and Remark 6.4]), we are currently not able to argue that the $\limsup$ in the previous theorem is actually a limit, nor to give a lower bound on the maximal time T based on the curvature of the initial metric g_0.

We have been led to the study of the Ricci–Bourguignon flow by a paper of Catino and Mazzieri, who, in [21] and [22] (with Mongodi), before our work studied the self–similar solutions of this flow, called *ρ–Einstein solitons* of gradient type, which are Riemannian manifolds (M,g) that satisfy

$$\mathrm{Ric}_g + \nabla^2 f = \rho \mathrm{R}_g g + \lambda g\,, \tag{0.5}$$

where λ is a real constant and f is a smooth function on M called *potential*. These metrics are the generalized fixed points of the RB–flow, modulo the action of the diffeomorphism group and scaling (see Theorem 3.5.3). In Section 3.5 we give an overview of the results in [21, 22], as the two points of view, the static one, expressed by the above equation, and the dynamic one, by looking at their motion under the RB–flow, are strictly intertwined.
Some results, already known for the Ricci gradient solitons (i.e. $\rho = 0$), are extended to other values of the parameter ρ. For example, by means of the strong maximum principle, Catino and Mazzieri in [21] prove that, for $\rho < 1/2(n-1)$, any steady ($\lambda = 0$) or expanding ($\lambda < 0$) compact gradient soliton is actually an Einstein manifold, that is, the Ricci tensor of the metric g is a (constant) multiple of the metric itself, while in the shrinking case ($\lambda > 0$) the metric g must have positive scalar curvature (see [21, Theorem 3.1], also recalled in this thesis as Theorem 3.5.8, for the complete *triviality* result). Moreover, the authors prove that any gradient ρ–Einstein soliton for $\rho \neq 0$, has a property, namely the *rectifiability* (see Definition 3.5.5), which implies strong constraints on the allowed geometric structures, leading for instance to a *rigidity* result for complete non compact shrinking solitons proven in [22] (Theorem 3.5.12 of this thesis).

As for the Ricci flow, the evolution equations of the Riemann and the Ricci tensor are quite involved in general dimension n, thus making hard to understand which geometric properties are preserved under the flow. Following again Hamilton's seminal paper [49], in Chapter 4 we restrict our analysis to the three–dimensional case, where such equations simplify considerably. First of all, we specialise the evolution equation of the Ricci tensor along the flow:

$$\begin{aligned}\frac{\partial}{\partial t}\mathrm{Ric}_g =\,& \Delta \mathrm{Ric}_g - \rho(\nabla^2 \mathrm{R} + \Delta \mathrm{R}_g g) \\ & - 6\mathrm{Ric}_g^2 + 3\mathrm{R}_g \mathrm{Ric}_g + (2|\mathrm{Ric}_g|^2 - \mathrm{R}_g^2) g\,,\end{aligned} \tag{0.6}$$

where $\mathrm{Ric}^2_{ij} = \mathrm{Ric}_{ik}\mathrm{Ric}_{jl}g^{kl}$. We observe that the 0–order term is ρ–independent, i.e. it is the same given by the Ricci flow, therefore it is not surprising that the RB–flow, provided the differential operator involved in equation (0.6) is still elliptic, preserves the same conditions of the Ricci flow.

THEOREM. *Let $(M, g(t))_{t\in[0,T)}$ be a solution of the RB–flow (3.1) on a compact three–manifold. Then, if $\rho < 1/4$,*

(i) *non negative Ricci curvature is preserved along the flow;*
(ii) *non negative sectional curvature is preserved along the flow;*
(iii) *the pinching inequality $\mathrm{Ric}_g \geq \varepsilon \mathrm{R}_g g$ is preserved along the flow for any $\varepsilon \leq 1/3$.*

Another remarkable property of the three–dimensional Ricci flow is the Hamilton–Ivey estimate [55, 58], which ultimately implies that ancient solutions of the Ricci flow have non negative sectional curvature. The strategy here is to look at the system of ODEs satisfied by the sectional curvatures, properly modified by means of Uhlenbeck's trick, introduced by Hamilton in [50] and recalled in Section 1.4.

THEOREM (Hamilton-Ivey Estimate). *Let $(M, g(t))$ be a solution of the Ricci–Bourguignon on a compact three–manifold such that the initial metric satisfies the normalizing assumption $\min_{p\in M} \nu_p(0) \geq -1$. If $\rho \in [0, 1/6])$, then at any point (p, t) where $\nu_p(t) < 0$ the scalar curvature satisfies*

$$\mathrm{R}_g \geq |\nu|\big(\log(|\nu|) + \log(1 + 2(1 - 6\rho)t) - 3\big) .$$

Moreover, with a similar computation, we are able to enlarge the range of the parameter ρ up to $1/4$ for compact ancient solutions of the RB–flow (Proposition 4.1.7).

Finally, in dimension three, the rectifiability property of ρ–Einstein gradient solitons has stronger consequences (see again [21, 22]) which we review in Section 4.2.
In light of all the previous results, at the end of Section 4.2, we discuss some conjectures which we intend to investigate in the future.

In the second part of Chapter 4, we study the evolution of a compact three–manifold whose initial metric has positive Ricci tensor and we prove that, in analogy with Hamilton's result for the Ricci flow, the RB–flow "pinches" the Ricci tensor towards a multiple of the metric whenever the scalar curvature blows up (see Proposition 4.3.1). Moreover, we prove that, since the maximal time of existence is finite by Proposition 3.3.1, the scalar curvature actually blows up somewhere for $t \to T$ (see Lemma 4.3.2). Given these facts, an estimate for the gradient of the scalar curvature is a key step in the proof of Hamilton's sphere theorem [49, Theorem 1.1], as it leads to a control of the ratio between the minimum and the maximum of the scalar curvature when $t \to T$, proving that the scalar curvature is blowing up at the same rate everywhere. At the present time we are able to prove only an integral version of such gradient estimate (Proposition 4.3.9), which does not seem sufficient to conclude, therefore we have to assume such control by hypothesis to prove a weaker version of Hamilton's theorem for the Ricci–Bourguignon flow.

THEOREM. *Let $\rho < 1/4$ and $(M, g(t))$, $t \in [0, T)$, be the maximal solution of the Ricci–Bourguignon flow with initial metric g_0 on the compact 3–manifold M. Suppose that g_0 has positive Ricci curvature and there holds*

$$\lim_{t\to T} \frac{\min_{p\in M} \mathrm{R}_{g(t)}}{\max_{p\in M} \mathrm{R}_{g(t)}} = \alpha > 0 . \tag{0.7}$$

Then there exist

- *a sequence $t_i \nearrow T$,*
- *a sequence $(p_i) \subset M$,*

such that the new Ricci–Bourguignon flows $g_i(t)$ defined for $t \leq 0$ by

$$g_i(t) = \mathrm{R}_{g(t_i)}(p_i)g\Big(t_i + \frac{t}{\mathrm{R}_{g(t_i)}(p_i)}\Big) ,$$

satisfy

$$(M, g_i(t), p_i) \to (M, (c - t)g_\infty, p)$$

on the time interval $(-\infty, 0]$, for some $c > 0$, where $p \in M$ and the metric g_∞ on M has constant positive sectional curvature. Hence (M, g_∞) is isometric to a quotient of the round sphere $\mathbb{S}^3$.

Since our research on both the RG and the RB–flows is far from being complete, we devote the last chapter to review open problems and discuss some future lines of investigation. In particular, in Section 5.2.1, we focus on the research of Perelman–type monotonicity formulas [69, Section 1–3], in view of their many applications to singularity analysis. At the moment we are not able to prove that the natural analogues of Perelman $\mathcal{F}$ and $\mathcal{W}$ functionals are monotone during the Ricci–Bourguignon flow, while we have some partial results on "modified" functionals, inspired by the paper of Li [60] for the Ricci flow. In particular, we prove the following theorem.

THEOREM . *Suppose* $(M, g(t))$ *is a compact solution of the Ricci–Bourguignon flow on [0,T) for* $\rho < 0$. *If* $g(0)$ *has non negative scalar curvature, then there exist a constant* $C = C(\rho, n)$, *such that the functional* $\mathcal{F}_C : \Gamma(S^2_+ M) \times C^\infty(M) \to \mathbb{R}$ *defined by*

$$\mathcal{F}_C(g, f) = \int_M (C\mathrm{R}_g + |\nabla f|^2) e^{-f} \, d\mu_g \, ,$$

is non decreasing along the coupled Ricci–Bourguignon flow

$$\begin{cases} \frac{\partial}{\partial t} g = -2\big(\mathrm{Ric}_g - \rho \mathrm{R}_g g\big) \\ \frac{\partial}{\partial t} f = -\Delta f - (1 - n\rho)\mathrm{R}_g + |\nabla f|^2 \end{cases} .$$

Moreover the monotonicity is strict unless the solution is a trivial Ricci–flat soliton.

For this purpose, in analogy with the Ricci flow, we hope in the future to be able to find monotone quantities along the RB–flow such that the ρ–Einstein solitons solve their Euler–Lagrange equations, thus linking the solitons with the singularity models arising from the blow–up procedure. Moreover, we are also looking for a monotonicity formula which allows a control on the injectivity radius during the blow–up procedure, as in Perelman *no local collapsing* theorem (see [69, Section 4]). This would imply the possibility to remove, for instance, hypothesis (0.7) and still get the convergence of the rescaled metrics in the three–dimensional Ricci positive case.

CHAPTER 1

Technical tools

In this chapter we recall some technical tools we are going to need in studying both the renormalization group flow and the Ricci–Bourguignon flow. In particular in the first section we fix the conventions on the Levi–Civita connection on a Riemannian manifold and the related curvature tensors.
Section 1.2 is devoted to non linear parabolic theory on manifolds: we give the basic definitions of linearized operator and principal symbol of a differential operator, together with a sufficient condition to get the short time existence of a smooth solution to the associated Cauchy problem. Since both flows involve geometric operators, i.e., they are defined in terms of the curvature tensors, there are always some degeneracies in the problem that prevent the direct application of standard theory. Therefore, in Section 1.3 we explain the so called DeTurck's trick, a method that allows to overcome such degeneracies and obtain existence an uniqueness of a short time solution.
In Section 1.4 we explain in detail how to simplify the evolution equations of the curvature tensors along a geometric flow. In Section 1.5 we adapt the maximum principle by Hamilton when the operator involved is no more the rough Laplacian, but a general quasilinear operator with non linearities only in the 0–th order terms.
Finally, in Section 1.6 we prove some interpolation inequalities, useful to get *a priori* estimates on the norm of the Riemann tensor during a geometric flow.

1.1. Notations and conventions

We fix first the conventions that, unless otherwise stated, we shall adopt throughout the thesis.
The pair (M, g) is the datum of an n–dimensional smooth manifold M, which we always assume compact, that is closed without boundary, and a smooth Riemannian metric g on M. We denote by $d\mu_g$ the canonical volume measure associated to the metric g and by ∇ the Levi–Civita connection of the metric g, that is the unique torsion–free connection compatible with g.
The *curvature tensor* of (M, g, ∇) is defined, as in [42], by

$$\mathrm{Rm}(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z\,,$$

while the associated $(4,0)$–tensor is defined by $\mathrm{Rm}(X,Y,Z,T) = g(\mathrm{Rm}(X,Y)Z,T)$. In a local coordinate system, we have

$$\mathrm{R}_{ijkl} = g\Big(\mathrm{Rm}\Big(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\Big)\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^l}\Big)\,.$$

The *Ricci tensor* is then obtained by tracing $\mathrm{R}_{ik} = g^{jl}\mathrm{R}_{ijkl}$ and $\mathrm{R} = g^{ik}\mathrm{R}_{ik}$ denotes the scalar curvature.
The *sectional curvature* of a plane $\pi \subset T_pM$ spanned by a pair of vectors $X, Y \in T_pM$ is defined as

$$\mathrm{K}(\pi) = \mathrm{K}(X,Y) = \frac{\mathrm{Rm}(X,Y,X,Y)}{g(X,X)g(Y,Y) - g(X,Y)^2}.$$

With this choice, for the sphere $\mathbb{S}^n$ we have $\mathrm{Rm}(v,w,v,w) = \mathrm{R}_{ijkl}v^iw^jv^kw^l > 0$.
The *curvature operator* $\mathcal{R} \in End(\Lambda^2(TM))$ is defined by

$$\langle \mathcal{R}(X\wedge Y), W\wedge Z\rangle = \mathrm{Rm}(X,Y,W,Z)\,, \tag{1.1}$$

where $\langle\, ,\rangle$ is the linear extension of g to the exterior powers of TM.
The so called *Weyl tensor* is defined by the following decomposition formula (see [42, Chapter 3, Section K]) in dimension $n \geq 3$,

$$\mathrm{W}_{ijkl} = \mathrm{R}_{ijkl} + \frac{\mathrm{R}}{2(n-1)(n-2)}(g \owedge g)_{ijkl} - \frac{1}{n-2}(\mathrm{Ric} \owedge g)_{ijkl}\,, \tag{1.2}$$

where $\owedge$ is the *Kulkarni–Nomizu* product, defined by

$$(p \owedge q)_{ijkl} = p_{ik}q_{jl} + p_{jl}q_{ik} - p_{il}q_{jk} - p_{jk}q_{il}\,.$$

The tensor W satisfies all the symmetries of the curvature tensor and all its traces with the metric are zero, as it can be easily seen by the above formula.
In dimension three W is identically zero for every Riemannian manifold (M^3, g), it becomes relevant instead when $n \geq 4$ since its nullity is a condition equivalent for (M, g) to be *locally conformally flat*, that is, around every point $p \in M$ there is a conformal deformation $\widetilde{g}_{ij} = e^f g_{ij}$ of the original metric g, such that the new metric is flat, namely, the Riemann tensor associated to $\widetilde{g}$ is zero in U_p (here $f : U_p \to \mathbb{R}$ is a smooth function defined in a open neighborhood U_p of p).

1.2. Non linear parabolic operators

The first question that we need to address in analysing a geometric flow is whether the flow actually exists, i.e. if there is a short time solution (unique? regular?) to its evolution equation. For this purpose the basic tool is the non linear PDEs theory for vector bundles, as the objects evolving are typically Riemannian metrics on a smooth manifold and their associated curvature tensors.
In this section we define the principal symbol of an operator, which is used to classify the type of a differential equation, paying attention to the elliptic and parabolic cases.
Let E and F be two vector bundles on the manifold M and let $\Gamma(E)$ and $\Gamma(F)$ be the vector spaces of the smooth sections.

DEFINITION 1.2.1. A linear map $L : \Gamma(E) \to \Gamma(F)$ is a *linear differential operator of order k* if for every $u \in \Gamma(E)$

$$L(u) = \sum_{|\alpha| \leq k} L^\alpha(D_\alpha u),$$

where α is a multiindex, D is a connection on the bundle E and $L^\alpha \in Hom(E, F)$. Moreover, L^α is also tensorial with respect to the index α, that is, for every $|\alpha| = s$, there exists a tensor $L^s \in \Gamma\big((TM)^s \otimes Hom(E, F)\big)$ such that L^α are its components in local coordinates (x^i) around $p \in M$.

REMARK 1.2.2. In the present context the connection D will always be the Levi–Civita connection associated to the (evolving) metric g extended to bundles of tensors; when we will state the maximum principles for parabolic operators the connection will be a different one. When we write the expression of a differential operator with respect to local frames of the bundles we use instead the ordinary derivation of the components, which is the common part in every connection and determines the order of the differential operator.

For example, if $(\eta_l)_{l=1,\dots,m}$ is a local frame for E around $p \in M$, a second order linear differential operator $L : \Gamma(E) \to \Gamma(E)$ can be expressed locally as

$$L(u^l \eta_l) = \Big((a^{ij})^l_k \partial^2_{ij} u^k + (b^i)^l_k \partial_i u^k + c^l_k\Big)\eta_l$$

where $a \in \Gamma(S^2(TM)) \otimes Hom(E, E))$, $b \in \Gamma(TM \otimes Hom(E, E))$ and $c \in \Gamma(Hom(E, E))$.

DEFINITION 1.2.3. Given $\xi \in T^*_p M$ the *principal symbol* of L in the direction ξ is given by

$$\sigma_\xi(L) = \sum_{|\alpha| = k} L^\alpha \xi_\alpha$$

where we have omitted the point dependence, ξ_i are the components of ξ in the basis dx^i of $T^*_p M$ and $\xi_\alpha = \prod_j \xi_{\alpha_j}$. As L^α are tensorial, the definition of the symbol is independent by the choice of the coordinates and when ξ is a section of the cotangent bundle, the pointwise expression above defines a section of $Hom(E, F)$ (for a free-coordinates definition of the symbol, see for instance [77]).

The principal symbol satisfies some simple but useful algebraic properties

$$\sigma_\xi(P+Q) = \sigma_\xi(P) + \sigma_\xi(Q)\,; \quad \sigma_\xi(P\cdot Q) = \sigma_\xi(P)\sigma_\xi(Q)\,; \quad \sigma_\xi(Q\circ P) = \sigma_\xi(Q)\circ\sigma_\xi(P)\,;$$

where in the first two cases $P, Q : \Gamma(E) \to \Gamma(F)$, while in the latter case $Q : \Gamma(F) \to \Gamma(G)$.

DEFINITION 1.2.4. We say that a linear differential operator L of order k is *elliptic* in $p \in M$ if for every $\xi \in T_p^*M$ $\sigma_\xi(L)$ is positive definite. This condition implies that E and F have the same rank so we identify the two vector bundles. Moreover we say that L is *uniformly strongly elliptic* if there exists $\delta > 0$ such that for every $u \in \Gamma(E)$, for every $\xi \in \Gamma(T^*M)$

$$\langle \sigma_\xi(L)u, u\rangle \geq \delta|\xi|^k|u|^2, \tag{1.3}$$

where $\langle\,,\rangle$ is a metric on the bundle E (see [3, 44]). This condition is equivalent to having all the eigenvalues of $\sigma_\xi(L)$ with uniformly (with respect to $p \in M$) positive real parts and it implies that k is even. Finally we say that L is *weakly elliptic* if the above condition holds with $\delta = 0$.

If L is a non linear differential operator of order k that we can write as

$$L(u) = F(u, Du, \dots, D^k u),$$

where F is a smooth function in its arguments, we define the *linearization* of L at $u_0 \in \Gamma(E)$ as the differential operator $DL_{u_0} : \Gamma(E) \to \Gamma(F)$ given by

$$DL_{u_0}(v) = \frac{d}{dt}F\big(u_0 + tv, D(u_0+tv), \dots, D^k(u_0+tv)\big)\Big|_{t=0}.$$

By applying the chain rule, we can see that

$$DL_{u_0}(v) = \sum_{|\alpha|\leq k} \frac{\partial F}{\partial(D_\alpha u)}(u_0, Du_0, \dots, D^k u_0)D_\alpha v,$$

hence the linearization of L at u_0 is a linear differential operator whose coefficients depend on u_0. We say that L is (uniformly, strongly) elliptic if its linearization at any u_0 is (uniformly, strongly) elliptic.

The principal symbol encodes the analytic properties of L that depend only on its highest derivatives, such as the existence and regularity of solutions to the equation $Lu = f$ and to the evolution problem associated.

The evolution equations systems we are dealing with in this thesis are of the form

$$\begin{cases} \partial_t g = Lg \\ g(0) = g_0 \end{cases} \tag{1.4}$$

where L is a (possibly) time–dependent differential operator involving the curvature tensors. If L is strongly elliptic (uniformly in space and time), then the PDE is *strictly parabolic* and the initial value problem (1.4) admits a unique short time solution by standard theory of partial differential equations (see [3, 41, 44, 64]).

As a major example we can consider the operator which associates to any metric g on M its Ricci curvature tensor Ric_g and the well–known Ricci flow introduced by Hamilton in [49]. In such paper Hamilton showed that the Ricci flow is not strictly parabolic, but this bad feature comes only from the diffeomorphism invariance of the Ricci tensor

$$\forall\varphi \in \mathcal{D}iff(M) \qquad \varphi^*(\mathrm{Ric}_g) = \mathrm{Ric}_{\varphi^* g}.$$

Hamilton proved the short time existence for the flow using the Nash–Moser implicit function theorem. Shortly afterwards, DeTurck in [33] found a simpler way to get the short time existence and uniqueness, by defining a stricly parabolic flow and showing that it is equivalent to the original Ricci flow. In the following years, many authors, including DeTurck [34] and Hamilton [55], have reformulated the original idea, in an attempt at explaining the deeper meaning of what sometimes appears as a piece of magic.

1.3. DeTurck's trick

In DeTurck's first paper [33], the original diffeomorphism invariant flow (the 3-dimensional Ricci flow) is modified via a proper vector field that depends on another flow and the coupled evolution equations become a strictly parabolic flow equivalent to the initial one. However, in greater dimension, other equations are necessary and the proof is more complicated. In the following, we present an approach of Hamilton [55] which uses DeTurck's vector field to modify the original flow and get the short time existence, while for the uniqueness part exploits the harmonic map flow (see also [29] for details). This argument is necessary and sufficient every time the involved differential operator is diffeomorphism invariant but has no other degeneracies.

LEMMA 1.3.1. *Let $L : \Gamma(S^2M) \to \Gamma(S^2M)$ be a* geometric *differential operator, that is for every smooth diffeomorphism $\psi : M \to M$ satisfying $L(\psi^* g) = \psi^*(Lg)$. Then the principal symbol of DL_g has a kernel which is at least n–dimensional. In particular, we have that*

$$Im(\sigma_\xi(\delta_g^*)) \subseteq Ker(\sigma_\xi(DL_g)), \tag{1.5}$$

where $\delta_g^* : \Gamma(T^*M) \to \Gamma(S^2M)$ is the L^2–adjoint of the divergence operator

$$(\delta_g^*(\alpha))_{ij} = \frac{1}{2}(\nabla_i \alpha_j + \nabla_j \alpha_i).$$

PROOF. Given a vector field $V \in \Gamma(TM)$, we will denote the Lie derivative along V with $\mathscr{L}_V$. Let α be a 1–form, $\alpha^\sharp$ the vector field associated via the metric g and $\varphi(t)$ the 1–parameter group of diffeomorphism generated by $\alpha^\sharp$. Then, by deriving at $t = 0$ the equation $L(\varphi(t)^*g) = \varphi(t)^*(Lg)$ we get

$$DL_g(\mathscr{L}_{\alpha^\sharp} g) = \mathscr{L}_{\alpha^\sharp}(Lg) \tag{1.6}$$

It is a simple computation that $\mathscr{L}_{\alpha^\sharp} g = \delta_g^*(\alpha)$. As a differential operator on the 1–form α, the left member of the equation above is of order $k+1$, where k is the order of L, while the right member is of order 1. Hence, the principal symbol (of order $k+1$) of the left member must be zero:

$$0 = \sigma_\xi(DL_g \circ \delta_g^*)(\alpha) = \sigma_\xi(DL_g) \circ \sigma_\xi(\delta_g^*)(\alpha). \tag{1.7}$$

Since the above equation holds for every $\alpha \in \Gamma(T^*M)$, this proves the lemma. □

REMARK 1.3.2. As observed by Hamilton, the degeneracy of the Ricci flow comes essentially from the contracted second Bianchi identity, which is a consequence of the diffeomorphism invariance of the Ricci tensor. There are several books in which one can find a detailed discussion on this topic, such as [1, 5, 29]. In particular, in [1] one can find the derivation of the first and the second Bianchi identities from the diffeomorphism invariance of the Riemann tensor. For a general geometric operator the computation above could lead to an equality which is a combination of the Bianchi identities, as in the case of the cross curvature flow (see [28, 25, Appendix B.3]).

Before we focus on DeTurck's trick, we compute two identities on time–dependent vector fields, tensors and generated diffeomorphisms, which will come in handy several times throughout this work.

LEMMA 1.3.3. *The following facts hold true.*

(i) *Let $V(t)$ be a smooth time–dependent vector field and $A(t)$ a smooth time–dependent covariant tensor. Then,*

$$\frac{\partial}{\partial t}(\theta_t^* A(t)) = \theta_t^* \Big(\frac{\partial}{\partial t} A(t) + \mathscr{L}_{V(t)} A(t) \Big) \,,$$

where θ_t is the 1–parameter family of diffeomorphisms generated by the vector field $V(t)$.

(ii) *Let now V be a smooth vector field, A a smooth covariant 2–tensor and ϕ a smooth diffeomorphism. Then,*

$$\phi^*(\mathscr{L}_V A) = \mathscr{L}_{(\phi_*^{-1} V)}(\phi^* A) \,.$$

PROOF. The relation at point (i) comes from the following straightforward computation.

$$\begin{aligned}\frac{\partial}{\partial t}(\theta_t^* A(t))\Big|_{t=t_0} &= \lim_{h\to 0}\frac{\theta^*_{t_0+h}A(t_0+h)-\theta^*_{t_0}A(t_0)}{h}\\ &= \lim_{h\to 0}\frac{\theta^*_{t_0+h}A(t_0+h)-\theta^*_{t_0+h}A(t_0)+\theta^*_{t_0+h}A(t_0)-\theta^*_{t_0}A(t_0)}{h}\\ &= \lim_{h\to 0}\theta^*_{t_0+h}\Big(\frac{A(t_0+h)-A(t_0)}{h}\Big)+\lim_{h\to 0}\frac{\theta^*_{t_0}(\theta_{t_0+h}\circ\theta^{-1}_{t_0})^*A(t_0)-\theta^*_{t_0}A(t_0)}{h}\\ &= \theta^*_{t_0}\Big(\frac{\partial}{\partial t}A(t)\Big|_{t=t_0}\Big)+\theta^*_{t_0}(\mathscr{L}_{V(t_0)}A(t_0))\,.\end{aligned}$$

About point (ii), let θ_t be the 1–parameter family of diffeomorphisms generated by V and $p\in M$. Then,

$$\frac{\partial}{\partial t}(\phi^{-1}\circ\theta_t)(p)=\phi_*^{-1}\Big(\frac{\partial}{\partial t}\theta_t(p)\Big)=\phi_*^{-1}(V_{\theta_t(p)}(t))=\big(\phi_*^{-1}V(t)\big)_{(\phi^{-1}\circ\theta_t(p))}\,,$$

therefore, $\psi_t=\phi^{-1}\circ\theta_t$ is the 1–parameter family of diffeomorphisms generated by $\phi_*^{-1}V$. Hence, the second equation comes from the following computation.

$$\begin{aligned}\phi^*(\mathscr{L}_V A) &= \phi^*\Big(\lim_{h\to 0}\frac{(\theta_{t+h}\circ\theta_t^{-1})^*A-A}{h}\Big)=\lim_{h\to 0}\frac{(\theta_{t+h}\circ\theta_t^{-1}\circ\phi)^*A-\phi^*A}{h}\\ &= \lim_{h\to 0}\frac{(\phi\circ\psi_{t+h}\circ\psi_t^{-1})^*A-\phi^*A}{h}=\lim_{h\to 0}\frac{(\psi_{t+h}\circ\psi_t^{-1})^*\phi^*A-\phi^*A}{h}\\ &= \mathscr{L}_{(\phi_*^{-1}V)}(\phi^*A)\,.\end{aligned}$$

□

In the following lemma, DeTurck's vector field is defined and its basic properties are proved in order to get existence and uniqueness.

LEMMA 1.3.4. *Let $V:\Gamma(S^2M)\to\Gamma(TM)$ be DeTurck's vector field, that is defined by*

$$V^j(g)=-g_0^{jk}g^{pq}\nabla_p\Big(\frac{1}{2}\mathrm{tr}_g(g_0)g_{qk}-(g_0)_{qk}\Big)=-\frac{1}{2}g_0^{jk}g^{pq}\big(\nabla_k(g_0)_{pq}-\nabla_p(g_0)_{qk}-\nabla_q(g_0)_{pk}\big)\,, \tag{1.8}$$

where g_0 is a Riemannian metric on M and g_0^{jk} is the inverse matrix of g_0.
The following facts hold true.

(i) *The linearization in g_0 of the Lie derivative in the direction V is given by*

$$\begin{aligned}(D\mathscr{L}_V)_{g_0}(h)_{ik} &= \frac{1}{2}g_0^{pq}\nabla^0_i\{\nabla^0_k h_{pq}-\nabla^0_p h_{qk}-\nabla^0_q h_{pk}\}\\ &\quad+\frac{1}{2}g_0^{pq}\nabla^0_k\{\nabla^0_i h_{pq}-\nabla^0_p h_{qi}-\nabla^0_q h_{pi}\}+\mathrm{LOT}\,,\end{aligned}$$

where ∇^0 is the Levi–Civita connection of the metric g_0 and LOT stands for lower order terms. Hence, its principal symbol in the direction ξ, with respect to an orthonormal basis $\{(\xi)^\flat, e_2,\dots,e_n\}$, is

$$\sigma_\xi((D\mathscr{L}_V)_{g_0})=\left(\begin{array}{c|c|c}\begin{matrix}-1&1&\dots&1\\0&0&\dots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\dots&0\end{matrix}&0&0\\\hline 0&-\mathrm{Id}_{(n-1)}&0\\\hline 0&0&0\end{array}\right), \tag{1.9}$$

expressed in the coordinates

$$(h_{11},h_{22},\dots,h_{nn},h_{12},h_{13},\dots,h_{1n},h_{23},h_{24},\dots,h_{n-1,n})$$

of any $h \in S^2M$.

(ii) *If* $\varphi : (M, g) \to (M, g_0)$ *is a diffeomorphism, then* $V((\varphi^{-1})^* g) = \Delta_{g,g_0}\varphi$*, where the harmonic map Laplacian with respect to* g *and* g_0 *is defined by*

$$\Delta_{g,g_0}\varphi = \mathrm{tr}_g(\nabla(\varphi_*))$$

with ∇ *the connection defined on* $T^*M \otimes \varphi^*TM$ *using the Levi–Civita connections of* g *and* g_0 *(see* [29, Chapter 3, Section 4] *for more details).*

PROOF. About point (i), from the variation formula of the Levi–Civita connection (see [5, Theorem 1.174] or [77]), it follows

$$DV_{g_0}(h)^j = \frac{1}{2} g_0^{jk} g_0^{pq} \{\nabla^0_k h_{pq} - \nabla^0_p h_{qk} - \nabla^0_q h_{pk}\} + \mathrm{LOT}\,,$$

where ∇^0 is the Levi–Civita connection of g_0. By the well–known formula $(\mathscr{L}_V g)_{ik} = \nabla_i V_k + \nabla_k V_i$, we obtain

$$(D\mathscr{L}_V)_{g_0}(h)_{ik} = \frac{1}{2} g_0^{pq} \nabla^0_i \{\nabla^0_k h_{pq} - \nabla^0_p h_{qk} - \nabla^0_q h_{pk}\} + \frac{1}{2} g_0^{pq} \nabla^0_k \{\nabla^0_i h_{pq} - \nabla^0_p h_{qi} - \nabla^0_q h_{pi}\} + \mathrm{LOT}\,,$$

hence, the principal symbol of the operator $(D\mathscr{L}_V)_{g_0}$ is given by

$$\begin{aligned} \sigma_\xi((D\mathscr{L}_V)_{g_0})(h)_{ik} &= \frac{1}{2}\{\xi_i g_0^{pq}(\xi_k h_{pq} - \xi_p h_{qk} - \xi_q h_{pk})\} + \frac{1}{2}\{\xi_k g_0^{pq}(\xi_i h_{pq} - \xi_p h_{qi} - \xi_q h_{pi})\} \\ &= \xi_i \xi_k \mathrm{tr}_{g_0}(h) - \xi_i \xi^p h_{pk} - \xi_k \xi^p h_{pi}\,. \end{aligned}$$

Then, with respect to an orthonormal basis $\{(\xi)^\flat, e_2, \ldots, e_n\}$, the matrix representing the principal symbol of the operator $(D\mathscr{L}_V)_{g_0}$ is the one of the statement.

About point (ii), let (x^i) and (y^α) be coordinates respectively around $p \in (M, g)$ and $\varphi(p) \in (M, g_0)$. Then the tangent map $\varphi_* \in \Gamma(T^*M \otimes \varphi^*TM)$ has local expression $\varphi_* = \frac{\partial\varphi^\gamma}{\partial x^i} dx^i \otimes \frac{\partial}{\partial y^\gamma}$. By using the canonical extension of the two Levi–Civita connection associated to g and g_0 to the above bundle and tracing with respect to g we obtain the local expression of $(\Delta_{g,g_0}\varphi) \in \Gamma(\varphi^*TM)$

$$(\Delta_{g,g_0}\varphi)^\gamma(p) = g^{ij}(p)\Big(\frac{\partial^2\varphi^\gamma}{\partial x^i \partial x^j}(p) - \frac{\partial\varphi^\gamma}{\partial x^k}(p)\Gamma(g)^k_{ij}(p) + \frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial y^j}\Gamma(g_0)^\gamma_{\alpha\beta}(\varphi(p))\Big)\,.$$

When φ is a diffeomorphism, the relation between the Christoffel symbols of the two metrics g and $(\varphi^{-1})^* g$ is a straightforward computation from the Koszul formula

$$\frac{\partial\varphi^\gamma}{\partial x^k}\Gamma(g)^k_{ij}(p) = \frac{\partial^2\varphi^\gamma}{\partial x^i \partial x^j}(p) + \frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial y^j}\Gamma((\varphi^{-1})^* g)^\gamma_{\alpha\beta}(\varphi(p))\,,$$

hence the local expression of the harmonic map Laplacian of a diffeomorphism can be rewritten as

$$\begin{aligned} (\Delta_{g,g_0}\varphi)^\gamma(p) &= g^{ij}(p)\frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial y^j}\Big(- \Gamma((\varphi^{-1})^* g)^\gamma_{\alpha\beta}(\varphi(p)) + \Gamma(g_0)^\gamma_{\alpha\beta}(\varphi(p))\Big) \\ &= ((\varphi^{-1})^* g)^{\alpha\beta}\Big(- \Gamma((\varphi^{-1})^* g)^\gamma_{\alpha\beta} + \Gamma(g_0)^\gamma_{\alpha\beta}\Big)(\varphi(p))\,. \end{aligned}$$

To finish the proof it is sufficient to consider a coordinates system y^α around $\varphi(p)$ such that the Christoffel symbols of $(\varphi^{-1})^* g$ are null. Then

$$\begin{aligned} V((\varphi^{-1})^* g)^\gamma &= -\frac{1}{2} g_0^{\gamma\delta}((\varphi^{-1})^* g)^{\alpha\beta}\big(\partial_\delta(g_0)_{\alpha\beta} - \partial_\alpha(g_0)_{\delta\beta} - \partial_\beta(g_0)_{\alpha\delta}\big) \\ &= ((\varphi^{-1})^* g)^{\alpha\beta}\Gamma(g_0)^\gamma_{\alpha\beta} = ((\varphi^{-1})^* g)^{\alpha\beta}\Big(- \Gamma((\varphi^{-1})^* g)^\gamma_{\alpha\beta} + \Gamma(g_0)^\gamma_{\alpha\beta}\Big)\,. \end{aligned}$$

As the difference between two Christoffel symbols is $C^\infty(M)$–linear, the last expression is independent of the choice of coordinates. □

The existence part of DeTurck's trick does not work only with the original vector field proposed by DeTurck; any vector field that breaks the diffeomorphism invariance and other degeneracies does the trick: see [8] for an example involving fourth order geometric flows.

PROPOSITION 1.3.5 (DeTurck's Trick: existence [33, 34]). *Let (M, g_0) be a compact Riemannian manifold.*
Let $L : \Gamma(S^2M) \to \Gamma(S^2M)$ and $V : \Gamma(S^2M) \to \Gamma(TM)$ be differential operators such that L is geometric. If the linearized operator $D(L - \mathscr{L}_V)_{g_0}$ is uniformly strongly elliptic, then the problem (1.4) admits a smooth solution on an open interval $[0, T)$, for some $T > 0$, for some $T > 0$, given by $g(t) = \phi_t^* \widetilde{g}(t)$, where $\widetilde{g}(t)$ is the unique smooth maximal solution of

$$\begin{cases} \frac{\partial}{\partial t}\widetilde{g} = L\widetilde{g} - \mathscr{L}_{V(\widetilde{g})}\widetilde{g} \\ \widetilde{g}(0) = g_0 \end{cases} \tag{1.10}$$

and ϕ_t is the smooth 1–parameter family of diffeomorphisms generated by the vector field $V(\widetilde{g}(t))$.

PROOF. As the linearized operator $D(L-\mathscr{L}_V)_{g_0}$ is uniformly strongly elliptic, problem (1.10) admits a unique smooth maximal solution $\widetilde{g}(t)$ in an open interval $[0, T)$, for some $T > 0$ (see the previous section or [3], for instance). Then, the metric $g(t) = \phi_t^* \widetilde{g}(t)$ clearly satisfies $g(0) = g_0$ and

$$\frac{\partial}{\partial t} g(t) = \frac{\partial}{\partial t}\left(\phi_t^* \widetilde{g}(t)\right) = \phi_t^* \left(\frac{\partial}{\partial t}\widetilde{g}(t) + \mathscr{L}_{V(\widetilde{g}(t))}\widetilde{g}(t)\right) = \phi_t^*\left(L\widetilde{g}(t)\right) = L(\phi_t^*\widetilde{g}(t)) = Lg(t)\,,$$

by point (i) of Lemma 1.3.3. □

PROPOSITION 1.3.6 (*Ad hoc* Uniqueness for DeTurck's vector field). *Under the hypothesis of Proposition 1.3.5, with V DeTurck's vector field, the solution of the Cauchy problem (1.4) is also unique.*

PROOF. Let $g_1(t)$ and $g_2(t)$ be solutions of the Cauchy problem (1.4) with the same initial data g_0. By parabolicity of the harmonic map flow, introduced by Eells and Sampson in [36], there exist $\varphi_1(t)$ and $\varphi_2(t)$ solutions of

$$\begin{cases} \partial_t \varphi_i = \Delta_{g_i, g_0}\varphi_i \\ \varphi(0) = Id_{M^3} \end{cases}$$

Now we define $\widetilde{g}_i = (\varphi_i^{-1})^* g_i$ and, using that $\frac{d}{dt}\varphi^{-1} = -(\varphi^{-1})_*(\frac{d}{dt}\varphi)$, it is easy to show, using the two formulas proven in Lemma 1.3.3, that both $\widetilde{g}_1$ and $\widetilde{g}_2$ are solutions of the Cauchy problem (1.10) associated to the strong elliptic operator $L - \mathscr{L}_V$ and starting at the same initial metric g_0

$$\begin{aligned} \frac{\partial}{\partial t}(\widetilde{g}_i) &= (\varphi_i^{-1})^*\left(\frac{\partial}{\partial t} g_i\right) + \mathscr{L}_{(\varphi_i)_*(\frac{d}{dt}\varphi_i^{-1})}(\varphi_i^{-1})^* g_i \\ &= L(\widetilde{g}_i) - \mathscr{L}_{\frac{d}{dt}\varphi_i}\widetilde{g}_i = (L - \mathscr{L}_{V(\widetilde{g}_i)})\widetilde{g}_i\,, \end{aligned}$$

hence they must coincide, by uniqueness. By point (ii) of Lemma 1.3.4, the diffeomorphisms φ_i also coincide because they are both the one–parameter group generated by $-V(\widetilde{g}_1) = -V(\widetilde{g}_2)$. Finally, $g_1 = \varphi_1^*(\widetilde{g}_1) = \varphi_2^*(\widetilde{g}_2) = g_2$ and this concludes the proof. □

1.4. Uhlenbeck's trick

In this section we present a trick attributed to Uhlenbeck and first used by Hamilton (see [50]) to deal with the Ricci flow in dimension greater than 3. This trick can be seen from several points of view and a detailed description (for the Ricci flow) can be found in [1, Chapter 5]. Briefly, we relate the curvature tensor of the evolving metric to an evolving tensor of an abstract bundle with the same symmetries of the curvature tensor (see Proposition 1.4.7); in a different but equivalent formulation we may find a precise orthonormal moving frame of $(TM, g(t))$ and write the evolution equation of the coefficients of the Riemann tensor with respect to this frame. In this last approach we deal with a system of scalar evolution equations and no more a tensorial equation, and we are going to use slightly different versions of the maximum principle. This trick requires a small conceptual complication but allows in many cases to simplify the evolution equations of geometric quantities related to the Riemann curvature tensor, as we shall see in Section 3.2.2 for the Ricci–Bourguignon flow.

1.4.1. Time–dependent bundle isometries. Let $\big(M, g(t)\big)_{t\in[0,T)}$ be the solution of the Cauchy problem (1.4) and consider on the tangent bundle TM the family of endomorphisms $\big(\varphi(t)\big)_{t\in[0,T')}$ defined by the following evolution equation

$$\begin{cases} \frac{\partial}{\partial t}\varphi(t) = -\frac{1}{2}(Lg(t))^\sharp \circ \varphi(t)\,, \\ \varphi(0) = \mathrm{Id}_{TM}\,, \end{cases} \tag{1.11}$$

where $(Lg(t))^\sharp$ is the endomorphism of the tangent bundle canonically associated to the symmetric tensor $Lg(t)$ by raising an index, i.e. $g(t)((Lg(t))^\sharp v, w) = Lg(t)(v,w)$ for every $p \in M$, $v, w \in T_pM$.

REMARK 1.4.1. For every point p of the manifold M, the evolution equation represents a system of linear ODEs on the fiber T_pM, therefore a unique solution exists as long as the solution of the Cauchy problem (1.4) exists.

NOTATION. We will omit the time–dependence of tensors etc., unless it is strictly necessary to avoid misunderstandings.

LEMMA 1.4.2. *Let $(h(t))_{t\in[0,T)}$ be the family of bundle metrics defined by $h(t) = \varphi(t)^*(g(t))$, where $\varphi(t)$ satisfies relations (1.11). Then $h(t) = g_0$ for every $t \in [0,T)$.*

PROOF. Let $X, Y \in \Gamma(TM)$ be vector fields independent of time, we compute

$$\begin{aligned} \frac{\partial}{\partial t}h(t)(X,Y) &= \frac{\partial}{\partial t}\Big(g\big(\varphi(X),\varphi(Y)\big)\Big) \\ &= \frac{\partial g}{\partial t}\big(\varphi(X),\varphi(Y)\big) + g\Big(\frac{\partial \varphi}{\partial t}(X),\varphi(Y)\Big) + g\Big(\varphi(X),\frac{\partial \varphi}{\partial t}(Y)\Big) \\ &= Lg\big(\varphi(X),\varphi(Y)\big) - \frac{1}{2}g\Big(Lg^\sharp\big(\varphi(X)\big),\varphi(Y)\Big) - \frac{1}{2}g\Big(\varphi(X),Lg^\sharp\big(\varphi(Y)\big)\Big) \\ &= 0\,. \end{aligned}$$

Therefore $h(t) = \varphi(0)^*\big(g(0)\big) = g_0$. □

REMARK 1.4.3. From the previous lemma we have that

$$\forall t \in [0,T) \qquad \varphi(t) : (TM, h) \to (TM, g(t))$$

is a bundle isometry. Therefore, the pull–back of the Levi–Civita connection is a connection on TM compatible with the metric h. More precisely, we denote with (V, h) the vector bundle (TM, h) in order to stress that we do not consider its Levi–Civita connection but the family of time–dependent connections defined via the bundle isometries $\varphi(t)$.

LEMMA 1.4.4. *Let $D(t) : \Gamma(TM, g(t)) \times \Gamma(V,h) \to \Gamma(V,h)$ be the pull–back connection defined by*

$$\varphi(t)\big(D(t)_X\zeta\big) = \nabla(t)_X\big(\varphi(t)(\zeta)\big)\,, \qquad \forall t \in [0,T)\,, \forall X \in \Gamma(TM, g(t))\,, \forall \zeta \in \Gamma(V,h)\,,$$

where $\nabla(t)$ is the Levi–Civita connection of $g(t)$.
Let again $D(t)$ be the canonical extension to the tensor powers and to the dual of (V,h) and $T \in \Gamma((TM)^s \otimes (T^*M)^t, g(t))$. Then

$$D(t)_X\big(\varphi(t)^*(T)\big) = \varphi(t)^*\big(\nabla(t)_X T\big) \qquad \forall t \in [0,T)\,, \forall X \in \Gamma(TM, g(t))\,.$$

In particular $D_X h = \varphi^*(\nabla_X g) = 0$, so every connection of the family $D(t)$ is compatible with the bundle metric h on V.

PROOF. We show the computation for $T \in \Gamma(T^*M, g(t))$, the general case is straightforward. For every $\zeta \in \Gamma(V,h)$ and $X \in \Gamma(TM, g(t))$ we have

$$\begin{aligned} \big(D_X(\varphi^*(T))(\zeta) &= X\big(\varphi^*T(\zeta)\big) - (\varphi^*T)(D_X\zeta) \\ &= X\Big(T\big(\varphi(\zeta)\big)\Big) - T\big(\varphi(D_X\zeta)\big) = X\Big(T\big(\varphi(\zeta)\big)\Big) - T\Big(\nabla_X\big(\varphi(\zeta)\big)\Big) \\ &= \nabla_X T\big(\varphi(\zeta)\big) = \varphi^*(\nabla_X T)(\zeta)\,. \end{aligned}$$

□

REMARK 1.4.5. A synthetic way to write the previous formula is $D(t)_X\big(\varphi(t)\big) = 0$. Indeed, we can consider φ as a section of $V^* \otimes TM$, as it is linear on the fibers, and extend the connection to the mixed product of V and TM in the natural way

$$D_X(\zeta \otimes Y) = (D_X\zeta) \otimes Y + \zeta \otimes (\nabla_X Y) \qquad \forall X, Y \in TM\,, \forall \zeta \in V\,.$$

This procedure holds in fact in a more general setting: if $\pi_V : V \to M$ and $\pi_W : W \to M$ are vector bundles over a manifold M and $\psi : V \to W$ is a bundle map (i.e. linear on the corresponding fibers) fiberwise invertible, given a connection ∇ on W, we can pull–back it to a connection D on V, extend it (denoted again D) to the mixed product and compute, for every $X \in \Gamma(TM), \zeta \in \Gamma(V), T \in \Gamma(W^*)$,

$$\begin{aligned}(D_X\psi)(\zeta, T) &= X\big(\psi(\zeta, T)\big) - \psi(D_X\zeta, T) - \psi(\zeta, \nabla_X T)\\ &= X\Big(T\big(\psi(\zeta)\big)\Big) - T\big(\psi(D_X\zeta)\big) - \nabla_X T\big(\psi(\zeta)\big)\\ &= X\Big(T\big(\psi(\zeta)\big)\Big) - T\Big(\nabla_X\big(\psi(\zeta)\big)\Big) - \nabla_X T\big(\psi(\zeta)\big) = 0\end{aligned}$$

by construction.

