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Research Article

On Perimeters and Volumes of Fattened Sets

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In this paper we analyze the shape of fattened sets; given a compact set $C \subset \mathbb{R}^N$ let C_r be its r−fattened set; we prove a general bound $rP(C_r) \le N\mathscr{L}(\{C_r \setminus C\})$ between the perimeter of C_r and the Lebesgue measure of $C_r \setminus C$. We provide two proofs: one elementary and one based on Geometric Measure Theory. Note that, by the Flemin–Rishel coarea formula, $P(C_r)$ is integrable for $r \in (0, a)$. We further show that for any integrable continuous decreasing function $\psi : (0, 1/2) \longrightarrow (0, \infty)$ there exists a compact set $C \subset \mathbb{R}^N$ such that $P(C_r) \ge \psi(r)$. These results solve a conjecture left open in (Mennucci and Duci, 2015) and provide new insight in applications where the fattened set plays an important role.

1. Introduction

For any $A \subseteq \mathbb{R}^N$ closed, let u_A be the *distance function* from A

$$u_A(x) \overset{\text{def}}{=} \inf_{y\in A} |y - x| . \tag{1}$$

Let $\{u_A \le r\} = \{x \in \mathbb{R}^N : u_A(x) \le r\}$ be the *fattened set* of A of radious $r > 0$. It is equal to the Minkowski sum of A and a closed ball $D_r(0)$ of radius r with center in the origin. It is also called *the parallel set* or the *tubolar neighborhood*.

Let $\mathscr{L}$ be the N-dimensional *Lebesgue measure*. Let $P(E)$ be the *perimeter* of a Borel set $E \subseteq \mathbb{R}^N$.

1.1. Main Results. In this short essay we will prove some geometrical properties of the function u_A and of the fattened set.

The main result is as follows.

Theorem 1. *Let $C \subset R^N$ be a compact set. For all $r > 0$ we have*

$$rP(\{u_C \le r\}) \le N\mathscr{L}(\{0 < u_C \le r\}). \tag{2}$$

Note that (2) is sharp (consider the case $C = \{0\}$; see relations (16)).

We will provide two proofs of Theorem 1. An elementary proof is in Section 3; it is based on simple geometrical properties of sets in $\mathbb{R}^N$. Another proof is in Section 5, it is based on semiconcavity of u_C^2 and the Gauss–Green formula. It may be appreciated that the first method of proof is simpler.

A corollary of the above theorem is that, for any $r > 0$, the perimeter of the fattened set $\{u_C \le r\}$ is finite—even when the perimeter of C is not finite. We elaborate on this fact further.

Remark 2. Let $\psi(r) = P(\{u_C \le r\})$ for convenience; by the Fleming–Rishel coarea formula (see Proposition 14 and Lemma 24 here)

$$\mathscr{L}(\{0 < u_C \le r\}) = \int_0^r \psi(s)\, ds < \infty \tag{3}$$

(so the function ψ is locally in L^1); then thesis (2) can be rewritten as

$$\psi(r) \le \frac{N}{r} \int_0^r \psi(s)\, ds; \tag{4}$$

the above implies

$$\psi(r) = o\left(\frac{1}{r}\right) \tag{5}$$

for $r \longrightarrow 0$.

The above properties (5) and (3) are again "sharp," in this sense.

Theorem 3. *If* $\psi : (0, 1/2) \longrightarrow (0, \infty)$ *is a continuous decreasing function satisfying* $\lim_{r\longrightarrow 0} \psi(r) = \infty$ *and* $\int_0^{1/2} \psi(s) \mathrm{d}s < \infty$ *(that is requirement (3) in the above remark); then* $\psi(r) = o(1/r)$ *(as in (5)), and we can construct a compact set C such that* $P(\{u_C \le r\}) \ge \psi(r)$ *for* $r \in (0, 1/2)$.

This holds in any $\mathbb{R}^N$ (for $N \ge 1$). The proof is in Section 4.1.

The interest on these results was spurred by the use of u_A in [1]. We will discuss the connection to [1] in Section 6. The main theorem will be as follows.

Theorem 4. *Let* $p \in [1, \infty)$ *and* $\varphi : [0, \infty) \longrightarrow [0, \infty)$ *a Borel function such that*

$$\int_0^\infty t^{N-1} \varphi(t)^p \, \mathrm{d}t < \infty, \tag{6}$$

$$\exists \varepsilon > 0, \quad \operatorname*{ess\text{-}sup}_{0<t<\varepsilon} \varphi(t) < \infty; \tag{7}$$

then for any compact $C \subseteq \mathbb{R}^N$ *we have* $\varphi \circ u_C \in L^p(\mathbb{R}^N)$.

Request (7) is justified by the existence of compact sets with properties as described in Theorem 3.

1.2. Motivation

1.2.1. Banach-Like Distances. Let $\mathscr{M}$ be the family of all nonempty compact subsets of $\mathbb{R}^N$. In 2015 Duci and Mennucci [1] studied a family of distances on $\mathscr{M}$, defined by means of the distance function u_A. Some natural questions came from that study; one of them was eventually answered by Theorem 4. See Section 6 for more details.

The study of fattened set though is quite wide and interesting; we provide two examples.

1.2.2. Steiner Formulas. In 2004 Hug et al. [2] generalized the Steiner formulas as follows. Let $S_1 = \{u \in \mathbb{R}^N : |u| = 1\}$. Fix $C \subseteq \mathbb{R}^N$ closed, and let ∂C be the topological boundary.

Definition 5. Define the *normal bundle* $N(C) \subseteq C \times S^1$ as the set of pairs $(x, u) \in C \times S^1$ for which there exists a $t > 0$ such that x is the unique point of C at minimum distance from $x + tu$ (this definition is equivalent to the definition in Section 2.1 in [2], but it is simplified to avoid introducing further definitions and notations that are not needed in this paper); in that case, let $\delta(x, u)$ be the supremum of such t; let $\delta(x, u) = 0$ for all $(x, u) \in (\mathbb{R}^N \times S^1) \setminus N(C)$.

$N(C)$ is a Borel subset of $(\partial C) \times S^1$. By Lemma 2.3 in [2], it is countably $(N-1)$-rectifiable. There are examples of compact sets such that $\mathscr{H}^{N-1}(N(C)) = \infty$ (the examples in this paper will do).

Definition 6. A *reach measure* is a real function μ with domain the Borel subsets of $\mathbb{R}^N \times S_1$, such that

(i) $\mu(A) = 0$ for all Borel subsets of $(\mathbb{R}^N \times S_1) \setminus N(C)$

(ii) at the same time, for any fixed $a > 0$, $\mu(A)$ is a signed measure (of bounded variation) when evaluated on the family of Borel sets A contained in

$$\left\{(x, u) \in N(C) : |x| \le a,\ \delta(x, u) \ge \frac{1}{a}\right\}. \tag{8}$$

Theorem 7 (Theorem 2.1 in [2]). *For any closed set C there exist reach measures* $\mu_0, \mu_1 \ldots \mu_{N-1}$ *such that*

$$\int_{x\in B, u\in S_1} (\delta(x, u) \wedge r)^{N-j} \, \mathrm{d}\left|\mu_j\right|(x, u) < \infty \tag{9}$$

for any $r > 0$ *and* $B \subseteq \mathbb{R}^N$ *compact, and, for any* $f : \mathbb{R}^N \longrightarrow \mathbb{R}$ *bounded Borel,*

$$\int_{\mathbb{R}^N \setminus C} f(x) \, \mathrm{d}x = \sum_{i=0}^{N-1} (N-i) \cdot \omega_{N-i} \int_{N(C)} \int_0^{\delta(x,u)} t^{N-1-i} f(x + tu) \, \mathrm{d}t \mathrm{d}\mu_i(x, u) \tag{10}$$

Remark 8. A closed set C is *a set of positive reach* [3] when there exists a $\rho > 0$ such that for any $x \in \mathbb{R}^N$ with $u_C(x) < \rho$ there exists a unique $y \in C$ at minimum distance from x; the *reach* is the largest such ρ (possibly infinite). For example, smooth manifolds embedded in $\mathbb{R}^N$ have positive reach, as well as convex subsets. It is easily verified that $\rho = \inf\{\delta(x, u) : (x, u) \in N(C)\}$, where $N(C), \delta$ were defined in Definition 5. When the set C has positive reach then for small $r > 0$ formula (2) can be verified (through relation (3)) using results in [3]: see Theorem 5.5 in [3] that provides an explicit formula for $\mathscr{L}(\{u_C \le r\}) = p(r)$, where $p(r)$ is a polynomial in r of degree at most N.

