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On Nonlinear Higher Spin Curvature

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ABSTRACT

We present the first nonlinear term of the higher spin curvature which is covariant with respect to deformed gauge transformations that are linear in the field. We consider in detail the case of spin 3 after presenting spin 2 as an example, and then construct the general spin s quadratic term of the deWit-Freedman curvature.

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1   Introduction

Geometry plays a very important role in field theory. General relativity is the most beautiful and obvious example to prove this. In higher spin gauge field theory the elements of geometry were studied first in the classical paper by Bernard de Witt and Daniel Freedman [1]. Geometrical interpretations of Yang-Mills theory and general relativity hint on possible geometric structures of higher spin gauge field theories in general [1], [2], [3], [4], [5]. For the well analyzed existing theories gauge symmetry itself has a hidden geometrical origin. Since the gauge symmetry is a redundancy in the (non-observable) variables of the physical system, the observables of any gauge theory are gauge invariant. For electrodynamics the observables are the components of the Maxwell tensor $F_{\mu\nu}$ (curvature of the electromagnetic field A_μ), for gravity these are the components of the Riemann curvature tensor $R_{\mu\nu\lambda\rho}$ (constructed from the dynamical field, the metric tensor $g_{\mu\nu}$). In analogy with these cases, in the higher spin gauge field theory the curvature of the higher spin field should be observable. The linear curvature as well as linearized "Cristoffel symbols" and connections for any higher spin field are known from [1]. Taking traces of the linearized curvature one can get geometrical free equations of motion for the unconstrained higher spin gauge field [2], which is nonlocal, but can be localized with the help of partial gauge fixing after which the geometrical equation coincides with Fronsdal's equation.

In this letter we construct a higher spin curvature that is of second order in the field, which can be used to find that part of the geometric equations of motion that are quadratic in the field, which, after the same partial gauge fixing (ruling out the trace of the gauge parameter and the double trace of the field), may or may not coincide with the corresponding second order of Fronsdal's equation of motion coming from the cubic Lagrangian [8]. The cubic (self)interaction Lagrangian and the corresponding gauge transformation laws for Fronsdal's fields that are of first order on field, are known [7] - [12]. The connection between these two independent constructions is a subject for future investigations. Now we are going to construct the first nonlinear term in the higher spin curvature independently, and with no use of Fronsdal's constraints on the fields involved. In order to get a gauge covariant curvature, we don't need to impose the constraint of tracelessness on the gauge parameter respectively the constraint of double tracelessness on the higher spin field. We don't make use of Fronsdal's theory in this letter, because the nonlinear curvature that we construct here, describes not only Fronsdal's (constrained) fields, but, if it exists to all orders in the field, gives us a possibility for the construction of theories of Higher Spin fields alternative to Fronsdal's (consider the nonlinear continuation of conformal higher spin theory [19, 20] as an example).

In Section 2 we discuss as a simple example linearized gravity and construct the second order of the Riemann curvature using the gauge covariance condition. In Section 3 we construct the curvature of second order in the field for the spin 3 gauge field. In the final Section 4 we derive that part of the nonlinear gauge covariant curvature that is of second order in the field for any higher spin field

which reproduces all lower spin cases and has the same symmetry properties as the linear curvature.

The most remarkable property of the curvature that is of second order in the field is that it is a sum of terms, which are all squares of the linear connection $\Gamma^{(s-1)}$, the s minus first member in the hierarchy of de Witt - Freedman - Christoffel symbols [1]. Using shorthands explained later, we can summarize the curvature up to second order in the field in the following compact way

$$R = d\Gamma + \Gamma \star \Gamma \tag{1.1}$$

for any higher spin gauge field.

2 Quadratic term of linearized Riemann curvature

We start from a consideration of the linearized Riemann curvature (spin 2 case). We use the following self consistency definitions for covariant derivatives, Christoffel symbols and curvature:

$$\nabla_\mu V_\nu = \partial_\mu V_\nu + \Gamma^\rho_{\mu\nu} V_\rho \tag{2.1}$$

$$[\nabla_\mu, \nabla_\nu] V_\lambda = R_{\mu\nu,\lambda}{}^\rho V_\rho \tag{2.2}$$

$$\Gamma_{\rho,\mu\nu} = g_{\rho\sigma} \Gamma^\sigma_{\mu\nu} = \frac{1}{2}(\partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} - \partial_\nu g_{\mu\rho}) \tag{2.3}$$

$$R_{\mu\nu,\lambda}{}^\rho = R_{\mu\nu,\lambda\delta} g^{\delta\rho} = \partial_\mu \Gamma^\rho_{\nu\lambda} - \partial_\nu \Gamma^\rho_{\mu\lambda} + \Gamma^\sigma_{\mu\lambda} \Gamma^\rho_{\nu\sigma} - \Gamma^\sigma_{\nu\lambda} \Gamma^\rho_{\mu\sigma} \tag{2.4}$$