The final purpose of this method is computing the evolution equation of $\varphi^*(\mathrm{Rm}) \in \Gamma((V^*)^4)$, which we will finalize for the Ricci–Bourguignon flow in Section 3.2.2. In order to do that, we need another lemma which provides the natural extension of the second covariant derivative:

LEMMA 1.4.6. *Let $D^2 : \Gamma(TM) \times \Gamma(TM) \times \Gamma(V) \to \Gamma(V)$ be the second covariant derivative defined by*

$$D^2_{X,Y}(\zeta) = D_X(D_Y\zeta) - D_{\nabla_X Y}\zeta\,, \qquad \forall X, Y \in \Gamma(TM)\,, \forall \zeta \in \Gamma(V)\,,$$

and the rough Laplacian defined by $\Delta_D = \mathrm{tr}_g(D^2)$. Then $\forall T \in \Gamma((TM)^s \otimes (T^*M)^t)$

$$D^2_{X,Y}\big(\varphi^*(T)\big) = \varphi^*(\nabla^2_{X,Y}T)\,, \qquad \forall X, Y \in \Gamma(TM)\,, \tag{1.12}$$

$$\Delta_D\big(\varphi^*(T)\big) = \varphi^*(\Delta_g T)\,. \tag{1.13}$$

PROOF. Analogous to the previous ones. □

In the next proposition with a simple computation we show that the pull–back of the curvatures via the bundle isometries $\{\varphi(t)\}$ have the same symmetry and positivity properties as the original ones.

PROPOSITION 1.4.7. *Let $\mathrm{Pm}(t) \in \Gamma((V^*)^4)$ be the pull–back of the Riemann curvature tensor via the family of isometries $\{\varphi(t)\}$. The followings hold true:*

1. *$\mathrm{Pm}(t)$ has the same symmetry properties of $\mathrm{Rm}_{g(t)}$, hence the associated algebraic operator defined by $\varphi(t) \circ \mathcal{P} = \mathcal{R} \circ \varphi(t)$ is a "curvature operator" $\mathcal{P}(p,t) \in \mathcal{C}_b(V_p)$ for every $p \in M$ and $t \in [0, T)$ (see Remark 1.4.8).*
2. *$\mathcal{P}(p,t)$ has the same eigenvalues of $\mathcal{R}(p,t)$. In particular, $\mathcal{P}$ is positive definite if and only if $\mathcal{R}$ is positive definite.*
3. *$\mathrm{Pic}(t) = \mathrm{tr}_h\big(\mathrm{Pm}(t)\big) = \varphi(t)^*(\mathrm{Ric}_{g(t)})\,,\ \mathrm{P} = \mathrm{tr}_h(\mathrm{Pic}(t)) = \mathrm{R}_{g(t)}$. The conditions on the Ricci and the scalar curvature can be seen on their algebraic version.*
4. *$B(\mathrm{Pm}) = \varphi^*\big(B(\mathrm{Rm})\big)$, where $B(T)_{ijkl} = h^{pq}h^{rs}T_{ipjr}T_{kqls}$ for every $(4,0)$–tensor on V.*

PROOF. (1) and (2) are obvious from the definition.
This computations in (3) and (4) are probably easier in local coordinates.
Therefore, let $\left\{\frac{\partial}{\partial x^i}\right\}_{i=1,\dots,n}$ be the local frame of coordinate fields in TM and $\{e_a\}_{a=1,\dots,n}$ a local frame for V (which can coincide with the coordinate frame as the bundles are the same). In this setting the bundle isometry φ has components φ^i_a and the following equations are satisfied:

$$h_{ab} = \varphi^*(g)(e_a, e_b) = g\big(\varphi(e_a), \varphi(e_b)\big) = g_{ij}\varphi^i_a\varphi^j_b\,, \quad g^{ij} = h^{ab}\varphi^i_a\varphi^j_b\,,$$

where the first one is simply the definition of pull–back metric and the second one is obtained by inverting the first one. Then we compute

$$\begin{aligned}\mathrm{Pic}_{ac} &= h^{bd}\mathrm{Pm}_{abcd} = h^{bd}\mathrm{Rm}_{ijkl}\varphi^i_a\varphi^j_b\varphi^k_c\varphi^l_d \\ &= g^{jl}\mathrm{Rm}_{ijkl}\varphi^i_a\varphi^k_c = \mathrm{Ric}_{ik}\varphi^i_a\varphi^k_c = (\varphi^*\mathrm{Ric})_{ac}\,, \\ \mathrm{P} &= h^{ac}\mathrm{Pic}_{ac} = h^{ac}\mathrm{Ric}_{ik}\varphi^i_a\varphi^k_c = g^{ik}\mathrm{Ric}_{ik} = \mathrm{R}\,, \\ B(\mathrm{Pm})_{abcd} &= h^{st}h^{uv}\mathrm{Pm}_{asbu}\mathrm{Pm}_{ctdv} \\ &= h^{st}h^{uv}\mathrm{Rm}_{iojq}\mathrm{Rm}_{kplr}\varphi^i_a\varphi^o_s\varphi^j_b\varphi^q_u\varphi^k_c\varphi^p_t\varphi^l_d\varphi^r_v \\ &= g^{op}g^{qr}\mathrm{Rm}_{iojq}\mathrm{Rm}_{kplr}\varphi^i_a\varphi^j_b\varphi^k_c\varphi^l_d = \varphi^*\big(B(\mathrm{Rm})\big)_{abcd}\,.\end{aligned}$$

□

REMARK 1.4.8. It must be noticed that, whereas for every $p \in M$ and $t \in [0,T)$ the operator $\mathcal{P}(p,t)$ belongs to the set of algebraic curvature operators $\mathcal{C}_b(V_p)$ (see Appendix A), it is not the curvature operator of the pull–back connection $D(t)$, since the notion of curvature operator as an element of $S^2(\Lambda^2(T^*_pM))$ makes sense only for affine connections and we have on purpose denoted with (V,h) the vector bundle (TM,g_0) to ignore its tangent structure.
Almost everywhere in the present literature, the pull–back $\varphi^*(\mathrm{Rm})$ tensor is again denoted by Rm and this abuse of notation is somehow misleading in suggesting the wrong impression that $\mathrm{Pm}(t) = \varphi(t)^*(\mathrm{Rm}_{g(t)}) = \mathrm{Rm}_{\varphi(t)^*(g(t))} = \mathrm{Rm}_h$, but this formula, which holds whenever $\varphi \in \mathcal{D}iff(M)$, is no longer true for general isomorphisms of the tangent bundle.

1.4.2. Alternative formulations. There is another way to see Uhlenbeck's trick, which is apparently less complicated, but it might hide some tricky aspects.
If $\big(M,g(t)\big)_{t\in[0,T)}$ is the solution of the Cauchy problem (1.4), we consider $\{e_a\}_{a=1,\dots,n}$ a local frame for TM which is orthonormal with respect to the initial data g_0. Let $\{e_a(t)\}_{a=1,\dots,n}$ be the time–dependent local frame that any point p is solution to the following ODEs system:

$$\begin{cases} \frac{d}{dt}e_a(t) = -\frac{1}{2}(Lg(t))^\sharp\big(e_a(t)\big)\,, \\ e_a(0) = e_a\,, a = 1,\dots,n \end{cases}$$

Since the system is linear, the solution exists as long as the solution $g(t)$ of the Cauchy problem exists.

LEMMA 1.4.9. *For every $t \in [0,T)$ $\{e_a(t)\}_{a=1,\dots,n}$ is a local frame of TM orthonormal with respect to the metric $g(t)$.*

PROOF. Since $g_0(e_a,e_b) = \delta_{ab}$, in order to prove the lemma it suffices to show that the scalar products $g(t)(e_a(t),e_b(t))$ are time–independent. From now on, we omit the space and time–dependence of the quantities. Therefore

$$\begin{aligned}\frac{d}{dt}\big(g(e_a,e_b)\big) &= \frac{\partial g}{\partial t}(e_a,e_b) + g\Big(\frac{de_a}{dt},e_b\Big) + g\Big(e_a,\frac{de_b}{dt}\Big) \\ &= Lg(e_a,e_b) - \frac{1}{2}g\big((Lg)^\sharp(e_a),e_b\big) - \frac{1}{2}g\big(e_a,(Lg)^\sharp(e_b)\big) = 0\,.\end{aligned}$$

□

The time–derivative of a tensor is defined as the tensor whose components, with respect to a fixed frame, are derived. In Section 3.2.2, instead of looking at the evolution of $\varphi^*(\mathrm{Rm})$, we could compute the evolution of the components of the Riemann tensor with respect to the orthonormal moving frame defined above. In other words, we could compute the time derivative of the functions $\mathrm{Rm}(e_a,e_b,e_c,e_d)$ for every $a,b,c,d = 1,\dots,n$ and this would lead to the same equations, with slightly different meaning.

REMARK 1.4.10. This formulation of Uhlenbeck's trick has been introduced by Hamilton in [53] and is fairly complicated to be completely described. The key idea is to work on the *frame bundle FM* of the manifold M: to any (p,q)–tensor on the manifold is associated a global function from FM to $\bigotimes^p(\mathbb{R})^* \otimes \bigotimes^q(\mathbb{R})$; differentiating with respect to a connection on M is equivalent to differentiating this vector–valued function with respect to some suitable vector fields on FM. Then, when the metric (and so the connection) changes in time, one shows that the construction can be made on the subbundle of $FM \times [0,T)$ of orthonormal moving frames and there is again a global vector field on this bundle that applied to the vector–valued function gives the time derivative of the components of the original tensor.

The two versions of the trick described above have a synthesis in a third formulation, which we first found in [11]: here one consider the "spatial" bundle E on $M \times [0,T)$ whose fiber at each point (p,t) is given by T_pM. There is a canonical way to extend the time–dependent Levi–Civita connection of TM to a compatible connection on E by defining the covariant time derivative for any section $X \in \Gamma(E)$

$$D_{\frac{\partial}{\partial t}}(X) = \frac{\partial}{\partial t}X + \frac{1}{2}(Lg(t))^\sharp(X).$$

Then one can compute the evolution equation (with the covariant time derivative) of the Riemann tensor seen as an element of $\Gamma(\bigotimes^4 E^*)$. As expected, the Laplacian operator here is the trace of the second covariant derivative only with respect to the spatial components.

1.5. Maximum principle for elliptic operators

After having established the short time existence of solutions to the Cauchy problem (1.4), we want to understand the behaviour of such solutions. If the geometric quantities related to the metric, such as the curvature tensors, satisfy parabolic equations, the maximum principle provides a powerful tool in order to find geometric properties that are pointwise preserved during the flow. Already known from the Euclidean setting, the maximum principle can be easily generalized to functions on a compact manifold that satisfy a heat–type equation, but a really strong extension was established by Hamilton in his first papers on the Ricci flow: in [49] he proved a maximum principle for symmetric $(2,0)$–tensors and in [50] he gave a further generalization to sections of vector bundles endowed with a time–dependent connection compatible with a fixed metric. At the present time there are many refinements of maximum principle and several applications to the Ricci flow: a very detailed exposition can be found in [26, Chapter 10].
In this section we briefly discuss some maximum principles which will be used to study the Ricci–Bourguignon flow in Chapters 3 and 4: the main adjustment is that the operator involved in the evolution of the curvature tensors is no more the metric Laplacian $\Delta_{g(t)}$, but a time–dependent second–order operator which in many cases is still uniformly strongly elliptic.
For completeness, we recall also the weak maximum principle for functions.

THEOREM 1.5.1 (Scalar Maximum Principle). *Let $u : M \times [0,T) \to \mathbb{R}$ be a smooth function solution of the following parabolic equation*

$$\frac{\partial}{\partial t}u = \Delta_{g(t)}u + g(t)(X(p,t,u,\nabla_{g(t)}u), \nabla_{g(t)}u) + F(u,t)\,, \tag{1.14}$$

where X is a continuous vector field and $F : \mathbb{R} \times [0,T) \to \mathbb{R}$ is a continuous function, locally Lipschitz in the last variable.
Let $c_1, c_2 \in \mathbb{R}$ be bounds for the solution at time 0, i.e. $c_1 \leq u(p,0) \leq c_2$ for every $p \in M$, and $U_1(t), U_2(t)$ the solutions of the associated ODE

$$\begin{cases} \frac{dU_i}{dt} &= F(U_i(t),t)\,, \\ U_i(0) &= c_i(0)\,. \end{cases} \tag{1.15}$$

Then $U_1(t) \leq u(p,t) \leq U_2(t)$, for every $p \in M$, $t \in [0,T)$, as long as the solutions of the ODE exist.

Now we want to focus on heat–type equations for sections of vector bundles over M.
Let M be a smooth compact manifold, $(g(t))$, $t \in [0,T)$, a family of Riemannian metrics on M and $(E, h(t))$ $t \in [0,T)$, be a real vector bundle on M, endowed with a (possibly time–dependent)

bundle metric. Let $D(t) : \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$ be a family of linear connections on E compatible at each time with the bundle metric $h(t)$. We have already seen in Section 1.4 how to define the second covariant derivative, using also the Levi-Civita connections $\nabla_{g(t)}$ associated to the Riemannian metrics on M.
We consider a second–order linear operator L on $\Gamma(E)$ that lacks its 0–order term, hence it can be written in a local frame field $\{e_i\}_{i=1,\dots,n}$ of TM

$$L = a^{ij} D^2_{e_i e_j} - b^i D_{e_i} \tag{1.16}$$

where $a = a^{ij} e_i \otimes e_j \in \Gamma(S^2(TM))$ is a symmetric $(0,2)$–tensor and $b = b^i e_i \in \Gamma(TM)$ is a smooth vector field.
The first maximum principle we present applies to symmetric 2–tensors: this is nothing but Hamilton's maximum principle for symmetric 2–tensors (see [49]) generalized to uniformly strongly elliptic operators.

DEFINITION 1.5.2. Let $S \in \Gamma(S^2(T^*M))$ be a symmetric $(2,0)$–tensor and let $F \in \Gamma(S^2(T^*M))$ be a polynomial in S formed by contracting products of S with itself using the metric. We say that F satisfies the *null–eigenvector condition* if whenever S_p is non negative at a point $p \in M$ and $v \in T_pM$ is a null eigenvector of S_p, then $F_p(v,v) \geq 0$.

THEOREM 1.5.3 (Maximum principle for symmetric 2–tensors). *Let $S : [0,T) \to \Gamma(S^2(T^*M))$ be a smooth solution of*

$$\frac{\partial}{\partial t} S = LS + F$$

where L is a uniformly strongly elliptic operator locally given by equation (1.16) with $D(t) = \nabla_{g(t)}$ and F is a polynomial function in S that satisfies the null–eigenvector condition. If $S(0) \geq 0$, then $S(t) \geq 0$ in $[0,T)$.

PROOF (SKETCH). We can rewrite Hamilton's proof in [49, Section 9], by observing that, if $v \in T_pM$ is a null unit eigenvector of S_p at time θ, we can extend v in a neighbourhood of (p,θ) to a time–independent vector field V such that $(\nabla_{g(\theta)})_i V = 0$ and $\Delta_{g(\theta)} V = 0$ in p for every $i = 1,\dots,n$. Then, in (p,θ), we have that $L(S)(v,v)_p = L\big(S(V,V)\big)_p \geq 0$, because the function $S(V,V)$ at time θ has a local spatial minimum in p and L is uniformly strongly elliptic, just like it happened for the Laplacian. ☐

This theorem is a special case of the vectorial maximum principle which we will see below in Theorem 1.5.5. However, such theorem requires a fixed bundle metric while in Theorem 1.5.3 the metric is the extension of $g(t)$ to $S^2(T^*M)$ and this can be useful when we want to study the evolution equation of symmetric $(2,0)$–tensors without applying Uhlenbeck's trick explained in Section 1.4.
Weinberger in [79] proved the maximum principle for systems of solutions of a time–dependent heat equation in the Euclidean space; Hamilton in [50] treated the general case of a vector bundle over an evolving Riemannian manifold. Here we present a slight generalization of Hamilton's theorem to uniformly strongly elliptic operators (see [74, Theorem A] for the "static" version proved by Savas–Halilaj and Smoczyk).
For this theorem we consider a fixed bundle metric h on the vector bundle E and a family of time–dependent connections $D(t)$, compatible at every time with h.

DEFINITION 1.5.4. Let $S \subset E$ be a subset and set $S_p = S \cap E_p$ for every $p \in M$. We say that S is *invariant under parallel translation* with respect to D if, for every curve $\gamma : [0,l] \to M$ and vector $v \in S_{\gamma(0)}$, the unique parallel (with respect to D) section $v(s) \in E_{\gamma(s)}$ along $\gamma(s)$ with $v(0) = v$ is contained in S.

THEOREM 1.5.5 (Vectorial Maximum Principle). *Let $u : [0,T) \to \Gamma(E)$ be a smooth solution of the following parabolic equation*

$$\frac{\partial}{\partial t} u = Lu + F(u,t)\,, \tag{1.17}$$

where L is a uniformly strongly elliptic operator as defined in (1.16) and $F : E \times [0,T) \to E$ is a continuos map, locally Lipschitz in the factor E, which is also fiber–preserving, i.e. for every $p \in M$, $v \in E_p$, $t \in [0,T)$ $F(v,t) \in E_p$.
Let $K \subset E$ be a closed subset, invariant under parallel translation with respect to $D(t)$, for every $t \in [0,T)$, and convex in the fibers, i.e. $K_p = K \cap E_p$ is convex for every $p \in M$.
Suppose that K is preserved by the ODE associated to equation (1.17), i.e. for every $p \in M$ and $U_0 \in K_p$, the solution $U(t)$, of

$$\begin{cases} \frac{dU}{dt} & = & F_p(U(t),t) \\ U(0) & = & U_0 \end{cases} \tag{1.18}$$

remains in K_p. Then, if u is contained in K at time 0, u remains in K, i.e. $u(p,t) \in K_p$ for every $p \in M$, $t \in [0,T)$, as long as the solution of the ODE exists.

We can follow exactly the detailed proof written in [26, Chapter 10, Section 3], provided that we generalize Lemma 10.34 there. In [6, Lemma 1.2] Bohm e Wilking prove that if $K \subset E$ satisfies all the hypothesis of the theorem and $u \in \Gamma(E)$ is a smooth section of E, if $u(p) \in K_p$ $\forall p \in M$, then $\forall v \in T_pM$ $D_v u(p) \in C_{u(p)}K_p$, the tangent cone of the convex set K_p at $u(p)$ and also the Hessian

$$D^2_v u(p) = D_v D_v u(p) - D_{D_v v} u(p)$$

belongs to $C_{u(p)}K_p$. Then, as L is uniformly strongly elliptic, it can be diagonalized pointwise by an orthonormal basis and the result follows by the convexity of the tangent cone.
There is a further generalization of this maximum principle which allows the subset K to be time–dependent.

THEOREM 1.5.6 (Vectorial Maximum Principle–Time–dependent Set). *Let $u : [0,T) \to \Gamma(E)$ be a smooth solution of the parabolic equation (1.17), with the notations of the previous theorem. For every $t \in [0,T)$, let $K(t) \subset E$ be a closed subset, invariant under parallel translation with respect to $D(t)$, convex in the fibers and such that the space–time track*

$$\mathcal{T} = \{(v,t) \in E \times \mathbb{R} : v \in K(t), t \in [0,T)\}$$

is closed in $E \times [0,T)$. Suppose that, for every $t_0 \in [0,T)$, $K(t_0)$ is preserved by the ODE associated, i.e. for any $p \in M$, any solution $U(t)$ of the ODE that starts in $K(t_0)_p$ will remain in $K(t)_p$ for all later times, as long as it exists. Then, if $u(0)$ is contained in $K(0)$, $u(p,t) \in K(t)_p$ for ever $p \in M$, $t \in [0,T)$, as long as the solution of the ODE exists.

The proof of this theorem, when K depends continuously on time and F does not depend on time is due to Bohm and Wilking (see [6, Theorem 1.1]). In the general case the proof can be found in [26, Chapter 10, Section 5], with the usual adaptation to the strongly uniformly elliptic case.

The following lemma is a collection of standard arguments that together give a simple way to find invariant convex subsets of E, when $E = End_{SA}(V)$ is the space of self–adjoint endomorphisms of a bundle V.

LEMMA 1.5.7. [26, Lemma 10.11] *Let (V,h) be a vector bundle on M of rank m, endowed of a bundle metric h and a connection D compatible with h. Let $E = End_{SA}(V)$ be the space of endomorphisms of V which are self–adjoint with respect to h.*

(1) *The canonical extension of D to $End(V)$ restricts to a metric connection on E.*
(2) *The eigenvalues of a parallel section $e \in \Gamma(E)$ do not depend on the points of M.*
(3) *The super(sub)–levels of a function of the ordered eigenvalues define invariant subsets of E.*
(4) *The convexity of such subsets is directly related to the convexity of the defining functions.*

PROOF. 1) Let $e \in \Gamma(E)$ be a self–adjoint endomorphism and $X \in \Gamma(TM)$. Then $D_X e$ is self–adjoint, because for every $v, v' \in \Gamma(V)$

$$\begin{aligned} h((D_X e)(v), v') &= X(h(e(v), v')) - h(e(D_X v), v') - h(e(v), D_X v') \\ &= X(h(v, e(v'))) - h(D_X v, e(v')) - h(v, e(D_X v')) \\ &= h(v, (D_X e)v') \end{aligned}$$

This computation shows that the sub–bundle of self–adjoint endomorphisms can be considered for applications of Theorem 1.5.5.
2) Let $p \in M$ be a point, $e_p \in E_p$ and $v_p \in V_p$ such that $e_p(v_p) = \lambda v_p$, for $\lambda \in \mathbb{R}$. If $q \in M$ is another point, γ a curve between p and q and e and v the parallel translations of respectively e_p and v_p along γ we have that both $e(v)$ and λv are parallel sections along γ that coincide in p. By uniqueness of parallel translation they must coincide along γ; in particular $e_q(v_q) = \lambda v_q$, that is λ is an eigenvalue of e at each point of the curve.
3) The previous argument can be extended to each eigenvalue of e, hence for a parallel section along γ is well defined the m–uple of its ordered eigenvalues along γ, that is $(\lambda_1(e), \dots, \lambda_m(e)) \in \Lambda$, where $\Lambda \subset \mathbb{R}^m$ is the set of ordered m-uple of real numbers.
If $G : \Lambda \to \mathbb{R}$ is a real function and $c \in \mathbb{R}$, then the subset

$$K = \{e \in \Gamma(E) : \forall p \in M \quad G((\lambda_i(e_p))) \leq c\} \tag{1.19}$$

is therefore invariant under parallel translation.
4) If K is defined as above by the function G and the map $e \mapsto G(\lambda_i(e_p))$ is convex for every $p \in M$, then K is convex. Indeed, let $e, e' \in K$ and $s \in (0,1)$, then for every $p \in M$

$$G(\lambda_i(se_p + (1-s)e'_p)) \leq sG(\lambda_i(e_p)) + (1-s)G(\lambda_i(e'_p)) \leq sc + (1-s)c = c\,, \tag{1.20}$$

by basic property of convex functions. If the previous map is concave and K is defined by the opposite inequality, then K is again convex. □

1.6. Interpolation inequalities

In this section we prove some inequalities between Sobolev norms of tensors that we will use to prove the curvature estimates for the Ricci–Bourguignon flow in Section 3.4.
Let $\|T\|_{H^p_k}$ be the Sobolev norm given by the sum of the L^p norms of the first k covariant derivatives of a tensor T.
First of all, we prove the proposition:

PROPOSITION 1.6.1. *Let $k \in \mathbb{N}$, $p \in [1, +\infty]$ and $q \in [1, +\infty)$. There exists a constant $C(n, k, p, q)$ such that for all $0 \leq j \leq k$ and all tensor T*

$$\|T\|_{H^{r_j}_j} \leq C \|T\|_p^{1-\frac{j}{k}} \|T\|_{H^q_k}^{\frac{j}{k}}\,,$$

where $\frac{1}{r_j} = \frac{1-\frac{j}{k}}{p} + \frac{\frac{j}{k}}{q}$.

To prove this proposition, we need several lemmas.

LEMMA 1.6.2 (Hamilton [49], Theorem 12.1). *Let $p \in [1, +\infty]$, $q \in [1, +\infty)$ and $r \in [2, +\infty)$ such that $\frac{2}{r} = \frac{1}{p} + \frac{1}{q}$. There exists a constant $C(n, r)$ such that for all tensor T*

$$\|\nabla T\|_r^2 \leq C \|T\|_p \|\nabla^2 T\|_q\,.$$

PROOF. We first observe that,

$$\nabla_i(|\nabla T|^{r-2}) = \frac{r-2}{2} |\nabla T|^{r-4} \nabla_i(|\nabla T|^2) \text{ and } |\nabla_i(|\nabla T|^2)| \leq 2|\nabla^2 T||\nabla T|\,.$$

Since M is compact without boundary, we get

$$\begin{aligned}
\|\nabla T\|_r^r &= \int_M \langle \nabla_i T, |\nabla T|^{r-2} \nabla^i T\rangle \, d\mu_g \\
&= -\int_M \langle T, \nabla_i(|\nabla T|^{r-2} \nabla^i T)\rangle \, d\mu_g \\
&= -\int_M \langle T, \frac{r-2}{2} |\nabla T|^{r-4} \nabla_i(|\nabla T|^2) \nabla^i T\rangle \, d\mu_g - \int_M \langle T, |\nabla T|^{r-2} \Delta T\rangle \, d\mu_g \\
&\leq C \int_M |T||\nabla^2 T||\nabla T|^{r-2} \, d\mu_g \\
&\leq C \|T\|_p \|\nabla^2 T\|_q \|\nabla T\|_r^{r-2}\,,
\end{aligned}$$

using Hölder's inequality with $\frac{r-2}{r} + \frac{1}{p} + \frac{1}{q} = 1$. This ends the proof of this lemma. □

LEMMA 1.6.3. *Let $k \in \mathbb{N}$, $p \in [1, +\infty]$ and $q \in [1, +\infty)$. There exists a constant $C(n, k, p, q)$ such that for all tensor T*

$$\|T\|^2_{H^r_{k+1}} \leq C\|T\|_{H^p_k}\|T\|_{H^q_{k+2}}\,,$$

where $\frac{2}{r} = \frac{1}{p} + \frac{1}{q}$.

PROOF. We apply Lemma 1.6.2 to $\nabla^k T$:

$$\|\nabla^{k+1}T\|^2_r \leq C\|\nabla^k T\|_p\|T\|_{H^q_{k+2}} \leq C\|T\|_{H^p_k}\|T\|_{H^q_{k+2}}.$$

On the other side, using Hölder's inequality

$$\|T\|^2_r \leq \|T\|_p\|T\|_q \leq \|T\|_{H^p_k}\|T\|_{H^q_{k+2}}\,.$$

The combination of both inequalities gives the result. □

LEMMA 1.6.4 (Hamilton [49], Corollary 12.5). *Let $k \in \mathbb{N}$. If $f : \{0, \dots, k\} \to \mathbb{R}$ satisfies for all $0 < j < k$*

$$f(j) \leq Cf(j-1)^{\frac{1}{2}} f(j+1)^{\frac{1}{2}}\,,$$

where C is a positive constant, then for all $0 \leq j \leq k$

$$f(j) \leq C^{j(k-j)} f(0)^{1-\frac{j}{k}} f(k)^{\frac{j}{k}}\,.$$

PROOF OF PROPOSITION 1.6.1. We apply Lemma 1.6.4 with $f(j) = \|T\|_{H^{r_j}_j}$. We observe that $\frac{2}{r_j} = \frac{1}{r_{j-1}} + \frac{1}{r_{j+1}}$, then Lemma 1.6.3 shows that there exists $C(n, k, p, q)$ such that

$$f(j) \leq Cf(j-1)^{\frac{1}{2}} f(j+1)^{\frac{1}{2}}\,.$$

Since $r_0 = p$ and $r_k = q$, Lemma 1.6.4 gives Proposition 1.6.1. □

REMARK 1.6.5. We observe that, in Proposition 1.6.1, there holds the same conclusion, if we substitute in both members of the inequality the full Sobolev space norm of T with the corresponding Lebesgue norm only of the higher covariant derivative of T: it is sufficient to apply Lemma 1.6.4 with $f(j) = \|\nabla^j \mathrm{Rm}\|_{r_j}$ and Lemma 1.6.2. In such case this proposition is a generalization of [49, Corollary 12.7]. Also the following Lemma 1.6.7 remains true with the same substitution made in Proposition 1.6.1.

NOTATION. Given two tensor S and T, we denote with $S * T$ any linear combination of expressions of the the form $S \otimes T \otimes g^{\otimes_k} \otimes (g^{-1})^{\otimes_j}$, i.e. any product of the two tensors with some indexes raised or lowered by the metric g.

LEMMA 1.6.6. *For all tensors of the form $S * T$, there exists C depending on the dimension and the coefficients in the expression such that*

$$|S * T| \leq C|S||T|\,.$$

PROOF. By the Cauchy–Schwarz inequality, $\mathrm{tr}_g(T)^2 = (g^{ij}T_{ij})^2 \leq n|T|^2$. Then

$$|S * T| \leq C|S \otimes T \otimes g^{\otimes j} \otimes (g^{-1})^{\otimes k}| \leq Cn^{\frac{j+k}{2}}|S||T|\,.$$

□

Let $k \in \mathbb{N}$, and set, for a tensor T, $F_g(T) = \sum_{j+l=k,\, j,l \geq 0} \nabla^j T * \nabla^l T * \nabla^k T.$

LEMMA 1.6.7. *Let $k \in \mathbb{N}$. Let $p \in [2, +\infty]$ and $q \in [2, +\infty)$ such that $\frac{1}{p} + \frac{2}{q} = 1$. There exists $C(n, k, p, q, F)$ such that for all tensor T,*

$$\int_M |F_g(T)|\, d\mu_g \leq C\|T\|_p\|T\|^2_{H^q_k}\,.$$

PROOF. Let us consider one term in $F_g(T)$ that can be written $\nabla^j T * \nabla^l T * \nabla^k T$, $j, l \geq 0$ such that $j + l = k$. We set

$$\frac{1}{r_j} = \frac{1 - \frac{j}{k}}{p} + \frac{\frac{j}{k}}{q}, \qquad \frac{1}{r_l} = \frac{1 - \frac{l}{k}}{p} + \frac{\frac{l}{k}}{q}.$$

Since $\frac{1}{r_j} + \frac{1}{r_l} + \frac{1}{q} = 1$, we apply Lemma 1.6.6 to get the first inequality, then Hölder's inequality and finally we replace each factor with the full Sobolev space norm.

$$\begin{aligned}\int_M |\nabla^j T * \nabla^l T * \nabla^k T| \, d\mu_g &\leq C' \int_M |\nabla^j T||\nabla^l T||\nabla^k T| \, d\mu_g \\ &\leq C' \|\nabla^j T\|_{r_j} \|\nabla^l T\|_{r_l} \|\nabla^k T\|_q \\ &\leq C' \|T\|_{H^{r_j}_j} \|T\|_{H^{r_l}_l} \|T\|_{H^q_k},\end{aligned}$$

Then we apply Proposition 1.6.1 to the first two factors and we obtain

$$\int_M |\nabla^r T * \nabla^s T * \nabla^k T| \, d\mu_g \leq C \|T\|_p \|T\|^2_{H^q_k}.$$

The result follows since $F_g(T)$ is a linear combination of such terms. □

CHAPTER 2

The renormalization group flow

In this chapter we introduce and study the renormalization group flow, the first geometric flow considered in this thesis. Unlike the Ricci flow, the mean curvature flow or the Ricci–Bourguignon flow which we will study in the next chapters, the renormalization group flow involves a quadratic expression of the curvature tensors, as the cross curvature flow introduced by Chow and Hamilton in [**28**]. The short time existence and uniqueness of a smooth evolution of every initial metric of a three–dimensional manifold with curvature not changing sign, was established for the cross curvature flow by Buckland in [**14**]. In the second section, we prove the analogous result for the renormalization group flow (Theorem 2.1.1), provided the sectional curvature of the initial metric is suitably bounded from below. In the third section we carry out the computation of the evolution equations for the Ricci, the scalar and the sectional curvature along the flow in dimension three. Unfortunately, due to the complexity of these equations, we are currently unable to establish if the condition assuring the short time existence is preserved along the flow and therefore to begin the study of the long time behaviour of solutions (see Remark 2.2.2). Finally in the last section we compute the principal symbols of similar quadratic flows.
Most of the contents of this chapter appeared in [**31**], written by the author and Mantegazza. Gimre, Guenther and Isenberg extended the short time existence of this flow to any dimensions in [**47**] (see also [**45**]). Since the computation of the principal symbol is the same, we also present their n–dimensional version of Theorem 2.1.1.

The *renormalization group* (RG) arises in modern theoretical physics as a method to investigate the changes of a system viewed at different distance scales. Since its introduction in the early ’50, this set of ideas has given rise to significant developments in *quantum field theory* (QFT) and opened connections between contemporary physics and Riemannian geometry. In spite of this, the RG still lacks a strong mathematical foundation.
In this chapter we deal with a particular example in string theory, the flow equation for the world–sheet non linear sigma–models, and we try to analyse the contribution given from its second order truncation. More precisely, let S be the classical (harmonic map) action

$$S(\varphi) = \frac{1}{4\pi a}\int_\Sigma \mathrm{tr}_h(\varphi^* g)\, d\mu_h$$

where $\varphi : \Sigma \to M$ is a smooth map between a surface (Σ, h) and a Riemannian manifold (M, g) of dimension $n \geq 3$. The quantity $a > 0$ is the so–called *string coupling constant*. Roughly speaking, in order to control the path integral quantization of the action S, one introduces a cut–off momentum Λ which parametrizes the spectrum of fluctuations of the theory as the distance scale is varied as $1/\Lambda \to 1/\Lambda'$. This formally generates a flow (the renormalization group flow) in the space of actions which is controlled by the induced scale–dependence in (M, g). Setting $\tau := -\ln(\Lambda/\Lambda')$, one thus considers the so–called *beta functions* β, associated with the renormalization group of the theory and defined by the formal flow $g(\tau)$ satisfying

$$\frac{\partial g_{ik}(\tau)}{\partial \tau} = -\beta_{ik}.$$

In the perturbative regime (that is, when $a|\mathrm{Rm}(g)| \ll 1$) the beta functions β_{ik} can be expanded in powers of a, with coefficients which are polynomial in the curvature tensor of the metric g and its derivatives. As the quantity $a|\mathrm{Rm}(g)|$ is supposed to be very small, the first order truncation should provide a good approximation of the full RG–flow

$$\frac{\partial g_{ik}}{\partial \tau} = -a\mathrm{R}_{ik} + o(a)\,,$$

as $a \to 0$.
Hence, the first order truncation (with the substitution $\tau = t/2a$) coincides with the *Ricci flow* $\partial_t g = -2\mathrm{Ric}$, as noted by Friedan [39, 40] and Lott [61], see also [17].
It is a well–known fact that generally the Ricci flow becomes singular in finite time and in [49] Hamilton proved that at a finite singular time $T > 0$, the Riemann curvature blows up. Then, near a singularity, the Ricci flow might no longer be a valid approximation of the behavior of the sigma–model. From the physical point of view, it appears then relevant to possibly consider the coupled flow generated by a more general action, as in [16, 68].
Another possibility may be to consider also the second order term in the expansion of the beta functions, whose coefficients are quadratic in the curvature and therefore are (possibly) dominating, even when $a|\mathrm{Rm}(g)| \to 0$. The resulting flow is called *two–loop* RG–flow

$$\frac{\partial g_{ik}}{\partial \tau} = -a\mathrm{R}_{ik} - \frac{a^2}{2}\mathrm{R}_{ijlm}\mathrm{R}_{kstu}g^{js}g^{lt}g^{mu}\,, \tag{2.1}$$

see [59]. We refer to it as $\mathrm{RG}^{2,a}$–flow.

After rescaling the flow parameter $\tau \to t/2a$ in equation (2.1), the $\mathrm{RG}^{2,a}$–flow is given by

$$\partial_t g_{ik} = -2\mathrm{R}_{ik} - a\mathrm{R}_{ijlm}\mathrm{R}_{kstu}g^{js}g^{lt}g^{mu}\,,$$

which can be seen as a sort of "perturbation" of the Ricci flow $\partial_t g_{ik} = -2\mathrm{R}_{ik}$.

2.1. Short time existence

In this section, we are going to consider the short time existence of this flow for an initial three–dimensional, smooth, compact Riemannian manifold.
In the special three–dimensional case the Weyl part is identically zero, hence the algebraic decomposition of the Riemann tensor (1.2) simplifies and the evolution equation has the following expression.

$$\partial_t g_{ik} = -2\mathrm{R}_{ik} - a(2\mathrm{R}\mathrm{R}_{ik} - 2\mathrm{R}^2_{ik} + 2|\mathrm{Ric}|^2 g_{ik} - \mathrm{R}^2 g_{ik})\,,$$

where $\mathrm{R}^2_{ik} = \mathrm{R}_{ij}\mathrm{R}_{lk}g^{jl}$. In particular we prove the following theorem

THEOREM 2.1.1. *Let (M, g_0) be a compact, smooth, three–dimensional Riemannian manifold and $a \in \mathbb{R}$. Assume that the sectional curvature K_0 of the initial metric g_0 satisfies*

$$1 + 2a\mathrm{K}_0(X, Y) > 0 \tag{2.2}$$

for every point $p \in M$ and vectors $X, Y \in T_pM$. Let

$$Lg_{ik} = -2\mathrm{R}_{ik} - a\mathrm{R}_{ijlm}\mathrm{R}_{kstu}g^{js}g^{lt}g^{mu}\,,$$

then, there exists some $T > 0$ such that the Cauchy problem

$$\begin{cases} \partial_t g = Lg \\ g(0) = g_0 \end{cases} \tag{2.3}$$

admits a unique smooth solution $g(t)$ for $t \in [0, T)$.

REMARK 2.1.2. Notice that, even if not physically relevant, in this theorem we also allow $a < 0$. In such case the condition on the initial metric becomes

$$\mathrm{K}_0(X, Y) < -\frac{1}{2a}\,,$$

which is clearly satisfied by every manifold with negative curvature.
Any manifold with positive curvature satisfies instead condition (2.2), for every $a > 0$.

The evolution problem involves a fully non linear second–order differential operator L and, from the general existence theory of non linear parabolic PDEs recalled in Section 1.2, it follows that the Cauchy problem (2.3) admits a unique smooth solution for short time if the linearized operator around the initial data $DL_g(h) = \frac{d}{ds}L(g+sh)\big|_{s=0}$ is strongly elliptic, that is, it satisfies condition (1.3) for any symmetric 2–tensor h and cotangent vector ξ.
It is trivial that the operator involved, as for the Ricci flow, can only be weakly elliptic, due to the invariance of the curvature tensors by the action of the group of diffeomorphisms of the manifold M (see Lemma 1.3.1). Under assumption (2.2), we will show that there are no other degeneracies and the other eigenvalues of the principal symbols are all strictly positive, then DeTurck's trick described in Section 1.3 will be sufficient to establish the existence of a short time solution of the $\mathrm{RG}^{2,a}$–flow.

We start computing the linearized operator DL_g of the operator L at a metric g in any dimension $n \geq 3$.
The Riemann and Ricci tensors have the following linearizations, see [5, Theorem 1.174] or [77].

$$\begin{aligned} D\mathrm{Rm}_g(h)_{ijkm} &= \frac{1}{2}\Big(-\nabla_j\nabla_m h_{ik} - \nabla_i\nabla_k h_{jm} + \nabla_i\nabla_m h_{jk} + \nabla_j\nabla_k h_{im}\Big) + \mathrm{LOT}\,, \\ D\mathrm{Ric}_g(h)_{ik} &= \frac{1}{2}\Big(-\Delta h_{ik} - \nabla_i\nabla_k \mathrm{tr}(h) + \nabla_i\nabla^t h_{tk} + \nabla_k\nabla^t h_{it}\Big) + \mathrm{LOT}\,, \end{aligned}$$

where, as usual, we use the metric g to lower and upper indices and LOT stands for *lower order terms*.
Then, the linearized of L around g, for every $h \in S^2M$, is given by

$$\begin{aligned} DL_g(h)_{ik} &= -2D\mathrm{Ric}_g(h)_{ik} - aD\mathrm{Rm}_g(h)_i{}^{stu}\mathrm{R}_{kstu} - a\mathrm{R}_{istu}D\mathrm{Rm}_g(h)_k{}^{stu} + \mathrm{LOT} \\ &= \Delta h_{ik} + \nabla_i\nabla_k\mathrm{tr}(h) - \nabla_i\nabla^t h_{tk} - \nabla_k\nabla^t h_{it} \\ &\quad -\frac{a}{2}\mathrm{R}_{kstu}\big(\nabla^s\nabla^t h_i^u + \nabla_i\nabla^u h^{st} - \nabla^s\nabla^u h_i^t - \nabla_i\nabla^t h^{su}\big) \\ &\quad -\frac{a}{2}\mathrm{R}_{istu}\big(\nabla^s\nabla^t h_k^u + \nabla_k\nabla^u h^{st} - \nabla^s\nabla^u h_k^t - \nabla_k\nabla^t h^{su}\big) + \mathrm{LOT} \\ &= \Delta h_{ik} + \nabla_i\nabla_k\mathrm{tr}(h) - \nabla_i\nabla^t h_{kt} - \nabla_k\nabla^t h_{it} \\ &\quad +a\mathrm{R}_{kstu}\big(\nabla_i\nabla^t h^{su} - \nabla^s\nabla^t h_i^u\big) + a\mathrm{R}_{istu}\big(\nabla_k\nabla^t h^{su} - \nabla^s\nabla^t h_k^u\big) + \mathrm{LOT}\,, \end{aligned}$$

where the last passage follows from the symmetries of the Riemann tensor (interchanging the last two indices makes it change sign).
Now we obtain the principal symbol of the linearized operator in the direction of an arbitrary cotangent vector ξ by replacing each covariant derivative ∇_α with the corresponding component ξ_α,

$$\begin{aligned} \sigma_\xi(DL_g)(h)_{ik} &= \xi^t\xi_t h_{ik} + \xi_i\xi_k\mathrm{tr}(h) - \xi_i\xi^t h_{kt} - \xi_k\xi^t h_{it} \\ &\quad +a\mathrm{R}_{kstu}\big(\xi_i\xi^t h^{su} - \xi^s\xi^t h_i^u\big) + a\mathrm{R}_{istu}\big(\xi_k\xi^t h^{su} - \xi^s\xi^t h_k^u\big)\,. \end{aligned}$$

Since the symbol is homogeneous, we can assume that $|\xi|_g = 1$, furthermore, we can assume to do all the following computations in an orthonormal basis $\{e_1, \dots, e_n\}$ of T_pM such that $\xi = g(e_1, \cdot)$, hence, $\xi_i = 0$ for $i \neq 1$.
Then, we obtain,

$$\begin{aligned} \sigma_\xi(DL_g)(h)_{ik} &= h_{ik} + \delta_{i1}\delta_{k1}\mathrm{tr}(h) - \delta_{i1}\delta^{t1}h_{tk} - \delta_{k1}\delta^{t1}h_{it} \\ &\quad +a\mathrm{R}_{kstu}\big(\delta_{i1}\delta^{t1}h^{su} - \delta^{s1}\delta^{t1}h_i^u\big) + a\mathrm{R}_{istu}\big(\delta_{k1}\delta^{t1}h^{su} - \delta^{s1}\delta^{t1}h_k^u\big) \\ &= h_{ik} + \delta_{i1}\delta_{k1}\mathrm{tr}(h) - \delta_{i1}h_{1k} - \delta_{k1}h_{i1} \\ &\quad +a\mathrm{R}_{ks1u}\delta_{i1}h^{su} - a\mathrm{R}_{k11u}h_i^u + a\mathrm{R}_{is1u}\delta_{k1}h^{su} - a\mathrm{R}_{i11u}h_k^u\,. \end{aligned} \tag{2.4}$$

So far the dimension n of the manifold was arbitrary, now we carry out the computation in the special case $n = 3$ (using again the symmetries of the Riemann tensor),

$$\sigma_\xi(DL_g)\begin{pmatrix} h_{11}\\ h_{12}\\ h_{13}\\ h_{22}\\ h_{33}\\ h_{23}\end{pmatrix} = \begin{pmatrix} h_{22}(1+2a\mathrm{R}_{1212}) + h_{33}(1+2a\mathrm{R}_{1313}) + h_{23}(4a\mathrm{R}_{1213})\\ h_{33}a\mathrm{R}_{1323} + h_{23}a\mathrm{R}_{1223}\\ h_{22}a\mathrm{R}_{1232} + h_{23}a\mathrm{R}_{1332}\\ h_{22}(1+2a\mathrm{R}_{1212}) + h_{23}2a\mathrm{R}_{1213}\\ h_{33}(1+2a\mathrm{R}_{1313}) + h_{23}2a\mathrm{R}_{1213}\\ h_{22}a\mathrm{R}_{1213} + h_{33}a\mathrm{R}_{1213} + h_{23}(1 + a\mathrm{R}_{1212} + a\mathrm{R}_{1313})\end{pmatrix}.$$

Then, we conclude that

$$\sigma_\xi(DL_g) = \begin{pmatrix} 0 & 0 & 0 & 1+2a\mathrm{R}_{1212} & 1+2a\mathrm{R}_{1313} & 4a\mathrm{R}_{1213}\\ 0 & 0 & 0 & 0 & a\mathrm{R}_{1323} & a\mathrm{R}_{1223}\\ 0 & 0 & 0 & a\mathrm{R}_{1232} & 0 & a\mathrm{R}_{1332}\\ 0 & 0 & 0 & 1+2a\mathrm{R}_{1212} & 0 & 2a\mathrm{R}_{1213}\\ 0 & 0 & 0 & 0 & 1+2a\mathrm{R}_{1313} & 2a\mathrm{R}_{1213}\\ 0 & 0 & 0 & a\mathrm{R}_{1213} & a\mathrm{R}_{1213} & 1 + a\mathrm{R}_{1212} + a\mathrm{R}_{1313}\end{pmatrix}.$$

As expected, in the kernel of the principal symbol there is at least the three–dimensional space of forms $h = \xi \otimes \nu + \nu \otimes \xi \in S^2(T_p^*M)$ where ν is any cotangent vector, that is the principal symbol of the operator δ^*, or, otherwise said, the variations of the metric which are tangent to the orbits of the group of diffeomorphisms (see Lemma 1.3.1).
Now we use the algebraic decomposition of the Riemann tensor in order to simplify the computation of the other eigenvalues.
We recall that, in dimension three, the Riemann tensor is fully determined by the Ricci tensor in the following way

$$\mathrm{R}_{ijkl} = (\mathrm{Ric} \owedge g)_{ijkl} - \frac{\mathrm{R}}{4}(g \owedge g)_{ijkl}\,.$$

We supposed to choose at every point $p \in M$ an orthonormal basis, therefore the principal symbol can be expressed in the simpler form

$$\sigma_\xi(DL_g) = \begin{pmatrix} 0 & 0 & 0 & 1+a(\mathrm{R}-2\mathrm{R}_{33}) & 1+a(\mathrm{R}-2\mathrm{R}_{22}) & 4a\mathrm{R}_{23}\\ 0 & 0 & 0 & 0 & a\mathrm{R}_{12} & -a\mathrm{R}_{13}\\ 0 & 0 & 0 & a\mathrm{R}_{13} & 0 & -a\mathrm{R}_{12}\\ 0 & 0 & 0 & 1+a(\mathrm{R}-2\mathrm{R}_{33}) & 0 & 2a\mathrm{R}_{23}\\ 0 & 0 & 0 & 0 & 1+a(\mathrm{R}-2\mathrm{R}_{22}) & 2a\mathrm{R}_{23}\\ 0 & 0 & 0 & a\mathrm{R}_{23} & a\mathrm{R}_{23} & 1+a\mathrm{R}_{11}\end{pmatrix}.$$

In order to apply the argument of DeTurck, we need the weak ellipticity of the linearized operator. To get that we have to compute the eigenvalues of the minor

$$A = \begin{pmatrix} 1+a(\mathrm{R}-2\mathrm{R}_{33}) & 0 & 2a\mathrm{R}_{23}\\ 0 & 1+a(\mathrm{R}-2\mathrm{R}_{22}) & 2a\mathrm{R}_{23}\\ a\mathrm{R}_{23} & a\mathrm{R}_{23} & 1+a\mathrm{R}_{11}\end{pmatrix}.$$

We claim that with a suitable orthonormal change of the basis of the plane $span\{e_2, e_2\} = e_1^\perp$ we can always get an orthonormal basis $\{e_1', e_2', e_3'\}$ of T_pM such that $e_1' = e_1$ and $\mathrm{R}_{23}' = \mathrm{Ric}(e_2', e_3') = 0$.
Indeed, if $\{e_2', e_3'\}$ is any orthonormal basis of $e_1^\perp$, we can write

$$e_2' = \cos\alpha e_2 + \sin\alpha e_3 \qquad e_3' = -\sin\alpha e_2 + \cos\alpha e_3$$

for some $\alpha \in [0, 2\pi)$. Plugging this into the expression of the Ricci tensor, we obtain

$$\begin{aligned}\mathrm{R}_{23}' &= \cos\alpha\sin\alpha(\mathrm{R}_{33} - \mathrm{R}_{22}) + (\cos^2\alpha - \sin^2\alpha)\mathrm{R}_{23}\\ &= \frac{1}{2}\sin(2\alpha)(\mathrm{R}_{33} - \mathrm{R}_{22}) + \cos(2\alpha)\mathrm{R}_{23}\,.\end{aligned}$$

Hence, in order to have $\mathrm{R}'_{23} = 0$, it is sufficient to choose

$$\alpha = \begin{cases} \frac{\pi}{4} & \text{if } \mathrm{R}_{22} = \mathrm{R}_{33}, \\ \frac{1}{2}\arctan\left(\frac{2\mathrm{R}_{23}}{\mathrm{R}_{22}-\mathrm{R}_{33}}\right) & \text{otherwise.} \end{cases}$$

The matrix written above represents the symbol $\sigma_\xi(DL_g)$ with respect to a generic orthonormal basis where the first vector coincides with $g(\xi, \cdot)$, so with this change of basis we obtain

$$\sigma_\xi(DL_g) = \begin{pmatrix} 0 & 0 & 0 & 1 + a(\mathrm{R} - 2\mathrm{R}_{33}) & 1 + a(\mathrm{R} - 2\mathrm{R}_{22}) & 0 \\ 0 & 0 & 0 & 0 & a\mathrm{R}_{12} & -a\mathrm{R}_{13} \\ 0 & 0 & 0 & a\mathrm{R}_{13} & 0 & -a\mathrm{R}_{12} \\ 0 & 0 & 0 & 1 + a(\mathrm{R} - 2\mathrm{R}_{33}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 + a(\mathrm{R} - 2\mathrm{R}_{22}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 + a\mathrm{R}_{11} \end{pmatrix} .$$

Hence, the other three eigenvalues of the matrix $\sigma_\xi(DL_g)$ are the diagonal elements of the matrix

$$A = \begin{pmatrix} 1 + a(\mathrm{R} - 2\mathrm{R}_{33}) & 0 & 0 \\ 0 & 1 + a(\mathrm{R} - 2\mathrm{R}_{22}) & 0 \\ 0 & 0 & 1 + a\mathrm{R}_{11} \end{pmatrix} ,$$

that is,

$$\lambda_1 = 1 + a(\mathrm{R} - 2\mathrm{R}_{33}) , \quad \lambda_2 = 1 + a(\mathrm{R} - 2\mathrm{R}_{22}) , \quad \lambda_3 = 1 + a\mathrm{R}_{11} .$$

Now we recall that, if $\{e_j\}_{j=1,\dots,n}$ is an orthonormal basis of the tangent space, the Ricci quadratic form is the sum of the sectional curvatures,

$$\mathrm{R}_{ii} = \sum_{j \neq i} \mathrm{K}(e_i, e_j)$$

and the scalar curvature R is given by

$$\mathrm{R} = \sum_{i=1}^{n} \mathrm{R}_{ii} = 2 \sum_{i<j} \mathrm{K}(e_i, e_j) .$$

Then, in dimension three, denoting by $\alpha = \mathrm{K}(e_2, e_3)$, $\beta = \mathrm{K}(e_1, e_3)$ and $\gamma = \mathrm{K}(e_1, e_2)$, we obtain that the above eigenvalues are

$$\lambda_1 = 1 + 2a\gamma , \qquad \lambda_2 = 1 + 2a\beta , \qquad \lambda_3 = 1 + a(\beta + \gamma) .$$

It is now easy to see, by the arbitrariness of the cotangent vector ξ, that these three eigenvalues are positive, hence, the operator L is weakly elliptic, if and only if all the sectional curvatures $\mathrm{K}(X, Y)$ of (M, g) satisfy $1 + 2a\mathrm{K}(X, Y) \geq 0$. If this expression is always positive, then there are exactly three zero eigenvalues, due to the diffeomorphism invariance of the operator L.
We are then ready to prove Theorem 2.1.1.

