Adding Cycles into the Neoclassical Growth Model

Abstract

We propose a stochastic Solow growth model where a cyclical component is added to the TFP process. Theoretically, an important feature of the model is that its main equation takes a state space representation where key parameters can be estimated via an unobserved component approach without involving capital stock measures. In addition, the dynamic properties of the model are mostly unaffected by the newly introduced cyclical component. Empirically, our novel framework is consistent with secular U.S. empirical evidence.

JEL CODES: O40, E32, C32

Key Words: Stochastic Growth Model, Cyclical Fluctuations, Unobserved Component Approach

The problem of combining long-run and short-run macroeconomics has still not be solved.

(Solow, 1988, p. 310)

1 Introduction

The standard Solow (1956) Growth Model (SGM) represents a very simple and naive framework to analyze countries' long-run economic growth. Despite its simplicity, it is still widely employed as a basic setting for both theoretical and empirical analyses (see, for instance, Young et al., 2013; Esfahani et al., 2014; Fernald and Jones, 2014).

The SGM is generally used to study exclusively the long-run economic features of an economy. A first attempt to couple long-run growth with short-run fluctuations is due to Brock and Mirman (1972) and Mirman (1973), who introduced stochastic elements in the aggregate technology process (i.e., TFP). In their stochastic frameworks, fluctuations (i.e., economic cycles) are merely the result of (periodic) disembodied technology shocks.[1] Solow (1988) argues that it is difficult to "*imagine shocks to taste and technology large enough on a quarterly or annual time scale to be responsible for the ups and downs of the business cycle*". Moreover, Solow (2007) is worried "*about the tendency of modern (American) macroeconomists to forget about the pathology of the business cycle*". Ramey and Ramey (1995) were among the first to empirically examine the link between growth and short-run fluctuations. They show that the above-mentioned relation is strongly supported by data. Therefore, ignoring this link may produce misleading results. Using U.S. data for the period 1870-2016 we support their findings. Moreover, our estimates fit the proposed framework and are in line with main stylized facts.

The ultimate goal of this paper is to provide a novel framework able to model cyclical fluctuations within a stochastic neoclassical growth framework.[2] To our knowledge no study bridges the theoretical and empirical literature and explicitly examine jointly the theoretical

[1] Note that the optimizing version of the stochastic Solow growth model proposed by Brock and Mirman's (1972) largely stimulated the subsequent RBC literature. However, Solow (1988) raised some concerns about the usefulness of this novel theoretical setting.

[2] Empirical attempts to account for macro-fluctuations in a long-run growth framework are Crespo (2008) and Fuentes and Morales (2011). However, both studies focus exclusively on the empirical determinants of TFP growth and do not focus on the theoretical implications and practical properties of adding a cyclical component to the neoclassical framework.

and empirical properties of a SGM featuring cycles. To fill this gap we improve the stochastic SGMs discussed in Lee et al. (1997) and Binder and Pesaran (1999) by adding a cyclical component to the TFP process. Within this novel framework we also aim at addressing some empirical regularities of the U.S. economy.

Two features of our study are noteworthy. Fist, its empirical representation can be easily estimated via unobserved component approach without involving any capital stock measures. This is an important advantage since it is well-known that capital stock (*i*) suffers from measurement errors (with the risk of having biases estimates), and (*ii*) is not available over long time span. Second, the dynamic properties of the proposed model do not differ from those implied by a standard stochastic SGM. For example, in our model, a permanent decrease in the saving rate does not alter the adjustment process toward the steady-state respect to a classical SGM. This clearly facilitates the discussion of policy implications of changes in variables influenced by the policymaker.

The paper is organized as follows. In Section 2 we present some important stylized facts about U.S. growth. In Section 3 we describe and discuss the model. Section 4 shows the empirical results. Section 5 considers the dynamic properties of the model. Section 6 concludes.

2 The empirical facts

The log of the U.S. GDP per capita over the period 1870-2016[3] can be decomposed in (i) a deterministic trend; (ii) two (trignometric) cycles and (iii) an irregular component.[4] The related GDP trend and the two cycles are depicted in Figure 1. Great Depression and post-WWII periods represent persistent but not permanent deviations from the long-run growth rate of around 2%.[5] Cycle 1 (2) exhibits a period of 7 (21) years. Note that the presence of short and long cycles are in line with the empirical evidence reported in Crespo (2008).

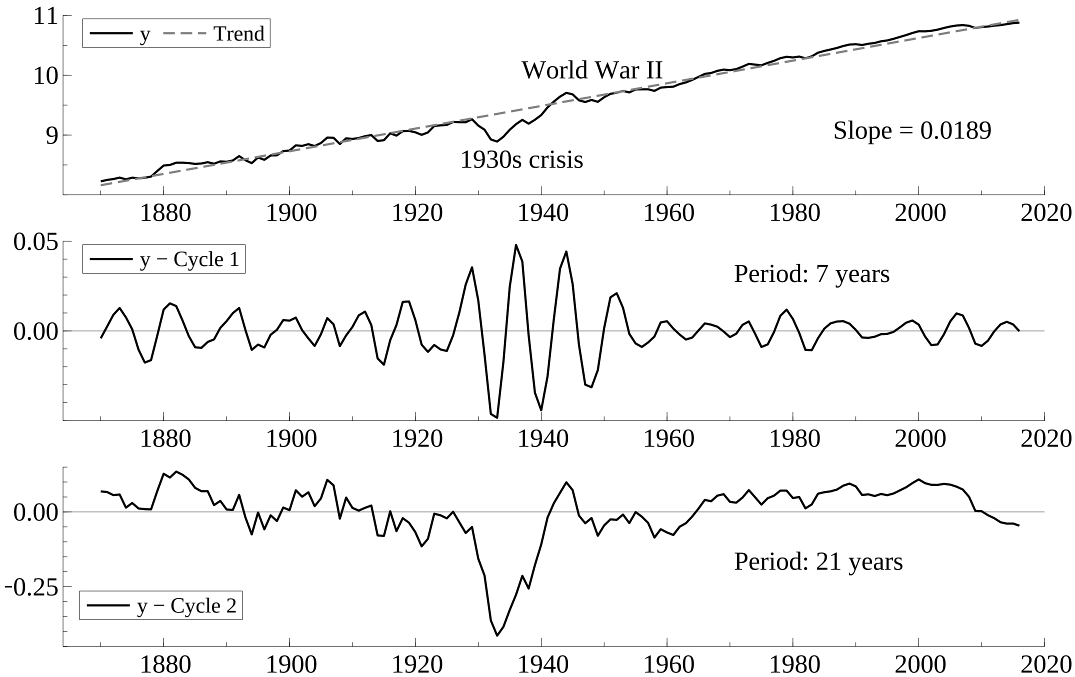
Within the SGM, the observed constant linear trend in the GDP per capita data comes