Then we note that expression (2.3) is very convenient for a linearization and we can obtain a linearized Christoffel symbol just replacing the metric $g_{\mu\nu}$ by the linearized field $h_{\mu\nu} = g_{\mu\nu} - \delta_{\mu\nu}$. To obtain the corresponding expression for the curvature ready for linearization we can use (2.3) and after some algebra write the full covariant curvature $R_{\mu\nu,\lambda\rho}$ in the form

$$R_{\mu\nu,\lambda\rho} = \partial_\mu \Gamma_{\rho,\nu\lambda} - \partial_\nu \Gamma_{\rho,\mu\lambda} - g^{\sigma\delta} \left(\Gamma_{\sigma,\mu\lambda} \Gamma_{\delta,\nu\rho} - \Gamma_{\sigma,\nu\lambda} \Gamma_{\delta,\mu\rho} \right) \tag{2.5}$$

Then substituting in (2.5)

$$g^{\mu\nu} = \delta^{\mu\nu} - h^{\mu\nu} + h^\mu_\sigma h^{\nu\sigma} - \dots \tag{2.6}$$

$$2\Gamma_{\rho,\mu\nu} = \partial_\rho h_{\mu\nu} - \partial_\mu h_{\nu\rho} - \partial_\nu h_{\mu\rho} \tag{2.7}$$

we arrive at the following expansion of the curvature up to the third order in the field:

$$R^h_{\mu\nu,\lambda\rho} = R^{(1)}_{\mu\nu,\lambda\rho} + R^{(2)}_{\mu\nu,\lambda\rho} + R^{(3)}_{\mu\nu,\lambda\rho} + \dots \tag{2.8}$$

$$2R^{(1)}_{\mu\nu,\lambda\rho} = \partial_\mu \partial_\rho h_{\nu\lambda} - \partial_\nu \partial_\rho h_{\mu\lambda} - \partial_\mu \partial_\lambda h_{\nu\rho} + \partial_\nu \partial_\lambda h_{\mu\rho} \tag{2.9}$$

$$R^{(2)}_{\mu\nu,\lambda\rho} = -\delta^{\sigma\delta} \left(\Gamma_{\sigma,\mu\lambda} \Gamma_{\delta,\nu\rho} - \Gamma_{\sigma,\nu\lambda} \Gamma_{\delta,\mu\rho} \right) \tag{2.10}$$

$$R^{(3)}_{\mu\nu,\lambda\rho} = h^{\sigma\delta} \left(\Gamma_{\sigma,\mu\lambda} \Gamma_{\delta,\nu\rho} - \Gamma_{\sigma,\nu\lambda} \Gamma_{\delta,\mu\rho} \right) \tag{2.11}$$

Finalizing this section we note that the same expansion could be recovered from the initial linearized curvature (2.9) and gauge invariance of order zero in the field $\delta_{(0)}h_{\mu\nu} = \partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu$ using the "covariant" (not invariant) Noether's equation:

$$\delta_{(1)}R^{(1)}_{\mu\nu,\lambda\rho} + \delta_{(0)}R^{(2)}_{\mu\nu,\lambda\rho} = \mathcal{L}_\epsilon R^{(1)}_{\mu\nu,\lambda\rho} \tag{2.12}$$

where $\mathcal{L}_\epsilon R^{(1)}_{\mu\nu,\lambda\rho}$ is the Lie derivative of the first order curvature

$$\begin{aligned}
\mathcal{L}_\epsilon R^{(1)}_{\mu\nu,\lambda\rho} = \quad & \varepsilon^\rho\partial_\rho R^{(1)}_{\alpha\beta,\mu\nu} + \partial_\alpha\varepsilon^\rho R^{(1)}_{\rho\beta,\mu\nu} + \partial_\beta\varepsilon^\rho R^{(1)}_{\alpha\rho,\mu\nu} \\
& +\partial_\mu\varepsilon^\rho R^{(1)}_{\alpha\beta,\rho\nu} + \partial_\nu\varepsilon^\rho R^{(1)}_{\alpha\beta,\mu\rho}
\end{aligned} \tag{2.13}$$