In this respect it is worth noting this result by Fu et al [4]. For any $r > 0$ that is a regular value for the distance function u_C, the set $\overline{\mathbb{R}^N \setminus C_r}$ is a set of positive reach (see Corollary 3.4 in [4]). When $N = 2$ or $N = 3$ a Sard–type result shows that the set of critical values is *small*, in an appropriate sense (see Thm. 4.1 in [4]).

1.2.3. Minkowski Content. The study of the fattened set is linked to the Minkowski content

$$\lim_{r\longrightarrow 0} \frac{\mathscr{L}(\{u_A \le r\})}{r} \tag{11}$$

which has wide applications in the theory of Stochastic Differential Equations. See Ambrosio et al. [2, 5] and references therein.

2. Notation

We will write $B_r(x)$ for the *open ball*

$$B_r(x) \stackrel{\text{def}}{=} \left\{y \in \mathbb{R}^N : |x - y| < r\right\} \tag{12}$$

of center x and radius $r > 0$ in $\mathbb{R}^N$; we will write B_r for $B_r(0)$. $D_r(x)$ will be the *disk*

$$D_r(x) \overset{\text{def}}{=} \left\{ y \in \mathbb{R}^N : |x - y| \le r \right\} \tag{13}$$

of center x and radius $r > 0$ in $\mathbb{R}^N$ and $D_r = D_r(0)$; $S_r(x)$ will be the *sphere*

$$S_r(x) \overset{\text{def}}{=} \left\{ y \in \mathbb{R}^N : |x - y| = r \right\} \tag{14}$$

of center x and radius $r > 0$ in $\mathbb{R}^N$.

For $A \subseteq \mathbb{R}^N$ closed, let u_A be the distance function from A (defined in (1)). Let $\{u_A \le r\}$ be the *fattened set* of A of radious $r > 0$.

Let d_H be the *Hausdorff distance* of compact sets in $\mathbb{R}^N$; it can be defined (as shown in Sec. C in Chap. 4 in [6] and Sec. 2.2 in Chap. 4 in [7]) as

$$d_H(A, B) \overset{\text{def}}{=} \sup_{x \in \mathbb{R}^N} \left| u_A(x) - u_B(x) \right|. \tag{15}$$

For $d \ge 0$ let $\mathscr{H}^d$ be the d–dimensional *Hausdorff measure*; let $\mathscr{L}$ be the N–dimensional *Lebesgue measure*, and $\omega_N \overset{\text{def}}{=} \mathscr{L}(B_1)$. Let $P(E)$ be the *perimeter* of a Borel set $E \subseteq \mathbb{R}^N$, as defined in Definition 1.6 in [8]. We define $\omega_N = \mathscr{L}(B_1)$ and consequently for $r > 0$

$$\begin{aligned} \mathscr{L}(B_r) &= \omega_N r^N, \\ P(B_r) &= \mathscr{H}^{N-1}(S_r) = N\omega_N r^{N-1}, \\ \mathscr{L}(B_r) &= \omega_N r^N = \frac{r}{N}\mathscr{H}^{N-1}(S_r). \end{aligned} \tag{16}$$

3. Area and Perimeter of Fattened Sets

These facts are known; see Sec. 4 and 5 in [1].

Proposition 9. *Let $A \subset \mathbb{R}^N$ be a compact set.*

(1) *Let $r > 0$. Let $F = \{u_A \le r\}$ and $E = \{u_A = r\}$ for convenience.*

  (i) *The boundary ∂F of F is contained in the set E. (Equality may fail, consider $A = \{|x| \in [1, 2]\}, r = 1$.)*

  (ii) *E is Lebesgue negligible (hence ∂F is as well).*

  (iii) *As a consequence, for any $r > 0$,*

$$P(\{u_C \le r\}) = P(\{u_C < r\}). \tag{17}$$

(2) *For any fixed $A \in \mathscr{M}$, the fattening map $\lambda \longmapsto \{u_A \le \lambda\}$ is Lipschitz (of constant one) as a map from $[0, \infty)$ to $(\mathscr{M}, d_H)$.*

(3) *For any fixed $\lambda > 0$, the "fattened area map" $L_\lambda : \mathscr{M} \longrightarrow \mathbb{R}$ defined by $L_\lambda(A) \overset{\text{def}}{=} \mathscr{L}(\{u_A \le \lambda\})$ is continuous on $(\mathscr{M}, d_H)$.*

We add a further result in the same spirit.

Lemma 10. *If A_n, A are compact and $A_n \longrightarrow A$ according the Hausdorff distance and $\widetilde{A} = \{u_A \le \lambda\}$, $\widetilde{A}_n = \{u_{A_n} \le \lambda\}$ are the fattened sets, then $\mathscr{L}(\widetilde{A}_n \Delta \widetilde{A}) \longrightarrow 0$ where Δ is the symmetric difference of sets.*

Proof. Let again $E = \{x : u_A(x) = \lambda\}$; then for any $x \in \widetilde{A}_n \Delta \widetilde{A}$ we have that

(i) if $u_A(x) > \lambda$ then $u_{A_n}(x) > \lambda$ for n large (by (15)), so $x \notin \widetilde{A}_n \Delta \widetilde{A}$ for n large;

(ii) if $u_A(x) < \lambda$ then $u_{A_n}(x) < \lambda$ for n large, so $x \notin \widetilde{A}_n \Delta \widetilde{A}$ for n large;

(iii) if $u_A(x) = \lambda$ then $x \in E$, that is negligible (by Proposition 9).

The proof then follows from the Lebesgue dominated convergence theorem. □

We will also need this simple inequality.

Lemma 11. *For each $r, R \ge 0$ and $N \ge 1$ integer we have*

$$(r + R)^N - R^N \ge r(r + R)^{N-1}. \tag{18}$$

Proof. We have

$$\begin{aligned} (r + R)^N - R^N &= \sum_{j=0}^{N-1} \binom{N}{j} r^{N-j} R^j, \\ r(r + R)^{N-1} &= \sum_{j=0}^{N-1} \binom{N-1}{j} r^{N-j} R^j, \\ \frac{\binom{N}{j}}{\binom{N-1}{j}} &= \frac{N}{N-j} \ge 1 \quad \forall j = 0, \dots, N-1. \end{aligned} \tag{19}$$

□

We now come to the proof of the main contribution of this paper, Theorem 1.

Proof. The proof is in three steps.