[3]Data on U.S. real GDP per capita are from Maddison Project Database, version 2018 (Bolt et al., 2018).

[4]Additional details on the model identification strategy are reported in Appendix A. Note that the residuals do not suffer from autocorrelation issues since the Q-statistic exhibits a p-value greater than 0.05 (at various lags).

[5]Similar conclusions can be found in Fernald and Jones (2014).

Figure 1: U.S. GDP DECOMPOSITION (1870-2016)



Notes: The trend-cycle decomposition is executed using an univariate unobserved components approach. Additional details on the identification scheme are reported in Appendix A.

from the TFP process. However, as pointed out by Malin (2006) and Crespo (2008), the TFP may exhibit also cycles. It is thus likely that the two cycles depicted in Figure 1 characterizing the GDP per capita are driven by TFP fluctuations. In the next Section, we rationalize this evidence by developing a SGM with cycle.

3 Theoretical framework

As in the standard SGM, final output, Y_t, is produced using a simple two-factors Cobb-Douglas technology:

$$Y_t = K_t^{\alpha} \left(A_t L_t\right)^{(1-\alpha)}, \text{ where } 0 < \alpha < 1. \tag{1}$$

In Eq. (1), K_t and L_t denote physical capital and labour, respectively, and A_t represents technology. Physical capital evolves as follows,

$$\begin{aligned} K_t &= I_{t-1} + (1-\delta)K_{t-1}, \\ I_t &= sY_t, \end{aligned} \tag{2}$$

where $\delta \in (0,1)$ is the usual depreciation rate of capital and s the saving rate. In per effective labour unit, $\hat{k}_t = K_t/(A_t L_t) = k_t/A_t$, we have:

$$\Delta \log(\hat{k}_t) = -\Delta \log(A_t L_t) + \log(s\hat{k}_{t-1}^{-(1-\alpha)} + 1 - \delta). \tag{3}$$

In our novel setting, technology is modeled as follows

$$\begin{aligned} &\log(A_t) = a_0 + gt + \psi_t + u_{a,t}, \\ &u_{a,t} = \phi_a u_{a,t-1} + \epsilon_{a,t},\ |\phi_a| < 1,^{6}\ \epsilon_{a,t} \sim \text{NID}\left(0, \sigma_a^2\right), \end{aligned} \tag{4}$$

where ψ_t represents a generic combination of zero-mean stationary cycle components.

The definition of the technology process defined in Eq. (4) allows us to write the log of GDP per capita as follows (see Appendix B.1):

$$\log(y_t) = \mu + \lambda \log(y_{t-1}) + g(1-\lambda)t + (1-\alpha)\psi_t + (\alpha - \lambda)\psi_{t-1} + e_t, \tag{5}$$

where μ and e_t are combinations of constant parameters and exogenous shocks, respectively.

Via iterative backward substitution it can be shown that Eq. (5) has the following state-space representation (see Appendix B.2):

$$\begin{cases} \log(y_t) = z_t + (1-\alpha)\psi_{1,t} + (1-\alpha)\psi_{2,t} \\ z_t = \mu + (1-\lambda)gt + \lambda z_{t-1} + (1-\alpha)\epsilon_{a,t} \\ \psi_{1,t} = \varphi_{1,1}\cos(\varsigma_1 \cdot t) + \varphi_{1,2}\sin(\varsigma_1 \cdot t) + \varepsilon_{1,t} \\ \psi_{2,t} = \varphi_{2,1}\cos(\varsigma_2 \cdot t) + \varphi_{2,2}\sin(\varsigma_2 \cdot t) + \varepsilon_{2,t}, \end{cases} \tag{6}$$

[6]This value is consistent with empirical evidence and, of course, ensures the existence of a steady-state.

where z_t is an unobserved component and $\psi_{1,t}$ and $\psi_{2,t}$ represent the two cycles of TFP process, consistent with the empirical evidence. We stress that the system of equations defined in (6) can be estimated using standard techniques. Moreover, it does not include capital stock as dependent variable. This aspect is important since this variable suffers from measurement errors that can produce misleading results (Stevens, 1994; Burda and Severgnini, 2014). Note also the capital stock series is usually not available for a relatively long time period (e.g., 100 years). Needless to mention, the absence of a capital stock measure in Eq. (6) facilitates empirical analyses.