It is easy to see that as a solution of the equation (2.12) we will find (2.10) and the following gauge transformation of first order in the field:

$$\delta_{(1)}h_{\mu\nu} = \epsilon^\lambda\Gamma_{\lambda,\mu\nu} + \partial_\mu f_\nu(h,\epsilon) + \partial_\nu f_\mu(h,\epsilon) \tag{2.14}$$

where $f_\mu(h,\epsilon)$ is at this stage an arbitrary vector function linear in the field h and the parameter ϵ. The linearized curvature (2.9) is invariant with respect to any gradient transformation $\delta h_{\mu\nu} = \partial_\mu f_\nu + \partial_\nu f_\mu$ with the arbitrary vector parameter f_μ. Nevertheless the next order of Noether's procedure fixes this ambiguity, and we obtain the well known gravitational gauge transformation $\delta h_{\mu\nu} = \epsilon^\rho\partial_\rho h_{\mu\nu} + \partial_\mu\epsilon^\rho h_{\nu\rho} + \partial_\nu\epsilon^\rho h_{\mu\rho}$ ($f_\mu = h_{\mu\rho}\epsilon^\rho$) in the first order on the field.

In the next sections we generalize this covariant Noether's procedure for the spin 3 and the general spin s case constructing the unknown quadratic part of the higher spin curvature.

3   The Case of Spin 3

To handle the spin 3 case we should introduce an additional nonabelian charge to avoid trivialization of the theory [16]. Then our spin three field $h^a_{\alpha\beta\gamma}$ and the symmetry parameter $\varepsilon^a_{\beta\gamma}$ carry an additional Lie algebra basis index. Introducing the corresponding zero order gauge transformation in the field

$$\delta^0_\varepsilon h^a_{\alpha\beta\gamma} = 3\partial_{(\alpha}\varepsilon^a_{\beta\gamma)} = \partial_\alpha\varepsilon^a_{\beta\gamma} + \partial_\beta\varepsilon^a_{\alpha\gamma} + \partial_\gamma\varepsilon^a_{\alpha\beta} \tag{3.1}$$

we define the first order curvature

$$\begin{aligned}
R^{a(1)}_{\alpha\alpha',\beta\beta',\gamma\gamma'} = \quad & \partial_\alpha\partial_\beta\partial_\gamma h^a_{\alpha'\beta'\gamma'} \\
& -\partial_{\alpha'}\partial_\beta\partial_\gamma h^a_{\alpha\beta'\gamma'} - \partial_\alpha\partial_{\beta'}\partial_\gamma h^a_{\alpha'\beta\gamma'} - \partial_\alpha\partial_\beta\partial_{\gamma'} h^a_{\alpha'\beta'\gamma} \\
& +\partial_{\alpha'}\partial_{\beta'}\partial_\gamma h^a_{\alpha\beta\gamma'} + \partial_{\alpha'}\partial_\beta\partial_{\gamma'} h^a_{\alpha\beta'\gamma} + \partial_\alpha\partial_{\beta'}\partial_{\gamma'} h^a_{\alpha'\beta\gamma} \\
& -\partial_{\alpha'}\partial_{\beta'}\partial_{\gamma'} h^a_{\alpha\beta\gamma}
\end{aligned} \tag{3.2}$$

from the standard condition of gauge invariance

$$\delta^0_\varepsilon R^{a(1)}_{\alpha\alpha',\beta\beta',\gamma\gamma'} = 0 \tag{3.3}$$

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Turning to the next step we should solve the following Noether's equation

$$\delta_\varepsilon^0 R_{\alpha\alpha',\beta\beta',\gamma\gamma'}^{a(2)} + \delta_\varepsilon^1 R_{\alpha\alpha',\beta\beta',\gamma\gamma'}^{a(1)} = 0 + O(R_{\alpha\alpha',\beta\beta',\gamma\gamma'}^{a(1)}, \varepsilon^{b\mu\nu}) \tag{3.4}$$

where $O(R_{\alpha\alpha',\beta\beta',\gamma\gamma'}^{a(1)}, \varepsilon^{b\mu\nu})$ represents some expression that is linear in the first order curvature and the gauge parameter and has two derivatives, which plays the role of the spin 3 generalization of the Lie derivative in (2.12).

To present a solution of (3.4) we should first introduce the spin 3 generalization of the Christoffel symbol (2.7):

$$
\begin{aligned}
\Gamma_{\mu\nu,\alpha\beta\gamma}^a \quad &= \partial_\mu\partial_\nu h_{\alpha\beta\gamma}^a \\
&- \frac{1}{2}\partial_\alpha\partial_\mu h_{\nu\beta\gamma}^a - \frac{1}{2}\partial_\alpha\partial_\nu h_{\mu\beta\gamma}^a - \frac{1}{2}\partial_\beta\partial_\mu h_{\alpha\nu\gamma}^a \\
&- \frac{1}{2}\partial_\beta\partial_\nu h_{\alpha\mu\gamma}^a - \frac{1}{2}\partial_\gamma\partial_\mu h_{\alpha\beta\nu}^a - \frac{1}{2}\partial_\gamma\partial_\nu h_{\alpha\beta\mu}^a \\
&+ \partial_\alpha\partial_\beta h_{\mu\nu\gamma}^a + \partial_\alpha\partial_\gamma h_{\mu\beta\nu}^a + \partial_\beta\partial_\gamma h_{\alpha\mu\nu}^a
\end{aligned} \tag{3.5}
$$