Proof of Theorem 2.1.1. Following the work of DeTurck [33, 34] we exposed in Section 1.3.5, we show that problem (2.3) is equivalent to a Cauchy problem for a strictly parabolic operator, modulo the action of the diffeomorphism group of M. Our operator $Lg_{ik} = -2\mathrm{R}_{ik} - a\mathrm{R}_{ijlm}\mathrm{R}_{kstu}g^{js}g^{lt}g^{mu}$ is clearly invariant under diffeomorphisms, hence, in order to show the smooth existence part in Theorem 2.1.1 we only need to check that $D(L - \mathscr{L}_V)_{g_0}$ is uniformly strongly elliptic, where V is the vector field defined in Lemma 1.3.4. By the same lemma, with respect to the orthonormal basis $\{e_1, e'_2, e'_3\}$ introduced above, we have

$$\sigma_\xi(D(L - \mathscr{L}_V)_{g_0}) = \begin{pmatrix} 1 & 0 & 0 & a(\mathrm{R} - 2\mathrm{R}_{33}) & a(\mathrm{R} - 2\mathrm{R}_{22}) & 0 \\ 0 & 1 & 0 & 0 & a\mathrm{R}_{12} & -a\mathrm{R}_{13} \\ 0 & 0 & 1 & a\mathrm{R}_{13} & 0 & -a\mathrm{R}_{12} \\ 0 & 0 & 0 & 1 + 2a\gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 + 2a\beta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 + a(\beta + \gamma) \end{pmatrix} .$$

Finally, by the discussion above, we conclude that a necessary and sufficient condition for the strong ellipticity of the linear operator $D(L-\mathscr{L}_V)_{g_0}$ is then that all the sectional curvatures of (M,g_0) satisfy

$$1+2a\mathrm{K}_0(X,Y)>0\,,$$

for every $p\in M$ and vectors $X,Y\in T_pM$.
The uniqueness of the solution, proven exactly in the same way as for the Ricci flow, is stated in Proposition 1.3.6. ☐

We also present here the generalization of the previous theorem to any dimension, which follows the same techniques.

THEOREM 2.1.3 (Gimre, Guenther, Isenberg, [47]). *Let (M,g_0) be a smooth n–dimensional compact Riemannian manifold, such that the sectional curvature K_0 of the initial metric g_0 satisfies*

$$1+2a\mathrm{K}_0(X,Y)>0$$

for every point $p\in M$ and vectors $X,Y\in T_pM$. Then there exists a unique smooth maximal solution to the RG^{2,a}–flow (2.3) starting at g_0.

PROOF. The method is clearly the same of the previous theorem, therefore we consider the expression of the principal symbol of the linearized of L given by equation (2.4) and we write it again as a matrix acting on the $n(n+1)/2$–vector

$$(h_{11},h_{12},\dots,h_{1n},h_{22},\dots,h_{nn},h_{23},\dots,h_{(n-1)n})$$

representing $h\in S^2(T_pM)$ without repetition of symmetric indexes that correspond to equal components of h. We have

$$\begin{aligned}
\sigma_\xi(DL_{g_0})(h)_{11}&=\sum_{s=2}^{n}(1+2a\mathrm{R}_{1s1s}h_{ss})+4\sum_{2\le s<u}^{n}\mathrm{R}_{1s1u}h_{su}\\
\sigma_\xi(DL_{g_0})(h)_{1k}&=\sum_{2\le s,u}^{n}\mathrm{R}_{ks1u}h_{su}\\
\sigma_\xi(DL_{g_0})(h)_{ik}&=(1+a\mathrm{R}_{1i1i}+a\mathrm{R}_{1k1k})h_{ik}\\
&\quad+\sum_{2\le s\neq i}^{n}\mathrm{R}_{1i1s}h_{sk}+\sum_{2\le s\neq k}^{n}\mathrm{R}_{1k1s}h_{is}
\end{aligned}$$

The matrix expression of the principal symbol is given by

$$\sigma_\xi(DL_{g_0})=\left(\begin{array}{c|c}0&A\\\hline 0&B\end{array}\right)\,,$$

where B is a square $\frac{n(n-1)}{2}$–matrix, whose diagonal elements corresponding to the components ik, $i,k\ge 2$ are $1+a\mathrm{K}(e_1,e_i)+a\mathrm{K}(e_1,e_k)$; while the non diagonal elements of B involve terms of the form R_{1s1u}, with $s\neq u$. Since (R_{1s1u}) is a symmetric form, we can as before find an orthonormal basis $\{e'_2,\dots,e'_n\}$ of $e_1^\perp$ such that $\mathrm{R}'_{1s1u}=0$ if $s\neq u$; with this choice B is diagonal and its eigenvalues depend only on the sectional curvatures.
By applying again DeTurck's Trick (Proposition 1.3.5) we obtain that, if the initial metric g_0 on M satisfies the condition of the statement, the RG^{2,a}–flow admits a smooth short time solution, which is also unique by Proposition 1.3.6. ☐

2.2. Evolution equations of the curvature tensors

In order to continue the study of this flow, some natural questions arise; for instance, one would like to find some Perelman–type entropy functionals which are monotone along the flow, as proposed by Tseytlin in [78]; another possibility is to investigate the evolution of the curvatures along the flow under the hypothesis of Theorem 2.1.1 and try to find (if there are) some preserved conditions in order to explore the long–time behaviour and the structure of the singularities at the maximal time of existence. In this direction a first step is to see if the bound (2.2) on the sectional

curvature is preserved in dimension 3. However, the evolution equation of the tensor $\mathrm{Sec} = \frac{1}{2}\mathrm{R}g - \mathrm{Ric}$, which is minus the Einstein tensor, whose eigenvalues are the sectional curvatures, (apparently) does not give any information about such desired preservation (see Remark 2.2.2).
Below we carry out the computation, in dimension three, of the evolution of the scalar and the Ricci curvature and we finally derive the equation for the sectional curvatures.
As the evolution equations along the Ricci flow are already known, we consider the perturbation term

$$\frac{\partial}{\partial t} g_{ik} = 2\mathrm{R}\mathrm{R}_{ik} - 2\mathrm{R}_{ij}\mathrm{R}^j_k + 2|\mathrm{Ric}|^2 g_{ik} - \mathrm{R}^2 g_{ik} \tag{2.5}$$

and afterwards we combine the Ricci flow with this perturbation to get the full evolution equations.
We recall the complete first variation formulas for the scalar curvature and the Ricci tensor under a generic flow $\frac{\partial}{\partial t} g = h$, that is

$$\begin{aligned}
\frac{\partial}{\partial t}\mathrm{R} &= -\Delta H + \delta^2 h - \langle h, \mathrm{Ric}\rangle \\
\frac{\partial}{\partial t}\mathrm{R}_{ik} &= -\frac{1}{2}\Big(\Delta h_{ik} + \nabla_i\nabla_k H - \nabla_i(\delta h)_k - \nabla_k(\delta h)_i + 2\mathrm{R}_{jikl}h^{jl} - 2\mathrm{R}_{ij}h^j_k - \mathrm{R}_{kj}h^j_i\Big)\,,
\end{aligned}$$

where as usual we used the metric g to raise the indexes, $H = \mathrm{tr}_g h$ and δ is the divergence operator. Starting with the scalar curvature, the most complex term is $\delta^2 h$ and to simplify the expression we use several times the twice–traced second Bianchi identity and the commutation formulas for derivatives of tensors:

$$\begin{aligned}
\delta^2 h =& \nabla^i\nabla^k h_{ik} \\
=& \nabla^i\Big(2\nabla^k\mathrm{R}\mathrm{R}_{ik} + \mathrm{R}\nabla_i\mathrm{R} - \nabla^k\mathrm{R}\mathrm{R}_{ik} - 2\nabla^k\mathrm{R}_{ij}\mathrm{R}^j_k + 2\nabla_i|\mathrm{Ric}|^2 - \nabla_i(\mathrm{R})^2\Big) \\
=& 2\langle\nabla^2\mathrm{R}, \mathrm{Ric}\rangle + |\nabla\mathrm{R}|^2 + |\nabla\mathrm{R}|^2 + \mathrm{R}\Delta\mathrm{R} - \frac{1}{2}|\nabla\mathrm{R}|^2 \\
&- \langle\nabla^2\mathrm{R}, \mathrm{Ric}\rangle - 2\mathrm{R}^j_k\nabla^i\nabla^k\mathrm{R}_{ij} - 2\nabla^k\mathrm{R}_{ij}\nabla^i\mathrm{R}^j_k + 2\Delta|\mathrm{Ric}|^2 - \Delta(\mathrm{R})^2 \\
=& 2\Delta|\mathrm{Ric}|^2 - \frac{1}{2}\Delta(\mathrm{R}^2) + \langle\nabla^2\mathrm{R}, \mathrm{Ric}\rangle + \frac{1}{2}|\nabla\mathrm{R}|^2 - 2\nabla^k\mathrm{R}_{ij}\nabla^i\mathrm{R}^j_k \\
&- 2\mathrm{R}^j_k\nabla^k\nabla^i\mathrm{R}_{ij} - 2\mathrm{R}^i_{kil}\mathrm{R}^l_j\mathrm{R}^{jk} - 2\mathrm{R}_{ikjl}\mathrm{R}^{li}\mathrm{R}^{kj} \\
=& 2\Delta|\mathrm{Ric}|^2 - \frac{1}{2}\Delta(\mathrm{R}^2) + \frac{1}{2}|\nabla\mathrm{R}|^2 - 2\nabla^k\mathrm{R}_{ij}\nabla^i\mathrm{R}^j_k - 2\mathrm{R}_{kl}\mathrm{R}^l_j\mathrm{R}^{jk} \\
&- 2\mathrm{R}^{li}\mathrm{R}^{kj}\Big(\mathrm{R}_{ij}g_{kl} + \mathrm{R}_{kl}g_{ij} - \mathrm{R}_{il}g_{kj} - \mathrm{R}_{kj}g_{il} - \frac{\mathrm{R}}{2}(g_{ij}g_{kl} - g_{il}g_{kl})\Big) \\
=& 2\Delta|\mathrm{Ric}|^2 - \frac{1}{2}\Delta(\mathrm{R}^2) + \frac{1}{2}|\nabla\mathrm{R}|^2 - 2\nabla^k\mathrm{R}_{ij}\nabla^i\mathrm{R}^j_k \\
&- 6\mathrm{tr}(\mathrm{Ric}^3) + 5\mathrm{R}|\mathrm{Ric}|^2 - \mathrm{R}^3\,.
\end{aligned}$$

Now we combine the computation above with these equations

$$H = |\mathrm{Rm}|^2 = 4|\mathrm{Ric}|^2 - \mathrm{R}^2\,, \qquad \langle h, \mathrm{Ric}\rangle = -2\mathrm{tr}(\mathrm{Ric}^3) + 4\mathrm{R}|\mathrm{Ric}|^2 - \mathrm{R}^3$$

and we obtain the perturbation terms in the evolution equation of the scalar curvature

$$\frac{\partial}{\partial t}\mathrm{R} = -\frac{1}{2}\Delta|\mathrm{Rm}|^2 + \frac{1}{2}|\nabla\mathrm{R}|^2 - 2\nabla^k\mathrm{R}_{ij}\nabla^i\mathrm{R}^j_k - 4\mathrm{tr}(\mathrm{Ric}^3) + \mathrm{R}|\mathrm{Ric}|^2\,.$$

Now we compute the perturbation terms in the evolution of the Ricci tensor. We have

$$\begin{aligned}
\nabla_i(\delta h)_k =& \nabla_i(\nabla^j\mathrm{R}\mathrm{R}_{kj}) + \nabla_i\mathrm{R}\nabla_k\mathrm{R} + \mathrm{R}\nabla_i\nabla_k\mathrm{R} \\
&- 2\nabla_i(\nabla^j\mathrm{R}_{kl}\mathrm{R}^l_j) + 2\nabla_i\nabla_k|\mathrm{Ric}|^2 - \nabla_i\nabla_k\mathrm{R}^2 \\
=& \nabla_i(\nabla^j\mathrm{R}\mathrm{R}_{kj} - 2\nabla^j\mathrm{R}_{kl}\mathrm{R}^l_j) + 2\nabla_i\nabla_k|\mathrm{Ric}|^2 - \frac{1}{2}\nabla_i\nabla_k\mathrm{R}^2\,,
\end{aligned}$$

where we have used that $\nabla_i\nabla_k\mathrm{R}^2 = 2\mathrm{R}\nabla_i\nabla_k\mathrm{R}+2\nabla_i\mathrm{R}\nabla_k\mathrm{R}$. Therefore, the second and first order terms in the evolution are

$$\begin{aligned}
&-\frac{1}{2}\Big(\Delta h_{ik} + \nabla_i\nabla_k H - \nabla_i(\delta h)_k - \nabla_k(\delta h)_i\Big)\\
&= -\frac{1}{2}\Big(\Delta \mathrm{Rm}^2_{ik} + 4\nabla_i\nabla_k|\mathrm{Ric}|^2 - \nabla_i\nabla_k\mathrm{R}^2\\
&\quad + \nabla_i(2\nabla^j\mathrm{R}_{kl}\mathrm{R}^l_j - \nabla^j\mathrm{R}\mathrm{R}_{kj}) - 2\nabla_i\nabla_k|\mathrm{Ric}|^2 + \frac{1}{2}\nabla_i\nabla_k\mathrm{R}^2\\
&\quad + \nabla_k(2\nabla^j\mathrm{R}_{il}\mathrm{R}^l_j - \nabla^j\mathrm{R}\mathrm{R}_{ij}) - 2\nabla_k\nabla_i|\mathrm{Ric}|^2 + \frac{1}{2}\nabla_k\nabla_i\mathrm{R}^2\Big)\\
&= -\frac{1}{2}\Big(\Delta \mathrm{Rm}^2_{ik} + \nabla_i(2\nabla^j\mathrm{R}_{kl}\mathrm{R}^l_j - \nabla^j\mathrm{R}\mathrm{R}_{kj}) + \nabla_k(2\nabla^j\mathrm{R}_{il}\mathrm{R}^l_j - \nabla^j\mathrm{R}\mathrm{R}_{ij})\Big)\,,
\end{aligned}$$

where we have used that the Hessian of a function is symmetric.
Next we compute the 0–th order terms, using again the decomposition of the Riemann tensor in dimension three. We have

$$\begin{aligned}
\mathrm{R}_{jikl}h^{jl} &= \mathrm{R}_{jikl}\Big(2\mathrm{R}\mathrm{R}^{jl} - 2\mathrm{R}^j_p\mathrm{R}^{pl} + 2|\mathrm{Ric}|^2g^{jl} - \mathrm{R}^2g^{jl}\Big)\\
&= 5\mathrm{R}\mathrm{Ric}^2_{ik} - 4\mathrm{Ric}^3_{ik} - 2\mathrm{R}^2\mathrm{R}_{ik} + 2\mathrm{tr}(\mathrm{Ric}^3)g_{ik} - 3\mathrm{R}|\mathrm{Ric}|^2g_{ik} + \mathrm{R}^2g_{ik}\,,\\
\mathrm{R}_{ij}h^j_k &= 2\mathrm{R}\mathrm{Ric}^2_{ik} - 2\mathrm{Ric}^3_{ik} + 2|\mathrm{Ric}|^2\mathrm{R}_{ik} - \mathrm{R}^2\mathrm{R}_{ik}\,,
\end{aligned}$$

hence, the perturbation terms in the evolution of the Ricci tensor are

$$\begin{aligned}
\frac{\partial}{\partial t}\mathrm{R}_{ik} = &-\frac{1}{2}\Big(\Delta \mathrm{Rm}^2_{ik} + \nabla_i(2\nabla^j\mathrm{R}_{kl}\mathrm{R}^l_j - \nabla^j\mathrm{R}\mathrm{R}_{kj}) + \nabla_k(2\nabla^j\mathrm{R}_{il}\mathrm{R}^l_j - \nabla^j\mathrm{R}\mathrm{R}_{ij})\Big)\\
&+ 6\mathrm{R}\mathrm{Ric}^2_{ik} - 4\mathrm{Ric}^3_{ik} - 4|\mathrm{Ric}|^2\mathrm{R}_{ik} - 2\mathrm{R}^2\mathrm{R}_{ik} + 4\mathrm{tr}(\mathrm{Ric}^3)g_{ik} - 6\mathrm{R}|\mathrm{Ric}|^2g_{ik} + 2\mathrm{R}^3g_{ik}\,.
\end{aligned}$$

Finally, we put the previous equations together and we get, for $\mathrm{Sec} = \frac{1}{2}\mathrm{R}g - \mathrm{Ric}$,

$$\begin{aligned}
\frac{\partial}{\partial t}\mathrm{Sec}_{ik} &= \frac{1}{2}\frac{\partial}{\partial t}\mathrm{R}g_{ik} + \frac{1}{2}\mathrm{R}\frac{\partial}{\partial t}g_{ik} - \frac{\partial}{\partial t}\mathrm{R}_{ik}\\
&= \frac{1}{2}\Big(-\frac{1}{2}\Delta|\mathrm{Rm}|^2 + \frac{1}{2}|\nabla\mathrm{R}|^2 - 2\nabla^j\mathrm{R}_{ls}\nabla^l\mathrm{R}^s_j\Big)g_{ik}\\
&\quad + \frac{1}{2}\Big(\Delta \mathrm{Rm}^2_{ik} + \nabla_i(2\nabla^j\mathrm{R}_{kl}\mathrm{R}^l_j - \nabla^j\mathrm{R}\mathrm{R}_{kj}) + \nabla_k(2\nabla^j\mathrm{R}_{il}\mathrm{R}^l_j - \nabla^j\mathrm{R}\mathrm{R}_{ij})\Big) + I_0\,,
\end{aligned}$$

where I_0 stands for the 0–th order term which we treat separately. We try to rewrite this expression in terms of Sec, noticing that

$$\mathrm{Sec}^2_{ik} = \frac{1}{4}\mathrm{R}^2g_{ik} + \mathrm{Ric}^2_{ik} - \mathrm{R}\mathrm{R}_{ik}\,,$$

therefore

$$-\frac{1}{4}\Delta|\mathrm{Rm}|^2g_{ik} + \frac{1}{2}\Delta\mathrm{Rm}^2_{ik} = -\frac{1}{4}\Delta\mathrm{R}^2 - \Delta(\mathrm{Ric}^2_{ik}) + \Delta(\mathrm{R}\mathrm{R}_{ik}) = -\Delta(\mathrm{Sec}^2_{ik})\,.$$

Then, applying again the twice contracted Bianchi identity (i.e. $\delta\mathrm{Sec} = 0$) we get

$$\begin{aligned}
\nabla^j(\mathrm{Sec}_{jl}\mathrm{Sec}^l_k) &= -\frac{1}{2}\nabla^j\mathrm{R}\mathrm{R}_{kj} + \nabla^j\mathrm{R}_{kl}\mathrm{R}^l_j\\
\nabla^j\mathrm{Sec}_{ls}\nabla^l\mathrm{Sec}^s_j &= -\frac{1}{4}|\nabla\mathrm{R}|^2 + \nabla^j\mathrm{R}_{ls}\nabla^l\mathrm{R}^s_j
\end{aligned}$$

and

$$\frac{\partial}{\partial t}\mathrm{Sec}_{ik} = -\Delta(\mathrm{Sec}^2_{ik}) - \nabla^j\mathrm{Sec}_{ls}\nabla^l\mathrm{Sec}^s_jg_{ik} + \nabla_i\nabla^j(\mathrm{Sec}_{jl}\mathrm{Sec}^l_k) + \nabla_k\nabla^j(\mathrm{Sec}_{jl}\mathrm{Sec}^l_i) + I_0\,.$$

To rewrite the 0–th order term we employ these simple equations

$$\begin{aligned}
\mathrm{Sec}^3_{ik} &= -\mathrm{Ric}^3_{ik} + \frac{3}{2}\mathrm{R}\mathrm{Ric}^2_{ik} - \frac{3}{4}\mathrm{R}^2\mathrm{R}_{ik} + \frac{1}{8}\mathrm{R}^3 g_{ik}\,,\\
\mathrm{tr}(\mathrm{Sec}^3) &= -\mathrm{tr}(\mathrm{Ric}^3) + \frac{3}{2}\mathrm{R}|\mathrm{Ric}|^2 - \frac{3}{8}\mathrm{R}^3\,,\\
\mathrm{S}\mathrm{Sec}^2_{ik} &= \frac{1}{2}\mathrm{R}\mathrm{Ric}^2_{ik} - \frac{1}{2}\mathrm{R}\mathrm{R}_{ik} + \frac{1}{8}\mathrm{R}^3 g_{ik}\,,\\
|\mathrm{Sec}|^2\mathrm{Sec}_{ik} &= |\mathrm{Ric}|^2\mathrm{R}_{ik} - \frac{1}{4}\mathrm{R}^2\mathrm{R}_{ik}\,,\\
\mathrm{S}|\mathrm{Sec}|^2 &= \frac{1}{2}\mathrm{R}|\mathrm{Ric}|^2 - \frac{1}{8}\mathrm{R}^3\,,\quad \mathrm{S}^3 = \frac{1}{8}\mathrm{R}^3\,.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
I_0 &= 4\mathrm{Ric}^3_{ik} - 6\mathrm{tr}(\mathrm{Ric}^3)g_{ik} - 7\mathrm{R}\mathrm{Ric}^2_{ik} + 4|\mathrm{Ric}|^2\mathrm{R}_{ik} + 3\mathrm{R}^2\mathrm{R}_{ik} + \frac{15}{2}\mathrm{R}|\mathrm{Ric}|^2 g_{ik} - \frac{5}{2}\mathrm{R}^3 g_{ik}\\
&= -4\mathrm{Sec}^3_{ik} + 6\mathrm{tr}(\mathrm{Sec}^3)g_{ik} - 2\mathrm{S}\mathrm{Sec}^2_{ik} + 4|\mathrm{Sec}|^2\mathrm{Sec}_{ik} - 3\mathrm{S}|\mathrm{Sec}|^2 g_{ik} + \mathrm{S}^3 g_{ik}\,.
\end{aligned}$$

In the following proposition we summarize the computations above and state the evolution equations along the full $\mathrm{RG}^{2,a}$–flow.

PROPOSITION 2.2.1. *During the $RG^{2,a}$–flow of a three dimensional manifold the scalar curvature, the Ricci tensor and the (negative) Einstein tensor satisfy the following equations.*

$$\begin{aligned}
\frac{\partial}{\partial t}\mathrm{R} &= \Delta\mathrm{R} + 2|\mathrm{Ric}|^2\\
&\quad - a\Big(-\frac{1}{2}\Delta|\mathrm{Rm}|^2 + \frac{1}{2}|\nabla\mathrm{R}|^2 - 2\nabla^k\mathrm{R}_{ij}\nabla^i\mathrm{R}^j_k - 4\mathrm{tr}(\mathrm{Ric}^3) + \mathrm{R}|\mathrm{Ric}|^2\Big)\,,
\end{aligned}\tag{2.6}$$

$$\begin{aligned}
\frac{\partial}{\partial t}\mathrm{R}_{ik} &= \Delta\mathrm{R}_{ik} - 6\mathrm{R}^2_{ik} + 3\mathrm{R}\mathrm{R}_{ik} + 2|\mathrm{Ric}|^2 g_{ik} - \mathrm{R}^2 g_{ik}\\
&\quad + \frac{a}{2}\Big(\Delta\mathrm{Rm}^2_{ik} + \nabla_i(2\nabla^j\mathrm{R}_{kl}\mathrm{R}^l_j - \nabla^j\mathrm{R}\mathrm{R}_{kj}) + \nabla_k(2\nabla^j\mathrm{R}_{il}\mathrm{R}^l_j - \nabla^j\mathrm{R}\mathrm{R}_{ij})\Big)\\
&\quad - a\Big(6\mathrm{R}\mathrm{Ric}^2_{ik} - 4\mathrm{Ric}^3_{ik} - 4|\mathrm{Ric}|^2\mathrm{R}_{ik} - 2\mathrm{R}^2\mathrm{R}_{ik} + 4\mathrm{tr}(\mathrm{Ric}^3)g_{ik} - 6\mathrm{R}|\mathrm{Ric}|^2 g_{ik} + 2\mathrm{R}^3 g_{ik}\Big)\,,
\end{aligned}\tag{2.7}$$

$$\begin{aligned}
\frac{\partial}{\partial t}\mathrm{Sec}_{ik} &= \Delta\mathrm{Sec}_{ik} - |\mathrm{Sec}|^2 g_{ik} + 6\mathrm{Sec}^2_{ik} - 4\mathrm{S}\mathrm{Sec}_{ik} + \mathrm{S}^2 g_{ik}\\
&\quad + a\Big(\Delta(\mathrm{Sec}^2_{ik}) + \nabla^j\mathrm{Sec}_{ls}\nabla^l\mathrm{Sec}^s_j g_{ik} - \nabla_i\nabla^j(\mathrm{Sec}_{jl}\mathrm{Sec}^l_k) - \nabla_k\nabla^j(\mathrm{Sec}_{jl}\mathrm{Sec}^l_i)\Big)\\
&\quad - a\Big(-4\mathrm{Sec}^3_{ik} + 6\mathrm{tr}(\mathrm{Sec}^3)g_{ik} - 2\mathrm{S}\mathrm{Sec}^2_{ik} + 4|\mathrm{Sec}|^2\mathrm{Sec}_{ik} - 3\mathrm{S}|\mathrm{Sec}|^2 g_{ik} + \mathrm{S}^3 g_{ik}\Big)\,.
\end{aligned}\tag{2.8}$$

REMARK 2.2.2. We observe that the evolution equation of the tensor Sec involves a second order quasilinear differential operator. In order to find out whether the condition (2.2) is preserved we would like to apply a version of Hamilton's maximum principle 1.5.3 for quasilinear elliptic operators to the symmetric tensor $\mathrm{Sec}_a = \mathrm{Sec} + \frac{1}{2a}g$. Since g is parallel with respect to its Levi–Civita connection, the differential operator on Sec_a is the same acting on Sec and, by computing the principal symbol, we can say that it is uniformly strongly elliptic if the bound (2.2) is fulfilled. However, under the same condition, the 0–th order term does not satisfy the null–eigenvector condition, therefore we cannot establish if such bound is preserved along the flow.

2.3. Similar quadratic flows

The analysis leading to Theorem 2.1.3 can be repeated step–by–step for the operator L_0, given by

$$L_0 g = -a\mathrm{R}_{ijlm}\mathrm{R}_{kstu}g^{js}g^{lt}g^{mu}\,,$$

with associated $\mathrm{RG}^{2,a}_0$–flow

$$\partial_t g_{ik} = -a\mathrm{R}_{ijlm}\mathrm{R}_{kstu}g^{js}g^{lt}g^{mu}\,.$$

In this case, along the same lines, the existence of a unique smooth evolution of an initial metric g_0 is guaranteed as long as

$$a\mathrm{K}_0(X, Y) > 0$$

for every point $p \in M$ and vectors $X, Y \in T_pM$. That is, if $a > 0$ when the initial manifold has positive curvature and if $a < 0$ when it has negative curvature.
For geometrical purposes, this flow could be more interesting than the $\mathrm{RG}^{2,a}$–flow, in particular because of its scaling invariance, which is not shared by the latter.
Another possibility in this direction is given by the *squared Ricci flow*, that is, the evolution of an initial metric g_0 according to

$$\partial_t g_{ik} = -a\mathrm{R}_{ij}\mathrm{R}_k^j\,,$$

which is scaling invariant and can be analyzed analogously, or a "mixing" with the Ricci flow (non scaling invariant)

$$\partial_t g_{ik} = -2\mathrm{R}_{ik} - a\mathrm{R}_{ij}\mathrm{R}_k^j\,,$$

for any constant $a \in \mathbb{R}$, as before.
Indeed, the principal symbol of the operator $H = \mathrm{R}_{ij}\mathrm{R}_k^j$ can be computed as in Section 2.1. The linearized of the operator H around a metric g, for every $h \in S^2M$, is given by

$$\begin{aligned}
DH_g(h)_{ik} &= \mathrm{R}_k^j D\mathrm{Ric}_g(h)_{ij} + \mathrm{R}_i^j D\mathrm{Ric}_g(h)_{jk} + \mathrm{LOT}\\
&= \frac{1}{2}\mathrm{R}_k^j\Big(-\Delta h_{ij} - \nabla_i\nabla_j \mathrm{tr}(h) + \nabla_i\nabla^m h_{mj} + \nabla_j\nabla^m h_{im}\Big)\\
&\quad + \frac{1}{2}\mathrm{R}_i^j\Big(-\Delta h_{jk} - \nabla_j\nabla_k \mathrm{tr}(h) + \nabla_j\nabla^m h_{mk} + \nabla_k\nabla^m h_{jm}\Big) + \mathrm{LOT}\,.
\end{aligned}$$

Hence, the principal symbol in the direction of the cotangent vector ξ, as before, is

$$\begin{aligned}
\sigma_\xi(DH_g)(h)_{ik} &= -\frac{1}{2}\mathrm{R}_k^j\Big(\xi^m\xi_m h_{ij} + \xi_i\xi_j \mathrm{tr}(h) - \xi_i\xi^m h_{jm} - \xi_j\xi^m h_{im}\Big)\\
&\quad -\frac{1}{2}\mathrm{R}_i^j\Big(\xi^m\xi_m h_{jk} + \xi_j\xi_k \mathrm{tr}(h) - \xi_j\xi^m h_{km} - \xi_k\xi^m h_{jm}\Big)\\
&= -\frac{1}{2}\mathrm{R}_k^j\Big(h_{ij} + \delta_{1i}\delta_{1j}\mathrm{tr}(h) - \delta_{1i}h_{1j} - \delta_{1j}h_{1i}\Big)\\
&\quad -\frac{1}{2}\mathrm{R}_i^j\Big(h_{jk} + \delta_{1j}\delta_{1k}\mathrm{tr}(h) - \delta_{1j}h_{1k} - \delta_{1k}h_{1j}\Big)\\
&= -\frac{1}{2}\Big(\mathrm{R}_{1k}(\delta_{1i}\mathrm{tr}(h) - h_{1i}) + \mathrm{R}_{1i}(\delta_{1k}\mathrm{tr}(h) - h_{1k})\Big)\\
&\quad -\frac{1}{2}\Big(\mathrm{R}_k^j(h_{ij} - \delta_{1i}h_{1j}) + \mathrm{R}_i^j(h_{jk} - \delta_{1k}h_{1j})\Big)\,,
\end{aligned}$$

where $\xi = g(e_1, \cdot)$ and $\{e_i\}$ is an orthonormal basis of T_pM.
Again, by specifying the initial metric to be g_0 and diagonalizing the restriction of the Ricci tensor to the hyperspace $e_1^\perp$, the principal symbol of the operator H, is described by

$$\begin{aligned}
\sigma_\xi(DH_{g_0})(h)_{11} &= -\frac{1}{2}\Big(2\mathrm{R}_{11}\sum_{j=2}^{n} h_{jj}\Big)\\
\sigma_\xi(DH_{g_0})(h)_{1k} &= -\frac{1}{2}\Big(2\mathrm{R}_{1k}h_{kk} + \mathrm{R}_{1k}\sum_{j\neq 1,\,k} h_{jj} + \sum_{j\neq 1,\,k}\mathrm{R}_{1j}h_{jk}\Big)\\
\sigma_\xi(DH_{g_0})(h)_{kk} &= -\frac{1}{2}\Big(2\mathrm{R}_{kk}h_{kk}\Big)\\
\sigma_\xi(DH_{g_0})(h)_{ik} &= -\frac{1}{2}\Big((\mathrm{R}_{kk} + \mathrm{R}_{ii})h_{ik}\Big)
\end{aligned}$$

for every $i, k \in \{2, \dots, n\}$ with $i \neq k$.
It is easy to see that the matrix associated to $\sigma_\xi(DH_{g_0})$ expressed in the coordinates

$$(h_{11}, h_{12}, \dots, h_{1n}, h_{22}, h_{33}, \dots, h_{nn}, h_{23}, h_{24}, \dots, h_{n-1,n})$$

of S^2M is upper triangular with n zeroes on the first n diagonal elements, then the next $(n-1)$ ones are the values $-\mathrm{R}_{kk}$ for $k \in \{2, \dots, n\}$ and finally, the last $(n-1)(n-2)/2$ ones are given by $-(\mathrm{R}_{ii} + \mathrm{R}_{kk})/2$ for every $i, k \in \{2, \dots, n\}$ with $i \neq k$.
Now, applying Propositions 1.3.5 and 1.3.6, the squared Ricci flow

$$\partial_t g_{ik} = -a\mathrm{R}_{ij}\mathrm{R}_{lk}g^{jl}\,,$$

has a unique smooth solution for short time, when $a > 0$ for every initial manifold (M, g_0) with positive Ricci curvature and when $a < 0$, for every initial manifold (M, g_0) with negative Ricci curvature.

CHAPTER 3

The Ricci–Bourguignon flow

In this chapter we begin to study the evolution of the metric of a compact Riemannian manifold (M, g_0) under the following flow equation

$$\frac{\partial}{\partial t} g = -2\mathrm{Ric} + 2\rho \mathrm{R} g = -2(\mathrm{Ric} - \rho \mathrm{R} g)\,, \tag{3.1}$$

where ρ is a real constant. This family of geometric flows contains, as a special case, the Ricci flow, setting $\rho = 0$. Moreover, by a suitable rescaling in time, when ρ is non positive they can be seen as an interpolation between the Ricci flow and the Yamabe flow (see [10, 75, 81], for instance), obtained as a limit flow when $\rho \to -\infty$.

It should be noted that for special values of the constant ρ the tensor $\mathrm{Ric} - \rho \mathrm{R} g$ appearing on the right side of the evolution equation is of special interest, in particular,

- $\rho = 1/2$ the Einstein tensor $\mathrm{Ric} - \mathrm{R}g/2$,
- $\rho = 1/n$ the *traceless* Ricci tensor $\mathrm{Ric} - \mathrm{R}g/n$,
- $\rho = 1/2(n-1)$ the Schouten tensor $\mathrm{Ric} - \frac{\mathrm{R}g}{2(n-1)}$,
- $\rho = 0$, the Ricci tensor Ric.

In dimension two the first three are 0, hence the flow is static, in higher dimension the values of ρ are strictly ordered as above in descending order.

Apart these special values of ρ, for which we call the associated flows as the name of the corresponding tensor, in general we will refer to the evolution defined by the PDEs system (3.1) as *Ricci–Bourguignon flow* (or shortly RB–flow).

The study of these flows was proposed by Jean–Pierre Bourguignon in [9, Question 3.24], inspired by some unpublished work of Lichnerowicz in the sixties and the paper of Aubin [2]. In 2003, Fischer [38] studied a conformal version of this problem where the scalar curvature is constrained along the flow. In 2011, Lu, Qing and Zheng [62] also proved some results on the conformal Ricci–Bourguignon flow.

We shall see that when ρ is larger than $1/2(n-1)$ the principal symbol of the elliptic operator in the right hand side of the second order quasilinear parabolic PDE (3.1) has one negative eigenvalue, not allowing even a short time existence result of the flow for general initial data (manifold M and initial metric g_0). On the contrary, the main task of Section 3.1 will be to prove that, for $\rho < 1/2(n-1)$, there exists a unique short time solution of the RB–flow with any initial metric on a compact manifold.

However, the problem of knowing whether the "critical" *Schouten flow*

$$\begin{cases} \dfrac{\partial}{\partial t} g = -2\,\mathrm{Ric} + \dfrac{\mathrm{R}}{n-1} g \\ g(0) = g_0 \end{cases} \tag{3.2}$$

when $\rho = 1/2(n-1)$, admits or not a short time solution for general initial manifolds and metrics remains open, when $n \geq 3$.

In Section 3.2 we compute the evolution equations of the curvature tensors under the RB–flow, in order to find geometric conditions preserved by the flow in Section 3.3 and to understand the long time behaviour of the solutions in Section 3.4. Section 3.5 is devoted to describing some results on self–similar solutions of the RB–flow, called solitons.

All the theorems of this chapter and some of the results of the next chapter, in Section 4.1, can be found in [18], written by the author with Catino, Djadli, Mantegazza and Mazzieri.

3.1. Short time existence

The evolution problem (3.1) involves a quasilinear second–order differential operator $Lg = -2(\mathrm{Ric} - \rho \mathrm{R} g)$, which, as for the Ricci flow and the renormalization group flow, has some degeneracies, due to the invariance of the curvature tensors by the action of the group of diffeomorphisms of the manifold M (see Lemma 1.3.1). In the present section we establish the short time existence, for $\rho < 1/2(n-1)$, using again DeTurck's trick presented in Section 1.3.
By the general theory of non linear parabolic PDEs, briefly recalled in Section 1.2, it follows that the evolution equation $\frac{\partial}{\partial t} g = Lg$ admits a unique smooth solution for short time if the linearized operator around the initial data is uniformly strongly elliptic, that is, if its principal symbol $\sigma_\xi(DL_{g_0})$ satisfies condition (1.3).
We will see that if $\rho < 1/2(n-1)$, then the principal symbol of the linearized operator is non negative definite, but the only zero eigenvalues are due to the diffeomorphism invariance of the geometric flow, while all the others are actually uniformly positive, hence we can cut out the degeneracies by means of DeTurck's trick. Instead, in the case of the Schouten flow $\rho = 1/2(n-1)$, the principal symbol contains an extra zero eigenvalue besides the ones due to the diffeomorphism invariance, preventing this argument to be sufficient to conclude and to give a general short time existence result.
We mention that the presence of this extra zero eigenvalue should be expected, as the Cotton tensor, which is obtained from the Schouten tensor as follows

$$\mathrm{C}_{ijk} = \nabla_k \mathrm{R}_{ij} - \nabla_j \mathrm{R}_{ik} = \nabla_k \mathrm{R}_{ij} - \nabla_j \mathrm{R}_{ik} - \frac{1}{2(n-1)} \big(\nabla_k \mathrm{R} g_{ij} - \nabla_j \mathrm{R} g_{ik} \big) \,,$$

satisfies the following invariance under the conformal change of metric $\widetilde{g} = e^{2u} g$,

$$e^{3u} \widetilde{\mathrm{C}}_{ijk} = \mathrm{C}_{ijk} + (n-2) \mathrm{W}_{ijkl} \nabla^l u$$

see [20, Equation 3.35]. Recently, Delay [32], following the work of Fischer and Marsden, gave some evidence on the fact that DeTurck's trick should fail for the Schouten tensor.

