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Notes on higher-spin algebras: minimal representations and structure constants

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Abstract: The higher-spin (HS) algebras relevant to Vasiliev's equations in various dimensions can be interpreted as the symmetries of the *minimal representation* of the isometry algebra. After discussing this connection briefly, we generalize this concept to any classical Lie algebra and consider the corresponding HS algebras. For $\mathfrak{sp}_{2N}$ and $\mathfrak{so}_N$, the minimal representations are unique so we get unique HS algebras. For $\mathfrak{sl}_N$, the minimal representation has one-parameter family, so does the corresponding HS algebra. The $\mathfrak{so}_N$ HS algebra is what underlies the Vasiliev theory while the $\mathfrak{sl}_2$ one coincides with the $3D$ HS algebra $hs[\lambda]$. Finally, we derive the explicit expression of the structure constant of these algebras — more precisely, their bilinear and trilinear forms. Several consistency checks are carried out for our results.

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JHEP05(2014)103

Contents

JHEP05(2014)103

1 Introduction

Gauge theories describing massless particles are naturally endowed with Lie algebras as the background preserving part of gauge symmetries, namely the global symmetries. Inversely, one may start with a proper Lie algebra as a global symmetry and then obtain the field theory by gauging it. One of the simplest non-trivial examples would be Gravity where the gauge symmetries — diffeomorphisms — give rise to the isometry algebras (Poincaré, AdS or dS) as global symmetries. The Cartan formulation of Gravity makes use of the inverse construction. Hence, in the study of massless higher-spin (HS) particles, it is also of primary importance to investigate the underlying global symmetries — HS algebras. Indeed, HS algebra is in the core of Vasiliev's equations [1–3] and recent developments of three-dimensional HS theories — see the review [4] and references therein.

HS algebra is, by definition, the Lie algebra of the global symmetries underlying a theory involving HS spectrum. However, in this paper, we shall use a looser definition for the term *HS algebra* which will be specified later in the text. So far, only one HS algebra is known to be fully consistent in each dimensions $D \geq 4$. The four-dimensional one [5] was considered by Fradkin and Vasiliev in the construction of HS cubic interactions [6], and then used for the interacting theory — Vasiliev's equations. Extensions to five and seven dimensions have been studied respectively in [7, 8] and [9]. Finally, the generalization to any dimensions of the algebra together with the equations was carried out in [2, 3]. These constructions were based on oscillators — spinors for four and five dimensions and vectors for any dimensions. It is important to note that this HS algebra is also isomorphic to the conformal HS symmetries of the free massless scalar in $D-1$ dimensions [10, 11]. Three-dimensional case is special: there is one-parameter family of HS algebras $hs[\lambda]$ [12–17] corresponding to the one-parameter family of backgrounds of the $3D$ interacting equations [18, 19].[1] Besides the explicit construction through oscillators, the aforementioned HS algebras can be also obtained as a quotient of universal enveloping algebra (UEA) of the isometry algebra.[2]

In fact, HS algebra as a coset of UEA has a deep relation to the theory of *minimal representations*. In 1974, Joseph [39] raised the question that what is the minimum number of Heisenberg pairs which are needed to represent a Lie algebra. To attack this question, he coined the notion of *Joseph ideal* which corresponds to the kernel of the aforementioned representation — minimal representation. The coset of UEA by Joseph ideal is an infinite-dimensional algebra including the original Lie algebra as subalgebra. Interestingly, HS algebras mentioned in the above paragraph all fall into these coset Lie algebras. Much progress has been made on the subject of minimal representations, see for example [40–50] and references therein. In physics literature, the minimal representations of isometry algebras are explored to a large extent by Gunaydin and collaborators [51–55].

In this paper, we first make a brief survey of the construction of HS algebras from the point of view of minimal representations. This construction is technically not much new compared to what was known in the HS literature, and we just attempt to make a link between two rather disconnected literatures. As we mentioned we shall use the term *HS algebra* loosely, as the symmetry algebra of the minimal representation of a finite-dimensional Lie algebra. In this article, we shall focus on the classical Lie algebras, $\mathfrak{sl}_N, \mathfrak{so}_N$ and $\mathfrak{sp}_{2N}$. The original part of the present paper, in a strict sense, is the presentation of the explicit structure constants of the HS algebras. In order to present them, let us first introduce the relevant notations. Let $T_{\boldsymbol{a}}$ denote the generators of a Lie algebra, say $\mathfrak{g}$. Then, the corresponding HS algebra, denoted by $hs(\mathfrak{g})$, can be given through an arbitrary function of the element $A^{\boldsymbol{a}}$ in $\mathfrak{g}^*$, the dual space of $\mathfrak{g}$:

$$T(A) = \sum_{n=0}^{\infty} \frac{1}{n!}\, T_{\boldsymbol{a}_1 \cdots \boldsymbol{a}_n}\, A^{\boldsymbol{a}_1} \cdots A^{\boldsymbol{a}_n}\,. \tag{1.1}$$

[1] Recently, the asymptotic symmetries of the $hs[\lambda] \oplus hs[\lambda]$ Chern-Simons theories are identified with the W-algebras $\mathcal{W}_\infty[\lambda] \oplus \mathcal{W}_\infty[\lambda]$ [20–23].

[2] This view has been discussed in [24], and pursuing the same idea a class of mixed-symmetry HS algebras has been considered in [25]. See also [26–32] for other discussions on HS algebras, and [33–36] and [37, 38] for generalizations.

The coefficients $T_{\boldsymbol{a}_1\cdots\boldsymbol{a}_n}$ correspond to the generators of $hs(\mathfrak{g})$. To be more precise, $A^{\boldsymbol{a}}$ are not arbitrary elements of $\mathfrak{g}^*$, but belong to its minimal orbit whose precise conditions will be presented later. Due to such conditions, the generators $T_{\boldsymbol{a}_1\cdots\boldsymbol{a}_n}$ have a smaller number of independent components, and they can be obtained from the Taylor expansion of $T(A)$. Hence, the algebraic structure of $hs(\mathfrak{g})$ can be studied in terms of $T(A)$:

- we first consider the bilinear form $\mathcal{B}(A) = \mathrm{Tr}[\,T(A_1)\star T(A_2)\,]$ where $\star$ and $\mathrm{Tr}[\,\cdot\,]$ denote respectively the associative product and the trace operation of $hs(\mathfrak{g})$ whose definitions will be provided later. This bilinear form serves as a HS generalization of the Killing form.
- Then, we move to the trilinear form $\mathcal{C}(A) = \mathrm{Tr}[\,T(A_1)\star T(A_2)\star T(A_3)\,]$ from which the structure constant of the algebra, consisting of totally symmetric and antisymmetric parts, can be obtained.

For the HS algebras associated with classical Lie algebras, we find

$\mathfrak{sp}_{2N}$ series:

$$\mathcal{B}(U) = \frac{1}{\sqrt{1+\frac{\langle U_1\,U_2\rangle}{4}}}\,,\qquad \mathcal{C}(U) = \frac{1}{\sqrt{1+\frac{\langle U_1\,U_2\rangle+\langle U_2\,U_3\rangle+\langle U_3\,U_1\rangle+\langle U_1\,U_2\,U_3\rangle}{4}}}\,. \tag{1.2}$$

$\mathfrak{sl}_N$ series:

$$\mathcal{B}(V) = {}_3F_2\left(\frac{N}{2}\,(1+\lambda)\,,\,\frac{N}{2}\,(1-\lambda)\,,\,1\,;\,\frac{N}{2}\,,\,\frac{N+1}{2}\,;\,-\frac{1}{4}\,\langle V_1\,V_2\rangle\right)\,, \tag{1.3}$$

$$\begin{aligned}\mathcal{C}(V) = \sum_{k=0}^{\infty}\sum_{\ell=0}^{k}(-1)^k\binom{k}{\ell}\,&\frac{\left(\frac{N(1+\lambda)}{2}\right)_{2\,k-\ell}\left(\frac{N(1-\lambda)}{2}\right)_{k+\ell}}{(N)_{3k}}\times\\ &\times\,\big[\langle V_1\,V_2\rangle+\langle V_2\,V_3\rangle+\langle V_3\,V_1\rangle+\langle V_1\,V_2\,V_3\rangle\big]^{k-\ell}\\ &\times\,\big[\langle V_1\,V_2\rangle+\langle V_2\,V_3\rangle+\langle V_3\,V_1\rangle-\langle V_3\,V_2\,V_1\rangle\big]^{\ell}\,.\end{aligned} \tag{1.4}$$

$\mathfrak{so}_N$ series:

$$\mathcal{B}(W) = {}_2F_1\left(2\,,\,\frac{N-4}{2}\,;\,\frac{N-1}{2}\,;\,-\frac{1}{8}\,\langle W_1\,W_2\rangle\right)\,, \tag{1.5}$$

$$\begin{aligned}\mathcal{C}(W) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^m}{m!}\,\frac{1}{n!}\,&\frac{\left(\frac{N-4}{2}\right)_{m+2n}(2)_{m+2n}}{\left(\frac{N-1}{2}\right)_{m+3n}\,8^{m+3n}}\times\\ &\times\,\big[\langle W_1\,W_2\rangle+\langle W_2\,W_3\rangle+\langle W_3\,W_1\rangle+\langle W_1\,W_2\,W_3\rangle\big]^m\\ &\times\,\big[\langle W_1\,W_2\rangle\langle W_2\,W_3\rangle\langle W_3\,W_1\rangle+2\,\langle W_1\,W_2\,W_3\rangle^2\big]^n\,.\end{aligned} \tag{1.6}$$

Here, U_i, V_i and W_i are respectively elements of ${\mathfrak{sp}_{2N}}^*$, ${\mathfrak{sl}_N}^*$ and ${\mathfrak{so}_N}^*$, and $\langle\,\cdot\,\rangle$ is the matrix trace. Various coincidence cases $\mathfrak{sp}_2\simeq\mathfrak{sl}_2$, $\mathfrak{sp}_4\simeq\mathfrak{so}_5$ and $\mathfrak{sl}_4\simeq\mathfrak{so}_6$ can be explicitly checked from the above formulas. Only $\mathfrak{sl}_N$ series admits an one-parameter family of HS algebras, and the appearance of ideals for particular values of λ is manifest from the expression of

JHEP05(2014)103

the bilinear form. It is worth to notice also that for $\mathfrak{sl}_2$, we recover the $3D$ algebra $hs[\lambda]$ with a particularly simple form of structure constant, since the trilinear form (1.4) can be considerably simplified.

The organization of the paper is as follows:

- in section 2, we review some generalities of HS algebras. First we show how they appear from a field theory of massless HS particles, and then present their relation to mathematical objects such as coadjoint orbits and minimal representations. We provide the definition of HS algebras and their realizations in terms of oscillators.

- In section 4, we derive explicit expressions for structure constants — the invariant bilinear and trilinear forms. For that, we introduce a trace for an element of HS algebra, defined as the identity piece of the element. We make explicit the latter definition showing that such trace can be given in fact through the $\mathfrak{gl}_1$ and $\mathfrak{sp}_2$ projectors previously introduced in [8, 56] and [57], respectively for $\mathfrak{sl}_4$ and $\mathfrak{so}_{D+1}$. With such trace formulas, we explicitly evaluate the bilinear and trilinear forms ending up with the results (1.2)–(1.6).

- In section 4, we discuss more about the HS algebras associated with $\mathfrak{sl}_N$. First, we discuss the formation of ideals and associated finite-dimensional coset algebras, which arise for particular values of λ. Then, we provide another description of these HS algebras, based on reduced number of oscillators which are free from equivalence relations. At the end of this section, we discuss in more details the $\mathfrak{sl}_2$ case, that is $hs[\lambda]$. Besides providing a relatively simple expression for the $\star$ product, we make a concrete link between the description used in this paper and that of the deformed oscillators.

- Finally, in section 5, we discuss some outlooks of the present work, while appendix A includes some technical details.

2 HS algebras and minimal representations

In order to keep the current paper as self-complete as possible, we review the definition and the construction of HS algebras collecting knowledge from the physics and mathematics literature. Our focus is on providing the precise definition and role of HS algebras in physics and introducing the notion of minimal representations.

2.1 HS algebras as global HS symmetries

A HS algebra is the global-symmetry counterpart of HS gauge symmetry. The latter depends on the description one chooses — frame-like, metric-like, etc. — whereas the global symmetry does not depend on such a choice. In the following, we briefly introduce HS algebra as the global symmetry of HS gauge fields in the metric-like description where the field content is a (infinite) set of symmetric tensor fields $\varphi_{\mu_1\cdots\mu_s}$ (with a certain multiplicity for a given spin field). The gauge transformation has a form,

$$\delta_\varepsilon\, \varphi_{\mu_1\cdots\mu_s} = \bar{\nabla}_{(\mu_1}\, \varepsilon_{\mu_2\cdots\mu_s)} + t_{\mu_1\cdots\mu_s}(\varphi,\varepsilon) + \mathcal{O}\left(\varphi^2\right)\,, \tag{2.1}$$

where $\bar{\nabla}$ is the (A)dS covariant derivative and $t_{\mu_1\cdots\mu_s}(\varphi,\varepsilon)$ is the *interaction-part* of transformation which is bilinear in gauge fields and parameters, denoted by φ and ε respectively.[3] To restrict ourselves to the global part of such symmetries, we impose the Killing equations:

$$0 = \left[\delta_\varepsilon\, \varphi_{\mu_1\cdots\mu_s}\right]_{\varphi=0} = \bar{\nabla}_{(\mu_1}\, \varepsilon_{\mu_2\cdots\mu_s)}\,. \tag{2.2}$$

The space of the solutions $\bar{\varepsilon}_{\mu_1\cdots\mu_r}$ — Killing tensors — defines the HS algebra as a vector space (see [60, 61] for related works). For more details, it is convenient to reiterate the discussion using auxiliary variables in the ambient-space formulation. The Killing equation (2.2) is then given by

$$U\cdot\partial_X\, E(X,U) = 0 \qquad \left[X, U \in \mathbb{R}^{D+1}\right]\,, \tag{2.3}$$

where the ambient-space gauge parameter E is related to the intrinsic one by

$$E(X,U) = \sum_{r=0}^{\infty} \frac{R^r}{r!}\, \bar{e}_{a_1}^{\ \mu_1}(x)\, U^{a_1}\, \cdots\, \bar{e}_{a_r}^{\ \mu_r}(x)\, U^{a_r}\, \varepsilon_{\mu_1\cdots\mu_r}(x)\,, \tag{2.4}$$

with R and x being the radial and (A)dS-intrinsic coordinate of the ambient space $\mathbb{R}^{D+1}$, and $\bar{e}_a^{\ \mu}$ the (A)dS background vielbein. The relation (2.4) is equivalent to imposing the tangentiality and homogeneity conditions on E as

$$X\cdot\partial_U\, E(X,U) = 0 = (X\cdot\partial_X - U\cdot\partial_U)\; E(X,U)\,. \tag{2.5}$$