This expression differs from the second generalized Christoffel symbol of deWit-Freedman [1] only by an additional Lie algebra index. Then we can present the following expressions for the second order curvature

$$
\begin{aligned}
R_{\alpha\alpha',\beta\beta',\gamma\gamma'}^{a(2)} \quad &= f^{abc}(\Gamma_{\alpha\beta\gamma}^{b\mu\nu}\Gamma_{\mu\nu,\alpha'\beta'\gamma'}^c \\
&- \Gamma_{\alpha'\beta\gamma}^{b\mu\nu}\Gamma_{\mu\nu,\alpha\beta'\gamma'}^c - \Gamma_{\alpha\beta'\gamma}^{b\mu\nu}\Gamma_{\mu\nu,\alpha'\beta\gamma'}^c - \Gamma_{\alpha\beta\gamma'}^{b\mu\nu}\Gamma_{\mu\nu,\alpha'\beta'\gamma}^c \\
&+ \Gamma_{\alpha'\beta'\gamma}^{b\mu\nu}\Gamma_{\mu\nu,\alpha\beta\gamma'}^c + \Gamma_{\alpha'\beta\gamma'}^{b\mu\nu}\Gamma_{\mu\nu,\alpha\beta'\gamma}^c + \Gamma_{\alpha\beta'\gamma'}^{b\mu\nu}\Gamma_{\mu\nu,\alpha'\beta\gamma}^c \\
&- \Gamma_{\alpha'\beta'\gamma'}^{b\mu\nu}\Gamma_{\mu\nu,\alpha\beta\gamma}^c)
\end{aligned} \tag{3.6}
$$

and the first order gauge transformation

$$
\begin{aligned}
\delta_\varepsilon^1 h_{\alpha\beta\gamma}^a = \quad &f^{abc}(\varepsilon^{b\mu\nu}\partial_\mu\partial_\nu h_{\alpha\beta\gamma}^c \\
&+ \partial_\alpha\varepsilon^{b\mu\nu}\partial_\mu h_{\nu\beta\gamma}^c + \partial_\beta\varepsilon^{b\mu\nu}\partial_\mu h_{\alpha\nu\gamma}^c + \partial_\gamma\varepsilon^{b\mu\nu}\partial_\mu h_{\alpha\beta\nu}^c \\
&+ \partial_\alpha\partial_\beta\varepsilon^{b\mu\nu}h_{\mu\nu\gamma}^c + \partial_\alpha\partial_\gamma\varepsilon^{b\mu\nu}h_{\mu\beta\nu}^c + \partial_\beta\partial_\gamma\varepsilon^{b\mu\nu}h_{\alpha\mu\nu}^c)
\end{aligned} \tag{3.7}
$$

This form of $\delta_\varepsilon^1 h_{\alpha\beta\gamma}^a$ is not unique at this stage of Noether's procedure and defined due to $\delta_\varepsilon^0 h_{\alpha\beta\gamma}^a$ with linearly field dependent gauge parameter. This can be easily seen comparing (3.4) and (3.3).

Inserting (3.6) and (3.7) into Noether's equation (3.4) we obtain the following

nice result

$$\delta_\varepsilon^0 R^{a(2)}_{\alpha\alpha',\beta\beta',\gamma\gamma'} + \delta_\varepsilon^1 R^{a(1)}_{\alpha\alpha',\beta\beta',\gamma\gamma'} = f^{abc}\big(\varepsilon^{b\mu\nu}\partial_\mu\partial_\nu R^{c(1)}_{\alpha\alpha',\beta\beta',\gamma\gamma'}$$

$$+\partial_\alpha\varepsilon^{b\mu\nu}\partial_\nu R^{c(1)}_{\mu\alpha',\beta\beta',\gamma\gamma'} + \partial_{\alpha'}\varepsilon^{b\mu\nu}\partial_\nu R^{c(1)}_{\alpha\mu,\beta\beta',\gamma\gamma'} + \partial_\beta\varepsilon^{b\mu\nu}\partial_\nu R^{c(1)}_{\alpha\alpha',\mu\beta',\gamma\gamma'}$$