(1) Let $z, w \in \mathbb{R}^N, r, \rho > 0$ and $D_r(z), D_\rho(w)$ two disjoint disks (as defined in (13)). The locus of points $x \in \mathbb{R}^n$ at the same distance from the two disks is (a connected sheet of) a quadric. Choose appropriate coordinates where $z = (-a, 0, \dots, 0)$, $w = (b, 0, \dots, 0)$, with $0 < r < a$, $0 < \rho < b < a$ and $a - r = b - \rho$ so that the locus contains the origin; then the locus can be written as

$$x_1 \ge 0 \wedge 4abx_1(x_1 + a - b) = (a - b)^2 \sum_{j=2}^{n} x_j^2. \tag{20}$$

See Figure 1

Suppose that $C = \bigcup_{i=1}^n D_{r_i}(z_i)$ is the union of finitely many disjoint disks. Let for simplicity

$$\delta_i(x) = u_{D_{r_i}(z_i)}(x) = \left(|x - z_i| - r_i\right)^+ \tag{21}$$

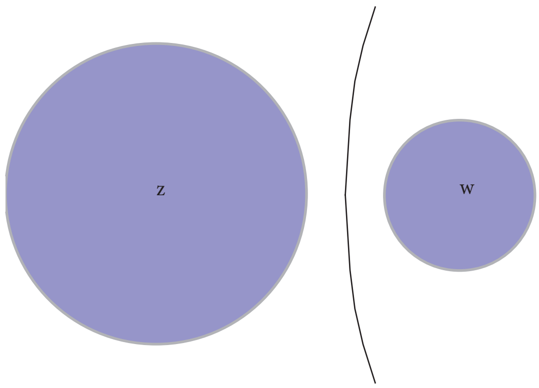

Figure 1

be the distance to one such disk. Note that

$$u_C(x) = \min_{i=1,\dots,n} \delta_i(x). \quad (22)$$

Fix $k \in 1, \dots n$ and then consider the region R_k of points x that are nearer to $D_{r_k}(z_k)$ than to any other disk; R_k is usually called a *Voronoi cell*. R_k is defined by the inequalities

$$R_k = \{x : \forall j \neq k,\ \delta_k(x) < \delta_j(x)\}. \quad (23)$$

These can be reduced to inequalities involving first and second-degree polynomials in $x = (x_1, \dots x_n)$. For example, the three inequalities

$$\begin{gathered} |x - z_j| > r_j, \\ |x - z_k| > r_k, \\ \delta_k(x) < \delta_j(x) \end{gathered} \quad (24)$$

when $r_k < r_j$ can be reduced (in appropriate coordinates as above, setting $z = z_k, w = z_j, \rho = r_k, r = r_j$) to

$$\begin{gathered} (x_1 - b)^2 + \sum_{j=2}^{n} x_j^2 > \rho^2 \wedge x_1 > 0 \\ \wedge\, 4abx_1(x_1 + a - b) > (a - b)^2 \sum_{j=2}^{n} x_j^2. \end{gathered} \quad (25)$$

So region R_k is an open semialgebraic set. Its boundary is a semialgebraic set. It is contained in the finite union of quadrics as described in (20).

The set $\{u_C = r\}$ is contained in the union of spheres $\bigcup_{i=1}^{n} S_{r+r_i}(z_i)$. A sphere can intersect a quadric such as (20) in a set of at most dimension $N - 2$. So when evaluating $\mathscr{H}^{N-1}(\{u_C = r\})$ we only consider the parts of $\{u_C = r\}$ that are inside the regions R_k.

Inside each region R_k the set $\{u_C = r\}$ is given by the equality $|x - z_i| = r + r_k$, so it is a part of a sphere. For each point $x \in \{u_C = r\} \cap R_k$, the projection of x to C is an unique point p contained in $S_{r_k}(z_k)$. The segment from x to z_k passes through p, and it is contained in R_k. This follows from well-known theory for distance functions, but in this case can also be easily checked with simple geometrical arguments. We denote by J the union of all segments xz_k for $x \in \{u_C = r\} \cap R_k$. See Figure 2.

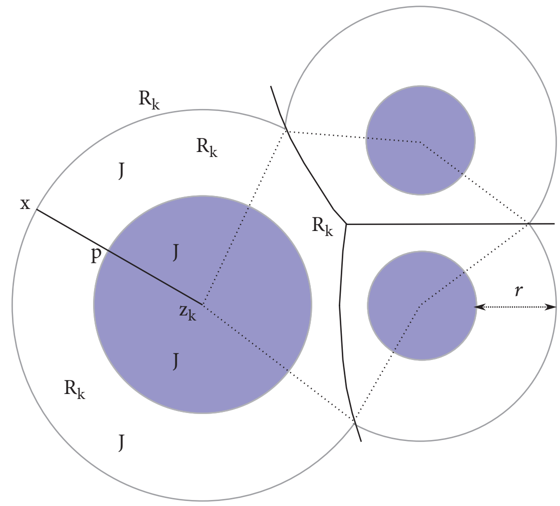

Figure 2: Voronoi cells, regions.

So we can establish the relations

$$\begin{gathered} \mathscr{L}(J \setminus C) = \frac{\alpha}{N}\left((r + r_k)^N - r_k^N\right), \\ \mathscr{H}^{N-1}(\{u_C = r\} \cap R_k) = (r + r_k)^{N-1}\alpha. \end{gathered} \quad (26)$$

($\alpha > 0$ is the solid angle under which z_k sees $\{u_C = r\} \cap R_k$). By Lemma 11 then

$$r\mathscr{H}^{N-1}(\{u_C = r\} \cap R_k) \leq N\mathscr{L}(J \setminus C). \quad (27)$$

Since

$$(J \setminus C) \subseteq (\{0 < u_C \leq r\} \cap R_k) \quad (28)$$

then *a fortiori*

$$r\mathscr{H}^{N-1}(\{u_C = r\} \cap R_k) \leq N\mathscr{L}(\{0 < u_C \leq r\} \cap R_k). \quad (29)$$

Summing in k we obtain the relation

$$r\mathscr{H}^{N-1}(\{u_C = r\}) \leq N\mathscr{L}(\{0 < u_C \leq r\}). \quad (30)$$

That is the same as (2) for the case when C is an union of spheres.

(2) Let C be a compact set. Define $A_n = \{u_C < 1/n\}$: this is a decreasing sequence of open sets A_n such that $A_n \supseteq C$, $\bigcap_{n=1}^{\infty} A_n = C$, $\mathscr{L}(A_n) \longrightarrow_n \mathscr{L}(A)$, and $d_H(A_n, C) \leq 1/n$.

Fix n; by Vitali's covering theorem we can choose finitely many disjoint disks inside A_n so that their union is C_n, and it satisfies $\mathscr{L}(A_n \setminus C_n) \leq 1/n$; we also have that $\mathrm{d}_H(A_n, C_n) \leq 1/n$.

Summarizing we obtain a sequence C_n such that each C_n is the union of finitely many disjoint disks, $d_H(C_n, C) \longrightarrow_n 0$ and $\mathscr{L}(C\Delta C_n) \longrightarrow_n 0$.

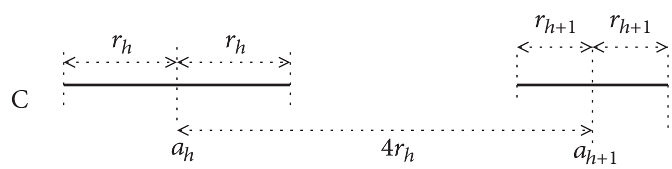

Figure 3

(3) Since $d_H(C_n, C) \longrightarrow 0$ then by Lemma 10

$$\mathscr{L}\left(\left\{u_{C_n} \le r\right\} \Delta \left\{u_C \le r\right\}\right) \longrightarrow 0. \tag{31}$$

Using moreover the fact that $\mathscr{L}(C\Delta C_n) \longrightarrow_n 0$ then

$$\mathscr{L}\left(\left\{0 < u_{C_n} \le r\right\} \Delta \left\{0 < u_C \le r\right\}\right) \longrightarrow 0. \tag{32}$$

Since $\mathscr{L}(C\Delta C_n) \longrightarrow_n 0$ then, by Thm. 1.9 in [8],

$$P(C) \le \liminf_n P(C_n). \tag{33}$$

So (2) follows.

□

4. Examples

We first present the simplest case, of a compact subset $C \subseteq \mathbb{R}$.