We start computing the linearized operator DL_{g_0} of the operator L at a metric g_0. The Ricci tensor and the scalar curvature have the following linearizations (see [5, Theorem 1.174] or [77]).

$$\begin{aligned} D\mathrm{Ric}_{g_0}(h)_{ik} &= \frac{1}{2} \Big(-\Delta h_{ik} - \nabla_i \nabla_k \mathrm{tr}(h) + \nabla_i \nabla^t h_{tk} + \nabla_k \nabla^t h_{it} \Big) + \mathrm{LOT} \,, \\ D\mathrm{R}_{g_0}(h) &= -\Delta(\mathrm{tr} h) + \nabla^s \nabla^t h_{st} + \mathrm{LOT} \,, \end{aligned}$$

where, as usual, we use the metric g_0 to lower and upper indices and to take traces and LOT stands for *lower order terms*.
Then, the linearization of L around g_0, for every bilinear form $h \in S^2(M)$, is given by

$$\begin{aligned} DL_{g_0}(h)_{ik} &= -2 \big(D\mathrm{Ric}_{g_0}(h)_{ik} - \rho D\mathrm{R}_{g_0}(h)(g_0)_{ik} \big) + \mathrm{LOT} \\ &= \Delta h_{ik} + \nabla_i \nabla_k \mathrm{tr}(h) - \nabla_i \nabla^t h_{tk} - \nabla_k \nabla^t h_{it} \\ &\quad -2\rho \big(\Delta(\mathrm{tr} h) - \nabla^s \nabla^t h_{st} \big) + \mathrm{LOT} \,. \end{aligned}$$

Now, we obtain the principal symbol of the linearized operator in the direction of an arbitrary cotangent vector ξ by replacing each covariant derivative ∇_α, appearing in the higher order terms, with the corresponding component ξ_α,

$$\begin{aligned} \sigma_\xi(DL_{g_0})(h)_{ik} &= \xi^t \xi_t h_{ik} + \xi_i \xi_k \mathrm{tr}_{g_0}(h) - \xi_i \xi^t h_{kt} - \xi_k \xi^t h_{it} \\ &\quad -2\rho \xi^t \xi_t \mathrm{tr}_{g_0}(h)(g_0)_{ik} + 2\rho \xi^t \xi^s h_{ts} (g_0)_{ik} . \end{aligned} \tag{3.3}$$

As usual, since the principal symbol is homogeneous, we assume that $|\xi|_{g_0} = 1$ and we perform all the computations in an orthonormal basis $\{e_i\}_{i=1,\dots,n}$ of T_pM such that $\xi = g_0(e_1, \cdot)$, that is $\xi_i = 0$ for $i \neq 1$.
Hence, we obtain,

$$\begin{aligned} \sigma_\xi(DL_{g_0})(h)_{ik} &= h_{ik} + \delta_{i1}\delta_{k1} \mathrm{tr}_{g_0}(h) - \delta_{i1} h_{k1} - \delta_{k1} h_{i1} \\ &\quad -2\rho \mathrm{tr}_{g_0}(h) \delta_{ik} + 2\rho h_{11} \delta_{ik} \,, \end{aligned} \tag{3.4}$$

that can be represented, in the coordinates system

$$(h_{11}, h_{22}, \dots, h_{nn}, h_{12}, \dots, h_{1n}, h_{23}, h_{24}, \dots, h_{n-1,n})$$

for any $h \in S^2M$, by the following matrix

$$\sigma_\xi(DL_{g_0}) = \left(\begin{array}{c|c|c} \begin{matrix} 0 & 1-2\rho & \dots & 1-2\rho \\ \vdots & & A[n-1] & \\ 0 & & & \end{matrix} & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & \mathrm{Id}_{(n-1)(n-2)/2} \end{array}\right), \tag{3.5}$$

where $A[m]$ is the $m \times m$ matrix given by

$$A[m] = \begin{pmatrix} 1-2\rho & -2\rho & \dots & -2\rho \\ -2\rho & 1-2\rho & \dots & -2\rho \\ \vdots & \vdots & \ddots & \vdots \\ -2\rho & -2\rho & \dots & 1-2\rho \end{pmatrix}. \tag{3.6}$$

As we can easily see, there are at least n null eigenvalues, as expected, and $(n-1)(n-2)/2$ eigenvalues equal to 1. The remaining $n-1$ eigenvalues can be computed by induction on the dimension of A.

LEMMA 3.1.1. *Let $A[m]$ be the matrix written above. Then there holds*

$$\det(A[m] - \lambda \mathrm{Id}_m) = (1-\lambda)^{(m-1)}(1-2m\rho-\lambda).$$

PROOF. If we substitute to the first column the difference between the second and the first column, we obtain the formula

$$\det(A[m] - \lambda \mathrm{Id}_m) = (1-\lambda)\det\left(A[m-1] - \lambda \mathrm{Id}_{(m-1)}\right) + (1-\lambda)\det(B[m-1]),$$

where $B[p]$ is the matrix

$$B[p] = \begin{pmatrix} -2\rho & -2\rho & \dots & -2\rho \\ -2\rho & 1-2\rho-\lambda & \dots & -2\rho \\ \vdots & \vdots & \ddots & \vdots \\ -2\rho & -2\rho & \dots & 1-2\rho-\lambda \end{pmatrix}.$$

Now if we apply the same process to B we obtain this other induction formula

$$\det(B[p]) = (1-\lambda)\det(B[p-1]).$$

Therefore, since $\det(B[2]) = -2\rho(1-\lambda)$, we can combine the two equations to get

$$\begin{aligned} \det(A[m] - \lambda \mathrm{Id}_m) &= (1-\lambda)\det\left(A[m-1] - \lambda \mathrm{Id}_{(m-1)}\right) + (1-\lambda)\det(B[m-1]) \\ &= (1-\lambda)\Big((1-\lambda)\det(A[m-2] - \lambda \mathrm{Id}_{(m-2)}) - 2\rho(1-\lambda)^{m-2}\Big) - 2\rho(1-\lambda)^{(m-1)} \\ &= (1-\lambda)^{(m-2)}\det(A[2] - \lambda \mathrm{Id}_2) - 2\rho(m-2)(1-\lambda)^{(m-1)} \\ &= (1-\lambda)^{(m-1)}(1-2m\rho-\lambda). \end{aligned}$$

□

By the previous lemma we deduce that the eigenvalues of the principal symbol of DL_{g_0} are 0 with multiplicity n, 1 with multiplicity $\frac{(n+1)(n-2)}{2}$ and $1-2(n-1)\rho$ with multiplicity 1.

Now we apply again DeTurck's trick to show that, if $\rho < 1/2(n-1)$, the RB–flow is equivalent to a Cauchy problem for a strictly parabolic operator, modulo the action of the diffeomorphism group of M and we prove the following theorem.

THEOREM 3.1.2. *Let $\rho < 1/2(n-1)$. Then, the evolution equation (3.1) admits a unique solution for a short time on any smooth, n–dimensional, compact Riemannian manifold (M, g_0).*

PROOF. Let $V : \Gamma(S^2M) \to \Gamma(TM)$ be DeTurck's vector field defined by equation (1.8). As the operator in the RB–flow $Lg = -2(\mathrm{Ric} - \rho \mathrm{R} g)$ is invariant under diffeomorphisms, by Proposition 1.3.5, in order to show the smooth existence part we only need to check that $D(L - \mathscr{L}_V)_{g_0}$ is uniformly strongly elliptic. By adding the two matrices representing the principal symbols of DL_{g_0} and $(D\mathscr{L}_V)_{g_0}$, given respectively by equations (3.5) and (1.9), we can easily see that the linearized DeTurck–Ricci–Bourguignon operator has principal symbol in the direction ξ, with respect to an orthonormal basis $\{(\xi)^\sharp, e_2, \dots, e_n\}$, equal to

$$\sigma_\xi((D(L - \mathscr{L}_V)_{g_0}) = \left(\begin{array}{cccc|c|c} 1 & -2\rho & \dots & -2\rho & & \\ \vdots & & A[n-1] & & 0 & 0 \\ 0 & & & & & \\ \hline & & 0 & & \mathrm{Id}_{(n-1)} & 0 \\ \hline & & 0 & & 0 & \mathrm{Id}_{(n-1)(n-2)/2} \end{array}\right),$$

expressed in the coordinates system

$$(h_{11}, h_{22}, \dots, h_{nn}, h_{12}, h_{13}, \dots, h_{1n}, h_{23}, h_{24}, \dots, h_{n-1,n})$$

for any $h \in S^2M$.

By Lemma 3.1.1, this matrix has $\frac{n(n+1)}{2} - 1$ eigenvalues equal to 1 and 1 eigenvalue equal to $1 - 2(n-1)\rho$; therefore a sufficient condition for the existence of a solution is $\rho < \frac{1}{2(n-1)}$. Moreover, by Proposition 1.3.6, we also have uniqueness of the smooth maximal solution of the Ricci–Bourguignon flow and this concludes the proof of Theorem 3.1.2. □

3.2. Evolution equations of the curvature tensors

In this section we first compute the evolution equation of the curvature tensors during the RB–flow and then we apply Uhlenbeck's trick described in Section 1.4 in order to get simpler evolution equations for the Riemann tensor and the curvature operator.

As the metric tensor evolves according to equation (3.1), it is easy to see, differentiating the identity $g_{ij}g^{jl} = \delta_i^l$, that

$$\frac{\partial}{\partial t} g^{jl} = 2(\mathrm{Ric}^{jl} - \rho \mathrm{R} g^{jl}) \,. \tag{3.7}$$

It follows that the canonical volume measure $d\mu_g$ satisfies

$$\frac{\partial}{\partial t} d\mu_g = \frac{\partial}{\partial t} \sqrt{\det g_{ij}}\, \mathscr{L}^n = \frac{\sqrt{\det g_{ij}}\, g^{ij} \frac{\partial}{\partial t} g_{ij}}{2} \mathscr{L}^n = (n\rho - 1)\mathrm{R} \sqrt{\det g_{ij}}\, \mathscr{L}^n = (n\rho - 1)\mathrm{R}\, d\mu_g \,. \tag{3.8}$$

Computing in a normal coordinates system, the evolution equation for the Christoffel symbols is given by

$$\begin{aligned}\frac{\partial}{\partial t}\Gamma^i_{jk} &= \frac{1}{2}g^{il}\left\{\frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial t}g_{kl}\right)+\frac{\partial}{\partial x_k}\left(\frac{\partial}{\partial t}g_{jl}\right)-\frac{\partial}{\partial x_l}\left(\frac{\partial}{\partial t}g_{jk}\right)\right\}\\ &+\frac{1}{2}\frac{\partial}{\partial t}g^{il}\left\{\frac{\partial}{\partial x_j}g_{kl}+\frac{\partial}{\partial x_k}g_{jl}-\frac{\partial}{\partial x_l}g_{jk}\right\}\\ &=\frac{1}{2}g^{il}\left\{\nabla_j\left(\frac{\partial}{\partial t}g_{kl}\right)+\nabla_k\left(\frac{\partial}{\partial t}g_{jl}\right)-\nabla_l\left(\frac{\partial}{\partial t}g_{jk}\right)\right\}\\ &=-\,g^{il}\left\{\nabla_j(\mathrm{R}_{kl}-\rho\mathrm{R}g_{kl})+\nabla_k(\mathrm{R}_{jl}-\rho\mathrm{R}g_{jl})-\nabla_l(\mathrm{R}_{jk}-\rho\mathrm{R}g_{jk})\right\}\\ &=-\,\nabla_j\mathrm{R}^i_k-\nabla_k\mathrm{R}^i_j-\nabla^i\mathrm{R}_{jk}+\rho(\nabla_j\mathrm{R}\delta^i_k+\nabla_k\mathrm{R}\delta^i_j+\nabla^i\mathrm{R}g_{jk})\,.\end{aligned}$$

PROPOSITION 3.2.1. *During the Ricci–Bourguignon flow of an n–dimensional Riemannian manifold $(M, g(t))$, the Riemann tensor, the Ricci tensor and the scalar curvature satisfy the following evolution equations.*

$$\begin{aligned}\frac{\partial}{\partial t}\mathrm{R}_{ijkl} \;=\;& \Delta\mathrm{R}_{ijkl}+2(B(\mathrm{Rm})_{ijkl}-B(\mathrm{Rm})_{ijlk}-B(\mathrm{Rm})_{iljk}+B(\mathrm{Rm})_{ikjl})\\ &-g^{pq}\big(\mathrm{R}_{pjkl}\mathrm{R}_{qi}+\mathrm{R}_{ipkl}\mathrm{R}_{qj}+\mathrm{R}_{ijpl}\mathrm{R}_{qk}+\mathrm{R}_{ijkp}\mathrm{R}_{ql}\big)\\ &-\rho\big(\nabla_i\nabla_k\mathrm{R}\,g_{jl}-\nabla_i\nabla_l\mathrm{R}\,g_{jk}-\nabla_j\nabla_k\mathrm{R}\,g_{il}+\nabla_j\nabla_l\mathrm{R}\,g_{ik}\big)\\ &+2\rho\mathrm{R}\,\mathrm{R}_{ijkl}\,,\end{aligned}\tag{3.9}$$

where the tensor B is defined as $B(\mathrm{Rm})_{ijkl} = g^{pq}g^{rs}\mathrm{R}_{ipjr}\mathrm{R}_{kqls}$.

$$\begin{aligned}\frac{\partial}{\partial t}\mathrm{R}_{ik} \;=\;& \Delta\mathrm{R}_{ik}+2g^{pq}g^{rs}\mathrm{R}_{pirk}\mathrm{R}_{qs}-2g^{pq}\mathrm{R}_{pi}\mathrm{R}_{qk}\\ &-(n-2)\rho\nabla_i\nabla_k\mathrm{R}-\rho\Delta\mathrm{R}\,g_{ik}\,.\end{aligned}\tag{3.10}$$

$$\frac{\partial}{\partial t}\mathrm{R} \;=\; \big(1-2(n-1)\rho\big)\Delta\mathrm{R}+2|\mathrm{Ric}|^2-2\rho\mathrm{R}^2\,.\tag{3.11}$$

PROOF. The following computation is the analogous to the one for the Ricci flow performed by Hamilton in [49].
From the first variation formula for the $(4,0)$–Riemann tensor (see [5, Theorem 1.174] or [77])

$$\begin{aligned}\frac{\partial}{\partial t}\mathrm{Rm}(X,Y,W,Z) \;=\;& \frac{1}{2}\big(h(\mathrm{Rm}(X,Y)W,Z)-h(\mathrm{Rm}(X,Y)Z,W)\big)\\ &+\frac{1}{2}\big(\nabla^2_{Y,W}h(X,Z)+\nabla^2_{X,Z}h(Y,W)-\nabla^2_{X,W}h(Y,Z)-\nabla^2_{Y,Z}h(X,W)\big)\,,\end{aligned}$$

where $X, Y, W, Z \in \Gamma(TM)$ are vector fields and $h = \frac{\partial}{\partial t}g$.
Along the RB–flow $h = -2(\mathrm{Ric} - \rho\mathrm{R}g)$, therefore

$$\begin{aligned}\frac{\partial}{\partial t}\mathrm{Rm}(X,Y,W,Z) \;=\;& -\mathrm{Ric}(\mathrm{Rm}(X,Y)W,Z)+\mathrm{Ric}(\mathrm{Rm}(X,Y)Z,W)+2\rho\mathrm{R}\mathrm{Rm}(X,Y,W,Z)\\ &-\nabla^2_{Y,W}\mathrm{Ric}(X,Z)-\nabla^2_{X,Z}\mathrm{Ric}(Y,W)+\nabla^2_{X,W}\mathrm{Ric}(Y,Z)+\nabla^2_{Y,Z}\mathrm{Ric}(X,W)\\ &+\rho\big(\nabla^2_{Y,W}\mathrm{R}g(X,Z)+\nabla^2_{X,Z}\mathrm{R}g(Y,W)-\nabla^2_{X,W}\mathrm{R}g(Y,Z)-\nabla^2_{Y,Z}\mathrm{R}g(X,W)\big)\,.\end{aligned}$$

By using the second Bianchi identity and the commutation formula for the covariant derivatives we obtain the following equation for the Laplacian of the Riemann tensor

$$\begin{aligned}\Delta\mathrm{Rm}(X,Y,W,Z) \;=\;& -\nabla^2_{Y,W}\mathrm{Ric}(X,Z)-\nabla^2_{X,Z}\mathrm{Ric}(Y,W)+\nabla^2_{X,W}\mathrm{Ric}(Y,Z)+\nabla^2_{Y,Z}\mathrm{Ric}(X,W)\\ &-\mathrm{Ric}(\mathrm{Rm}(W,Z)Y,X)+\mathrm{Ric}(\mathrm{Rm}(W,Z)X,Y)\\ &-2\big(B(\mathrm{Rm})(X,Y,W,Z)-B(\mathrm{Rm})(X,Y,Z,W)\\ &+B(\mathrm{Rm})(X,W,Y,Z)-B(\mathrm{Rm})(X,Z,Y,W)\big)\,.\end{aligned}$$

Plugging in the evolution equation this last formula we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\mathrm{Rm}(X,Y,W,Z) &= \Delta\mathrm{Rm}(X,Y,W,Z)+\rho\big(\nabla^2\mathrm{R}\owedge g)(X,Y,W,Z)\\ &\quad+2\big(B(\mathrm{Rm})(X,Y,W,Z)-B(\mathrm{Rm})(X,Y,Z,W)\\ &\quad+B(\mathrm{Rm})(X,W,Y,Z)-B(\mathrm{Rm})(X,Z,Y,W)\big)\\ &\quad-\mathrm{Ric}(\mathrm{Rm}(X,Y)W,Z)+\mathrm{Ric}(\mathrm{Rm}(X,Y)Z,W)\\ &\quad-\mathrm{Ric}(\mathrm{Rm}(W,Z)X,Y)+\mathrm{Ric}(\mathrm{Rm}(W,Z)Y,X)+2\rho\mathrm{R}\mathrm{Rm}(X,Y,W,Z)\,,\end{aligned}$$

which is (3.9) once written in coordinates.
For the Ricci tensor, the first variation formula is obtained from the previous one

$$\begin{aligned}\frac{\partial}{\partial t}\mathrm{Ric}(X,W) &= \mathrm{tr}\Big(\frac{\partial}{\partial t}\mathrm{Rm}(X,-,W,-)\Big)-\langle h(-,=),\mathrm{Rm}(X,-W,=)\rangle\\ &= -\frac{1}{2}\big(\nabla^2_{X,W}\mathrm{tr}(h)+\Delta h(X,W)-\nabla_X(\delta h)(W)-\nabla_W(\delta(h))(X)\big)\\ &\quad-\frac{1}{2}\mathrm{tr}\big(h((X,-)W,-)+h((X,-)-,W)+h((W,-)X,-)+h((W,-)-,X)\big)\,,\end{aligned}$$

where we have omitted the g–dependence in the trace tr and the divergence δ operator. Once plugged in the variation of the metric, the evolution becomes

$$\begin{aligned}\frac{\partial}{\partial t}\mathrm{Ric}(X,W) &= (1-n\rho)\nabla^2_{X,W}\mathrm{R}+\Delta\mathrm{Ric}(X,W)-\rho\Delta\mathrm{R}g(X,W)-(1-2\rho)\nabla^2_{X,W}\mathrm{R}\\ &\quad+2\langle\mathrm{Ric}(-,=),\mathrm{Rm}(X,-,W,=)\rangle-2\langle\mathrm{Ric}(X,-),\mathrm{Ric}(W,-)\rangle\,,\end{aligned}$$

where we have used the contracted Bianchi identity $\delta\mathrm{Ric}=\nabla\mathrm{R}/2$ and the symmetries of the Riemann tensor.
In the same way, the evolution equation (3.11) for the scalar curvature is obtained from the general first variation

$$\frac{\partial}{\partial t}\mathrm{R}=\mathrm{tr}\Big(\frac{\partial}{\partial t}\mathrm{Ric}\Big)-\langle h,\mathrm{Ric}\rangle=-\Delta\mathrm{tr}(h)+\delta^2h-\langle h,\mathrm{Ric}\rangle\,,$$

with the evolution of the metric of the RB–flow. □

3.2.1. The evolution of the Weyl tensor. By means of the evolution equations found for the curvatures, we are also able to write the equation satisfied by the Weyl tensor along the RB–flow (3.1). In [19] the authors compute the evolution equation of the Weyl tensor during the Ricci flow and we use most of their computations here.

PROPOSITION 3.2.2. *During the Ricci–Bourguignon flow of an n–dimensional Riemannian manifold (M,g) the Weyl tensor satisfies the following evolution equation*

$$\begin{aligned}\frac{\partial}{\partial t}\mathrm{W}_{ijkl} &= \Delta\mathrm{W}_{ijkl}+2(B(\mathrm{W})_{ijkl}-B(\mathrm{W})_{ijlk}-B(\mathrm{W})_{iljk}+B(\mathrm{W})_{ikjl})\\ &\quad+2\rho\mathrm{R}\mathrm{W}_{ijkl}-g^{pq}\big(\mathrm{W}_{pjkl}\mathrm{R}_{qi}+\mathrm{W}_{ipkl}\mathrm{R}_{qj}+\mathrm{W}_{ijpl}\mathrm{R}_{qk}+\mathrm{W}_{ijkp}\mathrm{R}_{ql}\big)\\ &\quad+\frac{2}{(n-2)^2}(\mathrm{Ric}^2\owedge g)_{ijkl}+\frac{1}{(n-2)}(\mathrm{Ric}\owedge\mathrm{Ric})_{ijkl}\\ &\quad-\frac{2\mathrm{R}}{(n-2)^2}(\mathrm{Ric}\owedge g)_{ijkl}+\frac{\mathrm{R}^2-|\mathrm{Ric}|^2}{(n-1)(n-2)^2}(g\owedge g)_{ijkl}\,,\end{aligned}\tag{3.12}$$

where we denote with $\owedge$ the Kulkarni–Nomizu product, defined in Section 1.1.

PROOF. By recalling the decomposition formula for the Weyl tensor (1.2) we have

$$\frac{\partial}{\partial t}\mathrm{W}=\frac{\partial}{\partial t}\mathrm{Rm}+\frac{1}{2(n-1)(n-2)}\Big(\frac{\partial}{\partial t}\mathrm{R}g\owedge g+2\frac{\partial}{\partial t}g\owedge g\Big)-\frac{1}{n-2}\Big(\frac{\partial}{\partial t}\mathrm{Ric}\owedge g+\mathrm{Ric}\owedge\frac{\partial}{\partial t}g\Big)=\mathcal{L}_{II}+\mathcal{L}_0\,,$$

where $\mathcal{L}_{II}$ is the second order term in the curvatures and $\mathcal{L}$ the 0–th one. We deal first with the higher order term; plugging in the evolution equations of $\mathrm{Rm}, \mathrm{Ric}$ and R we get

$$\begin{aligned}
\mathcal{L}_{II} &= \Delta\mathrm{Rm} - \rho(\nabla^2\mathrm{R} \owedge g) + \frac{1-2(n-1)\rho}{2(n-1)(n-2)}\Delta\mathrm{R} g \owedge g \\
&\quad -\frac{1}{n-2}\big(\Delta\mathrm{Ric} \owedge g - (n-2)\rho\nabla^2\mathrm{R} \owedge g - \rho\Delta\mathrm{R} g \owedge g\big) \\
&= \Delta\mathrm{Rm} + \frac{1-2(n-1)\rho + 2(n-1)\rho}{2(n-1)(n-2)}\Delta\mathrm{R} g \owedge g - \frac{1}{n-2}\Delta\mathrm{Ric} \owedge g \\
&= \Delta\mathrm{W}\,.
\end{aligned}$$

Then we consider the lower order terms

$$\begin{aligned}
(\mathcal{L}_0)_{ijkl} &= 2(B(\mathrm{Rm})_{ijkl} - B(\mathrm{Rm})_{ijlk} - B(\mathrm{Rm})_{iljk} + B(\mathrm{Rm})_{ikjl}) \\
&\quad -g^{pq}\big(\mathrm{R}_{pjkl}\mathrm{R}_{qi} + \mathrm{R}_{ipkl}\mathrm{R}_{qj} + \mathrm{R}_{ijpl}\mathrm{R}_{qk} + \mathrm{R}_{ijkp}\mathrm{R}_{ql}\big) \\
&\quad +2\rho\mathrm{R}\Big(\mathrm{W} - \frac{1}{2(n-1)(n-2)}\mathrm{R} g \owedge g + \frac{1}{n-2}\mathrm{Ric} \owedge g\Big)_{ijkl} \\
&\quad +\frac{1}{2(n-1)(n-2)}(2|\mathrm{Ric}|^2 g \owedge g - 2\rho\mathrm{R}^2 g \owedge g - 4\mathrm{R}\mathrm{Ric} \owedge g + 4\rho\mathrm{R}^2 g \owedge g)_{ijkl} \\
&\quad -\frac{1}{n-2}[2(\mathrm{Rm} * \mathrm{Ric}) \owedge g - 2\mathrm{Ric}^2 \owedge g - 2\mathrm{Ric} \owedge \mathrm{Ric} + 2\rho\mathrm{R}\mathrm{Ric} \owedge g]_{ijkl} \\
&= 2(B(\mathrm{Rm})_{ijkl} - B(\mathrm{Rm})_{ijlk} - B(\mathrm{Rm})_{iljk} + B(\mathrm{Rm})_{ikjl}) \\
&\quad -g^{pq}\big(\mathrm{R}_{pjkl}\mathrm{R}_{qi} + \mathrm{R}_{ipkl}\mathrm{R}_{qj} + \mathrm{R}_{ijpl}\mathrm{R}_{qk} + \mathrm{R}_{ijkp}\mathrm{R}_{ql}\big) + 2\rho\mathrm{R}\mathrm{W}_{ijkl} \\
&\quad -\frac{2}{n-2}[(\mathrm{Rm} * \mathrm{Ric}) \owedge g - \mathrm{Ric}^2 \owedge g - \mathrm{Ric} \owedge \mathrm{Ric}]_{ijkl} \\
&\quad -\frac{2\mathrm{R}}{(n-1)(n-2)}(\mathrm{Ric} \owedge g)_{ijkl} + \frac{|\mathrm{Ric}|^2}{(n-1)(n-2)}(g \owedge g)_{ijkl}\,,
\end{aligned}$$

where $(\mathrm{Rm} * \mathrm{Ric})_{ab} = \mathrm{R}_{apbq}\mathrm{R}_{st}g^{ps}g^{qt}$ and $(\mathrm{Ric}^2)_{ab} = \mathrm{R}_{ap}\mathrm{R}_{bq}g^{pq}$.
Now we deal separately with every term containing the full curvature Rm, using its decomposition formula, expanding the Kulkarni–Nomizu products and then contracting again. We have that

$$[(g \owedge g) * \mathrm{Ric}]_{ab} = 2[\mathrm{R}g - \mathrm{Ric}]_{ab}\,, [(\mathrm{Ric} \owedge g) * \mathrm{Ric}]_{ab} = [-2\mathrm{Ric}^2 + \mathrm{R}\mathrm{Ric} + |\mathrm{Ric}|^2 g]_{ab}\,.$$

Hence

$$\begin{aligned}
(\mathrm{Rm} * \mathrm{Ric}) \owedge g &= (\mathrm{W} * \mathrm{Ric}) \owedge g - \frac{2}{n-2}\mathrm{Ric}^2 \owedge g \\
&\quad +\frac{n\mathrm{R}}{(n-1)(n-2)}\mathrm{Ric} \owedge g + \frac{(n-1)|\mathrm{Ric}|^2 - \mathrm{R}^2}{(n-1)(n-2)} g \owedge g\,.
\end{aligned} \tag{3.13}$$

Then

$$\begin{aligned}
\mathrm{R}_{qi}\mathrm{R}_{pjkl}g^{pq} &= \mathrm{R}_{qi}\Big(W_{pjkl} - \frac{\mathrm{R}}{(n-1)(n-2)}(g_{pk}g_{jl} - g_{pl}g_{jk})\Big)g^{pq} \\
&\quad +\frac{1}{n-2}\mathrm{R}_{qi}(\mathrm{R}_{pk}g_{jl} + \mathrm{R}_{jl}g_{pk} - \mathrm{R}_{pl}g_{jk} - \mathrm{R}_{jk}g_{pl})g^{pq} \\
&= \mathrm{R}_{qi}\mathrm{W}_{pjkl}g^{pq} - \frac{\mathrm{R}}{(n-1)(n-2)}(\mathrm{R}_{ik}g_{jl} - \mathrm{R}_{il}g_{jk}) \\
&\quad +\frac{1}{n-2}(\mathrm{R}^2_{ik}g_{jl} - \mathrm{R}^2_{il}g_{jk} + \mathrm{R}_{ik}\mathrm{R}_{jl} - \mathrm{R}_{il}\mathrm{R}_{jk})\,.
\end{aligned}$$

Interchanging the indexes and using the symmetry properties we get

$$\begin{aligned} g^{pq}\big(\mathrm{R}_{pjkl}\mathrm{R}_{qi} + \mathrm{R}_{ipkl}\mathrm{R}_{qj} + \mathrm{R}_{ijpl}\mathrm{R}_{qk} + \mathrm{R}_{ijkp}\mathrm{R}_{ql}\big) &= g^{pq}\big(\mathrm{W}_{pjkl}\mathrm{R}_{qi} + \mathrm{W}_{ipkl}\mathrm{R}_{qj} + \mathrm{W}_{ijpl}\mathrm{R}_{qk} + \mathrm{W}_{ijkp}\mathrm{R}_{ql}\big) \\ &+ \frac{2}{n-2}(\mathrm{Ric}^2 \owedge g)_{ijkl} + \frac{2}{n-2}(\mathrm{Ric} \owedge \mathrm{Ric})_{ijkl} - \frac{2\mathrm{R}}{(n-1)(n-2)}(\mathrm{Ric} \owedge g)_{ijkl}\,. \end{aligned} \tag{3.14}$$

Finally the "B"–terms:

$$\begin{aligned} B(\mathrm{Rm})_{abcd} &= \Big(\mathrm{W} - \frac{\mathrm{R}}{2(n-1)(n-2)} g \owedge g + \frac{1}{n-2}\mathrm{Ric} \owedge g\Big)_{apbq} \\ &\cdot \Big(\mathrm{W} - \frac{\mathrm{R}}{2(n-1)(n-2)} g \owedge g + \frac{1}{n-2}\mathrm{Ric} \owedge g\Big)_{csdt} g^{ps}g^{qt}\,, \end{aligned}$$

$$\big(\mathrm{W}_{apbq}(g \owedge g)_{csdt} + (g \owedge g)_{apbq}\mathrm{W}_{csdt}\big)g^{ps}g^{qt} = -2\mathrm{W}_{adbc} - 2\mathrm{W}_{cbda}\,,$$

$$\begin{aligned} \big(\mathrm{W}_{apbq}(\mathrm{Ric} \owedge g)_{csdt} + (\mathrm{Ric} \owedge g)_{apbq}\mathrm{W}_{csdt}\big)g^{ps}g^{qt} &= (\mathrm{W} * \mathrm{Ric})_{ab}g_{cd} + (\mathrm{W} * \mathrm{Ric})_{cd}g_{ab} \\ &- (\mathrm{W}_{cbdp}\mathrm{R}_{aq} + \mathrm{W}_{cpda}\mathrm{R}_{bq} + \mathrm{W}_{adbp}\mathrm{R}_{cq} + \mathrm{W}_{apbd}\mathrm{R}_{dq})g^{pq}\,, \end{aligned}$$

$$(g \owedge g)_{apbd}(g \owedge g)_{csdt}g^{ps}g^{qt} = 4\big((n-2)g_{ab}g_{cd} + g_{ac}g_{bd}\big)\,,$$

$$\begin{aligned} &\big((\mathrm{Ric} \owedge g)_{apbq}(g \owedge g)_{csdt} + (\mathrm{Ric} \owedge g)_{csdt}(g \owedge g)_{apbq}\big)g^{ps}g^{qt} \\ &\qquad = 2\big((n-4)\mathrm{Ric}_{ab}g_{cd} + (n-4)\mathrm{Ric}_{cd}g_{ab} + 2\mathrm{Ric}_{ac}g_{bd} + 2\mathrm{Ric}_{bd}g_{ac}\big)\,, \end{aligned}$$

$$\begin{aligned} (\mathrm{Ric} \owedge g)_{abpq}(\mathrm{Ric} \owedge g)_{csdt}g^{ps}g^{qt} &= -2\mathrm{R}^2_{ab}g_{cd} - 2\mathrm{R}^2_{cd}g_{ab} + \mathrm{R}^2_{ac}g_{bd} + \mathrm{R}^2_{bd}g_{ac} \\ &+ (n-4)\mathrm{R}_{ab}\mathrm{R}_{cd} + 2\mathrm{R}_{ac}\mathrm{R}_{bd} + \mathrm{R}(\mathrm{R}_{ab}g_{cd} + \mathrm{R}_{cd}g_{ab}) + |\mathrm{Ric}|^2 g_{ab}g_{cd}\,. \end{aligned}$$

Now, adding the same type quantities for the different index permutations and using the symmetry properties of W we obtain

$$\begin{aligned} B(\mathrm{Rm})_{ijkl} - B(\mathrm{Rm})_{ijlk} - B(\mathrm{Rm})_{iljk} + B(\mathrm{Rm})_{ikjl} &= B(\mathrm{W})_{ijkl} - B(\mathrm{W})_{ijlk} - B(\mathrm{W})_{iljk} + B(\mathrm{W})_{ikjl} \\ &+ \frac{1}{n-2}\big((\mathrm{W} * \mathrm{Ric}) \owedge g\big)_{ijkl} - \frac{1}{(n-2)^2}(\mathrm{Ric}^2 \owedge g)_{ijkl} + \frac{1}{2(n-2)}(\mathrm{Ric} \owedge \mathrm{Ric})_{ijkl} \\ &+ \frac{\mathrm{R}}{(n-1)(n-2)^2}(\mathrm{Ric} \owedge g)_{ijkl} + \Big(\frac{|\mathrm{Ric}|^2}{2(n-2)^2} - \frac{\mathrm{R}^2}{2(n-1)(n-2)^2}\Big)(g \owedge g)_{ijkl}\,. \end{aligned} \tag{3.15}$$

We are ready to complete the computation of the 0–th order term in the evolution equation, using the previous formulas (3.13), (3.14), (3.15)

$$\begin{aligned} (\mathcal{L}_0)_{ijkl} \;=\; & 2(B(\mathrm{W})_{ijkl} - B(\mathrm{W})_{ijlk} - B(\mathrm{W})_{iljk} + B(\mathrm{W})_{ikjl}) + 2\rho\mathrm{R}\mathrm{W}_{ijkl} \\ & - g^{pq}\big(\mathrm{W}_{pjkl}\mathrm{R}_{qi} + \mathrm{W}_{ipkl}\mathrm{R}_{qj} + \mathrm{W}_{ijpl}\mathrm{R}_{qk} + \mathrm{W}_{ijkp}\mathrm{R}_{ql}\big) \\ & + \frac{2}{(n-2)^2}(\mathrm{Ric}^2 \owedge g)_{ijkl} + \frac{1}{(n-2)}(\mathrm{Ric} \owedge \mathrm{Ric})_{ijkl} \\ & - \frac{2\mathrm{R}}{(n-2)^2}(\mathrm{Ric} \owedge g)_{ijkl} + \frac{\mathrm{R}^2 - |\mathrm{Ric}|^2}{(n-1)(n-2)^2}(g \owedge g)_{ijkl}\,. \end{aligned}$$

□

3.2.2. Uhlenbeck's trick and the evolution of the curvature operator. In this subsection we want to compute the evolution equation of the curvature operator, as it was done for the Ricci flow by Hamilton in [50]. First of all, we simplify the expression of the reaction term in the evolution equation (3.9) of the Riemann tensor by means of the so called Uhlenbeck's trick. The trick is described in Section 1.4 from several points of view.

This method was first used by Hamilton in order to understand the long time behaviour of the Ricci flow starting from a compact 4–manifold with positive curvature operator. More recently, in [7], Bohm and Wilking have applied the same strategy to a compact n–dimensional manifold with positive curvature operator to find cones in the space of algebraic curvature operators (see

Appendix A) preserved along the Ricci flow and ultimately prove the convergence of the normalized Ricci flow to a metric of (positive) constant curvature.
Along the same lines, in [13] Brendle and Schoen proved the differentiable version of the Rauch–Berger–Klingenberg $1/4$–pinched sphere theorem. The major benefit of this approach is the powerful tool given by the Lie algebra structure of $\Lambda^2(T_pM)$, that allows to find new and more pinched preserved cones from the existing ones (see [7, Sections 2 and 3] and the PhD thesis of Nguyen [65, 66]).

As a first step, given a solution $g(t)$, $t \in [0,T)$ of the Ricci–Bourguignon flow with initial metric g_0, we recall the family of bundle isometries $\{\varphi(t)\} : (V, g_0) \to (TM, g(t))$ defined by the Cauchy problem (1.11), that, specialized to the RB–flow, becomes

$$\begin{cases} \frac{\partial}{\partial t}\varphi(t) = (\mathrm{Ric}_{g(t)})^\sharp \circ \varphi(t) - \rho \mathrm{R}_{g(t)}\varphi(t)\,, \\ \varphi(0) = \mathrm{Id}_{TM}\,. \end{cases} \tag{3.16}$$

Then we consider $\mathrm{Pm} \in \Gamma((V^*)^4)$, the pull–back of the Riemann curvature tensor via this family and we already know, by Proposition 1.4.7, that Pm has the same symmetry and positivity properties of Rm; in the following proposition we state the evolution equation of this pull–back tensor.

PROPOSITION 3.2.3. *Let $\mathrm{Pm}(t) = \varphi(t)^*(\mathrm{Rm}_{g(t)})$ during the RB–flow $(g(t))$. Then $\mathrm{Pm}(t)$ satisfies the following evolution equation*

$$\begin{aligned} \frac{\partial}{\partial t}(\mathrm{Pm})_{abcd} &= \Delta_D(\mathrm{Pm})_{abcd} - \rho(\varphi^*(\nabla^2\mathrm{R}) \owedge g_0)_{abcd} \\ &\quad + 2\big(B(\mathrm{Pm})_{abcd} - B(\mathrm{Pm})_{abdc} + B(\mathrm{Pm})_{acbd} - B(\mathrm{Pm})_{adbc}\big) - 2\rho\mathrm{P}(\mathrm{Pm})_{abcd}\,, \end{aligned} \tag{3.17}$$

where Δ_D is the rough Laplacian associated to the family of pull–back connections $D(t)$ (see Lemma 1.4.6), B is defined in the same way as in equation (3.9) for every $(4,0)$–tensor on V and $\mathrm{P}(t) = \mathrm{R}_{g(t)}$ is the double trace with respect to g_0 of $\mathrm{Pm}(t)$ (see Proposition 1.4.7, point (3)).

REMARK 3.2.4. On the right hand side of equation (3.17) there appears the term $\varphi^*(\nabla^2\mathrm{R})$, i.e. the pull–back of the Hessian of the scalar curvature, seen as a symmetric 2–form on the tangent bundle, that cannot be expressed in terms of the connection $D(t)$.

PROOF. Let $\zeta_1, \dots, \zeta_4$ be sections of V; then combining the evolution equations of the Riemann tensor (3.9) and of the bundle isometry φ (3.16) we obtain

$$\begin{aligned} \frac{\partial}{\partial t}(\mathrm{Pm})(\zeta_1,\dots,\zeta_4) &= \varphi^*\Big(\frac{\partial}{\partial t}\mathrm{Rm}\Big)(\zeta_1,\dots,\zeta_4) + \mathrm{Rm}\Big(\frac{\partial\varphi}{\partial t}(\zeta_1), \varphi(\zeta_2), \varphi(\zeta_3), \varphi(\zeta_4)\Big) + \dots \\ &= \varphi^*(\Delta_g\mathrm{Rm})(\zeta_1,\dots,\zeta_4) - \rho\varphi^*(\nabla^2\mathrm{R} \owedge g)(\zeta_1,\dots,\zeta_4) \\ &\quad + 2\varphi^*\big(B(\mathrm{Rm})\big)(\zeta_1,\dots,\zeta_4) + 2\rho\mathrm{R}\varphi^*(\mathrm{Rm})(\zeta_1,\dots,\zeta_4) \\ &\quad - \mathrm{Rm}\Big(\mathrm{Ric}^\sharp \circ \varphi(\zeta_1), \varphi(\zeta_2), \varphi(\zeta_3), \varphi(\zeta_4)\Big) + \dots \\ &\quad + \mathrm{Rm}\Big(\big(\mathrm{Ric}^\sharp \circ \varphi - \rho\mathrm{R}\varphi\big)(\zeta_1), \varphi(\zeta_2), \varphi(\zeta_3), \varphi(\zeta_4)\Big) + \dots \\ &= \Delta_D(\mathrm{Pm})(\zeta_1,\dots,\zeta_4) - \rho(\varphi^*(\nabla^2\mathrm{R}) \owedge g_0)(\zeta_1,\dots,\zeta_4) \\ &\quad + 2B(\mathrm{Pm})(\zeta_1,\dots,\zeta_4) - 2\rho\mathrm{P}\,\mathrm{Pm}(\zeta_1,\dots,\zeta_4)\,, \end{aligned}$$

where we used several computations done in Section 1.4. For $\zeta_1, \dots, \zeta_4$ belonging to a local frame we get the desired equation (3.17). ☐

Let now $\mathcal{P} \in \mathcal{C}_b(V)$ be the algebraic curvature operator (see Appendix A), defined by $\varphi \circ \mathcal{P} = \mathcal{R} \circ \varphi$. Again, by Proposition 1.4.7 we have that for every $p \in M$ and $t \in [0,T)$ the tensor $\mathcal{P}(p,t)$ has the same eigenvalues of the "real" curvature operator $\mathcal{R}(p,t)$; in particular $\mathcal{P}$ is positive definite if and only if $\mathcal{R}$ is positive definite. Moreover, by the computations carried out in Appendix A, the reaction term is nicely expressed in terms of $\mathcal{P}$. Therefore, combining the evolution equation (3.17) of Pm with the formulas proved in Appendix A, we find the evolution equation of $\mathcal{P}$.

$$\frac{\partial}{\partial t}\mathcal{P} = \Delta_D \mathcal{P} - 2\rho\varphi^*(\nabla^2 \mathrm{tr}_{g_0}(\mathcal{P})) \owedge g_0 + 2\mathcal{P}^2 + 2\mathcal{P}^\# - 4\rho \mathrm{tr}_{g_0}(\mathcal{P})\mathcal{P}, \tag{3.18}$$

where $\mathrm{tr}_{g_0}(\mathcal{P}(t)) = 1/2\mathrm{R}_{g(t)}$ is half of the scalar curvature of the metric $g(t)$, by Proposition 1.4.7, point (3) and equation (A.3).

REMARK 3.2.5. As we already said, the small conceptual complication arising from Uhlenbeck's trick is widely repaid: first of all the evolution equation (3.18) satisfied by $\mathcal{P}$ allows a simpler use of the maximum principle for tensors as the reaction term is nicer and the metric on $\Gamma(S^2(\Lambda^2 V^*))$ is independent from time. On the other hand, the subsets of $\Gamma(S^2(\Lambda^2 V^*))$ preserved by this PDE really correspond to curvature conditions preserved by the RB–flow.

3.3. Conditions preserved in any dimension

In this section we use the maximum principle, as stated in various formulations in Section 1.5, in order to find curvature conditions which are preserved by the Ricci–Bourguignon flow.
We begin by considering the evolution equation of the scalar curvature (3.11), which, under the usual hypothesis on ρ about short time existence of the RB–flow, behaves exactly like the scalar curvature evolving by the Ricci flow.

PROPOSITION 3.3.1. *Let $(M, g(t))_{t\in[0,T)}$ be a maximal solution of the RB–flow (3.1). If $\rho < \frac{1}{2(n-1)}$, the minimum of the scalar curvature is non decreasing along the flow. In particular if $\mathrm{R}_{g(0)} \geq \alpha$ for some $\alpha \in \mathbb{R}$, then $\mathrm{R}_{g(t)} \geq \alpha$ for every $t \in [0,T)$. Moreover if $\alpha > 0$ then $T < \frac{n}{2(1-n\rho)\alpha}$.*

PROOF. As $|\mathrm{Ric}|^2 \geq \mathrm{R}^2/n$, from equation (3.11) we obtain that the scalar curvature satisfies the following differential inequality:

$$\frac{\partial}{\partial t}\mathrm{R} \geq (1-2(n-1)\rho)\Delta \mathrm{R} + \frac{2}{n}(1-n\rho)\mathrm{R}^2 \,.$$

As $\rho < 1/2(n-1) < 1/n$, the second order operator above is a positive multiple of the metric Laplacian and the 0–order term is strictly positive, then by Hamilton's trick (see [50], [63, Lemma 2.1.3]) we obtain that the minimum of the scalar curvature is non decreasing along the flow, hence $\mathrm{R}_{g(t)} \geq \alpha$ for every $t \in [0,T)$. Moreover, by applying the weak scalar maximum principle 1.5.1 and comparing with the solution of the associated ODE, we obtain

$$\mathrm{R}_{g(t)} \geq \frac{n\alpha}{n - 2(1-n\rho)\alpha t} \tag{3.19}$$

that, for $\alpha > 0$, gives the estimate on the maximal time of existence. □

REMARK 3.3.2. In the special case of the Schouten flow (when $\rho = 1/2(n-1)$), actually there holds

$$\frac{\partial}{\partial t}\mathrm{R} \geq \frac{n-2}{n(n-1)}\mathrm{R}^2 \,,$$

at every point of the manifold, which implies that the scalar curvature is pointwise non decreasing and diverges in finite time.

REMARK 3.3.3. By means of the strong maximum principle, it follows that if the initial manifold has non negative scalar curvature then either $\mathrm{R}_{g(t)} > 0$ for every t or $\mathrm{R}_{g(t)} \equiv 0$ for every t. If the latter is the case, by the evolution equation of the scalar curvature (3.11) we get $|\mathrm{Ric}_{g(t)}| \equiv 0$, therefore either the manifold is Ricci–flat ($\mathrm{Ric} \equiv 0$) or the scalar curvature becomes positive for every positive time under any RB–flow with $\rho \leq \frac{1}{2(n-1)}$.

PROPOSITION 3.3.4. *Let $(M, g(t))$, $t \in (-\infty, 0]$ be a compact ancient solution to the RB–flow (3.1). If $\rho \leq \frac{1}{2(n-1)}$, then either $\mathrm{R}_{g(t)} > 0$ for all $t \in (-\infty, 0]$, or $\mathrm{Ric}_{g(t)} \equiv 0$ for all $t \in (-\infty, 0]$.*

PROOF. As $g(t)$ is an ancient solution, for every $c > 0$ we can define $g_c(s) = g(s - c)$, which is a solution of the RB–flow on $[0, c]$. As the scalar curvature is bounded on a compact manifold, we have $\mathrm{R}_{g_c(0)} \geq \alpha$ for some negative constant α, hence, from (3.19),

$$\mathrm{R}_{g_c(s)} \geq \frac{n}{n(\alpha)^{-1} - 2(1 - n\rho)s} > -\frac{n}{2(1 - n\rho)s}$$

on M for every $s \in (0, c]$. Since we only translated time by a constant, we obtain for every $t \in (-c, 0]$ and $p \in M$

$$\mathrm{R}_{g(t)} = \mathrm{R}_{g_c(t+c)} \geq -\frac{n}{2(1 - n\rho)(t + c)}\,.$$

By taking the limit as $c \to +\infty$, we conclude that $\mathrm{R}_{g(t)} \geq 0$ for every $t \in (-\infty, 0]$ and the previous remark implies the result. $\square$

The evolution equation (3.9) of the Riemann tensor has some mixed products of type $\mathrm{Rm} * \mathrm{Ric}$ which make difficult to understand the behaviour of the reaction term. As remarked before, if we perform Uhlenbeck's trick, the evolution equation (3.17) becomes a little nicer and can still be useful to understand how the RB–flow affects the geometry.
More precisely, we use the evolution equation (3.18) of the pull–back of the curvature operator $\mathcal{P} \in \Gamma(S^2(\Lambda^2 V^*))$ to prove that the cone of non negative curvature operators is preserved by the Ricci–Bourguignon flow, that is, the RB–flow preserves non negative sectional curvature in any dimension.