$$+\partial_{\beta'}\varepsilon^{b\mu\nu}\partial_\nu R^{c(1)}_{\alpha\alpha',\beta\mu,\gamma\gamma'} + \partial_\gamma\varepsilon^{b\mu\nu}\partial_\nu R^{c(1)}_{\alpha\alpha',\beta\beta',\mu\gamma'} + \partial_{\gamma'}\varepsilon^{b\mu\nu}\partial_\nu R^{c(1)}_{\alpha\alpha',\beta\beta',\gamma\mu}$$

$$+\partial_\alpha\partial_\beta\varepsilon^{b\mu\nu} R^{c(1)}_{\mu\alpha',\nu\beta',\gamma\gamma'} + \partial_\alpha\partial_\gamma\varepsilon^{b\mu\nu} R^{c(1)}_{\mu\alpha',\beta\beta',\nu\gamma'} + \partial_\beta\partial_\gamma\varepsilon^{b\mu\nu} R^{c(1)}_{\alpha\alpha',\mu\beta',\nu\gamma'}$$

$$+\partial_{\alpha'}\partial_{\beta'}\varepsilon^{b\mu\nu} R^{c(1)}_{\alpha\mu,\beta\nu,\gamma\gamma'} + \partial_{\alpha'}\partial_{\gamma'}\varepsilon^{b\mu\nu} R^{c(1)}_{\alpha\mu,\beta\beta',\gamma\nu} + \partial_{\beta'}\partial_{\gamma'}\varepsilon^{b\mu\nu} R^{c(1)}_{\alpha\alpha',\beta\mu,\gamma\nu}$$

$$+\partial_{\alpha'}\partial_\beta\varepsilon^{b\mu\nu} R^{c(1)}_{\alpha\mu,\nu\beta',\gamma\gamma'} + \partial_{\alpha'}\partial_\gamma\varepsilon^{b\mu\nu} R^{c(1)}_{\alpha\mu,\beta\beta',\nu\gamma'} + \partial_{\beta'}\partial_\gamma\varepsilon^{b\mu\nu} R^{c(1)}_{\alpha\alpha',\beta\mu,\nu\gamma'}$$

$$+\partial_{\beta'}\partial_\alpha\varepsilon^{b\mu\nu} R^{c(1)}_{\nu\alpha',\beta\mu,\gamma\gamma'} + \partial_{\gamma'}\partial_\alpha\varepsilon^{b\mu\nu} R^{c(1)}_{\mu\alpha',\beta\beta',\gamma\nu} + \partial_{\gamma'}\partial_\beta\varepsilon^{b\mu\nu} R^{c(1)}_{\alpha\alpha',\mu\beta',\gamma\nu}\big) \quad (3.8)$$

So we see that interpreting the right hand side of (3.7) as a spin 3 generalization of the Lie derivative of symmetric covariant tensors, we obtain the r.h.s. of (3.8) as constructed in the same way as "Lie derivative" of the first order curvature $R^{a(1)}_{\alpha\alpha',\beta\beta',\gamma\gamma'}$ with three pairs of antisymmetrized indices. Finally note also that the first order transformation (3.7) can be rewritten in the following form

$$\delta_\varepsilon^1 h^a_{\alpha\beta\gamma} = \quad f^{abc}\big[\partial_\alpha(\varepsilon^{b\mu\nu}\partial_\mu h^c_{\nu\beta\gamma} - \varepsilon^{b\mu\nu}\partial_\beta h^c_{\mu\nu\gamma} - \varepsilon^{b\mu\nu}\partial_\gamma h^c_{\mu\nu\beta})$$

$$+\partial_\beta(\varepsilon^{b\mu\nu}\partial_\mu h^c_{\nu\gamma\alpha} - \varepsilon^{b\mu\nu}\partial_\gamma h^c_{\mu\nu\alpha} - \varepsilon^{b\mu\nu}\partial_\alpha h^c_{\mu\nu\gamma})$$

$$+\partial_\gamma(\varepsilon^{b\mu\nu}\partial_\mu h^c_{\nu\alpha\beta} - \varepsilon^{b\mu\nu}\partial_\alpha h^c_{\mu\nu\beta} - \varepsilon^{b\mu\nu}\partial_\beta h^c_{\mu\nu\alpha})$$

$$+\partial_\alpha\partial_\beta(\varepsilon^{b\mu\nu} h^c_{\mu\nu\gamma}) + \partial_\beta\partial_\gamma(\varepsilon^{b\mu\nu} h^c_{\mu\nu\alpha}) + \partial_\gamma\partial_\alpha(\varepsilon^{b\mu\nu} h^c_{\mu\nu\beta})$$

$$+\varepsilon^{b\mu\nu}\Gamma^c_{\mu\nu,\alpha\beta\gamma}\big] \quad (3.9)$$

and therefore we can separate in (3.9) at this stage inessential symmetrized gradients (i.e. δ^0 with field dependent parameter) and obtain the essential part of $\delta_\varepsilon^1 h^a_{\alpha\beta\gamma}$ in the following elegant form:

$$\tilde\delta_\varepsilon^1 h^a_{\alpha\beta\gamma} = f^{abc}\varepsilon^{b\mu\nu}\Gamma^c_{\mu\nu,\alpha\beta\gamma} \quad (3.10)$$