Example 1 (Let $N = 1$). For $h \in \mathbb{N}, h \ge 1$ let $r_h > 0$ be a monotone nonincreasing sequence, such that $\beta = \sum_{h=1}^{\infty} r_h < \infty$. Let $a_1 = 0$ and $a_h = 4\sum_{j=1}^{h-1} r_j$ for $h \ge 2$. Consider a compact set

$$C = \{a_\infty\} \cup \bigcup_{h=1}^{\infty} [a_h - r_h, a_h + r_h]. \tag{34}$$

C is composed by countably many disjoint closed intervals spaced as in Figure 3, and the limit point $\{a_\infty\}$ is added. Then

$$\mathscr{L}(C) = 2\beta < \infty, \qquad P(C) = \infty. \tag{35}$$

Let $r > 0, r < r_1$, define

$$\chi(r) \stackrel{\text{def}}{=} \max\{h \ge 1 : r < r_h\}; \tag{36}$$

note that $\chi(r) < \infty$ since $\lim_{h\longrightarrow\infty} r_h = 0$. We can estimate for $r < r_1$ that

$$P(\{u_C \le r\}) \ge 2\chi(r), \tag{37}$$

and indeed (setting $k = \chi(r)$ for convenience)

$$\{u_C \le r\} \cap (-\infty, a_k] = [a_k - r_k - r, a_k] \cup \bigcup_{h=1}^{k-1} [a_h - r_h - r, a_h + r_h + r], \tag{38}$$

where the latter part is a disjoint union.

The example can be built in higher dimensions as well. Let $N = 2$ for simplicity (the case $N \ge 3$ is similar, by repeating spheres along the extra dimensions and changing some constants).

Example 2. Let $(m_h)_{h\ge1}$ be a monotone nondecreasing sequence of integers, such that $m_h \ge 1$ and $\sum_{h=1}^{\infty} 1/m_h < \infty$. Let $r_h = 4/(4m_h - 2)$. (Note that $2 \ge r_h m_h \ge 1$ and $\sum_{h=1}^{\infty} r_h < \infty$). Let $a_1 = r_1$ and $a_h = a_1 + 4\sum_{j=1}^{h-1} r_j$ for $h \ge 2$. Consider a compact set C_h union of a family of m_h disks with centers in $x_{h,j} = (a_h, r_h(1+4j))$ for $j = 0, 1, \ldots, m_h - 1$, each of radiouses r_h, as follows:

$$C_h \stackrel{\text{def}}{=} \bigcup_{j=0}^{m_h-1} D\left(x_{h,j}, r_h\right). \tag{39}$$

It is easily proven that

$$\mathscr{L}(C_h) = \pi r_h^2 m_h \le 2\pi r_h, \qquad P(C_h) = 2\pi r_h m_h \ge 2\pi. \tag{40}$$

Let then the compact set be

$$C \stackrel{\text{def}}{=} \{(a_\infty, t) : t \in [0, 4]\} \cup \bigcup_{h\ge1} C_h \tag{41}$$

C is compact since we include in C the line that contains limit points of $\bigcup_{h\ge1} C_h$. This set C is tightly contained in the rectangle of corners $(0, 0)$ and $(a_\infty, 4)$. See Figures 5 and 6 for two examples.

For any $h < k$ any two points $x \in C_h$, $y \in C_k$ in different families are at a distance $|x - y| \ge 2r_h$; moreover two different disks composing C_h are at a distance at least $2r_h$ (see Figure 4); hence

$$\mathscr{L}(C) < \infty, \qquad P(C) = \infty. \tag{42}$$

Let $r > 0, r < r_1$, and define (as before)

$$\chi(r) \stackrel{\text{def}}{=} \max\{h \ge 1 : r < r_h\}; \tag{43}$$

note that $\chi(r) < \infty$ since $\lim_{h\longrightarrow\infty} r_h = 0$. Let $M_h = \sum_{j=1}^{h} m_j$. We can estimate for $r < r_1$ that

$$P(\{u_C \le r\}) \ge 2\pi \sum_{h=1}^{\chi(r)} (r + r_h) m_h \ge 2\pi\chi(r), \tag{44}$$

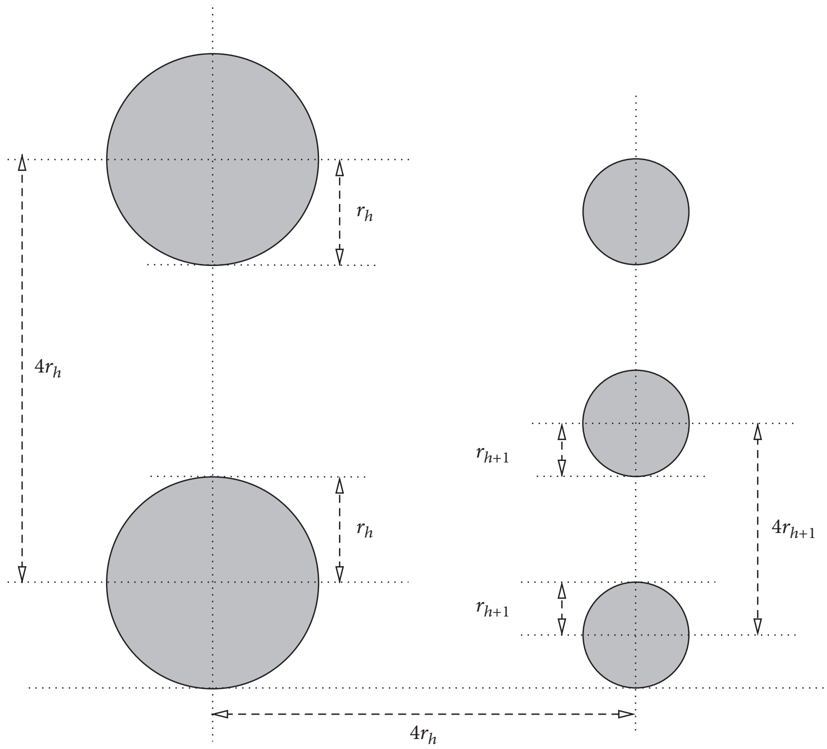

FIGURE 4: Distances between disks in C_h, C_{h+1}.

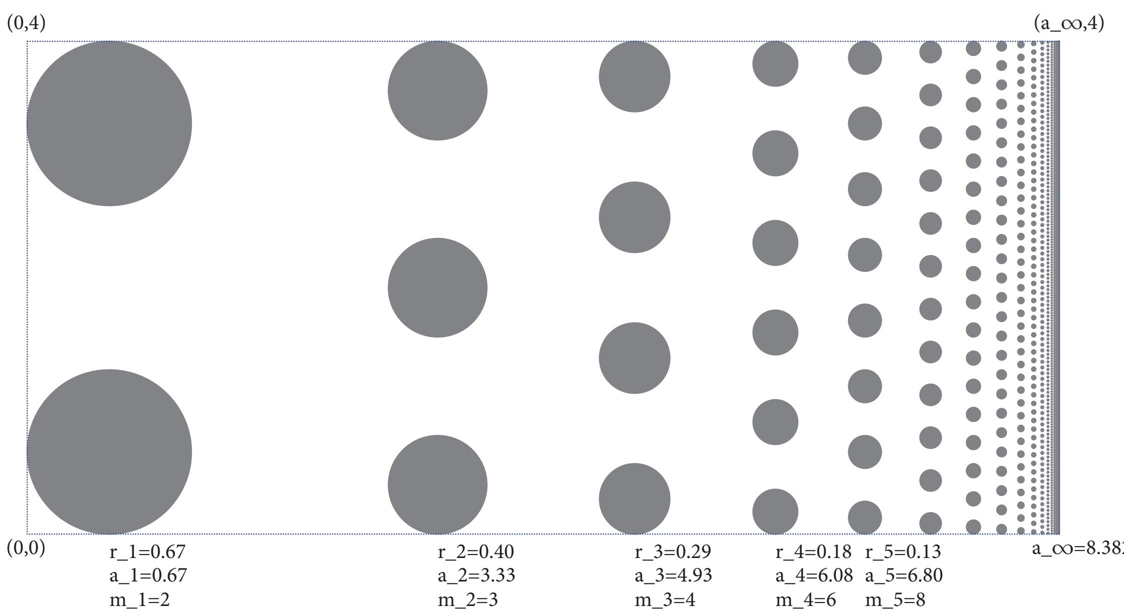

FIGURE 5: Compact set C (in gray) from Example 2 with $m_h = \lceil (3/2)^h \rceil$. (The blue, dotted rectangle has corners $(0,0)$ and $(a_\infty, 4)$ and is not part of C.)

and indeed for the first inequality we note that when $r < r_h$ and $k > h$, any two

$$x \in \{u_{C_k} \le r\},\qquad y \in \{u_{C_h} \le r\} \tag{45}$$

are at a distance $|x - y| \ge 2(r_h - r) > 0$. See Figure 4 again.