PROPOSITION 3.3.5. *Let $\rho < 1/2(n-1)$ and $(M, g(t))_{t \in [0,T)}$ be a solution of the RB–flow (3.1), such that the initial metric g_0 has non negative curvature operator. Then $\mathcal{R}_{g(t)} \geq 0$ for every $t \in [0, T)$.*

PROOF. We recall the evolution equation (3.18) for $\mathcal{P} = \varphi^{-1} \circ \mathcal{R} \circ \varphi$

$$\frac{\partial}{\partial t}\mathcal{P} = \Delta_D \mathcal{P} - 2\rho\varphi^*(\nabla^2 \mathrm{tr}_{g_0}(\mathcal{P})) \owedge g_0 + 2\mathcal{P}^2 + 2\mathcal{P}^{\#} - 4\rho\,\mathrm{tr}_{g_0}(\mathcal{P})\mathcal{P},$$

where $\mathrm{tr}_{g_0}(\mathcal{P}(t)) = 1/2\mathrm{R}_{g(t)}$ is half of the scalar curvature of the metric $g(t)$ (see Appendix A). By Proposition 1.4.7, it suffices to show that the non negativity of $\mathcal{P}$ is preserved by this equation. We want to apply the vectorial maximum principle 1.5.5, therefore we show that

$$L(Q) = \Delta_D Q - 2\rho\varphi^*(\nabla^2 \mathrm{tr}_{g_0}(Q)) \owedge g_0$$

is a uniformly strongly elliptic operator on the bundle $(S^2(\Lambda^2 V^*), g_0, D(t))$.
As L is a linear second order operator, we compute as usual its principal symbol in the arbitrary direction ξ. In order to simplify the computations, we choose opportune frames at every point $p \in M$ and time $t \in [0, T)$. Then, let $\{e_i\}_{i=1,\dots,n}$ be an orthonormal basis of $(V_p, (g_0)_p)$ such that $\xi = (g_0)_p(e_1, \cdot)$. According to Uhlenbeck's trick (Section 1.4) and the convention on algebraic curvature operators (Appendix A) we have that $\{e_i \wedge e_j\}_{i<j}$ is an orthonormal basis of $\Lambda^2 V_p$, $\{f_i = \varphi(t)_p(e_i)\}_{i_1,\dots,n}$ is an orthonormal basis of T_pM with respect to $(g(t))_p$ and the components of $\varphi(t)_p$ with these choices are $\varphi_i^a = \delta_i^a$. Hence, the principal symbol of the operator L written in these frames is

$$\begin{aligned}
\sigma_\xi(LQ)_{(ij)(kl)} &= \xi^p \xi_p Q_{(ij)(kl)} - 2\rho\delta_i^a \delta_j^b \delta_k^c \delta_l^d \mathrm{tr}_{g_0}(Q)(\xi \otimes \xi \owedge g_0)_{(ab)(cd)} \\
&= |\xi|^2 Q_{(ij)(kl)} - 2\rho\,\mathrm{tr}_{g_0}(Q)(\xi \otimes \xi \owedge g_0)_{(ij)(kl)} \\
&= Q_{(ij)(kl)} - 2\rho\Big(\sum_{p<q} Q_{(pq)(pq)}\Big)\delta_i^1 \delta_k^1 \delta_{jl}\,,
\end{aligned}$$

where we used that $|\xi| = 1$, $i < j$ and $k < l$ in the last passage. Now it is easy to see that the matrix representing the symbol has the following form

$$\sigma_\xi(L) = \left(\begin{array}{c|c|c} A[n-1] & \begin{matrix} -2\rho & \dots & 2\rho \\ \vdots & \ddots & \vdots \\ -2\rho & \dots & -2\rho \end{matrix} & 0 \\ \hline 0 & \mathrm{Id}_{(n-1)(n-2)/2} & 0 \\ \hline 0 & 0 & \mathrm{Id}_{N(N-1)/2} \end{array}\right),$$

where we have ordered the components as follows: first the $n-1$ ones of the form $(1j)(1j)$ with $j > 1$, then the $(n-1)(n-2)/2$ ones of the form $(ij)(ij)$ with $1 < i < j$, and last the $N(N-1)/2$ "non diagonal" ones, with $N = n(n-1)/2$ and A is the matrix defined in equation (3.6).
By Lemma 3.1.1, the eigenvalues of the symbol are 1 with multiplicity $N(N+1)/2 - 1$ and $1 - 2(n-1)\rho$ with multiplicity 1, hence, under the hypothesis of short time existence of the RB–flow $\rho < 1/2(n-1)$, the operator L is uniformly strongly elliptic.
In the second part of the proof we consider the reaction term $F(Q) = 2\big(Q^2 + Q^\# - 2\rho\,\mathrm{tr}_{g_0}(Q)Q\big)$. Clearly F is continuous, locally Lipschitz and fiber–preserving. Let $\Omega \subset \mathcal{C}_b(V)$ be the set of non negative algebraic curvature operators (see again Appendix A), we observe that $\Omega = \{Q : \lambda_N(Q_p) \geq 0\}$, where $N = n(n-1)/2$ and λ_N is the least eigenvalue of Q_p. Hence, Ω is clearly closed; by Lemma 1.5.7, it is invariant under parallel translation with respect to every connection $D(t)$ and it is convex, provided that the function $Q_p \mapsto \lambda_N(Q_p)$ is concave. We can rewrite

$$\lambda_N(Q_p) = \inf_{\{v \in \Lambda^2 V_p \,:\, |v|_{g_0} = 1\}} g_0(Q_p(v), v)\,;$$

so it is easy to conclude, by the bilinearity of the metric g_0 and the concavity of $\inf$, that the function defining Ω is actually concave and so its superlevels are convex. In order to finish the proof, we have to show that the ODE $dQ/dt = F(Q)$ preserves Ω. Now, by standard facts in convex analysis, we only need to prove that

$$F_p(Q_p) \in T_{Q_p}\Omega_p \qquad \text{for every } p \in M \text{ such that } Q_p \in \partial\Omega_p\,,$$

where $\partial\Omega_p = \{Q_p \in \Omega_p \,:\, \exists\, v \in \Lambda^2 V_p \text{ such that } Q_p(v,v) = 0\}$ and the tangent cone is

$$T_{Q_p}\Omega_p = \{S_p \in S^2(\Lambda^2 V_p^*) \,:\, S_p(v,v) \geq 0 \text{ for every } v \in \Lambda^2 V_p \text{ such that } Q_p(v,v) = 0\}\,.$$

Let $v \in \Lambda^2 V_p$ and $\{\theta_\alpha\}$ be respectively a null eigenvector of Q_p and an orthonormal basis of $\Lambda^2 V_p$ that diagonalizes Q_p. Clearly,

$$v = v^\alpha \theta_\alpha\,, \qquad (Q_p)_{\alpha\beta} = \lambda_\alpha \delta_{\alpha\beta}\,,$$

with $\lambda_\alpha \geq 0$. Then

$$(Q_p^2)_{\alpha\beta} = \lambda_\alpha^2 \delta_{\alpha\beta}\,, \quad (Q_p^\#)_{\alpha\beta} = \frac{1}{2}(c_\alpha^{\gamma\nu})^2 \lambda_\gamma \lambda_\nu \delta_{\alpha\beta}$$

and

$$F_p(Q_p)(v,v) = \lambda_\alpha^2 (v^\alpha)^2 + \frac{1}{2}(c_\alpha^{\gamma\nu})^2 \lambda_\gamma \lambda_\nu (v^\alpha)^2 \geq 0,$$

this completes the proof. □

As in the Ricci flow, there can be other preserved cones in the space of algebraic curvature operators (see [76] for a detailed survey of the subject) and this can be a future line of research to understand the long time behaviour of the RB–flow. At the same time, the evolution equation (3.10) for the Ricci tensor has the same reaction term as in the Ricci flow, therefore it contains some mixed products of type $\mathrm{Rm} * \mathrm{Ric}$ and it cannot be treated in any dimension without further hypotheses, while in dimension three it keeps the good features well known for the Ricci flow. The 3–dimensional case, in particular when the Ricci tensor of the initial metric is positive, will be addressed in the next chapter.

3.4. Curvature estimates and long time existence

In this section we prove that a compact solution of the RB–flow existing up to a finite maximal time T must have unbounded Riemann tensor when $t \to T$ (Theorem 3.4.4).
First of all we observe that, similarly to the Ricci flow, the RB–flow enjoys the property that, given a uniform bound on the norm of the Riemann tensor on $M \times [0, T)$, one has *a priori* estimates on the covariant derivatives of the Riemann tensor for any time $t > 0$. However, unlike the Ricci flow, the pointwise norm of the Riemann tensor does not satisfy a heat type equation (see computation below), which makes pointwise estimates on $|\nabla^k \mathrm{Rm}|$ difficult, so we have to look for integral bounds. Nonetheless, we are able to prove the following Bando–Bernstein–Shi type estimates.

THEOREM 3.4.1. *Assume $\rho < \frac{1}{2(n-1)}$. If $g(t)$ is a compact solution of the RB–flow for $t \in [0, T)$ such that*

$$\sup_{(x,t)\in M\times[0,T)} |\mathrm{Rm}(x,t)| \leq K\,,$$

then for all $k \in \mathbb{N}$ there exists a constant $C(n, \rho, k, K, T)$ such that for all $t \in (0, T]$

$$\|\nabla^k \mathrm{Rm}_{g(t)}\|_2^2 \leq \frac{C}{t^k} \sup_{t\in[0,T)} \|\mathrm{Rm}_{g(t)}\|_2^2\,.$$

In order to prove the above estimate we work out the evolution equations of the integrals of the squared norm of the covariant derivatives of the Riemann tensor and of the scalar curvature.

LEMMA 3.4.2. *Let $g(t)$ be a compact solution of the RB–flow for $t \in [0, T)$. Then, for any $k \in \mathbb{N}$ there holds*

$$\frac{d}{dt}\int_M |\nabla^k \mathrm{Rm}|^2\,d\mu_g = -2\int_M |\nabla^{k+1}\mathrm{Rm}|^2\,d\mu_g + 4\rho\int_M |\nabla^{k+1}\mathrm{R}|^2\,d\mu_g + \sum_{j+l=k,\,j,l\geq 0}\int_M \nabla^j \mathrm{Rm} * \nabla^l \mathrm{Rm} * \nabla^k \mathrm{Rm}\,d\mu_g \tag{3.20}$$

$$\frac{d}{dt}\int_M |\nabla^k \mathrm{R}|^2\,d\mu_g = -2(1-2(n-1)\rho)\int_M |\nabla^{k+1}\mathrm{R}|^2\,d\mu_g + \sum_{j+l=k,\,j,l\geq 0}\int_M \nabla^j \mathrm{Rm} * \nabla^l \mathrm{Rm} * \nabla^k \mathrm{Rm}\,d\mu_g\,. \tag{3.21}$$

PROOF. Starting from the evolution equations of the scalar curvature (3.11) and the Riemann tensor (3.9), by a direct computation we get

$$\begin{aligned}
\frac{\partial}{\partial t}\mathrm{R}^2 &= 2\mathrm{R}\,\frac{\partial}{\partial t}\mathrm{R} = 2(1-2(n-1)\rho)\mathrm{R}\Delta\mathrm{R} + 4\mathrm{R}(|\mathrm{Ric}|^2 - \rho\mathrm{R}^2)\\
&= (1-2(n-1)\rho)\Delta(\mathrm{R}^2) - 2(1-2(n-1)\rho)|\nabla\mathrm{R}|^2 + \mathrm{Rm} * \mathrm{Rm} * \mathrm{Rm}\,,\\
\frac{\partial}{\partial t}|\mathrm{Rm}|^2 &= 2\langle \mathrm{Rm}, \frac{\partial}{\partial t}\mathrm{Rm}\rangle + \mathrm{Rm} * \mathrm{Rm} * \mathrm{Rm}\\
&= 2\langle \mathrm{Rm}, \Delta\mathrm{Rm}\rangle - 2\rho\langle \nabla^2\mathrm{R} \owedge g, \mathrm{Rm}\rangle + \mathrm{Rm} * \mathrm{Rm} * \mathrm{Rm}\\
&= \Delta|\mathrm{Rm}|^2 - 2|\nabla\mathrm{Rm}|^2 - 8\rho\langle \nabla^2\mathrm{R}, \mathrm{Ric}\rangle + \mathrm{Rm} * \mathrm{Rm} * \mathrm{Rm}\,,
\end{aligned}$$

where the last term contains both the 0–th order term in the evolution of Rm and the term coming from the time derivative of the metric. It follows, by the divergence theorem and integration by parts, that

$$\begin{aligned}
\frac{d}{dt}\int_M \mathrm{R}^2\,d\mu_g &= -2(1-2(n-1)\rho)\int_M |\nabla\mathrm{R}|^2\,d\mu_g + \int_M \mathrm{Rm}*\mathrm{Rm}*\mathrm{Rm}\,d\mu_g\,,\\
\frac{d}{dt}\int_M |\mathrm{Rm}|^2\,d\mu_g &= -2\int_M |\nabla\mathrm{Rm}|^2\,d\mu_g - 8\rho\int_M \langle\nabla^2\mathrm{R},\mathrm{Ric}\rangle\,d\mu_g\\
&\quad + \int_M \mathrm{Rm}*\mathrm{Rm}*\mathrm{Rm}\,d\mu_g\\
&= -2\int_M |\nabla\mathrm{Rm}|^2\,d\mu_g + 4\rho\int_M |\nabla\mathrm{R}|^2\,d\mu_g + \int_M \mathrm{Rm}*\mathrm{Rm}*\mathrm{Rm}\,d\mu_g\,,
\end{aligned}$$

where in the last equation we used the contracted Bianchi identity $2\delta\mathrm{Ric} = \nabla\mathrm{R}$.
To prove the general evolution equations, we use several times the following commutation formulas which hold for a general tensor T during the RB–flow

$$\Big[\frac{\partial}{\partial t},\nabla\Big]T = \nabla\mathrm{Rm}*T\,,\quad [\Delta,\nabla]T = \mathrm{Rm}*\nabla T + \nabla\mathrm{Rm}*T\,,\quad [\nabla_i,\nabla_j]T = \mathrm{Rm}*T\,,$$

where we denote with $[\ ,\ ]$ the commutator between two differential operators, i.e. $[L,L'](T) = L(L'(T)) - L'(L(T))$.
We write in detail only the computation for the Riemann tensor and, at each step, we collect every possible term in the expression $\sum_{j+l=k,\,j,l\geq 0}\nabla^j\mathrm{Rm}*\nabla^l\mathrm{Rm}*\nabla^k\mathrm{Rm}$.

$$\begin{aligned}
\frac{\partial}{\partial t}|\nabla^k\mathrm{Rm}|^2 &= 2\Big\langle\frac{\partial}{\partial t}\nabla^k\mathrm{Rm},\nabla^k\mathrm{Rm}\Big\rangle + \mathrm{Rm}*\nabla^k\mathrm{Rm}*\nabla^k\mathrm{Rm}\\
&= 2\Big\langle\nabla^k\frac{\partial}{\partial t}\mathrm{Rm},\nabla^k\mathrm{Rm}\Big\rangle + \sum_{j+l=k,\,j,l\geq 0}\nabla^j\mathrm{Rm}*\nabla^l\mathrm{Rm}*\nabla^k\mathrm{Rm}\\
&= 2\langle\nabla^k\Delta\mathrm{Rm},\nabla^k\mathrm{Rm}\rangle - 2\rho\langle\nabla^k(\nabla^2\mathrm{R}\owedge g),\nabla^k\mathrm{Rm}\rangle\\
&\quad + \sum_{j+l=k,\,j,l\geq 0}\nabla^j\mathrm{Rm}*\nabla^l\mathrm{Rm}*\nabla^k\mathrm{Rm}\\
&= 2\langle\Delta\nabla^k\mathrm{Rm},\nabla^k\mathrm{Rm}\rangle - 8\rho\langle\nabla^k\nabla^2\mathrm{R},\nabla^k\mathrm{Ric}\rangle\\
&\quad + \sum_{j+l=k,\,j,l\geq 0}\nabla^j\mathrm{Rm}*\nabla^l\mathrm{Rm}*\nabla^k\mathrm{Rm}\\
&= \Delta|\nabla^k\mathrm{Rm}|^2 - 2|\nabla^{k+1}\mathrm{Rm}|^2\\
&\quad -8\rho\langle\nabla^2\nabla^k\mathrm{R},\nabla^k\mathrm{Ric}\rangle + \sum_{j+l=k,\,j,l\geq 0}\nabla^j\mathrm{Rm}*\nabla^l\mathrm{Rm}*\nabla^k\mathrm{Rm}\,.
\end{aligned}$$

To simplify the integral evolution we use again the commutation formulas, the divergence theorem and the contracted Bianchi identity.

$$\begin{aligned}
\frac{d}{dt}\int_M |\nabla^k\mathrm{Rm}|^2\,d\mu_g &= \int_M \frac{\partial}{\partial t}|\nabla^k\mathrm{Rm}|^2\,d\mu_g + \int_M \mathrm{Rm}*\nabla^k\mathrm{Rm}*\nabla^k\mathrm{Rm}\,d\mu_g\\
&= -2\int_M |\nabla^{k+1}\mathrm{Rm}|^2\,d\mu_g + 8\rho\int_M\langle\nabla\nabla^k\mathrm{R},\nabla\nabla^k\mathrm{Ric}\rangle\,d\mu_g\\
&\quad + \sum_{j+l=k,\,j,l\geq 0}\int_M \nabla^j\mathrm{Rm}*\nabla^l\mathrm{Rm}*\nabla^k\mathrm{Rm}\,d\mu_g\\
&= -2\int_M |\nabla^{k+1}\mathrm{Rm}|^2\,d\mu_g + 8\rho\int_M\langle\nabla^k\nabla\mathrm{R},\nabla^k\delta\mathrm{Ric}\rangle\,d\mu_g\\
&\quad + \sum_{j+l=k,\,j,l\geq 0}\int_M \nabla^j\mathrm{Rm}*\nabla^l\mathrm{Rm}*\nabla^k\mathrm{Rm}\,d\mu_g\,.
\end{aligned}$$

□

PROOF OF THEOREM 3.4.1. We follow [29, Chapter 7] and prove the theorem by induction on $k\in\mathbb{N}$. In the previous lemma we computed the (integral) evolution equation (3.20) of the k–th covariant derivative of the Riemann tensor. Since the $(k+1)$–th covariant derivative of the scalar curvature there appears, we cannot control directly the L^2–norm of $\nabla^k \mathrm{Rm}$, instead we consider

$$\mathcal{A}_k = \int_M |\nabla^k \mathrm{Rm}|^2\, d\mu_g + \frac{4|\rho|}{1-2(n-1)\rho}\int_M |\nabla^k \mathrm{R}|^2\, d\mu_g\,.$$

Now, we set $f_1(t) = \mathcal{A}_0 + \beta t \mathcal{A}_1$, where $\beta > 0$ is a constant to be determined later. By a straightforward computation, we have

$$\begin{aligned}
f_1'(t) &= \mathcal{A}_0' + \beta\mathcal{A}_1 + \beta t \mathcal{A}_1' \\
&= -2\int_M |\nabla \mathrm{Rm}|^2\, d\mu_g + (4\rho - 8|\rho|)\int_M |\nabla \mathrm{R}|^2\, d\mu_g + \int_M \mathrm{Rm} * \mathrm{Rm} * \mathrm{Rm}\, d\mu_g \\
&\quad +\beta\biggl(\int_M |\nabla \mathrm{Rm}|^2\, d\mu_g + \frac{4|\rho|}{1-2(n-1)\rho}\int_M |\nabla \mathrm{R}|^2\, d\mu_g\biggr) \\
&\quad +\beta t\biggl(-2\int_M |\nabla^2 \mathrm{Rm}|^2\, d\mu_g + (4\rho - 8|\rho|)\int_M |\nabla^2 \mathrm{R}|^2\, d\mu_g + \int_M \mathrm{Rm} * \nabla\mathrm{Rm} * \nabla\mathrm{Rm}\, d\mu_g\biggr) \\
&= -2\beta t\int_M |\nabla^2 \mathrm{Rm}|^2\, d\mu_g + \beta t(4\rho - 8|\rho|)\int_M |\nabla^2 \mathrm{R}|^2\, d\mu_g \\
&\quad (-2+\beta)\int_M |\nabla \mathrm{Rm}|^2\, d\mu_g + \Bigl(4\rho - 8|\rho| + \frac{4|\rho|\beta}{1-2(n-1)\rho}\Bigr)\int_M |\nabla \mathrm{R}|^2\, d\mu_g \\
&\quad + \int_M \mathrm{Rm} * \mathrm{Rm} * \mathrm{Rm}\, d\mu_g + \beta t \int_M \mathrm{Rm} * \nabla\mathrm{Rm} * \nabla\mathrm{Rm}\, d\mu_g\,.
\end{aligned}$$

Now, we observe that in the first row of the last equation there are two non positive quantities, while for the last row, using the assumption on the boundedness of $|\mathrm{Rm}|$, we have

$$\beta t \int_M \mathrm{Rm} * \nabla\mathrm{Rm} * \nabla\mathrm{Rm}\, d\mu_g \le c_1 K \beta T \int_M |\nabla \mathrm{Rm}|^2\, d\mu_g\,.$$

If we set $\beta = \min\Bigl\{(1-2(n-1)\rho), \frac{2}{c_1 K T + 1}\Bigr\}$, we obtain

$$f_1'(t) \le \int_M \mathrm{Rm} * \mathrm{Rm} * \mathrm{Rm}\, d\mu_g \le c_2(\rho, n, K)\|\mathrm{Rm}\|_2^2\,.$$

Finally, since $f_1(0) = \mathcal{A}_0 \le c_3(\rho, n)\|\mathrm{Rm}\|_2^2$, by integrating the previous inequality, we get

$$\|\nabla \mathrm{Rm}\|_2^2 \le \mathcal{A}_1 \le \frac{1}{\beta t}(f_1(0) + c_2(\rho,n,K)t\|\mathrm{Rm}\|_2^2) \le \frac{C(\rho,n,K,T)}{t}\|\mathrm{Rm}\|_2^2\,,$$

which is what we wanted to prove for $k = 1$.

More generally, we want to control all the derivatives of the curvature, hence we assume that the conclusion of the theorem is true up to $(k-1)$ and we define $f_k(t) = \sum_{i=0}^{k} \frac{\beta^i t^i}{i!}\mathcal{A}_i$. We have,

$$f_k'(t) = \sum_{i=0}^{k-2} \frac{\beta^i t^i}{i!}\left(\mathcal{A}_i' + \beta\mathcal{A}_{i+1}\right) + \frac{\beta^{k-1}t^{k-1}}{(k-1)!}\Bigl(\beta\mathcal{A}_k + \mathcal{A}_{k-1}' + \frac{\beta t}{k}\mathcal{A}_k'\Bigr)\,. \tag{3.22}$$

For the first $k-1$ terms in equation (3.22), we observe that

$$\begin{aligned}
\mathcal{A}_i' + \beta\mathcal{A}_{i+1} &= (-2+\beta)\|\nabla^{i+1}\mathrm{Rm}\|_2^2 \\
&\quad + \Bigl(4\rho - 8|\rho| + \frac{4|\rho|\beta}{1-2(n-1)\rho}\Bigr)\|\nabla^{i+1}\mathrm{R}\|_2^2 \\
&\quad + \sum_{j+l=i,\, j,l\ge 0}\int_M \nabla^j\mathrm{Rm} * \nabla^l \mathrm{Rm} * \nabla^i \mathrm{Rm}\, d\mu_g\,.
\end{aligned}$$

Provided that $\beta = \min\{2, (1-2(n-1)\rho)\}$, we only need to estimate the last term. By Lemma 1.6.7 with $p = +\infty$ and $q = 2$, for any $i = 0, \dots, k$ it follows,

$$\sum_{j+l=i,\, j,l\geq 0} \int_M \nabla^j \mathrm{Rm} * \nabla^l \mathrm{Rm} * \nabla^i \mathrm{Rm}\, d\mu_g \leq c_1(i) \|\mathrm{Rm}\|_\infty \|\mathrm{Rm}\|^2_{H^2_i} \leq c_1(i) K \|\mathrm{Rm}\|^2_{H^2_i}\,. \tag{3.23}$$

Moreover, by the inductive hypothesis, for every $i = 0, \dots, k-1$,

$$\begin{aligned}
\|\mathrm{Rm}\|^2_{H^2_i} &= \sum_{j=0}^{i} \|\nabla^j \mathrm{Rm}\|_2^2 \leq \sum_{j=0}^{i} \frac{C(j)}{t^j} \|\mathrm{Rm}\|_2^2 \\
&\leq \|\mathrm{Rm}\|_2^2 \max_j \{C(j)\} \sum_{j=0}^{i} \frac{1}{t^j} = \|\mathrm{Rm}\|_2^2 \frac{c_2(i)}{t^i} \sum_{j=0}^{i} t^j\,,
\end{aligned}$$

where we point out the dependence of the constant only from the index i.
For the last term in equation (3.22), likewise we did for f_1, we observe that

$$\begin{aligned}
\beta \mathcal{A}_k + \mathcal{A}'_{k-1} + \frac{\beta t}{k} \mathcal{A}'_k &= (-2+\beta) \|\nabla^k \mathrm{Rm}\|_2^2 + \left(4\rho - 8|\rho| + \frac{4|\rho|\beta}{1-2(n-1)\rho}\right) \|\nabla^k \mathrm{R}\|_2^2 \\
&\quad + \sum_{j+l=k-1,\, j,l\geq 0} \int_M \nabla^j \mathrm{Rm} * \nabla^l \mathrm{Rm} * \nabla^{k-1} \mathrm{Rm}\, d\mu_g \\
&\quad + \frac{\beta t}{k} \Bigg(-2 \|\nabla^{k+1} \mathrm{Rm}\|_2^2 + (4\rho - 8|\rho|) \|\nabla^{k+1} \mathrm{R}\|_2^2 \\
&\quad + \sum_{j+l=k,\, j,l\geq 0} \int_M \nabla^j \mathrm{Rm} * \nabla^l \mathrm{Rm} * \nabla^k \mathrm{Rm}\, d\mu_g \Bigg) \\
&\leq \left(-2 + \beta + \frac{\beta t}{k} c_1(k) K \right) \|\nabla^k \mathrm{Rm}\|_2^2 + \frac{\beta t}{k} c_1(k) K \|\mathrm{Rm}\|^2_{H^2_{k-1}} \\
&\quad + \sum_{j+l=k-1,\, j,l\geq 0} \int_M \nabla^j \mathrm{Rm} * \nabla^l \mathrm{Rm} * \nabla^{k-1} \mathrm{Rm}\, d\mu_g \\
&\leq \left(-2 + \beta + \frac{\beta t}{k} c_1(k) K \right) \|\nabla^k \mathrm{Rm}\|_2^2 \\
&\quad + K \|\mathrm{Rm}\|_2^2 \frac{c_3(k)}{t^{k-1}} \sum_{j=0}^{k-1} t^j\,,
\end{aligned}$$

where in the first inequality we used again equation (3.23) for $i = k$ and separate the norm of the k–th covariant derivative from the others, in the second one we used again the inductive hypothesis.
Then we set $\beta = \min\left\{2, (1-2(n-1)\rho), \frac{2k}{c_1(k)KT+k}\right\}$ and the first term in the previous inequality is non positive.
By means of the previous computations, we can estimate the right hand side of equation (3.22) and we finally obtain

$$\begin{aligned}
f'_k(t) &\leq K \|\mathrm{Rm}\|_2^2 \sum_{i=0}^{k-2} \frac{\beta^i t^i}{i!} \frac{c_2(i)}{t^i} \sum_{j=0}^{i} t^j + K \|\mathrm{Rm}\|_2^2 \frac{\beta^{k-1} t^{k-1}}{(k-1)!} \frac{c_3(k)}{t^{k-1}} \sum_{j=0}^{k-1} t^j \\
&= c_4(\rho, n, k, K, T) \|\mathrm{Rm}\|_2^2\,.
\end{aligned}$$

As we did for f_1, we integrate this last inequality and we get

$$\begin{aligned}\|\mathrm{Rm}\|_2^2 \leq \mathcal{A}_k &\leq \frac{k!}{\beta^k t^k} f_k(t)\\ &\leq \frac{k!}{\beta^k t^k}\Big(f_k(0) + tc_4(\rho,n,k,K,T)\|\mathrm{Rm}\|_2^2\Big) \leq \frac{C(\rho,n,k,K,T)}{t^k}\|\mathrm{Rm}\|_2^2\,,\end{aligned}$$

which completes the inductive step. The result follows by induction. □

REMARK 3.4.3. In [18], there is a proof of this theorem without recurring to induction, by using interpolation inequalities more similar to the ones proven by Hamilton in [49].

3.4.1. Long time existence.
In this subsection we use the global estimates of Theorem 3.4.1 to prove a result on long time behaviour of solutions of the Ricci–Bourguignon flow. While the proof follows exactly the one of Hamilton in [49, Theorem 14.1] for the Ricci flow, we are not able to show that the maximum of the L^∞–norm of the Riemann tensor goes to infinity for any sequence of times $t_i \to T$. This because, as already remarked, the squared norm of the Riemann tensor does not satisfy a heat equation, therefore we cannot control it pointwise in time, as in the so called "doubling time estimate" (see [29, Lemma 7.4 and Corollary 7.5], [30, Remark 6.4]). For the same reason, we are not able to give a lower bound on the maximal time T based on curvature of the initial metric g_0 (see [29, Corollary 7.7]).

THEOREM 3.4.4. *Assume $\rho < \frac{1}{1-2(n-1)\rho}$. If $g(t)$ is a compact solution of the RB–flow on a maximal time interval $[0,T)$, $T < +\infty$, then*

$$\limsup_{t\to T}\max_M |\mathrm{Rm}_{g(t)}| \;=\; +\infty\,.$$

PROOF. First of all we observe that, if the Riemann tensor is uniformly bounded as $t \to T$ and $T < +\infty$, then also its L^2–norm is uniformly bounded, since from the previous computations,

$$\mathcal{A}_0 = \|\mathrm{Rm}\|_2^2 + \frac{4|\rho|}{1-2(n-1)\rho}\|\mathrm{R}\|_2^2$$

satisfies $\mathcal{A}_0' \leq C\mathcal{A}_0$.
Then, by Theorem 3.4.1, we get, for any $j \in \mathbb{N}$

$$\|\nabla^j \mathrm{Rm}\|_2^2 \leq C_j\,.$$

By using the interpolation inequalities in Proposition 1.6.1, with $p = \infty$ and $q = 2$, we immediately get the estimates

$$\|\nabla^j \mathrm{Rm}\|_{\frac{2k}{j}} \;\leq\; C_{j,k}\,,$$

for all $j \in \mathbb{N}$ and $k \geq j$. Therefore, by interpolation, the same result holds for a generic esponent r, with the constant that depends on j and r.
Now, let $E_j := |\nabla^j \mathrm{Rm}|^2$. Then, for all $r < +\infty$ we have

$$\int_M \big(|E_j|^r + |\nabla E_j|^r\big)\,d\mu_g \;\leq\; C'_{j,r}\,.$$

Thus, by Sobolev inequality, if $r > j$, one has

$$\max_M |E_j|^r \;\leq\; C_t\int_M \big(|E_j|^r + |\nabla E_j|^r\big)\,d\mu_g\,.$$

Notice that the constant C_t depends on the metric $g(t)$ (in coordinates), but it does not depend on the derivatives of $g_{ij}(t)$. Moreover, by [49, Lemma 14.2] it follows that the metrics are all equivalent. Hence, the constant C_t is uniformly bounded as $t \to T$ and, from the previous estimates, if $|\mathrm{Rm}| \leq C$ on $M \times [0,T)]$, for every $j \in \mathbb{N}$ we obtain that

$$\max_M |\nabla^j \mathrm{Rm}| \;\leq\; C_j\,,$$

where the constant C_j depends only on the initial metric g_0 (in coordinates) and on the constant C. Arguing now as in [49, Section 14], it follows that the metrics $g(t)$ converge to some limit metric $g(T)$ in the C^∞ topology (with all their time/space standard partial derivatives, once written in local coordinates), hence, we can restart the flow with this initial metric $g(T)$, obtaining a smooth flow in some larger time interval $[0, T+\delta)$, in contradiction with the fact that T was the maximal time of smooth existence. This completes the proof of Theorem 3.4.4. □

3.5. Solitons

In this section we focus on self–similar solutions of the Ricci–Bourguignon flow. More precisely we state some results about gradient ρ–Einstein solitons, which arise as a natural modification of gradient solitons of the Ricci flow.

DEFINITION 3.5.1. A *gradient ρ–Einstein soliton* is a Riemannian manifold (M, g), $n \geq 3$, endowed with a smooth function $f : M \to \mathbb{R}$, such that the metric g satisfies the equation

$$\mathrm{Ric} + \nabla^2 f = \rho \mathrm{R}\, g + \lambda g\,, \tag{3.24}$$

for some constants $\rho, \lambda \in \mathbb{R}$, $\rho \neq 0$. The soliton is called *trivial* whenever ∇f is parallel. We say that the ρ–Einstein soliton is steady for $\lambda = 0$, shrinking for $\lambda > 0$ and expanding for $\lambda < 0$. The function f is called a *ρ–Einstein potential* of the gradient ρ–Einstein soliton.

REMARK 3.5.2. We point out that a trivial ρ–Einstein gradient soliton is an Einstein manifold by Schur Lemma, that is, the metric g satisfies the Einstein equation $\mathrm{Ric}_g = \mu\, g$, with $\mu = \frac{\lambda}{1-n\rho}$ when $\rho \neq 1/n$, or $\mu = 0$ when $\rho = 1/n$ (in this case it must be $\lambda = 0$).

The study of these objects can be carried out on its own, from a static point of view, starting from the geometric and analytic properties determined by equation (3.24) (see [37] for the Ricci flow case). This has been done by Catino and Mazzieri in [21] and their results motivated at first our systematic approach to the Ricci–Bourguignon flow, started in this thesis. In some cases, the authors extended properties already known for (gradient) Ricci solitons ($\rho = 0$) to other values of the parameter ρ, as in Theorem 3.5.8 below. Moreover, by proving an extra property for ρ–Einstein solitons not generally shared by the Ricci solitons, namely the rectifiability (see Theorem 3.5.6 and Remark 3.5.7 below), Catino and Mazzieri showed that their geometric structure is even more rigid, see Theorem 3.5.12.
All the results of this section may be found in [21] and [22].

First of all, it is easy to show that gradient ρ–Einstein solitons give rise to solutions to the RB–flow (3.1).

THEOREM 3.5.3 ([22]). *Let (M, g_0) be a complete gradient ρ–Einstein soliton with ρ–Einstein potential f_0 and Einstein constant λ. Then, for every t such that $\tau(t) := -2\lambda t + 1 > 0$, the family of metrics $g(t)$, given by*

$$g(t) = \tau(t)\phi_t^* g_0\,, \tag{3.25}$$

is a solution of the RB–flow with initial metric g_0, where ϕ_t is the 1–parameter family of diffeomorphisms generated by

$$\frac{\partial}{\partial t}\phi_t(x) = \frac{1}{\tau(t)}(\nabla^{g_0} f_0)(\phi_t(x))\,. \tag{3.26}$$

PROOF. We set $\tau(t) = -2\lambda t + 1$. As $\nabla^{g_0} f_0$ is a complete vector-field, there exists a 1–parameter family of diffeomorphisms $\phi_t : M \to M$ generated by the time dependent family of vector fields $X(t, \cdot) := \frac{1}{\tau(t)}(\nabla^{g_0} f_0)(\phi_t(\cdot))$, for every t such that $\tau(t) > 0$. We also set $f_t = f_0 \circ \phi_t$ and $g(t) = \tau(t)\,\phi_t^* g_0$. We compute

$$\frac{\partial}{\partial t} g(t) = -\frac{2\lambda}{\tau(t)} g(t) + \tau(t)\frac{\partial}{\partial t}(\phi_t^* g_0)\,.$$

By Lemma 1.3.3, we have $\frac{\partial}{\partial t}(\phi_t^* g_0) = \mathscr{L}_{(\phi_t^{-1})_* X(t)}(\phi_t^* g_0)$. Using the fact that $(\phi_t^{-1})_*(\nabla^{g_0} f_0) = \nabla^{\phi_t^* g_0}(\phi_t^* f_0) = \nabla^{g(t)} f_t$, we obtain

$$\frac{\partial}{\partial t} g(t) = -\frac{2\lambda}{\tau(t)} g(t) + \frac{1}{\tau(t)}\mathscr{L}_{\nabla^{g(t)} f_t}\, g(t)\,.$$

Having this at hand, we compute

$$\begin{aligned} -\mathrm{Ric}_{g(t)} &= \phi_t^*(-\mathrm{Ric}_{g_0}) = \phi_t^*\left(\frac{1}{2}\mathscr{L}_{\nabla^{g_0} f_0} g_0 - \lambda\, g_0 - \rho \mathrm{R}_{g_0}\, g_0\right) \\ &= \frac{1}{2}\left(\frac{1}{\tau(t)}\mathscr{L}_{\nabla^{g(t)} f_t} g(t) - \frac{2}{\tau(t)}\lambda\, g(t)\right) - \frac{\rho}{\tau(t)}\,\mathrm{R}_{(\tau(t)^{-1}g(t))}\, g(t) \\ &= \frac{1}{2}\frac{\partial}{\partial t} g(t) - \frac{\rho}{\tau(t)}\,\mathrm{R}_{(\tau(t)^{-1}g(t))}\, g(t) \end{aligned}$$

and we observe that $\mathrm{R}_{(\tau(t)^{-1}g(t))} = \tau(t)\,\mathrm{R}_{g(t)}$. In other words, we have obtained

$$\frac{\partial}{\partial t} g(t) = -2\left(\mathrm{Ric}_{g(t)} - \rho \mathrm{R}_{g(t)}\, g(t)\right),$$

and the proof is complete. □

Concerning the regularity of these structures we have the following result.

THEOREM 3.5.4 ([22]). *A gradient ρ–Einstein soliton is real analytic, provided $\rho \neq 1/n$.*

Now we recall some useful definitions to describe the geometric structure of ρ–Einstein solitons.

DEFINITION 3.5.5. We say that a smooth function $f : M \to \mathbb{R}$ is *rectifiable* in an open set $U \subset M$ if and only if $|\nabla f_{|U}|$ is constant along every regular connected component of the level sets of $f_{|U}$. In particular, it can be seen that $f_{|U}$ only depends on the signed distance r to the regular connected component of some of its level sets. If $U = M$, we simply say that f is rectifiable. Consequently, a gradient soliton is called *rectifiable* if and only if it admits a rectifiable potential function.

The rectifiability turns out to be one of the main properties of the ρ–Einstein solitons, as we state in the following theorem.

THEOREM 3.5.6 ([21]). *Every gradient ρ–Einstein soliton is rectifiable (for $\rho \neq 0$).*

Using the previous theorem, Catino and Mazzieri proved that, locally, the solitons can be foliated by hypersurfaces with constant mean curvature and constant induced scalar curvature, about a regular connected component of a level set of the potential f.

REMARK 3.5.7. It is worth noticing that Theorem 3.5.6 fails to be true in the case of gradient Ricci solitons. In fact, even though all of the easiest non trivial examples – such as the Gaussian soliton and the round cylinder in the shrinking case, or the Hamilton's cigar (also known in the physics literature as Witten's black hole) and the Bryant soliton in the steady case – are rectifiable, it is easy to check, for instance, that the Riemannian product of two rectifiable steady gradient Ricci solitons gives rise to a new steady soliton, which is generically not rectifiable.

Moreover, Catino and Mazzieri, using the elliptic maximum principle and some identities derived from the soliton equation (3.24), proved some classification results. First of all, in the compact case there is the following:

THEOREM 3.5.8 ([21]). *Let (M, g), $n \geq 3$, be compact gradient ρ–Einstein soliton. Then, the following cases occur.*

- (i) *If $\rho \leq 1/2(n-1)$, then either $\lambda > 0$ and $\mathrm{R} > 0$ or the soliton is trivial.*
- (i-bis) *If $\rho = 1/2(n-1)$, then the soliton is trivial.*
- (ii) *If $1/2(n-1) < \rho < 1/n$, then either $\lambda < 0$ and $\mathrm{R} < 0$ or the soliton is trivial.*
- (iii) *If $\rho \geq 1/n$, the soliton is trivial.*

In particular, for solitons corresponding to these special values of ρ, we get

COROLLARY 3.5.9. *Every compact gradient Einstein, Schouten or traceless Ricci soliton is trivial.*

We observe that the same statement as in case (i) of Theorem 3.5.8 was already known for Ricci solitons, which formally correspond to $\rho = 0$ (see [58, 55] or [25, Proposition 1.13]): every compact gradient Ricci soliton, if steady or expanding must be trivial, otherwise it must have positive scalar curvature.
In the general (also non compact) case the following results were proved. Here the rectifiability of the solitons plays a key role.

THEOREM 3.5.10 ([21]). *Let (M, g) be a complete n–dimensional, $n \geq 4$, locally conformally flat gradient ρ–Einstein soliton with $\rho < 0$ and $\lambda \leq 0$ or $\rho \geq 1/2$ and $\lambda \geq 0$. If (M, g) has positive sectional curvature, then it is rotationally symmetric.*

The same theorem also holds in the three–dimensional case, in which the Weyl part is always zero.

THEOREM 3.5.11 ([21]). *Let (M, g) be a three–dimensional gradient ρ–Einstein soliton with $\rho < 0$ and $\lambda \leq 0$ or $\rho \geq 1/2$ and $\lambda \geq 0$. If (M, g) has positive sectional curvature, then it is rotationally symmetric.*

We say that a Riemannian manifold is *rigid* if, for some $k \in \{0, \dots, (n-1)\}$, its universal cover, endowed with the lifted metric and the lifted potential function, is isometric to the Riemannian product $N^k \times \mathbb{R}^{n-k}$, where N^k is a k–dimensional Einstein manifold and $f = \frac{\lambda}{2}|x|^2$ on the Euclidean factor. We also recall that g has non negative radial sectional curvature if $\mathrm{Rm}(E, \nabla f, E, \nabla f) \geq 0$ for every vector field E. The following theorem is the analogous of a result by Petersen and Wylie [72] for the Ricci solitons, without the rectifiability assumpion, as this is always fulfilled, by Theorem 3.5.6.

THEOREM 3.5.12 ([22]). *Let (M, g) be a complete, non compact, gradient shrinking ρ–Einstein soliton with $0 < \rho \leq 1/2(n-1)$. If g has bounded curvature, non negative radial sectional curvature, and non negative Ricci curvature, then (M, g) is rigid.*

In particular, every complete, non compact, gradient shrinking ρ–Einstein soliton with $0 < \rho \leq 1/2(n-1)$ and non negative sectional curvature is rigid.

Among all the ρ–Einstein solitons, a class of particular interest is given by gradient Schouten solitons, namely Riemannian manifolds satisfying

$$\mathrm{Ric} + \nabla^2 f \;=\; \frac{\mathrm{R}}{2(n-1)}\, g \,+\, \lambda\, g\,,$$

for some smooth function f and some constant $\lambda \in \mathbb{R}$. In the steady case, we can prove the following triviality result, which holds true in every dimension without any curvature assumption.

THEOREM 3.5.13 ([21]). *Every complete gradient steady Schouten soliton is trivial, hence Ricci–flat.*

In particular, every complete three–dimensional gradient steady Schouten soliton is isometric to a quotient of $\mathbb{R}^3$. In analogy with Perelman's classification of three-dimensional gradient shrinking Ricci solitons [69], subsequently proved without any curvature assumption in [15], we have the following theorem.

THEOREM 3.5.14 ([21]). *Let (M, g) be a complete three–dimensional gradient shrinking Schouten soliton. Then, it is isometric to a finite quotient of either $\mathbb{S}^3$, or $\mathbb{R}^3$ or $\mathbb{R} \times \mathbb{S}^2$.*

CHAPTER 4

The Ricci–Bourguignon flow in dimension three

In [49] Hamilton introduced the Ricci flow in an attempt to prove the Poincaré conjecture, therefore, after having established the basic facts about the Ricci flow in any dimension, he turned his attention to the 3–dimensional case. Also in the case of the Ricci–Bourguignon flow it is natural to specialize our analysis to compact 3–manifolds for several reasons.
First of all, dimension three is the first significant case, as the RB–flow on surfaces is just a positive multiple of the Ricci flow for $\rho < 1/2$, hence the two flows behave the same way. Indeed, for every solution (in any dimension) of the Ricci flow and of the RB–flow, one can define a unique solution of the corresponding volume–normalized flow, which differs from the original one only by a time–dependent rescaling. In dimension two, the normalized flows both exist for all positive time and converge to a constant curvature metric conformal to the initial one (see [51]).
Moreover, in dimension three, the evolution equation of the Ricci tensor becomes treatable and useful to understand the behaviour of the RB–flow by means of the maximum principle, as we are going to see in Section 4.1. Finally, as we already said, our initial interest on the RB–flow came by the study of its gradient solitons carried out by Catino and Mazzieri in [21] and in the special 3–dimensional case, the rectifiability property (see Section 3.5) has strong consequences, which we review in Section 4.2.
In Section 4.3 we consider the case of a Ricci positive initial metric evolving by the RB–flow and we prove in Proposition 4.3.1 that, as the Ricci flow, the greater the scalar curvature becomes, the more pinched the sectional curvatures are. By Hamilton's Theorem [49, Theorem 1.1], we already know that a compact simply connected 3–manifold endowed with a Ricci positive metric is diffeomorphic to a sphere, because the normalized Ricci flow converges to a positive constant sectional curvature metric. Here, we try to prove that also the Ricci–Bourguignon, after parabolic rescaling, converges to a round metric. At the moment, we are able to prove this statement only assuming an extra hypothesis (see condition (4.8) in Theorem 4.3.3), since the integral estimate for the gradient of the scalar curvature proven in Section 4.3.1 is not sufficient to carry out Hamilton's argument. However, we hope that future results on Perelman–type functionals (see Section 5.2.1) could give an alternative way to prove such convergence by means of a blow–up procedure.

All the results proven in the previous chapter clearly remain true in dimension three. We summarize them in the following proposition.