Finally, one can express the equilibrium dynamics of GDP per capita growth as (see Appendix B.3):

$$\begin{aligned}\Delta \log(y_t) = g - (1-\lambda)\left[\left(\log(y_{t-1}) - \log(y_{ss|t})\right) - \left(\log(A_{t-1}) - \log(A_{ss|t})\right)\right] \\ + (1-\alpha)\Delta u_{a,t} + (1-\alpha)\Delta\psi_t - \alpha h.\end{aligned} \tag{7}$$

4 Empirical results

Table 1 reports estimation results of key parameters of the state-space model defined in Eq. (6).[7] Entries in Table 1 are in line with existing estimates on the magnitude of the rate of convergence $1-\lambda$ (Lee et al., 1997) and capital share α (Aiyar and Dalgaard, 2005). Note also that the value of g (close to 2%) is in line with the stylized facts depicted in Section 2.

Moreover, the cyclical factors ψ_1 and ψ_2 – extracted from the state-space model representation – fit well with the cyclical factors `Cycle 1` and `Cycle 2` extracted from the univariate trend-cycle decomposition of the U.S. GDP per capita (see Figure 2). This confirms that fluctuations in the $\log(y_t)$ are highly influenced by cyclical factors affecting the TFP growth process.

[7] As requested by a referee, to check the robustness of our analysis, we report in Section C the results for Australia. This country is characterized by similar long-run properties to those found in the U.S. (i.e., a stable growth pattern as shown in Fig. C.1) and results are in line with theoretical model of Eq. (6) (see Table C.1).

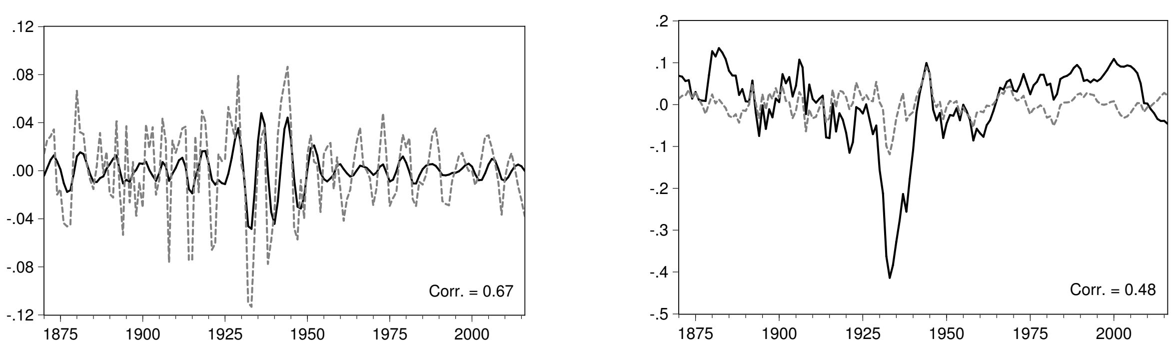
Table 1: Eq. (6): Estimated Parameters (1870-2016)

Parameter	Estimated Value
μ	0.512*
	(0.322)
λ	0.931***
	(0.040)
g	0.019***
	(0.001)
$1-\alpha$	0.619***
	(0.063)

Notes: The system defined in Eq. (6) is estimated over the period 1870-2016. The maximum likelihood – with Newton-Raphson optimization procedure and Marquardt step – is used as estimation method. Huber-White heteroskedasticity consistent standard errors are reported in paranthesis. ***, ** and * denote significance at the 1%, 5%, and 10% levels.

Figure 2: Comparing Cycles

Panel A: cycle 1 vs $\psi_{1,t}$ — Panel B: cycle 2 vs $\psi_{2,t}$



5 Dynamic properties

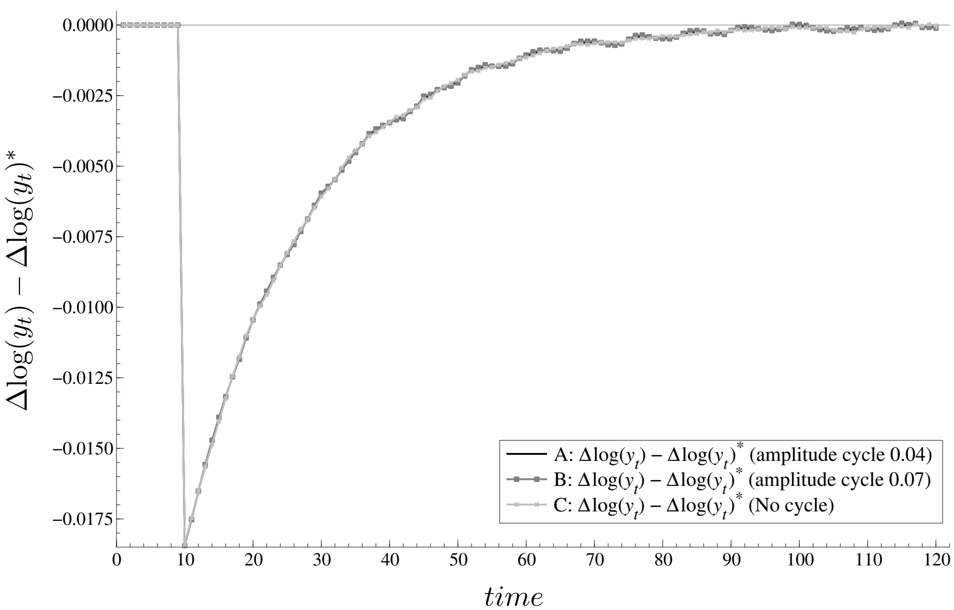
What about equilibrium dynamics in the presence of a cycle? Differently from the standard stochastic SGM, an additional term shows up in the dynamic equation of output, i.e., $(1-\alpha)\Delta\psi_t$. Apparently, this term does not alter the dynamic properties of the model. We show this by means of a simple simulation exercise.