4   The general spin s case

To start the work with general symmetric tensors we follow the notations of our previous papers (see [14] and ref. there) and introduce an additional formal vector variable a^μ to handle rank s symmetric tensors as the monomials

$$h^{(s)}(x;a) = h_{\mu_1\mu_2\dots\mu_s}(x)a^{\mu_1}a^{\mu_2}\dots a^{\mu_s} \quad (4.1)$$

In these notes we need to define only two operations to perform all calculations:

- Symmetrized gradient

$$\partial_{(\mu_{s+1}}h_{\mu_1\mu_2\dots\mu_s)} \Rightarrow (a\nabla)h^{(s)}(x;a) \quad (4.2)$$

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- Contraction inside the set of symmetrized indices (star product):

$$T(x)_{\mu_1\mu_2...\mu_s} H^{\mu_1\mu_2...\mu_s}(x) \Rightarrow T^{(s)}(x;a) *_a H^{(s)}(x;a).$$

$$\text{where} \quad *_a = \frac{1}{(s!)^2} \prod_{i=1}^{s} \overleftarrow{\partial}^{\mu_i}_a \overrightarrow{\partial}^a_{\mu_i}. \tag{4.3}$$

To distinguish easily between "a" and "x" spaces we introduce for space-time derivatives $\frac{\partial}{\partial x^\mu}$ the notation ∇_μ, $(a\nabla) = a^\mu \nabla_\mu$.

Then using these notations we can write a zero order gauge transformation (symmetrized gradient) in the following form:

$$\delta_{(0)} h^{(s)}(x;a) = (a\nabla)\epsilon^{(s-1)}(x;a) \tag{4.4}$$

where $\epsilon^{(s-1)}(x;a)$ is a rank $s-1$ symmetric tensor gauge parameter for the spin s gauge field[¶]. Other important expressions are the hierarchy of first order generalized Christoffel symbols introduced in [1]. These (n,s) bitensors

$$\Gamma^n_{(1)}(x;b,a) \equiv \Gamma^{(n)}_{(1)}(x)_{\rho_1...\rho_n,\mu_1...\mu_s} b^{\rho_1}...b^{\rho_n} a^{\mu_1}...a^{\mu_s} \tag{4.5}$$

can be written in our notation in an elegant form:

$$\Gamma^n_{(1)}(x;b,a) = \sum_{k=0}^{n} \frac{(-1)^k}{k!}(b\nabla)^{n-k}(a\nabla)^k(b\partial_a)^k h^{(s)}(x;a) \tag{4.6}$$

Inserting (4.4) into the latter we obtain the transformation law for these objects:

$$\delta_{(0)}\Gamma^n_{(1)}(x;b,a) = \frac{(-1)^n}{n!}(a\nabla)^{n+1}(b\partial_a)^n \epsilon^{(s-1)}(x;a), \tag{4.7}$$

So we see that last term of the hierarchy of the linearized curvature

$$R_{(1)}(x;b,a) = \Gamma^{(s)}_{(1)}(x;b,a) = \sum_{k=0}^{s} \frac{(-1)^k}{k!}(b\nabla)^{n-k}(a\nabla)^k(b\partial_a)^k h^{(s)}(x;a) \tag{4.8}$$

is invariant with respect to the gauge transformation (4.4):

$$\delta_{(0)}R_{(1)}(x;b,a) = 0 \tag{4.9}$$

Another important object for the present considerations is the last one before the curvature Christoffel symbol $\Gamma(x;b,a) = \Gamma^{s-1}_{(1)}(x;b,a)$ with the gauge transformation

$$\delta_{(0)}\Gamma(x;b,a) = (-1)^{s-1}(a\nabla)^s \epsilon^{(s-1)}(x;b) \tag{4.10}$$

[¶]We do not necessarily discuss constrained higher spin fields here (Fronsdal's formulation [6]), our discussion is relevant also for unconstrained higher spin fields representing general rank s symmetric tensors.