In this ambient-space description, the solution $\bar{E}$ of the Killing equation reads simply

$$\bar{E}(X,U) = \sum_{r=0}^{\infty} \frac{1}{2^r\,(r!)^2}\, \bar{E}_{a_1b_1,\,\dots\,,a_rb_r}\, X^{[a_1}\, U^{b_1]}\, \cdots\, X^{[a_r}\, U^{b_r]}\,, \tag{2.6}$$

and from the tracelessness of the gauge parameter, one can also conclude that the Killing tensors are completely traceless:

$$\partial_U^2\, \bar{E}(X,U) = 0\,, \qquad \partial_X^2\, \bar{E}(X,U) = 0\,, \qquad \partial_U\cdot\partial_X\, \bar{E}(X,U) = 0\,. \tag{2.7}$$

The generators of HS algebra are the duals of $\bar{E}_{a_1b_1,\,\dots\,,a_rb_r}$ and given by

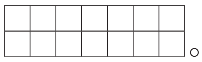
$$\begin{aligned}\left(M^{a_1b_1,\,\dots\,,a_rb_r}\right)(X,U) &= X^{[a_1}\, U^{b_1]}\, \cdots\, X^{[a_r}\, U^{b_r]} + X\cdot U\, S_1^{a_1b_1,\,\dots\,,a_rb_r} \\ &\quad + X^2\, S_2^{a_1b_1,\,\dots\,,a_rb_r} + U^2\, S_3^{a_1b_1,\,\dots\,,a_rb_r}\,,\end{aligned} \tag{2.8}$$

with arbitrary tensors S_i due to the tracelessness of $\bar{E}$: using such a freedom, one can choose traceless $M^{a_1b_1,\,\cdots\,,a_rb_r}$. So far, we have not used any information coming from interactions but just the field content, and we have determined only the vector-space structure of

[3] To be more precise, in Fronsdal's formulation of HS fields [58, 59], the gauge fields and parameters are subjected to trace conditions: $\bar{g}^{\mu_1\mu_2}\, \bar{g}^{\mu_3\mu_4}\, \varphi_{\mu_1\cdots\mu_s} = \mathcal{O}\left(\varphi^2\right)$ and $\bar{g}^{\mu\nu}\, \varepsilon_{\mu_1\cdots\mu_{s-1}} = \mathcal{O}(\varphi)$, where $\bar{g}^{\mu\nu}$ is the (A)dS inverse metric.

HS algebra: the basis elements have the symmetry of the rectangular two-row $O(D+1)$ diagrams,

$$M^{a_1b_1,\,\dots\,,a_rb_r} \sim \qquad (2.9)$$



$$_\circ\,,$$

that satisfy

$$M^{\cdots\,,a_ib_i,\,\dots\,,a_jb_j,\,\cdots} = M^{\cdots\,,a_jb_j,\,\dots\,,a_ib_i,\,\cdots}\,, \qquad M^{(a_1b_1),\,\cdots} = 0 = M^{[a_1b_1,a_2]b_2,\,\cdots}\,,$$
$$\eta_{a_1a_2}\, M^{a_1b_1,a_2b_2,\,\cdots} = 0\,. \tag{2.10}$$

The Lie-algebra bracket $[\![\,,\,]\!]$ of HS algebra is inherited from that of the gauge algebra as

$$\delta^{(0)}_{[\![\varepsilon_1,\varepsilon_2]\!]} = \delta^{(0)}_{\varepsilon_1}\,\delta^{(1)}_{\varepsilon_2} - \delta^{(0)}_{\varepsilon_2}\,\delta^{(1)}_{\varepsilon_1}\,, \tag{2.11}$$

where $\delta^{(0)}_{\varepsilon}$ and $\delta^{(1)}_{\varepsilon}$ are respectively the first and second terms of the gauge transformation (2.1). Hence, the bracket of HS algebra is entirely specified by the first-order interacting terms $t_{\mu_1\cdots\mu_s}$ of the gauge transformations, and they are in turn fixed by the cubic interaction terms of the Lagrangian — see [62] for a recent related discussion. It is important to note that the global symmetries close at the level of $\delta^{(1)}$:

$$\delta^{(1)}_{\bar{\varepsilon}_1}\,\delta^{(1)}_{\bar{\varepsilon}_2} - \delta^{(1)}_{\bar{\varepsilon}_2}\,\delta^{(1)}_{\bar{\varepsilon}_1} = \delta^{(1)}_{[\![\bar{\varepsilon}_1,\bar{\varepsilon}_2]\!]} + (\text{trivial part})\,, \tag{2.12}$$

so that $\delta^{(1)}_{\bar{\varepsilon}}$ provides the representation of HS algebra carried by the field content. Here, (trivial part) means the transformations, either of the form of free gauge symmetry or proportional to the free equations of motion. Moreover, this representation leaves the quadratic action $S^{(2)}$ invariant: $\delta^{(1)}_{\bar{\varepsilon}}\, S^{(2)}[\varphi] = 0$, so is endowed with an invariant scalar product, which is positive definite if the free action $S^{(2)}$ is unitary. Hence, for a unitary HS theory, the representation of HS algebra given by $\delta^{(1)}_{\bar{\varepsilon}}$ is also unitary. This condition, known as *admissibility condition* [63, 64], turns out to be quite tight for a quest of candidate HS algebras.

Another condition on HS algebra is the requirement that its spin-two part reproduce Gravity. This condition fixes certain brackets of HS algebra. First, the spin-two part gives the isometry algebra $\mathfrak{so}_{D+1}$:

$$[\![\, M_{ab}\,, M_{cd}\,]\!] = 2\left(\eta_{a[c}\, M_{d]b} - \eta_{b[c}\, M_{d]a}\right), \tag{2.13}$$

and other HS generators are subject to covariant transformation under isometry:

$$[\![\, M_{a_1b_1,\dots,a_rb_r}\,, M_{cd}\,]\!] = 2\sum_{k=1}^{r} \eta_{a_k[c}\, M_{\dots,d]b_k,\dots} - \eta_{b_k[c}\, M_{\dots,d]a_k,\dots}\,. \tag{2.14}$$

All in all, apart from the admissibility condition, any Lie algebra generated by Killing tensors (2.9) which transform covariantly under the isometry algebra $\mathfrak{so}_{D+1}$ is a candidate HS algebra.

JHEP05(2014)103

2.2 UEA construction of HS algebras

As discussed in [24, 25], the generators of HS algebra subject to $\mathfrak{so}_{D+1}$-covariant transformation can be constructed from the universal enveloping algebra (UEA) of $\mathfrak{so}_{D+1}$. In the following discussion, let us focus on the $D \geq 4$ cases. The UEA is defined as the quotient of the tensor algebra of $\mathfrak{so}_{D+1}$ with the two-sided ideal generated by

$$I_{abcd} = M_{ab} \otimes M_{cd} - M_{cd} \otimes M_{ab} - [\![\, M_{ab}\,,\, M_{cd}\,]\!]\,. \tag{2.15}$$

Hence, the class representatives can be taken as $GL(D+1)$ tensors,

$$M_{a_1 b_1} \odot \cdots \odot M_{a_n b_n} := \frac{1}{n!} \sum_{\sigma \in S_n} M_{a_{\sigma(1)} b_{\sigma(1)}} \otimes \cdots \otimes M_{a_{\sigma(n)} b_{\sigma(n)}}\,,$$

which are generically reducible under index-permutation symmetries. When decomposed into irreducible components, they contain Killing tensors $M_{a_1 b_1, \ldots, a_n b_n}$ as well as other elements. At the quadratic level, the $GL(D+1)$ decomposition gives

$$M^{(a_1}{}_{(b_1} \odot M^{a_2)}{}_{b_2)} \sim \yng(2,2)\,, \qquad M_{[a_1 b_1} \odot M_{a_2 b_2]} \sim \yng(1,1,1,1)\,. \tag{2.16}$$

The traceless part of $M^{(a_1}{}_{(b_1} \odot M^{a_2)}{}_{b_2)}$ gives a Killing tensor but its trace part and $M_{[a_1 b_1} \odot M_{a_2 b_2]}$ are not Killing tensors. However, they generate an ideal called *Joseph ideal* which we shall discuss more extensively in the next subsection. When the UEA is quotiented by this ideal, the coset is spanned only by Killing tensors so satisfies the condition of HS algebra. To summarize, the following two classes of the elements in $\mathfrak{so}_{D+1} \odot \mathfrak{so}_{D+1}$:

$$J_{ab} := M_{(a}{}^{c} \odot M_{b)c} - \frac{\eta_{ab}}{D+1}\, M^{cd} \odot M_{cd} \sim \yng(2)_{\circ}\,, \qquad J_{abcd} := M_{[ab} \odot M_{cd]} \sim \yng(1,1,1,1)\,, \tag{2.17}$$

generate the Joseph ideal which contains all the non-Killing-tensor elements. This construction fixes the values of all $\mathfrak{so}_{D+1}$ Casimir operators, and in particular the quadratic one is given by

$$C_2 := \frac{1}{2}\, M^{ab} \odot M_{ba} = -\frac{(D+1)(D-3)}{4}\,. \tag{2.18}$$

In [25], the authors considered another ideal generated only by J_{ab} — but not by J_{abcd}. In such a case, the quadratic Casimir remains arbitrary while the other Casimirs are determined as functions of the former. Hence, one can further take the quotient with $C_2 - \nu$. Then, the resulting coset algebra contains more generators than the original case, and additional generators correspond to the Killing tensors of certain types of mixed-symmetry fields. This algebra has one-parameter family with label ν, and it is denoted by $hs(\nu)$. In $D = 5$ case, $hs(\nu)$ can be decomposed into two parts which are isomorphic to each other: each part can be independently obtained from the ideal generated by the elements J_{ab} and $J^{\pm\lambda}_{abcd}$, where the latter element is given by

$$J^{\lambda}_{abcd} := M_{[ab} \odot M_{cd]} - i\, \frac{\lambda}{6}\, \epsilon_{abcdef}\, M^{ef}\,. \tag{2.19}$$

JHEP05(2014)103

The generators of the resulting coset algebra are all given by Killing tensors like the $\lambda = 0$ case, and the quadratic Casimir is given by

$$C_2 = \nu = 3\,(\lambda^2 - 1)\,. \tag{2.20}$$

Moreover, one can show that the algebra admits further ideals for any half-integer values of λ with $|\lambda| \geq 1$, and quotienting those ideals results in finite-dimensional algebras. Such finite-dimensional HS algebras have been considered recently in [65] — see section 4.1 for more details.

To conclude this section, let us also mention that the aforementioned HS algebra can be obtained by quotienting directly the tensor algebra of $\mathfrak{so}_{D+1}$ with the ideal generated by the following elements:

$$I^{\lambda}_{abcd} = M_{a[b} \otimes M_{cd]} - \eta_{a[b}\, M_{cd]} - i\,\frac{\lambda}{6}\,\epsilon_{abcdef}\, M^{ef}\,, \tag{2.21}$$

$$I^{\lambda}_{ab} = M_{c(a} \otimes M_{b)}{}^{c} + \frac{D-3}{2}\left(1-\lambda^2\right)\eta_{ab}\,, \tag{2.22}$$

where the $\lambda \neq 0$ cases are only for $D = 5$. From the above, it becomes more clear that Killing tensors can be taken as class representatives. The existence of one-parameter family for $D = 5$ is actually due to the fact that $\mathfrak{so}_6$ is isomorphic to $\mathfrak{sl}_4$. There, the elements (2.21) and (2.22) can be combined into

$$I^{\lambda}{}^{ac}_{bd} = L^{[a}{}_{b} \otimes L^{c]}{}_{d} + \delta^{[a}_{(b}\, L^{c]}{}_{d)} + \lambda\, \delta^{[a}_{[b}\, L^{c]}{}_{d]} + \frac{1}{4}\left(\lambda^2 - 1\right) \delta^{[a}_{[b}\, \delta^{c]}_{d]}\,, \tag{2.23}$$

where $L^{a}{}_{b}$ are the generators of $\mathfrak{sl}_4$ with $a, b = 1, \ldots, 4$ and $L^{a}{}_{a} = 0$. In fact, these elements generate an ideal in the UEA of $\mathfrak{sl}_N$ for any N. In particular for $\mathfrak{sl}_2$, the $3D$ HS algebra $hs[\lambda]$ can be obtained in this way. We will come back to this point later.

2.3 Minimal representations

In order to obtain HS algebra from the UEA, we quotient the UEA with an ideal corresponding to the tensors which are not of the Killing-tensor type (2.9). This quotienting procedure fixes all the Casimir operators as well as the underlying representation: the ideal coincides with the kernel of such representations. Indeed, this reduction of generators can be carried out by simply choosing the proper representation of the isometry algebra, which is small enough to project all the generators except for Killing tensors. It turns out that these representations are in fact the *smallest* ones, namely the minimal representations [39]. Here, for the self-completeness, we provide a brief introduction to the minimal representations by mainly focusing on the case of the classical Lie algebras over $\mathbb{C}$.

Let us first introduce the convention which we shall adopt in the following discussion:

- let us begin with $\mathfrak{sp}_{2N}$ which is generated by elements N_{AB}:

$$N_{[AB]} = 0\,, \qquad A, B = 1, 2, \ldots, 2N\,, \tag{2.24}$$

with the commutation relation,

$$[\![\, N_{AB}\,,\ N_{CD}\,]\!] = \Omega_{A(C}\, N_{D)B} + \Omega_{B(C}\, N_{D)A}\,. \tag{2.25}$$

Here, $\Omega_{AB} = -\Omega_{BA}$ is the symplectic matrix, with the inverse Ω^{AB}:

$$\Omega^{AB}\,\Omega_{BC} = \Omega_{CB}\,\Omega^{BA} = \delta_C^A\,, \tag{2.26}$$

which is used to lower the indices as $V_A = \Omega_{AB}\,V^B$.