For the sake of simplicity, let us assume a deterministic trigonometric cycle (e.g., $\psi_t = \varphi_1 \cos(\varsigma \cdot t) + \varphi_2 \sin(\varsigma \cdot t)$).[8] We then simulate the pattern generated by Eq. (7) to study the effect of a permanent drop in the saving rate, s. Three different parameter structures are analyzed. In particular, economies A and B are characterized by TFP processes featuring

[8]Note that adding and/or considering other cycle types do not alter the main results of this section.

different cycle's amplitude (i.e., 0.04 for A and 0.07 for B) and persistent exogenous shocks, whereas in economy C the TFP process features persistent exogenous shocks only. In each of the three considered economies, namely A, B, and C, the effects of a permanent drop in s are studied.

Figure 3: Dynamic Properties of Different Economic Structures



Notes: $\Delta\log(y_t)$ corresponds to Eq. (7) with $s = 0.2$ for all $t \geq 0$; $\Delta\log(y_t)^*$ corresponds to Eq. (7) with $s = 0.2$ for $t \in [0, 9]$ and $s = 0.1$ for $t \geq 10$. The steady-state is reached when $\Delta\log(y_t) - \Delta\log(y_t)^* = 0$. The following parameter values are used in the simulation exercises. Model A: $\alpha = 0.33$, $n = 0.01$, $g = 0.02$, $h = 0$, $\delta = 0.06$, $s = 0.2$, $\phi_a = 0.9$, $\sigma_{\varepsilon_a} = 0.03$, $\sqrt{\varphi_1^2 + \varphi_2^2} = 0.04$, $\varsigma = 0.809$. Model B: $\alpha = 0.33$, $n = 0.01$, $g = 0.02$, $h = 0$, $\delta = 0.06$, $s = 0.2$, $\phi_a = 0.9$, $\sigma_{\varepsilon_a} = 0.03$, $\sqrt{\varphi_1^2 + \varphi_2^2} = 0.07$, $\varsigma = 0.809$. Model C: $\alpha = 0.33$, $n = 0.01$, $g = 0.02$, $h = 0$, $\delta = 0.06$, $s = 0.2$, $\phi_a = 0.9$, $\sigma_{\varepsilon_a} = 0.03$.

We assume the saving rate to move from 0.2 to 0.1 at t=10. What is relevant in this exercise is the determination of time needed to reach the new steady-state. This can be easily computed, for each economy A, B, and C, by calculating the difference between the Eq. (7) assuming a value of $s = 0.2$ for each $t \geq 0$ ($\Delta\log(y_t)$) and Eq. (7) assuming a value of $s = 0.2$ for $t \in [0, 9]$ and $s = 0.1$ for $t \geq 10$ ($\Delta\log(y_t)^*$). Dynamics in Figure 3 suggest that the time needed to reach the new steady-state (i.e., $\Delta\log(y_t) - \Delta\log(y_t)^* = 0$) does not change across different economies. The inclusion of the cycle in the SGM produces permanent fluctuations in the economy dynamics but does not alter the adjustment process toward the steady-state.

6 Conclusions

We propose a novel Solow growth model. Precisely, in a stochastic version of the Solow framework, a cyclical component is added to the TFP process. This represents a first attempt to account for business cycle features within the Solow growth theory. The proposed novel framework is supported by U.S. empirical evidence for the period 1870-2016. In addition, we find that the long-run dynamic properties of this novel framework coincide with those of a standard stochastic SGM.

Our framework embodies also easy-to-use characteristics that may facilitate its empirical application. In practice, the model is represented by a system of equations that do not account for capital stock and can be estimated via a standard unobserved component approach. Therefore, any measurement error issue associated to the use of capital stock measures is avoided.

There are a number of directions in which this work could be fruitfully extended. First, this paper assumes constant population growth and saving rates. However, both rates can be assumed to be stochastic. Second, our empirical analysis relies on the U.S. The theoretical framework should be empirically validated using other countries' long-term data. Finally, there are good reasons to believe that the observed fluctuations are driven by several sources. Therefore, future works should focus on the identification of variables driving TFP fluctuations. Such extensions may have relevant policy implications and are left for future research.

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A Univariate trend-cycle decomposition of log of real GDP per capita

An unobserved components approach is used to model macroeconomic time series. The trend-cycle decomposition model is given by

$$y_t = \mu_t + \psi_t + \epsilon_t, \ \epsilon_t \sim \text{NID}(0, \sigma_\epsilon^2), \tag{A.1}$$

where y_t is the log of U.S. real GDP per capita, μ_t captures the trend, ψ_t represents the cycle, and ϵ_t is the irregular component.

Note that the following common specifications may be considered for the trend (Commandeur and Koopman, 2007):

1. Deterministic trend (DT): $\mu_t = \mu_{t-1} + \beta$.
2. Random walk with drift (RW): $\mu_t = \mu_{t-1} + \beta + \eta_t, \ \eta_t \sim \text{NID}(0, \sigma_\eta^2)$.
3. Local linear trend (LL): $\mu_t = \mu_{t-1} + \beta_{t-1} + \eta_t, \ \eta_t \sim \text{NID}(0, \sigma_\eta^2); \ \beta_t = \beta_{t-1} + \zeta_t, \ \zeta_t \sim \text{NID}(0, \sigma_\zeta^2)$.
4. Smooth trend (ST): $\mu_t = \mu_{t-1} + \beta_{t-1}; \ \beta_t = \beta_{t-1} + \zeta_t, \ \zeta_t \sim \text{NID}(0, \sigma_\zeta^2)$.