PROPOSITION 4.0.15. *Let $\rho < \frac{1}{4}$ and (M, g_0) be a 3–dimensional compact Riemannian manifold. Then:*

- *there exists a unique smooth solution of the Ricci–Bourguignon flow (3.1) with initial metric g_0 in a maximal time interval $[0, T)$.*
- *The minimum of the scalar curvature is non decreasing during the flow. Moreover, if $\min \mathrm{R}_{g_0} \geq \alpha > 0$, then the maximal time of existence of the flow is finite and satisfies $T < \frac{3}{2(1-3\rho)\alpha}$.*
- *If the initial metric g_0 has non negative sectional curvature, the same holds for $g(t)$, for every $t \in [0, T)$.*
- *If $|\mathrm{Ric}_{g(t)}| \leq K$ on $M \times [0, T)$, then for every $k \in \mathbb{N}$ there exists a constant $C(\rho, k, K, T)$ such that for all $t \in (0, T]$*

$$\|\nabla^k \mathrm{Ric}_{g(t)}\|_2^2 \leq \frac{C}{t^k} \sup_{t \in [0,T)} \|\mathrm{Ric}_{g(t)}\|_2^2 \, .$$

- *If $T < +\infty$, then*

$$\limsup_{t\to T} \max_M |\mathrm{Ric}_{g(t)}| = +\infty \,.$$

4.1. Conditions preserved in dimension three

In general dimension it is very hard to find curvature conditions preserved by the flow, besides the non negativity of the scalar curvature and of the curvature operator (see Section 3.3) and this is due principally to the involved structure of the reaction term; for instance in the evolution equation (3.10) satisfied by the Ricci tensor, the reaction term involves the full curvature tensor. However, if we restrict our attention to the 3–dimensional case, the Weyl part of the Riemann tensor vanishes and all the geometric informations are encoded in the Ricci tensor.
With a straightforward computation, by plugging in the decomposition of the Riemann tensor in dimension three,

$$\mathrm{R}_{ijkl} = (\mathrm{Ric} \owedge g)_{ijkl} - \frac{\mathrm{R}}{4}(g \owedge g)_{ijkl} \,,$$

we can rewrite the general evolution equation for the Ricci tensor (3.10) in the following way

$$\begin{aligned} \frac{\partial}{\partial t}\mathrm{Ric} = &\Delta\mathrm{Ric} - \rho(\nabla^2\mathrm{R} + \Delta\mathrm{R}g) \\ &- 6\mathrm{Ric}^2 + 3\mathrm{R}\mathrm{Ric} + (2|\mathrm{Ric}|^2 - \mathrm{R}^2)g \,. \end{aligned} \tag{4.1}$$

REMARK 4.1.1. As we can easily see from the evolution equation (3.10) in any dimension for the Ricci tensor, the reaction term is the same of the corresponding evolution equation during the Ricci flow (see [49, Corollary 7.3]). Therefore, it should not be surprising that the same curvature conditions are preserved in dimension three, provided that the second order operator in equation (4.1) stays uniformly strongly elliptic and this is exactly the content of the following proposition (see [49, Section 9] for the Ricci flow).

PROPOSITION 4.1.2. *Let $(M, g(t))_{t\in[0,T)}$ be a solution of the RB–flow (3.1) on a compact 3–manifold. Then, if $\rho < 1/4$,*

(i) *non negative Ricci curvature is preserved along the flow;*
(ii) *non negative sectional curvature is preserved along the flow;*
(iii) *the pinching inequality $\mathrm{Ric} \geq \varepsilon \mathrm{R} g$ is preserved along the flow for any $\varepsilon \leq 1/3$.*

PROOF. In the original spirit of Hamilton's work on three manifold, we want to apply Theorem 1.5.3, hence we express the curvature conditions as the non negativity of suitable symmetric 2–tensors.
(i) First we have to show that $L(\mathrm{Ric}) = \Delta\mathrm{Ric} - \rho(\nabla^2\mathrm{R} + \Delta\mathrm{R}g)$ is strongly uniformly elliptic on $S^2(TM^*)$, then we compute its symbol. Let ξ be a generic unit covector in $p \in M$ and $\{\xi^\#, e_2, e_3\}$ an orthonormal basis in p at time t. The symbol has the following form

$$\sigma_\xi(L(\mathrm{Ric}))_{ik} = \mathrm{R}_{ik} - \rho\mathrm{R}(\delta_{i1}\delta_{k1} + \delta_{ik}) = \left(\begin{array}{ccc|c} 1-2\rho & -2\rho & -2\rho & \\ -\rho & 1-\rho & -\rho & 0 \\ -\rho & -\rho & 1-\rho & \\ \hline & 0 & & \mathrm{Id}_3 \end{array}\right)\begin{pmatrix} \mathrm{R}_{11} \\ \mathrm{R}_{22} \\ \mathrm{R}_{33} \\ \mathrm{R}_{12} \\ \mathrm{R}_{13} \\ \mathrm{R}_{23} \end{pmatrix} .$$

Hence, the eigenvalues are 1 with multiplicity 5 and $1 - 4\rho$ with multiplicity 1, which are all positive in the hypothesis of the proposition.
Then, we have to control the reaction term $F(\mathrm{Ric})$, which is clearly a polynomial in Ric. Therefore, let v be a unit null–eigenvector of Ric_p; let $\lambda, \mu \geq 0$ be the other eigenvalues and $\{v, e_2, e_3\}$ an orthonormal basis which diagonalizes Ric_p. Hence,

$$\begin{aligned} F_p(\mathrm{Ric}_p)(v,v) = &- 6\mathrm{Ric}_p^2(v,v) + 3\mathrm{R}_p\mathrm{Ric}_p(v,v) + 2|\mathrm{Ric}_p|^2|v|^2 - (\mathrm{R}_p)^2|v|^2 \\ = &\,2(\lambda^2+\mu^2) - (\lambda+\mu)^2 = (\lambda-\mu)^2 \geq 0 \,, \end{aligned}$$

as the norms of Ric_p and v are taken with respect to g_p and v is a null eigenvector of Ric_p^2 too.
(ii) Even if this condition is preserved in any dimension (Proposition 3.3.5), we give here a proof peculiar of the 3–dimensional case. It is easy to see that, if the sectional curvatures are $k_1 = \mathrm{K}(e_2, e_3)$, $k_2 = \mathrm{K}(e_1, e_3)$ and $k_3 = \mathrm{K}(e_1, e_2)$ with respect to an orthonormal basis of T_pM then the tensor $\mathrm{Sec} = \frac{1}{2}\mathrm{R}g - \mathrm{Ric}$ is diagonalized by the same basis $\{e_i\}$ with eigenvalues respectively $\{k_i\}$. Then we have to look at the evolution equation satisfied by Sec in order to control the sectional curvatures.
Therefore, since we have $\mathrm{S} = \mathrm{tr}(\mathrm{Sec}) = \mathrm{R}/2$ and $\mathrm{Ric} = \mathrm{S}g - \mathrm{Sec}$, we compute

$$\begin{aligned}\frac{\partial}{\partial t}\mathrm{Sec} = &-\frac{\partial}{\partial t}\mathrm{Ric} + \frac{1}{2}\Big(\frac{\partial}{\partial t}\mathrm{R}g + \mathrm{R}\frac{\partial}{\partial t}g\Big)\\
= &-\Delta\mathrm{Ric} + \rho(\nabla^2\mathrm{R} + \Delta\mathrm{R}g) + \frac{1}{2}(1-4\rho)\Delta\mathrm{R}g\\
&+ 6\mathrm{Ric}^2 - 3\mathrm{R}\mathrm{Ric} - (2|\mathrm{Ric}|^2 - \mathrm{R}^2)g + (|\mathrm{Ric}|^2 - \rho\mathrm{R}^2)g - \mathrm{R}(\mathrm{Ric} - \rho\mathrm{R}g)\\
= &\,\Delta\mathrm{Sec} + 2\rho(\nabla^2\mathrm{S} - \Delta\mathrm{S}g)\\
&\,6\mathrm{Sec}^2 - 4\mathrm{S}\mathrm{Sec} + (\mathrm{S}^2 - |\mathrm{Sec}|^2)g\,.\end{aligned}$$

As usual, we compute the symbol of the second order operator in the direction $\xi \in T_pM^*$ with respect to an orthonormal basis $\{\xi^\#, e_2, e_3\}$

$$\sigma_\xi(L(\mathrm{Sec}))_{ik} = \mathrm{S}_{ik} + 2\rho\mathrm{S}(\delta_{i1}\delta_{k1} - \delta_{ik}) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ -2\rho & 1-2\rho & -2\rho & 0\\ -2\rho & -2\rho & 1-2\rho & \\ \hline & 0 & & \mathrm{Id}_3\end{array}\right)\begin{pmatrix}\mathrm{S}_{11}\\ \mathrm{S}_{22}\\ \mathrm{S}_{33}\\ \mathrm{S}_{12}\\ \mathrm{S}_{13}\\ \mathrm{S}_{23}\end{pmatrix}.$$

Again, the eigenvalues are 1 with multiplicity 5 and $1-4\rho$ with multiplicity 1, then the operator is uniformly strongly elliptic. For the reaction term, if $\mathrm{Sec}_p \in S^2(T_pM^*)$ is diagonalized by the orthonormal basis $\{v, e_2, e_3\}$ with respect to the metric g_p with eigenvalues respectively $0, \lambda, \mu \geq 0$, we have

$$\begin{aligned}F_p(\mathrm{Sec}_p)(v,v) &= 6\mathrm{Sec}_p^2(v,v) - 4\mathrm{S}_p\mathrm{Sec}_p(v,v) + \mathrm{S}_p^2|v|^2 - |\mathrm{Sec}_p|^2|v|^2\\
&= (\lambda+\mu)^2 - (\lambda^2+\mu^2) = 2\lambda\mu \geq 0\,.\end{aligned}$$

Then the non negativity of Sec is preserved by the RB–flow, that is if $g(0)$ has non negative sectional curvatures, so has $g(t)$ as long as it exists.
(iii) In this last case we have to control if the non negativity of $\mathrm{Pin} = \mathrm{Ric} - \varepsilon\mathrm{R}g$ is preserved along the RB–flow.
First of all we observe that, for $\varepsilon = 1/3$, the condition $\mathrm{Pin}(0) \geq 0$ implies that the Ricci tensor of the initial metric is pointwise a multiple of the metric itself, hence $g(0)$ is an Einstein metric by Schur Lemma, that is, $\mathrm{Ric}(0) = \lambda g(0)$ for some constant $\lambda \in \mathbb{R}$. It is clear now that $g(t) = (1-2(1-3\rho)\lambda t)g(0)$ satisfies the flow equation (3.1) and so it is the unique maximal solution with initial data $g(0)$. Then $g(t)$ is Einstein and $\mathrm{Pin}(t) = \mathrm{Ric}(t) - 1/3\mathrm{R}(t)g(t) = \lambda g(0) - \lambda g(0) = 0$ for every t such that the solution exists, that is the condition is preserved.
For $\varepsilon > 1/3$ the tensor Pin has always at least one negative eigenvalue.

Now, for $\varepsilon < 1/3$, as $\mathrm{R} = \frac{1}{1-3\varepsilon}\mathrm{P}$ and $\mathrm{Ric} = \mathrm{Pin} + \frac{\varepsilon}{1-3\varepsilon}\mathrm{P}g$, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\mathrm{Pin} &= \frac{\partial}{\partial t}\mathrm{Ric} - \varepsilon\Big(\frac{\partial}{\partial t}\mathrm{R}g + \mathrm{R}\frac{\partial}{\partial t}g\Big)\\ &= \Delta\mathrm{Ric} - \rho(\nabla^2\mathrm{R} + \Delta\mathrm{R}g) - \varepsilon(1-4\rho)\Delta\mathrm{R}g\\ &\quad - 6\mathrm{Ric}^2 + 3\mathrm{R}\mathrm{Ric} + (2|\mathrm{Ric}|^2 - \mathrm{R}^2)g - 2\varepsilon(|\mathrm{Ric}|^2 - \rho\mathrm{R}^2)g + 2\varepsilon(\mathrm{R}\mathrm{Ric} - \rho\mathrm{R}^2 g)\\ &= \Delta\mathrm{Pin} - \rho\Big(\frac{1}{1-3\varepsilon}\nabla^2\mathrm{P} + \frac{1-4\varepsilon}{1-3\varepsilon}\Delta\mathrm{P}g\Big)\\ &\quad - 6\mathrm{Pin}^2 + \frac{3-10\varepsilon}{1-3\varepsilon}\mathrm{P}\mathrm{Pin} + \Big(2(1-\varepsilon)|\mathrm{Pin}|^2 - \frac{2\varepsilon^2-4\varepsilon+1}{1-3\varepsilon}\mathrm{P}^2\Big)g\,.\end{aligned}$$

Also in this case, we compute the symbol of the second order operator in the direction $\xi \in T_pM^*$ with respect to an orthonormal basis $\{\xi^\#, e_2, e_3\}$

$$\sigma_\xi(L(\mathrm{Pin}))_{ik} = \mathrm{P}_{ik} - \frac{\rho}{1-3\varepsilon}\mathrm{P}(\delta_{i1}\delta_{k1} + (1-4\varepsilon)\delta_{ik}) = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & \mathrm{Id}_3\end{array}\right)\begin{pmatrix}\mathrm{P}_{11}\\ \mathrm{P}_{22}\\ \mathrm{P}_{33}\\ \mathrm{P}_{12}\\ \mathrm{P}_{13}\\ \mathrm{P}_{23}\end{pmatrix},$$

where A is the matrix

$$A = \begin{pmatrix} 1-2\rho\frac{1-2\varepsilon}{1-3\varepsilon} & -2\rho\frac{1-2\varepsilon}{1-3\varepsilon} & -2\rho\frac{1-2\varepsilon}{1-3\varepsilon}\\ -\rho\frac{1-4\varepsilon}{1-3\varepsilon} & 1-\rho\frac{1-4\varepsilon}{1-3\varepsilon} & -\rho\frac{1-4\varepsilon}{1-3\varepsilon}\\ -\rho\frac{1-4\varepsilon}{1-3\varepsilon} & -\rho\frac{1-4\varepsilon}{1-3\varepsilon} & 1-\rho\frac{1-4\varepsilon}{1-3\varepsilon}\end{pmatrix}$$

Even if the computation is a bit more complicated, A has the usual structure and its eigenvalues are again 1 with multiplicity 2 and $1-4\rho$ with multiplicity 1, so the operator is uniformly strongly elliptic.

Finally, for the reaction term, if $\mathrm{Pin}_p \in S^2(T_pM^*)$ is diagonalized by the orthonormal basis $\{v, e_2, e_3\}$ with eigenvalues respectively $0, \lambda, \mu \geq 0$, we can estimate $|\mathrm{Pin}_p|^2 \geq \frac{1}{2}(\mathrm{P}_p)^2$ then

$$\begin{aligned}F_p(\mathrm{Pin}_p)(v,v) &= -6\mathrm{Pin}_p^2(v,v) + \frac{3-10\varepsilon}{1-3\varepsilon}\mathrm{P}_p\mathrm{Pin}_p(v,v) + 2(1-\varepsilon)|\mathrm{Pin}_p|^2|v|^2 - \frac{2\varepsilon^2-4\varepsilon+1}{1-3\varepsilon}\mathrm{P}_p^2|v|^2\\ &\geq \Big(1-\varepsilon-\frac{2\varepsilon^2-4\varepsilon+1}{1-3\varepsilon}\Big)P_p^2 = \frac{\varepsilon^2}{1-3\varepsilon}\mathrm{P}_p^2 \geq 0\,,\end{aligned}$$

and this concludes the proof. ☐

We can use the evolution equation (3.18) of the pull–back of the curvature operator, obtained by means of Uhlenbeck's trick in Section 3.2.2, to obtain more refined conditions preserved, since, in dimension 3, we can rewrite the tensorial ODE associated to the evolution of $\mathcal{P}$ as a system of ODEs in the eigenvalues of $\mathcal{P}$ that, by Proposition 1.4.7, are nothing but the sectional curvatures of $\mathcal{R}$. This point of view has been introduced for the Ricci flow by Hamilton in [56] and can be easily generalized to the Ricci–Bourguignon flow as follows.

LEMMA 4.1.3. *If $n = 3$, then $\mathcal{P}_p$ has 3 eigenvalues λ, μ, ν and the ODE fiberwise associated to equation (3.18) can be written as the following system*

$$\begin{cases}\frac{d\lambda}{dt} = 2\lambda^2 + 2\mu\nu - 4\rho\lambda(\lambda+\mu+\nu)\,,\\ \frac{d\mu}{dt} = 2\mu^2 + 2\lambda\nu - 4\rho\mu(\lambda+\mu+\nu)\,,\\ \frac{d\nu}{dt} = 2\nu^2 + 2\lambda\mu - 4\rho\nu(\lambda+\mu+\nu)\,.\end{cases} \tag{4.2}$$

In particular, if we assume $\lambda(0) \geq \mu(0) \geq \nu(0)$, then $\lambda(t) \geq \mu(t) \geq \nu(t)$ as long as the solution of the system exists.

PROOF. We can pointwise identify V_p endowed of an orthonormal frame to $\mathbb{R}^3$ with the standard basis. Then $\Lambda^2 V_p \simeq \mathfrak{so}(3)$ with the standard structure constants and if an algebraic operator Q_p is diagonal, both Q_p^2 and $Q_p^\sharp$ are diagonal with respect to the same basis (for the detailed computation of this fact, see [29, Chapter 6.4]). Hence the ODE $\frac{d}{dt}Q_p = F_p(Q_p)$, fiberwise associated to (3.18) preserves the eigenvalues of Q_p, that is, if $Q_p(0)$ is diagonal with respect to an orthonormal basis, $Q_p(t)$ stays diagonal with respect to the same basis and the ODE can be rewritten as the system (4.2) in its eigenvalues.
To prove the last statement, it is sufficient to observe that

$$\begin{aligned} \frac{d}{dt}(\lambda-\mu) &= 2(\lambda-\mu)\big((1-2\rho)(\lambda+\mu)-(1+2\rho)\nu\big)\,, \\ \frac{d}{dt}(\mu-\nu) &= 2(\mu-\nu)\big((1-2\rho)(\mu+\nu)-(1+2\rho)\lambda\big)\,. \end{aligned}$$

□

REMARK 4.1.4. We already proved that the differential operator in the evolution equation of $\mathcal{P}$ is elliptic if $\rho < 1/2(n-1)$, that is, $\rho < 1/4$ in dimension 3. Therefore any geometric condition expressed in terms of the eigenvalues is preserved along the RB–flow if the cone identified by the condition is closed, convex and preserved by the system (4.2). By using this method, we can reprove Proposition 3.3.1 and Proposition 4.1.2
(0) If $\mathrm{R}_{g_0} \geq 0$, then $\mathrm{R}_{g(t)} \geq 0$.
By Proposition 1.4.7 $(\mathrm{R}_{g(t)})_p = 2\mathrm{tr}_{g_0}\mathcal{P}_p(t) = 2(\lambda+\mu+\nu)(t)$. The cone

$$K_p = \{Q_p : (\lambda+\mu+\nu)(Q_p) \geq 0\}$$

is closed as the inequality is large, convex as the function $Q_p \mapsto \mathrm{tr}(Q_p)$ is linear and preserved by the system (4.2)

$$\begin{aligned} \frac{d}{dt}\mathrm{tr}(Q_p) &= 2(\lambda^2+\mu^2+\nu^2+\lambda\mu+\lambda\nu+\mu\nu) - 4\rho\,\mathrm{tr}(Q_p)^2 \\ &= (\lambda+\mu)^2+(\lambda+\nu)^2+(\mu+\nu)^2 - 4\rho\,\mathrm{tr}(Q_p)^2 \\ &\geq \frac{1}{3}(\lambda+\mu+\lambda+\nu+\mu+\nu)^2 - 4\rho\,\mathrm{tr}(Q_p)^2 \\ &= \frac{4}{3}(1-3\rho)\mathrm{tr}(Q_p)^2 \end{aligned}$$

By integrating the last inequality we recover also the estimate on the minimum of the scalar curvature

$$\begin{aligned} \min_M \mathrm{R}_{g(t)} &= 2\mathrm{tr}_{g_0}\mathcal{P}_p(t) \geq 2\frac{3\mathrm{tr}_{g_0}\mathcal{P}_p(0)}{3-4(1-3\rho)\mathrm{tr}_{g_0}\mathcal{P}_p(0)} \\ &= \frac{3(\mathrm{R}_{g_0})_p}{3-2(1-3\rho)(\mathrm{R}_{g_0})_p t} \geq \frac{3\min_M \mathrm{R}_{g_0}}{3-2(1-3\rho)\min_M \mathrm{R}_{g_0} t}\,. \end{aligned}$$

(i) If $\mathrm{Ric}_{g_0} \geq 0$, then $\mathrm{Ric}_{g(t)} \geq 0$.
The eigenvalues of Ric are the pairwise sums of the sectional curvatures, hence the condition is identified by the cone

$$K_p = \{Q_p : (\mu+\nu)(Q_p) \geq 0\}\,.$$

The closedness is obvious; in order to see that K_p is convex, we observe that the greatest eigenvalue can be characterized by $\lambda(Q_p) = \max\{Q_p(v,v) : v \in V_p \text{ such that } |v|_h = 1\}$, hence it is convex. Then the function $Q_p \mapsto \mu(Q_p)+\nu(Q_p) = \mathrm{tr}(Q_p) - \lambda(Q_p)$ is concave and this implies that its superlevels are convex. By system (4.2), we obtain

$$\frac{d}{dt}(\mu+\nu) = 2\mu^2+2\nu^2+2\lambda(\mu+\nu) - 4\rho(\mu+\nu)\mathrm{tr}(Q_p)\,.$$

We observe that $\mu \equiv 0 \equiv \nu$ is a stationary solution. Otherwise, whenever $\mu(t_0)+\nu(t_0) = 0$ with $\mu(t_0) \neq 0 \neq \mu(t_0)$, $\frac{d}{dt}(\mu+\nu)(t_0) = 2(\mu^2+\nu^2)(t_0) > 0$, then K_p is preserved.
(ii) If $\mathrm{Sec}_{g_0} \geq 0$, then $\mathrm{Sec}_{g(t)} \geq 0$.

This condition is the non negativity of $\mathcal{P}$, identified by the cone $K_p = \{Q_p : \nu(Q_p) \geq 0\}$, which is convex being a superlevel of a concave function. We suppose that $\nu(t_0) = 0$, then

$$\frac{d}{dt}\nu(t_0) = 2\lambda(t_0)\mu(t_0) \geq 0$$

because the order between the eigenvalues is preserved and therefore $\lambda(t_0) \geq \mu(t_0) \geq 0$.
(iii) For every $\varepsilon \in (0, 1/3]$, if $\mathrm{Ric}_{g_0} - \varepsilon \mathrm{R}_{g_0} g_0 \geq 0$, then $\mathrm{Ric}_{g(t)} - \varepsilon \mathrm{R}_{g(t)} g(t) \geq 0$.
Translating in terms of eigenvalues of $\mathcal{P}$, the condition means $\mu(Q_p) + \nu(Q_p) - 2\varepsilon \mathrm{tr}(Q_p) \geq 0$, that is $\lambda(Q_p) \leq \frac{1-2\varepsilon}{2\varepsilon}(\mu(Q_p) + \nu(Q_p))$, which determines the following cone

$$K_p = \{Q_p : \lambda(Q_p) - C(\varepsilon)\big(\mu(Q_p) + \nu(Q_p)\big) \leq 0\},$$

where $C(\varepsilon) \in [1/2, +\infty)$. The defining function is the sum of two convex function, hence its sublevels are convex. Now, for $C = 1/2$, that corresponds to $\varepsilon = 1/3$, we have $\lambda(0) = \mu(0) = \nu(0)$ at each point of M, that is the initial metric g_0 has constant sectional curvature and this condition is preserved along the flow.
For $C > 1/2$, we suppose $\lambda(t_0) = C(\mu(t_0) + \nu(t_0))$, then

$$\begin{aligned}\frac{d}{dt}(\lambda - C(\mu+\nu))(t_0) &= 2\big[\lambda^2 + \mu\nu - C(\mu^2 + \nu^2 + \lambda(\mu+\nu)) - 2\rho \mathrm{tr}(Q_p)(\lambda - C(\mu+\nu))\big](t_0)\\ &= 2\big[C^2(\mu(t_0)+\nu(t_0))^2 + \mu(t_0)\nu(t_0) - C(\mu(t_0)^2 + \nu(t_0)^2) - C^2(\mu(t_0)+\nu(t_0))^2\big]\\ &\leq (1-2C)(\mu(t_0)^2 + \nu(t_0)^2) \leq 0\,.\end{aligned}$$

In Section 4.3, we will use the system (4.2) to prove that the pinching inequality is not only preserved by the Ricci–Bourguignon flow, but improves, in the sense that, the larger the scalar curvature gets, the smaller the traceless Ricci tensor becomes, as it happens also for the Ricci flow (see [49, Theorem 10.1]).

4.1.1. Hamilton-Ivey estimate. A remarkable property of the 3–dimensional Ricci flow is the pinching estimate, independently proved by Hamilton in [55] and Ivey in [58], which says that positive sectional curvature "dominates" negative sectional curvature.
We generalize this pinching estimate and some consequences for positive values of the parameter ρ. In the same notation used before, let $\lambda \geq \mu \geq \nu$ be the ordered eigenvalues of the curvature operator.

THEOREM 4.1.5 (Hamilton-Ivey Estimate). *Let $(M, g(t))$ be a solution of the Ricci–Bourguignon on a compact 3–manifold such that the initial metric satisfies the normalizing assumption $\min_{p\in M} \nu_p(0) \geq -1$. If $\rho \in [0, 1/6])$, then at any point (p, t) where $\nu_p(t) < 0$ the scalar curvature satisfies*

$$\mathrm{R} \geq |\nu|\big(\log(|\nu|) + \log(1 + 2(1-6\rho)t) - 3\big)\,. \tag{4.3}$$

PROOF. We want to apply the maximum principle for time–dependent sets (Theorem 1.5.6), hence we need to express condition (4.3) in terms of a family of closed, convex, invariant subsets of $S^2(\Lambda^2 V^*)$, where $(V, h(t), D(t))$ is the usual bundle isomorphic to the tangent bundle defined via Uhlenbeck's Trick (Section 1.4). Following [29, Theorem 9.4], we consider the time–dependent subset $k(t)$ of $S^2(\Lambda^2 V^*)$ defined at every $p \in M$ by

$$K_p(t) = \left\{ Q_p : \begin{array}{l} \mathrm{tr}(Q_p) \geq -\dfrac{3}{1+2(1-6\rho)t} \\[2ex] \text{and if } \nu(Q_p) \leq -\dfrac{1}{1+2(1-6\rho)t} \text{ then} \\[2ex] \mathrm{tr}(Q_p) \geq |\nu(Q_p)|\big(\log(|\nu(Q_p)|) + \log(1+2(1-6\rho)t) - 3\big) \end{array} \right\}.$$

By Lemma 1.5.7, we altready know that, for any $t \in [0, T)$, $K(t)$ defines an invariant subset of $S^2(\Lambda^2 V^*)$. Since, for $\rho \in [0, 1/6)]$, $K(t)$ depends continuously on time, the space–time track of $K(t)$ is closed in $S^2(\Lambda^2 V^*)$.

Now we show that $K_p(t)$ is convex for every $p \in M$ and $t \in [0,T)$. Following [29, Lemma 9.5], we consider the map

$$\Phi : S^2(\Lambda^2 V_p^*) \to \mathbb{R}^2\,, \qquad \Phi(Q_p) = (|\nu(Q_p)|, \mathrm{tr}(Q_p))$$

Clearly, we have that $Q_p \in K_p(t)$ if and only if $\Phi(Q_p) \in A(t)$, where

$$A(t) = \left\{ (x,y) \in \mathbb{R}^2 : \begin{array}{c} y \geq -\dfrac{3}{1+2(1-6\rho)t}\,; \quad y \geq -3x\,; \\ \text{if } x \geq \dfrac{1}{1+2(1-6\rho)t} \text{ then} \\ y \geq x\big(\log x + \log(1+2(1-6\rho)t) - 3\big) \end{array} \right\}$$

is a convex subset of $\mathbb{R}^2$. Then, in order to show that $K_p(t)$ is convex is sufficient to show that the segment between any two algebraic operators in $K_p(t)$ is sent by the map Φ into $A(t)$.
Therefore let $Q_p, Q'_p \in K_p(t)$, $s \in [0,1]$ and $Q_p(s) = sQ_p + (1-s)Q'_p$. About the first defining condition for $A(t)$, the trace is a linear functional, hence it is obviously fullfilled by $Q_p(s)$, while the second condition is satisfied by any algebraic operator.
The third condition is a bit tricky. If $\nu(Q_p), \nu(Q'_p) > -\frac{1}{1+(1-6\rho)t}$ then the condition is empty for every point of the segment because ν is a concave function. By continuity we can assume, without loss of generality, that $\nu(Q_p(s)) \leq -\frac{1}{1+(1-6\rho)t}$, for every $s \in [0,1]$, hence $x(Q_p(s)) = -\nu(Q_p(s))$ is a convex function and $x(Q_p(s)) \leq sx(Q_p) + (1-s)x(Q'_p)$. On the other hand the second condition implies that $x(Q_p(s)) \geq -y(Q_p(s))/3 = -\frac{1}{3}(sy(Q_p) + (1-s)y(Q'_p))$. Then $\Phi(Q_p(s))$ belongs to the trapezium of vertices

$$\Phi(Q_p),\ \Big(-\frac{1}{3}y(Q_p), y(Q_p)\Big),\ \Phi(Q'_p),\ \Big(-\frac{1}{3}y(Q'_p), y(Q'_p)\Big)\,,$$

contained in $A(t)$, as its vertexes are and $A(t)$ is convex.
Now we prove that $K_p(t)$ is preserved by the system (4.2). By taking the sum of the three equations in the system (see also Remark 4.1.4) we get

$$\frac{d}{dt}\mathrm{tr}(Q_p) \geq \frac{4}{3}(1-3\rho)\mathrm{tr}(Q_p)^2\,.$$

By hypothesis, $\nu(Q_p)(0) \geq -1$, hence $\mathrm{tr}(Q_p)(0) \geq -3$ for every $p \in M$ and by integrating the previous inequality,

$$\mathrm{tr}(Q_p)(t) \geq -\frac{3}{1+4(1-3\rho)t} \geq -\frac{3}{1+2(1-6\rho)t}\,,$$

which holds for any ρ.
In order to prove that the second inequality is preserved too, we consider, for every $p \in M$ such that $\nu(Q_p)(0) < 0$, the function

$$f(t) = \frac{\mathrm{tr}(Q_p)}{-\nu(Q_p)} - \log(-\nu(Q_p)) - \log(1+2(1-6\rho)t) \tag{4.4}$$

and we compute its derivative along the flow.

$$\begin{aligned}
\frac{d}{dt}f &= \frac{1}{\nu^2}\big[(-2\nu)\big(\lambda^2+\mu^2+\nu^2+\lambda\mu+\lambda\nu+\mu\nu-2\rho(\lambda+\mu+\nu)^2\big) \\
&\quad + 2(\lambda+\mu+\nu)\big(\nu^2+\lambda\mu-2\rho\nu(\lambda+\mu+\nu)\big)\big] - \frac{2}{\nu}\left(\nu^2+\lambda\mu-2\rho\nu(\lambda+\mu+\nu)\right) - \frac{2(1-6\rho)}{1+2(1-6\rho)t} \\
&= \frac{2}{\nu^2}\big[-\nu(\lambda^2+\mu^2+\lambda\mu)+\lambda\mu(\lambda+\mu)-\nu^3+2\rho\nu^2(\lambda+\mu+\nu)\big] - \frac{2(1-6\rho)}{1+2(1-6\rho)t}
\end{aligned}$$

As for the Ricci flow, it is easy to see that the quantity $-\nu(\lambda^2+\mu^2+\lambda\mu)+\lambda\mu(\lambda+\mu)$ is always non negative if $\nu<0$, hence, we get

$$\frac{d}{dt}f(t)\geq -2\nu+4\rho(\lambda+\mu+\nu)-\frac{2(1-6\rho)}{1+2(1-6\rho)t} \tag{4.5}$$

If $\rho\geq 0$ we can estimate $\lambda+\mu+\nu\geq 3\nu$ and obtain

$$\frac{d}{dt}f(t)\geq -2(1-6\rho)\big(\nu+\frac{1}{1+2(1-6\rho)t}\big)\geq 0$$

whenever $\nu\leq -\frac{1}{1+2(1-6\rho)t}$ and $\rho\leq 1/6$.
Hence, if (λ,μ,ν) is a solution of system (4.2) in $[0,T)$ with $(\lambda(0),\mu(0),\nu(0))\in K_p(0)$, we suppose that there is $t_1>0$ such that $\nu(t_1)<-\frac{1}{1+2(1-6\rho)t_1}$. Then, either $\nu(t)<-\frac{1}{1+2(1-6\rho)t}$ for any $t\in[0,t_1]$, either there exists $t_0<t_1$ such that $\nu(t_0)=-\frac{1}{1+2(1-6\rho)t_0}$ and $\nu(t)<-\frac{1}{1+2(1-6\rho)t}$ for any $t\in(t_0,t_1]$. In the first case, by hypothesis we obtain $f(0)\geq -3$ and $\frac{d}{dt}f(t)\geq 0$ for any $t\in[0,t_1]$, therefore $f(t_1)\geq -3$; in the second case $f(t_0)=\frac{(\lambda+\mu+\nu)(t_0)}{-\nu(t_0)}\geq -3$ and $\frac{d}{dt}f(t)\geq 0$ for any $t\in[t_0,t_1]$, therefore again $f(t_1)\geq -3$, which is equivalent to the second inequality. □

REMARK 4.1.6. The extra term $4\rho(\lambda+\mu+\nu)$ in equation (4.5) requires strong assumptions on the parameter ρ since we have no information on the sign of the trace. However, combining the previous proof with Proposition 3.3.4, we can enlarge the range of ρ to $[0,1/4)$ and conclude that an ancient solution to the RB–flow on a compact 3–manifold has non negative sectional curvature for any value of $\rho\in[0,1/4)$. This is the content of the following proposition, analogous to [29, Corollary 9.8] for the Ricci flow, which holds also in the complete non compact case, with the extra hypothesis of bounded curvature on each fixed time slice.

PROPOSITION 4.1.7. *Let $(M,g(t))$ be an ancient solution of the RB–flow on a compact 3–manifold with $\rho\in[0,1/4)$. Then $g(t)$ has non negative sectional curvature as long as it exists.*

PROOF. For $\rho\in[0,1/6)$, the scalar curvature satisfies inequality (4.3) by the previous proposition.
For $\rho\in[1/6,1/4)$, we need to consider a slightly different time–dependent set $K(t)$ defined pointwise by:

$$K_p(t)=\left\{Q_p:\ \begin{array}{l}\mathrm{tr}(Q_p)\geq -\dfrac{3}{1+4(1-3\rho)t}\\[2ex] \text{and if } \nu(Q_p)\leq -\dfrac{1}{1+4(1-3\rho)t}\ \text{then}\\[2ex] \mathrm{tr}(Q_p)\geq|\nu(Q_p)|\big(\log(|\nu(Q_p)|)+\log(1+4(1-3\rho)t)-3\big)\end{array}\right\}.$$

Clearly, as before, $K(t)$ defines a convex, closed, invariant subset of $S^2(\Lambda^2V^*)$. Moreover, we already proved that the first defining condition on the trace is preserved.
For the second inequality, we consider

$$f(t)=\frac{\mathrm{tr}(Q_p)}{-\nu(Q_p)}-\log(-\nu(Q_p))-\log(1+4(1-3\rho)t)$$

and, with the same computation as before, we obtain

$$\frac{d}{dt}f(t)\geq -2\nu+4\rho(\lambda+\mu+\nu)-\frac{4(1-3\rho)}{1+4(1-3\rho)t}\,.$$

Since $g(t)$ is an ancient solution, by Proposition 3.3.4, the scalar curvature is non negative, then we can drop the extra term $4\rho(\lambda+\mu+\nu)\geq 2\rho\mathrm{R}\geq 0$. Hence we get, for $\nu\leq-\frac{1}{1+4(1-3\rho)t}$,

$$\frac{d}{dt}f(t)\geq 2\Big(-\nu-\frac{4(1-3\rho)}{1+4(1-3\rho)t}\Big)\geq 2\frac{1-2+6\rho}{1+4(1-3\rho)t}\geq 0\,,$$

and we can conclude that the scalar curvature satisfies the following inequality

$$\mathrm{R} \geq |\nu|\big(\log(|\nu|) + \log(1 + 4(1-3\rho)t) - 3\big)\,.$$

In both cases, we can repeat the proof of [29, Corollary 9.8]. We suppose that $\nu_0 = \inf_M \nu(0) < 0$ and we consider the usual parabolic rescaling

$$\tilde{g}(t) = |\nu_0| g\Big(\frac{t}{|\nu_0|}\Big)\,,$$

which is a solution of the Ricci–Bourguignon flow satisfying the normalizing assumption $\tilde{\nu}(0) \geq -1$. Hence the scalar curvature satisfies, at every $(x, |\nu_0|t)$ such that $\tilde{\nu} \leq 0$,

$$\widetilde{\mathrm{R}} \geq |\tilde{\nu}|\big(\log(|\tilde{\nu}|) + \log(1 + C|\nu|t) - 3\big)\,,$$

where C is a positive constant. Since $\widetilde{\mathrm{R}} = |\nu_0|\mathrm{R}$ and $\tilde{\nu} = |\nu_0|\nu$, we get, for $g(t)$,

$$\mathrm{R} \geq |\nu|\big(\log(|\nu|) + \log(|\nu_0|^{-1} + Ct) - 3\big)\,.$$

By translating backward in time, since the solution is ancient and exists at least for a small positive time interval, we get, for every $\alpha \geq 0$,

$$\mathrm{R} \geq |\nu|\big(\log(|\nu|) + \log(C(t+\alpha)) - 3\big)\,,$$

which leads to a contradiction for $\alpha \to +\infty$, as the curvature is bounded on a compact manifold. □

4.2. Solitons

In Section 3.5 we have already listed some results about ρ–Einstein gradient solitons, studied by Catino and Mazzieri in [21, 22]. In this section we focus on the 3–dimensional case, since the results in this peculiar case have originally motivated the study of the Ricci–Bourguignon flow.

THEOREM 4.2.1. [21, Theorem 3.4] *Let (M, g) a 3–dimensional gradient ρ–Einstein soliton with $\rho < 0$ and $\lambda \leq 0$ or $\rho \geq 1/2$ and $\lambda \geq 0$. If (M, g) has positive sectional curvature, then it is rotationally symmetric.*

Indeed, by the soliton equation (3.24), for $\rho \geq 1/2$ and $\lambda \geq 0$ the potential function f is strictly convex (strictly concave in the other case), M is diffeomorphic to $\mathbb{R}^3$ and the potential function has exactly one critical point O. Moreover, for $\rho \neq 0$, the potential function f is always rectifiable (in any dimension) and the manifold is foliated by constant mean curvature surfaces around every regular connected component of a level set of f. This implies that every level set, besides $\{O\}$, is isometric to $(\mathbb{S}^2, g_{\mathbb{S}^2})$, up to a constant factor and the metric g, on $M \setminus \{O\}$, has the form

$$g = dr \otimes dr + \omega(r)^2 g_{\mathbb{S}^2}\,,$$

where $r(\cdot) = dist(O, \cdot)$ and $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ is a smooth positive function.
Moreover, by studying complete, non compact, gradient steady ρ–Einstein solitons which are warped product in [21, Theorem 4.3], the authors proved the following theorem.

THEOREM 4.2.2. [21, Corollary 4.4] *Up to homotheties, there is only one complete 3–dimensional gradient steady ρ–Einstein soliton, with $\rho < 0$ or $\rho \geq 1/2$ and positive sectional curvature.*

We remark that, in the k–noncollapsed case (see [69, Definition 4.2]), for $\rho = 0$, this is Perelman's claim [69], proved by Brendle in [12], who showed that the Bryant soliton is the only complete non compact k–noncollapsed gradient steady Ricci soliton with positive sectional curvature in dimension three. For $\rho = 1/2$, the only admissible 3–dimensional gradient steady Einstein soliton with positive sectional curvatures turns out to be asymptotically cylindrical with linear volume growth, the natural generalization of the 2–dimensional Hamilton's cigar.
For the Schouten case, $\rho = 1/4$, the authors prove the following result

THEOREM 4.2.3. [21] *Every complete 3–dimensional gradient steady Schouten soliton is isometric to a quotient of $\mathbb{R}^3$. Every complete 3–dimensional gradient shrinking Schouten soliton is isometric to a finite quotient of either $\mathbb{S}^3$ or $\mathbb{R}^3$ or $\mathbb{R} \times \mathbb{S}^2$.*

The last statement is analogous to Perelman's classification of three–dimensional gradient shrinking Ricci solitons [69], subsequently proved without any curvature assumption in [15].

After the Ricci and the Schouten solitons, we focus on the intermediate values of the parameter, i.e. $\rho \in (0, 1/4)$. By Theorem 3.5.8, we already know that every compact steady or expanding soliton is Einstein. By soliton equation (3.24), this implies that a steady soliton is Ricci–flat, hence in dimension three is flat, isometric to a finite quotient of $\mathbb{R}^3$. For $\lambda < 0$, it follows that a 3–dimensional expanding soliton has constant negative curvature, hence it is isometric to a finite quotient of the iperbolic space $\mathbb{H}^3$.

To complete the picture in the compact setting, there is left the case $\lambda > 0$. Again, by the triviality result, we know that a compact shrinking soliton has positive scalar curvature and also non negative sectional curvature, as it is an ancient solution, thus Proposition 4.1.7 applies.
For the Ricci solitons, Ivey in [58] proved that in this case the soliton is isometric to a finite quotient of the sphere $\mathbb{S}^3$ and we would like to extend this result to $\rho \in (0, 1/4)$.
The proof by Ivey mainly goes as follows: if the Ricci tensor is strictly positive, the (normalized) Ricci flow converges to a (positive) constant sectional curvature metric, by Hamilton's sphere theorem in [49], while the case when the Ricci tensor has a null eigenvalue is ruled out by means of a splitting result proven by Hamilton in [50].
In Section 4.3 we study the RB–flow starting from a Ricci positive metric on a compact 3–manifold and we prove the convergence of the rescaled flow (after a blow–up procedure) to a round metric, but we have to assume an extra hypothesis (see Theorem 4.3.3). Therefore, in order to classify 3–dimensional compact shrinking ρ–Einstein solitons, it seems more promising to follow (instead of Ivey's line) the static point of view and try to adapt the arguments of [37, Proposition 3.7].

For complete non compact shrinking solitons, there is the rigidity Theorem 3.5.12 proven by Catino, Mazzieri and Mongodi in [22], assuming non negative radial sectional curvature and non negative Ricci curvature. Therefore, one could try to prove a local version of Hamilton–Ivey estimate (Theorem 4.1.5), as it has been done by Chen in [24, Section 2], to conclude that any complete non compact 3–dimensional shrinking soliton with bounded curvature is rigid.

4.3. The Ricci positive case

In Section 4.1 we have seen that the Ricci–Bourguignon flow, as the Ricci flow, prefers positive curvature. Thanks to the system (4.2), we can also prove that, in the positive Ricci case, the metric is "almost" Einstein when the scalar curvature is large, that is the traceless Ricci tensor becomes smaller and smaller compared to R. This is the content of the following proposition, which is the analogous of [50, Thm. 5.3], proven by Hamilton for the Ricci flow.

PROPOSITION 4.3.1. *Let $\rho < 1/4$ and $(M, g(t))$ for $t \in [0, T)$ be a solution of the RB–flow on a compact 3–dimensional manifold M with initial metric g_0. If $\mathrm{Ric}_{g_0} > 0$, then there exist two constants $\delta \in (0, 1)$ and $C > 0$, depending on the initial metric g_0 and on the parameter ρ, such that*

$$\left|\mathrm{Ric}_{g(t)} - \frac{1}{3}\mathrm{R}_{g(t)}g(t)\right|_{g(t)} \leq C\mathrm{R}^{1-\delta}_{g(t)}$$

for every $t \in [0, T)$.

PROOF. We start observing that, in dimension three, to control the traceless Ricci tensor, it is sufficient to control the difference between the greatest and the smallest sectional curvatures. Indeed, as the eigenvalues of Ric are the pairwise sums of the sectional curvatures $\lambda \geq \mu \geq \nu$, we have

$$\begin{aligned}
\left|\mathrm{Ric} - \frac{1}{3}\mathrm{R}g\right|^2 &= |\mathrm{Ric}|^2 - \frac{1}{3}\mathrm{R}^2 \\
&= (\lambda+\mu)^2 + (\lambda+\nu)^2 + (\mu+\nu)^2 - \frac{1}{3}\big(2(\lambda+\mu+\nu)\big)^2 \\
&= \frac{1}{3}\big[(\lambda-\mu)^2 + (\lambda-\nu)^2 + (\mu-\nu)^2\big] \leq (\lambda-\nu)^2 .
\end{aligned}$$

Then, we only need to prove that there exist $C > 0$ and $\delta \in (0,1)$ such that

$$\lambda - \nu \leq C(\lambda + \mu + \nu)^{1-\delta}\,.$$

Since the manifold M is compact and the Ricci tensor of the initial metric is positive, we have that

- the scalar curvature of g_0 is bounded on both sides by positive constants, therefore there exist two constants $C' \geq c' > 0$ such that $c' \leq \lambda(0) + \mu(0) + \nu(0) \leq C'$;
- the Ricci tensor of g_0 is bounded below by some positive multiple of its trace–part, that is, by Remark 4.1.4, there exists $C'' \geq 1/2$ such that $\lambda(0) \leq C''(\mu(0) + \nu(0))$.