Now, we move to $\mathfrak{sl}_N$ and $\mathfrak{so}_N$ which we shall describe as subalgebras of $\mathfrak{sp}_{2N}$. For that, it is convenient to organize the $\mathfrak{sp}_{2N}$ indices A as

$$A = \alpha\,a\,, \qquad \alpha = \pm\,, \quad a = 1, 2, \dots, N\,, \tag{2.27}$$

with which the symplectic matrix becomes

$$\Omega_{\alpha a\,\beta b} = \epsilon_{\alpha\beta}\,\eta_{ab}\,, \qquad \epsilon_{\pm\mp} = \pm 1\,. \tag{2.28}$$

- The $\mathfrak{sl}_N$ is generated by the traceless elements $L^a{}_b := N_-{}^a{}_{+b} - \frac{1}{N}\,\delta^a_b\,N_-{}^c{}_{+c}$ with the commutation relation,
$$\llbracket\, L^a{}_b\,,\ L^c{}_d\,\rrbracket = \delta^c_b\,L^a{}_d - \delta^a_d\,L^c{}_b\,. \tag{2.29}$$
- The $\mathfrak{so}_N$ is generated by the antisymmetric elements $M_{ab} := N_{-a\,+b} - N_{-b\,+a}$ with the commutation relation (2.13).

The dual vector space of these Lie algebras are the spaces of matrices U^{AB}, $V_a{}^b$ and W^{ab} with $U^{[AB]} = 0$, $V_a{}^a = 0$ and $W^{(ab)} = 0$. It is convenient to introduce

$$N(U) = \frac{1}{2}\,N_{AB}\,U^{AB}\,, \qquad L(V) = L^a{}_b\,V_a{}^b\,, \qquad M(W) = \frac{1}{2}\,M_{ab}\,W^{ab}\,, \tag{2.30}$$

in terms of which the commutation relations of the algebras can be also given by

$$\llbracket\, T(A_1)\,,\, T(A_2)\,\rrbracket = T(A_1\,A_2 - A_2\,A_1)\,. \tag{2.31}$$

Here, (T, A) are $(N, U), (L, V)$ or (M, W), while the products of the dual matrices are given by $(U_1\,U_2)^{AB} = U_1{}^{AC}\,\Omega_{CD}\,U_2{}^{DB}$, $(V_1\,V_2)_a{}^b = V_{1a}{}^c\,V_{2c}{}^b$ and $(W_1\,W_2)^{ab} = W_1{}^{ac}\,\eta_{cd}\,W_2{}^{db}$.

Now let us come back to the introduction to minimal representations for classical Lie algebras. There are several different approaches to minimal representations.[4] Here, we follow the coadjoint orbit method where minimal representation is given as the quantization of the minimal nilpotent orbit. The coadjoint action of a Lie group $\mathcal{G}$ on the dual space $\mathfrak{g}^*$ of its Lie algebra $\mathfrak{g}$ is defined by

$$(\mathrm{Coad}_g\,A)(T) = A\left(g^{-1}\,T\,g\right)\,, \tag{2.32}$$

where g, T and A are arbitrary elements of $\mathcal{G}, \mathfrak{g}$ and $\mathfrak{g}^*$, respectively. Each orbit under such actions — coadjoint orbit — is an even dimensional subspace of $\mathfrak{g}^*$ with $\mathcal{G}$-invariant symplectic form. While there exists a continuum of semi-simple orbits, the number of nilpotent orbits is finite. The semi-simple orbits and the principal nilpotent orbit — the unique dense orbit of the nilpotent orbits — are given by a set of equations involving the

[4] See e.g. [40–50] and references therein for general introduction to minimal representation.

JHEP05(2014)103

dual of Casimir operators. Hence, their dimension is $\dim\mathfrak{g} - \operatorname{rank}\mathfrak{g}$. The other nilpotent orbits have smaller dimensions as they are defined by a larger number of polynomial equations. The nilpotent orbit with minimum dimension — apart from the trivial orbit $\{0\}$ — is also unique and called the minimal orbit $\mathcal{O}_{\rm min}(\mathfrak{g})$, which is what we are interested in. The minimal orbits of classical Lie algebras are determined by the following quadratic equations [48]:

$$\begin{aligned} \mathcal{O}_{\rm min}(\mathfrak{sp}_{2N}): &\quad U^{A[B}\,U^{D]C} = 0\,,\\ \mathcal{O}_{\rm min}(\mathfrak{sl}_{N}): &\quad V_{[a}{}^{b}\,V_{c]}{}^{d} = 0\,,\\ \mathcal{O}_{\rm min}(\mathfrak{so}_{N}): &\quad W^{a[b}\,W^{cd]} = 0 = W^{ab}\,W_{b}{}^{c}\,, \end{aligned} \tag{2.33}$$

and can be parameterized by

$$\begin{aligned} U^{AB} &= u^{A}\,u^{B}\,,\\ V_{a}{}^{b} &= v_{+a}\,v_{-}^{b} \qquad [v_{+}\cdot v_{-} = 0]\,,\\ W^{ab} &= w_{+}^{[a}\,w_{-}^{b]} \qquad [w_{\alpha}\cdot w_{\beta} = 0]\,. \end{aligned} \tag{2.34}$$

From the above, one can deduce the dimensions of these minimal orbits as

$\mathfrak{g}$	$\dim\mathfrak{g}$	$\operatorname{rank}\mathfrak{g}$	$\dim\mathcal{O}_{\rm prin}(\mathfrak{g})$	$\dim\mathcal{O}_{\rm min}(\mathfrak{g})$
$\mathfrak{sp}_{2N}$	$N(2N+1)$	N	$2\,N^2$	$2N$
$\mathfrak{sl}_{N}$	N^2-1	$N-1$	$N(N-1)$	$2(N-1)$
$\mathfrak{so}_{N}$	$\frac{N(N-1)}{2}$	$\lfloor\frac{N}{2}\rfloor$	$\lceil\frac{N(N-2)}{2}\rceil$	$2(N-3)$

where $\lfloor x\rfloor = \max\{m\in\mathbb{Z}\,|\,m\le x\}$ and $\lceil x\rceil = \min\{n\in\mathbb{Z}\,|\,n\ge x\}$. The kernel of the minimal representation in the UEA is the Joseph ideal $\mathcal{J}(\mathfrak{g})$ (the characteristic variety of the Joseph ideal is the closure of the minimal orbit $\mathcal{O}_{\rm min}(\mathfrak{g})$). Joseph ideal is the ideal of UEA generated by certain elements in $\mathfrak{g}\odot\mathfrak{g}$. In the case of classical Lie algebras, one can equivalently consider the ideals of the tensor algebra generated by the relations [48]:

$$\begin{aligned} \mathcal{J}(\mathfrak{sp}_{2N}): &\quad N_{A[B}\otimes N_{C]D} + \frac{\hbar}{2}\left(\Omega_{A[B}\,N_{C]D} + \Omega_{D[B}\,N_{C]A} - \Omega_{BC}\,N_{AD}\right)\\ &\qquad + \frac{\hbar^2}{2}\left(\Omega_{A[B}\,\Omega_{C]D} - \Omega_{BC}\,\Omega_{AD}\right) \sim 0\,,\\ \mathcal{J}(\mathfrak{sl}_{N}): &\quad L^{[a}{}_{b}\otimes L^{c]}{}_{d} + \hbar\left(\delta^{[a}_{(\beta}\,L^{c]}{}_{d)} + \lambda\,\delta^{[a}_{[b}\,L^{c]}{}_{d]}\right) + \hbar^2\,\frac{\lambda^2-1}{4}\,\delta^{[a}_{[b}\,\delta^{c]}_{d]} \sim 0\,,\\ \mathcal{J}(\mathfrak{so}_{N}): &\quad M_{a[b}\otimes M_{cd]} - \hbar\,\eta_{a[b}\,M_{cd]} \sim 0 \sim M_{c(a}\otimes M_{b)}{}^{c} - \hbar^2\,\frac{N-4}{2}\,\eta_{ab}\,, \end{aligned} \tag{2.35}$$

which are dual analogs of the ones (2.33) on $\mathfrak{g}^*$. Here, we have introduced the deformation parameter $\hbar$ — which are taken as $\hbar=1$ in the rest of the paper — in order to make manifest that the above relations provide quantizations of the orbits given in (2.33). Inversely, an irreducible representation of $\mathfrak{g}$ associated with a coadjoint orbit can be obtained by quantizing the orbit. Notice also that for the $\mathfrak{sl}_N$ series, the quantization or equivalently the minimal representation is not unique but is of one-parameter family labeled by λ. For the $\mathfrak{sl}_4$ case, the minimal representation has been studied also in [52].

Let us note that the minimal representations are the irreducible representations with the minimum non-zero Gelfand-Kirillov (GK) dimension [66]. The GK dimension of a vector space can be understood roughly as the number of continuous variables necessary to describe the vector space. In case of the minimal representation, it is half of the dimension of minimal orbit. For one-particle Hilbert space, it corresponds to the dimension of the space — not the spacetime — where the wave function lives. Several conclusions can be drawn from this perspective:

- a particle in $D \geq 4$ dimensions has GK dimension $D-1$. Collecting finitely many particles cannot increase the GK dimension, since it amounts to introducing some finite-range discrete variables which label the particles. On the contrary, an infinite collection of particles may have a bigger GK dimension: for example, Kaluza-Klein compactification decomposes a particle in higher dimensions — therefore, of higher GK dimension — into an infinite set of particles in lower dimensions.

- $(A)dS_D$ has isometry (a real form of) $\mathfrak{so}_{D+1}$, whose minimal representation has GK dimension $D-2$. Hence, a particle in $(A)dS_D$ cannot be minimal but a representation associated with the next-to-minimal orbit of $\mathfrak{so}_{D+1}$.

- The particles corresponding to conformal fields, namely singletons, in d dimensions have GK dimension $d-1$, the same as the minimal representation of the conformal algebra $\mathfrak{so}_{d+2}$. However, apart from the scalar, the other conformal-field representations require additional helicity labels. The spinor in $d=3$ and helicity-h representations in $d=4$ are exceptions as they have only one helicity component. Actually, such representations underlie the $D=4$ and $D=5$ HS algebras, respectively: in the former case the scalar and spinor singletons, namely Rac and Di, give the same HS algebra, while in the latter case the helicity h is related to the λ deformation (2.19) by $h=\lambda$.

- The Flato-Fronsdal theorem [67] — as well as its generalization [57] — corresponding to the HS AdS_{d+1}/CFT_d duality can be also understood in this way. If the conformal field theory on the boundary has a finite content of fields, then its GK dimension is $d-1$. The space of bilinear operators, which corresponds to the tensor product of two singletons, has doubled GK dimension which can be viewed as

$$2(d-1) = d + (d-2)\,. \tag{2.36}$$

  Here, d is the number of space variables of $(A)dS_{d+1}$, and $d-2$ is the GK dimension corresponding to the helicity variables. For instance, in the scalar singleton case, the corresponding dual $(A)dS_{d+1}$ fields are massless symmetric fields of all integer spins. The number of their helicity states up to spin s is $\sim s^{d-2}$, from which we can deduce the corresponding GK dimension.

- Finally, suppose we consider a D-dimensional field theory with a finite field content which carries a representation of a global-symmetry Lie algebra $\mathfrak{g}$. Then, the GK dimension of the Hilbert space cannot be smaller than that of the minimal representation of $\mathfrak{g}$:

$$D - 1 \geq \frac{1}{2} \dim(\mathcal{O}_{\rm min}(\mathfrak{g}))\,. \tag{2.37}$$

Among classical Lie algebras, $\mathfrak{sl}_N$ with $N \geq D+1$, $\mathfrak{so}_N$ with $N \geq D+3$ and $\mathfrak{sp}_{2N}$ with $N \geq D$ are already excluded with this condition. Together with the requirement that $\mathfrak{g}$ contains the D-dimensional isometry algebra $\mathfrak{so}_{D+1}$, one gets much stronger restrictions: for instance, the only possible orthogonal algebras $\mathfrak{g}$ are $\mathfrak{so}_{D+1}$ itself and $\mathfrak{so}_{D+2}$.[5]

2.4 HS algebras and reductive dual pairs

The HS algebra defined in section 2.2 is the symmetry (algebra) of the minimal representation of $\mathfrak{so}_N$. Abusing this terminology to the other Lie algebras, let us consider *HS algebra* associated with a Lie algebra $\mathfrak{g}$:

$$hs(\mathfrak{g}) = \mathrm{U}(\mathfrak{g})/\mathcal{J}(\mathfrak{g})\,, \tag{2.38}$$

where $\mathrm{U}(\mathfrak{g})$ and $\mathcal{J}(\mathfrak{g})$ are respectively the UEA and Joseph ideal of $\mathfrak{g}$. As a vector space, $hs(\mathfrak{g})$ corresponds to the space of polynomials in elements of $\mathcal{O}_{\rm min}(\mathfrak{g})$:

$$N(U) = \sum_{n=0}^{\infty} N^{(n)}(U)\,, \qquad L(V) = \sum_{n=0}^{\infty} L^{(n)}(V)\,, \qquad M(W) = \sum_{n=0}^{\infty} M^{(n)}(W)\,, \tag{2.39}$$

with

$$N^{(n)}(U) = \frac{1}{2^n\,n!}\,N_{A_1B_1,\dots,A_nB_n}\,U^{A_1B_1}\cdots U^{A_nB_n}\,, \tag{2.40}$$

$$L^{(n)}(V) = \frac{1}{n!}\,L^{a_1\cdots a_n}_{b_1\cdots b_n}\,V_{a_1}{}^{b_1}\cdots V_{a_n}{}^{b_n}\,, \tag{2.41}$$

$$M^{(n)}(W) = \frac{1}{2^n\,n!}\,M_{a_1b_1,\dots,a_nb_n}\,W^{a_1b_1}\cdots W^{a_nb_n}\,. \tag{2.42}$$

Therefore, the expansion coefficients,

$$N_{A_1B_1,\dots,A_nB_n}\,, \qquad L^{a_1\cdots a_n}_{b_1\cdots b_n}\,, \qquad M_{a_1b_1,\dots,a_nb_n}\,, \tag{2.43}$$

are the generators of $hs(\mathfrak{sp}_{2N})$, $hs_\lambda(\mathfrak{sl}_N)$ and $hs(\mathfrak{so}_N)$, respectively. Due to the properties of minimal orbits (2.33), these generators can be chosen to be traceless:

$$\Omega^{A_1A_2}\,N_{A_1B_1,\dots,A_nB_n} = 0\,, \qquad \delta^{b_1}_{a_1}\,L^{a_1\cdots a_n}_{b_1\cdots b_n} = 0\,, \qquad \eta^{a_1a_2}\,M_{a_1b_1,\dots,a_nb_n} = 0\,. \tag{2.44}$$

[5] In $3D$, massless HS particles have GK dimension 1 corresponding to the would-be gauge mode on the asymptotic boundary, and the global symmetry is rather the asymptotic symmetry than the bulk isometry one. In case of $\mathfrak{sl}_N \oplus \mathfrak{sl}_N$ Chern-Simons theory, the asymptotic symmetry is given by $\mathcal{W}_N \oplus \mathcal{W}_N$ [21]. Interestingly, $\mathcal{W}_N$ does not contain $\mathfrak{sl}_N$ as subalgebra — at least, not manifestly. If it did, the GK dimension of the Hilbert space would be bigger or equal to $N-1$, which is not the case for $N \geq 3$.