In general for $N \ge 1$ the above examples will satisfy

$$P(\{u_C \le r\}) \ge N\omega_N \chi(r). \tag{46}$$

We will need a lemma.

Lemma 12. *Suppose that $f : (0,1) \longrightarrow (0,\infty)$ is continuous and decreasing and $\int_0^1 f(t)\mathrm{d}t < \infty$ then $\lim_{r\longrightarrow 0} rf(r) = 0$.*

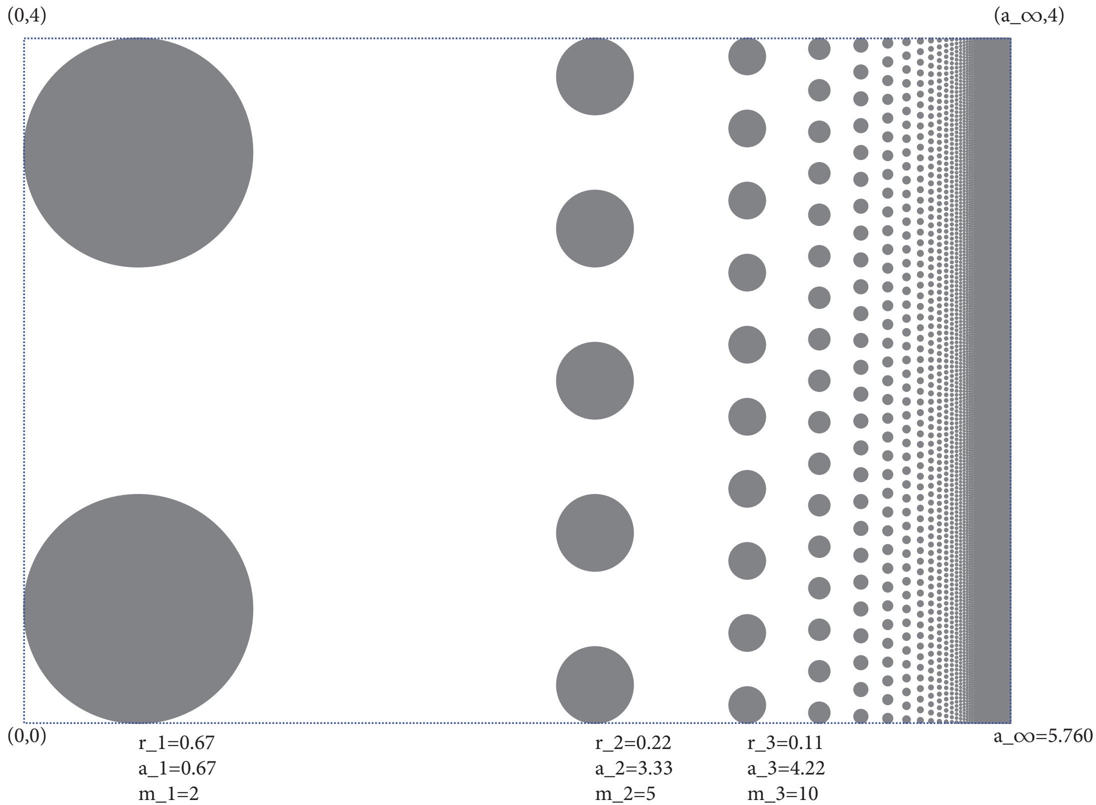

Figure 6: Compact set C (in gray) from Example 2 with $m_h = 1 + h^2$. (The blue, dotted rectangle has corners $(0, 0)$ and $(a_\infty, 4)$ and is not part of C.)

Equivalently (setting $f(t) = g(1/t)$), suppose that $g : (1, \infty) \longrightarrow (0, \infty)$ is continuous and increasing and $\int_1^\infty g(t)t^{-2}\mathrm{d}t < \infty$ and then $\lim_{t\longrightarrow\infty} g(t)/t = 0$.

Proof. Suppose instead that $\limsup_{t\longrightarrow\infty} g(t)/t > 0$ and let then $\varepsilon > 0$ and $t_j > 0$ be an increasing sequence such that $t_j \longrightarrow_j \infty$ and for all j we have $g(t_j) > \varepsilon t_j$: then

$$\begin{aligned}\frac{1}{\varepsilon}\int_1^\infty g(t)\,t^{-2}\mathrm{d}t &\geq \sum_{j=1}^\infty \int_{t_j}^{t_{j+1}} t_j t^{-2}\mathrm{d}t \\ &= \sum_{j=1}^\infty t_j\left(\frac{1}{t_j} - \frac{1}{t_{j+1}}\right) \\ &= \sum_{j=1}^\infty \left(\frac{t_{j+1} - t_j}{t_{j+1}}\right) = \infty\end{aligned} \tag{47}$$

where the last step is explained in exercise 11 in Section 3 in [9]. □

The above example can be finetuned as follows.

Proposition 13. *Let $\sigma : [1, \infty) \longrightarrow [1, \infty)$ be a continuous increasing function such that $\sigma(1) = 1$, $\lim_{t\longrightarrow\infty} \sigma(t) = \infty$, and*

$$\int_0^1 \sigma\left(\frac{1}{t}\right)\mathrm{d}t = \int_1^\infty \sigma(t)\,t^{-2}\mathrm{d}t < \infty, \tag{48}$$

and we will build a compact set, following the previous example, such that for $r < 1/2$

$$P\left(\{u_C \leq r\}\right) \geq N\omega_N\sigma\left(\frac{1}{r} - 1\right) - 1. \tag{49}$$

Proof. Let $\tau : [1, \infty) \longrightarrow [1, \infty)$ be the inverse $\tau = \sigma^{-1}$, and then $\tau(1) = 1$ and $\lim_{t\longrightarrow\infty} \tau(t)/t = \infty$. The subgraph

$$\left\{(s, t) : 1 < s < \infty,\ 0 < t < \frac{1}{\tau(s)}\right\} \tag{50}$$

coincides with the subgraph

$$\left\{(s, t) : 0 < t < 1,\ 1 < s < \sigma\left(\frac{1}{t}\right)\right\} \tag{51}$$

so

$$\int_1^\infty \frac{1}{\tau(s)}\mathrm{d}s = \int_0^1 \sigma\left(\frac{1}{t}\right)\mathrm{d}t < \infty. \tag{52}$$

Let $m_h = \lceil\tau(h)\rceil$; note that $m_1 = 1$. Since $1/\tau(t) \geq 1/m_h$ for $h \geq 2$ and $t \in [h-1, h]$ so $\sum_{h=1}^\infty 1/m_h < \infty$; so we can build C as in the previous examples.

When $N \geq 2$ we let $r_h = 4/(4m_h - 2)$ as before; note that $r_h m_h \geq 1$. We define χ as in (43) and note that $k = \chi(r)$ is characterized by

$$r_{k+1} \leq r < r_k. \tag{53}$$

Since $\tau(h)+1\geq m_h\geq\tau(h)$ then

$$\tau(k+1)+1\geq m_{k+1}\geq\frac{1}{r_{k+1}}\geq\frac{1}{r}\tag{54}$$

and hence for $r<1/2$

$$\chi(r)\geq\sigma\left(\frac{1}{r}-1\right)-1.\tag{55}$$

Combining this with relation (46) we obtain (49).