It is worth to note that $\Gamma(x; b, a)$ reproduces correctly (2.7) for $s = 2$ and (3.5) for $s = 3$. The expression (4.8) can be written in another useful form:

$$R_{(1)}(x; b, a) = \frac{1}{(s!)^2} [(b\partial_d)(a\partial_e) - (b\partial_e)(a\partial_d)]^s \, (d\nabla)^s h^{(s)}(x; e) \tag{4.11}$$

Finally using (4.6), (4.8) and (4.11) we present the connection between $\Gamma(x; b, a)$ and curvature $R_{(1)}(x; b, a)$

$$[(b\nabla)(a\partial_e) - (a\nabla)(b\partial_e)]\,\Gamma(x; c, e) = \frac{1}{s+1} [(b\partial_c)(a\partial_e) - (a\partial_c)(b\partial_e)]\, R_{(1)}(x; c, e) \tag{4.12}$$

Now we are ready to derive the second order curvature. First we propose the following generalization of (3.10) and (2.14) for the essential part of the first order spin s gauge transformation:

$$\delta_{(1)} h^{(s)}(x; a) = \epsilon^{(s-1)}(x; c) *_c \Gamma(x; c, a) \tag{4.13}$$

Then we calculate the first order variation of the first order curvature (4.11)

$$
\begin{aligned}
\delta_{(1)} R_{(1)}(x; b, a) \;=\;& \frac{1}{(s!)^2} [(b\partial_d)(a\partial_e) - (b\partial_e)(a\partial_d)]^s \left\{ \left[(d\nabla)^s \epsilon^{(s-1)}(x; c) \right] *_c \Gamma(x; c, e) \right. \\
&+ \left. \sum_{k=1}^{s} \binom{s}{k} \left[(d\nabla)^{s-k} \epsilon^{(s-1)}(x; c) \right] *_c (d\nabla)^k \Gamma(x; c, e) \right\}
\end{aligned} \tag{4.14}
$$

Applying (4.10) and (4.12) we can rewrite the latter variation in the following way:

$$
\begin{aligned}
\delta_{(1)} R_{(1)}(x; b, a) \;=\;& \frac{(-1)^{s-1}}{(s!)^2} A(b, a; \partial_d, \partial_e)^s \delta_{(0)} \Gamma(x; c, d) *_c \Gamma(x; c, e) \\
&+ \frac{1}{(s+1)!} \sum_{k=1}^{s} \binom{s}{k} [A(b, a; \nabla, \partial_e)]^{s-k} \epsilon^{(s-1)}(x; c) *_c \\
& [A(b, a; \nabla, \partial_e)]^{k-1} A(b, a; \partial_c, \partial_e) R_{(1)}(x; c, e)
\end{aligned} \tag{4.15}
$$

where

$$A(b, a; \nabla, \partial_e) = (b\nabla)(a\partial_e) - (a\nabla)(b\partial_e) \tag{4.16}$$
$$A(b, a; \partial_c, \partial_e) = (b\partial_c)(a\partial_e) - (a\partial_c)(b\partial_e) \tag{4.17}$$

So looking at (4.15) we see that after functional integration in the first line we obtain a solution of the expected covariant Noether's equation:

$$\delta_{(1)} R_{(1)}(x; b, a) + \delta_{(0)} R_{(2)}(x; b, a) = \mathcal{L}_\epsilon R_{(1)}(x; b, a) \tag{4.18}$$

where

$$R_{(2)}(x; b, a) = \frac{1}{2(s!)^2} [(b\partial_d)(a\partial_e) - (b\partial_e)(a\partial_d)]^s \, \Gamma(x; c, d) *_c \Gamma(x; c, e) \tag{4.19}$$

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is the second order curvature and

$$\mathcal{L}_\epsilon R_{(1)}(x; b, a) = \frac{1}{(s+1)!} \sum_{k=1}^{s} \binom{s}{k} [A(b, a; \nabla, \partial_e)]^{s-k} \epsilon^{(s-1)}(x; c)$$
$$*_c [A(b, a; \nabla, \partial_e)]^{k-1} A(b, a; \partial_c, \partial_e) R_{(1)}(x; c, e) \qquad (4.20)$$

is the generalized Lie derivative or spin s reparametrization. The formulas (4.12) and (4.19) make (1.1) obvious. It is also obvious from (4.19) that odd spin fields require additional Yang-Mills like internal symmetry indices for nontriviality.

To compare the results of this paper to the ones obtained in AdS space by Vasiliev and company (see [5] and references therein), one should continue this results from flat space to the AdS as it was done for linear curvatures in [17, 18]. The AdS continuation of the obtained second order curvature is relevant also to AdS/CFT tasks (see [13], [14], [15], [21]), and we hope to be able to do it in near future.

References

[1] B. deWit and D.Z. Freedman, "Systematics of higher spin gauge fields," Phys. Review D **21** (1980), 358-367.