Let us note that in order to derive (2.37), we have required the Hilbert space to carry a representation of the global symmetry. As we have seen in the $3D$ case, the presence of the asymptotic boundary may provoke a deformation of global symmetry invalidating this condition. This phenomenon is in principle possible in any odd D dimensions, so may provide a chance for a consistent HS theory with a finite field content — see [65] for a recent attempt.

JHEP05(2014)103

We will use the symbol $\bigstar$ for the product of $hs(\mathfrak{g})$, which is defined by $\bigstar := \otimes/\sim$. Here, $\sim$ is the equivalence relation (2.35).

For a classical Lie algebra $\mathfrak{g}$, instead of using the explicit form of Joseph ideals, one can rely on the reductive dual pairs to handle the algebraic structure of $hs(\mathfrak{g})$: see [7, 8] and [2] for the $\mathfrak{sl}_4$ and $\mathfrak{so}_{D+1}$ cases, respectively, and for more generalities see e.g. [45, 49] and references therein. A reductive dual pair in the symplectic group Sp_{2N} is a pair of subgroups,

$$(\, G_1\,,\, G_2\,) \subset Sp_{2N}\,, \tag{2.45}$$

which are centralizers of each other. Then, there is a bijection between two irreducible representations π_1 and π_2 of G_1 and G_2 so that for any π_1 (or π_2) there exists at most one π_2 (or π_1). The minimal representations of $\mathfrak{sl}_N$ and $\mathfrak{so}_N$ can be obtained from that of $\mathfrak{sp}_{2N}$ by considering the dual pairs,

$$(\, G_1\,,\, G_2\,) = (\, GL_1\,,\, GL_N\,) \quad \text{and} \quad (\, Sp_2\,,\, O_N\,)\,, \tag{2.46}$$

respectively. For the former case, we take the representation of GL_1 labeled by λ — so we can see again that the minimal representation of $\mathfrak{sl}_N$ has one-parameter family. For the latter case, we take the trivial representation of Sp_2. In the following, we review how one can deal with the explicit structures of HS algebras using such dual pair correspondences.

$hs(\mathfrak{sp}_{2N})$. Notice first that the minimal representation of $\mathfrak{sp}_{2N}$ is the metaplectic representation described by oscillators y_A:

$$N_{AB} = y_A\, y_B\,, \tag{2.47}$$

endowed with the Moyal $\star$ product,

$$(f \star g)(y) = \exp\left(\frac{1}{2}\,\Omega_{AB}\,\partial_{y_A}\,\partial_{z_B}\right) f(y)\; g(z)\Big|_{z=y}. \tag{2.48}$$

Hence, $hs(\mathfrak{sp}_{2N})$ is generated by polynomials of $y_A\, y_B$, that is, the space of all even-order polynomials in y_A:

$$N_{A_1B_1,\dots,A_nB_n} = y_{A_1}\, y_{B_1}\, \cdots\, y_{A_n}\, y_{B_n}\,, \tag{2.49}$$

and the generating function $N(U)$ (2.39) becomes a Gaussian,

$$N(U) = \exp\left(\frac{1}{2}\; y_A\, U^{AB}\, y_B\right). \tag{2.50}$$

In this case, the product of $hs(\mathfrak{sp}_{2N})$ coincides with the Moyal product: $\bigstar = \star$.

$hs_\lambda(\mathfrak{sl}_N)$. We consider the $\mathfrak{gl}_1$-center of $hs(\mathfrak{sp}_{2N})$, that is, the set of elements satisfying

$$\big[\, y_+ \cdot y_-\,,\; f(y)\,\big]_\star = \left(y_- \cdot \partial_{y_-} - y_+ \cdot \partial_{y_+}\right) f(y) = 0\,. \tag{2.51}$$

The solution space is generated by

$$\tilde{L}^{a_1\,\cdots\,a_n}_{b_1\,\cdots\,b_n} = y_-{}^{a_1}\, y_{+b_1} \cdots\, y_-{}^{a_n}\, y_{+b_n}\,, \tag{2.52}$$

whose traceless part can be identified with the generator $L^{a_1\cdots a_n}_{b_1\cdots b_n}$ of $hs_\lambda(\mathfrak{sl}_N)$. It is convenient for later use to generalize the definitions (2.39) and (2.41) to $\tilde{L}^{a_1\cdots a_n}_{b_1\cdots b_n}$ getting

$$\tilde{L}\left(\tilde{V}\right) = \exp\left(y_-\cdot\tilde{V}\cdot y_+\right), \tag{2.53}$$

where the matrix-variable $\tilde{V}_a^b$ satisfies

$$\tilde{V}_{[a}{}^b\,\tilde{V}_{c]}{}^d = 0 \quad\Leftrightarrow\quad \tilde{V}_a{}^b = \tilde{v}_{+a}\,\tilde{v}_-^b\,. \tag{2.54}$$

This space is also endowed with the $\star$ product, and we will refer to this algebra as $hs(\mathfrak{gl}_N)$. In order to get the HS algebra of $\mathfrak{sl}_N$, we take an irreducible representation of $\mathfrak{gl}_1$, and this amounts to quotienting $hs(\mathfrak{gl}_N)$ by the relation,

$$K_\lambda := y_+\cdot y_- - \frac{N}{2}\,\lambda \sim 0\,. \tag{2.55}$$

Let us make a brief remark here: consider, before taking the quotient by K_λ, the following isomorphism:

$$\begin{aligned} \rho_\lambda:\ hs(\mathfrak{gl}_N) \quad &\to \quad hs_\lambda(\mathfrak{gl}_N)\,, \\ f(y) \quad &\mapsto \quad \rho_\lambda(f)(y) = e^{\frac{\lambda}{2}\,\partial_{y_+}\cdot\,\partial_{y_-}}\,f(y)\,. \end{aligned} \tag{2.56}$$

Then, the image $hs_\lambda(\mathfrak{gl}_N)$ admits a deformed $\star$ product,

$$\begin{aligned} (f\star_\lambda g)(y) &= \rho_\lambda\left(\rho_\lambda^{-1}(f)\star\rho_\lambda^{-1}(g)\right)(y) \\ &= \exp\left[\frac{1}{2}\left(\partial_{y_+}\cdot\partial_{z_-} - \partial_{z_+}\cdot\partial_{y_-}\right) + \frac{\lambda}{2}\left(\partial_{y_+}\cdot\partial_{z_-} + \partial_{z_+}\cdot\partial_{y_-}\right)\right] f(y)\ g(z)\Big|_{z=y}\,. \end{aligned} \tag{2.57}$$

The algebra $hs_\lambda(\mathfrak{sl}_N)$ can be equivalently obtained by quotienting $hs_\lambda(\mathfrak{gl}_N)$ by the relation,

$$\rho_\lambda(K_\lambda) = K_0 \sim 0\,. \tag{2.58}$$

Hence, although $hs_\lambda(\mathfrak{gl}_N)$ are all equivalent for different λ, after quotienting, they become distinct algebras $hs_\lambda(\mathfrak{sl}_N)$. In the following, we shall use the description $hs_\lambda(\mathfrak{sl}_N) = hs(\mathfrak{gl}_N)/\langle K_\lambda\sim 0\rangle$ for explicit computations.

An element of $hs_\lambda(\mathfrak{sl}_N)$ is a class representative $[\![\,a\,]\!]$ for elements $a\in hs(\mathfrak{gl}_N)$, and the product of $hs_\lambda(\mathfrak{sl}_N)$ is defined by

$$[\![\,a\,]\!]\bigstar[\![\,b\,]\!] := [\![\,a\star b\,]\!]\,. \tag{2.59}$$

In order to get an explicit expression for the product $\bigstar$, we need to choose a class representative for a generic element of $hs(\mathfrak{gl}_N)$. We can do this for $\tilde{L}^{(n)}$, and for concreteness let us consider $\tilde{L}^{(2)}$, which can be decomposed into $L^{(n)}$ with $n=0,1,2$ as

$$\tilde{L}^{(2)}\left(\tilde{V}\right) = L^{(2)}\left(\tilde{V}\right) + \frac{4}{N+2}\,y_+\cdot y_-\ \tilde{v}_+\cdot\tilde{v}_-\,L^{(1)}\left(\tilde{V}\right) + \frac{2}{N(N+1)}(y_+\cdot y_-)^2\,(\tilde{v}_+\cdot\tilde{v}_-)^2\,. \tag{2.60}$$

From this example, one can notice that $\tilde{L}^{(n)}$ and $L^{(n)}$ do not belong to the same equivalence class since the trace part of $\tilde{L}^{(n\geq 2)}$ cannot be written as $K_\lambda\star a$ for an element $a\in hs(\mathfrak{gl}_N)$.

We can remove all the $y_+\cdot y_-$ terms in the traceless decomposition of $\tilde{L}^{(n)}$ using $\star$ product and the relation (2.55). Since this procedure is unambiguous, it can be served to choose a class representative. For $n=2$ case, we get

$$\tilde{L}^{(2)}\left(\tilde{V}\right) \sim L^{(2)}\left(\tilde{V}\right) + \frac{2N\lambda}{N+2}\,\tilde{v}_+\cdot\tilde{v}_-\;L^{(1)}\left(\tilde{V}\right) + \frac{N\lambda^2+1}{2(N+1)}(\,\tilde{v}_+\cdot\tilde{v}_-)^2 = [\![\,\tilde{L}^{(2)}\left(\tilde{V}\right)\,]\!]\,. \tag{2.61}$$

In general, the class representative of $\tilde{L}^{(n)}$ has the following form of series:

$$[\![\,\tilde{L}^{(n)}\left(\tilde{V}\right)\,]\!] = \sum_{m=0}^{n} s^{(n)}_m\,\frac{\langle\tilde{V}\rangle^m}{m!}\,L^{(n-m)}\left(\tilde{V}\right)\,, \tag{2.62}$$

where coefficients $s^{(n)}_m$ are fixed ones, in principle calculable, but their explicit expressions are not necessary for our purpose. Let us comment that in the above series only the structure $\langle\tilde{V}\rangle^m$ can appear as the coefficient of $L^{(n-m)}$ since it is the unique m-th order scalar in $\tilde{V}$ due to the property (2.54).

***$hs(\mathfrak{so}_N)$*.** Similarly to the $\mathfrak{sl}_N$ case, we first consider the $\mathfrak{sp}_2$ center of $hs(\mathfrak{sp}_{2N})$, that is, the set of elements satisfying

$$\big[\,y_\alpha\cdot y_\beta\,,f(y)\,\big]_\star = \big(y_\alpha\cdot\partial_{y^\beta} + y_\beta\cdot\partial_{y^\alpha}\big)\,f(y) = 0\,. \tag{2.63}$$

The solution space is again endowed with the $\star$ product, and we refer to this algebra as $\widetilde{hs}(\mathfrak{so}_N)$. It is generated by

$$\tilde{M}_{a_1b_1\,\cdots\,a_nb_n} = 2^n\,y_{[-a_1}\,y_{+]b_1}\,\cdots\,y_{[-a_n}\,y_{+]b_n}\,, \tag{2.64}$$

and we identify its traceless part with the generators $M_{a_1b_1\,\cdots\,a_nb_n}$ of $hs(\mathfrak{so}_N)$. Again generalizing the definitions (2.39) and (2.42) to $\tilde{M}_{a_1b_1\,\cdots\,a_nb_n}$, we get

$$\tilde{M}(\tilde{W}) = \exp\Big(\,y_{-a}\,\tilde{W}^{ab}\,y_{+b}\Big)\,, \tag{2.65}$$

with the matrix-variable $\tilde{W}$ satisfying

$$\tilde{W}^{a[b}\,\tilde{W}^{cd]} = 0 \quad\Leftrightarrow\quad \tilde{W}^{ab} = \tilde{w}_+^{[a}\,\tilde{w}_-^{b]}\,. \tag{2.66}$$

The HS algebra, $hs(\mathfrak{so}_N)$, is the quotient of $\widetilde{hs}(\mathfrak{so}_N)$ by the relation,

$$K_{\alpha\beta} := y_\alpha\cdot y_\beta \sim 0\,, \tag{2.67}$$

which corresponds to taking the trivial representation of $\mathfrak{sp}_2$. The class representative is given, analogously to $hs_\lambda(\mathfrak{sl}_N)$, by a series,

$$[\![\,\tilde{M}^{(n)}\left(\tilde{W}\right)\,]\!] = \sum_{m=0}^{[n/2]} t^{(n)}_m\,\frac{\langle\tilde{W}^2\rangle^m}{m!}\,M^{(n-2m)}\left(\tilde{W}\right)\,. \tag{2.68}$$

Remark that the structure $\langle\tilde{W}^2\rangle^m$ in front of $M^{(n-2m)}$ is again the unique possibility due to the property (2.66).