When $N=1$ then we just set $r_h=1/m_h$ and proceed similarly. □

4.1. Proof. We eventually prove Theorem 3.

Proof. Possibly adding a constant to ψ and large disks to C, we assume that $\psi(1/2)=N\omega_N-1$. We relate

$$\begin{aligned}t&=\frac{1}{r}-1,\\ r&=\frac{1}{t+1}\end{aligned}\tag{56}$$

for $t\geq 1$ and $0<r\leq 1/2$; we let

$$\sigma(t)=\frac{1+\psi(1/(t+1))}{N\omega_N}\tag{57}$$

and then $\sigma:[1,\infty)\longrightarrow[1,\infty)$ is continuous and increasing, $\sigma(1)=1$, $\lim_{t\longrightarrow\infty}\sigma(t)=\infty$, and

$$\begin{aligned}\int_0^1\sigma\left(\frac{1}{t}\right)\mathrm{d}t&=\int_0^1\frac{1+\psi(t/(t+1))}{N\omega_N}\mathrm{d}t\\ &=\frac{1}{N\omega_N}+\frac{1}{N\omega_N}\int_0^{1/2}\psi(s)\frac{1}{(s-1)^2}\mathrm{d}s\\ &<\infty\end{aligned}\tag{58}$$

so (48) is satisfied. By Lemma 12 $\lim_{r\longrightarrow 0}r\psi(r)=0$. We then build C using the previous examples; by (55) and (46)

$$\begin{aligned}P(\{u_C\leq r\})&\geq N\omega_N\chi(r)\geq N\omega_N\sigma\left(\frac{1}{r}-1\right)-1\\ &=\psi(r).\end{aligned}\tag{59}$$

□

5. Alternative Proof

We now sketch a different proof of Theorem 1.

5.1. Standard Results. We will need these standard results in Geometric Measure Theory. In the following $\mathbb{1}_E$ will be the characteristic function of a set E.

Proposition 14 (Fleming–Rishel coarea formula [10]). *Let $\Omega\subseteq\mathbb{R}^N$ be open and $f:\Omega\longrightarrow\mathbb{R}$ be locally integrable, and then*

$$\|Df\|_\Omega=\int_{\mathbb{R}}P(\{f\leq t\},\Omega)\,\mathrm{d}t\tag{60}$$

where

$$\begin{aligned}\|Df\|_\Omega&\overset{\text{def}}{=}\sup\Big\{\int_\Omega f\operatorname{div}(g)\,\mathrm{d}x:\\ &\quad g\in C_c^\infty(\Omega\longrightarrow\mathbb{R}^n),\ |g(x)|\leq 1\ \forall x\in\mathbb{R}^N\Big\}\end{aligned}\tag{61}$$

is the total mass of the distribution Df and

$$P(E,\Omega)\overset{\text{def}}{=}\|D\mathbb{1}_E\|_\Omega\tag{62}$$

is the perimeter of a Borel set E inside Ω.

(For a proof, see also Theorem 3.40 in [11]). We will write $P(E)$ for $P(E,\mathbb{R}^N)$.

Proposition 15 (Federer coarea formula [3]). *Let $\Omega\subseteq\mathbb{R}^N$ be a Borel set and $f,g:\Omega\longrightarrow\mathbb{R}$ with f Lipschitzian and g integrable, and then*

$$\begin{aligned}&\int_\Omega g(x)|\nabla f(x)|\,\mathrm{d}x\\ &\quad=\int_{\mathbb{R}}\left(\int_{\{f=t\}\cap\Omega)}g(y)\,\mathrm{d}\mathscr{H}^{N-1}(y)\right)\mathrm{d}t.\end{aligned}\tag{63}$$

In particular

$$\int_\Omega|\nabla f(x)|\,\mathrm{d}x=\int_{\mathbb{R}}\mathscr{H}^{N-1}(\{f=t\}\cap\Omega)\,\mathrm{d}t.\tag{64}$$

Proposition 16. *Let $\Omega\subseteq\mathbb{R}^N$ be an open set; let E be Borel and $P(E,\Omega)<\infty$. Let ∂^*E be the reduced boundary of E and ν be the inward normal, and then*

$$D\mathbb{1}_E=\nu|D\mathbb{1}_E|.\tag{65}$$

For definitions and proofs see, e.g., those in Section 3.5 in [11] or in Section 3 in [8].

Proposition 17. *Let $\Omega\subseteq\mathbb{R}^N$ be an open set; let E be Borel and $P(E,\Omega)<\infty$. Let ∂^*E be the reduced boundary of E, and then*

(i) $\overline{\partial^*E}=\partial E$;

(ii)

$$P(E,A)=\int_A|D\mathbb{1}_E|=\mathscr{H}^{n-1}(A\cap\partial^*E)\tag{66}$$

for any $A\subseteq\Omega$ open.

These results are due to De Giorgi [12]; for a proof see, e.g., in Section 3.5 in [11] or in Section 4 in [8]. Combining the above with the Fleming–Rishel coarea formula we obtain the following.

Proposition 18 (structure theorem). *If E is a Borel set such that $P(E)<\infty$ then*

$$\int_E\partial_{x_i}g(x)\,\mathrm{d}x=-\int_{\partial^*E}g(x)\nu_i(x)\,\mathrm{d}\mathscr{H}^{n-1}(x)\tag{67}$$

for all $g\in C_0^1(\mathbb{R}^N)$ and $i=1,\ldots,n$, where ∂^*E is the reduced boundary and $\nu(x)$ is the inward normal. (For a definition, see also Theorem 3.54 in [11]).

5.2. Lemmas

Lemma 19. *Let Ω be open; suppose that $f : \Omega \longrightarrow \mathbb{R}$ is Lipschitz; let $A \subseteq \Omega$ open. For almost all $r > 0$ we have that*

$$P(\{f \le r\}, A) = \mathscr{H}^{N-1}(\partial^* \{f = t\} \cap A) = \mathscr{H}^{N-1}(\{f = t\} \cap A). \tag{68}$$

The proof follows comparing the Fleming–Rishel and the Federer coarea formula.

Lemma 20. *Let $\Phi : \mathbb{R}^N \longrightarrow [0,\infty)$ be an integrable compactly supported function. Let $\phi : \mathbb{R}^N \longrightarrow \mathbb{R}$ be convex. Then the convolution $\Phi * \phi$ is convex.*

This is easily proved, e.g., by showing that the derivatives of $t \in \mathbb{R} \longmapsto (\Phi * \phi)(x + tv)$ are monotone nondecreasing.

Let C be compact. It is well known that u_C^2 is semiconcave; we provide a simple quantitative proof.

Lemma 21. *$u_C^2(x) - |x|^2$ is concave.*

Proof. Indeed

$$\varphi(x) = u_C^2(x) - |x|^2 = \min_{y\in C}\left(|x-y|^2 - |x|^2\right) = \min_{y\in C}|x-y|^2 - |x|^2 = \min_{y\in C}\left(-2x\cdot y + |y|^2\right) \tag{69}$$

but

$$x \longmapsto \left(-2x\cdot y + |y|^2\right) \tag{70}$$

is affine hence φ is concave. □

Lemma 22. *Let again $\psi(x) = u_C^2(x)$; it is well known that*

(i) *$\nabla u_C(x)$ and $\psi(x)$ are locally Lipschitz (by direct proof)*

(ii) *hence for almost any x the differentials exist,*

(iii) *when these functions are differentiable and $x \notin C$ then*

(a) *$|\nabla u_C(x)| = 1$ and $|\nabla\psi(x)| = 2\psi(x)$,*

(b) *the point y of C closest to x is unique and is $y = x - \nabla\psi(x)/2$;*

for these last two, see in [13] (in particular Theorem 3.1 and Remark 3.6).

5.3. Proof. Here is then a "geometric measure theory proof" of Theorem 1.

Proof. In the first part of this proof we assume that $P(C) < \infty$.