Three main specifications of the cycle can be considered:

(a) High order trigonometric cycle (HTC) as proposed by Harvey and Trimbur (2003). The generalized kth order cycle is given by $\psi_t = \psi_t^{(k)}$ where

$$\begin{bmatrix} \psi_t^{(j)} \\ \psi_t^{*(j)} \end{bmatrix} = \rho \begin{bmatrix} \cos(\varsigma) & \sin(\varsigma) \\ -\sin(\varsigma) & \cos(\varsigma) \end{bmatrix} \begin{bmatrix} \psi_{t-1}^{(j)} \\ \psi_{t-1}^{*(j)} \end{bmatrix} + \begin{bmatrix} \psi_t^{(j-1)} \\ \psi_t^{*(j-1)} \end{bmatrix}, \tag{A.2}$$

for $j = 1, ..., k$,[9] and where ς is the frequency (in radiants), ρ a damping factor and $\psi_t^{(0)}, \ \psi_t^{*(0)} \sim \text{NID}(0, \sigma_\kappa^2)$.

(b) AR(2) process as proposed by Clark (1987):

$$\psi_t = \phi_1 \psi_{t-1} + \phi_2 \psi_{t-2} + v_t, \ v_t \sim \text{NID}(0, \sigma_v^2) \tag{A.3}$$

(c) Multiple cycles (MC) as suggested by Crespo (2008):

$$\psi_t = \sum_{j=1}^{J} \psi_{j,t}, \tag{A.4}$$

where each $\psi_{j,t}$ can be modelled as an independent trigonometric cycle process with different frequencies.

[9]The value of k determines the smoothness of the cycle. In our case we have $k = 3$.

For completeness, the following 12 combinations of trend and cycle specification are considered: DT + HTC; DT + AR(2); DT + MC; RW + HTC; RW + AR(2); RW + MC; LL + HTC; LL + AR(2); LL + MC; ST + HTC; ST + AR(2); ST + MC. In all specifications we assume that disturbances driving each of the components are mutually uncorrelated. The performance of different specifications are compared on the basis of Akaike information criterion (AIC). This criterion is based on the formula:

$$\log PEV + 2 \cdot m/T, \tag{A.5}$$

where PEV is the prediction error variance and m is the number of parameters plus the number of non-stationary components in the state vector. Table A.1 summarizes results together with the slope coefficient of different specifications.

Table A.1: Model comparison

Model	**Slope**	**PEV**	**AIC**
DT + HTC	0.0186	0.2263	-6.0355
DT + AR(2)	0.0186	0.2279	-6.0285
DT + MC	0.0187	0.2183	-6.0298
RW + HTC	0.0184	0.2301	-5.9912
RW + AR(2)	0.0184	0.2336	-5.9759
RW + MC	0.0184	0.2221	-6.0025
LL + HTC	$0.0179^{\mp}$	0.2726	-5.8633
LL + AR(2)	$0.0185^{\mp}$	0.2322	-5.9819
LL + MC	$0.0185^{\mp}$	0.2297	-5.9500
ST + HTC	$0.0197^{\mp}$	0.2332	-5.9777
ST + AR(2)	$0.0192^{\mp}$	0.2322	-5.9818
ST + MC	$0.0194^{\mp}$	0.2258	-5.9821

Notes: $^{\mp}$ Slope value at the end of period. PEV values are multiplied by 100.

All results show that slope $\approx 2\%$ and that the best specification (i.e. the model with lower AIC) is the specification with the deterministic trend. Among the three specifications with deterministic trend we prefer the one having two cycles (DT + MC) since it appears to be closer to the economic theory of cycles[10] and, in particular, with the idea that over the long-run different cycles (with different periodicity) may take place (Maddison, 1991; Malin, 2006).

[10] See, for example, Mazzi and Ozyildirim (2017).

B Proofs

B.1 Derivation of Eq. (5)

The output equations used in our empirical analysis are derived in the spirit of Lee et al. (1997) and Binder and Pesaran (1999). First, $\Delta \log(A_t)$ and $\Delta u_{a,t}$ can be written as follows:

$$\begin{aligned} \Delta \log(A_t) &= g + \Delta\psi_t + \Delta u_{a,t} \\ \Delta u_{a,t} &= -(1-\phi_a)u_{a,t-1} + \epsilon_{a,t}. \end{aligned} \tag{B.1}$$

Second, we impose

$$\Delta \log(L_t) = n. \tag{B.2}$$

Using Eqs. (B.1) and (B.2) in Eq. (3) yields:

$$\Delta \log(\hat{k}_t) = -(n+g) - \Delta\psi_t - \Delta u_{a,t} + \log(s\hat{k}_{t-1}^{-(1-\alpha)} + 1 - \delta). \tag{B.3}$$

We acknowledge that in the proposed framework the steady-state of the economy is obtained by assuming that each stochastic process and all (exogenous) variables are equal to their long-average value (i.e., $u_{a,t} = \psi_t = 0$, $t \in \{1, \ldots, T\}$). We then linearise Eq. (B.3) around $\mathbb{E}\left[\log(\hat{k}_\infty)\right]$, where $\hat{k}_\infty$ is the random variable that underlies the steady-state distribution of $\hat{k}_t$. By taking expectation on both sides of Eq. (B.3) we obtain:

$$(n+g) = \mathbb{E}\left[\log(se^{-(1-\alpha)\log(\hat{k}_\infty)} + 1 - \delta)\right]. \tag{B.4}$$