JHEP05(2014)103

3 Trace and structure constants of HS algebras

In this section, we shall derive explicit form of the structure constants of the previously defined HS algebras associated with classical Lie algebras. Let us begin with recalling that the structure constant $C_{\boldsymbol{ab}}{}^{\boldsymbol{c}}$ of HS algebra $hs(\mathfrak{g})$ is defined by

$$T_{\boldsymbol{a}} \star T_{\boldsymbol{b}} = C_{\boldsymbol{ab}}{}^{\boldsymbol{c}}\, T_{\boldsymbol{c}}\,, \tag{3.1}$$

where $T_{\boldsymbol{a}}$ is one of the generators (2.43) of $hs(\mathfrak{g})$, and $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are the collective indices. A convenient way to address structure constant is by making use of *trace* of HS algebra, defined as the identity piece of given element — see e.g. [57] for more details:

$$\mathrm{Tr}\,[\, c_0 + c^{\boldsymbol{a}}\, T_{\boldsymbol{a}}] = c_0\,. \tag{3.2}$$

From the existence of the antiautomorphism $T^{(n)} \mapsto (-1)^n\, T^{(n)}$, one can show that the bilinear form,

$$B_{\boldsymbol{ab}} = \mathrm{Tr}\,[\, T_{\boldsymbol{a}} \star T_{\boldsymbol{b}}\,]\,, \tag{3.3}$$

is symmetric and invariant. The trilinear form is simply related to the structure constant and the bilinear form as

$$C_{\boldsymbol{abc}} = \mathrm{Tr}\,[\, T_{\boldsymbol{a}} \star T_{\boldsymbol{b}} \star T_{\boldsymbol{c}}\,] = C_{\boldsymbol{ab}}{}^{\boldsymbol{d}}\, B_{\boldsymbol{dc}}\,. \tag{3.4}$$

In the notation introduced in the previous section, the trace is given simply by

$$\mathrm{Tr}\,[\, T(A)\,] = T(0)\,, \tag{3.5}$$

while the bilinear and trilinear forms read

$$\begin{aligned} \mathcal{B}(A_1, A_2) &= \mathrm{Tr}\,[\, T(A_1) \star T(A_2)\,]\,, \\ \mathcal{C}(A_1, A_2, A_3) &= \mathrm{Tr}\,[\, T(A_1) \star T(A_2) \star T(A_3)\,]\,. \end{aligned} \tag{3.6}$$

In the following, for each of $hs(\mathfrak{sp}_{2N})$, $hs_\lambda(\mathfrak{sl}_N)$ and $hs(\mathfrak{so}_N)$, we shall work out the trace and the bi-/trilinear forms. For $hs(\mathfrak{so}_N)$ and $hs_\lambda(\mathfrak{sl}_2)$ the bilinear forms have been obtained respectively in [68] and [14]. Let us remark as well that the structure constants of $hs(\mathfrak{so}_5)$ and $hs_\lambda(\mathfrak{sl}_2)$ have been proposed respectively in [5] and [15–17].

Actually, in the case of $hs(\mathfrak{sp}_{2N})$, the algebraic structure is already explicit at the level of $\star$ product since there is no quotienting process to perform. However, we decide to treat the algebras $hs(\mathfrak{sp}_{2N})$, $hs_\lambda(\mathfrak{sl}_N)$ and $hs(\mathfrak{so}_N)$ in the equal footing, for the sake of remarking the similar algebraic properties they possess and making manifest the relations between them.

3.1 $hs(\mathfrak{sp}_{2N})$

Let us begin with $hs(\mathfrak{sp}_{2N})$, whose essential ingredients can be found e.g. in [69].

Trace. In this case, the trace defined as (3.5) is equivalent simply to

$$\mathrm{Tr}\,[f(y)] = f(0)\,, \tag{3.7}$$

for an element $f(y)$ of the algebra $hs(\mathfrak{sp}_{2N})$.

JHEP05(2014)103

Structure constant. Let us consider the bilinear and trilinear forms (3.6). For that, we need to first evaluate the product of generating functions (2.50) $N(U_1) \bigstar N(U_2)$ and $N(U_1) \bigstar N(U_2) \bigstar N(U_3)$. Since $\bigstar = \star$ for $hs(\mathfrak{sp}_{2N})$, we can rely on the composition property of the $\star$ product. For the Gaussian functions of type,

$$\mathcal{G}(S) = \frac{1}{\sqrt{\det\left(\frac{1+S}{2}\right)}} \exp\left[y_A \left(\frac{S-1}{S+1}\right)^{AB} y_B\right], \tag{3.8}$$

the $\star$ product admits a manifestly associative form:

$$\mathcal{G}(S_1) \star \mathcal{G}(S_2) = \mathcal{G}(S_1\, S_2)\,. \tag{3.9}$$

The connection between $\mathcal{G}(S)$ and $N(U)$ involves a Cayley transformation [69]:

$$\mathscr{C}(U) = \frac{2+U}{2-U}\,, \qquad \mathscr{C}^{-1}(S) = 2\,\frac{S-1}{S+1}\,, \tag{3.10}$$

and using the rule (3.9) and the trace formula (3.7), one gets the n-linear forms as

$$\frac{1}{\sqrt{G^{(n)}(U_1,\ldots,U_n)}} := \mathrm{Tr}\left[\,N(U_1) \star \cdots \star N(U_n)\,\right]\,, \tag{3.11}$$

where the function $G^{(n)}$ is given by

$$G^{(n)}(U) = \frac{\det_{2N}\left(\frac{1}{2}\prod_{k=1}^{n}\frac{2+U_k}{2-U_k} + \frac{1}{2}\right)}{\prod_{k=1}^{n}\det_{2N}\left(\frac{1}{2}\,\frac{2+U_k}{2-U_k}+\frac{1}{2}\right)} = \det_{2N}\left[\frac{1}{2}\prod_{k=1}^{n}(1+U_k)+\frac{1}{2}\right]. \tag{3.12}$$

Here, for the second equality we have used the fact that $U_k^2 = 0$. The $n = 2$ case can be obtained immediately using $U_1\, U_2\, U_1 = \langle U_1\, U_2\rangle\, U_1$ as

$$G^{(2)}(U_1,U_2) = 1 + \frac{1}{4}\,\langle\, U_1\, U_2\,\rangle\,. \tag{3.13}$$

The $n = 3$ case requires more calculations — see section 3.4 — and the result reads

$$G^{(3)}(U) = 1 + \frac{1}{4}\,\Lambda(U)\,, \tag{3.14}$$

where $\Lambda(U)$ is defined by

$$\Lambda(U) := \langle\, U_1\, U_2\,\rangle + \langle\, U_2\, U_3\,\rangle + \langle\, U_3\, U_1\,\rangle + \langle\, U_1\, U_2\, U_3\,\rangle\,. \tag{3.15}$$

Notice that, due to $U_i^{AB} = U_i^{BA}$ and $\Omega_{AB} = -\Omega_{BA}$, the following identity is satisfied:

$$\langle\, U_1\, U_2\, U_3\,\rangle + \langle\, U_3\, U_2\, U_1\,\rangle = 0\,. \tag{3.16}$$

Finally, we obtain the bilinear and trilinear forms as

$$\mathcal{B}(U) = \mathrm{Tr}[N(U_1) \bigstar N(U_2)] = \frac{1}{\sqrt{1+\frac{1}{4}\,\langle U_1\, U_2\rangle}}\,, \tag{3.17}$$

$$\mathcal{C}(U) = \mathrm{Tr}[N(U_1) \bigstar N(U_2) \bigstar N(U_3)] = \frac{1}{\sqrt{1+\frac{1}{4}\,\Lambda(U)}}\,. \tag{3.18}$$

Notice that these bilinear and trilinear forms are given by the same function $1/\sqrt{1+z}$, but with different arguments.

3.2 $hs_\lambda(\mathfrak{sl}_N)$

Now we move to the $hs_\lambda(\mathfrak{sl}_N)$ case, where we need to handle the $\mathfrak{gl}_1$ coset.

Trace. In order to conveniently deal with the coset structure, we extend the definition of the trace (3.5) to $hs(\mathfrak{gl}_N)$ supplementing it with the condition,

$$\mathrm{Tr}\,(K_\lambda \star a) = 0\,, \quad \forall a \in hs(\mathfrak{gl}_N)\,, \tag{3.19}$$

so that we get

$$\mathrm{Tr}(a_1 \bigstar \cdots \bigstar a_n) = \mathrm{Tr}(a_1 \star \cdots \star a_n)\,, \qquad \forall a_i \in hs_\lambda(\mathfrak{sl}_N)\,. \tag{3.20}$$

Since $a_1 \star \cdots \star a_n$ belongs to $hs(\mathfrak{gl}_N)$, we would like to have a trace formula for a generic element of $hs(\mathfrak{gl}_N)$. For that, we first consider the trace of the generating function $\tilde{L}\left(\tilde{V}\right)$ introduced in (2.53). Using (2.62) and (3.5), we get

$$\mathrm{Tr}\left[\tilde{L}\left(\tilde{V}\right)\right] = \mathrm{Tr}\left[\exp\left(y_-\cdot\tilde{V}\cdot y_+\right)\right] = s\left(\langle\tilde{V}\rangle\right)\,, \qquad s(z) = \sum_{n=0}^{\infty} s^{(n)}_n\,\frac{z^n}{n!}\,. \tag{3.21}$$

Hence, the trace of $\tilde{L}(\tilde{V})$ is encoded in the function $s(z)$, which requires the coefficients $s^{(n)}_n$. They can be obtained by taking the maximal trace of (2.62) as

$$s^{(n)}_n = \frac{n!}{(N)_n}\,[\![\,(y_+\cdot y_-)^n\,]\!]\,, \tag{3.22}$$

where $(N)_n = N(N+1)\cdots(N+n-1)$ is the Pochhammer symbol. The sequence $\sigma_n := [\![\,(y_+\cdot y_-)^n]\!]$ can be obtained from the recurrence relation,

$$\left[\!\!\left[\left(y_+\cdot y_- - \frac{1}{2}N\,\lambda\right)\star(y_+\cdot y_-)^n\right]\!\!\right] = \sigma_{n+1} - \frac{N}{2}\,\lambda\,\sigma_n - \frac{n\,(N+n-1)}{4}\,\sigma_{n-1} = 0\,. \tag{3.23}$$

Packing the σ_n as $\sigma(z) = \sum_{n=0}^\infty \sigma_n\, z^n/n!$, the relation (3.23) becomes a differential equation:

$$\left[\left(1+\frac{z}{2}\right)\left(1-\frac{z}{2}\right)\partial_z - \frac{N}{2}\left(\frac{z}{2}+\lambda\right)\right]\sigma(z) = 0\,, \tag{3.24}$$

whose solution can be easily obtained as

$$\sigma(z) = \left(1-\frac{z}{2}\right)^{-P}\left(1+\frac{z}{2}\right)^{-Q}\,, \tag{3.25}$$

with

$$P := N\,\frac{1+\lambda}{2}\,, \qquad Q := N\,\frac{1-\lambda}{2}\,. \tag{3.26}$$

However, we need $s(z)$ rather than $\sigma(z)$, and the former can be obtained from the latter as

$$s(z) = (N-1)\int_0^1 dw\,(1-w)^{N-2}\,\sigma(w\,z)\,. \tag{3.27}$$

JHEP05(2014)103

Rewriting $\sigma(z)$ in an integral form,

$$\sigma(z) = \frac{\Gamma(P+Q)}{\Gamma(P)\,\Gamma(Q)} \int_0^1 dx\, \frac{x^{P-1}\,(1-x)^{Q-1}}{\left[1+\frac{z}{2}\,(1-2\,x)\right]^{P+Q}}\,, \tag{3.28}$$

and evaluating the w-integral first in (3.27) with (3.28), we get

$$s(z) = \frac{\Gamma(N)}{\Gamma(P)\,\Gamma(Q)} \int_0^1 dx\, \frac{x^{P-1}\,(1-x)^{Q-1}}{1+(1-2x)\,\frac{z}{2}}\,, \tag{3.29}$$

where we used $P+Q=N$.

After obtaining the trace of the generating function $\tilde{L}\left(\tilde{V}\right)$, we can also compute the trace of any Gaussian element of $hs(\mathfrak{gl}_N)$, that is, $\exp(y_+\cdot B\cdot y_-)$ with an arbitrary matrix B. Using the identities,

$$\exp(y_-\cdot B\cdot y_+) = g(\partial_{\tilde{v}_+}\cdot B\cdot\partial_{\tilde{v}_-})\,\exp\left(y_-\cdot\tilde{V}\cdot y_+\right)\,\Big|_{\tilde{v}=0}\,, \tag{3.30}$$

$$g(\partial_{\tilde{v}_+}\cdot B\cdot\partial_{\tilde{v}_-})\,\frac{1}{1-c\,\tilde{v}_-\cdot\tilde{v}_+}\Big|_{\tilde{v}=0} = \frac{1}{\det_N\,(1-c\,B)}\,, \tag{3.31}$$

with $g(z)=\sum_{n=0}^{\infty} z^n/(n!)^2$, we get

$$\mathrm{Tr}\big[\exp(y_-\cdot B\cdot y_+)\big] = \frac{\Gamma(N)}{\Gamma(P)\,\Gamma(Q)} \int_0^1 dx\, \frac{x^{P-1}\,(1-x)^{Q-1}}{\det_N\left[1+\frac{1}{2}(1-2x)\,B\right]}\,. \tag{3.32}$$

From this, and using $\star$ product formula for Gaussian functions, one can deduce the trace formula for a generic element $f(y)$ in $hs(\mathfrak{gl}_N)$ as

$$\mathrm{Tr}\big[\,f(y)\,\big] = (\Delta_\lambda\star f)(0)\,, \tag{3.33}$$

where Δ_λ is given by

$$\Delta_\lambda(y) = \frac{\Gamma(N)}{\Gamma\left(\frac{N(1+\lambda)}{2}\right)\,\Gamma\left(\frac{N(1-\lambda)}{2}\right)} \int_0^1 dx\; x^{\frac{N(1+\lambda)}{2}-1}\,(1-x)^{\frac{N(1-\lambda)}{2}-1}\,e^{2(1-2x)\,y_+\cdot y_-}\,. \tag{3.34}$$

Notice that Δ_λ is nothing but the deformed version of the $\mathfrak{gl}_1$ projector introduced in [8, 56].[6] Hence, retrospectively, the formula (3.33) is very natural extension of the $hs(\mathfrak{sp}_{2N})$ trace (3.7) to $hs_\lambda(\mathfrak{sl}_N)$.