Thanks to the vectorial maximum principle 1.5.5, the proof is complete is we show that there exist $\delta = \delta(C'', \rho) \in (0,1)$ and $C = C(C', C'', \rho)$ such that the cone

$$K_p = \left\{ \begin{array}{ll} & \lambda(Q_p) + \mu(Q_p) + \nu(Q_p) \geq c' \\ Q_p : & \lambda(Q_p) - C''(\mu(Q_p) + \nu(Q_p)) \leq 0 \\ & \lambda(Q_p) - \nu(Q_p) - C(\mathrm{tr}(Q_p))^{1-\delta} \leq 0 \end{array} \right\}$$

is closed, convex and preserved by the system (4.2), where above and in the following we write $\mathrm{tr}(Q_p)$ for $\lambda + \mu\nu$. We already know, by Proposition 4.1.2, that the first two conditions are closed, convex and preserved along the flow, then we focus on the third inequality. The closedness is again obvious since the inequality is large; about the convexity, by Lemma 1.5.7, we have to observe that λ and $-\nu$ are convex functions; moreover, the trace is linear and $\delta \in (0,1)$, then also $-C(\mathrm{tr}(Q_p))^{1-\delta}$ is convex and the sum of convex functions is convex.
If we suppose that at some point and time (p, t_0) there holds $\lambda(Q_p)(t_0) = \nu(Q_p)(t_0)$, then all the sectional curvature are equal in p for every $t \geq t_0$ and the inequality is trivially satisfied. Otherwise, we can assume $\lambda(Q_p)(t) > \nu(Q_p)(t)$ and, following the analogous computation for the Ricci flow (see [50, Thm. 5.3]), we show that there exists δ such that the logarithm of the ratio between $\lambda - \nu$ and $\mathrm{tr}(Q_p)^{1-\delta}$ is non increasing along the flow.

$$\begin{aligned}
\frac{d}{dt}\log\left(\frac{\lambda-\nu}{\mathrm{tr}(Q_p)^{1-\delta}}\right) &= \frac{2}{\lambda-\nu}\big(\lambda^2 - \nu^2 + \mu\nu - \lambda\mu - 2\rho(\lambda-\nu)\mathrm{tr}(Q_p)\big) \\
&\quad - \frac{2(1-\delta)}{\mathrm{tr}(Q_p)}\big(\lambda^2 + \mu^2 + \nu^2 + \lambda\mu + \lambda\nu + \mu\nu - 2\rho\,\mathrm{tr}(Q_p)^2\big) \\
&= 2\delta\lambda + 2\nu - 2\mu - 2(1-\delta)\frac{(\mu^2 + \nu^2 + \mu\nu)}{\mathrm{tr}(Q_p)} - 4\rho\delta\mathrm{tr}(Q_p) \\
&= 2\delta(\lambda + \nu - \mu) - 2(1-\delta)\frac{2\mu^2 + \lambda(\mu-\nu) + \mu\nu}{\mathrm{tr}(Q_p)} - 4\rho\delta\mathrm{tr}(Q_p)\,.
\end{aligned}$$

To deal with the second term we use that $2\mu^2 \geq (\mu+\nu)^2/2$, $\lambda(\mu-\nu) \geq 0$, $\mu\nu \geq 0$ and $\mathrm{tr}(Q_p) \leq 3\lambda$, hence,

$$\frac{2\mu^2 + \lambda(\mu-\nu) + \mu\nu}{\mathrm{tr}(Q_p)} \geq \frac{(\mu+\nu)^2}{6\lambda} \geq \frac{\mu+\nu}{6C''}\,,$$

by the second inequality defining the cone. Putting these estimates together we have

$$\begin{aligned}
\frac{d}{dt}\log\left(\frac{\lambda-\nu}{\mathrm{tr}(Q_p)^{1-\delta}}\right) &\leq 2\delta(\lambda+\nu-\mu) - \frac{1-\delta}{3C''}(\mu+\nu) - 4\rho\delta\mathrm{tr}(Q_p) \\
&\leq \left(2C''\delta - \frac{1-\delta}{3C''}\right)(\mu+\nu) - 4\rho\delta\mathrm{tr}(Q_p)\,.
\end{aligned}$$

If $\rho \geq 0$ the last term is non positive, hence the ratio is decreasing along the flow if $\delta < 1/(1 + 6(C'')^2)$. If ρ is negative we estimate again $\mathrm{tr}(Q_p) \leq 3\lambda \leq 3C''(\mu+\nu)$ and the ratio is negative if $\delta < 1/(1 + 6(1-6\rho)(C'')^2)$.
By this computation, in order to conclude the proof we only have to show that the third inequality is satisfied for some C at the initial time and this can be done by estimating

$$\lambda(0) - \nu(0) \leq \lambda(0) \leq C''(\mu(0) + \nu(0)) \leq C''\mathrm{tr}(Q_p)(0) \leq C''(C')^{\delta}\mathrm{tr}(Q_p)^{1-\delta}(0)\,.$$

□

The previous estimate can be written as

$$\frac{|\overset{\circ}{\mathrm{Ric}}_{g(t)}|_{g(t)}}{\mathrm{R}_{g(t)}} \leq \frac{C}{\mathrm{R}^{\delta}_{g(t)}}\,, \tag{4.6}$$

where the left hand side is invariant under homotheties of the metric while the right hand side goes to zero where the scalar curvature blows up.
The next lemma shows that, starting with a Ricci positive metric, actually the scalar curvature of the solution must blow up somewhere for $t \to T$, the finite maximal time of existence.

LEMMA 4.3.2. *Let $\rho < 1/4$ and $(M, g(t))$ be the maximal solution of the RB–flow on the compact 3–dimensional manifold M with initial metric g_0. If $\mathrm{Ric}_{g_0} > 0$, then the maximum of the scalar curvature satisfies the following inequality during the RB–flow*

$$\mathrm{R}_{\max}(t) = \max_{q\in M}(\mathrm{R}_{g(t)})_q \geq \frac{1}{2(1-\rho)(T-t)}\,, \tag{4.7}$$

where T is the maximal time of existence of the solution. In particular

$$\int_0^T \mathrm{R}_{\max}(t)dt = +\infty$$

PROOF. Since the condition $\mathrm{Ric} > 0$ is preserved along the flow, there holds $|\mathrm{Ric}|^2 \leq \mathrm{R}^2$. Therefore, by the evolution equation (3.11) of the scalar curvature, we get

$$\frac{\partial}{\partial t}\mathrm{R} \leq (1-4\rho)\Delta\mathrm{R} + 2(1-\rho)\mathrm{R}^2\,.$$

Hence, using again Hamilton's trick (see [50], [63, Lemma 2.1.3]), at every differentiable time $t \in (0, T)$, we have that $\mathrm{R}_{\max}(t)$ satisfies

$$\frac{d}{dt}\mathrm{R}_{\max}(t) \leq 2(1-\rho)\mathrm{R}^2_{\max}(t)\,.$$

By integrating the previous inequality in $[s, s']$ we obtain

$$\frac{1}{\mathrm{R}_{\max}(s)} - \frac{1}{\mathrm{R}_{\max}(s')} \leq 2(1-\rho)(s'-s)\,.$$

To end the proof we recall that, since the initial metric has positive scalar curvature, by Proposition 3.3.1, the maximal solution has a finite time of existence T; hence, by Theorem 3.4.4, $\limsup_{t\to T}\sup_M|\mathrm{Rm}| = +\infty$.
In dimension three the Riemann tensor is controlled by the Ricci tensor, which is positive and so controlled by the scalar curvature $|\mathrm{Rm}| \leq C|\mathrm{Ric}| \leq C\mathrm{R}$, therefore we obtain $\limsup_{t\to T}\mathrm{R}_{\max} = +\infty$. Plugging in the parameter s' of the previous inequality a sequence s_i that realizes this $\limsup$ and letting $i \to \infty$ we get inequality (4.7). □

We have proven in Proposition 4.3.1 that where the scalar curvature is getting large the Ricci curvature gets close to a multiple of the metric. Now, we recall Schur Lemma, which states that, in dimension at least three, if the Ricci tensor is a pointwise multiple of the metric, the same holds globally, hence, the metric has constant curvature. By Theorem 3.4.4 and Lemma 4.3.2 the scalar curvature must blow up when t is approaching the maximal time T, which, by Proposition 3.3.1, is finite.
The next step in the proof of Hamilton's sphere theorem [49, Theorem 1.1] is to show that the scalar curvature must blow up at the same rate in any point of the manifold, proving an estimate on the gradient of the scalar curvature (see [49, Section 11]. This allowed Hamilton to prove that the normalized flow exists for every positive time and it converges, for $t \to +\infty$, to a positive constant curvature metric on M.

At the moment, we are not able to prove a pointwise estimate for the quantity $|\nabla\mathrm{R}|^2$, but only an integral one. In Section 4.3.1 we explain the main obstacle to get the precise analogous of Hamilton's estimate and the integral alternative. Unfortunately, the estimates proven so far are not sufficient to get a control on the quantity $\mathrm{R}_{\min}/\mathrm{R}_{\max}$, when $t \to T$, therefore we assume such control and we prove a weaker version of Hamilton's sphere theorem for the Ricci–Bourguignon flow (see [77, Theorem 10.1.1] in the Ricci flow case).

THEOREM 4.3.3. *Let $\rho < 1/4$ and $(M, g(t))$, $t \in [0, T)$, be the maximal solution of the Ricci–Bourguignon flow with initial metric g_0 on the compact 3–manifold M. Suppose that g_0 has positive Ricci curvature and there holds*

$$\lim_{t\to T} \frac{\min_{p\in M} \mathrm{R}_{g(t)}}{\max_{p\in M} \mathrm{R}_{g(t)}} = \alpha > 0\,. \tag{4.8}$$

Then there exist

- *a sequence $t_i \nearrow T$,*
- *a sequence $(p_i) \subset M$,*

such that the new Ricci–Bourguignon flows $g_i(t)$ defined for $t \le 0$ by

$$g_i(t) = \mathrm{R}_{g(t_i)}(p_i) g\Big(t_i + \frac{t}{\mathrm{R}_{g(t_i)}(p_i)}\Big)\,,$$

satisfy

$$(M, g_i(t), p_i) \to (M, (c - t)g_\infty, p)$$

on the time interval $(-\infty, 0]$, for some $c > 0$, where $p \in M$ and the metric g_∞ on M has constant positive sectional curvature. Hence (M, g_∞) is isometric to a quotient of the round sphere $\mathbb{S}^3$.

PROOF. By Proposition 4.3.2, there exists a sequence $(p_i, t_i) \in M \times [0, T)$, such that $t_i \nearrow T$ and

$$M_i = \mathrm{R}_{g(t_i)}(p_i) = \max_{p\in M, t\in[0,t_i]} \mathrm{R}_{g(t)}(p) \xrightarrow{i\to\infty} +\infty\,.$$

Now we perform, for each index $i \in \mathbb{N}$, a parabolic rescaling of the flow by the quantity M_i

$$g_i(t) = M_i g\Big(t_i + \frac{t}{M_i}\Big)\,.$$

and we want to apply the compactness theorem for a sequence of Ricci flows proved by Hamilton (see [54] and also [25, Theorem 3.10]), adapted to the Ricci–Bourguignon case, in the time interval $[-a, 0]$, for each $a > 0$.

The sequence of metrics g_i satisfies the following properties.

i) g_i is defined in $[-T_i, TM_i - T_i)$, where $T_i = t_i M_i \to +\infty$ and $TM_i - T_i \ge 1/2(1 - \rho) > 0$, by Proposition 4.3.2.

ii) g_i is again a solution of the RB–flow. Indeed, by observing the basic scaling properties of the Ricci tensor and the scalar curvature for every $\lambda \in \mathbb{R}^+$

$$\mathrm{Ric}_{\lambda g} = \mathrm{Ric}_g\,, \qquad \mathrm{R}_{\lambda g} = \frac{1}{\lambda}\mathrm{R}_g\,,$$

we have that

$$\begin{aligned}
\frac{\partial}{\partial t} g_i(t) &= M_i \frac{\partial}{\partial t} g\Big(t_i + \frac{t}{M_i}\Big) \\
&= -2\Big(\mathrm{Ric}_{g(t_i+t/M_i)} - \rho \mathrm{R}_{g(t_i+t/M_i)} g(t_i + t/M_i)\Big) \\
&= -2\big(\mathrm{Ric}_{g_i(t)} - \rho \mathrm{R}_{g_i(t)} g_i(t)\big)\,.
\end{aligned}$$

iii) For each $a > 0$ there exist $\varepsilon > 0$ and i_0 such that for every $i \ge i_0$ and $t \in [-a, 0]$, there holds

$$\varepsilon \le \mathrm{R}_{g_i(t)} \le 1\,.$$

Indeed, by the parabolic rescaling, the right inequality is true for every $i \in \mathbb{N}$, and $t \in [-T_i, 0]$. For the lower bound, let i_0 such that $T_{i_0} \ge a$. Since the minimum of the scalar curvature is non decreasing during the RB–flow by Proposition 3.3.1, we obtain

$$\min_{p\in M} \mathrm{R}_{g_i(t)} \ge \min_{p\in M} \mathrm{R}_{g_i(-a)} = \frac{\min_{p\in M} \mathrm{R}_{g(t_i - a/M_i)}}{M_i} \ge (1 - \varepsilon)\alpha \ge \varepsilon\,,$$

where we used condition (4.8) and the fact that $t_i - a/M_i \to T$ for $i \to \infty$.
iv) g_i has the infinite norm of the Riemann tensor uniformly bounded

$$\sup_i \max_{t\in[-a,0]} \|\mathrm{Rm}_{g_i(t)}\|_\infty \leq C\,.$$

Indeed, since each metric is only a parabolic rescaling of the original RB–flow solution $g(t)$, the Ricci tensor is again positive and controlled by the scalar curvature along each flow g_i and we already saw that in dimension three this is sufficient to have an upper bound for the norm of the Riemann tensor, therefore

$$\sup_i \max_{t\in[-a,0]} \|\mathrm{Rm}_{g_i(t)}\|_\infty \leq C \max_{p\in M, t\in[-a,0]} \mathrm{R}_{g_i(t)}(p) \leq C$$

by point (iii).
v) g_i has the L^2–norm of the Riemann tensor uniformly bounded in $[-a,0]$

$$\sup_i \max_{t\in[-a,0]} \|\mathrm{Rm}_{g_i(t)}\|_2 \leq C'\,.$$

indeed, by point (iv), we only need to prove that the volume is uniformly bounded in $[-a,0]$. Therefore, we recall the scaling properties of the standard volume form

$$\mathrm{Vol}(\lambda g) = \lambda^{3/2}\mathrm{Vol}(g)\,.$$

By Bishop–Gromov volume comparison result (see, for instance, [73, Section 9.1.2]), comparing $g_i(t)$, which has positive Ricci curvature for each $t \in [-a,0]$, with a Ricci–flat space

$$\mathrm{Vol}(g_i(t)) \leq \frac{\pi}{6}\mathrm{diam}(g_i(t))^3\,.$$

Since the initial metric g_0 is Ricci positive and the manifold M is compact, by Proposition 4.1.2, there exists $\beta \in (0,1/3]$ such that for every $s \in [0,T)$, $g(s)$ satisfies the pinching inequality $\mathrm{Ric}_{g(s)} \geq 2\beta\mathrm{R}_{g(s)}g(s)$. But this inequality is scaling invariant, therefore it holds for every g_i in $[-a,0]$. Hence, by Bonnet–Myers diameter's estimate (see [73, Section 6.4.1]),

$$\mathrm{diam}(g_i(t)) \leq \frac{\pi}{\sqrt{\beta \min \mathrm{R}_{g_i(t)}}}\,.$$

Combining the previous inequalities we get

$$\mathrm{Vol}(g_i(t)) \leq \frac{\pi}{6}\mathrm{diam}(g_i(t))^3 \leq \frac{\pi^2}{(\beta \min \mathrm{R}_{g_i(t)})^{3/2}} \leq \frac{\pi^2}{(\beta\varepsilon)^{3/2}}\,.$$

(vi) By Theorem 3.4.1 and point (v), the covariant derivatives of the Riemann tensor are uniformly bounded in $(-a,0]$

$$\sup_i \sup_{t\in(-a,0]} \|\nabla^k \mathrm{Rm}_{g_i(t)}\|_2 \leq C(k)\,.$$

(vii) There is a uniform lower bound on the injectivity radius for the metric g_i at point p_i and time 0. Indeed, by Proposition 4.3.1

$$\frac{\big|\overset{\circ}{\mathrm{Ric}}_{g_i(0)}\big|_{g_i(0)}(p_i)}{\mathrm{R}_{g_i(0)}(p_i)} = \frac{\big|\overset{\circ}{\mathrm{Ric}}_{g(t_i)}\big|_{g(t_i)}(p_i)}{\mathrm{R}_{g(t_i)}(p_i)} \leq \frac{C''}{\mathrm{R}^\delta_{g(t_i)}(p_i)} = \frac{C''}{{M_i}^\delta}\,.$$

The right member of the inequality goes to 0 as $i \to \infty$, hence the traceless Ricci part of $g_i(0)$ is uniformly close to zero, definitively in i. In dimension 3 with positive Ricci curvature we already know that the sectional curvatures are therefore uniformly close to each other (definitively in i). Hence, by Klingenberg's estimate (see [73, Section 6.6.2]) also the injectivity radius is uniformly bounded by below (definitively in i).
We can finally apply the compactness theorem of Hamilton and get, by passing to a subsequence, that $(M, g_i(t), p_i)$ converges locally smoothly to a complete (pointed) solution of Ricci–Bourguignon flow $(N, \widehat{g}(t), p)$ in $(-\infty, 0]$, with $\mathrm{R}_{\widehat{g}(0)}(p) = 1$.

Moreover, at every point $q \in M$, by Proposition 4.3.1 we have that

$$\frac{\big|\overset{\circ}{\mathrm{Ric}}_{g_i(0)}\big|_{g_i(0)}(q)}{\mathrm{R}_{g_i(0)}(q)} = \frac{\big|\overset{\circ}{\mathrm{Ric}}_{g(t_i)}\big|_{g(t_i)}(q)}{\mathrm{R}_{g(t_i)}(q)} \leq \frac{C''}{\mathrm{R}^{\delta}_{g(t_i)}(q)} = \frac{C''}{(M_i \mathrm{R}_{g_i(0)}(q))^{\delta}} \leq \frac{C''}{\varepsilon^{\delta} M_i^{\delta}}\,,$$

where we used the uniform lower bound proved in point (iii) for the scalar curvature of the metrics $g_i(0)$. Thus, by letting $i \to \infty$, we obtain that the limit metric $\widehat{g}(0)$ has the traceless Ricci tensor equal to 0, hence by Schur Lemma $(N, \widehat{g}(0))$ is an Einstein manifold with positive curvature, which is equivalent, in dimension three, to have constant sectional curvature.
By Bonnet–Myers again, N is compact and by definition of convergence of manifolds we have that $N = M$. Moreover, by uniqueness of the solution of the Ricci–Bourguignon flow, $\widehat{g}(t) = (t-c)g_\infty$, where g_∞ is some positive constant multiple of $\widehat{g}(0)$ and this concludes the proof. □

REMARK 4.3.4. We think that the previous convergence result can be improved as follows: condition (4.8) is used in the proof essentially to get uniform lower bounds of the scalar curvature and of the injectivity radius of the rescaled metrics. By proving a local version of Bando–Bernstein–Shi estimates (Theorem 3.4.1), we need the first control only locally around the points p_i, which holds true by hypothesis, therefore we can substitute condition (4.8) with the more "natural" hypothesis of a uniform lower bound of the injectivity radius, which might come from a monotonicity formula *à la* Perelman (see also Section 5.2.1).

4.3.1. Gradient bound for the scalar curvature. In this section we prove an integral estimate for the gradient of the scalar curvature. As in Proposition 4.3.1 and following Hamilton [49, Section 11], we try to prove an upper bound for the gradient of the scalar curvature with a quantity that rescales differently (less) when the metric is dilated.

LEMMA 4.3.5. *Let $(M, g(t))$ be a solution of the Ricci–Bourguignon flow with initially positive scalar curvature. Then*

$$\begin{aligned}\frac{\partial}{\partial t}\Big(\frac{|\nabla \mathrm{R}|^2}{\mathrm{R}}\Big) =&\,(1-2(n-1)\rho)\Delta\Big(\frac{|\nabla \mathrm{R}|^2}{\mathrm{R}}\Big) - 2(1-2(n-1)\rho)\mathrm{R}\Big|\nabla\Big(\frac{\nabla \mathrm{R}}{\mathrm{R}}\Big)\Big|^2 \\ &- 2|\nabla \mathrm{R}|^2\Big(\frac{|\mathrm{Ric}|^2}{\mathrm{R}^2} + 4\rho\Big) + \frac{4}{\mathrm{R}}\Big(\langle \nabla|\mathrm{Ric}|^2, \nabla \mathrm{R}\rangle + (n-1)\rho\mathrm{Ric}(\nabla \mathrm{R}, \nabla \mathrm{R})\Big) \qquad (4.9)\end{aligned}$$

PROOF. We start computing the evolution of the square norm of the gradient. By equations (3.1), (3.11)

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla \mathrm{R}|^2 =&\, \frac{\partial}{\partial t}(g^{ij}\nabla_i \mathrm{R}\nabla_j \mathrm{R}) = 2\mathrm{Ric}(\nabla \mathrm{R}, \nabla \mathrm{R}) - 2\rho \mathrm{R}|\nabla \mathrm{R}|^2 + 2\langle \nabla \mathrm{R}, \frac{\partial}{\partial t}\nabla \mathrm{R}\rangle \\ =&\, 2\mathrm{Ric}(\nabla \mathrm{R}, \nabla \mathrm{R}) - 2\rho \mathrm{R}|\nabla \mathrm{R}|^2 + 2\Big\langle \nabla \mathrm{R}, \nabla\Big((1-2(n-1)\rho)\Delta \mathrm{R} + 2|\mathrm{Ric}|^2 - 2\rho \mathrm{R}^2\Big)\Big\rangle \\ =&\, (1-2(n-1)\rho)\Delta|\nabla \mathrm{R}|^2 - 2(1-2(n-1)\rho)|\nabla^2 \mathrm{R}|^2 + 4\langle \nabla|\mathrm{Ric}|^2, \nabla \mathrm{R}\rangle \\ &+ \rho\Big(4(n-1)\mathrm{Ric}(\nabla \mathrm{R}, \nabla \mathrm{R}) - 10\mathrm{R}|\nabla \mathrm{R}|^2\Big)\,,\end{aligned}$$

where we used the fact that for functions the covariant derivative coincides with the usual derivative and the Bochner formula for functions $u \in C^\infty(M)$ reads

$$\Delta|\nabla u|^2 = 2|\nabla^2 u| + 2\langle \nabla \Delta u, \nabla u\rangle + 2\mathrm{Ric}(\nabla u, \nabla u)\,, \qquad (4.10)$$

applied to the scalar curvature. As the positivity of the scalar curvature is preserved along the flow, we can compute the evolution of the ratio between $|\nabla \mathrm{R}|^2$ and R, by recalling that for two functions $u, v \in C^\infty(M)$

$$\Delta\Big(\frac{u}{v}\Big) = \frac{\Delta u}{v} - \frac{u\Delta v}{v^2} - \frac{2\langle \nabla u, \nabla v\rangle}{v^2} + \frac{2u|\nabla v|^2}{v^3}\,,$$

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{|\nabla \mathrm{R}|^2}{\mathrm{R}}\right) &= \frac{1}{\mathrm{R}}\frac{\partial}{\partial t}(|\nabla \mathrm{R}|^2) - \frac{|\nabla \mathrm{R}|^2}{\mathrm{R}^2}\frac{\partial}{\partial t}\mathrm{R} \\
&= \frac{1-2(n-1)\rho}{\mathrm{R}}\Delta|\nabla \mathrm{R}|^2 - \frac{2(1-2(n-1)\rho)}{\mathrm{R}}|\nabla^2 \mathrm{R}|^2 + \frac{4}{\mathrm{R}}\langle \nabla|\mathrm{Ric}|^2, \nabla \mathrm{R}\rangle \\
&\quad + \rho\left(\frac{4(n-1)}{\mathrm{R}}\mathrm{Ric}(\nabla \mathrm{R}, \nabla \mathrm{R}) - 10|\nabla \mathrm{R}|^2\right) \\
&\quad - \frac{|\nabla \mathrm{R}|^2}{\mathrm{R}^2}\big((1-2(n-1)\rho)\Delta \mathrm{R} + 2|\mathrm{Ric}|^2 - 2\rho \mathrm{R}^2\big) \\
&= (1-2(n-1)\rho)\Delta\left(\frac{|\nabla \mathrm{R}|^2}{\mathrm{R}}\right) - 2(1-2(n-1)\rho)\left(\frac{|\nabla \mathrm{R}|^4}{\mathrm{R}^3} - \frac{\langle \nabla|\nabla \mathrm{R}|^2, \nabla \mathrm{R}\rangle}{\mathrm{R}^2} + \frac{|\nabla^2 \mathrm{R}|^2}{\mathrm{R}}\right) \\
&\quad - 2|\nabla \mathrm{R}|^2\left(\frac{|\mathrm{Ric}|^2}{\mathrm{R}^2} + 4\rho\right) + \frac{4}{\mathrm{R}}\Big(\langle \nabla|\mathrm{Ric}|^2, \nabla \mathrm{R}\rangle + (n-1)\rho \mathrm{Ric}(\nabla \mathrm{R}, \nabla \mathrm{R})\Big).
\end{aligned}
$$

The proof is completed by computing the terms of $\left|\nabla\left(\frac{\nabla \mathrm{R}}{\mathrm{R}}\right)\right|^2$. □

REMARK 4.3.6. Besides the fact that the elliptic operator involved is here $(1-4\rho)\Delta$, since we are interested in dimension 3, our purpose is again to control the potentially positive terms in the above evolution equation. We observe that, as in the Ricci flow case, there are the (potentially bad) term $4/\mathrm{R}\langle \nabla|\mathrm{Ric}|^2, \nabla \mathrm{R}\rangle$ and the (surely good) term $-2|\nabla \mathrm{R}|^2|\mathrm{Ric}|^2/\mathrm{R}^2$, and other two terms, according to the sign of ρ.
Since we have $\mathrm{R} > 0$ and also $\mathrm{Ric} > 0$, we can bound the square norm of the Ricci tensor, $1/3\mathrm{R}^2 \leq |\mathrm{Ric}|^2 \leq \mathrm{R}^2$, therefore, if ρ is non negative, we can estimate

$$
-8\rho|\nabla \mathrm{R}|^2 + \frac{8\rho}{\mathrm{R}}\mathrm{Ric}(\nabla \mathrm{R}, \nabla \mathrm{R}) \leq -8\rho|\nabla \mathrm{R}|^2 + 8\rho|\nabla \mathrm{R}|^2 = 0\,,
$$

while, for $\rho < 0$, the second term is non positive and for the first term we use Hamilton's estimate in [49, Lemma 11.6]

$$
|\nabla \mathrm{R}|^2 \leq \frac{20}{7}|\nabla \mathrm{Ric}|^2\,. \tag{4.11}
$$

Using the above estimate also on the ρ–independent term we obtain

$$
\frac{4}{\mathrm{R}}\langle \nabla|\mathrm{Ric}|^2, \nabla \mathrm{R}\rangle \leq \frac{4}{\mathrm{R}}2|\mathrm{Ric}||\nabla \mathrm{Ric}||\nabla \mathrm{R}| \leq 8\sqrt{\frac{20}{7}}|\nabla \mathrm{Ric}|^2 \leq 16|\nabla \mathrm{Ric}|^2\,.
$$

In conclusion, we have that

$$
\left(\frac{\partial}{\partial t} - (1-4\rho)\Delta\right)\left(\frac{|\nabla \mathrm{R}|^2}{\mathrm{R}}\right) \leq C(\rho)|\nabla \mathrm{Ric}|^2 - \frac{2|\mathrm{Ric}|^2|\nabla \mathrm{R}|^2}{\mathrm{R}^2}\,,
$$

where $C(\rho) = 16$ when $\rho \geq 0$ and $C(\rho) = 16\Big(1 - \frac{10}{7}\rho\Big)$ when $\rho < 0$.

Now, by a straightforward computation we get

$$
\frac{\partial}{\partial t}(\mathrm{R}^2) = (1-4\rho)\Delta(\mathrm{R}^2) - 2(1-4\rho)|\nabla \mathrm{R}|^2 + 4\mathrm{R}|\mathrm{Ric}|^2 - 4\rho \mathrm{R}^3\,,
$$

hence, for $\eta \in \big(0, \frac{1}{3(1-4\rho)}\big)$,

$$
\begin{aligned}
\left(\frac{\partial}{\partial t} - (1-4\rho)\Delta\right)\left(\frac{|\nabla \mathrm{R}|^2}{\mathrm{R}} - \eta \mathrm{R}^2\right) &\leq C(\rho)|\nabla \mathrm{Ric}|^2 - \frac{2|\mathrm{Ric}|^2|\nabla \mathrm{R}|^2}{\mathrm{R}^2} \\
&\quad + 2\eta(1-4\rho)|\nabla \mathrm{R}|^2 - 4\eta \mathrm{R}|\mathrm{Ric}|^2 + 4\eta\rho \mathrm{R}^3 \\
&\leq C(\rho)|\nabla \mathrm{Ric}|^2 - \frac{4}{3}\eta(1-3\rho)\mathrm{R}^3\,,
\end{aligned}
$$

where we used again the bound on $|\mathrm{Ric}|^2$.
The next step in Hamilton's proof is to exploit the fact that the evolution equation of $|\mathring{\mathrm{Ric}}|^2$ has a good term $-2/21|\nabla \mathrm{Ric}|^2$ and here is where we find a substantial difference between the Ricci flow and the RB–flow.

LEMMA 4.3.7. *In $(M, g(t))$ is a solution of the RB–flow on a compact 3–manifold, then the traceless Ricci tensor $|\mathring{\mathrm{Ric}}|^2$ satisfies the following equation*

$$\frac{\partial}{\partial t}|\mathring{\mathrm{Ric}}|^2 = \Delta|\mathring{\mathrm{Ric}}|^2 + \rho\Big(\frac{1}{3}\Delta(\mathrm{R}^2) - 2\langle \mathrm{Ric}, \nabla^2\mathrm{R}\rangle\Big) - 2|\nabla\mathrm{Ric}|^2 + \frac{2}{3}(1-\rho)|\nabla\mathrm{R}|^2$$
$$- 8\mathrm{tr}(\mathrm{Ric}^3) + \left(\frac{26}{3} - 4\rho\right)\mathrm{R}|\mathrm{Ric}|^2 - 2\left(1 - \frac{2}{3}\rho\right)\mathrm{R}^3 .$$

PROOF. By using the evolution equations of the Ricci tensor in dimension three (4.1) and of the inverse of the metric (3.7), we have

$$\begin{aligned}
\frac{\partial}{\partial t}(|\mathrm{Ric}|^2) &= \frac{\partial}{\partial t}(\mathrm{R}_{ij}\mathrm{R}_{kl}g^{ik}g^{jl}) \\
&= 2\langle \mathrm{Ric}, \frac{\partial}{\partial t}\mathrm{Ric}\rangle + 2\frac{\partial}{\partial t}(g^{ik})\mathrm{R}_{ij}\mathrm{R}_{kl}g^{jl} \\
&= 2\langle \mathrm{Ric}, \Delta\mathrm{Ric} - \rho(\nabla^2\mathrm{R} + \Delta\mathrm{R}) - 6\mathrm{Ric}^2 + 3\mathrm{R}\mathrm{Ric} + (2|\mathrm{Ric}|^2 - \mathrm{R}^2)g\rangle \\
&\quad + 4\mathrm{tr}(\mathrm{Ric}^3) - 4\rho\mathrm{R}|\mathrm{Ric}|^2 \\
&= \Delta|\mathrm{Ric}|^2 - 2|\nabla\mathrm{Ric}|^2 - 2\rho\langle \mathrm{Ric}, \nabla^2\mathrm{R}\rangle - 2\rho\mathrm{R}\Delta\mathrm{R} \\
&\quad - 8\mathrm{tr}(\mathrm{Ric}^3) + (10 - 4\rho)\mathrm{R}|\mathrm{Ric}|^2 - 2\mathrm{R}^3 \\
&= \Delta|\mathrm{Ric}|^2 - 2\rho\langle \mathrm{Ric}, \nabla^2\mathrm{R}\rangle - \rho\Delta\mathrm{R}^2 - 2|\nabla\mathrm{Ric}|^2 + 2\rho|\nabla\mathrm{R}|^2 \\
&\quad - 8\mathrm{tr}(\mathrm{Ric}^3) + (10 - 4\rho)\mathrm{R}|\mathrm{Ric}|^2 - 2\mathrm{R}^3 .
\end{aligned}$$

The proof is completed by using the equality $|\mathring{\mathrm{Ric}}|^2 = |\mathrm{Ric}|^2 - \frac{1}{3}\mathrm{R}^2$ and the evolution equation of R^2 written above. ☐

We underline that, even if we collect the terms differently, in order to highlight the operator $(1-4\rho)\Delta$ acting on $|\mathring{\mathrm{Ric}}|^2$, there are some second order terms that cannot be expressed in this way nor they have a definite sign. Therefore, as we did in order to get Bando–Bernstein–Shi estimates in Theorem 3.4.1, we prove the following integral estimate.

PROPOSITION 4.3.8. *Let $(M, g(t))$ a solution of the RB–flow on a compact 3–manifold. If g_0 has positive Ricci–curvature and $\rho < 1/10$, then for every $\eta < 1/3$ there exists $K(g_0, \eta, \rho) < +\infty$ such that*

$$\int_M \frac{|\nabla\mathrm{R}|^2}{\mathrm{R}}\, d\mu_g \leq \eta \int_M \mathrm{R}^2\, d\mu_g + K \tag{4.12}$$

as long as the solution of the RB–flow with initial data g_0 exists.

PROOF. Since M is a compact manifold without boundary, by the divergence theorem, for every $f \in C^\infty(M)$, $\int_M \Delta f\, d\mu_g = 0$.
By using the previous computations, the evolution of the volume form $d\mu_g$ along the flow and the fact that M is compact without boundary, we get

$$\begin{aligned}
\frac{d}{dt}\int_M \frac{|\nabla\mathrm{R}|^2}{\mathrm{R}}\, d\mu_g &= \int_M \frac{\partial}{\partial t}\left(\frac{|\nabla\mathrm{R}|^2}{\mathrm{R}}\right) d\mu_g - (1-3\rho)\int_M |\nabla\mathrm{R}|^2\, d\mu_g \\
&= -2(1-4\rho)\int_M \mathrm{R}\left|\nabla\left(\frac{\nabla\mathrm{R}}{\mathrm{R}}\right)\right|^2 d\mu_g + 4\int_M \frac{\langle \nabla|\mathrm{Ric}|^2, \nabla\mathrm{R}\rangle}{\mathrm{R}}\, d\mu_g - 2\int_M \frac{|\nabla\mathrm{R}|^2|\mathrm{Ric}|^2}{\mathrm{R}^2} d\mu_g \\
&\quad + 8\rho\int_M \left(\frac{\mathrm{Ric}(\nabla\mathrm{R}, \nabla\mathrm{R})}{\mathrm{R}} - |\nabla\mathrm{R}|^2\right) d\mu_g - (1-3\rho)\int_M |\nabla\mathrm{R}|^2\, d\mu_g .
\end{aligned}$$

Therefore, for any $\eta < \frac{1}{3}$, if $\rho \geq 0$,

$$\begin{aligned}\frac{d}{dt}\int_M \Big(\frac{|\nabla \mathrm{R}|^2}{\mathrm{R}} - \eta \mathrm{R}^2\Big)\,d\mu_g &\leq 4\int_M \frac{\langle \nabla |\mathrm{Ric}|^2, \nabla \mathrm{R}\rangle}{\mathrm{R}}\,d\mu_g - 2\int_M \frac{|\nabla \mathrm{R}|^2 |\mathrm{Ric}|^2}{\mathrm{R}^2}\,d\mu_g \\ &\quad + (2\eta(1-4\rho) - (1-3\rho))\int_M |\nabla \mathrm{R}|^2\,d\mu_g \\ &\quad - 4\eta\int_M \mathrm{R}|\mathrm{Ric}|^2 d\mu_g + \eta(1+\rho)\int_M \mathrm{R}^3\,d\mu_g \\ &\leq 16\int_M |\nabla \mathrm{Ric}|^2\,d\mu_g - \Big(1-\frac{1}{3}\rho\Big)\int_M |\nabla \mathrm{R}|^2\,d\mu_g \\ &\quad - \frac{1}{3}\eta(1-3\rho)\int_M \mathrm{R}^3\,d\mu_g \\ &\leq 16\int_M |\nabla \mathrm{Ric}|^2\,d\mu_g - \frac{1}{3}\eta(1-3\rho)\int_M \mathrm{R}^3\,d\mu_g\,,\end{aligned}$$

while, for $\rho < 0$,

$$\begin{aligned}\frac{d}{dt}\int_M \Big(\frac{|\nabla \mathrm{R}|^2}{\mathrm{R}} - \eta \mathrm{R}^2\Big)\,d\mu_g &\leq 16\int_M |\nabla \mathrm{Ric}|^2\,d\mu_g - \Big(1+\frac{22}{3}\rho\Big)\int_M |\nabla \mathrm{R}|^2\,d\mu_g \\ &\quad - \frac{1}{3}\eta(1-3\rho)\int_M \mathrm{R}^3\,d\mu_g \\ &\leq \Big(16 - \frac{22\cdot 20}{3}\rho\Big)\int_M |\nabla \mathrm{Ric}|^2\,d\mu_g - \frac{1}{3}\eta(1-3\rho)\int_M \mathrm{R}^3\,d\mu_g\,,\end{aligned}$$

where, in the last inequality,we used again Hamilton's estimate (4.11) to get rid of the positive coefficient before $\int_M |\nabla \mathrm{Ric}|^2\,d\mu_g$.

Now, we draw our attention to the evolution of the integral norm of the traceless Ricci tensor, using Lemma 4.3.7 and again the divergence theorem in the first equality, while in the second equality we integrate by parts $\int_M \langle \mathrm{Ric}, \nabla^2 \mathrm{R}\rangle\,d\mu_g$ and use the contracted Bianchi identity $2\delta(\mathrm{Ric}) = \nabla \mathrm{R}$.

$$\begin{aligned}\frac{d}{dt}\int_M |\mathring{\mathrm{Ric}}|^2\,d\mu_g &= \int_M \frac{\partial}{\partial t}\Big(|\mathrm{Ric}|^2 - \frac{1}{3}\mathrm{R}^2\Big)\,d\mu_g - (1-3\rho)\int_M \mathrm{R}|\mathring{\mathrm{Ric}}|^2\,d\mu_g \\ &= \int_M \Big(-2\rho\langle \mathrm{Ric}, \nabla^2 \mathrm{R}\rangle - 2|\nabla \mathrm{Ric}|^2 + \frac{2}{3}(1-\rho)|\nabla \mathrm{R}|^2\Big)\,d\mu_g \\ &\quad + \int_M \Big(-8\mathrm{tr}(\mathrm{Ric}^3) + \Big(\frac{26}{3} - 4\rho\Big)\mathrm{R}|\mathrm{Ric}|^2 - 2\Big(1-\frac{2}{3}\rho\Big)\mathrm{R}^3 - (1-3\rho)\mathrm{R}|\mathring{\mathrm{Ric}}|^2\Big)\,d\mu_g \\ &\leq \int_M \Big(-2|\nabla \mathrm{Ric}|^2 + \frac{2+\rho}{3}|\nabla \mathrm{R}|^2\Big)\,d\mu_g \\ &\quad + \int_M \Big(-8\mathrm{tr}(\mathrm{Ric}^3) + \frac{26}{3}\mathrm{R}|\mathrm{Ric}|^2 - 2\mathrm{R}^3 - (1+\rho)\mathrm{R}|\mathring{\mathrm{Ric}}|^2\Big)\,d\mu_g\end{aligned}$$

By [49, Lemma 11.5], we can estimate the last row as follows

$$-8\mathrm{tr}(\mathrm{Ric}^3) + \frac{26}{3}\mathrm{R}|\mathrm{Ric}|^2 - 2\mathrm{R}^3 - (1+\rho)\mathrm{R}|\mathring{\mathrm{Ric}}|^2 \leq (3-\rho)\mathrm{R}|\mathring{\mathrm{Ric}}|^2$$

Now, if $\rho \geq 0$, the coefficient of $\int_M |\nabla \mathrm{R}|^2\,d\mu_g$ is positive, hence we apply Hamilton's estimate (4.11) and obtain

$$-2|\nabla \mathrm{Ric}|^2 + \frac{2+\rho}{3}|\nabla \mathrm{R}|^2 \leq \frac{-2+20\rho}{21}|\nabla \mathrm{Ric}|^2\,,$$

hence, in order to have a "good" non positive term, we must impose $\rho < 1/10$, otherwise the evolution of $|\mathring{\mathrm{Ric}}|^2$ is useless. By this assumption, we choose a constant $C \geq \frac{21\cdot 16}{2-20\rho}$ and obtain

$$\frac{d}{dt}\int_M \Big(\frac{|\nabla \mathrm{R}|^2}{\mathrm{R}} - \eta \mathrm{R}^2 + C|\mathring{\mathrm{Ric}}|^2\,d\mu_g\Big)\,d\mu_g \leq -\frac{1}{3}\eta(1-3\rho)\int_M \mathrm{R}^3\,d\mu_g + C(3-\rho)\int_M \mathrm{R}|\mathring{\mathrm{Ric}}|^2\,d\mu_g\,.$$

If $\rho < 0$, we use Hamilton's estimate (4.11) only for the positive part and obtain

$$-2|\nabla \mathrm{Ric}|^2 + \frac{2+\rho}{3}|\nabla \mathrm{R}|^2 \leq -\frac{2}{21}|\nabla \mathrm{Ric}|^2 \,,$$

therefore, choosing a constant $C \geq \frac{21}{2}\Big(16 - \frac{22\cdot 20}{3}\rho\Big)$, we obtain the same inequality above.
By using Proposition 4.3.1, we can estimate the traceless Ricci tensor part with $\mathrm{R}^{2-2\delta}$, hence, depending on η, C and δ, there exists C' such that

$$\frac{d}{dt}\int_M \left(\frac{|\nabla \mathrm{R}|^2}{\mathrm{R}} - \eta \mathrm{R}^2 + C|\mathring{\mathrm{Ric}}|^2 \, d\mu_g\right) d\mu_g \leq C' \,,$$

and, by integrating the inequality with respect to the time parameter and using that the flow exists only up to a finite maximal time T, we finally get

$$\int_M \frac{|\nabla \mathrm{R}|^2}{\mathrm{R}} \, d\mu_g \leq \eta \int_M \mathrm{R}^2 \, d\mu_g + K \,,$$

where K depends on η, ρ and the initial metric g_0, since again we have an upper bound for T depending on $\min \mathrm{R}_{g_0}$. □

Along the same lines, we can prove a slightly different estimate.

PROPOSITION 4.3.9. *Let $(M, g(t))$ a solution of the RB–flow on a compact 3–manifold. If g_0 has positive Ricci-curvature and $\rho < 1/10$, then there exists $\alpha(g_0) \in (0, 1/2)$ such that for every $\beta > 0$ the solution of the RB–flow with initial data g_0 satisfies*

$$\int_M \frac{|\nabla \mathrm{R}|^2}{\mathrm{R}} \, d\mu_g \leq \beta \int_M \mathrm{R}^{2-\alpha} \, d\mu_g + K \,,$$

where $K < +\infty$ depends on g_0, β and ρ.

PROOF. With a straightforward computation we get

$$\begin{aligned}\frac{\partial}{\partial t}(\mathrm{R}^{2-\alpha}) &= (2-\alpha)\mathrm{R}^{1-\alpha}\Big((1-4\rho)\Delta \mathrm{R} + 2|\mathrm{Ric}|^2 - 2\rho \mathrm{R}^2\Big) \\ &= (2-\alpha)\Delta(\mathrm{R}^{2-\alpha}) - (2-\alpha)(1-\alpha)\mathrm{R}^{-\alpha}|\nabla \mathrm{R}|^2 + 2(2-\alpha)(|\mathrm{Ric}|^2 - \rho \mathrm{R}^2) \,.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{d}{dt}\int_M \mathrm{R}^{2-\alpha} \, d\mu_g &= \int_M \frac{\partial}{\partial t}(\mathrm{R}^{2-\alpha}) \, d\mu_g - (1-3\rho)\int_M \mathrm{R}^{3-\alpha} \, d\mu_g \\ &= -(1-4\rho)(2-\alpha)(1-\alpha)\int_M \mathrm{R}^{-\alpha}|\nabla \mathrm{R}|^2 \, d\mu_g + 2(2-\alpha)\int_M \mathrm{R}^{1-\alpha}|\mathrm{Ric}|^2 \, d\mu_g \\ &\quad - \big((1-3\rho) + 2\rho(2-\alpha)\big)\int_M \mathrm{R}^{3-\alpha} \, d\mu_g \\ &\geq -(1-4\rho)(2-\alpha)(1-\alpha)\int_M \mathrm{R}^{-\alpha}|\nabla \mathrm{R}|^2 \, d\mu_g + \frac{1}{3}(1-3\rho)(1-2\alpha)\int_M \mathrm{R}^{3-\alpha} \, d\mu_g \,.\end{aligned}$$

Using again Hamilton's estimate (4.11) and the fact that $\min \mathrm{R}_{g(t)}$ is non decreasing during the flow, we get

$$(1-4\rho)(2-\alpha)(1-\alpha)\mathrm{R}^{-\alpha}|\nabla \mathrm{R}|^2 \leq c'(\rho,\alpha)(\min \mathrm{R}_{g_0})^{-\alpha}|\nabla \mathrm{Ric}|^2 \,.$$

If $\rho \in [0, 1/10)$, we can choose a constant $C(\rho, \alpha, \beta)$ such that

$$16 + \beta c'(\rho,\alpha)(\min \mathrm{R}_{g_0})^{-\alpha} + C\frac{-2+20\rho}{21} \leq 0 \,.$$

Therefore

$$\begin{aligned}\frac{d}{dt}\int_M\Big(\frac{|\nabla \mathrm{R}|^2}{\mathrm{R}}-\beta\mathrm{R}^{2-\alpha}+C|\mathring{\mathrm{Ric}}|^2\Big)\,d\mu_g\leq\,&16\int_M|\nabla\mathrm{Ric}|^2\,d\mu_g+\int_M\beta(1-4\rho)(2-\alpha)(1-\alpha)\mathrm{R}^{-\alpha}|\nabla\mathrm{R}|^2\,d\mu_g\\&+C\int_M\Big(-2|\nabla\mathrm{Ric}|^2+\frac{2+\rho}{3}|\nabla\mathrm{R}|^2\Big)\,d\mu_g\\&+\int_M\Big(C(3-\rho)\mathrm{R}|\mathring{\mathrm{Ric}}|^2-\frac{1}{3}(1-3\rho)(1-2\alpha)\beta\mathrm{R}^{3-\alpha}\Big)\,d\mu_g\\\leq\,&\int_M\Big(C(3-\rho)\mathrm{R}|\mathring{\mathrm{Ric}}|^2-\frac{1}{3}(1-3\rho)(1-2\alpha)\beta\mathrm{R}^{3-\alpha}\Big)\,d\mu_g\,.\end{aligned}$$

As before, the inequality above holds even if $\rho < 0$ (for another constant C).
Regarding the last term, again by Proposition 4.3.1, with $\alpha < \min\{1/2, 2\delta\}$, where δ is the exponent of the proposition, we obtain

$$\frac{d}{dt}\int_M\Big(\frac{|\nabla \mathrm{R}|^2}{\mathrm{R}}-\beta\mathrm{R}^{2-\alpha}+C|\mathring{\mathrm{Ric}}|^2\Big)\,d\mu_g\leq C'\,,$$

and we are done by integrating the last inequality. □

CHAPTER 5

Present and future works

In this chapter we briefly recall some questions posed throughout the thesis, together with some open problems.