[2] D. Francia and A. Sagnotti, "Free geometric equations for higher spins," Phys. Lett. B **543** (2002) 303 [arXiv:hep-th/0207002].

[3] T. Damour and S. Deser, "Geometry of spin 3 gauge theories," Annales Poincare Phys. Theor. **47**, 277 (1987);

[4] D. Francia and A. Sagnotti, "On the geometry of higher-spin gauge fields," Class. Quant. Grav. **20** (2003) S473 [arXiv:hep-th/0212185]. N. Bouatta, G. Compere and A. Sagnotti, "An Introduction to Free Higher-Spin Fields"; [arXiv:hep-th/0409068].

[5] M. A. Vasiliev, " Nonlinear equations for symmetric massless higher spin fields in $(A)dS_d$." Phys. Lett. B **567** (2003) 139-151, arXiv:hep-th/0304049.

[6] C. Fronsdal, "Singletons And Massless, Integral Spin Fields On De Sitter Space (Elementary Particles In A Curved Space Vii)," Phys. Rev. D **20**, (1979) 848;"Massless Fields With Integer Spin," Phys. Rev. D **18** (1978) 3624.

[7] R. Manvelyan, K. Mkrtchyan and W. Rühl, "A generating function for the cubic interactions of higher spin fields," Phys. Lett. B **696** (2011) 410-415. arXiv:1009.1054 [hep-th]; K. Mkrtchyan, "On generating functions of Higher Spin cubic interactions," arXiv:1101.5643 [hep-th].

[8] R. Manvelyan, K. Mkrtchyan and W. Rühl, "General trilinear interaction for arbitrary even higher spin gauge fields," Nucl. Phys. B **836** (2010) 204 [arXiv:1003.2877 [hep-th]].

[9] R. Manvelyan, K. Mkrtchyan and W. Rühl, "Direct construction of a cubic selfinteraction for higher spin gauge fields," Nucl. Phys. B **844** (2011) 348 [arXiv:1002.1358 [hep-th]].

[10] R. Manvelyan, K. Mkrtchyan and W. Rühl, "Off-shell construction of some trilinear higher spin gauge field interactions," Nucl. Phys. B **826** (2010) 1 [arXiv:0903.0243 [hep-th]]. K. Mkrtchyan, "Linearized interactions of scalar and vector fields with the higher spin field in AdSD," Armenian J. Phys. **3** (2010) 98.

[11] R. Manvelyan and K. Mkrtchyan, "Conformal invariant interaction of a scalar field with the higher spin field in AdS_D," Mod. Phys. Lett. A **25** (2010) 1333 [arXiv:0903.0058 [hep-th]].

[12] K. Mkrtchyan, "Higher Spin Interacting Quantum Field Theory and Higher Order Conformal Invariant Lagrangians," arXiv:1011.0160 [hep-th].

[13] I. R. Klebanov and A. M. Polyakov, "AdS dual of the critical O(N) vector model," Phys. Lett. B **550** (2002) 213; [arXiv:hep-th/0210114].

[14] R. Manvelyan, K. Mkrtchyan and W. Rühl, "Ultraviolet behaviour of higher spin gauge field propagators and one loop mass renormalization," Nucl. Phys. B **803** (2008) 405 [arXiv:0804.1211 [hep-th]].

[15] W. Rühl, "The masses of gauge fields in higher spin field theory on AdS(4)," Phys.Lett. B **605** (2005) 413; [arXiv:hep-th/0409252]; the results presented here are based on extensive calculations performed by K. Lang and W. Rühl, Nucl. Phys. B **400** (1993) 597.

[16] F. A. Berends, G. J. H. Burgers and H. Van Dam, "On Spin Three Selfinteractions," Z. Phys. C **24** (1984) 247;

[17] R. Manvelyan and W. Rühl, "Generalized Curvature and Ricci Tensors for a Higher Spin Potential and the Trace Anomaly in External Higher Spin Fields in AdS_4 Space," Nucl. Phys. B **796** (2008) 457; [arXiv:0710.0952 [hep-th]].

[18] R. Manvelyan and W. Rühl, "The Generalized Curvature and Christoffel Symbols for a Higher Spin Potential in AdS_{d+1} Space," Nucl. Phys. B **797**, 371 (2008) [arXiv:0705.3528 [hep-th]].

[19] E. S. Fradkin and A. A. Tseytlin, "Conformal Supergravity," Phys. Rept. **119** (1985) 233.

[20] A. Y. Segal, "Conformal higher spin theory," Nucl. Phys. B **664** (2003) 59 [arXiv:hep-th/0207212].

[21] S. Giombi and X. Yin, "Higher Spins in AdS and Twistorial Holography," arXiv:1004.3736 [hep-th].