Structure constant. Let us come back to the relation (3.20). Since the $\bigstar$ product can be replaced with the $\star$ product inside of the trace, for the n-linear forms, it is sufficient to compute $a_1\star\cdots\star a_n$ where a_i are generating functions of $hs_\lambda(\mathfrak{sl}_N)$ generators. The generating function admits again a simple Gaussian form due to (2.34):

$$L(V) = \exp\left(y_-\cdot V\cdot y_+\right). \tag{3.35}$$

Hence, we need to compute

$$\frac{1}{G^{(n)}(\rho,V_1,\ldots,V_n)} := e^{2\rho\,y_+\cdot y_-}\star L(V_1)\star\cdots\star L(V_n)\Big|_{y=0}\,. \tag{3.36}$$

[6]See also [70] for an attempt of a different formulation for the coset algebra.

Here we introduced a $\star$ product of $e^{2\rho\, y_+ \cdot y_-}$ and evaluated at the end with $y=0$, since this is exactly the necessary information to compute the trace using (3.33). For the evaluation of the $\star$ product in (3.36), we can simply use the rule of $hs(\mathfrak{sp}_{2N})$ with $U^{\alpha a\,\beta b} = \epsilon^{\alpha\beta}\, V^a{}_c\, \eta^{cb}$ since $\mathfrak{sl}_N$ is a subalgebra of $\mathfrak{sp}_{2N}$. In this way, we obtain

$$\begin{aligned} G^{(n)}(\rho, V) &= \frac{\det_N \left(\frac{1}{2}\,\frac{1+\rho}{1-\rho}\,\prod_{k=1}^{n}\frac{2+V_k}{2-V_k} + \frac{1}{2}\right)}{\det_N\left(\frac{1}{2}\,\frac{1+\rho}{1-\rho}+\frac{1}{2}\right)\prod_{k=1}^{n}\det_N\left(\frac{1}{2}\,\frac{2+V_k}{2-V_k}+\frac{1}{2}\right)} \\ &= \det_N\left[\frac{1+\rho}{2}\,\prod_{k=1}^{n}(1+V_k) + \frac{1-\rho}{2}\right], \end{aligned} \tag{3.37}$$

where we used the condition $V^2 = 0$ for the last equality. Again the evaluation of the determinant for $n=2$ is immediate and gives

$$G^{(2)}(\rho, V) = 1 + \left(\frac{1-\rho}{2}\right)\left(\frac{1+\rho}{2}\right)\langle V_1\, V_2\rangle\,. \tag{3.38}$$

Using the trace formula (3.33), (3.34), we end up with the following expression of the bilinear form:

$$\begin{aligned} \mathcal{B}(V) &= \frac{\Gamma(N)}{\Gamma(P)\,\Gamma(Q)}\int_0^1 dx\, \frac{x^{P-1}\,(1-x)^{Q-1}}{1+\,x\,(1-x)\,\langle V_1\,V_2\rangle} \\ &= {}_3F_2\left(\frac{N}{2}\,(1+\lambda)\,,\ \frac{N}{2}\,(1-\lambda)\,,\ 1\,;\ \frac{N}{2}\,,\ \frac{N+1}{2}\ ;\ -\frac{1}{4}\,\langle V_1\,V_2\rangle\right). \end{aligned} \tag{3.39}$$

In order to compute the trilinear form, we need to evaluate first $G^{(3)}(\rho, V)$. After some computations described in section 3.4, we get

$$G^{(3)}(\rho, V) = 1 + \left(\frac{1-\rho}{2}\right)\left(\frac{1+\rho}{2}\right)\left[\frac{1-\rho}{2}\,\Lambda_+(V) + \frac{1+\rho}{2}\,\Lambda_-(V)\right], \tag{3.40}$$

where $\Lambda_\pm(V)$ are defined by

$$\begin{aligned} \Lambda_+(V) &:= \langle V_1\,V_2\rangle + \langle V_2\,V_3\rangle + \langle V_3\,V_1\rangle + \langle V_1\,V_2\,V_3\rangle\,, \\ \Lambda_-(V) &:= \langle V_1\,V_2\rangle + \langle V_2\,V_3\rangle + \langle V_3\,V_1\rangle - \langle V_3\,V_2\,V_1\rangle\,. \end{aligned} \tag{3.41}$$

Again, using the trace formula (3.33), (3.34), we get the trilinear form as

$$\mathcal{C}(V) = \frac{\Gamma(N)}{\Gamma(P)\,\Gamma(Q)}\int_0^1 dx\, \frac{x^{P-1}\,(1-x)^{Q-1}}{1+\,x\,(1-x)\,\big[\,x\,\Lambda_+(V) + (1-x)\,\Lambda_-(V)\,\big]}\,. \tag{3.42}$$

The integral can be evaluated by expanding the denominator and one gets

$$\mathcal{C}(V) = \sum_{k=0}^{\infty}\sum_{\ell=0}^{k}(-1)^k\binom{k}{\ell}\frac{\left(\frac{N(1+\lambda)}{2}\right)_{2\,k-\ell}\left(\frac{N(1-\lambda)}{2}\right)_{k+\ell}}{(N)_{3k}}\,\big[\Lambda_+(V)\big]^{k-\ell}\,\big[\Lambda_-(V)\big]^{\ell}, \tag{3.43}$$

which is a double series in $\Lambda_+(V)$ and $\Lambda_-(V)$. One can see that all the results are symmetric in $\lambda \to -\lambda$ and this shows $hs_{-\lambda}(\mathfrak{sl}_N) \simeq hs_\lambda(\mathfrak{sl}_N)$. Incidentally, let us recall that in $5D$, $hs(\nu) = hs_{+\lambda}(\mathfrak{sl}_4) \oplus hs_{-\lambda}(\mathfrak{sl}_4)$ with (2.20).

3.3 $hs(\mathfrak{so}_N)$

Finally, we consider $hs(\mathfrak{so}_N)$, the most relevant case for physics, where we need to handle the $\mathfrak{sp}_2$ coset.

Trace. We begin again with the general definition (3.5) of the trace. As in the $hs_\lambda(\mathfrak{sl}_N)$ case, we first consider the trace of the generating function $\tilde{M}(\tilde{W})$ (2.65) of $\widetilde{hs}(\mathfrak{so}_N)$:

$$\mathrm{Tr}\left[\tilde{M}\left(\tilde{W}\right)\right] = \mathrm{Tr}\left[\exp\!\left(y_- \cdot \tilde{W} \cdot y_+\right)\right] = t\left(\langle \tilde{W}^2 \rangle\right), \qquad t(z) = \sum_{n=0}^{\infty} t_n^{(2n)} \frac{z^n}{n!}, \tag{3.44}$$

which is given by the function $t(z)$ that has the Taylor expansion coefficients $t_n^{(2n)}$ appearing in (2.68). By taking the maximal trace of (2.68), we get the relation,

$$\begin{aligned} &t_n^{(2n)} \left(\partial_{\tilde{w}_+} \cdot \partial_{\tilde{w}_{[-}}\, \partial_{\tilde{w}_{+]}} \cdot \partial_{\tilde{w}_-}\right)^n \frac{\left(\tilde{w}_+ \cdot \tilde{w}_{[-}\, \tilde{w}_{+]} \cdot \tilde{w}_-\right)^n}{n!} \\ &= \left[\!\!\left[\left(\partial_{\tilde{w}_+} \cdot \partial_{\tilde{w}_{[-}}\, \partial_{\tilde{w}_{+]}} \cdot \partial_{\tilde{w}_-}\right)^n \frac{\left(y_+ \cdot \tilde{w}_{[-}\, \tilde{w}_{+]} \cdot y_-\right)^{2n}}{(2n)!} \right]\!\!\right], \end{aligned} \tag{3.45}$$

whose simplification reads

$$t_n^{(2n)} = \frac{n!}{\left(\frac{N}{2}\right)_n \left(\frac{N-1}{2}\right)_n}\, \tau_n\,, \qquad \tau_n := \left[\!\!\left[\left(\frac{y_+ \cdot y_{[-}\, y_{+]} \cdot y_-}{4}\right)^n \right]\!\!\right]. \tag{3.46}$$

The sequence τ_n can be determined by setting up a recurrence relation using (2.67) as

$$\left[\!\!\left[\frac{1}{4}\, y_{[+} \cdot y_{[-} \star y_{+]} \cdot y_{-]} \left(\frac{y_+ \cdot y_{[-}\, y_{+]} \cdot y_-}{4}\right)^n \right]\!\!\right] = \tau_{n+1} - \frac{1}{8}\left(\frac{3}{2} + n\right)\left(\frac{N}{2} + n\right)\tau_n = 0\,, \tag{3.47}$$

and finally we obtain the coefficients $t_n^{(2n)}$ as

$$t_n^{(2n)} = \frac{n!\,\left(\frac{3}{2}\right)_n}{8^n \left(\frac{N-1}{2}\right)_n}\,. \tag{3.48}$$

Coming back to the function $t(z)$ in (3.44), we get a hypergeometric function which can be represented by the following integral:

$$t(z) = {}_2F_1\left(1\,, \frac{3}{2}\,; \frac{N-1}{2}\,; \frac{z}{8}\right) = \frac{\Gamma\!\left(\frac{N-1}{2}\right)}{\Gamma\!\left(\frac{3}{2}\right)\Gamma\!\left(\frac{N-4}{2}\right)} \int_0^1 dx\, \frac{x^{\frac{1}{2}}\,(1-x)^{\frac{N-6}{2}}}{1 - x\,\frac{z}{8}}\,. \tag{3.49}$$

Now let us move to the trace of a Gaussian element $\exp(y_- \cdot C \cdot y_+)$ of $\widetilde{hs}(\mathfrak{so}_N)$ given by an arbitrary antisymmetric matrix C. Using the identities,

$$\exp(\,y_- \cdot C \cdot y_+) = h\left(\partial_{\tilde{w}_+} \cdot C \cdot \partial_{\tilde{w}_-}\right)\, \exp\!\left(\,y_+ \cdot \tilde{w}_{[-}\, \tilde{w}_{+]} \cdot y_-\right)\Big|_{u=0}, \tag{3.50}$$

$$h\left(\partial_{\tilde{w}_+} \cdot C \cdot \partial_{\tilde{w}_-}\right) \frac{1}{1 - \frac{c^2}{2}\, \tilde{w}_+ \cdot \tilde{w}_{[-}\, \tilde{w}_{+]} \cdot \tilde{w}_-}\Bigg|_{\tilde{w}=0} = \frac{1}{\det_N\left(1 - c\, C\right)}\,, \tag{3.51}$$

with $h(z)=\sum_{n=0}^{\infty}(2\,z)^n/[(n+1)!\,n!]$, we obtain its trace as

$$\mathrm{Tr}\Big[\exp\big(\,y_-\cdot C\cdot y_+\big)\Big]=\frac{\Gamma\big(\frac{N-1}{2}\big)}{\Gamma\big(\frac{3}{2}\big)\,\Gamma\big(\frac{N-4}{2}\big)}\int_0^1 dx\,\frac{x^{\frac{1}{2}}\,(1-x)^{\frac{N-6}{2}}}{\det_N\Big(1-\frac{\sqrt{x}}{2}\,C\Big)}\,. \tag{3.52}$$

This formula can be also recast into more intuitive form as

$$\mathrm{Tr}\big[\,f(y)\,\big]=(\Delta\star f)(0)\,, \tag{3.53}$$

where Δ corresponds this time to the $\mathfrak{sp}_2$ projector,

$$\Delta(y)=\frac{\Gamma\big(\frac{N-1}{2}\big)}{\Gamma\big(\frac{3}{2}\big)\,\Gamma\big(\frac{N-4}{2}\big)}\int_0^1 dx\,x^{\frac{1}{2}}\,(1-x)^{\frac{N-6}{2}}\,e^{-2\sqrt{x}\,y_+\cdot\,y_-}\,. \tag{3.54}$$

Notice however the above expression of the projector Δ differs from the original one given in [57]. The latter, denoted here by $\tilde{\Delta}$, has the form,

$$\tilde{\Delta}(y)=\frac{\Gamma\left(\frac{N-1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{N-2}{2}\right)}\int_{-1}^1 ds\,(1-s^2)^{\frac{N-4}{2}}\,\cosh\left(s\,\sqrt{2\,y_\alpha\cdot y_\beta\,y^\alpha\cdot y^\beta}\right). \tag{3.55}$$

A noticeable difference is that the expression (3.54) does not have a $\mathfrak{sp}_2$-invariant form involving only $y_+\cdot y_-$ — not $y_\alpha\cdot y_\beta\;y^\alpha\cdot y^\beta$. However, as shown in appendix A, the two expressions are equivalent in the sense of

$$(\Delta\star f)(0)=\left(\tilde{\Delta}\star f\right)(0)\qquad \forall f\in\widetilde{hs}(\mathfrak{so}_N)\,. \tag{3.56}$$

In the following, we shall use the expression (3.54) as it leads to simpler computations.