5.1. The renormalization group flow

A natural issue is to give a rigorous mathematical foundation to the derivation of the renormalization group flow and of its expression in the perturbative regime. This is actually out of our reach at the moment.

As of the two–loop RG–flow, studied in Chapter 2, in their recent work [45], Gimre, Guenther and Isenberg present some known and new results about special classes of solutions for the $\mathrm{RG}^{2,a}$–flow, such as fixed points, constant curvature solutions and solitons. For instance, they show the existence of a unique rotationally symmetric 2–dimensional steady gradient soliton with constant positive curvature, analogous to *Hamilton's cigar* for the Ricci flow. In the last section of the same work one can find a comprehensive list of open problems, among which the investigation on some Perelman–type entropy functionals, both for the first order truncation (the Ricci flow) coupled with some other flows, as proposed by Tseytlin in [78], and for the second order truncation, for which quadratic Riemannian functionals might be useful (see also [68]).

In Section 2.2, we approached another question which is fundamental for the future research on this topic: since the existence (and uniqueness) of solutions is only guaranteed in presence of a suitable condition on the sectional curvatures of the metric, it is crucial to find out whether this condition is preserved by the flow, to begin the analysis of long time behaviour of solutions. We restricted our attention to the three–dimensional case, as in general dimension the evolution equations of the geometric quantities are quite involved. At the present time, we know that the quasilinear differential operator in the evolution equation of (minus) the Einstein tensor remains elliptic as long as the short time condition is fulfilled, but the reaction term is not well behaved in general. Therefore, as future investigation, we will try to see whether subsets of initial data defined by stronger conditions can be preserved by the $\mathrm{RG}^{2,a}$–flow.

5.2. The Ricci–Bourguignon flow

The basic properties concerning this family of flows established in the present thesis, together with the results on their solitons proven by Catino and Mazzieri in [21, 22], open several lines of research.

First of all, we briefly review some questions already stated in the previous chapters.

- The short time existence of the Schouten flow, corresponding to the value $1/2(n-1)$ of the parameter ρ, is open, because of the presence of an extra degeneracy, besides the ones due to geometric invariance, in the principal symbol of the linearized operator (see Section 3.1).
- The research of geometric conditions preserved by the RB–flow in general dimension other than the ones found in Section 3.3, possibly applying the set of ideas of Bohm and Wilking in [7].
- In dimension three, most of the conditions preserved by the Ricci flow are also preserved by the RB–flow (see Section 4.1), since the evolution equation of the Ricci tensor has a ρ–independent reaction term. Nevertheless, the presence of the ρ–dependent term in

the system of ODEs associated to the curvature operator might lead to other preserved cones.

- A Hamilton–Ivey type estimate is currently available only for non negative values of the parameter ρ and we do not expect to generalize it to negative values of ρ. In this case, it should be interesting to explore whether the RB–flow develops a different behaviour, maybe for ρ below a certain range.
- Besides the previous questions, all related to various applications of the maximum principle, in some cases the quantities involved do not satisfies a reaction–diffusion equation, hence we no longer have pointwise estimates and we look for integral bounds. For instance, the integral *a priori* estimates proven in Section 3.4 are sufficient to prove that a maximal solution existing up to a finite time has unbounded curvature, but we actually lack a pointwise estimate on the square norm of the Riemann tensor. Another example is the integral estimate for the gradient of the scalar curvature proven in Section 4.3.1: the pointwise analogue for the Ricci flow proven by Hamilton in [**49**] is the key tool to get convergence of the normalized flow to a constant curvature metric on a compact three manifold with initial positive Ricci curvature. We are therefore looking for stronger integral estimates to be used to generalise Hamilton's argument.
- A local version of Bando–Bernstein–Shi estimates of Theorem 3.4.1 could enable us to improve the convergence result in the 3–dimensional Ricci positive case (Theorem 4.3.3), by assuming only a control of the injectivity radius (see the final remark of Section 4.3).
- Many facts have already been established on ρ–Einstein gradient solitons in the three–dimensional case by Catino and Mazzieri in [**21, 22**]. The known results are recalled in Section 4.2, where we discuss also possible ways to continue the classification, in particular in the shrinking case for positive values of the parameter ρ. Anyway, in the compact case, it seems more promising a static point of view, as the one exploited by Eminenti, La Nave and Mantegazza in [**37**], rather than Ivey's argument carried out for the Ricci flow in [**58**]. In the complete non compact case, a local version of Hamilton–Ivey estimates, as the one proven by Chen in [**24**, Section 2] for the Ricci flow, would be sufficient to have a rigidity results for any $\rho \in [0, 1/4]$.

We conclude with some partial results on another direction of research, which could certanly help, if successfully carried out, to answer some of the previous questions.

5.2.1. The functionals $\mathcal{F}_C$. The first of the several results due to Perelman is the fact that the Ricci flow is of "gradient type" in a more general space than the one of Riemannian metrics on M. Therefore, it is not surprising that we try, for the Ricci–Bourguignon flow, to find an analogous structure, or at least some *entropy* type functionals, that is, monotone quantities along the flow.
In this section we present partial results on some monotone functionals under the RB–flow similar to the functional of Perelman $\mathcal{F} : \Gamma(S^2_+M) \times C^\infty(M) \to \mathbb{R}$, defined by

$$\mathcal{F}(g, f) = \int_M (\mathrm{R} + |\nabla f|^2)e^{-f}\, d\mu_g\,. \tag{5.1}$$

We start observing that the precise analogue of Perelman's functional for the Ricci–Bourguignon flow is the following one,

$$\mathcal{F}(g, f) = \int_M \big((1 - n\rho)\mathrm{R} + |\nabla f|^2\big)e^{-f}\, d\mu_g\,. \tag{5.2}$$

This expression comes from the steady gradient ρ–Einstein solitonic equation (3.24), traced and then integrated by parts, with respect to the measure $d\mu = e^{-f}\, d\mu_g$. However, taking inspiration from Li's paper [**60**], who showed a general class of monotone functionals under the Ricci flow, we relax the definition in the following way.
Given a constant $C \in \mathbb{R}$, let $\mathcal{F}_C : \Gamma(S^2_+M) \times C^\infty(M) \to \mathbb{R}$ be the functional

$$\mathcal{F}_C(g, f) = \int_M (C\mathrm{R} + |\nabla f|^2)e^{-f}\, d\mu_g\,, \tag{5.3}$$

where M is a fixed n–dimensional, smooth, compact, differential manifold, g is a Riemannian metric on M and f is any smooth function.

We start computing the first variation of the functional $\mathcal{F}_C$, letting the metric g and the function f vary and setting $\frac{\partial}{\partial t}g = h$ and $\frac{\partial}{\partial t}f = k$ to be respectively the variations of g and f. We recall the first variation formulas for the standard measure and the scalar curvature of the metric g

$$\frac{\partial}{\partial t}(d\mu_g) = \frac{1}{2}\mathrm{tr}(h)d\mu_g\,, \quad \frac{\partial}{\partial t}\mathrm{R} = -\Delta\mathrm{tr}(h) + \delta^2 h - \langle h, \mathrm{Ric}\rangle\,,$$

where, as usual, we have omitted the g–dependence in the trace tr and the divergence δ operator. Then the first variation of $\mathcal{F}_C$ can be computed as follows

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_C &= \int_M \Big(C\frac{\partial}{\partial t}\mathrm{R} + \frac{\partial}{\partial t}(|\nabla f|^2) + (C\mathrm{R} + |\nabla f|^2)\big(\mathrm{tr}(h)/2 - k\big)\Big)e^{-f}\,d\mu_g \\ &= \int_M \Big(C\big(-\Delta\mathrm{tr}(h) + \delta^2 h - \langle h, \mathrm{Ric}\rangle\big) - h(\nabla f, \nabla f) + 2g(\nabla k, \nabla f) \\ &\qquad + \big(C\mathrm{R} + |\nabla f|^2\big)\big(\mathrm{tr}(h)/2 - k\big)\Big)e^{-f}\,d\mu_g\,.\end{aligned}$$

Since M is compact with no boundary, by the divergence theorem, for every function $\varphi \in C^\infty(M)$ and for every symmetric two–form $\sigma \in \Gamma(S^2M)$ there hold

$$\begin{aligned}\int_M \Delta\varphi\, e^{-f}\,d\mu_g &= \int_M \varphi\big(|\nabla f|^2 - \Delta f\big)e^{-f}\,d\mu_g\,, \\ \int_M \delta^2\sigma\, e^{-f}\,d\mu_g &= \int_M \big(\sigma(\nabla f, \nabla f) - \langle\sigma, \nabla^2 f\rangle\big)e^{-f}\,d\mu_g\,, \\ \int_M g(\nabla\varphi, \nabla f)\, e^{-f}\,d\mu_g &= \int_M \varphi\big(|\nabla f|^2 - \Delta f\big)e^{-f}\,d\mu_g\,.\end{aligned}$$

Hence, the first variation of the functional becomes

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_C &= \int_M \Big(-C\langle h, \mathrm{Ric} + \nabla^2 f\rangle + (C-1)h(\nabla f, \nabla f) \\ &\qquad + \big(-C\mathrm{tr}(h) + 2k\big)\big(|\nabla f|^2 - \Delta f\big) \\ &\qquad + \big(C\mathrm{R} + |\nabla f|^2\big)\big(\mathrm{tr}(h)/2 - k\big)\Big)e^{-f}\,d\mu_g \\ &= \int_M \Big(-C\langle h, \mathrm{Ric} + \nabla^2 f - \rho\mathrm{R}g\rangle - C\rho\mathrm{R}\mathrm{tr}(h) + (C-1)h(\nabla f, \nabla f) \\ &\qquad + \big(-C\mathrm{tr}(h) + 2k\big)\big(|\nabla f|^2 - \Delta f\big) \\ &\qquad + \big(C\mathrm{R} + |\nabla f|^2\big)\big(\mathrm{tr}(h)/2 - k\big)\Big)e^{-f}\,d\mu_g\,.\end{aligned}$$

Now, following Perelman, we consider the variation of the metric given by the Ricci–Bourguignon flow, modified by the family of diffeomorphisms generated by the vector field $-\nabla f$, hence,

$$h = -2(\mathrm{Ric} + \nabla^2 f - \rho\mathrm{R}g)\,, \quad \mathrm{tr}(h) = -2\big((1-n\rho)\mathrm{R} + \Delta f\big)$$

and we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_C &= \int_M \Big(2C|\mathrm{Ric} + \nabla^2 f - \rho\mathrm{R}g|^2 + 2C\rho\mathrm{R}\big((1-n\rho)\mathrm{R} + \Delta f\big) \\ &\qquad - 2(C-1)\big(\mathrm{Ric} + \nabla^2 f - \rho\mathrm{R}g\big)(\nabla f, \nabla f) \\ &\qquad + 2\big(C(1-n\rho)\mathrm{R} + C\Delta f + k\big)\big(|\nabla f|^2 - \Delta f\big) \\ &\qquad - \big(C\mathrm{R} + |\nabla f|^2\big)\big((1-n\rho)\mathrm{R} + \Delta f + k\big)\Big)e^{-f}\,d\mu_g\,.\end{aligned}$$

We now want to use the square in the first line in order to control the terms in the second line, hence we rewrite the formula as follows,

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_C &= \int_M \Big(2|\mathrm{Ric}+\nabla^2 f-\rho\mathrm{R}g|^2+2(C-1)|\mathrm{Ric}-\rho\mathrm{R}g|^2 \\ &\quad+2(C-1)|\nabla^2 f|^2+4(C-1)\langle\nabla^2 f,\mathrm{Ric}-\rho\mathrm{R}g\rangle \\ &\quad+2C\rho(1-n\rho)\mathrm{R}^2+2C\rho\mathrm{R}\Delta f \\ &\quad-2(C-1)(\mathrm{Ric}+\nabla^2 f-\rho\mathrm{R}g)(\nabla f,\nabla f) \\ &\quad+2\big(C(1-n\rho)\mathrm{R}+C\Delta f+k\big)\big(|\nabla f|^2-\Delta f\big) \\ &\quad-\big(C\mathrm{R}+|\nabla f|^2\big)\big((1-n\rho)\mathrm{R}+\Delta f+k\big)\Big)e^{-f}\,d\mu_g \\ &= \int_M \Big(2|\mathrm{Ric}+\nabla^2 f-\rho\mathrm{R}g|^2+2(C-1)|\mathrm{Ric}-\rho\mathrm{R}g|^2 \\ &\quad+2(C-1)|\nabla^2 f|^2+4(C-1)\langle\nabla^2 f,\mathrm{Ric}\rangle \\ &\quad+2C\rho(1-n\rho)\mathrm{R}^2-2(C-2)\rho\mathrm{R}\Delta f \\ &\quad-2(C-1)(\mathrm{Ric}+\nabla^2 f-\rho\mathrm{R}g)(\nabla f,\nabla f) \\ &\quad+2\big(C(1-n\rho)\mathrm{R}+C\Delta f+k\big)\big(|\nabla f|^2-\Delta f\big) \\ &\quad-\big(C\mathrm{R}+|\nabla f|^2\big)\big((1-n\rho)\mathrm{R}+\Delta f+k\big)\Big)e^{-f}\,d\mu_g\,.\end{aligned} \tag{5.4}$$

Using again the divergence theorem and the contracted Bianchi identity $\delta\mathrm{Ric}=\nabla\mathrm{R}/2$, integrating by parts, we have

$$\begin{aligned}\int_M\langle\nabla^2 f,\mathrm{Ric}\rangle\,e^{-f}\,d\mu_g &= \int_M\Big(\mathrm{Ric}(\nabla f,\nabla f)-g(\nabla f,\delta\mathrm{Ric})\Big)e^{-f}\,d\mu_g \\ &= \int_M\Big(\mathrm{Ric}(\nabla f,\nabla f)-\frac{1}{2}g(\nabla f,\nabla\mathrm{R})\Big)e^{-f}\,d\mu_g \\ &= \int_M\Big(\mathrm{Ric}(\nabla f,\nabla f)+\frac{1}{2}\mathrm{R}\big(\Delta f-|\nabla f|^2\big)\Big)e^{-f}\,d\mu_g\,,\end{aligned}$$

and

$$\int_M\nabla^2 f(\nabla f,\nabla f)e^{-f}\,d\mu_g=\frac{1}{2}\int_M|\nabla f|^2\big(|\nabla f|^2-\Delta f\big)e^{-f}\,d\mu_g=\frac{1}{2}\int_M\Delta|\nabla f|^2e^{-f}\,d\mu_g\,.$$

Hence, substituting these terms in formula (5.4) and collecting, we get

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_C &= \int_M \Big(2|\mathrm{Ric}+\nabla^2 f-\rho\mathrm{R}g|^2+2(C-1)|\mathrm{Ric}-\rho\mathrm{R}g|^2 \\ &\quad+(C-1)\big(2\mathrm{Ric}(\nabla f,\nabla f)+2|\nabla^2 f|^2-\Delta|\nabla f|^2\big) \\ &\quad+2\rho\mathrm{R}\big(C(1-n\rho)\mathrm{R}-(C-2)\Delta f+(C-1)|\nabla f|^2\big) \\ &\quad+2\big((1-Cn\rho)\mathrm{R}+C\Delta f+k\big)\big(|\nabla f|^2-\Delta f\big) \\ &\quad-\big(C\mathrm{R}+|\nabla f|^2\big)\big((1-n\rho)\mathrm{R}+\Delta f+k\big)\Big)e^{-f}\,d\mu_g\,.\end{aligned}$$

Using now the Bochner formula for $|\nabla f|^2$ (see [30, Section 4.3])

$$\Delta|\nabla f|^2 = 2|\nabla^2 f|^2 + 2\mathrm{Ric}(\nabla f, \nabla f) + 2\langle \nabla\Delta f, \nabla f\rangle\,,$$

we obtain

$$\begin{aligned}
\frac{d}{dt}\mathcal{F}_C &= \int_M \Big(2|\mathrm{Ric} + \nabla^2 f - \rho \mathrm{R} g|^2 + 2(C-1)|\mathrm{Ric} - \rho\mathrm{R} g|^2 \\
&\quad - 2(C-1)\langle\nabla\Delta f, \nabla f\rangle \\
&\quad + 2\rho\mathrm{R}\big(C(1-n\rho)\mathrm{R} - (C-2)\Delta f + (C-1)|\nabla f|^2\big) \\
&\quad + 2\big((1-Cn\rho)\mathrm{R} + C\Delta f + k\big)\big(|\nabla f|^2 - \Delta f\big) \\
&\quad - \big(C\mathrm{R} + |\nabla f|^2\big)\big((1-n\rho)\mathrm{R} + \Delta f + k\big)\Big) e^{-f}\, d\mu_g \\
&= \int_M \Big(2|\mathrm{Ric} + \nabla^2 f - \rho \mathrm{R} g|^2 + 2(C-1)|\mathrm{Ric} - \rho\mathrm{R} g|^2 \qquad (5.5)\\
&\quad + 2\rho\mathrm{R}\big(C(1-n\rho)\mathrm{R} - (C-2)\Delta f + (C-1)|\nabla f|^2\big) \\
&\quad + 2\big((1-Cn\rho)\mathrm{R} + \Delta f + k\big)\big(|\nabla f|^2 - \Delta f\big) \\
&\quad - \big((1-n\rho)\mathrm{R} + \Delta f + k\big)\big(C\mathrm{R} + |\nabla f|^2\big)\Big) e^{-f}\, d\mu_g\,,
\end{aligned}$$

where we integrated by parts the term in the second line and collected the result.

REMARK 5.2.1. For $C = 1$, we are looking at the behaviour of the original Perelman's functional (5.1) along the RB–flow, then the computation simplifies considerably and we get directly from equation (5.4) or from equation (5.5)

$$\begin{aligned}
\frac{d}{dt}\mathcal{F}_1 &= \int_M \Big(2|\mathrm{Ric} + \nabla^2 f - \rho\mathrm{R} g|^2 + 2\rho\mathrm{R}\big((1-n\rho)\mathrm{R} + \Delta f\big) \\
&\qquad + \big(\mathrm{R}(1-n\rho) + \Delta f + k\big)\big(|\nabla f|^2 - 2\Delta f - \mathrm{R}\big)\Big) e^{-f}\, d\mu_g\,,
\end{aligned}$$

If we consider now the coupled variation of f that preserves the measure $d\mu = e^{-f} d\mu_g$, that is, $k = -\big((1-n\rho)\mathrm{R} + \Delta f\big)$, we get

$$\frac{d}{dt}\mathcal{F}_1 = \int_M \Big((2|\mathrm{Ric} + \nabla^2 f - \rho\mathrm{R} g|^2 + 2\rho(1-n\rho)\mathrm{R}^2 + 2\rho\mathrm{R}\Delta f\Big) e^{-f}\, d\mu_g\,.$$

The first term is a square hence non negative, the second one is quadratic in R, hence it is non negative if ρ is non negative, the third term does not have a definite sign, but we could use the square to get rid of it in the following way. By means of the Cauchy–Schwarz inequality

$$|\sigma|^2 \geq \frac{[\mathrm{tr}(\sigma)]^2}{n}\,, \qquad (5.6)$$

which holds for every symmetric 2–form σ, applied to $\mathrm{Ric} + \nabla^2 f - \rho\mathrm{R}g$, we look for a positive constant $A \in (0, 2]$ such that

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_1 &\geq \int_M \Big((2-A)|\mathrm{Ric} + \nabla^2 f - \rho\mathrm{R}g|^2 \\ &\qquad + \frac{A}{n}\big((1-n\rho)\mathrm{R} + \Delta f\big)^2 + 2\rho(1-n\rho)\mathrm{R}^2 + 2\rho\mathrm{R}\Delta f\Big)e^{-f}\,d\mu_g \\ &\geq \int_M \Big((2-A)|\mathrm{Ric} + \nabla^2 f - \rho\mathrm{R}g|^2 \\ &\qquad + \frac{1-n\rho}{n}\big((1-n\rho)A + 2n\rho\big)\mathrm{R}^2 + \frac{2}{n}\big((1-n\rho)A + n\rho\big)\mathrm{R}\Delta f\Big)e^{-f}\,d\mu_g\,.\end{aligned}$$

The suitable constant is therefore $A = -n\rho/(1-n\rho) < 1$, which is negative for positive values of ρ, while it is positive and less than 1 for negative values of ρ, but in this last case the coefficient before the square of the scalar curvature becomes negative. Hence, for $\rho \neq 0$, we are not able to establish whether Perelman's functional is monotone along the Ricci–Bourguignon flow.

Therefore, as a first possibility we allow a more general variation of f, putting

$$k = -\Delta f - \mathrm{R}(1-n\rho) + 2\rho\mathrm{R} \qquad \text{and} \qquad C = \frac{n}{n-1}$$

in formula (5.5). We obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_{\frac{n}{n-1}} &= \int_M \Big(2|\mathrm{Ric} + \nabla^2 f - \rho\mathrm{R}g|^2 + \frac{2}{n-1}|\mathrm{Ric} - \rho\mathrm{R}g|^2 \\ &\qquad + 2\rho\mathrm{R}\Big(\frac{n}{n-1}(1-n\rho)\mathrm{R} + \frac{n-2}{n-1}\Delta f + \frac{1}{n-1}|\nabla f|^2\Big) \\ &\qquad + 2\rho\mathrm{R}\Big(2 - \frac{n}{n-1}\Big)\big(|\nabla f|^2 - \Delta f\big) \\ &\qquad - 2\rho\mathrm{R}\Big(\frac{n}{n-1}\mathrm{R} + |\nabla f|^2\Big)\Big)e^{-f}\,d\mu_g \\ &= 2\int_M \Big(|\mathrm{Ric} + \nabla^2 f - \rho\mathrm{R}g|^2 + \frac{1}{n-1}|\mathrm{Ric} - \rho\mathrm{R}g|^2 \\ &\qquad + 2\rho\mathrm{R}^2\Big(\frac{n}{n-1}(1-n\rho) - \frac{n}{n-1}\Big)\Big)e^{-f}\,d\mu_g \\ &= 2\int_M \Big(|\mathrm{Ric} + \nabla^2 f - \rho\mathrm{R}g|^2 + \frac{1}{n-1}|\mathrm{Ric} - \rho\mathrm{R}g|^2 - \frac{2n^2}{n-1}\rho^2\mathrm{R}^2\Big)e^{-f}\,d\mu_g\,.\end{aligned}$$

In order to obtain a sufficient condition for the monotonicity of $\mathcal{F}_{\frac{n}{n-1}}$, we consider again the Cauchy–Schwarz inequality (5.6) and apply it to $\sigma = \mathrm{Ric} - \rho\mathrm{R}g$ obtaining

$$|\mathrm{Ric} - \rho\mathrm{R}g|^2 \geq \frac{(1-n\rho)^2}{n}\mathrm{R}^2\,.$$

Hence, by a straightforward computation, if ρ satisfies the condition

$$\rho \in \left[\frac{-1-\sqrt{2n}}{n(2n-1)}, \frac{-1+\sqrt{2n}}{n(2n-1)}\right],$$

then,

$$\frac{d}{dt}\mathcal{F}_{\frac{n}{n-1}} \geq 2\int_M |\mathrm{Ric} + \nabla^2 f - \rho\mathrm{R}g|^2 e^{-f}\,d\mu_g\,,$$

that is, the functional $\mathcal{F}_{\frac{n}{n-1}}$ is monotone during the RB–flow.
We summarize this computation in the following proposition.

PROPOSITION 5.2.2. *Under the coupled system*

$$\begin{cases} \frac{\partial}{\partial t}g = -2(\mathrm{Ric} + \nabla^2 f - \rho\mathrm{R}g) \\ \frac{\partial}{\partial t}f = -\Delta f - (1-n\rho)\mathrm{R} + 2\rho\mathrm{R} \end{cases} \tag{5.7}$$

the functional $\mathcal{F}_{\frac{n}{n-1}}$*, defined by formula* (5.3) *with* $C = \frac{n}{n-1}$*, satisfies the following variation formula*

$$\frac{d}{dt}\mathcal{F}_{\frac{n}{n-1}} = 2\int_M \Big(|\mathrm{Ric} + \nabla^2 f - \rho \mathrm{R} g|^2 + \frac{1}{n-1}|\mathrm{Ric} - \rho \mathrm{R} g|^2 - \frac{2n^2}{n-1}\rho^2 \mathrm{R}^2\Big) e^{-f}\, d\mu_g. \tag{5.8}$$

Therefore, $\mathcal{F}_{\frac{n}{n-1}}$ *is non decreasing along the coupled flow* (5.7)*, provided that*

$$\rho \in \left[\frac{-1-\sqrt{2n}}{n(2n-1)}, \frac{-1+\sqrt{2n}}{n(2n-1)}\right].$$

A second possibility arises from equation (5.5) if we suppose that the initial manifold has non negative scalar curvature. Since we proved, in Proposition 3.3.1, that this condition is preserved during the RB–flow, we can argue that the terms in $\mathrm{R}|\nabla f|^2$ of equation (5.5) have the sign of their coefficients. Hence, imposing the preserving measure variation of f, $k = -\Delta f - (1-n\rho)\mathrm{R}$, we get

$$\begin{aligned} \frac{d}{dt}\mathcal{F}_C &= \int_M \Big(2|\mathrm{Ric} + \nabla^2 f - \rho \mathrm{R} g|^2 + 2(C-1)|\mathrm{Ric} - \rho \mathrm{R} g|^2 \\ &\qquad + 2\rho \mathrm{R}\big(C(1-n\rho)\mathrm{R} - (C-2)\Delta f + (C-1)|\nabla f|^2\big) \\ &\qquad - 2\rho(C-1)n\mathrm{R}\big(|\nabla f|^2 - \Delta f\big)\Big) e^{-f}\, d\mu_g\,, \\ &= \int_M \Big(2|\mathrm{Ric} + \nabla^2 f - \rho \mathrm{R} g|^2 + 2(C-1)|\mathrm{Ric} - \rho \mathrm{R} g|^2 \\ &\qquad - 2\rho(n-1)(C-1)\mathrm{R}|\nabla f|^2 \\ &\qquad + 2\rho\big(n(C-1) - C + 2\big)\mathrm{R}\Delta f + 2\rho(1-n\rho)C\mathrm{R}^2\Big) e^{-f}\, d\mu_g\,, \end{aligned} \tag{5.9}$$

Since $C \geq 1$, the term in the second line is non negative for non positive values of ρ. As before, the term containing $\mathrm{R}\Delta f$ has not a definite sign and we use the Cauchy–Schwarz inequality (5.6)

$$2|\mathrm{Ric} + \nabla^2 f - \rho \mathrm{R} g|^2 \geq \frac{2}{n}\big((1-n\rho)^2\mathrm{R}^2 + (\Delta f)^2 + 2(1-n\rho)\mathrm{R}\Delta f\big)$$

to make null its coefficient, obtaining the following equation for the constant C

$$0 = \frac{2}{n}\big(2 - 2n\rho + n^2(C-1)\rho - n(C-2)\rho\big)\,,$$

hence,

$$C = 1 - \frac{2-n\rho}{n(n-1)\rho}\,, \tag{5.10}$$

which is always greater than 1 for negative values of ρ. Finally, by applying the Cauchy–Schwarz inequality also to the second square in the first line and collecting the terms in R^2 we obtain

$$\frac{d}{dt}\mathcal{F}_C \geq \frac{2C(1-n\rho)}{n}\int_M \mathrm{R}^2\, e^{-f}\, d\mu_g \geq 0\,. \tag{5.11}$$

By means of the previous computation we are then able to prove the following proposition.

PROPOSITION 5.2.3. *Suppose* $(M, g(t))$ *is a compact solution of the Ricci–Bourguignon flow on [0,T) for* $\rho < 0$*. If* $g(0)$ *has non negative scalar curvature, then there exists a constant* $C = C(\rho, n)$*, defined by equation* (5.10)*, such that the functional* $\mathcal{F}_C$ *is non decreasing along the coupled Ricci–Bourguignon flow*

$$\begin{cases} \frac{\partial}{\partial t} g = -2(\mathrm{Ric} - \rho \mathrm{R} g) \\ \frac{\partial}{\partial t} f = -\Delta f - (1-n\rho)\mathrm{R} + |\nabla f|^2 \end{cases}. \tag{5.12}$$

Moreover the monotonicity is strict unless the solution is a trivial Ricci–flat soliton.

PROOF. To prove the first part of the proposition, it is sufficient to follow Perelman's work [69, Proposition 1.2] (for instance, see [25, Sections 5.2] for details). Indeed, by modifying the variations of g and f by means of the 1–parameter group of diffeomorphisms generated by the vector field ∇f, we recover the RB–flow evolution equation for the metric g, while f satisfies

$$\frac{\partial}{\partial t} f = -\Delta f - (1 - n\rho)\mathrm{R} + |\nabla f|^2 \,.$$

Since the functional $\mathcal{F}_C$ is invariant by this diffeomorphism action on both g and f, the inequality (5.11) gives the monotonicity.
In order to prove the last statement, we observe, from the same inequality, that if the variation at time s is 0, then the metric $g(s)$ has null scalar curvature, and the variation formula (at time s) of equation (5.9) becomes

$$0 = \frac{d}{dt}\mathcal{F}_C(s) = \int_M \Big(2|\mathrm{Ric} + \nabla^2 f|^2 + 2(C-1)|\mathrm{Ric}|^2\Big) e^{-f}\, d\mu_g \,.$$

Hence, the metric $g(s)$ is Ricci–flat and $f(s)$ is constant since it is an harmonic function on a compact manifold. By uniqueness of the solution of the Ricci–Bourguignon flow for $\rho < 0$, we conclude that $g(0)$ is Ricci–flat and $g(t) = g(0)$ for every time t. □

REMARK 5.2.4. We observe that the "natural" functional (5.2), arising from the soliton equation (3.24), corresponds to $\mathcal{F}_C$, for $C = (1 - n\rho) > 1$ for negative values of ρ. But, even under the hypotheses of non negative scalar curvature, we are not able to control the term containing $\mathrm{R}\Delta f$ in the variation formula by means of the squares of the first line.

REMARK 5.2.5. Another way, non yet explored, to relax Perelman's original idea is to consider the class of functionals

$$\mathcal{F}_\beta(g, f) = \int_M \big((1 - n\rho)\mathrm{R} + \beta|\nabla f|^2\big) e^{-\beta f}\, d\mu_g \,,$$

where β is a real constant. The expression above comes again from the steady soliton equation, but integrated with respect to the measure $e^{-\beta f}\, d\mu_g$.

The investigation of functionals of type $\mathcal{W}$–entropy, either scaling invariant *à la* Perelman, or in the relaxed way of Li, is at the beginning. A monotonicity formula for these quantities might lead, for instance, to a lower bound for the injectivity radius, allowing us to take limits after the blow–up procedure, in presence of a singularity of the flow.
More generally, we are also investigating if some variational techniques, useful for the Yamabe flow, can be employed successfully in the study of the Ricci–Bourguignon flow.

Finally, we mention that an application of the present work can be found in the recent paper [23] by Chen, He and Zeng, who studied monotonicity of eigenvalues of Laplacian–type operators $-\Delta + c\mathrm{R}$, where c is a constant, along the Ricci–Bourguignon flow.

APPENDIX A

Lie algebra structure of $\Lambda^2(T_pM)$ and algebraic curvature operators

In this section we fix the notations about the Lie algebra structure that we put on $\Lambda^2(T_pM)$ which allows us to see the algebraic properties of the curvature operator and of its evolution equation.
We recall that, thanks to the symmetries of the Riemann tensor Rm, the associated curvature operator $\mathcal{R}$ can be seen as an element of $End_{SA}(\Lambda^2(TM)) \simeq S^2(\Lambda^2 TM^*)$, where SA means self–adjont with respect to the extension of g to exterior powers of TM. There is one more algebraic property satisfied by a curvature operator, the first Bianchi identity, which define a subspace of $S^2(\Lambda^2 TM^*)$ which we will denote with $\mathcal{C}_b(M)$. Following [42, Chapter 3, Section K], given a vector space with a scalar product $(V, \langle\,,\,\rangle)$ we define the space of *algebraic curvature operators* $\mathcal{C}_b(V)$ as the subspace of $S^2(\Lambda^2(V^*))$ of elements that satisfy the first Bianchi identity.
Here we do not follow the original convention of Hamilton on the curvature operator, afterward used also in [29] and [26], as the convention used, for example, in [1] seems more natural, even if the difference between the two notations is only a matter of constants. More precisely, with Hamilton's convention the eigenvalues of the curvature operator are twice the sectional curvatures, while here the eigenvalues are the sectional curvatures.
The construction below is the same for every point $p \in M$ at any time, so to simplify the notation we denote with $\langle\,,\rangle$ the scalar product on T_pM given by the metric g at a generic time t. If $\{e_i\}$ is an orthonormal basis of T_pM, then $\{e_i \wedge e_j\}_{i<j}$ is an orthonormal basis of $\Lambda^2(T_pM)$, as the scalar product is extended canonically by

$$\langle X \wedge Y, W \wedge Z\rangle = \det \begin{pmatrix} \langle X, W\rangle & \langle X, Z\rangle \\ \langle Y, W\rangle & \langle Y, Z\rangle \end{pmatrix} .$$

Now, in order to understand the reaction term in equation (3.9), we have to identify $\Lambda^2(T_pM)$ with the Lie algebra $\mathfrak{so}(n)$, by mapping an element $X \wedge Y$ to the linear endomorphism of $T_pM \simeq \mathbb{R}^n$ given by

$$X \wedge Y(W) = \langle Y, W\rangle X - \langle X, W\rangle Y ,$$

that is, the generic element of the orthonormal basis $e_i \wedge e_j$ is identified with the matrix $E_{ij} - E_{ji}$, where E_{ij} is the matrix with all null entries, except the (i, j)–th which is 1.
The structure of Lie algebra is thus inherited by $\Lambda^2(T_pM)$ with the bracket of $\mathfrak{so}(n)$

$$\begin{aligned}
[e_i \wedge e_j, e_k \wedge e_l] &= (E_{ij} - E_{ji})(E_{kl} - E_{lk}) - (E_{kl} - E_{lk})(E_{ij} - E_{ji}) \\
&= \delta_{jk}E_{il} - \delta_{ik}E_{jl} - \delta_{jl}E_{ik} + \delta_{il}E_{jk} - \delta_{il}E_{kj} + \delta_{ik}E_{lj} + \delta_{jl}E_{ki} - \delta_{jk}E_{li} \\
&= \delta_{jk}e_i \wedge e_l - \delta_{ik}e_j \wedge e_l - \delta_{jl}e_i \wedge e_k + \delta_{il}e_j \wedge e_k .
\end{aligned}$$

Using the scalar product $\langle\,,\rangle$ we can also canonically identify $\Lambda^2(T_pM)$ with $\Lambda^2(T_p^*M)$, therefore any $\varphi \in \Lambda^2(T_p^*M)$ can be written as

$$\varphi = \sum_{i<j} \varphi_{ij} e^i \wedge e^j = \frac{1}{2}\sum_{i,j} \varphi_{ij} e^i \wedge e^j ,$$

where $\{e^i\}_{i=1,\dots,n}$ is the dual basis of T_p^*M and $e^i \wedge e^j(e_k, e_l) = e^i \wedge e^j(e_k \wedge e_l) = \langle e_i \wedge e_j, e_k \wedge e_l\rangle$ from the canonical identification. In this way the coefficients of φ are

$$\varphi(e_k, e_l) = \sum_{i<j} \varphi_{ij} e^i \wedge e^j(e_k, e_l) = \varphi_{kl} = \langle \varphi, e_k \wedge e_l\rangle .$$

i.e. the notation is consistent.
It is a simple computation to show the structure of the Lie algebra $\Lambda^2(T_pM)$ (and the corresponding one of $\Lambda^2(T_p^*M)$) with respect to the basis $\{e_i \wedge e_j\}_{i<j}$.

$$[\varphi, \psi]_{ij} = \sum_p \varphi_{ip}\psi_{pj} - \psi_{ip}\varphi_{pj}$$

To simplify the notation, we can consider a generic orthonormal basis $\{\theta_\alpha\}_{\alpha=1,\dots,n(n-1)/2}$ of $\Lambda^2(T_pM)$; then the Lie algebra is encoded in the structure constants $c^\gamma_{\alpha\beta}$ defined by

$$[\theta_\alpha, \theta_\beta] = \sum_\gamma c^\gamma_{\alpha\beta}\theta_\gamma.$$

These structure constants satisfy the following properties

(1) $c^\gamma_{\alpha\beta} = \langle[\theta_\alpha, \theta_\beta], \theta_\gamma\rangle$ as the basis is orthonormal;
(2) $c^\gamma_{\alpha\beta} = -c^\gamma_{\beta\alpha} = -c^\alpha_{\gamma\beta}$;
(3) if $\{\varphi^\alpha\}$ is the dual basis of $\Lambda^2(T_p^*M)$ with corresponding constant structures $c_\gamma^{\alpha\beta}$, then $c_\gamma^{\alpha\beta} = c^\gamma_{\alpha\beta}$.

We can now define the two following symmetric inner products of $S^2(\Lambda^2(T_p^*M))$ that we will use to rewrite the reaction term in equation (3.9). Given $P, Q \in S^2(\Lambda^2(T_p^*M))$, we set

$$(PQ)_{\alpha\beta} = \frac{1}{2}\sum_\gamma (P_{\alpha\gamma}Q_{\gamma\beta} + Q_{\alpha\gamma}P_{\gamma\beta}) \tag{A.1}$$

$$(P\#Q)_{\alpha\beta} = \frac{1}{2}c_\alpha^{\gamma\eta}c_\beta^{\delta\nu}P_{\gamma\delta}Q_{\eta\nu} \tag{A.2}$$

Finally, we write some useful relations between the curvature operator and the Riemann tensor: with respect to the orthonormal basis $\{e_i \wedge e_j\}_{i<j}$ we have

$$\mathcal{R}(e_i \wedge e_j) = \sum_{k<l} \mathcal{R}^{(kl)}_{(ij)} e_k \wedge e_l = \frac{1}{2}\sum_{k,l} \mathcal{R}^{(kl)}_{(ij)} e_k \wedge e_l\,,$$

where, by definition, $\mathcal{R}^{(kl)}_{(ij)} = \mathrm{R}_{ijkl}$. The relation between the scalar curvature R and the coefficients of the curvature operator w.r.t. this frame is the following

$$\mathrm{R} = \mathrm{R}_{ijkl}g^{ik}g^{jl} = \sum_{i,k=1}^n \mathrm{R}_{ikik} = 2\sum_{i<k}\mathrm{R}_{ikik} = 2\sum_{i<k}\mathcal{R}^{(ik)}_{(ik)}\,. \tag{A.3}$$

By the symmetries of the curvature operator, it is easy to see that $\mathcal{R}$ is self–adjoint with respect to $\langle\,,\rangle$, therefore it can be considered as an element of $S^2(\Lambda^2(T_p^*M))$, that is $\mathcal{R} = \sum_{\alpha,\beta}\mathcal{R}_{\alpha\beta}\varphi^\alpha \otimes \varphi^\beta$, with

$$\mathcal{R}_{\alpha\beta} = \mathcal{R}(\theta_\alpha, \theta_\beta) = \mathcal{R}(\sum_{i<j}\theta^{ij}_\alpha e_i \wedge e_j, \sum_{k<l}\theta^{kl}_\beta e_k \wedge e_l) = \sum_{i<j,k<l}\theta^{ij}_\alpha\theta^{kl}_\beta \mathrm{R}_{ijkl}\,.$$

where $\{\varphi^\alpha\}$ and $\{\theta_\alpha\}$ are dual orthonormal basis of $\Lambda^2(T_p^*M)$ and $\Lambda^2(T_pM)$.

LEMMA A.0.6. *The following equations hold true*

$$(\mathcal{R}^2)_{ijkl} = B(\mathrm{Rm})_{ijkl} - B(\mathrm{Rm})_{ijlk}\,; \tag{A.4}$$

$$(\mathcal{R}\#\mathcal{R})_{ijkl} = B(\mathrm{Rm})_{ikjl} - B(\mathrm{Rm})_{iljk}\,, \tag{A.5}$$

where $B(T)_{ijkl} = g^{pq}g^{rs}T_{ipjr}T_{kqls}$ *for every* $(4,0)$*–tensor.*

PROOF. Both the equations follow by straightforward computations, where we use several times the previous formulas. By definition, we have

$$\begin{aligned}
\mathcal{R}^2 &= (\mathcal{R}^2)_{\alpha\beta}\varphi^\alpha\otimes\varphi^\beta = \sum_\gamma \mathcal{R}_{\alpha\gamma}\mathcal{R}_{\gamma\beta}\varphi^\alpha\otimes\varphi^\beta\\
&= \sum_\gamma\sum_{i<j,p<q}\sum_{s<t,k<l}\theta^{ij}_\alpha\theta^{pq}_\gamma\theta^{st}_\gamma\theta^{kl}_\beta \mathrm{R}_{ijpq}\mathrm{R}_{stkl}\varphi^\alpha\otimes\varphi^\beta\\
&= \sum_\gamma\sum_{i<j,p<q}\sum_{s<t,k<l}\theta^{pq}_\gamma\theta^{st}_\gamma \mathrm{R}_{ijpq}\mathrm{R}_{stkl}(e^i\wedge e^j)\otimes(e^k\wedge e^l)\\
&= \sum_{i<j,p<q}\sum_{s<t,k<l} < e^p\wedge e^q, e^s\wedge e^t > \mathrm{R}_{ijpq}\mathrm{R}_{stkl}(e^i\wedge e^j)\otimes(e^k\wedge e^l)\\
&= \sum_{i<j,k<l,p<q}\mathrm{R}_{ijpq}\mathrm{R}_{pqkl}(e^i\wedge e^j)\otimes(e^k\wedge e^l)\,.
\end{aligned}$$

Therefore, using the first Bianchi identity,

$$\begin{aligned}
\mathcal{R}^2_{ijkl} &= \sum_{p<q}\mathrm{R}_{ijpq}\mathrm{R}_{klpq} = \sum_{p<q}(\mathrm{R}_{ipqj}+\mathrm{R}_{iqjp})(\mathrm{R}_{kpql}+\mathrm{R}_{kqlp})\\
&= \frac{1}{2}(B_{ijkl}-B_{ijlk}-B_{ijlk}+B_{ijkl}) = B_{ijkl}-B_{ijlk}\,.
\end{aligned}$$

For the second equation, we get

$$\begin{aligned}
\mathcal{R}\#\mathcal{R} &= (\mathcal{R}\#\mathcal{R})_{\alpha\beta}\varphi^\alpha\otimes\varphi^\beta = \frac{1}{2}c^{\gamma\eta}_\alpha c^{\delta\nu}_\beta\mathcal{R}_{\gamma\delta}\mathcal{R}_{\eta\nu}\\
&= \frac{1}{2}\sum_{i<j,k<l}c^{\gamma\eta}_\alpha c^{\delta\nu}_\beta\mathcal{R}_{\gamma\delta}\mathcal{R}_{\eta\nu}\varphi^\alpha_{ij}\varphi^\beta_{kl}(e^i\wedge e^j)\otimes(e^k\wedge e^l)\,.
\end{aligned}$$

Now, by definition of the structure constants, it follows

$$\begin{aligned}
c^{\gamma\eta}_\alpha\varphi^\alpha_{ij} &= [\varphi^\gamma,\varphi^\eta](\theta_\alpha)\varphi^\alpha_{ij} = [\varphi^\gamma,\varphi^\eta](e_i\wedge e_j)\\
&= \sum_p(\varphi^\gamma_{ip}\varphi^\eta_{pj}-\varphi^\eta_{ip}\varphi^\gamma_{pj})\,;
\end{aligned}$$

and in the same way

$$c^{\delta\nu}_\beta\varphi^\beta_{kl} = \sum_q(\varphi^\delta_{kq}\varphi^\nu_{ql}-\varphi^\nu_{kq}\varphi^\delta_{ql})\,.$$

Plugging these two last formulas in the expression of $(\mathcal{R}\#\mathcal{R})_{ijkl} = \mathcal{R}^\#_{ijkl}$, we obtain

$$\begin{aligned}
\mathcal{R}^\#_{ijkl} &= \frac{1}{2}\sum_{p,q}(\varphi^\gamma_{ip}\varphi^\eta_{pj}-\varphi^\eta_{ip}\varphi^\gamma_{pj})(\varphi^\delta_{kq}\varphi^\nu_{ql}-\varphi^\nu_{kq}\varphi^\delta_{ql})\mathcal{R}_{\gamma\delta}\mathcal{R}_{\eta\nu}\\
&= \frac{1}{2}\sum_{p,q}\varphi^\gamma_{ip}\varphi^\eta_{pj}\varphi^\delta_{kq}\varphi^\nu_{ql}\mathcal{R}_{\gamma\delta}\mathcal{R}_{\eta\nu} - \frac{1}{2}\sum_{p,q}\varphi^\gamma_{ip}\varphi^\eta_{pj}\varphi^\nu_{kq}\varphi^\delta_{ql}\mathcal{R}_{\gamma\delta}\mathcal{R}_{\eta\nu}\\
&\quad -\frac{1}{2}\sum_{p,q}\varphi^\eta_{ip}\varphi^\gamma_{pj}\varphi^\delta_{kq}\varphi^\nu_{ql}\mathcal{R}_{\gamma\delta}\mathcal{R}_{\eta\nu} + \frac{1}{2}\sum_{p,q}\varphi^\eta_{ip}\varphi^\gamma_{pj}\varphi^\nu_{kq}\varphi^\delta_{ql}\mathcal{R}_{\gamma\delta}\mathcal{R}_{\eta\nu}\\
&= \frac{1}{2}\sum_{p,q}(\mathcal{R}_{ipkq}\mathcal{R}_{pjql}-\mathcal{R}_{ipql}\mathcal{R}_{pjkq}-\mathcal{R}_{pjkq}\mathcal{R}_{ipql}+\mathcal{R}_{pjql}\mathcal{R}_{ipkq})\\
&= \frac{1}{2}(B_{ikjl}-B_{iljk}-B_{iljk}+B_{ikjl}) = B_{ikjl}-B_{iljk}
\end{aligned}$$

by the symmetries of the curvature operator. □

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