The function $f(\log(\hat{k}_\infty)) \doteq \log(se^{-(1-\alpha)\log(\hat{k}_\infty)} + 1 - \delta)$ is a convex function of $\log(\hat{k}_\infty)$. Then, Jensen's inequality implies:

$$(n+g) = \log(se^{-(1-\alpha)\mathbb{E}\left[\log(\hat{k}_\infty)\right]} + 1 - \delta) + h. \tag{B.5}$$

The parameter h is a strictly positive number which depends on the degree of the curvature of the function f. From Eq. (B.5) we easily obtain an expression for $\mathbb{E}\left[\log(\hat{k}_\infty)\right]$:

$$\mathbb{E}\left[\log(\hat{k}_\infty)\right] = \frac{1}{1-\alpha}\left[\log(s) - \log(e^{n+g-h} - 1 + \delta)\right] \tag{B.6}$$

which can be used to linearise Eq. (B.3). Specifically, let ξ_t be the approximation error. Then, the expansion of the non-linear term in Eq. (B.3) around $\mathbb{E}\left[\log(\hat{k}_\infty)\right] \doteq \tilde{k}_{ss}$ yields

$$\log(se^{-(1-\alpha)\log(\hat{k}_{t-1})} + 1 - \delta) = \gamma - (1-\lambda)\log(k_{t-1}) + \xi_t, \tag{B.7}$$

where ξ_t is an approximation error and

$$(1-\lambda) = \frac{s(1-\alpha)e^{-(1-\alpha)\tilde{k}_{ss}}}{se^{-(1-\alpha)\tilde{k}_{ss}} + 1 - \delta} \tag{B.8}$$

and

$$\gamma = \log(se^{-(1-\alpha)\tilde{k}_{ss}} + 1 - \delta) + (1-\lambda)\tilde{k}_{ss}. \tag{B.9}$$

Using Eq. (B.6), $(1-\lambda)$ and γ simplify as follows:

$$(1-\lambda) = (1-\alpha)\left[1 - (1-\delta)\, e^{-(n+g-h)}\right] \tag{B.10}$$

and

$$\gamma = n + g - h - \left[1 - (1-\delta)e^{-(n+g-h)}\right]\left[\log(e^{n+g-h} - 1 + \delta) - \log(s)\right]. \tag{B.11}$$

Note also that for small values of n, g, δ, and h Eqs. (B.10) and (B.11) take the following form:

$$(1-\lambda) \approx (1-\alpha)(n + g + \delta - h) \tag{B.12}$$

$$\gamma \approx (n + g - h) + (n + g + \delta + h)\left[\log(s) - \log(n + g + \delta - h)\right]. \tag{B.13}$$

We derive now the univariate representation for the output per capita. The production function, $Y_t = K_t^{\alpha}(A_t L_t)^{(1-\alpha)}$, can be expressed in terms of the logarithm of output per capita, $\log(Y_t/L_t) \doteq \log(y_t)$, as

$$\log(y_t) = \alpha \log(\hat{k}_t) + \log(A_t). \tag{B.14}$$

Using Eqs. (B.3) and (B.1) – and the related approximations in Eqs. (B.12) and (B.13) – jointly with Eq. (B.14) and assuming that the error of approximation is relatively unimportant (see also Lee et al. (1997)), we can write down the equation for $\Delta\log(y_t)$. Precisely, we have

$$
\begin{aligned}
\Delta \log(y_t) &= \alpha \Delta \log(\hat{k}_t) + \Delta \log(A_t) \\
&= \alpha \left(-(n+g) - \Delta\psi_t - \Delta u_{a,t} + \gamma - (1-\lambda)\log(\hat{k}_{t-1}) \right) \\
&+ g + \Delta\psi_t + \Delta u_{a,t} \\
&= \alpha(\gamma - (n+g)) + (1-\alpha)\Delta\psi_t + (1-\lambda)a_0 + (1-\lambda)gt - (1-\lambda)g \\
&+ (1-\lambda)\psi_{t-1} + (1-\lambda)u_{a,t-1} - (1-\lambda)\log(y_{t-1}) + g + (1-\alpha)\Delta u_{a,t} + (1-\lambda)u_{a,t-1} \\
&\approx -\alpha h - (1-\lambda)\frac{\alpha}{1-\alpha}\left[\log(n+g+\delta-h) - \log(s)\right] + (1-\lambda)a_0 + \lambda g + (1-\lambda)gt - (1-\lambda)\log(y_{t-1}) \\
&+ (1-\alpha)\Delta\psi_t + (1-\lambda)\psi_{t-1} + (1-\alpha)\Delta u_{a,t} + (1-\lambda)u_{a,t-1}.
\end{aligned} \tag{B.15}
$$

Rearranging terms yields:

$$
\log(y_t) = \mu + \lambda\log(y_{t-1}) + g(1-\lambda)t + (1-\alpha)\psi_t + (\alpha-\lambda)\psi_{t-1} + e_t, \tag{B.16}
$$

where

$$
\mu = -\alpha h + \lambda g + (1-\lambda)\left[a_0 - \frac{\alpha}{1-\alpha}\left[\log(n+g+\delta-h) - \log(s)\right]\right] \tag{B.17}
$$

and

$$
e_t = \left[(1-\lambda) - (1-\alpha)(1-\phi_a)\right] u_{a,t-1} + (1-\alpha)\epsilon_{a,t}. \tag{B.18}
$$

B.2 Derivation of Eq. (6)

Let us start from (B.18) and then substitute (B.12). This yields

$$
e_t = (1-\alpha)\left[(n+g+\delta-h) - (1-\phi_a)\right] u_{a,t} + (1-\alpha)\epsilon_{a,t}. \tag{B.19}
$$