Structure constant. In order to obtain the bilinear and trilinear forms, we need to compute the $\star$ product of the generating function of HS generators, which admit again a simple form:

$$M(W)=\exp\left(y_-\cdot W\cdot y_+\right), \tag{3.57}$$

as a function of the minimal orbit element W (2.33). For the use of the trace formula (3.53), (3.54), the Gaussian factor $e^{2\rho\,y_+\cdot\,y_-}$ should be again inserted in the computation, hence we consider

$$\frac{1}{G^{(n)}(\rho,W_1,\dots,W_n)}:=e^{2\rho\,y_+\cdot\,y_-}\star M(W_1)\star\cdots\star M(W_n)\Big|_{y=0}. \tag{3.58}$$

Since $\mathfrak{so}_N$ is a subalgebra of $\mathfrak{sl}_N$, we can use the formula (3.37) with the simple replacement $V^{ab}=W^{ab}$ ending up with

$$G^{(n)}(\rho,W)=\det_N\left[\frac{1+\rho}{2}\,\prod_{k=1}^n(1+W_k)+\frac{1-\rho}{2}\right]. \tag{3.59}$$

Again, the $n=2$ case can be evaluated simply as

$$G^{(2)}(\rho,W)=\left[1+\frac{1-\rho^2}{8}\,\langle W_1\,W_2\rangle\right]^2, \tag{3.60}$$

and gives the following expression for the bilinear form:

$$\begin{aligned}\mathcal{B}(W) &= \frac{\Gamma\big(\frac{N-1}{2}\big)}{\Gamma\big(\frac{3}{2}\big)\,\Gamma\big(\frac{N-4}{2}\big)} \int_0^1 dx\, \frac{(1-x)^{\frac{1}{2}}\, x^{\frac{N-6}{2}}}{\big(1+\frac{x}{8}\,\langle W_1\, W_2\rangle\big)^2} \\ &= {}_2F_1\left(2\,,\, \frac{N-4}{2}\,;\, \frac{N-1}{2}\,;\, -\frac{1}{8}\,\langle W_1\, W_2\rangle\right), \end{aligned} \tag{3.61}$$

which has been obtained in [68] in a different notation. The $n = 3$ case requires more involved computations — see section 3.4 — and we get in the end

$$G^{(3)}(\rho, W) = \left[1 + \frac{1-\rho^2}{8}\,\Lambda(W)\right]^2 - \frac{\rho^2\,(1-\rho^2)^2}{32}\,\Sigma(W)\,, \tag{3.62}$$

where $\Lambda(W)$ and $\Sigma(W)$ are given by

$$\Lambda(W) := \langle W_1\, W_2\rangle + \langle W_2\, W_3\rangle + \langle W_3\, W_1\rangle + \langle W_1\, W_2\, W_3\rangle\,, \tag{3.63}$$

$$\Sigma(W) := \big\langle (W_1\, W_{(2}\, W_{3)})^2\big\rangle = \frac{1}{2}\,\langle W_1\, W_2\, W_3\rangle^2 + \frac{1}{4}\,\langle W_1\, W_2\rangle\langle W_2\, W_3\rangle\langle W_3\, W_1\rangle\,. \tag{3.64}$$

Finally, the trilinear form is given by

$$\mathcal{C}(W) = \frac{\Gamma\big(\frac{N-1}{2}\big)}{\Gamma\big(\frac{3}{2}\big)\,\Gamma\big(\frac{N-4}{2}\big)} \int_0^1 dx\, \frac{(1-x)^{\frac{1}{2}}\, x^{\frac{N-6}{2}}}{\big(1+\frac{x}{8}\,\Lambda(W)\big)^2 - \frac{(1-x)\,x^2}{32}\,\Sigma(W)}\,, \tag{3.65}$$

and the integral can be evaluated, by expanding the denominator, as

$$\mathcal{C}(W) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\big(\frac{N-4}{2}\big)_{m+2n}\,(2)_{m+2n}}{\big(\frac{N-1}{2}\big)_{m+3n}\;8^{m+3n}}\,\frac{[-\Lambda(W)]^m}{m!}\,\frac{[4\,\Sigma(W)]^n}{n!}\,. \tag{3.66}$$

With this, we have completed the computations of the bilinear and trilinear forms of the HS algebras associated with classical Lie algebras. In the next subsections, we provide an important element left out in the previous computations — the evaluation of determinant — and examine certain consistency conditions for our results.

3.4 Evaluation of determinant

In the previous sections 3.1, 3.2 and 3.3, we faced the evaluation of the determinant,

$$G^{(n)}(\rho, A) = \det\left[\frac{1+\rho}{2}\prod_{i=1}^{n}(1+A_i) + \frac{1-\rho}{2}\right], \tag{3.67}$$

for the computations of the n-linear forms. Here, the matrices A_i are in the minimal coadjoint orbit, so either U_i, V_i or W_i depending on whether we consider $\mathfrak{sp}_{2N}$, $\mathfrak{sl}_N$ or $\mathfrak{so}_N$. Because the minimal orbit matrices A_i admit the parameterizations (2.34), they enjoy the following property:

$$\langle\, A_{i_1}\,\cdots\, A_{i_p}\rangle = \langle\, \bar{A}_{i_1}\cdots \bar{A}_{i_p}\rangle\,, \tag{3.68}$$

where $(\bar{A}_i)_{jk} = \delta_{ij}\,A_{jk}$, and A_{ij} are given by

$$U_{ij} = \Omega_{AB}\,(u_i)^A\,(u_j)^B\,, \qquad V_{ij} = (v_i)_+\cdot(v_j)_-\,, \qquad (W_{ij})_{\alpha\beta} = \frac{1}{2}\,(w_i)_\alpha\cdot(w_j)_\beta\,. \tag{3.69}$$

Hence, the $2N \times 2N$ or $N \times N$ matrices A_i can be replaced by the $n \times n$ ones $\bar{A}_i$. Notice that only for the $\mathfrak{so}_N$ case, the components A_{ij} are again 2×2 matrices. After some computations, one can show that this determinant can be recast into

$$G^{(n)}(\rho, A) = \det\left[1 + \frac{1-\rho}{2}\,\mathsf{Up}(A) + \frac{1+\rho}{2}\,\mathsf{Lo}(A)\right], \tag{3.70}$$

in terms of upper and lower triangular matrices $\mathsf{Up}(A)$ and $\mathsf{Lo}(A)$ with components,

$$\big[\mathsf{Up}(A)\big]_{ij} = \delta_{i<j}\,A_{ij}\,, \qquad \big[\mathsf{Lo}(A)\big]_{ij} = \delta_{i>j}\,A_{ij}\,. \tag{3.71}$$

This expression makes simple the evaluation of the determinant. Focusing on the $n=3$ case, we get

$$\begin{aligned} G^{(3)}(\rho, A) = \det\bigg[1 + \left(\frac{1-\rho}{2}\right)\left(\frac{1+\rho}{2}\right)\Big\{&A_{12}\,A_{21} + A_{23}\,A_{32} + A_{31}\,A_{13} \\ &+ \frac{1-\rho}{2}\,A_{12}\,A_{23}\,A_{31} - \frac{1+\rho}{2}\,A_{13}\,A_{32}\,A_{21}\Big\}\bigg]. \end{aligned} \tag{3.72}$$

For $\mathfrak{sp}_{2N}$ and $\mathfrak{sl}_N$, this is the end of the computation, and using

$$A_{ij}\,A_{ji} = \langle A_i\,A_j\rangle\,, \qquad A_{ij}\,A_{jk}\,A_{ki} = \langle A_i\,A_j\,A_k\rangle\,, \tag{3.73}$$

we obtain the results (3.14) and (3.40). For $\mathfrak{so}_N$, one still needs to evaluate the determinant of 2×2 matrix, and a straightforward computation gives (3.62).

3.5 Isomorphisms between HS algebras

Let us conclude this section by examining the HS algebras associated with isomorphic classical Lie algebras. These can be considered as consistency checks for our result.

$\mathfrak{sl}_2 \simeq \mathfrak{sp}_2$ case. To begin with, we consider the case $\mathfrak{sl}_2 \simeq \mathfrak{sp}_2$. The bilinear and trilinear forms of $hs(\mathfrak{sp}_2)$ are both given by the function $1/\sqrt{1+z}$ but with different arguments — see (3.18). In the case of $\mathfrak{sl}_N$, they are given in general by different functions, but for $N=2$ due to the identity,

$$\langle V_1\,V_2\,V_3\rangle = -\langle V_3\,V_2\,V_1\rangle\,, \tag{3.74}$$

they do admit expressions through the same function. In particular, this function coincides to that of $\mathfrak{sp}_{2N}$ when the deformation parameter is $\lambda = 1/2$:

$${}_3F_2\left(\frac{N}{2}\,(1+\lambda)\,,\,\frac{N}{2}\,(1-\lambda)\,,\,1\,;\,\frac{N}{2}\,,\,\frac{N+1}{2}\,;\,-z\right) = \frac{1}{\sqrt{1+z}} \qquad \left[N\!=\!2, \lambda\!=\!\frac{1}{2}\right]. \tag{3.75}$$

Hence, this shows the isomorphism $hs_{\frac{1}{2}}(\mathfrak{sl}_2) \simeq hs(\mathfrak{sp}_2)$.

$\mathfrak{so}_5 \simeq \mathfrak{sp}_4$ case. Let us move to the cases involving $\mathfrak{so}_N$. For that, let us first note that $\Sigma(W)$ in (3.64) can be also written as

$$\Sigma(W) = -45\,(W_1)^{a_1}{}_{[a_1}\,(W_2)^{a_2}{}_{a_2}\,(W_3)^{a_3}{}_{a_3}\,(W_1)^{a_4}{}_{a_4}\,(W_2)^{a_5}{}_{a_5}\,(W_3)^{a_6}{}_{a_6]}\,, \tag{3.76}$$

therefore vanishes for N smaller than 6. Consequently, in such cases the bilinear and trilinear forms are both given by the same function. In particular, for $N = 5$, this function coincides with that of $\mathfrak{sp}_4$:

$${}_2F_1\left(2\,,\,\frac{N-4}{2}\,;\,\frac{N-1}{2}\,;\,-z\right) = \frac{1}{\sqrt{1+z}} \qquad \big[N=5\big]\,. \tag{3.77}$$

This demonstrates the isomorphism $hs(\mathfrak{so}_5) \simeq hs(\mathfrak{sp}_4)$.

$\mathfrak{so}_6 \simeq \mathfrak{sl}_4$ case. The bilinear form and the trilinear form of $hs(\mathfrak{so}_6)$ are given by different functions, and both of them have to coincide with those of $hs(\mathfrak{sl}_4)$. In order to check these, we need first to establish the link $\mathfrak{so}_6 \simeq \mathfrak{sl}_4$ using the chiral spinor representation Σ_{ab} as

$$M_{ab} = -L^{\alpha}{}_{\beta}\,(\Sigma_{ab})^{\beta}{}_{\alpha}\,, \qquad V_{\alpha}{}^{\beta} = -\frac{1}{2}\,W_{ab}\left(\Sigma^{ab}\right)^{\beta}{}_{\alpha}\,. \tag{3.78}$$

From this, we get the relation between the arguments of the bilinear forms (3.61), (3.39):

$$\langle\, W_1\, W_2\,\rangle = 2\,\langle\, V_1\, V_2\,\rangle\,, \tag{3.79}$$

and the function appearing in the bilinear form of $hs(\mathfrak{so}_6)$ coincides with that of $hs_\lambda(\mathfrak{sl}_4)$ when $\lambda = 0$:

$${}_2F_1\left(2\,,\,\frac{N-4}{2}\,;\,\frac{N-1}{2}\,;\,-z\right) = {}_3F_2\left(\frac{N'}{2}\,(1+\lambda)\,,\,\frac{N'}{2}\,(1-\lambda)\,,\,1\,;\,\frac{N'}{2}\,,\,\frac{N'+1}{2}\,;\,-z\right)$$
$$\big[N=6,\; N'=4,\; \lambda=0\big]\,. \tag{3.80}$$

For the trilinear forms, we have the following relations between the arguments of the trilinear forms (3.65) and (3.42):

$$\Lambda(W) = \Lambda_+(V) + \Lambda_-(V)\,, \qquad \Sigma(W) = \frac{1}{2}\,\big(\Lambda_+(V) - \Lambda_-(V)\big)^2\,, \tag{3.81}$$

and the coincidence of the trilinear forms can be shown thanks to the identity:

$$\int_0^1 dx\,\frac{(1-x)^{\frac{1}{2}}}{(1+x\,z)^2 - (1-x)\,x^2\,\omega^2} = \int_0^1 dx\,\frac{4\,x\,(1-x)}{1+4\,x\,(1-x)\,\big[\,z+(2x-1)\,\omega\,\big]}\,, \tag{3.82}$$

which can be proven by expanding both integrands in z and ω and evaluating the integrals. This demonstrates the isomorphism $hs(\mathfrak{so}_6) \simeq hs_0(\mathfrak{sl}_4)$.

4 More on $hs_\lambda(\mathfrak{sl}_N)$

Differently from $hs(\mathfrak{sp}_{2N})$ and $hs(\mathfrak{so}_N)$, the HS algebra $hs_\lambda(\mathfrak{sl}_N)$ has one-parameter family. This gives rise to an interesting consequence: for certain values of λ, the algebra develops an ideal, with a finite-dimensional algebra as the corresponding coset one. This is what happens to the 3D algebra $hs[\lambda] \simeq hs_\lambda(\mathfrak{sl}_2)$, and to the 5$D$ algebra $hs_\lambda(\mathfrak{sl}_4)$. In this section we look into these points more closely.

4.1 Ideals and finite-dimensional HS algebras

From the expression (3.39) of the bilinear form, one can notice that the hypergeometric function ${}_3F_2$ becomes a polynomial when $N(1\pm\lambda)/2$ takes negative integer values. In such a case,

$$N(1\pm\lambda) = -2\,M \qquad [\,M\in\mathbb{N}\,]\,, \tag{4.1}$$

the invariant bilinear form becomes degenerate for the generators $L^{(n)}$ with $n>M$ implying that they form an ideal. This ideal itself can be considered as a HS algebra although it does not contain the generators corresponding to the fields of spin $s\le M+1$ — however, if needed, one can simply include the spin-two generators to this algebra with standard commutation relations analogous to (2.14).

On the other hand, one can also consider the coset of the original algebra by this ideal. The resulting algebra is then composed of a finite number of generators,

$$L^{a_1\cdots a_n}_{b_1\cdots b_n} \qquad [\,n=0,1,\dots,M\,]\,, \tag{4.2}$$

therefore the associated spins are bounded by $M+1$. All these generators can be packed into the traceful generators (2.52),

$$\tilde{L}^{a_1\cdots a_M}_{b_1\cdots b_M}\,, \tag{4.3}$$

in terms of which one can easily realize that this algebra is isomorphic to

$$\mathfrak{gl}_{\binom{N+M-1}{M}}\,, \tag{4.4}$$

where $\binom{N+M-1}{M}$ corresponds simply to the number of possible values that the symmetrized indices $(a_1,\dots,a_M)$ can take. For the $N=4$ case of $\mathfrak{sl}_4\simeq\mathfrak{so}_6$, the $5D$ finite-dimensional HS algebras have been obtained in [65] making use of the decomposition of $\mathfrak{sl}_{\binom{M+3}{M}}$ generators into traceless tensors of $\mathfrak{sl}_4$.