Let $F_t = \{u_C \le t\}$ and $E_t = \{u_C = t\}$ in the following. The idea of the proof is as follows. Set $\psi(x) = u_C^2(x)$. Let $K = \{0 < u_C \le r\} = F_r \setminus C$, supposing that the boundary of K is smooth (in this case $\partial F_t = E_t$.), and we note that $\partial K = \partial C \cup \partial F_r$; we have that $\nabla\psi(x) = 0$ for $x \in \partial C$ whereas $\nabla\psi(x) = -2u_C(x)\nu = -2r\nu(x)$ for $x \in \partial F_r$, where $\nu(x)$ is the inward normal to ∂F_r; so we use the Gauss–Green formula and write

$$2N\mathscr{L}(K) \ge \int_K \Delta\psi(x)\,\mathrm{d}x = -\int_{E_t}\sum_{i=1}^{n}\partial_{x_i}\psi(x)\,\nu_i(x)\,\mathrm{d}\mathscr{H}^{N-1}(x) = 2r\mathscr{H}^{n-1}(E_t), \tag{71}$$

where the first inequality follows from Lemma 21 that implies that $\Delta(u_C^2) \le 2N$ in the sense of distributions.

We now provide the general proof.

We sketch the following facts:

(1) For almost all $t > 0$ we have that $P(F_t) < \infty$, by the coarea formula. (Note that, by the proof in Section 3, this is actually true *for every* $t > 0$.)

(2) $\partial F_t \subseteq E_t$ by Proposition 9, so $\partial^* F_t \subseteq E_t$ for any $t > 0$ such that $P(F_t) < \infty$.

(3) By Lemma 19 for almost any $r > 0$ we have $\mathscr{H}^{n-1}(E_t \setminus \partial^* F_t) = 0$.

(4) Hence for almost any $r > 0$ and $\mathscr{H}^{n-1}$–almost any $x \in E_r$ we have that

$$\nabla u_C(x) = \frac{\nabla\psi(x)}{(2\psi(x))} = -\nu(x) \tag{72}$$

that is the outward normal. This follows from the existence of a "weak" tangent space of $\partial^* F_r$ near x; see Theorem 3.8 in [8].

(5) Let $\Phi_h : \mathbb{R}^N \longrightarrow [0,\infty)$ a family of C^∞ mollifiers, with support in $D_{1/h}$ and $\int_{\mathbb{R}^N}\Phi_h(x)\mathrm{d}x = 1$. Let $\psi_h \stackrel{\text{def}}{=} \psi * \Phi_h$: then (possibly passing to a subsequence) for almost all $t > 0$ we have

$$\lim_{h\longrightarrow\infty}\int_{\partial^* F_t}\left|\nabla\psi_h - \nabla\psi\right|\mathrm{d}\mathscr{H}^{n-1}(x) = 0. \tag{73}$$

Indeed we know that $\lim_{h\longrightarrow\infty}\nabla\psi_h = \nabla\psi$ in $L^1(F_R)$ (for any R large), and we use the coarea formula

$$\lim_{h\longrightarrow\infty}\int_0^R g_h(t)\,\mathrm{d}t = 0, \quad g_h(t) = \int_{\partial^* F_t}\left|\nabla\psi_h - \nabla\psi\right|\mathrm{d}\mathscr{H}^{n-1}(x)) \tag{74}$$

and then (up to a subsequence) $g_h(t) \longrightarrow_h 0$ for almost all $t > 0$.

Let $r > 0$ be such that all above properties are satisfied. Let $K = \{0 < u_C \le r\} = F_r \setminus C$. Note that $P(K) < \infty$ (since the boundaries of C and F_r are separated); and the essential boundary of K is the union of the essential boundaries of F_r

and of C. For any point y in the topological boundary of C, $\lim_{x\longrightarrow y}\nabla\psi(x)=0$, and then

$$\int_{\partial^* C}\left|\nabla\psi_h(x)\right| \mathrm{d}\mathscr{H}^{n-1}(x)\longrightarrow_h 0. \tag{75}$$

By Proposition 18

$$\begin{aligned}\int_K \Delta\psi_h(x)\,\mathrm{d}x &= \sum_{i=1}^n \int_K \partial_{x_i x_i}\psi_h(x)\,\mathrm{d}x \\ &= -\int_{\partial^* K}\sum_{i=1}^n \partial_{x_i}\psi_h(x)\,\nu_i(x)\,\mathrm{d}\mathscr{H}^{n-1}(x);\end{aligned} \tag{76}$$

the right hand side passes to the limit (by (75) and by (73))

$$\begin{aligned}\lim_{h\longrightarrow\infty} & -\int_{\partial^* K}\sum_{i=1}^n \partial_{x_i}\psi_h(x)\,\nu_i(x)\,\mathrm{d}\mathscr{H}^{n-1}(x) \\ &= -\int_{\partial^* F_r}\sum_{i=1}^n \partial_{x_i}\psi(x)\,\nu_i(x)\,\mathrm{d}\mathscr{H}^{n-1}(x) \\ &= \int_{\partial^* F_r} 2r\mathrm{d}\mathscr{H}^{n-1}(x) = 2r\mathscr{H}^{n-1}\left(\partial^* F_r\right) \\ &= 2rP\left(F_r\right).\end{aligned} \tag{77}$$

For the left hand side of (76) we proceed as follows. By Lemma 21, $\varphi(x)=\psi(x)-|x|^2$ is concave, and let

$$\varphi_h = \Phi_h * \varphi = \psi_h - \Phi_h * |x|^2; \tag{78}$$

by Lemma 20 φ_h is concave and smooth so $\nabla\varphi_h \le 0$

$$0 \ge \int_K \Delta\varphi_h \mathrm{d}x = \int_K \Delta\psi_h \mathrm{d}x - \int_K \Delta\left(\Phi_h * |x|^2\right)\mathrm{d}x; \tag{79}$$

then

$$\begin{aligned}\lim_{h\longrightarrow\infty} & \int_K \Delta\left(\Phi_h * |x|^2\right)\mathrm{d}x \\ &= 2N \lim_{h\longrightarrow\infty}\int_{\mathbb{R}^N}\Phi_H * \mathbb{1}_K \mathrm{d}x = 2N\mathscr{L}(K)\end{aligned} \tag{80}$$

so

$$\limsup_{h\longrightarrow\infty}\int_K \Delta\psi_h \mathrm{d}x \le 2N\mathscr{L}(K). \tag{81}$$

This ends the first part of the proof. Assuming that $P(C)<\infty$, we have proved the required result (2); namely,

$$rP\left(\{u_C \le r\}\right) \le N\mathscr{L}\left(\{0<u_C\le r\}\right), \tag{82}$$

for almost any $r>0$, i.e., for all $r>0, r\notin N$ where $N\subseteq\mathbb{R}$ is a negligible set. Let then $r>0, r\in N$, and then there is a sequence $r_h \searrow_h r$ with $r_h\notin N$; by Proposition 9 (point (2)) and the lower semicontinuity of the perimeter we deduce that (2) holds for r as well.

We eventually extend (2) to the general case. Suppose C is compact and $P(C)=\infty$ and then there is a sequence C_h of compact sets with smooth boundary such that $d_H(C_h,C)\longrightarrow_h 0$. Indeed we may define $C_h=\{x:\Phi_h * \mathbb{1}_K(x)\ge\tau\}$ where τ is a regular value and then use Sard's theorem. By Lemma 10 and the lower semicontinuity of the perimeter we deduce that (2) holds. □

5.4. Almgren Taylor Wang. It is worth mentioning that the above proof follows closely part of the proof by Ambrosio (lectures notes *Minimizing Movements* held in Padova, June 20-24, 1994) of this result by Almgren, Taylor, and Wang [14].

Theorem 23. *Let $C\subset\mathbb{R}^N$ be a compact set; suppose that there are $\theta>0,\tau>0$ such that*

$$\mathscr{H}^{n-1}\left(C\cap B_r(x)\right)\ge\theta\rho^{n-1}\quad \forall x\in C,\ \rho\in(0,\tau). \tag{83}$$

Then for all $R>\tau$

$$\operatorname*{ess-sup}_{0<r<R}\mathscr{H}^{N-1}\left(\{u_C=r\}\right)\le\alpha R^{N-1}\mathscr{H}^{N-1}(C) \tag{84}$$

where $\alpha=\gamma\theta^{-1}\tau^{1-N}$ and $\gamma>0$ is an appropriate positive constant dependent only on the dimension N.