Since in developed economies, over long-run periods, we have $(n+g+\delta-h) \approx (1-\phi_a)$ (B.19) simplifies to

$$
e_t \approx (1-\alpha)\epsilon_{a,t}. \tag{B.20}
$$

Eq. (B.16) can then be written as

$$
\log(y_t) \approx \mu + \lambda\log(y_{t-1}) + g(1-\lambda)t + (1-\alpha)\psi_t + (\alpha-\lambda)\psi_{t-1} + (1-\alpha)\epsilon_{a,t}. \tag{B.21}
$$

Applying iterative backward substitution for $\log(y_{t-1})$, (B.21) can be rewritten as

$$\log(y_t) \approx \mu \sum_{j=0}^{\infty} \lambda^j + g(1-\lambda)t \sum_{j=0}^{\infty} \lambda^j - g(1-\lambda)\lambda \sum_{j=1}^{\infty} j\lambda^{j-1} + (1-\alpha)\psi_t$$
$$+ (1-\alpha)\lambda \sum_{j=0}^{\infty} \lambda^j \psi_{t-j-1} + (\alpha-\lambda) \sum_{j=0}^{\infty} \lambda^j \psi_{t-j-1} + (1-\alpha) \sum_{j=0}^{\infty} \lambda^j \epsilon_{a,t-j}. \tag{B.22}$$

Since $\lambda < 1$, we have

$$\log(y_t) \approx \frac{\mu - g\lambda}{(1-\lambda)} + gt + (1-\alpha)\psi_t + (1-\alpha)\lambda \sum_{j=0}^{\infty} \lambda^j \psi_{t-j-1}$$
$$+ (\alpha-\lambda) \sum_{j=0}^{\infty} \lambda^j \psi_{t-j-1} + (1-\alpha) \sum_{j=0}^{\infty} \lambda^j \epsilon_{a,t-j}. \tag{B.23}$$

The quantity $(1-\alpha)\lambda \sum_{j=0}^{\infty} \lambda^j \psi_{t-j-1} + (\alpha-\lambda) \sum_{j=0}^{\infty} \lambda^j \psi_{t-j-1}$ can be written as

$$[(\lambda - \alpha\lambda) + (\alpha - \lambda)] \sum_{j=0}^{\infty} \lambda^j \psi_{t-j-1} = \alpha(1-\lambda) \sum_{j=0}^{\infty} \lambda^j \psi_{t-j-1}. \tag{B.24}$$

As a first approximation, since $(1-\lambda) \approx (1-\alpha)(n+g+\delta-h)$ is a very small number, the quantity (B.24) can be approximated to zero. Then, (B.23) can be written as

$$\log(y_t) \approx \frac{\mu - g\lambda}{(1-\lambda)} + gt + (1-\alpha)\psi_t + (1-\alpha) \sum_{j=0}^{\infty} \lambda^j \epsilon_{a,t-j}. \tag{B.25}$$

By imposing $z_t = \log(y_t) - (1-\alpha)\psi_t$, we can write the following

$$z_t = \frac{\mu - g\lambda}{(1-\lambda)} + gt + (1-\alpha) \sum_{j=0}^{\infty} \lambda^j \epsilon_{a,t-j}$$
$$= \frac{\mu - g\lambda}{(1-\lambda)} + gt + (1-\alpha)(1-\lambda L)^{-1} \epsilon_{a,t} \tag{B.26}$$

$$(1-\lambda L)z_t = (1-\lambda L)\left[\frac{\mu - g\lambda}{(1-\lambda)} + gt\right] + (1-\alpha)\epsilon_{a,t} \tag{B.27}$$

$$z_t - \lambda z_{t-1} = \mu - g\lambda + gt - \lambda g(t-1) + (1-\alpha)\epsilon_{a,t}. \tag{B.28}$$

Given the (B.26) and the (B.28), and the fact that two cycles are present in the $\log(y_t)$ process, the following system of equations can be easily estimated via unobserved component models:

$$\begin{cases} \log(y_t) = z_t + (1-\alpha)\psi_{1,t} + (1-\alpha)\psi_{2,t} \\ z_t = \mu + (1-\lambda)gt + \lambda z_{t-1} + (1-\alpha)\epsilon_{a,t} \\ \psi_{1,t} = \varphi_{1,1}\cos(\varsigma_1 \cdot t) + \varphi_{1,2}\sin(\varsigma_1 \cdot t) + \varepsilon_{1,t} \\ \psi_{2,t} = \varphi_{2,1}\cos(\varsigma_2 \cdot t) + \varphi_{2,2}\sin(\varsigma_2 \cdot t) + \varepsilon_{2,t}. \end{cases} \tag{B.29}$$

B.3 Derivation of Eq. (7)

Using Eqs. (B.3) and Eq. (B.7), and assuming negligible the error ξ_t, the law of motion for the logarithm of the capital per effective labor around the steady-state $\tilde{k}_{ss}$ can be rewritten as:

$$\log(k_t) = k_{ss|t} + \lambda(\log(k_{t-1}) - \tilde{k}_{ss}) - \Delta u_{a,t} - \Delta\psi_t - h. \tag{B.30}$$

Hence, making usage of Eq. (B.14) we obtain:

$$\log(y_t) = \alpha\left(\tilde{k}_{ss} + \lambda(\log(\hat{k}_{t-1}) - \tilde{k}_{ss}) - \Delta u_{a,t} - \Delta\psi_t - h\right) + \log(A_t). \tag{B.31}$$