So far, we have only considered complex Lie algebras, but for these finite-dimensional algebras, it would be also interesting to find the corresponding real form induced by that of $\mathfrak{sl}_N$. We consider here only a particular case $\mathfrak{su}(N_1,N_2)$, a real form of $\mathfrak{sl}_{N_1+N_2}$, since it is the most relevant in physics: $\mathfrak{su}(1,1)\simeq\mathfrak{sl}(2,\mathbb{R})\simeq\mathfrak{so}(1,2)$ and $\mathfrak{su}(2,2)\simeq\mathfrak{so}(4,2)$. To deal with $\mathfrak{su}(N_1,N_2)$, we simply divide the indices into two groups $\hat{a},\hat{b}=1,\dots,N_1$ and $\check{a},\check{b}=1,\dots,N_2$. Then, the reality conditions of $\mathfrak{su}(N_1,N_2)$ read

$$\left(L^{\hat{a}}_{\hat{b}}\right)^\dagger = L^{\hat{b}}_{\hat{a}}\,,\qquad \left(L^{\check{a}}_{\check{b}}\right)^\dagger = L^{\check{b}}_{\check{a}}\,,\qquad \left(L^{\hat{a}}_{\check{b}}\right)^\dagger = -L^{\check{b}}_{\hat{a}}\,. \tag{4.5}$$

From the above, one can deduce

$$\left(\tilde{L}^{\hat{a}_1\cdots\hat{a}_k\check{a}_{k+1}\cdots\check{a}_M}_{\hat{b}_1\cdots\hat{b}_\ell\check{b}_{\ell+1}\cdots\check{b}_M}\right)^\dagger = (-1)^{k+\ell}\,\tilde{L}^{\hat{b}_1\cdots\hat{b}_\ell\check{b}_{\ell+1}\cdots\check{b}_M}_{\hat{a}_1\cdots\hat{a}_k\check{a}_{k+1}\cdots\check{a}_M}\,. \tag{4.6}$$

This reality condition can be also understood in terms of the oscillators as

$$\left(y_{\pm\hat{a}}\right)^\dagger = -y_{\mp\hat{a}}\,,\qquad \left(y_{\pm\check{a}}\right)^\dagger = y_{\mp\check{a}}\,. \tag{4.7}$$

Hence, the real HS algebra associated with $\mathfrak{su}(N_1,N_2)$ is

$$\mathfrak{u}\big(\mathcal{N}_{\rm even}\,,\,\mathcal{N}_{\rm odd}\,\big)\,,^{7} \tag{4.8}$$

where

$$\mathcal{N}_{\rm even/odd}=\sum_{0\le\,{\rm even/odd}\,k\,\le M}\binom{N_1+k-1}{k}\binom{N_2+M-k-1}{M-k}. \tag{4.9}$$

For $\mathfrak{su}(1,1)$ we get

$$\mathfrak{u}\left(\frac{M}{2}+1\,,\,\frac{M}{2}\right)\quad[\,\text{even }M\,]\,,\qquad \mathfrak{u}\left(\frac{M+1}{2}\,,\,\frac{M+1}{2}\right)\quad[\,\text{odd }M\,]\,, \tag{4.10}$$

and, for $\mathfrak{su}(2,2)$ we get

$$\begin{aligned}&\mathfrak{u}\left(\frac{(M+2)(M^2+4\,M+6)}{12}\,,\,\frac{M(M+2)(M+4)}{12}\right)\qquad[\,\text{even }M\,]\,,\\&\mathfrak{u}\left(\frac{(M+1)(M+2)(M+3)}{12}\,,\,\frac{(M+1)(M+2)(M+3)}{12}\right)\quad[\,\text{odd }M\,]\,.\end{aligned} \tag{4.11}$$

We arrive to these real forms when we repeat all the constructions of the present paper in the real vector space starting from $\mathfrak{su}(N_1,N_2)$. However, let us note that there is no reason to give preference to these real forms.

4.2 Reduced set of oscillators

In section 2.4, the algebras $hs_\lambda(\mathfrak{sl}_N)$ and $hs(\mathfrak{so}_N)$ are constructed as cosets making use of N sets of oscillators which are subject to certain equivalence relations. In fact, in the case of $hs_\lambda(\mathfrak{sl}_N)$, there exists yet another description where all the generators can be given by certain polynomials of $N-1$ sets of oscillators [39]. Since these oscillators are not subject to any condition, it is sufficient to know how $\mathfrak{sl}_N$ is represented by them: they are simply given by

$$L^N{}_j=y_{+j}\,,\qquad L^i{}_j=y_-{}^i\,y_{+j}-\frac{\lambda}{N}\,\delta^i_j\,,\qquad L^i{}_N=-\,y_-{}^i\left(y_-{}^j\,y_{+j}-\lambda\right), \tag{4.12}$$

with $i,j=1,\dots,N-1$. Then, the HS generators are given by all possible $\star$ polynomials of $\mathfrak{sl}_N$ generators in the above representation — which is again the minimal representation.

4.3 $N=2$ case: the $3D$ HS algebra

The $N=2$ case is of particular interest since $hs_\lambda(\mathfrak{sl}_2)$ coincides with the $3D$ HS algebra $hs[\lambda]$. The latter algebra, also known as $Aq(2;\nu)$ [14], has been investigated by many authors — see [71, 72] for recent works. In this subsection, we show how the known structures of $hs[\lambda]$ can be derived from the results obtained in this paper for $hs_\lambda(\mathfrak{sl}_N)$.

[7] The $\mathfrak{u}(1)$ part of $\mathfrak{u}(\mathcal{N}_{\rm even},\mathcal{N}_{\rm odd})$ corresponds to the center of the algebra. It might be interpreted as the spin-one generator.

Lone-star product. The associative product of $hs[\lambda]$, namely Lone-star product [16], has been derived in the $\mathcal{V}^s_n$ basis, which is proportional in our case to

$$L_{\underbrace{1\cdots 1}_{p}\,\underbrace{2\cdots 2}_{q}} \qquad [\,p+q=2(s-1)\,,\ p-q=n\,]\,, \tag{4.13}$$

with $L_{a_1\cdots a_{2n}} := L^{b_1\cdots b_n}_{a_1\cdots a_n}\,\epsilon_{b_1 a_{n+1}}\cdots\epsilon_{b_n a_{2n}}$. The precise expression for such product is quite lengthy and we refer to [15–17]. Instead, we show how this $\star$ product can be obtained in a relatively simple form from the bilinear and trilinear forms (3.39), (3.42). In the $N=2$ case, such forms are simplified using (3.74) into[8]

$$\mathcal{B}(V) = {}_2F_1\left(1+\lambda\,,\,1-\lambda\,;\,\frac{3}{2}\,;\,-\frac{1}{4}\,\langle V_1\,V_2\rangle\right), \tag{4.15}$$

$$\mathcal{C}(V) = {}_2F_1\left(1+\lambda\,,\,1-\lambda\,;\,\frac{3}{2}\,;\,-\frac{1}{4}\,\Lambda(V)\right). \tag{4.16}$$

Let us mention that the bilinear form (4.15) has been initially obtained in [14] making use of the deformed oscillators. From (4.15) and (4.16), we can derive the explicit expression of the $\star$ product for the generating elements:

$$L(V_1)\star L(V_2)=\sum_{n=0}^{\infty}{}_2F_1\left(n{+}1{+}\lambda,n{+}1{-}\lambda;n{+}\frac{3}{2};-\frac{1}{4}\,\langle V_1\,V_2\rangle\right)L^{(n)}(V_1{+}V_2{+}V_1\,V_2)\,. \tag{4.17}$$

From this, one can extract the contribution of each generators to the $\star$ product.

Deformed oscillators. Another convenient description of $hs[\lambda]$ is the deformed oscillators — see e.g. [14]. In the following, we provide a link of the description presented in section 2.4 to the deformed oscillator one. Let us first notice that the former description $hs_\lambda(\mathfrak{sl}_N)$ requires N sets of oscillators (y_{+a}, y_{-a}) — two sets for $N=2$. On the other hand, in the latter description through deformed oscillators, one needs only one pair of oscillators. Despite of this discrepancy, one can establish an explicit link between two descriptions by introducing a single set *matrix-valued* oscillators out of two sets of usual oscillators as

$$\hat{y}_a := 2\begin{pmatrix} 0 & y_{+a} \\ y_-{}^c\,\epsilon_{ca} & 0\end{pmatrix}. \tag{4.18}$$

Then, we can define the product between two such oscillators as the matrix product,

$$\hat{y}_a\,\hat{y}_b := 2\begin{pmatrix} 0 & y_{+a} \\ y_-{}^c\,\epsilon_{ca} & 0\end{pmatrix}\star 2\begin{pmatrix} 0 & y_{+b} \\ y_-{}^d\,\epsilon_{db} & 0\end{pmatrix}, \tag{4.19}$$

where the multiplications of matrix entities are with respect to the $\star$ product. The commutator of such product is readily calculated and gives

$$[\,\hat{y}_a\,,\,\hat{y}_b\,] = 2\,\epsilon_{ab}\left(\hat{1}+\hat{\nu}\,\hat{k}\right), \tag{4.20}$$

[8]Moreover, the hypergeometric function admits another simple expression,

$${}_2F_1\left(1+\lambda\,,\,1-\lambda\,;\,\frac{3}{2}\,;\,-z\right) = \frac{\sinh\left(2\,\lambda\,\sinh^{-1}(\sqrt{z})\right)}{2\,\lambda\,\sqrt{1+z}\,\sqrt{z}} = \frac{\left(\sqrt{1+z}+\sqrt{z}\right)^{2\lambda}-\left(\sqrt{1+z}-\sqrt{z}\right)^{2\lambda}}{4\,\lambda\,\sqrt{1+z}\,\sqrt{z}}\,. \tag{4.14}$$

where $\hat{k}$ and $\hat{\nu}$ are defined by

$$\hat{k} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \hat{\nu} := 2\,\lambda\,\hat{1} + \hat{k}\,, \tag{4.21}$$

so satisfy

$$\hat{k}^2 = 1\,, \qquad \left\{\hat{k}\,,\,\hat{y}_a\right\} = 0\,, \qquad \left[\hat{\nu}\,,\,\hat{y}_a\right] = 0\,, \qquad \left[\hat{\nu}\,,\,\hat{k}\right] = 0\,. \tag{4.22}$$

Notice that the equations (4.20) and (4.22) are the defining relations of the deformed oscillators, and one can see that $\hat{\nu}$ is a constant diagonal matrix:

$$\hat{\nu} = \begin{pmatrix} 2\,\lambda + 1 & 0 \\ 0 & 2\,\lambda - 1 \end{pmatrix}, \tag{4.23}$$

and can be treated as a constant number $\hat{\nu} = 2\,\lambda \pm 1$ in the $\pm$ eigenspace of $\hat{k}$.

5 Outlook

Finally, let us conclude the present paper with a few remarks:

- first of all, the results obtained here may be applied to the construction of n-point correlation functions of the Vasiliev theory, along the line of [73–75]: the necessary ingredients there are the trace formula and the boundary-to-bulk propagator in the unfolded formulation. The former is provided in this paper, while the latter has been investigated in [76].

- In this paper, we considered the HS algebras associated only with symmetric tensor fields. However in higher dimensions, there exist many other massless fields of the mixed-symmetry tensor type, and their understanding is important for the generalization of the currently understood version of HS theory to wider context. In this respect, it would be interesting to generalize our results to the cases of mixed-symmetry HS algebras. In particular, examining possibilities to interpret the works [57] and [25] within this picture would be tempting.

- Another direction of generalizing HS gauge theory and the corresponding global symmetry is the study of partially-massless HS fields. Actually these have been already explored by a number of authors: see [33–35] for the mathematics literature and [36] for the physics one. Again, it would be interesting to reformulate such results in the language presented in this paper, in particular by making use of a certain reductive dual pair correspondence.

We are currently investigating the latter two issues, and hope to report on them in the near future.

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A Two representations of the trace projector

In [57], the $\mathfrak{sp}_2$ projector has been determined in the form of (3.55). In order to see its equivalence to the other form (3.54), we begin with the following transformation:

$$\cosh\left(s\sqrt{2\,K_{\alpha\beta}\,K^{\alpha\beta}}\right)=T\left[e^{2\,\omega^2\,K_{\alpha\beta}\,K^{\alpha\beta}}\right](s)\,, \tag{A.1}$$

where the linear map T is defined by

$$T\left[\omega^{2n}\right](s)=\frac{n!}{(2n)!}\,s^{2n}\,. \tag{A.2}$$

Then, using the following identity:

$$e^{2\,\omega^2\,K_{\alpha\beta}\,K^{\alpha\beta}}=\int\frac{d^3\vec{z}}{(2\pi)^{3/2}}\,e^{-\frac{1}{2}\,\vec{z}^2}\cosh\left(\sqrt{2}\,\omega\,z^{\alpha\beta}\,K_{\alpha\beta}\right)\,, \tag{A.3}$$

with $z^{\pm\pm}=\pm z_1+i\,z_2$, and $z^{\pm\mp}=z_3$, we get cosh function with $K_{\alpha\beta}$-linear argument. It can be shown by a straightforward computation that

$$\left(\cosh\left(\sqrt{2}\,\omega\,z^{\alpha\beta}\,K_{\alpha\beta}\right)\star f\right)(0)=\left(\cosh\left(2\sqrt{2}\,\omega\,|\vec{z}\,|\,y_+\cdot y_-\right)\star f\right)(0)\,, \tag{A.4}$$

which allows us to replace the $\mathfrak{sp}_2$ element $K_{\alpha\beta}$ by $y_+\cdot y_-$ in (A.3). The next steps are the $z^{\alpha\beta}$-integral and the T-transformation: first, the $z^{\alpha\beta}$-integral gives

$$\int\frac{d^3\vec{z}}{(2\pi)^{3/2}}\,e^{-\frac{1}{2}\,\vec{z}^2}\,\cosh\left(2\sqrt{2}\,\omega\,|\vec{z}\,|\,y_+\cdot y_-\right)=\sum_{n=0}^{\infty}\frac{(2n+1)!}{n!}\,\omega^{2n}\,\frac{(2\,y_+\cdot y_-)^{2n}}{(2n)!}\,, \tag{A.5}$$

whose T-transformation reads

$$\sum_{n=0}^{\infty}(2n+1)\,s^{2n}\,\frac{\left(\sqrt{2}\,y_+\cdot y_-\right)^{2n}}{(2n)!}\,. \tag{A.6}$$

Finally, evaluating the s-integral (3.55), we get

$$(\tilde{\Delta}\star f)(0)=\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n}{\left(\frac{N-1}{2}\right)_n}\left(\frac{(2\,y_+\cdot y_-)^{2n}}{(2n)!}\star f\right)(0)\,. \tag{A.7}$$

On the other hand, it is straightforward to obtain the above expression starting from Δ (3.54): first we expand Δ in $y_+\cdot y_-$ (again only even powers contribute to the trace) and then evaluate the x-integral of (3.54).

JHEP05(2014)103

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JHEP05(2014)103