6. Banach-Like Metric Spaces of Compact Sets

Let $\mathscr{M}$ be the family of all nonempty compact subsets of $\mathbb{R}^N$. In 2015 Duci and Mennucci [1] studied a family of distances $d_{p,\varphi}$ on $\mathscr{M}$, where $p\in[1,\infty)$ ([1] also includes the case $p=\infty$, which will not be considered here) and $\varphi:[0,\infty)\longrightarrow(0,\infty)$ is a fixed given function. The distance was then defined, for $A,B\in\mathscr{M}$, as

$$d_{p,\varphi}(A,B)\stackrel{\text{def}}{=}\left\|\varphi\circ u_A-\varphi\circ u_B\right\|_{L^p(\mathbb{R}^N)}. \tag{85}$$

Lemma 24 (Lemma 6.2 in [1]). *Suppose that φ is monotonically strictly decreasing and*

$$\varphi(|x|)\in L^p\left(\mathbb{R}^N\right). \tag{86}$$

Let $C\subseteq\mathbb{R}^N$ be closed and nonempty; then the following are equivalent:

(a) $\varphi\circ u_C\in L^p(\mathbb{R}^N)$.

(b) *C is bounded (and then C is compact).*

So, for any φ satisfying the above hypotheses, the formula in (85) would properly define a distance on $\mathscr{M}$.

6.1. Further Results on $(\mathscr{M},d_{p,\varphi})$. Assuming appropriate hypotheses on φ, [1] proved many interesting properties of the metric space $(\mathscr{M},d_{p,\varphi})$.

Hypotheses 25 (Hypotheses 6.1 in [1]). Suppose that φ is monotonically strictly decreasing and of class C^1, such that

$$\varphi(|x|)\in L^p\left(\mathbb{R}^N\right). \tag{87}$$

The resulting metric spaces enjoy many interesting properties.

(i) $d_{p,\varphi}$ is Euclidean invariant.

(ii) The topology induced by $d_{p,\varphi}$ over the space $\mathscr{M}$ coincides with the topology induced by the Hausdorff distance (Theorem 6.11 in [1]).

(iii) The metric space $(\mathscr{M},d_{p,\varphi})$ is complete (Theorem 6.12 in [1]).

6.2. Induced Distance and Geodesics. Given a metric space (M, d), we define the *length*$\text{Len}^d\gamma$ of a continuous path $\gamma : [\alpha, \beta] \longrightarrow M$ by using the *total variation*

$$\text{Len}^d\gamma \stackrel{\text{def}}{=} \sup_T \sum_{i=1}^{n} d\left(\gamma\left(t_{i-1}\right), \gamma\left(t_i\right)\right), \tag{88}$$

where the supremum is computed over all finite subsets $T = \{t_0, \ldots, t_n\}$ of $[\alpha, \beta]$ and $t_0 \leq \cdots \leq t_n$. We then define the *induced distance*d^g by

$$d^g\left(x, y\right) \stackrel{\text{def}}{=} \inf_\gamma \text{Len}^d\gamma, \tag{89}$$

where the infimum is taken in the class of all continuous paths γ connecting x to y. If the infimum is a minimum, the path providing the minimum is called a *geodesic*.

With further hypotheses on $\varphi(t)$, more can be said on the metric space $(\mathscr{M}, d_{p,\varphi})$.

Hypotheses 26 (Hypotheses 6.17 in [1]). Let φ be as defined in Hypotheses 25. We moreover suppose that there is a constant $T > 0$ such that $\varphi(t)$ is convex for $t \geq T$, and

$$\varphi'\left(|x|\right) \in L^p\left(\mathbb{R}^{\mathrm{N}}\right). \tag{90}$$

Then these results follow.

(i) The space $(\mathscr{M}, d_{p,\varphi})$ is Lipschitz–arc connected (Prop. 6.18 in [1]).

(ii) For any $\rho > 0$, $A \in \mathscr{M}$,

$$\mathbb{D}^g\left(A, \rho\right) \stackrel{\text{def}}{=} \left\{B \in \mathscr{M} \mid d^g_{p,\varphi}\left(A, B\right) \leq \rho\right\} \tag{91}$$

is compact in the $(\mathscr{M}, d)$ topology (where $d^g_{p,\varphi}$ is the distance induced by $d_{p,\varphi}$) (Theorem 6.20 in [1]).

(iii) Hence any two compact sets are connected by a geodesic in $(\mathscr{M}, d_{p,\varphi})$.

(iv) When $p \in (0, \infty)$ there is a variational description of the geodesics (Sec. 6.4 in [1]), a weak formulation of a *"tangent bundle"* (Sec. 6.5 in [1]).

(v) For $p = 2$ there is an interpretation of the metric $d^g_{p,\varphi}$ as a *"Riemannian metric"* when smoothly deforming smooth boundaries of compact sets (Sec. 6.6 in [1]).

6.3. Open Problem. All the above properties are quite interesting for their applications in Shape Analysis and Optimization. In this context, the set of Hypotheses 25 and Hypotheses 26 can be satisfied by simple functions such as $\varphi(t) = \exp(-t)$ or $\varphi(t) = (1 + t)^{-\alpha}$ for $\alpha > N/p$.

From a theoretical point of view, it may be interesting instead to understand more in detail which hypotheses are strictly needed to prove each of the above statements. This requires a deeper understanding of the properties of the distance function and the fattened sets. The theorems in this paper are a step in this direction.

In proving Lemma 24 it was assumed that φ be *monotonically strictly decreasing*. This was a useful ingredient in writing a simple direct proof of the implication $b \Longrightarrow a$. Some geometrical and analytical considerations though suggested that this hypothesis was not strictly needed.

Indeed as a consequence of Theorem 1 we can now prove Theorem 4.

Proof. By Lemma 19 and Theorem 1 for almost any $r > 0$,

$$r\mathscr{H}^{N-1}\left(\left\{u_C = r\right\}\right) \leq N\mathscr{L}^N\left(\left\{u_C \leq r\right\}\right). \tag{92}$$

Let then $\overline{r} > 0$ be such that $C \subseteq D_{\overline{r}}(0)$, and then $\{u_C = r\} \subseteq D_{(r+\overline{r})}$; so for almost all $r > 0$

$$\mathscr{H}^{N-1}\left(\left\{u_C = r\right\}\right) \leq \frac{N}{r}\left(r + \overline{r}\right)^N. \tag{93}$$

Let $\varphi : [0, \infty) \longrightarrow [0, \infty)$ be as in hypotheses; we know (Lemma 22) that $|\nabla u_C(x)| = 1$ for almost all $x \in \mathbb{R}^N \setminus C$, and then by the Federer coarea formula Proposition 15

$$\begin{aligned}
&\int_{\{u_C(x)>\varepsilon\}} \varphi\left(u_C\left(x\right)\right)^p \, \mathrm{d}x \\
&= \int_{\{u_C(x)>\varepsilon\}} \varphi\left(u_C\left(x\right)\right)^p \left|\nabla u_C\left(x\right)\right| \mathrm{d}x \\
&= \int_\varepsilon^\infty \left(\int_{\{u_C=t\}\cap\Omega)} \varphi\left(u_C\left(x\right)\right)^p \, \mathrm{d}\mathscr{H}^{N-1}\left(y\right)\right) \mathrm{d}t \\
&= \int_\varepsilon^\infty \varphi\left(t\right)^p \mathscr{H}^{N-1}\left(\left\{u_C = t\right\}\right) \mathrm{d}t \\
&\leq \int_\varepsilon^\infty \varphi\left(t\right)^p \frac{N}{t}\left(t + \overline{r}\right)^N \mathrm{d}t < \infty
\end{aligned} \tag{94}$$

by relation (6), whereas

$$\int_{\{u_C(x)\leq\varepsilon\}} \varphi\left(u_C\left(x\right)\right)^p \, \mathrm{d}x < \infty \tag{95}$$

by (7); so $\varphi \circ u_C \in L^p(\mathbb{R}^N)$. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares no conflicts of interest.

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