In order to derive the law of motion for the logarithm of per capita output we make use of the Eq. (B.14) to obtain expressions for $\log(\hat{k}_{t-1})$, $\tilde{k}_{ss}$, and hence for $\log(\hat{k}_{t-1}) - \tilde{k}_{ss}$:

$$\log(\hat{k}_{t-1}) = \frac{\log(y_{t-1}) - \log(A_{t-1})}{\alpha} \tag{B.32}$$

$$\tilde{k}_{ss} = \frac{\log(y_{ss|t}) - \log(A_{ss|t})}{\alpha} \tag{B.33}$$

$$\log(\hat{k}_{t-1}) - \tilde{k}_{ss} = \frac{\log(y_{t-1} - \log(y_{ss|t}) + \log(A_{ss|t}) - \log(A_{t-1})}{\alpha}. \tag{B.34}$$

So, Eq. (B.31) reads as follow:

$$\begin{aligned} &\log(y_t) = \log(y_{ss|t}) - \log(A_{ss|t}) + \lambda\left(\log(y_{t-1}) - \log(y_{ss|t}) + \log(A_{ss|t}) - \log(A_{t-1})\right) \\ &- \alpha\Delta u_{a,t} - \alpha\Delta\psi_t - \alpha h + \log(A_t) \\ &= \log(y_{ss|t}) + \lambda\left(\log(y_{t-1}) - \log(y_{ss|t})\right) + (1-\lambda)\left(\log(A_{t-1}) - \log(A_{ss|t})\right) - \alpha\Delta u_{a,t} \\ &- \alpha\Delta\psi_t - \alpha h + \Delta\log(A_t) \\ &= g + \log(y_{ss|t}) + \lambda\left(\log(y_{t-1}) - \log(y_{ss|t})\right) + (1-\lambda)\left(\log(A_{t-1}) - \log(A_{ss|t})\right) + (1-\alpha)\Delta u_{a,t} \\ &+ (1-\alpha)\Delta\psi_t - \alpha h. \end{aligned} \tag{B.35}$$

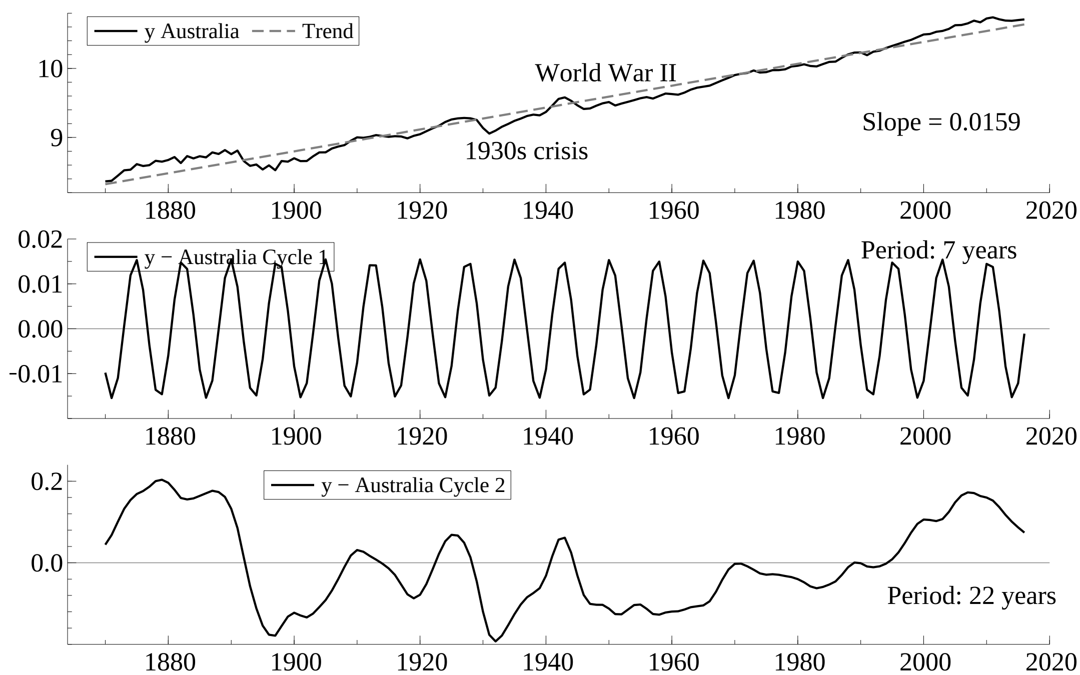
Subtracting $\log(y_{t-1})$ from both sides of Eq. (B.35), we get:

$$\Delta \log(y_t) = g - (1-\lambda)\left[(\log(y_{t-1}) - \log(y_{ss|t})) - (\log(A_{t-1}) - \log(A_{ss|t}))\right] \\ + (1-\alpha)\Delta u_{a,t} + (1-\alpha)\Delta\psi_t - \alpha h. \tag{B.36}$$

Eq. (7) helps to understand economic dynamics in the case of a change in the steady-state conditions generated, for instance, by a change in s and/or n.

C Additional results: Estimates for Australia

Figure C.1: Australia GDP decomposition (1870-2016)



Notes: The trend-cycle decomposition is executed using an univariate unobserved components approach as for the U.S.

Table C.1: Eq. (6): Estimated Parameters for Australia (1870-2016)

Parameter	Estimated Value
μ	0.276*
	(0.143)
λ	0.959***
	(0.017)
g	0.017***
	(0.001)
$1-\alpha$	0.478***
	(0.039)

Notes: The system defined in Eq. (6) is estimated over the period 1870-2016. The maximum likelihood – with Newton-Raphson optimization procedure and Marquardt step – is used as estimation method. Huber-White heteroskedasticity consistent standard errors are reported in paranthesis. ***, ** and * denote significance at the 1%, 5%, and 10% levels.